Lifting Tropical Intersections

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Abstract. We show that points in the intersection of the tropicalizations of subvarieties of a torus lift to algebraic intersection points with expected multiplicities, provided that the tropicalizations intersect in the expected dimension. We also prove a similar result for intersections inside an ambient subvariety of the torus, when the tropicalizations meet inside a facet of multiplicity 1. The proofs require not only the geometry of compactified tropicalizations of subvarieties of toric varieties, but also new results about the geometry of finite type schemes over non-noetherian valuation rings of rank 1. In particular, we prove subadditivity of codimension and a principle of continuity for intersections in smooth schemes over such rings, generalizing well-known theorems over regular local rings. An appendix on the topology of finite type morphisms may also be of independent interest.

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1. Introduction

Tropical geometry studies the valuations of solutions to polynomial equations, and may be thought of as a generalization of the theory of Newton polygons to multiple polynomials in multiple variables. It is natural to consider what conditions guarantee that if two closed subvarieties X and X' in a torus T have points with the same valuation, then $X \cap X'$ contains a point of the given valuation. In the context of tropical geometry, this is the question of when tropicalization commutes with intersection. The tropicalization of the intersection of two closed subvarieties of T over a nonarchimedean field is contained in the intersection of their tropicalizations, but this containment is sometimes strict. For instance, if X is smooth in characteristic zero, and X' is the translate of X by a general torsion point t in T, then X and X' are either disjoint or meet transversely, but their tropicalizations are equal. The dimension of $X \cap X'$ is then strictly less than the dimension of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, so there are tropical intersection points that do not lift to algebraic intersection points. In other cases, such as Example 6.2 below, the tropicalizations meet in a positive dimensional set that does not contain the tropicalization of any positive dimensional variety, so at most finitely many of the tropical intersection points can lift to algebraic intersection points. Our main result, in its most basic form, says that the dimension of the tropical intersection is the only obstruction to such lifting.

Following standard terminology from intersection theory in algebraic geometry, we say that $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ meet *properly* at a point w if $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ has codimension $\operatorname{codim} X + \operatorname{codim} X'$ in a neighborhood of w. We say that $\mathrm{Trop}(X)$ meets $\mathrm{Trop}(X')$ properly if they meet properly at every point of their intersection, which may be empty.[1]

Theorem 1.1. *Suppose* $\mathrm{Trop}(X)$ *meets* $\mathrm{Trop}(X')$ *properly at* w. *Then* w *is contained in the tropicalization of* $X \cap X'$.

In other words, tropicalization commutes with intersections when the intersections have the expected dimension. This generalizes a well-known result of Bogart, Jensen, Speyer, Sturmfels, and Thomas, who showed that tropicalization commutes with intersection when the tropicalizations meet transversely [BJS+07, Lemma 3.2]. Our proof, and the proofs of all of the other main results below, involves a reduction to the case where the base field is complete with respect to its nonarchimedean norm and the point w is rational over the value group. The reduction is based on the results of Appendix A, and should become a standard step in the rigorous application of tropical methods to algebraic geometry. We emphasize that the theorems hold in full generality, over an arbitrary algebraically closed nonarchimedean field, and at any real point in the intersection of the tropicalizations.

Although Theorem 1.1 improves significantly on previously known results, it is still too restrictive for the most interesting potential applications. Frequently, X and X' are closed subvarieties of an ambient variety Y inside the torus. In this case, one cannot hope that $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ will meet properly in the above sense. Instead, we say that the tropicalizations $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ meet properly at a point w in $\mathrm{Trop}(Y)$ if the intersection

$$\mathrm{Trop}(X) \cap \mathrm{Trop}(X') \subset \mathrm{Trop}(Y)$$

has pure codimension $\mathrm{codim}_Y X + \mathrm{codim}_Y X'$ in a neighborhood of w. We extend Theorem 1.1 to proper intersections at suitable points of $\mathrm{Trop}(Y)$, as follows.

Recall that $\mathrm{Trop}(Y)$ is the underlying set of a polyhedral complex of pure dimension $\dim Y$, with a positive integer multiplicity assigned to each facet, by which we mean a maximal face. We say that a point in $\mathrm{Trop}(Y)$ is simple if it is in the interior of a facet of multiplicity 1.

Theorem 1.2. *Suppose* $\mathrm{Trop}(X)$ *meets* $\mathrm{Trop}(X')$ *properly at a simple point* w *in* $\mathrm{Trop}(Y)$. *Then* w *is contained in the tropicalization of* $X \cap X'$.

Every point in the tropicalization of the torus is simple, so Theorem 1.1 is the special case of Theorem 1.2 where Y is the full torus T. The hypothesis that w be simple is necessary; see Section 6. We strengthen Theorem 1.2 further by showing that where $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ meet properly, the facets of the tropical intersection appear with the expected multiplicities, suitably interpreted. See Theorem 5.1.1 for a precise statement.

[1] Codimension is subadditive for intersections of tropicalizations, which means that $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ has codimension at most $j + j'$ at every point, so $\mathrm{Trop}(X)$ meets $\mathrm{Trop}(X')$ properly if the intersection is either empty or has the smallest possible dimension. See Proposition 3.2.1.

The proof of Theorem 1.2 is in two steps. Roughly speaking, we lift first from tropical points to points in the initial degeneration, and then from the initial degeneration to the original variety. Recall that the tropicalization of a closed subvariety X of T is the set of weight vectors w such that the initial degeneration X_w is nonempty in the torus torsor T_w over the residue field.

Theorem 1.3. *Suppose* $\mathrm{Trop}(X)$ *meets* $\mathrm{Trop}(X')$ *properly at a simple point* w *in* $\mathrm{Trop}(Y)$. *Then* X_w *and* X'_w *have nonempty proper intersection in the smooth variety* Y_w.

If w is a simple point of $\mathrm{Trop}(Y)$ then standard arguments show that X_w meets X'_w properly at every point in the intersection; the main content of the theorem is that $X_w \cap X'_w$ is nonempty. The proof, given in Section 3, uses extended tropicalizations and the intersection theory of toric varieties. In Section 4, we develop geometric techniques over valuation rings of rank 1 that permit lifting of such proper intersections at smooth points from the special fiber to the generic fiber.

Theorem 1.4. *Suppose* X_w *meets* X'_w *properly at a smooth point* x *of* Y_w. *Then* x *is contained in* $(X \cap X')_w$.

In particular, if X_w meets X'_w properly at a smooth point of Y_w then w is contained in $\mathrm{Trop}(X \cap X')$. If w is defined over the value group of the nonarchimedean field then surjectivity of tropicalization says that x can be lifted to a point of $X \cap X'$. The dimension theory over valuation rings of rank 1 developed in Section 4 also gives a new proof of surjectivity of tropicalization, as well as density of tropical fibers.

Theorem 1.4 is considerably stronger than Theorems 1.1 and 1.2. It often happens that initial degenerations meet properly even when tropicalizations do not. The proof of this theorem follows standard arguments from dimension theory, but the dimension theory that is needed is not standard because the valuation rings we are working with are not noetherian. We develop the necessary dimension theory systematically in Section 4, using noetherian approximation to prove a version of the Krull principal ideal theorem and then deducing subadditivity of codimension for intersections in smooth schemes of finite type over valuation rings of rank 1. Furthermore, we prove a principle of continuity, showing that intersection numbers are well-behaved in families over valuation rings of rank 1. In Section 5, we apply this principle of continuity to prove a stronger version of Theorem 1.2 with multiplicities and then extend all of these results to proper intersections of three or more subvarieties of T. One special case of these lifting results with multiplicities, for complete intersections, has been applied by Rabinoff in an arithmetic setting to construct canonical subgroups for abelian varieties over p-adic fields [Rab12a].

Remark 1.5. In addition to their intrinsic appeal, our results are motivated by the possibility of applications to proving correspondence theorems such as the one proved by Mikhalkin for plane curves [Mik05]. Mikhalkin considers curves of fixed degree and genus subject to constraints of passing through specified points. He shows, roughly speaking, that a tropical plane curve that moves in a family of the expected dimension passing through specified points lifts to a predictable number of algebraic curves passing through prescribed algebraic points. Here, we consider points subject to the constraints of lying inside closed subvarieties X and X' and prove the analogous correspondence—if a tropical point moves in a family of the expected dimension inside $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ then it lifts to a predictable number of points in $X \cap X'$. One hopes that Mikhalkin's correspondence theorem will eventually be reproved and generalized using Theorem 1.2 on suitable tropicalizations of moduli spaces of curves. Similarly, one hopes that Schubert problems can be answered tropically using Theorem 1.2 on suitable tropicalizations of Grassmannians and flag varieties. The potential for such applications underlines the importance of having the flexibility to work with intersections inside ambient subvarieties of the torus.

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2. Preliminaries

Let K be an algebraically closed field, and let $\nu : K^* \to \mathbb{R}$ be a valuation with value group G. Let R be the valuation ring, with maximal ideal $\mathfrak{m}$, and residue field $k = R/\mathfrak{m}$. Since K is algebraically closed, the residue field k is algebraically closed and the valuation group G is divisible. In particular, G is dense in $\mathbb{R}$ unless the valuation ν is trivial, in which case G is zero. Typical examples of such nonarchimedean fields in equal characteristic are given by the generalized power series fields $K = k((t^G))$, whose elements are formal power series with coefficients in the algebraically closed field k and exponents in G, where the exponents occurring in any given series are required to be well-ordered [Poo93].

Let $\mathcal{T}$ be an algebraic torus of dimension n over R, with character lattice $M \cong \mathbb{Z}^n$, and let $N = \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice. We write N_G for $N \otimes G$, and $N_\mathbb{R}$ for $N \otimes \mathbb{R}$, the real vector space of linear functions on the character lattice. We treat M and N additively and write x^u for the character associated to a lattice point u in M, which is a monomial in the Laurent polynomial ring $K[M]$. We write T for the associated torus over K.

In this section, we briefly review the basic properties of initial degenerations and tropicalizations, as well as the relationship between tropical intersections and intersections in suitable toric compactifications of T.

2.1. Initial degenerations. Each vector w in $N_{\mathbb{R}}$ determines a weight function on monomials, where the w-weight of ax^u is $\nu(a) + \langle u, w\rangle$. The tilted group ring $R[M]^w \subset K[M]$ consists of the Laurent polynomials $\sum a_u x^u$ in which every monomial has nonnegative w-weight. We then define $\mathcal{T}^w = \operatorname{Spec} R[M]^w$. If ν is nontrivial, then $\mathcal{T}^w$ is an integral model of T, which means that it is a scheme over R whose generic fiber $\mathcal{T}^w \times_{\operatorname{Spec} R} \operatorname{Spec} K$ is naturally identified with T. In general, the scheme $\mathcal{T}^w$ carries a natural $\mathcal{T}$-action, and is a $\mathcal{T}$-torsor if w is in N_G.

Remark 2.1.1. If ν is nontrivial and w is not in N_G, or if ν is trivial and w does not span a rational ray, then $\mathcal{T}^w$ is not of finite type over $\operatorname{Spec} R$. Some additional care is required in handling these schemes, but no major difficulties arise for the purposes of this paper. The special fiber $\mathcal{T}^w_k$ is still finite type over k, and is a torsor over a quotient torus of T_k. The basic properties of the schemes $\mathcal{T}^w$ and their closed subschemes may be understood by passing to a valued extension field with value group G' such that w is in $N_{G'}$ and analyzing how these schemes and their special fibers transform under such extensions. This analysis is carried out in Appendix A. See, in particular, Theorems A.3 and A.4, and Remark A.5.

For arbitrary w in $N_{\mathbb{R}}$, the scheme $\mathcal{T}^w$ is reduced and irreducible, flat over $\operatorname{Spec} R$, and contains T as a dense open subscheme. We define T_w to be the closed subscheme cut out by monomials of strictly positive w-weight. If the valuation is nontrivial, then T_w is the special fiber of $\mathcal{T}^w$.

Let X be a closed subscheme of T, of pure dimension d. Let $\mathcal{X}^w$ denote the closure of X in $\mathcal{T}^w$.

Definition 2.1.2. The initial degeneration X_w is the closed subscheme of T_w obtained by intersecting with $\mathcal{X}^w$.

The terminology reflects the fact that X_w is cut out by residues of initial terms (lowest-weight monomials) of Laurent polynomials in the ideal of X. If the valuation is nontrivial, then $\mathcal{X}^w$ is an integral model of X, and X_w is the special fiber of this model.

One consequence of Gröbner theory is that the space of weight vectors $N_{\mathbb{R}}$ can be decomposed into finitely many polyhedral cells so that the initial degenerations are essentially invariant at points of N_G on the relative interior of each cell, as discussed in more detail in the following subsection. Roughly speaking, the cells are cut out by inequalities whose linear terms have integer coefficients, and whose constant terms are in the value group G. More precisely, the cells are integral G-affine polyhedra, defined as follows.

Definition 2.1.3. An integral G-affine polyhedron in $N_{\mathbb{R}}$ is the solution set of a finite number of inequalities

$$\langle u, v\rangle \leq b,$$

with u in the lattice M and b in the value group G.

An integral G-affine polyhedral complex Σ is a polyhedral complex consisting entirely of integral G-affine polyhedra. In other words, it is a finite collection of integral G-affine polyhedra such that every face of a polyhedron in Σ is itself in Σ, and the intersection of any two polyhedra in Σ is a face of each. Note that if G is zero, then an integral G-affine polyhedron is a rational polyhedral cone, and an integral G-affine polyhedral complex is a fan.

2.2. Tropicalization. Following Sturmfels, we define the tropicalization of X to be

$$\mathrm{Trop}(X) = \{w \in N_{\mathbb{R}} \mid X_w \text{ is nonempty}\}.$$

The foundational theorems of tropical geometry, due to the work of many authors, are the following.[2]

(1) The tropicalization $\mathrm{Trop}(X)$ is the underlying set of an integral G-affine polyhedral complex of pure dimension d.
(2) The integral G-affine polyhedral structure on $\mathrm{Trop}(X)$ can be chosen so that the initial degenerations X_w and $X_{w'}$ are $\mathcal{T}_k$-affinely equivalent for any w and w' in N_G in the relative interior of the same face.
(3) The image of $X(K)$ under the natural tropicalization map $\mathrm{trop} : T(K) \to N_{\mathbb{R}}$ is exactly $\mathrm{Trop}(X) \cap N_G$.

Here, two subschemes of the $\mathcal{T}_k$-torsors T_w and $T_{w'}$ are said to be $\mathcal{T}_k$-affinely equivalent if they are identified under some $\mathcal{T}_k$-equivariant choice of isomorphism $T_w \cong T_{w'}$. In fact, a stronger statement holds: for any points w and w' in N_G, there exists a $\mathcal{T}_k$-equivariant isomorphism $T_w \cong T_{w'}$ which sends X_w to $X_{w'}$ for all X such that w and w' are in the relative interior of the same face of $\mathrm{Trop}(X)$. If the valuation is nontrivial, then it follows from (3) that $\mathrm{Trop}(X)$ is the closure of the image of $X(K)$ under the tropicalization map.

For any extension of valued fields $L|K$, the tropicalization of the base change $\mathrm{Trop}(X_L)$ is exactly equal to $\mathrm{Trop}(X)$ [Pay09a, Proposition 6.1]. In particular, for any algebraically closed extension $L|K$ with nontrivial valuation, $\mathrm{Trop}(X)$ is the closure of the image of $X(L)$. Furthermore, if we extend to some L that is complete with respect to its valuation then the tropicalization map on $X(L)$ extends naturally to a continuous map on the nonarchimedean analytification of X_L, in the sense of Berkovich [Ber90], whose image is exactly $\mathrm{Trop}(X)$ [Pay09a]. It follows, by reducing to the case of a complete field, that $\mathrm{Trop}(X)$ is connected if X is connected, since the analytification of a connected scheme over a complete nonarchimedean field is connected.

Remark 2.2.1. The natural tropicalization map from $T(K)$ to $N_{\mathbb{R}}$ takes a point t to the linear function $u \mapsto \nu \circ \mathrm{ev}_t\, x^u$, and can be understood as a coordinatewise valuation map, as follows. The choice of a basis for M induces isomorphisms

[2] See [BG84], [SS04], and Propositions 2.1.4, Theorem 2.2.1, and Proposition 2.4.5 of [Spe05] for proofs of (1) and (2). The first proposed proof of (3), due to Speyer and Sturmfels, contained an essential gap that is filled by Theorem 4.2.5, below. Other proofs of (3) have appeared in [Dra08, Theorem 4.2], [Gub12, Proposition 4.14] and [Pay09b, Pay12]. See also Remark 4.2.6.

$T \cong (K^*)^n$ and $N_{\mathbb{R}} \cong \mathbb{R}^n$. In such coordinates, the tropicalization map sends $(t_1, \ldots, t_n)$ to $(\nu(t_1), \ldots, \nu(t_n))$.

There is no canonical choice of polyhedral structure on $\mathrm{Trop}(X)$ satisfying (2) in general, but any refinement of such a complex again satisfies (2). Throughout the paper, we assume that such a polyhedral complex with underlying set $\mathrm{Trop}(X)$ has been chosen. We refer to its faces and facets as faces and facets of $\mathrm{Trop}(X)$, and refine the complex as necessary.

The tropicalization of a closed subscheme of a torus torsor over K is well-defined up to translation by N_G. In particular, if the value group is zero, then the tropicalization is well-defined, and is the underlying set of a rational polyhedral fan. An important special case is the tropicalization of an initial degeneration. The valuation ν induces the trivial valuation on the residue field k, and there is a natural identification of $\mathrm{Trop}(X_w)$ with the star of w in $\mathrm{Trop}(X)$ [Spe05, Proposition 2.2.3]. This star is, roughly speaking, the fan one sees looking out from w in $\mathrm{Trop}(X)$; it is constructed by translating $\mathrm{Trop}(X)$ so that w is at the origin and taking the cones spanned by faces of $\mathrm{Trop}(X)$ that contain w.

2.3. Tropical multiplicities. If w and w' are points of N_G in the relative interior of the same cone σ, then X_w and X'_w are $\mathcal{T}_k$-affinely equivalent. In particular, they are isomorphic as schemes, so the sum of the multiplicities of their irreducible components are equal. By Theorem A.4, the sum of the multiplicities of the irreducible components of the initial degeneration at w is invariant under extensions of valued fields, for any w in $N_{\mathbb{R}}$. Since any point in $N_{\mathbb{R}}$ becomes rational over the value group after a suitable extension, it follows that this sum is independent of the choice of w in the relative interior of σ. We will be concerned with this sum only in the case where σ is a facet. For applications of tropical multiplicities at points that are not in the relative interior of a facet, see [BPR11].

Definition 2.3.1. The tropical multiplicity $m(\sigma)$ of a facet σ in $\mathrm{Trop}(X)$ is the sum of the multiplicities of the irreducible components of X_w, for w in the relative interior of σ.

For $w \in N_G$, the multiplicities on the facets of $\mathrm{Trop}(X_w)$ agree with those on the facets of $\mathrm{Trop}(X)$ that contain w. This is because the initial degeneration of X_w at a point in the relative interior of a given facet is $\mathcal{T}_k$-affinely equivalent to the initial degeneration of X at a point in the relative interior of the corresponding facet of $\mathrm{Trop}(X)$. See [Spe05, Proposition 2.2.3] and [Gub12, Proposition 10.9].

Points in the relative interiors of facets of multiplicity 1 will be particularly important for our purposes.

Definition 2.3.2. A simple point in $\mathrm{Trop}(X)$ is a point in the relative interior of a facet of multiplicity 1.

If $w \in N_G$ is a simple point, then the initial degeneration X_w is isomorphic to a d-dimensional torus [Spe05, Proposition 2.2.4], and initial degenerations at arbitrary simple points are isomorphic to tori of dimension at most d. The latter follows from the case where w is in N_G by choosing a suitable extension of valued fields and applying Theorem A.4. In particular, the initial degeneration at an arbitrary simple point is smooth.

The importance of simple points, roughly speaking, is that intersections of tropicalizations of closed subvarieties of X at simple points behave just like intersections in $N_{\mathbb{R}}$ of tropicalizations of closed subvarieties of T. See, for instance, the proof of Theorem 1.2 in Section 5.

2.4. Minkowski weights and the fan displacement rule. We briefly review the language of Minkowski weights and their ring structure, in which the product is given by the fan displacement rule. These tools are the basis of both toric intersection theory, as developed in [FS97], and the theory of stable tropical intersections [RGST05, Mik06], discussed in the following sections. We refer the reader to these original papers for further details. Although the theory of Minkowski weights extends to arbitrary complete fans, we restrict to unimodular fans, corresponding to smooth toric varieties, which suffice for our purposes. Fix a complete unimodular fan Σ in $N_{\mathbb{R}}$.

Definition 2.4.1. A *Minkowski weight* of codimension j on Σ is a function c that assigns an integer $c(\sigma)$ to each codimension j cone σ in Σ, and satisfies the following balancing condition at every codimension $j+1$ cone.

Balancing Condition. *Let τ be a cone of codimension $j+1$ in Σ. Say $\sigma_1, \ldots, \sigma_r$ are the codimension j cones of Σ that contain τ, and let v_i be the primitive generator of the image of σ_i in $N_{\mathbb{R}}/\operatorname{span}(\tau)$. Then c is balanced at τ if*

$$c(\sigma_1)v_1 + \cdots + c(\sigma_r)v_r = 0$$

in $N_{\mathbb{R}}/\operatorname{span}(\tau)$.

Tropicalizations are one interesting source of Minkowski weights. If X is a closed subscheme of pure codimension j in T and $\operatorname{Trop}(X)$ is a union of cones in Σ, then the function c given by

$$c(\sigma) = \begin{cases} m(\sigma) & \text{if } \sigma \subset \operatorname{Trop}(X). \\ 0 & \text{otherwise.} \end{cases}$$

satisfies the balancing condition, and hence is a Minkowski weight of codimension j. This correspondence between tropicalizations and Minkowski weights is compatible with intersection theory in the toric variety $Y(\Sigma)$. See Section 2.7, below.

We write $\operatorname{Mink}^j(\Sigma)$ for the group of Minkowski weights of codimension j on Σ. The direct sum

$$\operatorname{Mink}^*(\Sigma) = \bigoplus_j \operatorname{Mink}^j(\Sigma)$$

naturally forms a graded ring, whose product is given by the fan displacement rule, as follows. Let v be a vector in $N_{\mathbb{R}}$ that is sufficiently general so that for any two cones σ and σ' in Σ, the displaced cone $\sigma' + v$ meets σ properly. The set of such vectors is open and dense in $N_\mathbb{R}$; it contains the complement of a finite union of linear spaces. For each cone σ in Σ, let N_σ denote the sublattice of N generated by $\sigma \cap N$. Note that if σ intersects the displaced cone $\sigma' + v$ then σ and σ' together span $N_\mathbb{R}$, so the index $[N : N_\sigma + N_{\sigma'}]$ is finite.

Fan Displacement Rule. *The product $c \cdot c'$ is the Minkowski weight of codimension $j + j'$ given by*

$$(c \cdot c')(\tau) = \sum_{\sigma,\sigma'} [N : N_\sigma + N_{\sigma'}] \cdot c(\sigma) \cdot c'(\sigma'),$$

where the sum is over cones σ and σ' containing τ, of codimensions j and j' respectively, such that the intersection $\sigma \cap (\sigma' + v)$ is nonempty.

The balancing conditions on c and c' ensure that the product is independent of the choice of displacement vector v, and also satisfies the balancing condition.

2.5. Toric intersection theory. We briefly recall the basics of intersection theory in smooth complete toric varieties. As in the previous section, we fix a complete unimodular fan Σ in $N_\mathbb{R}$. Let $Y(\Sigma)$ be the associated smooth complete toric variety. A codimension j cycle in $Y(\Sigma)$ is a formal sum of codimension j closed subvarieties, with integer coefficients, and we write $A^j(Y(\Sigma))$ for the Chow group of codimension j cycles modulo rational equivalence. Let X be a closed subscheme of pure codimension j in $Y(\Sigma)$. We write $[X]$ for the class it represents in $A^j(Y(\Sigma))$; by definition, this is the sum over the irreducible components Z of X of the length of X along Z times the class $[Z]$.
Intersection theory gives a natural ring structure on the direct sum

$$A^*(Y(\Sigma)) = \bigoplus_j A^j(Y(\Sigma)).$$

If X and X' have complementary dimension, which means that $\dim X + \dim X' = \dim Y(\Sigma)$, then the product $[X] \cdot [X']$ is a zero-dimensional cycle class. The degree of this class, which is the sum of the coefficients of any representative cycle, is denoted $\deg([X] \cdot [X'])$. In particular, if σ is a cone of codimension j in Σ, with $V(\sigma)$ the associated closed T-invariant subvariety, then X and $V(\sigma)$ have complementary dimension. We write c_X for the induced function on codimension j cones, given by

$$c_X(\sigma) = \deg([X] \cdot [V(\sigma)]).$$

The fact that the intersection product respects rational equivalence ensures that c_X satisfies the balancing condition and is therefore a Minkowski weight of codimension j on Σ. The main result of [FS97] then says that there is a natural isomorphism of rings

$$A^*(Y(\Sigma)) \xrightarrow{\sim} \mathrm{Mink}^*(\Sigma)$$

taking the Chow class $[X]$ to the Minkowski weight c_X.

The theory of refined intersections says that the product $[X] \cdot [X']$ of the classes of two pure-dimensional closed subschemes of a smooth variety Y is not only a well-defined cycle class in that smooth variety, but also the Gysin push-forward of a well-defined cycle class in the intersection $X \cap X'$. If X and X' meet properly, then this class has codimension zero, and hence can be written uniquely as a formal sum of the components of $X \cap X'$. The coefficient of a component Z of the intersection is called the INTERSECTION MULTIPLICITY of X and X' along Z, and is denoted $i(Z, X \cdot X'; Y)$. It is a positive integer, less than or equal to the length of the scheme theoretic intersection of X and X' along the generic point of Z, by [Ful98, Proposition 7.1], and can be computed by Serre's alternating sum formula

$$i(Z, X \cdot X'; Y) = \sum_j (-1)^j \operatorname{length}_{\mathcal{O}_{Y,Z}} \operatorname{Tor}^j_{\mathcal{O}_{Y,Z}}(\mathcal{O}_X, \mathcal{O}_{X'}).$$

See [Ful98, Chapter 8] for further details on the theory of refined intersections in smooth varieties.

Definition 2.5.1. If X and X' meet properly in a smooth variety Y, then the REFINED INTERSECTION CYCLE is

$$X \cdot X' = \sum_Z i(Z, X \cdot X'; Y)\, Z,$$

where the sum is over all irreducible components Z of $X \cap X'$.

We will consider such refined intersections inside the torus T, as well as in its smooth toric compactifications. Finally, it is useful to work not only with tropicalizations of closed subschemes of T, but also with tropicalizations of these refined intersection cycles.

Definition 2.5.2. Let $a_1 Z_1 + \cdots + a_r Z_r$ be a pure-dimensional cycle in T, with positive integer coefficients a_i. The TROPICALIZATION of this cycle is the union

$$\operatorname{Trop}(a_1 Z_1 + \cdots + a_r Z_r) = \operatorname{Trop}(Z_1) \cup \cdots \cup \operatorname{Trop}(Z_r)$$

with multiplicities

$$m(\tau) = a_1 m_{Z_1}(\tau) + \cdots + a_r m_{Z_r}(\tau),$$

where $m_{Z_i}(\tau)$ is defined to be zero if $\operatorname{Trop}(Z_i)$ does not contain τ.

2.6. Stable tropical intersections. Many papers have introduced and studied analogues of intersection theory in tropical geometry, including [AR10, BLdM11, Kat12, Mik06, RGST05, Tab08]. Here we are interested in only one basic common feature of these theories, the stable tropical intersection, and its compatibility with intersection theory in toric varieties. The key point is that stable tropical intersections are defined combinatorially, depending only on the polyhedral geometry of the tropicalizations, by a local displacement rule. Compatibility with the multiplication rule for Minkowski weights and algebraic intersections of generic translates is discussed in the following section.

Given closed subschemes X and X' of T, of pure codimensions j and j', respectively, the stable intersection $\operatorname{Trop}(X) \cdot \operatorname{Trop}(X')$ is a polyhedral complex of pure codimension $j + j'$, with support contained in the set-theoretic intersection $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, and with appropriate multiplicities on its facets, satisfying the balancing condition. Many different closed subschemes of T may have the same tropicalization, but the stable intersection depends only on the tropicalizations.

Suppose the valuation is trivial, so $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ are fans, and let τ be a face of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ of codimension $j + j'$ in $N_{\mathbb{R}}$. The TROPICAL INTERSECTION MULTIPLICITY of $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ along τ, is given by the fan displacement rule

$$i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X')) = \sum_{\sigma, \sigma'} [N : N_\sigma + N_{\sigma'}] \cdot m(\sigma) \cdot m(\sigma'),$$

where the sum is over facets σ and σ' of $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$, respectively, such that the intersection $\sigma \cap (\sigma' + v)$ is nonempty, and $v \in N$ is a fixed generic displacement vector, as in Section 2.4.

In the general case, where the valuation may be nontrivial, the tropical intersection multiplicities are defined similarly, by a local displacement rule, as follows. Let τ be a face of codimension $j + j'$ in $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, and for each face σ containing τ, let N_σ be the sublattice of N parallel to the affine span of σ.

Definition 2.6.1. The tropical intersection multiplicity $i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'))$ is given by the LOCAL DISPLACEMENT RULE

$$i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X')) = \sum_{\sigma, \sigma'} [N : N_\sigma + N_{\sigma'}] \cdot m(\sigma) \cdot m(\sigma'),$$

where $v \in N$ is a generic displacement vector, as above, and the sum is over facets σ and σ' of $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$, respectively, such that the intersection $\sigma \cap (\sigma' + \epsilon v)$ is nonempty for ϵ sufficiently small and positive.

Definition 2.6.2. The STABLE TROPICAL INTERSECTION, denoted $\mathrm{Trop}(X) \cdot \mathrm{Trop}(X')$, is the union of those faces τ such that $i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'))$ is positive, weighted by their tropical intersection multiplicities.

2.7. Compatibility of toric and stable tropical intersections.

As discussed in Section 2.4, tropical multiplicities satisfy the balancing condition, and hence give Minkowski weights. The stable tropical intersections are compatible with multiplication of Minkowski weights, and algebraic intersections of generic translates in the torus, as follows.

Let X be a closed subscheme of pure codimension j in T. Suppose the valuation is trivial, so $\mathrm{Trop}(X)$ is a fan. After subdividing, we may assume this fan is unimodular, and extends to a complete unimodular fan Σ in $N_{\mathbb{R}}$. Let $Y(\Sigma)$ be the associated smooth complete toric variety, and let $\overline{X}$ be the closure of X in $Y(\Sigma)$. Because Σ contains $\mathrm{Trop}(X)$ as a subfan, the closure $\overline{X}$ meets every torus orbit O_τ properly, and the intersection is nonempty if and only if τ is contained in $\mathrm{Trop}(X)$. This follows from the basic properties of extended tropicalizations, given in [Pay09a, Proposition 3.7].

Remark 2.7.1. Indeed the closure $\overline{X}$ is proper and meets every torus orbit properly if and only if Σ contains a subfan whose support is $\mathrm{Trop}(X)$. See [Gub12, Section 14].

Furthermore, for any facet σ of $\mathrm{Trop}(X)$, the tropical multiplicity $m(\sigma)$ is equal to the intersection number of the closure $\overline{X}$ with the torus invariant subvariety corresponding to σ, that is

$$m(\sigma) = \deg([\overline{X}] \cdot V(\sigma)).$$

In other words,

$$c_{\overline{X}}(\sigma) = \begin{cases} m(\sigma) & \text{if } \sigma \subset \mathrm{Trop}(X). \\ 0 & \text{otherwise.} \end{cases}$$

See [ST08, Lemma 3.2] for the case of a tropical compactification and [KP11, Lemma 2.3] for the general case.

LEMMA 2.7.2. *Suppose the valuation is trivial, and let X be a pure-dimensional closed subscheme of T. Then the tropicalization* $\mathrm{Trop}(X)$ *is equal to the tropicalization of the fundamental cycle* $\mathrm{Trop}([X])$.

Recall that the multiplicities of facets in the tropicalization of a cycle are defined by linearity; see Definition 2.5.2.

Proof. The tropicalizations agree set-theoretically, because both are the closures of the images of $X(L)$ where $L|K$ is an algebraically closed valued extension field with nontrivial valuation. The multiplicities agree because the tropical multiplicity of a facet σ in $\mathrm{Trop}(X)$ is $m(\sigma) = \deg([\overline{X}] \cdot V(\sigma))$, and this degree is a linear function of the cycle $[\overline{X}]$. $\square$

Remark 2.7.3. The tropicalization of a pure-dimensional closed subscheme of T also agrees with the tropicalization of its underlying cycle in the case of a nontrivial valuation, but this requires considerably more work. This is deduced from our general results on intersection multiplicities over valuation rings of rank 1 in Corollary 4.4.6. See also [Gub12, Section 13] for an analytic proof using nonarchimedean GAGA and Gubler's theory of cycles and Cartier divisors on affinoid spaces.

Now consider the general case, where the valuation may be nontrivial, and let $w \in N_G$ be a point of $\mathrm{Trop}(X)$. After subdividing, we may assume that the star of w in $\mathrm{Trop}(X)$ is unimodular, and extends to a complete unimodular fan Σ. We write $\overline{X}_w$ for the closure of X_w in $Y(\Sigma)$.

DEFINITION 2.7.4. If σ is a facet of $\mathrm{Trop}(X)$ that contains w then let

$$\sigma_w = \mathbb{R}_{\geq 0}(\sigma - w)$$

be the corresponding cone in Σ.

Recall that the star of w in $\mathrm{Trop}(X)$ is the tropicalization of the initial degeneration X_w and, as discussed in Section 2.3, the initial degeneration of X_w at a point in the relative interior of σ_w is $\mathcal{T}_k$-affinely equivalent to the initial degeneration of X at a point of N_G in the relative interior of σ. In particular, the tropical multiplicity $m(\sigma_w)$ in $\mathrm{Trop}(X_w)$ is equal to the tropical multiplicity $m(\sigma)$.

Lemma 2.7.5. *Let γ be a face of codimension j in Σ. Then*

$$c_{\overline{X}_w}(\gamma) = \begin{cases} m(\sigma) & \text{if } \gamma = \sigma_w, \text{ for some } \sigma \subset \mathrm{Trop}(X). \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows from the case of the trivial valuation and the fact that the tropical multiplicity $m(\sigma)$ in $\mathrm{Trop}(X)$ is equal to the tropical multiplicity $m(\sigma_w)$ in $\mathrm{Trop}(X_w)$. □

This correspondence between tropical multiplicities and Minkowski weights is compatible with intersections; stable tropical intersections corresponding to products of Minkowski weights, as follows. Suppose X' is a closed subscheme of pure codimension j' in T, and τ is a face of codimension $j+j'$ in $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ that contains w. We may assume that the star of w in $\mathrm{Trop}(X')$ is a subfan of the unimodular fan Σ, and we write $\overline{X}'_w$ for the closure of X'_w in $Y(\Sigma)$.

Proposition 2.7.6. *Let γ be a cone of codimension $j+j'$ in Σ. Then*

$$(c_{\overline{X}_w} \cdot c_{\overline{X}'_w})(\gamma) = \begin{cases} i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X')) & \text{if } \gamma = \tau_w, \\ 0 & \text{otherwise,} \end{cases}$$

Proof. Suppose $\gamma = \tau_w$. We claim that the displacement rules giving $(c_{\overline{X}_w} \cdot c_{\overline{X}'_w})(\gamma)$ and $i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'))$ agree, term by term. If σ and σ' are facets of $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$, respectively, that contain τ, then σ meets $\sigma' + \epsilon v$ for small positive ϵ if and only if σ_w meets $\sigma'_w + v$. Then we have an equality of summands in the displacement rules,

$$[N : N_\sigma + N_{\sigma'}] \cdot m(\sigma) \cdot m(\sigma') = [N : N_{\sigma_w} + N_{\sigma'_w}] \cdot c_{\overline{X}_w}(\sigma_w) \cdot c_{\overline{X}'_w}(\sigma'_w),$$

because $N_\sigma + N'_\sigma$, $m(\sigma)$, and $m(\sigma')$ are equal to $N_{\sigma_w} + N_{\sigma'_w}$, $c_{\overline{X}_w}(\sigma_w)$, and $c_{\overline{X}'_w}(\sigma'_w)$, respectively. The Minkowski weights $c_{\overline{X}_w}$ and $c_{\overline{X}'_w}$ vanish on cones that do not come from facets of $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$, respectively. Therefore, all nonzero summands in the displacement rules are accounted for in the equality above, and the proposition follows. □

It follows that the star of w in the stable tropical intersection $\mathrm{Trop}(X) \cdot \mathrm{Trop}(X')$ is exactly the union of the faces of Σ on which $(c_{\overline{X}_w} \cdot c_{\overline{X}'_w})$ is positive, with multiplicities given by this product of Minkowski weights.
We now return to the case of the trivial valuation. As above, we choose Σ to be a complete unimodular fan in $N_\mathbb{R}$ that contains $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ as subfans.

Proposition 2.7.7. *Suppose the valuation is trivial. Assume X meets X' properly in T and, moreover, $\overline{X} \cap V(\sigma)$ meets $\overline{X}' \cap V(\sigma)$ properly in $V(\sigma)$ for each $\sigma \in \Sigma$.*

Then, for any face τ of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ *of codimension $j + j'$ in $N_{\mathbb{R}}$, the tropical intersection multiplicity along τ is equal to the weighted sum of multiplicities of tropicalizations*

$$i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X')) = \sum_Z i(Z, X \cdot X'; T) m_Z(\tau),$$

where the sum is over components Z of $X \cap X'$ such that $\mathrm{Trop}(Z)$ *contains τ, with tropical multiplicity $m_Z(\tau)$.*

Proof. First of all, by the preceding proposition,

$$i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X')) = (c_{\overline{X}} \cdot c_{\overline{X}'})(\tau).$$

Next, the isomorphism from $\mathrm{Mink}^*(\Sigma)$ to $A^*(Y(\Sigma))$ identifies $(c_{\overline{X}} \cdot c_{\overline{X}'})(\tau)$ with the intersection number $\deg([\overline{X}] \cdot [\overline{X}'] \cdot V(\tau))$. By hypothesis, $\overline{X}$ meets $\overline{X}'$ properly in $Y(\Sigma)$, and the generic point of each component of the intersection lies in T, so we have the refined intersection

$$\overline{X} \cdot \overline{X}' = \sum_Z i(Z, X \cdot X'; T)[\overline{Z}],$$

where the sum is taken over components Z of $X \cap X'$. Therefore,

$$\deg([\overline{X}] \cdot [\overline{X}'] \cdot [V(\tau)]) = \sum_Z i(Z, X \cdot X'; T) \deg([\overline{Z}] \cdot [V(\tau)]).$$

Now, since $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ are subfans of Σ, the closures $\overline{X}$ and $\overline{X}'$ meet $V(\sigma)$ properly in $Y(\Sigma)$, for all $\sigma \in \Sigma$. Furthermore, by hypothesis, $\overline{X} \cap V(\sigma)$ and $\overline{X}' \cap V(\sigma)$ meet properly in $V(\sigma)$ for all σ. It follows that Z meets all orbits of $Y(\Sigma)$ properly. Therefore $\mathrm{Trop}(Z)$ is a union of cones of codimension $j + j'$ in Σ (see Remark 2.7.1) and the intersection number $\deg([\overline{Z}] \cdot [V(\tau)])$ is equal to the tropical multiplicity $m_Z(\tau)$. □

The following proposition may be seen as a geometric explanation for the existence of well-defined stable tropical intersections, in the case of the trivial valuation; they are exactly the tropicalizations of intersections of generic translates in the torus.

Proposition 2.7.8. *Suppose the valuation is trivial, and let t be a general point in T. Then X meets tX' properly and the tropicalization of the intersection scheme $X \cap tX'$ is exactly the stable tropical intersection* $\mathrm{Trop}(X) \cdot \mathrm{Trop}(X')$.

Proof. First, an easy Bertini argument shows that X meets tX' properly when t is general. Furthermore, since $Y(\Sigma)$ is smooth, subadditivity of codimension says that every component of $\overline{X} \cap t\overline{X}'$ has codimension at most $j + j'$. Say Z is a component of this intersection, and let O_σ be the torus orbit in $Y(\Sigma)$ that contains the generic point of Z. Now, $\overline{X}$ and $t\overline{X}'$ meet O_σ properly, since their tropicalizations are subfans of Σ, and $\overline{X} \cap O_\sigma$ meets $t\overline{X}' \cap O_\sigma$ properly in O_σ, since t is general. Therefore Z has codimension $j + j' + \operatorname{codim} O_\sigma$, and hence O_σ must be the dense torus T.

Next, the Cohen-Macaulay locus in an excellent scheme is open [Gro65, Scholium 7.8.3(iv)], and contains every generic point, so the Cohen-Macaulay loci of X and X' are open and dense. Therefore, since t is general, the generic point of every component of $X \cap tX'$ must lie in the Cohen-Macaulay loci of both X and X'. Then, by [Ful98, Example 8.2.7], the intersection multiplicity $i(Z, X \cdot tX'; T)$ is equal to the length of the intersection of X and tX' along the generic point of Z. Proposition 2.7.7 then says that

$$i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X')) = \sum_Z \mathrm{length}(\mathcal{O}_{Z, X \cap tX'})\, m_Z(\tau).$$

Now the right hand side is the multiplicity of τ in the tropicalization of the cycle $[X \cap tX']$, and the proposition follows, by Lemma 2.7.2. □

Remark 2.7.9. We apply this proposition in the proof of Lemma 3.3.3. All that is needed for that application is the standard fact that the intersection multiplicity at a point of proper intersection is strictly positive. In particular, the equality of the intersection multiplicity $i(Z, X \cdot tX'; T)$ with the length of the scheme-theoretic intersection of X and X' along Z is not needed for the main lifting theorems.

In Section 5, we extend Proposition 2.7.8 to the general case, where the valuation may be nontrivial, as an application of tropical lifting theorems. See Theorem 5.3.3.

3. The trivial valuation case

The first step in our approach to tropical lifting theorems is to understand how the intersection of $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ near a point w relates to the intersection of the initial degenerations X_w and X'_w. If w is in N_G then the stars of w in $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ are the tropicalizations of X_w and X'_w, respectively, with respect to the trivial valuation, so this amounts to understanding how tropicalization relates to intersections when the tropicalizations meet properly and the valuation is trivial. Our proof that tropicalization commutes with intersection in this case uses both the theory of extended tropicalizations as projections of nonarchimedean analytifications and a new result about the support of stable intersections of tropicalizations (Lemma 3.3.3). We prove two necessary lemmas about the topology of extended tropicalizations in the general case, where the valuation is not necessarily trivial, since the arguments are identical to those in the case where the valuation is trivial.

3.1. Topology of extended tropicalizations. The extended tropicalizations of $Y(\Sigma)$ and its closed subschemes were introduced by Kajiwara [Kaj08], and their basic properties were studied further in Section 3 of [Pay09a], to which we refer the reader for further details. We recall the definition and prove that the extended tropicalization of a closed subscheme X in which the torus T is dense is the closure of the ordinary tropicalization of $X \cap T$. We also characterize the closure in the extended tropicalization $\mathbf{Trop}(Y(\Sigma))$ of a cone σ in Σ.

Let Σ be a fan in $N_{\mathbb{R}}$, and let $Y(\Sigma)$ be the associated toric variety. Recall that each cone σ in Σ corresponds to an affine open subvariety U_σ of $Y(\Sigma)$ whose coordinate ring is the semigroup ring $K[\sigma^\vee \cap M]$ associated to the semigroup of lattice points in the dual cone $\sigma^\vee$ in $M_{\mathbb{R}}$. Let $\mathbf{R}$ be the real line extended in the positive direction

$$\mathbf{R} = \mathbb{R} \cup \{+\infty\},$$

which is a semigroup under addition, with identity zero. Then

$$\mathbf{Trop}(U_\sigma) = \operatorname{Hom}(\sigma^\vee \cap M, \mathbf{R})$$

is the space of all semigroup homomorphisms taking zero to zero, with its natural topology as a subspace of $\mathbf{R}^{\sigma^\vee \cap M}$. For each face τ of Σ, let $N(\tau)$ be the real vector space

$$N(\tau) = N_{\mathbb{R}} / \operatorname{span}(\tau).$$

Then $\operatorname{Trop}(U_\sigma)$ is naturally a disjoint union of the real vector spaces $N(\tau)$, for $\tau \preceq \sigma$, where $N(\tau)$ is identified with the subset of semigroup homomorphisms that are finite exactly on the intersection of $\tau^\perp$ with $\sigma^\vee \cap M$.

The tropicalization $\mathbf{Trop}(Y(\Sigma))$ is constructed by gluing the spaces $\mathbf{Trop}(U_\sigma)$ for $\sigma \in \Sigma$ along the natural open inclusions $\mathbf{Trop}(U_\tau) \subset \mathbf{Trop}(U_\sigma)$ for $\tau \preceq \sigma$. It is a disjoint union

$$\mathbf{Trop}(Y(\Sigma)) = \bigsqcup_{\sigma \in \Sigma} N(\sigma),$$

just as $Y(\Sigma)$ is the disjoint union of the torus orbits O_σ. Points in $N(\sigma)$ may be seen as weight vectors on monomials in the coordinate ring of O_σ, and the tropicalization of a closed subscheme X in $Y(\Sigma)$ is defined to be

$$\mathbf{Trop}(X) = \bigsqcup_{\sigma \in \Sigma} \{w \in N(\sigma) \mid (X \cap O_\sigma)_w \text{ is not empty}\}.$$

In other words, $\mathbf{Trop}(X)$ is the disjoint union of the tropicalizations of its intersections with the T-orbits in $Y(\Sigma)$. This space is compact when Σ is complete.

LEMMA 3.1.1. *If every component of X meets the dense torus T, then $\mathbf{Trop}(X)$ is the closure in $\mathbf{Trop}(Y(\Sigma))$ of the ordinary tropicalization* $\operatorname{Trop}(X \cap T)$.

Proof. Since tropicalizations are invariant under extensions of valued fields [Pay09a, Proposition 6.1], and the completion of the algebraically closed field K is algebraically closed [BGR84, Proposition 3.4.1.3], we may assume the field K is complete with respect to its valuation.

The tropicalization map from $X(K)$ to $\mathbf{Trop}(Y(\Sigma))$ extends to a proper continuous map on the nonarchimedean analytification X^{an} whose image is $\mathbf{Trop}(X)$ [Pay09a, Section 2]. Since the open subset $(X \cap T)^{\mathrm{an}}$ is dense in X^{an} [Ber90, Corollary 3.4.5] and maps onto $\operatorname{Trop}(X \cap T)$, the extended tropicalization $\mathbf{Trop}(X)$ is contained in the closure of $\operatorname{Trop}(X \cap T)$. The lemma follows, since $\mathbf{Trop}(X)$ is closed. □

If σ is a face of Σ that contains another face τ, then we write σ_τ for the image of σ in $N(\tau)$. It is a cone of dimension $\dim(\sigma) - \dim(\tau)$. We will use the following lemma in the proof of Proposition 3.3.2.

Lemma 3.1.2. *Let σ and τ be faces of Σ, and let $\overline{\sigma}$ be the closure of σ in $\mathbf{Trop}(Y(\Sigma))$. Then*

$$\overline{\sigma} \cap N(\tau) = \sigma_\tau$$

if σ contains τ, and $\overline{\sigma}$ is disjoint from $N(\tau)$ otherwise.

Proof. Let v be a point in $N(\tau)$, and let $\pi : \mathbf{Trop}(U_\tau) \to N(\tau)$ be the continuous map that restricts to the canonical linear projections from $N(\gamma)$ onto $N(\tau)$ for $\gamma \preceq \tau$. If v is not in σ_τ, then the preimage under π of $N(\tau) \smallsetminus \sigma_\tau$ is an open neighborhood of v that is disjoint from σ, so v is not in $\overline{\sigma}$. Similarly, if σ does not contain τ then $\pi^{-1}(N(\tau)) \smallsetminus \sigma$ is an open neighborhood of v that is disjoint from σ.
It remains to show that $\overline{\sigma} \cap N(\tau)$ contains σ_τ in the case where σ contains τ. Suppose v is in σ_τ and let w be a point in σ that projects to v. Let w' be a point in the relative interior of τ. Then $w + \mathbb{R}_{\geq 0} w'$ is a path in σ whose limit is v, so v is in $\overline{\sigma}$. □

3.2. Tropical subadditivity. Although not strictly necessary for the main results of the paper, it is helpful to know that codimension is subadditive for intersections of tropicalizations in $N_\mathbb{R}$, so $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ never intersect in less than the expected dimension. The proof is by a diagonal projection argument, similar to the proof of Lemma 3.3.3, below.

Proposition 3.2.1. *Let X and X' be closed subschemes of pure codimension j and j', respectively, in T. If $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ is nonempty then it has codimension at most $j + j'$ at every point.*

Proof. Let w be a point in $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$. Then, in a neighborhood of w, the intersection can be identified with a neighborhood of zero in $\mathrm{Trop}(X_w) \cap \mathrm{Trop}(X'_w)$. Therefore, replacing X and X' by their initial degenerations at w, we may assume the valuation is trivial, and it will suffice to show that the global codimension of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X)$ is at most $j + j'$.
Suppose the valuation is trivial. Let T' be the quotient of $T \times T$ by the diagonal subtorus, with $\pi : X \times X' \to T'$ the induced projection, and $p : \mathrm{Trop}(X) \times \mathrm{Trop}(X') \to N'_\mathbb{R}$ the tropicalization of π. Since $\mathrm{Trop}(X)$ meets $\mathrm{Trop}(X')$, the point zero is in the image of p, which is exactly the tropicalization of the closure of the image of π.
Let y be a point in the image of π. The fiber $\pi^{-1}(y)$ has dimension at least $\dim X + \dim X' - \dim T'$ and, since the valuation is trivial, its tropicalization is contained in $p^{-1}(0)$. Since $p^{-1}(0)$ is naturally identified with $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, it follows that the tropical intersection has dimension at least $\dim X' + \dim X - \dim T$, as required. □

3.3. Lower bounds on multiplicities. For the remainder of this section, we assume the valuation ν is trivial. Let X and X' be closed subschemes of pure codimension j and j' in T, respectively. After subdividing the tropicalizations, we choose a complete unimodular fan Σ in $N_{\mathbb{R}}$ such that each face of $\mathrm{Trop}(X)$ and each face of $\mathrm{Trop}(X')$ is a face of Σ. We write $\overline{X}$ and $\overline{X}'$ for the closures of X and X' in the smooth complete toric variety $Y(\Sigma)$.
Our goal is to prove the following version of Theorem 1.1 in the special case of the trivial valuation, with lower bounds on the multiplicities of the facets in $\mathrm{Trop}(X \cap X')$. These lower bounds are extended to the general case in Section 5.

Theorem 3.3.1. *Suppose ν is trivial and* $\mathrm{Trop}(X)$ *meets* $\mathrm{Trop}(X')$ *properly. Then*

$$\mathrm{Trop}(X \cap X') = \mathrm{Trop}(X) \cap \mathrm{Trop}(X'),$$

and the multiplicity of any facet τ is bounded below by the tropical intersection multiplicity

$$m_{X\cap X'}(\tau) \geq i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X')).$$

Furthermore, the tropical intersection multiplicity is equal to the weighted sum of algebraic intersection multiplicities

$$i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X')) = \sum_{Z} i(Z, X \cdot X'; Y(\Sigma))\, m_Z(\tau),$$

where the sum is over components Z of $X \cap X'$ whose tropicalizations contain τ.

Given the compatibility of toric and stable tropical intersections in Proposition 2.7.7, the main step in the proof of Theorem 3.3.1 is showing that every component of $\overline{X} \cap \overline{X}'$ is the closure of a component of $X \cap X'$. This step can fail spectacularly when $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ do not meet properly. In such cases, $\overline{X} \cap \overline{X}'$ may contain components of larger than expected dimension, even if X and X' meet properly in T, and it is difficult to predict what $\mathrm{Trop}(X \cap X')$ will look like.

Proposition 3.3.2. *Suppose* $\mathrm{Trop}(X)$ *meets* $\mathrm{Trop}(X')$ *properly, and let τ be a face of Σ. Then*

1. *The tropicalizations of $\overline{X} \cap O_\tau$ and $\overline{X}' \cap O_\tau$ meet properly in $N(\tau)$.*
2. *The subschemes $\overline{X} \cap V(\tau)$ and $\overline{X}' \cap V(\tau)$ meet properly in $V(\tau)$.*
3. *The closures $\overline{X}$ and $\overline{X}'$ meet properly in $Y(\Sigma)$, and every component of $\overline{X} \cap \overline{X}'$ is the closure of a component of $X \cap X'$.*

Proof. First, if the tropicalizations of $\overline{X} \cap O_\tau$ and $\overline{X}' \cap O_\tau$ meet properly in $N(\tau)$, then the intersections themselves must meet properly in O_τ, by part (1) of the foundational theorem in Section 2.2. If this holds for all cones σ containing τ, then $\overline{X} \cap V(\tau)$ meets $\overline{X}' \cap V(\tau)$ properly in $V(\tau)$. Therefore (1) implies (2).

Now, since $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ are subfans of Σ, the closures $\overline{X}$ and $\overline{X}'$ meet all orbits of $Y(\Sigma)$ properly (see Section 2.7). Hence, if (2) holds then every component of $\overline{X} \cap \overline{X}' \cap V(\tau)$ has codimension $j + j'$ in $V(\tau)$. In particular, if O_τ is not the dense torus T, then $V(\tau)$ does not contain a component of $\overline{X} \cap \overline{X}'$. If this holds for all τ, then every component of $\overline{X} \cap \overline{X}'$ meets the dense torus T. This shows that (2) implies (3). We now prove (1).

The tropicalization of $\overline{X} \cap O_\tau$ is the intersection of the extended tropicalization $\mathbf{Trop}(\overline{X})$ with $N(\tau)$ and, by Lemma 3.1.1, this extended tropicalization is the closure of $\mathrm{Trop}(X)$. Therefore, by Lemma 3.1.2, $\mathrm{Trop}(\overline{X} \cap O_\tau)$ is the union of the projected faces σ_τ such that σ is in $\mathrm{Trop}(X)$ and contains τ. Similarly, $\mathrm{Trop}(\overline{X}' \cap O_\tau)$ is the union of the projected faces σ'_τ such that σ' is in $\mathrm{Trop}(X')$ and contains τ. Since the intersection of σ_τ and σ'_τ is exactly $(\sigma \cap \sigma')_\tau$, the codimension of $\sigma_\tau \cap \sigma'_\tau$ in $N(\tau)$ is equal to the codimension of $\sigma \cap \sigma'$ in $N_{\mathbb{R}}$. In particular, if $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ meet properly at τ, then $\mathrm{Trop}(\overline{X} \cap O_\tau)$ and $\mathrm{Trop}(\overline{X}' \cap O_\tau)$ meet properly in $N(\tau)$, as required. □

The following lemma says that if $\mathrm{Trop}(X)$ meets $\mathrm{Trop}(X')$ properly then the support of their stable tropical intersection is equal to their set-theoretic intersection.

Lemma 3.3.3. *Suppose* $\mathrm{Trop}(X)$ *and* $\mathrm{Trop}(X')$ *meet properly and let τ be a facet of their intersection. Then the tropical multiplicity* $i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'))$ *is strictly positive.*

Proof. First, we reduce to the case where X and X' have complementary dimension. Recall that the tropical multiplicity $m(\sigma)$ of a facet σ in $\mathrm{Trop}(X)$ is equal to the algebraic intersection number $\deg(\overline{X} \cdot V(\sigma))$ in $Y(\Sigma)$, as discussed in Section 2.7, and the intersection number $\deg(\overline{X} \cdot \overline{X}' \cdot V(\tau))$ in $Y(\Sigma)$ is equal to the intersection number

$$\deg\left((\overline{X} \cdot V(\tau)) \cdot (\overline{X}' \cdot V(\tau))\right)$$

in $V(\tau)$ [Ful98, Example 8.1.10], because $V(\tau)$ is smooth. Because $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ are subfans of Σ, the closures $\overline{X}$ and $\overline{X}'$ in $Y(\Sigma)$ meet all torus orbits properly, and it follows that the tropical intersection multiplicity $i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'))$ is equal to the tropical intersection multiplicity at zero of $\mathrm{Trop}(\overline{X} \cdot O_\tau)$ and $\mathrm{Trop}(\overline{X}' \cdot O_\tau)$ inside $N(\tau)$. By Proposition 3.3.2, the intersection cycles $\overline{X} \cdot O_\tau$ and $\overline{X}' \cdot O_\tau$ have complementary dimension in O_τ and their tropicalizations meet properly in $N(\tau)$. The same is then true for each of the reduced components of $\overline{X} \cdot O_\tau$ and $\overline{X}' \cdot O_\tau$, and it suffices to prove the proposition in this case.

Assume X and X' have complementary dimension. Since their tropicalizations meet properly, by hypothesis, their intersection must be the single point zero. Consider the quotient T' of $T \times T$ by its diagonal subtorus, and the induced projection

$$\pi : X \times X' \to T'.$$

By Proposition 2.7.8, the tropical intersection number $\mathrm{Trop}(X) \cdot \mathrm{Trop}(X')$ is equal to the length of $X \cap tX'$ for a general point $t \in T$, which is equal to the length of the general fiber of π. In particular, to prove the proposition, it is enough to show that π is dominant. Now, since the valuation is trivial, the tropicalization of any fiber of π is contained in the preimage of zero under the induced projection

$$\mathrm{Trop}(X) \times \mathrm{Trop}(X') \to N'_{\mathbb{R}},$$

which is the single point zero, by hypothesis. In particular, every nonempty fiber of π is zero-dimensional, and hence π is dominant, as required. □

We can now easily prove the desired result.

Proof of Theorem 3.3.1. Let τ be a facet of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$. By Propositions 2.7.7 and 3.3.2, the tropical intersection multiplicity $i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'))$ is a weighted sum over components of $X \cap X'$

$$i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X')) = \sum_Z i(Z, X \cdot X'; T) m_Z(\tau).$$

By Lemma 3.3.3, this multiplicity is strictly positive, so there is at least one component Z of $X \cap X'$ such that $\mathrm{Trop}(Z)$ contains τ. This proves that τ is contained in $\mathrm{Trop}(X \cap X')$, and hence $\mathrm{Trop}(X \cap X')$ is equal to $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, set-theoretically.

Finally, to see the inequality on multiplicities, recall that the multiplicity of τ in $\mathrm{Trop}(X \cap X')$ is

$$m_{X \cap X'}(\tau) = \sum_Z \mathrm{length}_{\mathcal{O}_{T,Z}}(\mathcal{O}_{X \cap X', Z}) m_Z(\tau),$$

by Lemma 2.7.2. The inequality $m_{X \cap X'}(\tau) \geq i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'))$ follows by comparing the two summation formulas term by term, because $\mathrm{length}_{\mathcal{O}_{T,Z}}(\mathcal{O}_{X \cap X', Z})$ is at least as large as the intersection multiplicity $i(Z, X \cdot X'; T)$ [Ful98, Proposition 8.2]. □

4. Geometry over valuation rings of rank 1

In [Nag66], Nagata began extending certain results from dimension theory to non-noetherian rings. Here we continue in Nagata's spirit with a view toward tropical geometry, focusing on valuation rings of rank 1 and showing that these rings have many of the pleasant properties of regular local rings.

Recall that a valuation ring of rank 1 is an integral domain R whose field of fractions K admits a nontrivial valuation $\nu : K^* \to \mathbb{R}$ such that the nonzero elements of R are exactly the elements of K^* of nonnegative valuation. It is a local ring with exactly two prime ideals, the zero ideal and the maximal ideal. If K is algebraically closed or, more generally, if the valuation is not discrete, then R is not noetherian, because the maximal ideal is not finitely generated.

Throughout this section, R is a valuation ring of rank 1, with fraction field K and residue field k. In this section only, we allow the possibility that K

is not algebraically closed, because the results of this section hold equally for nonclosed fields and the greater generality creates no additional difficulties. We fix

$$S = \operatorname{Spec} R,$$

and use calligraphic notation for schemes over S. If $\mathcal{X}$ is a scheme over S, we write $\mathcal{X}_K = \mathcal{X} \times_S \operatorname{Spec} K$ for the generic fiber and $\mathcal{X}_k = \mathcal{X} \times_S \operatorname{Spec} k$ for the special fiber.

A module is flat over R if and only if it is torsion-free [Mat89, Exercise 10.2], so an integral scheme $\mathcal{X}$ is flat over S if and only if it is dominant. Furthermore, a scheme that is flat and locally of finite type over an integral scheme is necessarily locally of finite presentation [RG71, Theorem 3.4.6 and Corollary 3.4.7]. In particular, if $\mathcal{X}$ is an integral scheme that is surjective and locally of finite type over S then the special fiber $\mathcal{X}_k$ is of pure dimension equal to the dimension of the generic fiber $\mathcal{X}_K$. This follows from [Gro66, Theorem 12.1.1(i)], applied to the generic point of each component of the special fiber. We will use these technical facts repeatedly throughout the section, without further mention.

Dimension theory over valuation rings of rank 1 involves some subtleties. For instance, if R is not noetherian then the formal power series ring $R[[x]]$ has infinite Krull dimension [Arn73]. Because dimensions are well-behaved under specialization over R, such pathologies are always visible in at least one of the fibers. For $\mathcal{X} = \operatorname{Spec} R[[x]]$, the special fiber is $\operatorname{Spec} k[[x]]$, which is one dimensional, but $R[[x]] \otimes K$ is the subring of $K[[x]]$ consisting of formal power series whose coefficients have valuations bounded below. This ring, and hence the generic fiber of $\operatorname{Spec} R[[x]]$, is infinite dimensional. Nevertheless, for a scheme of finite type over S, the generic fiber and special fiber are schemes of finite type over K and k, respectively, and such pathologies do not occur.

Many of the results of this section have natural generalizations to more general base schemes. See Appendix B for details.

4.1. Subadditivity and lifting over valuation rings of rank 1. Serre famously proved that codimension is subadditive under intersections in regular schemes [Ser65, Theorem V.3]. In other words, for any irreducible closed subschemes X and X' of a regular scheme Y, and any irreducible component Z of $X \cap X'$, we have

$$\operatorname{codim}_Y Z \leq \operatorname{codim}_Y X + \operatorname{codim}_Y X'.$$

The same is then necessarily true for schemes smooth over Y. We extend Serre's theorem to smooth schemes over rank 1 valuation rings, as follows. Recall that we have fixed $S = \operatorname{Spec} R$, where R is a valuation ring of rank 1.

Theorem 4.1.1. *Let $\mathcal{Y}$ be smooth over S. Then codimension is subadditive under intersection in $\mathcal{Y}$.*

In this respect, valuation rings of rank 1 behave like regular local rings.

Remark 4.1.2. Subadditivity of codimension is often used to deform points in families using dimension counting arguments. Such techniques are essential, for instance, in the theory of limit linear series developed by Eisenbud and Harris [EH86]. Here, we use subadditivity of codimension to lift intersection points from the special fiber to the generic fiber in schemes of finite type over valuation rings of rank 1.[3]

As an application of Theorem 4.1.1 we prove the following.

Theorem 4.1.3. *Let $\mathcal{Y}$ be smooth over S with closed subschemes $\mathcal{X}$ and $\mathcal{X}'$ of pure codimension j and j', respectively. Suppose the special fibers $\mathcal{X}_k$ and $\mathcal{X}'_k$ intersect in codimension $j + j'$ at a point x in $\mathcal{Y}_k$. Then*

(1) *There is a point in $\mathcal{X}_K \cap \mathcal{X}'_K$ specializing to x.*
(2) *If x is closed then there is a closed point in $\mathcal{X}_K \cap \mathcal{X}'_K$ specializing to x. More generally, if $k(x)$ denotes the residue field of x, there is a point $x' \in \mathcal{X}_K \cap \mathcal{X}'_K$ specializing to x and satisfying*

$$\operatorname{trdeg} k(x')/K = \operatorname{trdeg} k(x)/k.$$

In Section 4.4 we also prove a principle of continuity for intersection multiplicities in families over S.

Remark 4.1.4. The theorem reduces easily to the case where $\mathcal{X}$ and $\mathcal{X}'$ are reduced and irreducible and then, using subadditivity, it is not difficult to show that both must meet the generic fiber $\mathcal{Y}_K$ and are therefore flat over S. Our methods also give a more general lifting theorem for intersections of flat subschemes over an arbitrary base scheme (Theorem B.4), even when subadditivity fails. Here we proceed through subadditivity because it eliminates the need for a flatness hypothesis and makes the lifting arguments particularly transparent, especially for those familiar with similar arguments over regular local rings.

4.2. A principal ideal theorem. Recall that Krull's principal ideal theorem, translated into geometric terms, says that every component of a Cartier divisor on a noetherian scheme has codimension 1. Here we prove the following generalization to valuation rings of rank 1.

Theorem 4.2.1. *Let $\mathcal{X}$ be of finite type over S, and let $\mathcal{Z}$ be a locally principal closed subscheme of $\mathcal{X}$. Then every irreducible component of $\mathcal{Z}$ has codimension at most* 1 *in each component of $\mathcal{X}$ that contains it.*

The same is not true for valuation rings that are not of rank 1. If A is such a ring then $\operatorname{Spec} A$ has principal subschemes of codimension greater than 1. A key step in the proof of the theorem is the following technical proposition, which we prove by noetherian approximation.

[3] The assumption that schemes are of finite type over S is crucial for the main results of this section. In the tropical setting, if ν is nontrivial and w is not in N_G, then closed points of X_w never lift to closed points of X. However, this does not contradict Theorem 4.2.5 because in this case $\mathcal{X}^w$ is not of finite type. See Appendix A.

Proposition 4.2.2. *Let $\mathcal{X}$ be irreducible, locally of finite type, and flat over S. Suppose that $\mathcal{D}$ is a locally principal closed subscheme in $\mathcal{X}$ that does not meet the generic fiber $\mathcal{X}_K$. Then every irreducible component of $\mathcal{D}_k$ is an irreducible component of $\mathcal{X}_k$.*

Proof. Since $\mathcal{X}$ is flat and locally of finite type over S, it is locally of finite presentation over S. The question is local, so we may assume that $\mathcal{X}$ is affine, and in particular of finite presentation over S. Therefore, there exists a finitely generated subring $R' \subset R$, with models X' and D' over $S' = \operatorname{Spec} R'$ of $\mathcal{X}$ and $\mathcal{D}$, respectively. Then S' is irreducible by construction, and we may assume X' is irreducible as well.

Let s' be the image in S' of the closed point in S. Since $\mathcal{X}_K$ and $\mathcal{X}_k$ are both pure of the same dimension d, the fibers of X' over the generic point and s' are also pure of dimension d. Similarly, $\mathcal{D}_k$ is pure of dimension d if and only if $D'_{s'}$ is, so to prove the proposition it is enough to show that $D'_{s'}$ is pure of dimension d.

Let Z be a component of D' that meets the fiber over s'. We will show that $Z_{s'}$ has pure dimension d. Let s'' be the image in S' of the generic point of Z. Then s'' is a specialization of the generic point of S' and specializes to s', so upper semicontinuity of fiber dimension [Gro66, Theorem 13.1.3] implies that $X'_{s''}$ has pure dimension d. We claim that $Z_{s''}$ is a component of $X'_{s''}$. To see this, note that s'' is not the generic point of S', since $\mathcal{D}$ does not meet the generic fiber of $\mathcal{X}$, and hence every component of $X'_{s''}$ has codimension at least 1 in X'. Since X' is noetherian, Krull's principal ideal theorem says that every component of D' has codimension at most 1 in X', and hence $Z_{s''}$ is a component of $X'_{s''}$, as claimed. In particular, $Z_{s''}$ has pure dimension d. Since s'' specializes to s' and $X'_{s'}$ has pure dimension d, it follows by upper semicontinuity that $Z_{s'}$ has pure dimension d, as required. □

The following two lemmas are special cases of a more general altitude formula over valuation rings of finite rank [Nag66, Theorem 2]. These codimension formulas are used in the proof of the principal ideal theorem, and again in the proof of subadditivity of codimension over valuation rings of rank 1. We include proofs for the reader's convenience.

Lemma 4.2.3. *Suppose $\mathcal{X}$ is irreducible, finite type, and flat over S with $\mathcal{Z} \subset \mathcal{X}_k$ an irreducible closed subset of the special fiber. Then*

$$\operatorname{codim}_{\mathcal{X}} \mathcal{Z} = \operatorname{codim}_{\mathcal{X}_k} \mathcal{Z} + 1.$$

Proof. A maximal chain of irreducible closed subsets between $\mathcal{Z}$ and $\mathcal{X}_k$ can be extended by adding $\mathcal{X}$ at the end, so $\operatorname{codim}_{\mathcal{X}} \mathcal{Z}$ is at least $\operatorname{codim}_{\mathcal{X}_k} \mathcal{Z} + 1$. We now show that $\operatorname{codim}_{\mathcal{X}} \mathcal{Z}$ is at most $\operatorname{codim}_{\mathcal{X}_k} \mathcal{Z} + 1$.

If $W \subset W'$ is a strict inclusion of irreducible closed subsets of $\mathcal{X}$ that meet the special fiber then the dimension of W_k is less than or equal to the dimension of W'_k. This inequality is strict if W and W' are both contained in the special fiber. Similarly, if W and W' both meet the generic fiber then $\dim W_K$ is less than $\dim W'_K$, and the dimensions of the special fibers are equal to the dimensions of the respective generic fibers, by flatness. Therefore, if $\dim W_k$ is equal to $\dim W'_k$ then W is contained in the special fiber and W' is not. This can happen at most once in any chain of inclusions, so any chain between $\mathcal{Z}$ and $\mathcal{X}$ has length at most $\dim \mathcal{X}_k - \dim Z + 1$, as required. □

LEMMA 4.2.4. *Suppose $\mathcal{X}$ is irreducible, finite type, and flat over S, and let $\mathcal{Z} \subset \mathcal{Z}'$ be irreducible closed subsets. Then*

$$\operatorname{codim}_{\mathcal{X}} \mathcal{Z} = \operatorname{codim}_{\mathcal{X}} \mathcal{Z}' + \operatorname{codim}_{\mathcal{Z}'} \mathcal{Z}.$$

Proof. If $\mathcal{Z}$ meets the generic fiber of $\mathcal{X}$ then this is a classical formula for codimension of varieties over K. Suppose $\mathcal{Z}$ is contained in the special fiber. If $\mathcal{Z}'$ meets the generic fiber then $\operatorname{codim}_{\mathcal{X}} \mathcal{Z}' = \dim \mathcal{X}_K - \dim \mathcal{Z}'_K$, and $\operatorname{codim}_{\mathcal{Z}'} \mathcal{Z} = \dim \mathcal{Z}'_k - \dim \mathcal{Z} + 1$, by Lemma 4.2.3. Now $\dim \mathcal{X}_K$ and $\dim \mathcal{Z}'_K$ are equal to $\dim \mathcal{X}_k$ and $\dim \mathcal{Z}'_k$, respectively, so adding these two equations gives

$$\operatorname{codim}_{\mathcal{X}} \mathcal{Z}' + \operatorname{codim}_{\mathcal{Z}'} \mathcal{Z} = \dim \mathcal{X}_k - \dim \mathcal{Z} + 1,$$

and Lemma 4.2.3 says that the right hand side is equal to $\operatorname{codim}_{\mathcal{X}} Z$. The proof in the case where $\mathcal{Z}'$ is contained in the special fiber is similar. □

We now prove the principal ideal theorem over valuation rings of rank 1.

Proof of Theorem 4.2.1. We may assume that $\mathcal{X}$ is integral, and since the statement is local about the generic points of $\mathcal{Z}$, we may assume that $\mathcal{Z}$ is irreducible. If $\mathcal{X}$ is supported in the special fiber, the theorem reduces to the classical principal ideal theorem over k. Otherwise $\mathcal{X}$ is dominant, and hence flat, over S. If $\mathcal{Z}$ meets the generic fiber $\mathcal{X}_K$, then the theorem follows from the classical principal ideal theorem for $\mathcal{X}_K$. On the other hand, if $\mathcal{Z}$ is contained in the special fiber, then it must be a union of components of the special fiber, by Proposition 4.2.2, and hence has codimension 1 in X, by Lemma 4.2.3. □

We apply the principal ideal theorem to prove the following result on lifting closed points in the special fiber to closed points in the generic fiber. In fact, we prove a more general result, for points of arbitrary transcendence degree. Our argument is in the spirit of Katz's proof of [Kat09, Lemma 4.15].

THEOREM 4.2.5. *Let $\mathcal{X}$ be an irreducible scheme locally of finite type over S, and let x be a closed point of the special fiber $\mathcal{X}_k$. If x is in the closure of $\mathcal{X}_K$ then the set of closed points x' in $\mathcal{X}_K$ specializing to x is Zariski dense in $\mathcal{X}_K$. More generally, if x is not necessarily closed in $\mathcal{X}_k$, and $k(x)$ denotes the residue field of x, we can choose x' to satisfy the identity*

$$\operatorname{trdeg} k(x')/K = \operatorname{trdeg} k(x)/k,$$

and again the choices of x' are Zariski dense in $\mathcal{X}_K$.

Proof. Note that the statement for closed points is a special case of the statement involving transcendence degrees. We may assume $\mathcal{X}$ is integral and in particular flat over S, and also affine.

By hypothesis, x is a specialization of the generic point of $\mathcal{X}$. Observe that given $x' \in \mathcal{X}_K$ specializing to x, the closure of x' is flat over S, so we obtain the inequality $\operatorname{trdeg} k(x')/K \geq \operatorname{trdeg} k(x)/k$. Now, if $\dim \mathcal{X}_K = \operatorname{trdeg} k(x)/k$, then we may take x' to be the generic point of $\mathcal{X}_K$. We thus proceed by induction on $d := \dim \mathcal{X}_K - \operatorname{trdeg} k(x)/k$.
Suppose now $d > 0$, and note that this implies that x is not the generic point of any component of $\mathcal{X}_k$. Let W be a closed subset properly contained in $\mathcal{X}_K$; by affineness, we can choose a regular function f on $\mathcal{X}$ that vanishes at x, but not on any component of $W \cup \mathcal{X}_k$. Then let $\mathcal{D} \subset \mathcal{X}$ be the principal subscheme cut out by f. Passing to a smaller neighborhood of x, if necessary, we may assume every component of $\mathcal{D}$ contains x. By Proposition 4.2.2, some component $\mathcal{Z}$ of $\mathcal{D}$ meets the generic fiber, and hence $\dim \mathcal{Z}_K - \operatorname{trdeg} k(x)/k = d - 1$. By induction, $\mathcal{Z}_K$ contains a point x' specializing to x, not contained in W, and with $\operatorname{trdeg} k(x')/K = \operatorname{trdeg} k(x)/k$. We conclude the desired statement. $\square$

Remark 4.2.6. For the initial degeneration of a closed subscheme of a torus T over K associated to a weight vector $w \in N_G$, in the special case where x is closed, Theorem 4.2.5 says that every point in $X_w(k)$ lifts to a point in $X(K)$, and the set of such lifts is Zariski dense. In particular, the theorem gives an algebraic proof of surjectivity of tropicalization, along the lines suggested by Speyer and Sturmfels in [SS04], as well as a proof of the density of tropical fibers stated as Theorem 4.1 and Corollary 4.2 in [Pay09b]. A gap in the proof of the former paper is explained in a footnote on the first page of the latter. The proof of Theorem 4.1 in the latter also contains a serious error, discovered by W. Buczynska and F. Sottile, which is explained and corrected in [Pay12].

Remark 4.2.7. When applied to non-closed points, Theorem 4.2.5 says that every closed subvariety of X_w is a component of the special fiber of the closure in $\mathcal{X}^w$ of some closed subvariety of X. So, at least in this weak sense, every curve in X_w lifts to a curve in X, every surface in X_w lifts to a surface in X, and so on.

4.3. Proofs of subadditivity and lifting. We now use the principal ideal theorem to prove subaddivity of codimension under intersection and deduce a lifting theorem for proper intersections, in smooth schemes over a valuation ring of rank 1. Our proof follows the classical argument for smooth varieties, which is simpler than Serre's proof for regular schemes.

Remark 4.3.1. Codimension is not subadditive under intersection in smooth schemes over valuation rings of rank greater than 1, as shown by the following example. Nevertheless, this failure of subadditivity can be understood and controlled, and proper intersections in special fibers still lift to more general fibers over arbitrary valuation rings and, more generally, for flat families of subschemes in a smooth scheme over an arbitrary base. See Appendix B.

Example 4.3.2. Let A be a valuation ring of finite rank r greater than 1, and let a be an element of A that generates an ideal of height r. Then $A[x]$ has Krull dimension $r + 1$, by [Nag66, Theorem 2]. The ideals of $A[x]$ generated by x and $x - a$ each have height 1, but the ideal $(x, x - a) = (x, a)$ has height $r + 1$, which is greater than 2.

Our proof of Theorem 4.1.1 involves a reduction to the diagonal and requires the following lemma on dimensions of fiber products over S.

LEMMA 4.3.3. *Let $\mathcal{Y}$ be irreducible and finite type over S, and let $\mathcal{X}$ and $\mathcal{X}'$ be irreducible closed subschemes. Then for every irreducible component $\mathcal{Z}$ of $\mathcal{X} \times_S \mathcal{X}'$,*

$$\operatorname{codim}_{\mathcal{Y}\times_S\mathcal{Y}} \mathcal{Z} \leq \operatorname{codim}_{\mathcal{Y}} \mathcal{X} + \operatorname{codim}_{\mathcal{Y}} \mathcal{X}'.$$

Furthermore, the inequality is strict if and only if $\mathcal{Y}$ is dominant over S and $\mathcal{X}$ and $\mathcal{X}'$ are both contained in the special fiber.

Proof. If $\mathcal{X}$ and $\mathcal{X}'$ both meet the generic fiber then they are flat over S. In this case, $\mathcal{Z}$ must also meet the generic fiber, and the proposition holds with equality, by classical dimension theory over K. Similarly, if $\mathcal{Y}$ is not dominant over S, then the proposition holds with equality by dimension theory over k. Suppose $\mathcal{Y}$ is dominant and $\mathcal{X}$ is contained in the special fiber. Then $\mathcal{Z}$ is also contained in the special fiber. If, furthermore, $\mathcal{X}'$ is contained in the special fiber, then $\mathcal{Z}$ has dimension $\dim \mathcal{X} + \dim \mathcal{X}'$. Therefore, by Lemma 4.2.3,

$$\operatorname{codim}_{\mathcal{Y}\times_S\mathcal{Y}} \mathcal{Z} = 2 \dim \mathcal{Y}_K - \dim \mathcal{X} - \dim \mathcal{X}' + 1.$$

The right hand side is equal to $\operatorname{codim}_{\mathcal{Y}} \mathcal{X} + \operatorname{codim}_{\mathcal{Y}} \mathcal{X}' - 1$, so the inequality is strict by exactly one in this case. A similar argument shows that the proposition holds with equality if $\mathcal{X}'$ meets the generic fiber. □

We will prove subadditivity of codimension using the previous proposition and a reduction to the diagonal.

LEMMA 4.3.4. *If $\mathcal{Y}$ is smooth over S then the diagonal Δ in $\mathcal{Y} \times_S \mathcal{Y}$ is a local complete intersection subscheme.*

Here, by local complete intersection we mean that the number of local generators for $\mathcal{I}_\Delta$ is equal to the codimension.

Proof. First note that the diagonal morphism is locally of finite presentation, by [Gro64, Corollary 1.4.3.1]. The lemma follows, since a locally finite presentation immersion of a smooth scheme in another smooth scheme over an arbitrary base is a local complete intersection. See Proposition 7 of Section 2.2 in [BLR90]. □

We now proceed with the proof of subadditivity of codimension and lifting of proper intersections in the special fiber, over valuation rings of rank 1.

Proof of Theorem 4.1.1. The intersection $\mathcal{X} \cap \mathcal{X}'$ can be realized as the intersection of $\mathcal{X} \times_S \mathcal{X}'$ with the diagonal in $\mathcal{Y} \times_S \mathcal{Y}'$. Now, the codimension of any component $\mathcal{Z}$ of $\mathcal{X} \times_S \mathcal{X}'$ is at most

$$\operatorname{codim}_{\mathcal{Y}\times_S\mathcal{Y}} \mathcal{Z} \leq \operatorname{codim}_{\mathcal{Y}} \mathcal{X} + \operatorname{codim}_{\mathcal{Y}} \mathcal{X}',$$

by Lemma 4.3.3. Then the principal ideal theorem (Theorem 4.2.1) and Lemma 4.2.4 together imply that codimension can only decrease when intersecting with a local complete intersection subscheme of $\mathcal{Y} \times_S \mathcal{Y}$. Lemma 4.3.4 says that the diagonal is a local complete intersection, and the theorem follows. □

Proof of Theorem 4.1.3. Let $\mathcal{Z}$ be a component of $\mathcal{X} \cap \mathcal{X}'$ that contains x. By Theorem 4.1.1,

$$\operatorname{codim}_{\mathcal{Y}} \mathcal{Z} \leq \operatorname{codim}_{\mathcal{Y}} \mathcal{X} + \operatorname{codim}_{\mathcal{Y}} \mathcal{X}'.$$

The codimension of the special fiber $\mathcal{Z}_k$ in $\mathcal{Y}$ is equal to $\operatorname{codim}_{\mathcal{Y}} \mathcal{X} + \operatorname{codim}_{\mathcal{Y}} \mathcal{X}' + 1$, by Lemma 4.2.3, so $\mathcal{Z}$ must meet the generic fiber. Therefore, the generic point of $\mathcal{Z}_K$ is a point of $\mathcal{X}_K \cap \mathcal{X}'_K$ specializing to x. If x is closed in its fiber, we can then find a closed point of $\mathcal{Z}_K$ specializing to x, by Theorem 4.2.5, and similarly for the assertion on transcendence degrees. □

4.4. Intersection multiplicities over valuation rings of rank 1. The principle of continuity says that intersection numbers are constant when cycles vary in flat families. See [Ful98, Section 10.2] for a precise statement and proof when the base is a smooth variety. For applications to tropical lifting with multiplicities, we need to apply a principle of continuity over the spectrum S of a possibly non-noetherian valuation ring of rank 1. Lacking a suitable reference, we include a proof.

Definition 4.4.1. Suppose Y is smooth over a field k, and let X and X' be closed subschemes of Y whose intersection $X \cap X'$ is finite. Then the intersection number $i(X \cdot X'; Y)$ of X and X' in Y is

$$\sum_{i=0}^{\dim Y} (-1)^i \dim_k \operatorname{Tor}^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_{X'}).$$

In other words, the intersection number is the sum of the local intersection multiplicities at the finitely many points of $X \cap X'$, weighted by degrees of extension fields,

$$i(X \cdot X'; Y) = \sum_{P \in X \cap X'} [k(P) : k] i(P, X \cdot X'; Y).$$

While we are primarily interested in the case where X and X' have complementary dimension, so the finiteness of $X \cap X'$ means that X and X' meet properly in Y, this hypothesis is not technically necessary, since the local intersection multiplicities vanish when the intersection is not proper [Ser65, Theorem V.C.1].

Theorem 4.4.2. *Let $\mathcal{Y}$ be smooth and quasiprojective over S, and let $\mathcal{X}$ and $\mathcal{X}'$ be closed subschemes of $\mathcal{Y}$ that are flat over S. If $\mathcal{X} \cap \mathcal{X}'$ is finite over S then*

$$i(\mathcal{X}_K \cdot \mathcal{X}'_K; \mathcal{Y}_K) = i(\mathcal{X}_k \cdot \mathcal{X}'_k; \mathcal{Y}_k).$$

In other words, intersection numbers are invariant under specialization over S. We now give a direct proof of this equality, by first establishing coherence properties for the structures sheaves of these schemes and then using free resolutions to compute the Tor groups. An alternative approach, using noetherian approximation and specialization properties of intersection theory over a noetherian base, is sketched in Remark 4.4.4.
In general, the structure sheaf of a non-noetherian scheme is not necessarily coherent. The following proposition asserts that such pathologies do not occur on the schemes we are considering.

Proposition 4.4.3. *Let $\mathcal{X}$ be locally of finite presentation over S. Then $\mathcal{O}_{\mathcal{X}}$ is coherent.*

Proof. We first claim that $\mathcal{O}_{\mathcal{X}}$ is coherent in the special case where $\mathcal{X}$ is the affine space $\operatorname{Spec} R[t_1, \ldots, t_n]$. Since the question is local, it is enough to show that for any affine subset $\mathcal{U}$ of $\mathcal{X}$, and any homomorphism $f : \mathcal{O}_{\mathcal{X}}^m|_{\mathcal{U}} \to \mathcal{O}_{\mathcal{X}}|_{\mathcal{U}}$, the kernel of f over $\mathcal{U}$ is finitely generated. Now, the image of f is finitely generated, and is the quasicoherent ideal sheaf $\tilde{I}$ associated to some ideal I in $\mathcal{O}_{\mathcal{X}}(\mathcal{U})$. The ideal I is torsion-free and hence flat over S, so it is finitely presented by [RG71, Theorem 3.4.6]. Therefore, by [Mat89, Theorem 2.6], the kernel of the given presentation $\mathcal{O}_{\mathcal{X}}(\mathcal{U})^m \to I$ is finitely generated, which proves the claim.
For the general case, since coherence is local, we may again assume that $\mathcal{X}$ is affine, so $\mathcal{X} = \operatorname{Spec} A[t_1, \ldots, t_n]/I$ for some finitely generated ideal I. The same argument as in the special case above shows that I is finitely presented. Therefore $\tilde{I}$ is coherent on $\operatorname{Spec} R[t_1, \ldots, t_n]$, and hence $\mathcal{O}_{\mathcal{X}}$ is coherent [Gro60, Chapter 0, 5.3.10]. □

Proof of Theorem 4.4.2. Since $\mathcal{Y}$ is quasiprojective and $\mathcal{X} \cap \mathcal{X}'$ is finite over S, there is an affine open subset of $\mathcal{Y}$ containing the intersection, and hence we may assume $\mathcal{Y}$ is affine. Now $\mathcal{O}_{\mathcal{Y}}$ is coherent, by Proposition 4.4.3, and $\mathcal{O}_{\mathcal{X}}$ and $\mathcal{O}_{\mathcal{X}'}$, being flat and finitely generated, are finitely presented and hence coherent as $\mathcal{O}_{\mathcal{Y}}$-modules. Let $\mathcal{P}_\bullet$ be a resolution of $\mathcal{O}_{\mathcal{X}}$ by free $\mathcal{O}_{\mathcal{Y}}$-modules of finite rank. Then, since $\mathcal{O}_{\mathcal{X}}$ is flat over S, the restrictions $\mathcal{P}_\bullet \otimes K$ and $\mathcal{P}_\bullet \otimes k$ to the generic and special fibers remain exact, giving free resolutions of $\mathcal{O}_{\mathcal{X}_K}$ and $\mathcal{O}_{\mathcal{X}_k}$, respectively. Therefore, not only does the homology of

$$\mathcal{Q}_\bullet = \mathcal{P}_\bullet \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}'}$$

compute $\operatorname{Tor}_{\mathcal{Y}}(\mathcal{X}, \mathcal{X}')$, but also the homology of the base changes $\mathcal{Q}_\bullet \otimes K$ and $\mathcal{Q}_\bullet \otimes k$ computes $\operatorname{Tor}_{\mathcal{Y}_K}(\mathcal{X}_K, \mathcal{X}'_K)$ and $\operatorname{Tor}_{\mathcal{Y}_k}(\mathcal{X}_k, \mathcal{X}'_k)$, respectively.
Now, $\operatorname{Tor}^i_{\mathcal{Y}}(\mathcal{X}, \mathcal{X}')$ is coherent, and supported on $\mathcal{X} \cap \mathcal{X}'$ [Gro63, 6.5.1]. It follows from [Gro64, Proposition 1.4.7] that the push forward of a quasicoherent, finitely presented module under a finite morphism of finite presentation is still quasicoherent and finitely presented. Applying this to $\pi : \mathcal{X} \cap \mathcal{X}' \to S$ it follows that the push forward $\pi_* \operatorname{Tor}^i_{\mathcal{Y}}(\mathcal{X}, \mathcal{X}')$ is coherent, and since π is affine,

this push forward can be computed as the homology of the complex

$$\mathcal{L}_\bullet = \pi_* \mathcal{Q}_\bullet.$$

As above, since $\mathcal{O}_\mathcal{X}$ is flat over S, the homology of $\mathcal{L}_\bullet \otimes K$ and $\mathcal{L}_\bullet \otimes k$ compute $\mathrm{Tor}^i_{\mathcal{Y}_K}(\mathcal{X}_K, \mathcal{X}'_K)$ and $\mathrm{Tor}^i_{\mathcal{Y}_k}(\mathcal{X}_k, \mathcal{X}'_k)$, respectively. Note also that since $\mathcal{O}_{\mathcal{X}'}$ is flat over S, the terms of $\mathcal{Q}_\bullet$ and hence $\mathcal{L}_\bullet$ are flat $\mathcal{O}_S$-modules.

Since the homology of $\mathcal{L}_\bullet$ is coherent, there is a quasi-isomorphic bounded below complex $\mathcal{M}_\bullet$ of free $\mathcal{O}_S$-modules of finite rank, and since $\mathcal{L}_\bullet$ is flat over S, the restrictions $\mathcal{M}_\bullet \otimes K$ and $\mathcal{M}_\bullet \otimes k$ are quasi-isomorphic to $\mathcal{L}_\bullet \otimes K$ and $\mathcal{L}_\bullet \otimes k$, respectively [Gro61b, Chapter 0, Proposition 11.9.1; see also Remark 11.9.3]. We claim that the homology of $\mathcal{M}_\bullet$ vanishes in high degree: indeed, the homology of $\mathcal{M}_\bullet \otimes k$ computes Tor on the smooth $\mathcal{Y}_k$, so vanishes in high degree, and the claim then follows from flatness of $\mathcal{M}_\bullet$ and Nakayama's lemma. Thus, we can truncate the complex, replacing some $\mathcal{M}_i$ by the image of $\mathcal{M}_{i+1}$, which is coherent and torsion-free and hence free of finite rank. In particular, we may assume that $\mathcal{M}_\bullet$ is also bounded above. It is then clear that

$$i(\mathcal{X}_s \cdot \mathcal{X}'_s, \mathcal{Y}_s) = \sum_i (-1)^i \operatorname{rk} \mathcal{M}_i,$$

for s equal to either the generic or closed point of S, and the theorem follows. □

Remark 4.4.4. Theorem 4.4.2 can also be proved over an arbitrary base scheme as follows. It is easy to see that intersection numbers are invariant under extension of the base field, so the result behaves well under base change. Passing to the closure of the more general point, and then taking an affine neighborhood around the special point, we reduce to the case where the base is affine and irreducible. This ensures that X and X' are finitely presented over the base S, because they are flat and finite type, and then we can proceed by noetherian approximation. Say the affine base is $\operatorname{Spec} A$. Then there is a finitely generated subalgebra A_0 of A over which X, X', and Y and all of the relevant morphisms are defined, and all can be chosen so that the relevant geometric properties are preserved, including that the models of X and X' are flat [Gro66, Sections 11 and 12], and the model of Y is quasiprojective [Gro66, Theorem 8.10.5] and smooth [Gro67, Section 17.7]. In particular, we may assume the base is noetherian. By [Gro61a, Proposition 7.1.9] we can thus reduce to the case of a DVR, and the theorem follows from the well-known fact that intersection numbers are preserved by specialization over a DVR [Ful98, Section 20.3].

In the geometric case that we are interested in, where the fraction field K is algebraically closed and hence the residue field k is algebraically closed as well, we apply Theorem 4.4.2 on intersection numbers to get the following result on individual intersection multiplicities along components of the intersection of the special fibers. Note that this result is of a local nature, and holds for arbitrary expected dimension for the intersection.

Theorem 4.4.5. *Assume K is algebraically closed. Let $\mathcal{Y}$ be smooth over S, and let $\mathcal{X}$ and $\mathcal{X}'$ be closed subschemes that are flat over S. Suppose $\mathcal{X}_k$ and $\mathcal{X}'_k$ meet properly along a component Z of their intersection. Then*

$$i(Z, \mathcal{X}_k \cdot \mathcal{X}'_k; \mathcal{Y}_k) = \sum_{\widetilde{Z}} m(Z, \widetilde{Z}) \cdot i(\widetilde{Z}, \mathcal{X}_K \cdot \mathcal{X}'_K; \mathcal{Y}_K),$$

where the sum is over irreducible components $\widetilde{Z}$ of $\mathcal{X}_K \cap \mathcal{X}'_K$ whose closures contain Z, and $m(Z, \widetilde{Z})$ is the multiplicity of Z in the special fiber of the closure of $\widetilde{Z}$ inside $\mathcal{Y}$.

Proof. We first prove the special case where $\mathcal{X}$ and $\mathcal{X}'$ have complementary dimension. Suppose Z is a point. We claim that there is an affine open neighborhood $\mathcal{U}$ of Z such that $\mathcal{X} \cap \mathcal{X}' \cap \mathcal{U}$ is finite over S and has special fiber exactly Z. By upper semicontinuity of fiber dimension [Gro66, Theorem 13.1.3], the union $\mathcal{W}_1$ of the positive dimensional components of $\mathcal{X}_K \cap \mathcal{X}'_K$ and $\mathcal{X}_k \cap \mathcal{X}'_k$ is closed in $\mathcal{Y}$. The intersection of $\mathcal{X}_K$ and $\mathcal{X}'_K$ in the complement of $\mathcal{W}_1$ is a finite set of K-points. Let $\mathcal{W}_2$ be the closure of those points in this set that do not specialize to Z. We claim that any affine neighborhood of Z contained in $\mathcal{Y} \smallsetminus (\mathcal{W}_1 \cup \mathcal{W}_2)$ is the desired neighborhood. Now $\mathcal{X} \cap \mathcal{X}' \cap \mathcal{U}$ is separated, quasifinite over S and has special fiber Z, by construction, and it is locally of finite presentation, since $\mathcal{X}$ and $\mathcal{X}'$ are locally of finite type and flat, and hence locally of finite presentation. By [Gro66, Theorem 8.11.1], to show that this intersection is finite it only remains to check that it is proper over S. The generic fiber consists of finitely many K-points, each of which extends to an R-point, and one can check that this implies the valuative criterion for universal closedness [Gro61a, Theorem 7.3.8], which proves the claim. Now, note that $m(\widetilde{Z}, Z)$ is 1 for every $\widetilde{Z}$, since each $\widetilde{Z}$ is a K-point whose closure is a section. The required equality of intersection numbers then follows immediately from Theorem 4.4.2.

We now prove the general case, where the intersection may have positive dimension, by reducing to the case of a zero-dimensional intersection. Since the statement is local on $\mathcal{Y}$, we may assume $\mathcal{Y}$ is affine, that Z is the only component of $\mathcal{X}_k \cap \mathcal{X}'_k$, and that every component of $\mathcal{X}_K \cap \mathcal{X}'_K$ contains Z in its closure. Let $\widetilde{Z}_1, \ldots, \widetilde{Z}_m$ be the irreducible components of $\mathcal{X}_K \cap \mathcal{X}'_K$. Localizing further, we may also assume that Z and the $\widetilde{Z}_i$ are all smooth (although we cannot assume that the closures of the $\widetilde{Z}_i$ are smooth, as Z may appear with multiplicity greater than 1).

Now, let $\mathcal{L}_k$ be a general linear subspace of complementary dimension in the special fiber, so $\mathcal{L}_k$ meets Z transversely at finitely many points. Choose a general lift $\mathcal{L}$ of $\mathcal{L}_k$ to S; by the Zariski density of Theorem 4.2.5 applied to the Grassmannian over S, the generic fiber $\mathcal{L}_K$ meets each component of $\mathcal{X}_K \cap \mathcal{X}'_K$ transversely, at isolated points. After further localizing, we may assume $\mathcal{L}_k$ meets Z at a single point z, and every point $\widetilde{z}$ of $\mathcal{X}_K \cap \mathcal{X}'_K \cap \mathcal{L}_K$ specializes to z. Since $\mathcal{L}$ is a complete intersection, applying [Ful98, Example 8.1.10] in the special fiber, we have

$$i(Z, \mathcal{X}_k \cdot \mathcal{X}'_k; \mathcal{Y}_k) = i(z, (\mathcal{X}_k \cap \mathcal{L}_k) \cdot (\mathcal{X}'_k \cap \mathcal{L}_k); \mathcal{L}_k).$$

Now, note that an inductive application of [Gro66, Proposition 11.3.7] implies that $\mathcal{X} \cap \mathcal{L}$ and $\mathcal{X}' \cap \mathcal{L}$ remain flat over S so, by the zero-dimensional case treated above, the right hand side is equal to

$$\sum_{\widetilde{z}} i(\widetilde{z}, (\mathcal{X}_K \cap \mathcal{L}_K) \cdot (\mathcal{X}'_K \cap \mathcal{L}_K); \mathcal{L}_K).$$

If $\widetilde{Z}_i$ is the component of $\mathcal{X}_K \cap \mathcal{X}'_K$ containing $\widetilde{z}$, then a similar argument in the generic fiber gives

$$i(\widetilde{z}, (\mathcal{X}_K \cap \mathcal{L}_K) \cdot (\mathcal{X}'_K \cap \mathcal{L}_K); \mathcal{L}_K) = i(\widetilde{Z}_i, \mathcal{X}_K \cdot \mathcal{X}'_K; \mathcal{Y}_K).$$

The theorem follows since, by the zero-dimensional case above, the multiplicity $m(\widetilde{Z}_i, Z)$ is the number of points $\widetilde{z}$ in $\widetilde{Z}_i \cap \mathcal{L}_K$ specializing to z. $\square$

Corollary 4.4.6. *Let X be a pure-dimensional closed subscheme of T. Then* $\operatorname{Trop}(X)$ *is equal to the tropicalization of the fundamental cycle* $\operatorname{Trop}([X])$.

Proof. We may assume the valuation is nontrivial. It is clear that the tropicalizations agree set-theoretically, since both are the closure of the image of $X(K)$. To see that the multiplicities agree, let w be a point in the relative interior of a facet of $\operatorname{Trop}(X)$, and apply Theorem 4.4.5 in the special case where $\mathcal{Y}$ and $\mathcal{X}'$ are both equal to $\mathcal{T}^w$. $\square$

5. Lifting tropical intersections with multiplicities

We now use the results of Sections 3 and 4 to prove the main tropical lifting theorems, both as stated in the introduction and in refined forms with multiplicities. We also prove generalizations to intersections of three or more subschemes and discuss weaker lifting results when the tropicalizations do not meet properly.

5.1. Intersections of two subschemes. Let Y be a closed subvariety of T, and let X and X' be closed subschemes of pure codimension j and j', respectively, in Y.

Proof of Theorem 1.3. Suppose $\operatorname{Trop}(X)$ meets $\operatorname{Trop}(X')$ properly at a simple point w of $\operatorname{Trop}(Y)$. Let $\widetilde{K}|K$ be an extension of valued fields such that $w \in N_{\widetilde{G}}$, where $\widetilde{G}$ is the value group of G. Then the tropicalizations after base change $\operatorname{Trop}(\widetilde{X})$ and $\operatorname{Trop}(\widetilde{X}')$ meet properly at w, and Theorem A.4 implies furthermore that X_w meets X'_w properly at a smooth point of Y_w if and only if $\widetilde{X}_w$ meets $\widetilde{X}'_w$ properly at a smooth point of $\widetilde{Y}_w$. Therefore, after an extension of valued fields, we may assume $w \in N_G$.
The initial degeneration Y_w is a torus torsor containing X_w and X'_w, and the tropicalizations $\operatorname{Trop}(X_w)$ and $\operatorname{Trop}(X'_w)$ with respect to the trivial valuation

are the stars of w in $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$, respectively, and hence meet properly in the vector space $\mathrm{Trop}(Y_w)$. Theorem 3.3.1 then says that

$$\mathrm{Trop}(X_w \cap X'_w) = \mathrm{Trop}(X_w) \cap \mathrm{Trop}(X'_w).$$

In particular, $X_w \cap X'_w$ is nonempty of codimension $j + j'$, and the theorem follows, since Y_w is smooth. $\square$

Proof of Theorem 1.4. Suppose X_w meets X'_w properly at a smooth point x of Y_w. After extending the valued field we may assume the valuation is nontrivial and $w \in N_G$. In this case, the integral model $\mathcal{Y}_w$ is of finite type over $\mathrm{Spec}\, R$, by Proposition A.1. We can then pass to an open neighborhood of x in $\mathcal{Y}_w$ in which the special fiber is smooth and apply Theorem 4.1.3 to deduce that there is a point of $X \cap X'$ specializing to x. Therefore x is contained in $(X \cap X')_w$. $\square$

As noted in the introduction, Theorem 1.2 follows immediately from Theorems 1.3 and 4.1.3, and Theorem 1.1 is the special case where Y is the torus T. So this concludes the proof of the theorems stated in the introduction.

We now state and prove a refined version of Theorem 1.2 with multiplicities. If $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ meet properly at a simple point w of $\mathrm{Trop}(Y)$, and τ is a facet of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ containing w, then the tropical intersection multiplicity $i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'); \mathrm{Trop}(Y))$ along τ in $\mathrm{Trop}(Y)$ is defined by a local displacement rule, as in Section 2.6, but the displacement vector is required to be in the subspace of $N_{\mathbb{R}}$ parallel to the affine span of the facet of $\mathrm{Trop}(Y)$ containing w. Then $i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'); \mathrm{Trop}(Y))$ is the multiplicity of τ_w in the stable tropical intersection of the stars of w in $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$, inside the star of w in $\mathrm{Trop}(Y)$.

Theorem 5.1.1. *Suppose* $\mathrm{Trop}(X)$ *meets* $\mathrm{Trop}(X')$ *properly along a face* τ *of codimension* $j+j'$ *that contains a simple point of* $\mathrm{Trop}(Y)$. *Then* τ *is contained in* $\mathrm{Trop}(X \cap X')$ *with multiplicity bounded below by the tropical intersection multiplicity*

$$m_{X \cap X'}(\tau) \geq i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'); \mathrm{Trop}(Y)),$$

and both are strictly positive. Furthermore, the tropical intersection multiplicity is equal to the weighted sum of algebraic intersection multiplicities

$$i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'); \mathrm{Trop}(Y)) = \sum_{Z} i(Z, X \cdot X'; Y)\, m_Z(\tau),$$

where the sum is over components Z *of* $X \cap X'$ *whose tropicalizations contain* τ.

Although the global intersection product $X \cdot X'$ need not be defined, the intersection multiplicities $i(Z, X \cdot X'; Y)$ (as defined in terms of Tor in Section 2.5) appearing in the statement of Theorems 5.1.1 and Theorem 5.2.3, below, are nonetheless well-defined. Indeed, for every component Z of $X \cap X'$ whose tropicalization contains τ, we have that Y must be smooth on a non-empty open subset that intersects Z by the simple point hypothesis, and because $\mathrm{Trop}(X)$

is assumed to meet $\mathrm{Trop}(X')$ properly along τ, it also follows that X meets X' properly along Z.

Proof. First, by Theorem 1.2, the face τ is contained in $\mathrm{Trop}(X \cap X')$. We prove the identity between the tropical intersection multiplicity and the weighted sum of algebraic intersection multiplicities and then deduce the inequality for $m_{X\cap X'}(\tau)$, as follows. Let w be a point in τ that is a simple point of $\mathrm{Trop}(Y)$. After extending the valued field, we may assume $w \in N_G$. Then the tropical intersection multiplicity along τ is equal to the local tropical intersection multiplicity at w

$$i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'); \mathrm{Trop}(Y)) = i(\tau_w, \mathrm{Trop}(X_w) \cdot \mathrm{Trop}(X'_w); \mathrm{Trop}(Y_w)),$$

where $\tau_w = \mathbb{R}_{\geq 0}(\tau - w)$, as in Section 2.7. Then, since Y_w is a torus torsor, Theorem 3.3.1 says that this local tropical intersection multiplicity is given by

$$i(\tau_w, \mathrm{Trop}(X_w) \cdot \mathrm{Trop}(X'_w); \mathrm{Trop}(Y_w)) \;=\; \sum_{Z_0} i(Z_0, X_w \cdot X'_w; Y_w) m_{Z_0}(\tau_w),$$

where the sum is over components Z_0 of $X_w \cap X'_w$ whose tropicalizations contain τ_w. By Theorem 4.4.5, for each such Z_0,

$$i(Z_0, X_w \cdot X'_w; Y_w) = \sum_Z m(Z, Z_0)\, i(Z, X \cdot X'; Y),$$

where the sum is over components Z of $X \cap X'$ and $m(Z, Z_0)$ is the multiplicity of Z_0 in the special fiber of the closure of Z in $\mathcal{Y}_w$. Combining these identities with the results of Section 2.7 gives the required equality, since $m_Z(\tau) = \sum_{Z_0} m(Z, Z_0) m_{Z_0}(\tau_w)$, where the sum ranges over components Z_0 of the special fiber of the closure of Z.
The inequality follows immediately from this equality, since

$$m_{X\cap X'}(\tau) = \sum_Z \mathrm{length}(\mathcal{O}_{Z,X\cap X'}) m_Z(\tau),$$

by Corollary 4.4.6, and $\mathrm{length}(\mathcal{O}_{Z,X\cap X'}) \geq i(Z, X \cdot X'; Y)$ for each Z [Ful98, Proposition 7.1]. □

If $\mathrm{Trop}(X)$ meets $\mathrm{Trop}(X')$ properly in $N_\mathbb{R}$, then part (2) of Theorem 5.1.1 has a particularly simple statement equating the tropicalization of the refined intersection cycle with the stable tropical intersection.

Corollary 5.1.2. *Suppose* $\mathrm{Trop}(X)$ *meets* $\mathrm{Trop}(X')$ *properly in* $N_\mathbb{R}$. *Then*

$$\mathrm{Trop}(X \cdot X') = \mathrm{Trop}(X) \cdot \mathrm{Trop}(X').$$

We also note that the inequality in Theorem 5.1.1 can be replaced by an equality if X and X' are smooth, or mildly singular.

Corollary 5.1.3. *Under the hypotheses of Theorem 5.1.1, if X and X' are Cohen-Macaulay then*

$$m_{X\cap X'}(\tau) = i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'); \mathrm{Trop}(Y)).$$

Proof. If X and X' are Cohen-Macaulay, then the length of the scheme-theoretic intersection of X and X' along each component Z of expected dimension is equal to the intersection multiplicity $i(Z, X \cdot X'; Y)$ [Ful98, Example 8.2.7]. The result then follows from the equality in Theorem 5.1.1, since

$$m_{X \cap X'}(\tau) = \sum_Z \mathrm{length}(\mathcal{O}_{Z, X\cap X'}) m_Z(\tau),$$

by Lemma 2.7.2, where the sum is over components Z of $X \cap X'$ such that $\mathrm{Trop}(Z)$ contains τ. □

5.2. Intersections of three or more subschemes. In applications, and particularly in the context of enumerative geometry, one frequently wants to intersect more than two subschemes. The algebraic intersection product of several closed subschemes $X_1 \cdots X_r$ may be treated either by induction from results on intersection of pairs or by a standard reduction to the diagonal argument. For the reduction to the diagonal, one uses the facts that $X_1 \cap \cdots \cap X_r$ is canonically identified with the intersection $X_1 \times \cdots \times X_r \cap \Delta$ in Y^r, where Δ is the diagonal subscheme, and that $X_1 \times \cdots \times X_r \cdot \Delta$ is the push forward of $X_1 \cdots X_r$ under the diagonal embedding.

Definition 5.2.1. Let $X_1, \ldots, X_r$ be closed subschemes in T, of pure codimension $j_1, \ldots, j_r$, respectively. Then the stable tropical intersection $\mathrm{Trop}(X_1) \cdots \mathrm{Trop}(X_r)$ is the iterated pairwise stable tropical intersection

$$\mathrm{Trop}(X_1) \cdots \mathrm{Trop}(X_r) = (\cdots ((\mathrm{Trop}(X_1) \cdot \mathrm{Trop}(X_2)) \cdot \mathrm{Trop}(X_3)) \cdots \mathrm{Trop}(X_r)).$$

The following proposition shows that stable tropical intersections for three or more closed subschemes can be computed as a pairwise intersection with the tropicalization of the diagonal. In particular, the stable tropical intersection is independent of the order of the factors.

Proposition 5.2.2. *The stable tropical intersection* $\mathrm{Trop}(X_1 \times \cdots \times X_r) \cdot \mathrm{Trop}(\Delta)$ *is the image of* $\mathrm{Trop}(X_1) \cdots \mathrm{Trop}(X_r)$ *under the diagonal embedding of* $N_{\mathbb{R}}$ *in* $N_{\mathbb{R}}^r$.

Proof. Let Σ be a complete unimodular fan that contains $\mathrm{Trop}(X_1), \ldots, \mathrm{Trop}(X_r)$ as subfans. By Proposition 2.7.6, the stable tropical intersections $\mathrm{Trop}(X_1 \times \cdots \times X_r) \cdot \mathrm{Trop}(\Delta)$ and $\mathrm{Trop}(X_1) \cdots \mathrm{Trop}(X_r)$ correspond to the products of Minkowski weights $c_{\overline{X_1 \times \cdots \times X_r}} \cdot c_{\overline{\Delta}}$ and $c_{\overline{X}_1} \cdots c_{\overline{X}_r}$, on Σ^r and Σ, respectively. These products of Minkowski weights are equal to $c_{\overline{X}_1 \cdots \overline{X}_r}$ and $c_{\overline{X_1 \times \cdots \times X_r} \cdot \overline{\Delta}}$, by the identification of rings of Minkowski weights with Chow rings, discussed in Section 2.5. The proposition follows, since $\overline{X_1 \times \cdots \times X_r} \cdot \overline{\Delta}$ is the push forward of $\overline{X}_1 \cdots \overline{X}_r$ under the diagonal embedding. □

As in the case of pairwise intersections, when $X_1, \ldots, X_r$ are subschemes of pure codimension $j_1, \ldots, j_r$ in an irreducible variety Y, we define the stable tropical intersection multiplicity along a face τ of codimension $j_1 + \cdots + j_r$ in $\mathrm{Trop}(Y)$, provided that τ contains a simple point w of $\mathrm{Trop}(Y)$, as the multiplicity of τ_w

in the stable tropical intersection of the stars of w in $\mathrm{Trop}(X_1), \ldots, \mathrm{Trop}(X_r)$ inside the star of w in $\mathrm{Trop}(Y)$.

We now generalize Theorem 5.1.1 to intersections of several closed subschemes by reducing to the case of a pairwise intersection with the diagonal in Y^r.

Theorem 5.2.3. *Let $X_1, \ldots, X_r$ be closed subschemes of pure codimension $j_1, \ldots, j_r$ in Y, respectively. Suppose τ is a facet of $\mathrm{Trop}(X_1) \cap \cdots \cap \mathrm{Trop}(X_r)$ of codimension $j_1 + \cdots + j_r$ in $\mathrm{Trop}(Y)$ that contains a simple point of $\mathrm{Trop}(Y)$. Then the tropicalization $\mathrm{Trop}(X_1 \cap \cdots \cap X_r)$ contains τ with multiplicity bounded below by the tropical intersection multiplicity*

$$m_{X_1 \cap \cdots \cap X_r}(\tau) \geq i(\tau, \mathrm{Trop}(X_1) \cdots \mathrm{Trop}(X_r); \mathrm{Trop}(Y)),$$

and both are strictly positive. Furthermore, the tropical intersection multiplicity is equal to the weighted sum of algebraic intersection multiplicities

$$i(\tau, \mathrm{Trop}(X_1) \cdots \mathrm{Trop}(X_r); \mathrm{Trop}(Y)) = \sum_Z i(Z, X_1 \cdots X_r; Y) m_Z(\tau),$$

where the sum is over components of $X_1 \cap \cdots \cap X_r$ such that $\mathrm{Trop}(Z)$ contains τ.

Proof. Let τ be a facet of $\mathrm{Trop}(X_1) \cap \cdots \cap \mathrm{Trop}(X_r)$ of codimension $j_1 + \cdots + j_r$ in $\mathrm{Trop}(Y)$ that contains a simple point w of $\mathrm{Trop}(Y)$. Then the image of τ under the diagonal embedding δ of $\mathrm{Trop}(Y)$ in $\mathrm{Trop}(Y)^r$ is a facet of $\mathrm{Trop}(X_1 \times \cdots \times X_r) \cap \mathrm{Trop}(\Delta)$ of codimension $(r-1)\dim(Y) + j_1 + \cdots + j_r$ in $\mathrm{Trop}(Y^r)$, where Δ is the diagonal subscheme in Y^r, containing a simple point of $\mathrm{Trop}(Y^r)$. By Theorem 5.1.1, $\delta(\tau)$ is a face of $\mathrm{Trop}(X_1 \times \cdots \times X_r \cap \Delta)$. Furthermore, the multiplicity of $\delta(\tau)$ in the stable tropical intersection $\mathrm{Trop}(X_1 \times \cdots \times X_r) \cdot \mathrm{Trop}(\Delta)$ inside $\mathrm{Trop}(Y^r)$ is strictly positive, equal to

$$\sum_Z i(Z, X_1 \times \cdots \times X_r \cdot \Delta; Y^r) m_Z(\delta(\tau)),$$

and less than or equal to $m_{X_1 \times \cdots \times X_r \cap \Delta}(\delta(\tau))$. Now $X_1 \times \cdots \times X_r \cap \Delta$ is canonically identified with $X_1 \cap \cdots \cap X_r$, and both the tropical multiplicities and the local intersection multiplicities agree. In other words, we have

$$i(Z, X_1 \times \cdots \times X_r \cdot \Delta; Y^r) = i(Z, X_1 \cdots X_r; Y)$$

and

$$m_Z(\delta(\tau)) = m_Z(\tau).$$

The theorem then follows, by Proposition 5.2.2. □

The following application of Theorem 5.2.3, giving a formula for counting points in zero-dimensional complete intersections, has been applied by Rabinoff to construct canonical subgroups of abelian varieties over p-adic fields, via a suitable generalization for power series [Rab12a, Rab12b].

Let $f_i = \sum a_i(u) x^u$ be an equation defining X_i, with $P_i = \mathrm{conv}\{u | a_i \text{ is nonzero}\}$ its Newton polytope. Projecting the lower faces of the lifted Newton polytope $\mathrm{conv}\{(u, \nu(a_i(u)))\}$ in $M_{\mathbb{R}} \times \mathbb{R}$ gives the Newton

SUBDIVISION of P_i. This Newton subdivision is dual to $\mathrm{Trop}(X_i)$, in the sense that there is a natural polyhedral structure on $\mathrm{Trop}(X_i)$ whose faces are in order-reversing bijection with the positive dimensional faces of the Newton subdivision. A face τ of $\mathrm{Trop}(X_i)$ corresponds to the convex hull of the lattice points u such that $a_i(u)x^u$ is a monomial of minimal w-weight, for w in the relative interior of τ.

Our formula is phrased in terms of mixed volumes of faces of the Newton subdivision. Recall that, for lattice polytopes $Q_1, \ldots, Q_n$ in $M_{\mathbb{R}} \cong \mathbb{R}^n$, the euclidean volume of the Minkowski sum $b_1Q_1 + \cdots + b_nQ_n$ is a polynomial of degree n in $b_1, \ldots, b_n$, and the mixed volume $V(Q_1, \ldots, Q_n)$ is the coefficient of $b_1 \cdots b_n$ divided by $n!$. If Σ is the smallest common refinement of the inner normal fans of the Q_i, then each Q_i corresponds to a nef line bundle L_i on the toric variety $Y(\Sigma)$, and the mixed volume is equal to the intersection number, divided by $n!$,

$$V(Q_1, \ldots, Q_n) = (c_1(L_1) \cdots c_1(L_n))/n!.$$

See [Ful93, p. 116] for further details on mixed volumes and their relation to toric intersection theory[4] and [KK12, Section 4] for generalizations to arbitrary projective varieties via Newton-Okounkov bodies. Mixed volume formulas for tropical stable complete intersections are standard in the case of the trivial valuation. The earliest reference we know of relating mixed volumes to tropical complete intersections for a nontrivial (discrete) valuation is due to Smirnov [Smi96].

Corollary 5.2.4. *Let $X_1, \ldots, X_n$ be hypersurfaces in T, and suppose w is an isolated point in $\mathrm{Trop}(X_1) \cap \cdots \cap \mathrm{Trop}(X_n)$. Let Q_i be the face of the Newton subdivision corresponding to the minimal face of $\mathrm{Trop}(X_i)$ that contains w. Then the number of points in $X_1 \cap \cdots \cap X_n$ with tropicalization w, counted with multiplicities, is exactly $n!V(Q_1, \ldots, Q_n)$.*

Proof. Fix a complete unimodular fan Σ in $N_{\mathbb{R}}$ such that, for $i = 1, ..., n$, every face of $\mathrm{Star}_w \mathrm{Trop}(X_i)$ is a union of faces of Σ. Since $\mathrm{Star}_w \mathrm{Trop}(X_i)$ is the codimension 1 skeleton of the possibly degenerate inner normal fan of Q_i, it follows that Q_i corresponds to a nef line bundle L_i on $Y(\Sigma)$. Furthermore, the Minkowski weight of codimension 1 on Σ given by the tropical multiplicities on $\mathrm{Star}_w \mathrm{Trop}(X_i)$ corresponds to $c_1(L_i)$. It follows, by the compatibility of toric and stable tropical intersections that the tropical intersection multiplicity

$$i(w, \mathrm{Trop}(X_1) \cdots \mathrm{Trop}(X_n)) = (c_1(L_1) \cdots c_1(L_n)),$$

and the latter is $n!V(Q_1, \ldots, Q_n)$. By Theorem 5.2.3, this tropical intersection multiplicity is equal to the number of points in $X_1 \cap \cdots \cap X_n$ with tropicalization w, counted with multiplicities, as required. □

[4] There is a minor misstatement in the definition of mixed volumes, in the text in between displayed formulas (1) and (2) of [Ful93, p. 116]. The mixed volume is the coefficient of $\nu_1 \cdots \nu_n$ divided by $n!$, not multiplied by $n!$. The displayed formulas (1), (2), and (3) are correct. Formulas (1) and (2) uniquely determine the mixed volumes of rational polytopes, as does (3), which also characterizes mixed volumes of arbitrary convex bodies.

Remark 5.2.5. In Corollary 5.2.4, the multiplicities on the points x in $X_1 \cap \cdots \cap X_n$ with tropicalization w are equal to the lengths of the scheme-theoretic intersection of $X_1, \ldots, X_n$ along x, as in Corollary 5.1.3, since complete intersections are Cohen-Macaulay.

5.3. NON-PROPER INTERSECTIONS. Our main tropical lifting results require the tropicalizations to meet properly. Nevertheless, our results on lifting from the special fiber to the generic fiber of $\mathcal{Y}^w$, such as Theorem 1.4, still yield nontrivial statements for nonproper tropical intersections. In many cases, such as Example 6.2, the initial degenerations meet properly even when the tropicalizations do not.

PROPOSITION 5.3.1. *Suppose that for each $w \in \mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, we have that Y_w is smooth and X_w meets X'_w properly in Y_w. Then X meets X' properly in Y. Furthermore, the set of w such that $X_w \cap X'_w$ is nonempty is either empty or the underlying set of a polyhedral complex of pure codimension $j + j'$ in $\mathrm{Trop}(Y)$.*

Proof. If w is in $\mathrm{Trop}(X \cap X')$, then X_w meets X'_w along $(X \cap X')_w$. Conversely, if X_w meets X'_w properly at some smooth point x of Y_w then x is contained in $(X \cap X')_w$, by Theorem 1.4, and hence w is in $\mathrm{Trop}(X \cap X')$. Therefore, the hypotheses of the proposition ensure that the set of w such that X_w meets X'_w is exactly $\mathrm{Trop}(X \cap X')$, and $(X \cap X')_w$ is equal to $X_w \cap X'_w$, for all w. If $X_w \cap X'_w$ is nonempty then it has pure codimension $j + j'$ in Y_w. It follows that $X \cap X'$ has pure codimension $j + j'$ in Y, and hence $\mathrm{Trop}(X \cap X')$ is a polyhedral complex of pure codimension $j + j'$ in $\mathrm{Trop}(Y)$, as required. □

In the special case where Y is the ambient torus T, one can say even more. If all initial degenerations meet properly, then the set of weight vectors where the intersection is nonempty is exactly the stable tropical intersection, as was suggested to us by J. Rau.

PROPOSITION 5.3.2. *Suppose X_w meets X'_w properly in T_w for all w. Then the set of $w \in N_{\mathbb{R}}$ such that $X_w \cap X'_w$ is nonempty is exactly the underlying set of the stable tropical intersection $\mathrm{Trop}(X) \cdot \mathrm{Trop}(X')$.*

Proof. Let Σ be a complete unimodular fan that contains the stars of w in $\mathrm{Trop}(X)$ and in $\mathrm{Trop}(X')$ as subfans. First we claim that, for any w in $N_{\mathbb{R}}$, the closures $\overline{X}_w$ and $\overline{X}'_w$ meet properly in $Y(\Sigma)$ and, furthermore, $\overline{X}_w \cap O_\sigma$ and $\overline{X}'_w \cap O_\sigma$ meet properly in O_σ for every σ in Σ. Indeed, after choosing an extension of valued fields such that w is rational over the value group and subdividing Σ, we may assume that it contains tropical fans, in the sense of [Tev07], for X_w and X'_w as subfans. Then, for $v \in N_G$ in the relative interior of σ and $\epsilon \in G$ sufficiently small and positive, the initial degenerations $X_{w+\epsilon v}$ and $X'_{w+\epsilon v}$ agree with the initial degenerations of X_w and X'_w, respectively, for the weight vector v [Gub12, Corollary 10.12]. The images of these initial degenerations under projection to O_σ are identified with $\overline{X}_w \cap O_\sigma$ and $\overline{X}'_w \cap O_\sigma$, up to simultaneous translation in O_σ [Gub12, Remark 12.7]. Since $X_{w+\epsilon v}$ and

$X'_{w+\epsilon v}$ meet properly in $T_{w+\epsilon v}$, it follows that $\overline{X}_w \cap O_\sigma$ and $\overline{X}'_w \cap O_\sigma$ meet properly in O_σ, as claimed.
Since $\overline{X}_w \cap O_\sigma$ and $\overline{X}'_w \cap O_\sigma$ meet properly in O_σ for every σ in Σ, Proposition 2.7.7 says that a cone τ_w of codimension $j + j'$ in Σ appears in the tropicalization of $X_w \cdot X'_w$ with multiplicity equal to the stable tropical intersection multiplicity $i(\tau_w, \mathrm{Trop}(X_w) \cdot \mathrm{Trop}(X'_w))$ which agrees with the stable tropical intersection multiplicity $i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'))$, by Proposition 2.7.6. In particular, $X_w \cap X'_w$ is nonempty if and only if w is contained in the stable tropical intersection $\mathrm{Trop}(X) \cdot \mathrm{Trop}(X')$. □

For closed subschemes of the torus T, when a tropical intersection is not proper we can translate the subschemes by a suitable element of $T(K)$ so that the initial degenerations meet properly, without changing the tropicalizations. The following theorem extends Proposition 2.7.8 to the general case, where the valuation may not be trivial, and gives a geometric meaning to the stable tropical intersection, as the tropicalization of an intersection with a translate by a general point t such that $\mathrm{Trop}(t)$ is zero.
If t is in $T(K)$ then the tropicalization of the translate tX is the translate of $\mathrm{Trop}(X)$ by the vector $\mathrm{Trop}(t)$ in N_G. In particular, the tropicalization is invariant under translation by points t such that $\mathrm{Trop}(t)$ is zero. The set of such points is Zariski dense in $T(K)$ [Pay09b, Corollary 4.2] (see also [Pay12, Remark 2]), but not Zariski open if the valuation is nontrivial. Nevertheless, we say that a property holds for a general point t such that $\mathrm{Trop}(t)$ is zero if there is a Zariski open subset U_0 of the initial degeneration T_0 such that the property holds for all t with $t_0 \in U_0$. This set is again Zariski dense in $T(K)$.

Theorem 5.3.3. *Let X and X' be pure-dimensional closed subschemes of T. Then, for a general point t such that* $\mathrm{Trop}(t)$ *is zero,*

$$\mathrm{Trop}(X \cap tX') = \mathrm{Trop}(X) \cdot \mathrm{Trop}(X').$$

Proof. If the valuation is trivial then the theorem is given by Proposition 2.7.8. Assume the valuation is nontrivial. Let τ be a face of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, and let $w \in N_G$ be a point in τ. The initial degeneration $(tX')_w$ is the translate of X'_w by t_0. Since t_0 is general in T_0, Proposition 2.7.8 says that

$$\mathrm{Trop}(X_w \cap (tX')_w) = \mathrm{Trop}(X_w) \cdot \mathrm{Trop}(X'_w).$$

In particular, if w is not in the stable tropical intersection then the initial degenerations X_w and $(tX')_w$ are disjoint, so w is not in $\mathrm{Trop}(X \cap tX')$. On the other hand, if w is in the stable intersection then X_w meets $(tX')_w$ properly in T_w, by Proposition 2.7.8, and all points of $X_w \cap (tX')_w$ lift to the generic fiber $X \cap tX'$, by Theorem 4.1.3. Note that for w and w' in the relative interior of τ, because the isomorphisms $X_w \cong X_{w'}$ and $tX'_w \cong tX'_{w'}$ can be simultaneously induced by a single isomorphism $T_w \cong T_{w'}$, if X_w meets tX'_w then $X_{w'}$ likewise meets $tX'_{w'}$ properly. Since there are only finitely many faces of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, the point t can be chosen sufficiently general so that this holds in every face.

This shows that the underlying sets of $\mathrm{Trop}(X \cap tX')$ and $\mathrm{Trop}(X) \cdot \mathrm{Trop}(X')$ are equal. Now, let τ be a facet of $\mathrm{Trop}(X) \cdot \mathrm{Trop}(X')$ and let $w \in N_G$ be a point in the relative interior of τ. The tropical intersection multiplicity along τ is equal to the local tropical intersection multiplicity at w

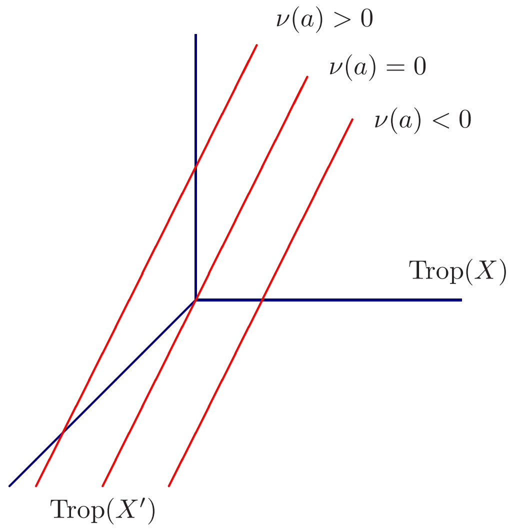
$$i(\tau, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'); \mathrm{Trop}(T)) = i(\tau_w, \mathrm{Trop}(X_w) \cdot \mathrm{Trop}(X'_w); \mathrm{Trop}(T_w)),$$

and, by Proposition 2.7.8, the right hand side is equal to $m_{X_w \cap tX'_w}(\tau_w)$. Applying Theorem 4.4.5 to the intersection of $\mathcal{X}^w$ and $(t\mathcal{X}')^w$ in $\mathcal{T}^w$, as in the proof of Theorem 5.1.1, then shows that $m_{X_w \cap tX'_w}(\tau_w)$ is equal to $m_{X \cdot tX'}(\tau)$. As in the proof of Proposition 2.7.8, the genericity of t guarantees that the cycle $X \cdot tX'$ is equal to the fundamental cycle of $X \cap tX'$. Therefore, $m_{X \cdot tX'}(\tau)$ is equal to $m_{X \cap tX'}(\tau)$, and the theorem follows. □

6. Examples

Here we give a number of examples illustrating our tropical lifting theorems and the necessity of their hypotheses. The first example involves tropicalizations that meet properly, but not necessarily in the interiors of maximal faces.

Example 6.1. Inside $Y = (K^*)^2$, let X and X' be given by $y = x + 1$ and $y = ax^2$, respectively, with $a \in K^*$. We consider three cases, according to whether or not $\nu(a)$ is zero, and its sign if it is nonzero.

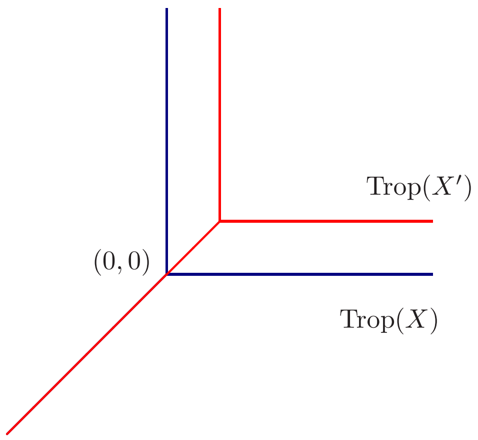


If $\nu(a)$ is positive then $\mathrm{Trop}(X)$ meets $\mathrm{Trop}(X')$ at two points, each with multiplicity 1. In this case, Theorem 5.1.1 says that X meets X' at two points, each with multiplicity 1, and one of these intersection points lies over each of the points of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$. It is also straightforward to check this directly. If $\nu(a)$ is negative, then $\mathrm{Trop}(X)$ meets $\mathrm{Trop}(X')$ at a single point, but with tropical intersection multiplicity 2. In this case, Theorem 5.1.1 says that X meets X' at either a single point with multiplicity 2, or at two points of

multiplicity 1, and one can check that the intersection is always two points of multiplicity 1. In both of these cases, the nonemptiness of $X \cap X'$ also follows from the transverse tropical lifting result of [BJS+07], since the tropicalizations meet properly in the interiors of maximal faces.
Suppose $\nu(a)$ is zero. Then $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ meet at a single point with tropical multiplicity 2, but the intersection is in a nonmaximal face of $\mathrm{Trop}(X)$, so transverse lifting results do not apply. Nevertheless, Theorem 5.1.1 still says that X and X' meet at either a single point with multiplicity 2, or at two points of multiplicity 1. Either possibility can occur; the algebraic intersection is a single point of multiplicity 2 when the characteristic is not 2 and a is equal to $-1/4$, and two points of multiplicity 1 otherwise.

In the following example, the tropicalizations meet nonproperly along a positive dimensional set that does not contain the the tropicalization of any curve.

Example 6.2. Inside $Y = (K^*)^2$, let X and X' be given by $y = x + 1$ and $y = ax + b$, respectively, with a and b in K^*. Assume a and b are not both 1, so the closures of X and X' are distinct lines in K^2. In particular, X and X' intersect in at most one point.
Suppose $\nu(a)$ is zero and $\nu(b)$ is positive. Then $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ intersect nonproperly, along the ray $\mathbb{R}_{\leq 0} \cdot (1, 1)$.



If a is 1, then the closures of X and X' are parallel lines in K^2, so none of the tropical intersection points lift.
Suppose a is not 1. Then the unique algebraic intersection point is

$$X \cap X' = ((1 - b)/(a - 1), (a - b)/(a - 1)),$$

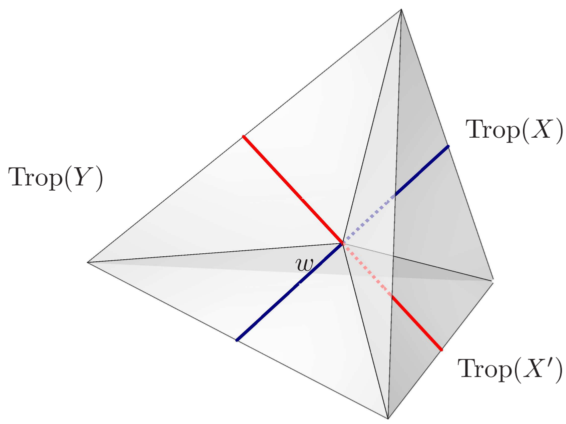
and the unique point of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ that lifts is $(-\nu(a-1), -\nu(a-1))$. For a suitable choice of a congruent to 1 modulo $\mathfrak{m}$, any nonzero G-rational point in the tropical intersection can lift. For such a, the initial degenerations X_w and X'_w coincide for all nonzero w in $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, and are disjoint otherwise. In particular, none of the initial degenerations meet properly. On the other hand, if a is not congruent to 1 modulo $\mathfrak{m}$, then the initial degenerations X_w and X'_w meet transversely at a single point for $w = (0,0)$, and are disjoint otherwise. In this case $(0,0)$ is the unique tropical intersection point that lifts, as it must be by Theorem 1.4. Note that, even though $\mathrm{Trop}(X \cap X')$ and the stable tropical intersection are both zero-dimensional, neither is necessarily contained in the other.

In the remaining examples, we assume the characteristic of K is not 2 and consider tropicalizations of skew lines inside a smooth quadric surface. These examples demonstrate the necessity of requiring the point of proper intersection to be a simple point of $\mathrm{Trop}(Y)$ or a smooth point of Y_w, in Theorems 1.2, 1.3, and 1.4.

Example 6.3. Let Y be the surface in $(K^*)^3$ given by

$$z^2 + xy + x + y = 0.$$

Then Y contains the curves X and X' given by $x + 1 = z - 1 = 0$ and $x + 1 = z + 1 = 0$, respectively. Now, $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ still meet properly at $w = (0,0,0)$, as shown.

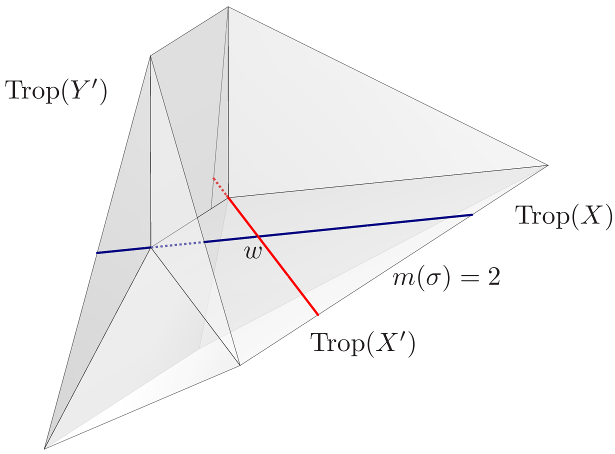


However, this tropical intersection point does not lift because the closures of X and X' are skew lines in K^3. This is consistent with Theorems 1.2 and 1.3 because w is not in the relative interior of a facet, and the initial degenerations X_w and X'_w are disjoint. The closures of X_w and X'_w in k^3 are skew lines in the closure of Y_w, which is a smooth quadric surface.

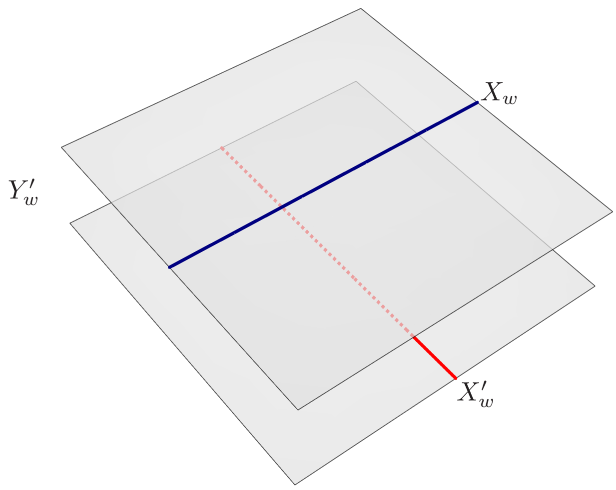
Example 6.4. Let $a \in K^*$ be an element of positive valuation, and let Y' be the surface given by

$$z^2 - 1 + a(xy + x + y + 1) = 0,$$

and let X and X' be as in Example 6.3. Then X and X' are contained in Y', and their tropicalizations still meet properly in $\mathrm{Trop}(Y')$ at the origin w, which is in the relative interior of a facet σ of $\mathrm{Trop}(Y')$, as shown.



The tropical intersection point w does not lift to $X \cap X'$, but this is still consistent with Theorems 1.2 and 1.3 because the multiplicity of the facet σ is 2, and hence w is not a simple point of $\mathrm{Trop}(Y')$. The initial degeneration Y'_w has two disjoint components, each of which contains the initial degeneration of one of the curves.

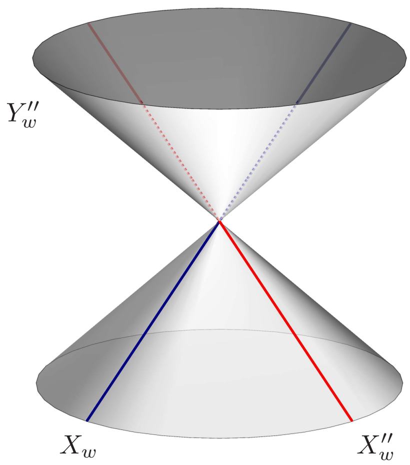


In particular, this tropical intersection point does not lift even to the intersection of the initial degenerations.

Example 6.5. Let $a \in K^*$ be an element of positive valuation, with Y'' be the surface in $(K^*)^3$ given by

$$(x+1)(y+1) + (x+z)(y+z+a) = 0.$$

Let X again be as in Example 6.3, and let X'' be the curve given by $y+1 = z-1+a = 0$. Once again, $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X'')$ meet properly at $w = (0,0,0)$. Furthermore, the initial degenerations X_w and X''_w meet properly at a single point in Y''_w.



This intersection point lifts to the initial degenerations but not to the general fiber, because the closures of X and X'' are skew lines in K^3. This is still consistent with Theorem 1.4 because Y''_w is a cone, and X_w meets X''_w at the singular point.

Appendix A. Initial degenerations, value groups, and base change

Here we study how initial degenerations behave with respect to arbitrary extensions of valued fields. These basic results are used throughout the paper to reduce our main lifting theorems to the case where w is in N_G, by extending the ground field.

Let $M_{G,w}$ be the maximal sublattice of M on which w is G-rational. In other words,

$$M_{G,w} = \{u \in M \mid \langle u, w \rangle \text{ is in } G\}.$$

Since K is algebraically closed, the value group G is divisible, and hence $M_{G,w}$ is saturated in M. If the weight of a monomial ax^u is zero, then $\langle u, w \rangle = -\nu(a)$, and hence u is in $M_{G,w}$. In particular, $M_{G,w}$ contains all exponents of monomials that restrict to nonzero functions on T_w, and T_w is a torsor over the torus associated to $M_{G,w}$.

Proposition A.1. *Suppose the valuation is nontrivial. Then the integral model $\mathcal{T}^w$ is of finite type over* $\operatorname{Spec} R$ *if and only if w is in N_G. Furthermore, if w is in N_G, then $\mathcal{T}^w$ is of finite presentation over* $\operatorname{Spec} R$.

Proof. Suppose w is in N_G. Let $u_1, \ldots, u_r$ be a basis for M, and choose $a_1, \ldots, a_r$ in K^* such that $\nu(a_i) = -\langle u_i, w\rangle$. Then $R[M]^w$ is generated over R by

$$\left\{a_1 x^{u_1}, (a_1 x^{u_1})^{-1}, \ldots, a_r x^{u_r}, (a_r x^{u_r})^{-1}\right\},$$

and hence is of finite type. Furthermore, the relations are generated by

$$\{1 - a_i x^{u_i} \cdot (a_i x^{u_i})^{-1}\},$$

so $R[M]^w$ is of finite presentation.

For the converse, suppose w is not in N_G. Then, $M_{G,w}$ is a proper sublattice of M, and hence the special fiber T_w has dimension strictly less than the dimension of T. Since $\mathcal{T}^w$ is irreducible and fiber dimension is semicontinuous in irreducible families of finite type, it follows that $\mathcal{T}^w$ is not of finite type. □

Remark A.2. Suppose the valuation is trivial, and let $\mathcal{S}_w$ be the set of lattice points $u \in M$ such that $\langle u, w\rangle \geq 0$. Then $\mathcal{S}_w$ is a subsemigroup of M, and $R[M]^w$ is naturally identified with the semigroup ring $K[\mathcal{S}_w]$. The semigroup $\mathcal{S}_w$ is finitely generated if and only if the ray spanned by w has rational slope, and it follows that $\mathcal{T}^w$ is of finite type over K if and only if w is a scalar multiple of a lattice point.

When the schemes $\mathcal{T}^w$ and $\mathcal{X}^w$ are not of finite type, there are many technical difficulties in handling them directly. These technical difficulties can be overcome by extending scalars, since these schemes become finite type after a suitable base change, as follows.

Suppose the value group G is a proper subgroup of $\mathbb{R}$, and b is a real number that is not in G. Then there is a unique valuation $\tilde{\nu}$ on the function field $K(t)$ such that $\tilde{\nu}(t) = b$. This valuation extends to the algebraic closure $\widetilde{K}$ of $K(t)$. Iterating this procedure finitely many times, we can ensure that an arbitrary w is rational over the value group of a suitable extension of K. In particular, for a suitable choice of extension $\widetilde{K}|K$, the scheme $\widetilde{\mathcal{T}}_w$ is of finite type over the valuation ring $\widetilde{R}$.

Let $\widetilde{K}|K$ be an arbitrary extension of valued fields, and let $\widetilde{G}$ be the value group of $\widetilde{K}$, with $\widetilde{R}$ the valuation ring in $\widetilde{K}$. For any closed subscheme X of T over K, the tropicalization of the base change $\widetilde{X}$ is equal to the tropicalization of X [Pay09a, Proposition 6.1], so the initial degeneration $\widetilde{X}_w$ is nonempty if and only if X_w is nonempty. Here we give a more precise geometric relationship between these initial degenerations. First we treat the associated schemes over the valuation rings. Let

$$\varphi : \widetilde{\mathcal{T}}_w \to \mathcal{T}_w$$

be the natural map induced by the inclusion of tilted group rings, which is equivariant over the projection of tori induced by the inclusions $M_{G,w} \hookrightarrow M_{\widetilde{G},w}$ and $R \hookrightarrow \widetilde{R}$.

Theorem A.3. *The scheme $\widetilde{\mathcal{X}}^w$ is the preimage of $\mathcal{X}^w$ under φ.*

Proof. It is clear that φ maps $\widetilde{\mathcal{X}}^w$ into $\mathcal{X}^w$. To show that $\widetilde{\mathcal{X}}^w$ is the full preimage of $\mathcal{X}^w$, we must prove that any Laurent polynomial in the tilted group ring $\widetilde{R}[M]^w$ that vanishes on $\widetilde{X}$ is in the ideal generated by $I_X \cap R[M]^w$. Let $f = \sum_u \alpha(u)x^u$ be a nonzero Laurent polynomial over $\widetilde{K}$ in $I_{\widetilde{X}}$. Then f can be written as

$$f = \alpha_1 f_1 + \cdots + \alpha_n f_n,$$

with $f_1, \ldots, f_n$ in I_X and linearly independent over K, and α_i in $\widetilde{K}$ all nonzero. Say $a_i(u)$ is the coefficient of x^u in f_i, so the coefficient of x^u in f is

$$\alpha(u) = \alpha_1 a_1(u) + \cdots + \alpha_n a_n(u).$$

Now, suppose f is in $\widetilde{R}[M]^w$. If each $\alpha_i f_i$ is in $\widetilde{R}[M]^w$, then it is easy to see that f is in the ideal generated by $I_X \cap R[M]^w$, by applying the case $n = 1$, below. The difficulty is that there may be some cancellation of leading terms in the above expression for f, by which we mean that the valuation of some coefficient $\alpha(u)$ may be strictly larger than $\min_i\{\widetilde{\nu}(\alpha_i a_i(u))\}$. Roughly speaking, this means that the vector $(\alpha_1, \ldots, \alpha_n)$ is nearly orthogonal to $(a_1(u), \ldots, a_n(u))$. When n is greater than 1, we proceed by carefully eliminating one term in the summation, writing f as a $\widetilde{K}$-linear combination of $f_1, ..., f_{n-1}$ plus a single element of $\widetilde{K}$ times a K-linear combination of $f_1, ..., f_n$. A suitable choice in the elimination procedure ensures that each of these two terms is in $\widetilde{R}[M]^w$, and we deduce that they are in the ideal generated by $I_X \cap R[M]^w$, by induction on n. Roughly speaking, we choose $u_1, \ldots, u_{n-1}$ so that $(a_1(u_i), \ldots, a_n(u_i))$ are as close to orthogonal as possible to $(\alpha_1, \ldots, \alpha_n)$, for $1 \leq i \leq n-1$. Then we replace $(\alpha_1, \ldots, \alpha_n)$ by the unique vector in K^n that is orthogonal to $(a_1(u_i), \ldots, a_n(u_i))$ for $1 \leq i \leq n-1$ and with nth coordinate 1. The corresponding K-linear combination of $f_1, \ldots, f_n$ is then multiplied by α_n and subtracted from f. The details are as follows.

Suppose $n = 1$. Let $m = a_1(u)x^u$ be a monomial of lowest weight in f_1, and let $g = f_1/m$. Then f can be expressed as

$$f = \alpha_1 m g,$$

with g in $I_X \cap R[M]^w$, and $\alpha_1 m$ in $\widetilde{R}[M]^w$, as required.

We proceed by induction on n. Given $u_1, \ldots, u_{n-1}$ in M, consider the matrix whose (i, j)th entry is $a_i(u_j)$, and let δ_i be the ith maximal minor

$$\delta_i = \begin{vmatrix} a_1(u_1) & \cdots & a_1(u_{n-1}) \\ \vdots & & \vdots \\ \widehat{a_i(u_1)} & \cdots & \widehat{a_i(u_{n-1})} \\ \vdots & & \vdots \\ a_n(u_1) & \cdots & a_n(u_{n-1}) \end{vmatrix}.$$

Since $f_1, \ldots, f_{n-1}$ are linearly independent over K, we may choose $u_1, \ldots, u_{n-1}$ so that δ_n is nonzero and

$$\tilde{\nu}(\alpha(u_1)) + \cdots + \tilde{\nu}(\alpha(u_{n-1})) - \nu(\delta_n)$$

is as large as possible. This choice is essential in the proof of the following claim.

We claim that

$$h = \frac{\alpha_n}{\delta_n} \sum_{i=1}^{n} (-1)^{n-i} \delta_i f_i$$

is in $\widetilde{R}[M]^w$. The claim implies that h is in the ideal generated by $I_X \cap R[M]^w$, by the case $n = 1$, above, and also that the difference $f - h$ is in $\widetilde{R}[M]^w$. The coefficient of f_n in the above expression is equal to α_n, by construction, so the difference $f - h$ can be written as a $\widetilde{K}$-linear combination of $f_1, \ldots, f_{n-1}$. It follows by induction that $f - h$ is also in the ideal generated by $I_X \cap R[M]^w$, and this proves the theorem.

It therefore remains to show that $h = \sum \beta(u) x^u$ is in $\widetilde{R}[M]^w$, which means that the w-weight of each monomial $\beta(u)x^u$ is nonnegative. Fix one such monomial, write $u_n = u$ for its exponent, and let A be the square $n \times n$ matrix whose (i, j)th entry is $a_i(u_j)$,

$$A = \begin{pmatrix} a_1(u_1) & \cdots & a_1(u_n) \\ \vdots & \ddots & \vdots \\ a_n(u_1) & \cdots & a_n(u_n) \end{pmatrix}.$$

Let A_{ij} be the (i, j)th minor of A, the determinant of the submatrix obtained by deleting the ith row and jth column. So $\delta_i = A_{in}$. Expanding $\det A$ in the last column shows that

$$\beta(u_n) = \frac{\alpha_n}{A_{nn}} \det A.$$

Therefore, the valuation of $\beta(u_n)$ is

$$\tilde{\nu}(\beta(u_n)) = \nu(\det A) - \nu(A_{nn}) + \tilde{\nu}(\alpha_n).$$

Since f is in $\widetilde{R}[M]^w$ by hypothesis, it will be enough to show that $\tilde{\nu}(\beta(u_n))$ is at least as large as $\tilde{\nu}(\alpha(u_n))$. To compare $\beta(u_n)$ with $\alpha(u_n)$, we consider the matrix

$$A' = \begin{pmatrix} \alpha_1 a_1(u_1) & \cdots & \alpha_1 a_1(u_n) \\ \vdots & \ddots & \vdots \\ \alpha_{n-1} a_{n-1}(u_1) & \cdots & \alpha_{n-1} a_{n-1}(u_n) \\ \alpha(u_1) & \cdots & \alpha(u_n) \end{pmatrix}.$$

Recall that the coefficient $\alpha(u_j)$ in the bottom row is $\alpha_1 a_1(u_j) + \cdots + \alpha_n a_n(u_j)$, so the determinant of A' is $\alpha_1 \cdots \alpha_n \det A$. Expanding $\det A'$ in the last row gives also

$$\det A' = \alpha_1 \cdots \alpha_{n-1} \left(\sum_{i=1}^{n} (-1)^{n-i} \alpha(u_i) A_{ni} \right),$$

and comparing these two expressions for $\det A'$ yields

$$\det A = \frac{1}{\alpha_n}\left(\sum_{i=1}^{n}(-1)^{n-i}\alpha(u_i)A_{ni}\right),$$

Therefore,

$$\nu(\det A) \geq \min_i\{\nu(\alpha(u_i)) + \nu(A_{ni})\} - \nu(\alpha_n).$$

Now $u_1, \ldots, u_{n-1}$ were chosen so that this minimum occurs at $i = n$. Substituting the resulting inequality for $\nu(\det A)$ into the expression for $\tilde{\nu}(\beta(u_n))$ above shows that $\tilde{\nu}(\beta(u_n))$ is greater than or equal to $\tilde{\nu}(\alpha(u_n))$. This proves the claim, and the theorem follows. $\square$

We now pass to the initial degenerations. Let

$$\phi : \widetilde{T}_w \to T_w$$

be the natural projection of torus torsors induced by the inclusions of tilted group rings, modulo monomials of strictly positive w-weight.

Theorem A.4. *The initial degeneration $\widetilde{X}_w$ is the preimage of X_w under ϕ.*

Proof. It is clear that ϕ maps $\widetilde{X}_w$ into X_w. Any element f_w of the ideal of $\widetilde{X}_w$ is the residue of a Laurent polynomial f in $I_{\widetilde{X}} \cap \widetilde{R}[M]^w$, which is then in the ideal generated by $I_X \cap R[M]^w$, by Theorem A.3. Taking residues shows that f_w in the ideal generated by the pullback of I_{X_w}, and it follows that $\widetilde{X}_w$ is the full preimage of X_w. $\square$

Remark A.5. Since ϕ is smooth and has connected fibers, it follows that many geometric properties of initial degenerations are preserved under extensions of valued fields. For instance, the sum of the multiplicities of the irreducible components of X_w is equal to that of $\widetilde{X}_w$, which is helpful for defining tropical multiplicities. Most importantly for our purposes, a point $\tilde{x}$ is smooth in $\widetilde{Y}_w$ if and only if $\phi(\tilde{x})$ is smooth in Y_w, $\widetilde{X}_w$ meets $\widetilde{X}'_w$ properly at $\tilde{x}$ if and only if X_w meets X'_w properly at $\phi(\tilde{x})$, and $\tilde{x}$ is in $(\widetilde{X} \cap \widetilde{X}')_w$ if and only if $\phi(\tilde{x})$ is in $(X \cap X')_w$.

Appendix B. Topology of finite type morphisms

Because the results may be of independent interest, we explain how Theorem 4.2.5 on the existence of closed points in fibers, and Theorem 4.1.3 on lifting points of intersection, can both be extended to an arbitrary base scheme. The proof of Proposition 4.2.2 does not use that the base scheme is the spectrum of a valuation ring of rank 1, and in fact yields the following result.

Proposition B.1. *Let $X \to S$ be a flat morphism of finite type of irreducible schemes, and suppose that D is a locally principal closed subscheme of X that does not meet the generic fiber. Then, for every $s \in S$, every irreducible component of the fiber D_s is an irreducible component of X_s.*

Remark B.2. Some hypothesis such as flatness is necessary for such results on locally principal subschemes over general base schemes, as shown by the following example. Suppose $X \to S$ is the blowup of the affine plane at the origin. Then the strict transform D of a line through the origin is locally principal and does not meet the generic fiber, but its intersection with the exceptional fiber is a single point.

We can now prove the first stated result.

Theorem B.3. *Let $X \to S$ be a morphism locally of finite type, and let s be a specialization of s' in S. Suppose x' is a point in $X_{s'}$ specializing to a closed point x in X_s. Then there is a closed point x'' in $X_{s'}$ which specializes to x, and such that x' specializes to x''. Moreover, the set of such x'' is Zariski dense in the closure of x' inside $X_{s'}$. More generally, if x is not necessarily closed in X_s, and $k(x)$ denotes the residue field of x, we can choose x'' to satisfy the inequality*

$$\text{(B.1)} \qquad \operatorname{trdeg} k(x'')/k(s') \leq \operatorname{trdeg} k(x)/k(s),$$

and again the choices of x'' are Zariski dense in the closure of x' inside $X_{s'}$. Moreover, if S is the spectrum of a valuation ring, we have equality in (B.1).

Note that the example of Remark B.2 shows that we cannot do better than the inequality (B.1) for a general base scheme.

Proof. First, in light of the generalized Proposition B.1, the argument of Theorem 4.2.5 goes through to prove the desired result in the case that S is an arbitrary valuation ring, with s' the generic point of S. The only subtlety is that in the general case, the components of $\mathcal{D}$ not containing x do not necessarily form a closed subset. Nonetheless, we can pass to an open neighborhood of x such that every component of $\mathcal{D}$ meeting the generic fiber (if there are any) must contain x, and this suffices for the argument. To remove the restriction that s' be the generic point of S, we simply note that the closure of s' is again the spectrum of a valuation ring.

We thus wish to reduce to the valuation ring case. Replacing X by the closure of x', we may assume X is integral and x' is its generic point. By [Gro61a, Proposition 7.1.4(ii)], there is a valuation ring A in $k(x')$ with a dominant morphism

$$\operatorname{Spec} A \to X$$

mapping the closed point to x. Let R be the valuation ring in the residue field $k(s')$ given by intersecting with A in $k(x')$, and let $X' = X \times_S \operatorname{Spec} R$. Then the following diagram is commutative.

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] & S \end{tikzcd}$$

By construction the map from $\operatorname{Spec} A$ to X factors through X'. So, by the case we have already handled, there is a point x'' in the generic fiber of X' over $\operatorname{Spec} R$ specializing to the image $\tilde{x}$ in X' of the closed point of $\operatorname{Spec} A$ and satisfying the desired equality of residue field extensions; moreover, such points are Zariski dense in the generic fiber. Since $k(s')$ is identified with the fraction field of R, by construction, the generic fiber of X' maps isomorphically to the generic fiber of X. Finally, observing that we have the inequality

$$\operatorname{trdeg} k(\tilde{x})/(R/\mathfrak{m}_R) \leq \operatorname{trdeg} k(x)/k(s),$$

we conclude the desired statement. □

Naive statements about global codimension and subadditivity do not extend from valuation rings of rank 1 to valuation rings of higher rank, as shown by Example 4.3.2. Nevertheless, hypotheses on codimension of intersection can still yield lifting results even when subadditivity of codimension fails, as demonstrated by the following theorem.

THEOREM B.4. *Let $Y \to S$ be a smooth morphism, and let X and X' be closed subschemes of Y, flat over S, such that the codimension in Y of every component of X and every component of X' is less than or equal to j and j', respectively. Suppose that, for some $s \in S$, the fibers X_s and X'_s meet in codimension $j + j'$ at a point x in Y_s. Then for any $s' \in S$ specializing to s:*

(1) *There is a point x' in $X_{s'} \cap X'_{s'}$ specializing to x.*
(2) *If x is closed in Y_s, then x' may be chosen to be closed in $Y_{s'}$. More generally, we may choose x' so that we have*

$$\operatorname{trdeg} k(x')/k(s') \leq \operatorname{trdeg} k(x)/k(s).$$

Proof. In light of Theorem B.3, the second assertion follows immediately from the first. Observe that the flatness hypotheses mean that the hypotheses of the theorem are preserved under arbitrary base change. In particular, we reduce to the case that S is the spectrum of a valuation ring, with s' the generic point. We then prove the desired statement with a reduction to the diagonal and inductive application of Proposition B.1, using that S is the spectrum of a valuation ring to preserve the flatness hypothesis. □

Remark B.5. In Example 4.3.2, the special fibers of the two subschemes coincide, and the generic fibers are disjoint. This does not contradict Theorem B.4 because the special fibers do not meet properly, and the failure of subadditivity is for simple numerical reasons. The intersection that does not lift has dimension one larger than expected, but is supported in a fiber of codimension r, which is greater than 1.

APPENDIX C. AN APPLICATION TO TROPICAL ELIMINATION THEORY

Let X be an irreducible closed subscheme of T and let $\varphi : T \to T'$ be a homomorphism of tori that induces a generically finite morphism from X to the closure of its image, which we denote X'. Then, set theoretically, $\operatorname{Trop}(X')$

is the image of $\mathrm{Trop}(X)$ under the induced linear map $\phi : N_{\mathbb{R}} \to N'_{\mathbb{R}'}$. The fundamental problem of tropical elimination theory, solved by Sturmfels and Tevelev for the special case where the valuation is trivial, is to determine the multiplicities on the facets of $\mathrm{Trop}(X')$. Here we use tropical lifting theorems to generalize [ST08, Theorem 1.1] to the case of a nontrivial valuation. See also [BPR11, Section 8] for an analytic proof of this result and applications to curves.

After subdividing, we may assume that ϕ maps each face of $\mathrm{Trop}(X)$ onto a face of $\mathrm{Trop}(X')$.

Theorem C.1. *The multiplicity of a facet σ' in* $\mathrm{Trop}(X')$ *is*

$$m(\sigma') = \frac{1}{\delta} \sum_{\phi(\sigma)=\sigma'} m(\sigma) \cdot [N'_{\sigma'} : \phi(N_\sigma)],$$

where δ is the degree of φ.

Theorem C.1 and Corollary 4.4.6 together imply that tropicalization of cycles commutes with push forward. See [Gub12, Theorem 13.17].

Proof. We prove the theorem by intersecting X and X' with suitable translates of subtori and then counting points using tropical intersection theory and lifting theorems. First, we choose the translated subtori to ensure that these intersections occur in a locus where α is well-behaved.

Since φ is generically finite, there is a dense open subset $U' \subset X'$ such that the induced map $\varphi^{-1}(U') \to U'$ is finite of degree δ [Har77, Exercise II.3.7]. Shrinking U' further, if necessary, we may assume that U' is smooth and $\varphi^{-1}(U')$ is flat over U', so the preimage of a zero-dimensional subscheme of length m in U is a zero-dimensional subscheme of length $\delta \cdot m$.

Let Λ' be a sublattice of N' complementary to $N'_{\sigma'}$, so N' splits as a direct sum

$$N' = N'_{\sigma'} \oplus \Lambda'.$$

We write $T'_{\Lambda'}$ for the subtorus of T' whose lattice of one-parameter subgroups is Λ'. Let $\Lambda \subset N$ be the preimage of Λ', with T_Λ the associated subtorus of T. Let $\widetilde{T}_\Lambda$ be the preimage of $T'_{\Lambda'}$ which is the product of T_Λ with a zero-dimensional scheme of length

$$\ell = \frac{[N' : \phi(N)]}{[\Lambda' : \phi(\Lambda)]}.$$

If the characteristic of K is zero, or prime to ℓ, then $\widetilde{T}_\Lambda$ is a union of translates of T_Λ by ℓ distinct torsion points.

We claim that, for any nonempty open subset $U \subset X$ there is an open dense set of $t \in T$ such that $t\widetilde{T}_\Lambda \cap X$ is contained in U. To see this, consider the incidence subscheme W in $T \times X$ parametrizing pairs (t, x) such that x is in $t\widetilde{T}_\Lambda$. Then the first projection is dominant and generically finite, while the second projection is flat and maps W surjectively onto X. Therefore, the preimage of $X \smallsetminus U$ has positive codimension in W and hence projects into a set of positive codimension in T. Therefore the complement of the closure of $p_1(p_2^{-1}(X \smallsetminus U))$

is an open dense subset of T consisting of points t such that $t\widetilde{T}_\Lambda \cap X$ is contained in U.

We fix

$$U = \varphi^{-1}(U') \cap X^{\mathrm{sm}}$$

and choose $v \in N_G$ such that $v' = \phi(v)$ is in the relative interior of σ'. Since $\mathrm{Trop}^{-1}(v)$ is Zariski dense in T, we can choose $t \in \mathrm{Trop}^{-1}(v)$ in the open dense subset of T described above, such that $t\widetilde{T}_\Lambda \cap X$ is contained in U. Note that, since v' is in the relative interior of the maximal face σ' of $\mathrm{Trop}(X')$, it has finitely many preimages $v_1, \ldots, v_r$ in $\mathrm{Trop}(X)$, one in each maximal face σ_i mapping onto σ'. Let σ_i be the maximal face of $\mathrm{Trop}(X)$ containing v_i.

Let $t' = \varphi(t)$. We now consider $t'T'_{\Lambda'} \cap X'$ and $t\widetilde{T}_\Lambda \cap X$, and especially the parts of these intersections that live in $\mathrm{Trop}^{-1}(v')$ and $\mathrm{Trop}^{-1}(v_i)$, for $1 \leq i \leq r$, respectively. By construction, $\mathrm{Trop}(t'T'_{\Lambda'})$ is the affine linear space $\Lambda'_{\mathbb{R}} + v'$ with multiplicity 1, and meets $\mathrm{Trop}(X')$ transversally at v'. The translation of $t'T'_{\Lambda'}$ by a sufficiently small vector in $N'_{\mathbb{R}}$ also meets σ' transversally at a single point, so the fan displacement rule gives the local tropical intersection multiplicity as

$$i(v', \mathrm{Trop}(X') \cdot \mathrm{Trop}(t'T'_{\Lambda'})) = m(\sigma') \cdot [N' : N'_{\sigma'} + \Lambda'].$$

By the choice of t, the intersection of $t'T'_{\Lambda'}$ with X' is contained in the smooth locus of X'. In particular, both are Cohen-Macaulay along their intersection in $\mathrm{Trop}^{-1}(v')$. Therefore, Corollary 5.1.3 says that the intersection of X' with $t'T'_{\Lambda'}$ in $\mathrm{Trop}^{-1}(v')$ is a zero-dimensional scheme Z' of length $i(v', \mathrm{Trop}(X') \cdot \mathrm{Trop}(t'T'_{\Lambda'}))$. Similarly, $\mathrm{Trop}(t\widetilde{T}_\Lambda)$ is the affine linear space $\Lambda_{\mathbb{R}} + v$ with multiplicity ℓ, and meets $\mathrm{Trop}(X)$ transversally at v_i with local tropical intersection multiplicity

$$i(v, \mathrm{Trop}(X) \cdot \mathrm{Trop}(t\widetilde{T}_\Lambda)) = \ell \cdot m(\sigma_i) \cdot [N : N_{\sigma_i} + \Lambda].$$

Both $t\widetilde{T}_\Lambda$ and X are smooth and hence Cohen-Macaulay along their intersection, so the intersection of $t\widetilde{T}_\Lambda$ with X in $\mathrm{Trop}^{-1}(v_i)$ is a zero-dimensional scheme Z_i of length $i(v, \mathrm{Trop}(X) \cdot \mathrm{Trop}(t\widetilde{T}_\Lambda))$.

By the choice of t, the map φ is finite of degree δ in a neighborhood of Z'. Furthermore, the preimage $\varphi^{-1}(Z')$ is exactly $Z_1 \cup \cdots \cup Z_r$. Therefore,

$$\mathrm{length}(Z') = \frac{1}{\delta}(\mathrm{length}(Z_1) + \cdots + \mathrm{length}(Z_r)).$$

Substituting the above tropical intersection multiplicities for these lengths gives the identity

$$m(\sigma') \cdot [N' : N_{\sigma'} + \Lambda'] = \frac{\ell}{\delta} \cdot \sum_{i=1}^{r} m(\sigma_i) \cdot [N : N_{\sigma_i} + \Lambda].$$

Now $\sigma_1, \ldots, \sigma_r$ are exactly the faces of $\mathrm{Trop}(X)$ that map onto σ, and $\ell = [N' : \phi(N)]/[\Lambda' : \phi(\Lambda)]$. By rearranging terms, one then sees that, to prove the theorem, it suffices to show

$$[N' : \phi(N)] \cdot [N : N_{\sigma_i} + \Lambda] = [N'_{\sigma'} : \phi(N_{\sigma_i})] \cdot [\Lambda' : \phi(\Lambda)],$$

for $1 \leq i \leq r$. Both sides are equal to $[N' : \phi(N_{\sigma_i} + \Lambda)]$, and the theorem follows. $\square$

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