A Criterion for Flatness of Sections of Adjoint Bundle of a Holomorphic Principal Bundle over a Riemann Surface

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Abstract. Let E_G be a holomorphic principal G–bundle over a compact connected Riemann surface, where G is a connected reductive affine algebraic group defined over $\mathbb{C}$, such that E_G admits a holomorphic connection. Take any $\beta \in H^0(X, \mathrm{ad}(E_G))$, where $\mathrm{ad}(E_G)$ is the adjoint vector bundle for E_G, such that the conjugacy class $\beta(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x. We give a sufficient condition for the existence of a holomorphic connection on E_G such that β is flat with respect to the induced connection on $\mathrm{ad}(E_G)$.

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1. Introduction

A holomorphic vector bundle E over a compact connected Riemann surface X admits a holomorphic connection if and only if every indecomposable component of E is of degree zero [We], [At]. This criterion generalizes to the holomorphic principal G–bundles over X, where G is a connected reductive affine algebraic group defined over $\mathbb{C}$ [AB].

Let E_G be a holomorphic principal G–bundle over X, where X and G are as above. Let $\mathfrak{g}$ denote the Lie algebra of G. Let β be a holomorphic section of the adjoint vector bundle $\mathrm{ad}(E_G) = E_G \times^G \mathfrak{g}$. Our aim here is to find a criterion for the existence of a holomorphic connection on E_G such that β is flat with respect to the induced connection on $\mathrm{ad}(E_G)$. A sufficient condition is obtained in Theorem 3.4.

For $G = \mathrm{GL}(r, \mathbb{C})$, Theorem 3.4 says the following:

Let E be a holomorphic vector bundle of rank r on X, and $\beta \in H^0(X, End(E))$. Let

$$E = \bigoplus_{i=1}^{\ell} E_i$$

be the generalized eigen-bundle decomposition for β. So $\beta|_{E_i} = \lambda_i \cdot \mathrm{Id}_{E_i} + N_i$, where $\lambda_i \in \mathbb{C}$, and either $N_i = 0$ or N_i is nilpotent. If $N_i \neq 0$, then assume that the section $N^{r_i - 1}$ is nowhere vanishing, where r_i is the rank of the vector bundle E_i. Also, assume that E admits a holomorphic connection. Then Theorem 3.4 says that E admits a holomorphic connection D such that β is flat with respect to the connection on $End(E)$ induced by D.
One may ask whether the above condition that N^{r_i-1} is nowhere vanishing whenever $N_i \neq 0$ can be replaced by the weaker condition that the conjugacy class of $\beta(x)$, $x \in X$, is independent of x. As example constructed by the referee shows that this cannot be done (see Example 3.6).

2. Flat sections of the adjoint bundle

Let X be a compact connected Riemann surface. Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. The Lie algebra of G will be denoted by $\mathfrak{g}$. The set of all conjugacy classes in $\mathfrak{g}$ will be denoted by $\mathfrak{g}/G$.
Let

$$f : E_G \longrightarrow X \tag{2.1}$$

be a holomorphic principal G–bundle. Define the adjoint vector bundle

$$\mathrm{ad}(E_G) := E_G \times^G \mathfrak{g}\,.$$

In other words, $\mathrm{ad}(E_G)$ is the quotient of $E_G \times \mathfrak{g}$ where any $(z\,, v) \in E_G \times \mathfrak{g}$ is identified with $(zg\,, \mathrm{Ad}(g)(v))$, $g \in G$; here $\mathrm{Ad}(g)$ is the automorphism of $\mathfrak{g}$ corresponding to the automorphism of G defined by $g' \longmapsto g^{-1}g'g$. Therefore, we have a set-theoretic map

$$\phi : \mathrm{ad}(E_G) \longrightarrow \mathfrak{g}/G \tag{2.2}$$

that sends any $(z\,, v) \in E_G \times \mathfrak{g}$ to the conjugacy class of v.
Let

$$\mathrm{At}(E_G) := (f_* TE_G)^G \subset f_* TE_G$$

be the Atiyah bundle for E_G, where f is the projection in (2.1), and TE_G is the holomorphic tangent bundle of E_G (the action of G on E_G produces an action of G on the direct image f_*TE_G). The Atiyah bundle fits in a short exact sequence of vector bundles

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G) \longrightarrow TX \longrightarrow 0\,; \tag{2.3}$$

the above projection $\mathrm{At}(E_G) \longrightarrow TX$, where TX is the holomorphic tangent bundle of X, is defined by the differential $df : TE_G \longrightarrow f^*TX$ of f. A *holomorphic connection* on E_G is a holomorphic splitting of the sequence in (2.3) [At].

A holomorphic connection D on E_G induces a holomorphic connection on each holomorphic fiber bundle associated to E_G. In particular, the vector bundle $\mathrm{ad}(E_G)$ gets a holomorphic connection from D. A section β of $\mathrm{ad}(E_G)$ is said to be *flat* with respect to D if β is flat with respect to the connection on $\mathrm{ad}(E_G)$ induced by D.

Lemma 2.1. *Take a holomorphic connection D on E_G, and let $\beta \in H^0(X, \mathrm{ad}(E_G))$ be flat with respect to D. Then the element $\phi \circ \beta(x) \in \mathfrak{g}/G$, where $x \in X$, is independent of x.*

Proof. Any holomorphic connection on X is flat because $\Omega_X^2 = 0$. Using the flat connection D, we may holomorphically trivialize E_G on any connected simply connected open subset of X. With respect to such a trivialization, the section β is a constant one because it is flat with respect to D. This immediately implies that $\phi \circ \beta(x) \in \mathfrak{g}/G$ is independent of $x \in X$. □

3. Holomorphic connections on Principal G-bundles

A nilpotent element v of the Lie algebra of a complex semisimple group H is called *regular nilpotent* if the dimension of the centralizer of v in H coincides with the rank of H [Hu, p. 53].
As before, G is a connected reductive affine algebraic group defined over $\mathbb{C}$. Take E_G as in (2.1).

Proposition 3.1. *Take any $\beta \in H^0(X, \mathrm{ad}(E_G))$. Assume that*

- *E_G admits a holomorphic connection,*
- *the element $\phi \circ \beta(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x, where ϕ is defined in* (2.2), *and*
- *for every adjoint type simple quotient H of G, the section of the adjoint bundle* $\mathrm{ad}(E_H)$ *given by β, where $E_H := E_G \times^G H$ is the principal H–bundle over X associated to E_G for the projection $G \longrightarrow H$, has the property that it is either zero or it is regular nilpotent at some point of X.*

Then the principal G–bundle E_G admits a holomorphic connection for which β is flat.

Proof. Let $Z := G/[G\,,G]$ be the abelian quotient. It is a product of copies of $\mathbb{C}^*$. There are quotients $H_1\,,\cdots\,,H_\ell$ of G such that

(1) each H_i is simple of adjoint type (the center is trivial), and
(2) the natural homomorphism

$$\varphi\,:\,G\longrightarrow Z\times\prod_{i=1}^{\ell}H_i \tag{3.1}$$

is surjective, and the kernel of φ is a finite group.

Let

$$E_Z\,:=\,E_G\times^G Z \quad \text{and} \quad E_{H_i}\,:=\,E_G\times^G H_i\,,\ i\,\in\,[1\,,\ell]\,,$$

be the holomorphic principal Z–bundle and principal H_i–bundle associated to E_G for the quotient Z and H_i respectively. Let $\text{ad}(E_Z)$ and $\text{ad}(E_{H_i})$ be the adjoint vector bundles for E_Z and E_{H_i} respectively. Since the homomorphism φ in (3.1) induces an isomorphism of Lie algebras, we have

$$\text{ad}(E_G) = \text{ad}(E_Z) \oplus (\bigoplus_{i=1}^{\ell} \text{ad}(E_{H_i}))\,. \tag{3.2}$$

Let β_Z (respectively, β_i) be the holomorphic section of $\text{ad}(E_Z)$ (respectively, $\text{ad}(E_{H_i})$) given by β using the decomposition in (3.2). Since the conjugacy class of $\beta(x)$ is independent of $x \in X$ (the second condition in the proposition), we conclude that the conjugacy class of $\beta_i(x)$ is also independent of $x \in X$.

A holomorphic connection on E_G induces a holomorphic connection on E_Z. Since E_Z admits a holomorphic connection, and Z is a product of copies of $\mathbb{C}^*$, there is a unique holomorphic connection D^Z on E_Z whose monodromy lies inside the maximal compact subgroup of Z. The connection on $\text{ad}(E_Z)$ induced by this connection D^Z has the property that any holomorphic section of $\text{ad}(E_Z)$ is flat with respect to it. In particular, the section β_Z is flat with respect to this induced connection on $\text{ad}(E_Z)$.

Now take any $i \in [1\,,\ell]$. A holomorphic connection on E_G induces a holomorphic connection on E_{H_i}. If the section β_i is zero at some point, then β_i is identically zero because the conjugacy class of $\beta_i(x)$ is independent of x. Hence, in that case β_i is flat with respect to any connection on $\text{ad}(E_{H_i})$. Therefore, assume that β_i is not zero at any point of X.

By the assumption in the proposition, β_i is regularly nilpotent over some point of X. Since the conjugacy class of $\beta_i(x)$, $x \in X$, is independent of x, we conclude that β_i is regular nilpotent over every point of X. We will now show that the holomorphic principal H_i–bundle E_{H_i} is semistable.

For each point $x \in X$, from the fact that $\beta_i(x)$ is regular nilpotent we conclude that there is a unique Borel subalgebra $\widetilde{\mathfrak{b}}_x$ of $\text{ad}(E_{H_i})_x$ such that $\beta_i(x) \in \widetilde{\mathfrak{b}}_x$ [Hu, p. 62, Theorem]. Let

$$\widetilde{\mathfrak{b}} \subset \text{ad}(E_{H_i})$$

be the Borel subalgebra bundle such that for every point x the fiber $(\widetilde{\mathfrak{b}})_x$ is $\widetilde{\mathfrak{b}}_x$. Fix a Borel subgroup $B \subset H_i$. Using $\widetilde{\mathfrak{b}}$, we will construct a holomorphic reduction of structure group of E_{H_i} to the subgroup B.

Let $\mathfrak{b}$ be the Lie algebra of B. The Lie algebra of H_i will be denoted by $\mathfrak{h}_i$. We recall that $\text{ad}(E_{H_i})$ is the quotient of $E_{H_i} \times \mathfrak{h}_i$ where two points $(z_1\,, v_1)$ and $(z_2\,, v_2)$ of $E_{H_i} \times \mathfrak{h}_i$ are identified if there is an element $h \in H_i$ such that $z_2 = z_1 h$ and $v_2 = \text{Ad}(h)(v_1)$, where $\text{Ad}(h)$ is the automorphism of $\mathfrak{h}_i$ corresponding to the automorphism $y \longmapsto h^{-1}yh$ of H_i. For any point $x \in X$, let $E_{B,x} \subset (E_{H_i})_x$ be the complex submanifold consisting of all $z \in (E_{H_i})_x$ such that for all $v \in \mathfrak{b}$, the image of $(z\,, v)$ in $\text{ad}(E_{H_i})_x$ lies in $\widetilde{\mathfrak{b}}_x$. Since any two Borel subalgebras of $\mathfrak{h}_i$ are conjugate, it follows that $E_{B,x}$ is nonempty. The normalizer of $\mathfrak{b}$ in H_i coincides with B. From this it follows that $E_{B,x}$ is preserved by the action of B on $(E_{H_i})_x$, with the action of B on $E_{B,x}$ being transitive. Let

$$E_B \subset E_{H_i}$$

be the complex submanifold such that $E_B \bigcap (E_{H_i})_x = E_{B,x}$ for every $x \in X$. From the above properties of $E_{B,x}$ it follows immediately that E_B is a holomorphic reduction of structure group of the principal H_i–bundle E_{H_i} to the subgroup B.

Consider the adjoint action of B on $\mathfrak{b}_1 := \mathfrak{b}/[\mathfrak{b}\,,\mathfrak{b}]$. Let

$$E_B(\mathfrak{b}_1) := E_B \times^B \mathfrak{b}_1 \longrightarrow X$$

be the holomorphic vector bundle associated to E_B for the B–module $\mathfrak{b}_1$. Since β_i is everywhere regular nilpotent, it follows that the vector bundle $E_B(\mathfrak{b}_1)$ is trivial. Consequently, for any character χ of B which is a nonnegative integral combination of simple roots, the line bundle $E_B(\chi) \longrightarrow X$ associated to E_B for the character χ is trivial [AAB, p. 708, Theorem 5]. Therefore, for any character χ of B, the line bundle $E_B(\chi)$ associated to E_B for χ is trivial.

Let d be the complex dimension of $\mathfrak{h}_i$. Consider the adjoint action on B on $\mathfrak{h}_i$. Note that $\mathrm{ad}(E_{H_i})$ is identified with the vector bundle associated to the principal B–bundle E_B for this B–module $\mathfrak{h}_i$. Since B is solvable, there is a filtration of B–modules

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{d-1} \subset V_d = \mathfrak{h}_i$$

such that $\dim V_j = j$ for all $j \in [1\,,d]$. The corresponding filtration of vector bundles associated to E_B is a filtration of $\mathrm{ad}(E_{H_i})$ such that the successive quotients are the line bundles $E_B(V_j/V_{j-1})$, $i \in [1\,,d]$, associated to E_B for the B–modules V_j/V_{j-1}. We noted above that the line bundles associated to E_B for the characters of B are trivial.

Therefore, we get a filtration of $\mathrm{ad}(E_{H_i})$ such that each successive quotient is the trivial line bundle. This immediately implies that the vector bundle $\mathrm{ad}(E_{H_i})$ is semistable. Hence the holomorphic principal H_i–bundle E_{H_i} is semistable [AAB, p. 698, Lemma 3].

Since H_i is simple, and E_{H_i} is semistable, there is a natural holomorphic connection on E_{H_i} [BG, p. 20, Theorem 1.1] (set the Higgs field in [BG, Theorem 1.1] to be zero). Let D^{H_i} denote this connection. The vector bundle $\mathrm{ad}(E_{H_i})$ being semistable of degree zero has a natural holomorphic connection [Si, p. 36, Lemma 3.5]. See also [BG, p. 20, Theorem 1.1]. (In both [Si, Lemma 3.5] and [BG, Theorem 1.1] set the Higgs field to be zero.) Let D^{ad} denote this holomorphic connection on $\mathrm{ad}(E_{H_i})$. This connection D^{ad} coincides with the one induced by D^{H_i} (see the construction of the connection in [BG]).

Any holomorphic section of $\mathrm{ad}(E_{H_i})$ is flat with respect to D^{ad}. To see this, let

$$\phi : \mathcal{O}_X \longrightarrow \mathrm{ad}(E_{H_i})$$

be the homomorphism given by a nonzero holomorphic section of $\mathrm{ad}(E_{H_i})$. Since $\mathrm{image}(\phi)$ is a semistable subbundle of $\mathrm{ad}(E_{H_i})$ of degree zero, the connection D^{ad} preserves $\mathrm{image}(\phi)$, and, moreover, the restriction of D^{ad} to $\mathrm{image}(\phi)$ coincides with the canonical connection of $\mathrm{image}(\phi)$ [Si, p. 36, Lemma 3.5].

The canonical connection on the trivial holomorphic line bundle image(ϕ) is the trivial connection (the monodromy is trivial).
In particular, the connection on $\text{ad}(E_{H_i})$ induced by D^{H_i} has the property that the section β_i is flat with respect to it.
Since the homomorphism of Lie algebras corresponding to φ (in (3.1)) is an isomorphism, if we have holomorphic connections on E_Z and E_{H_i}, $[1\,,\ell]$, then we get a holomorphic connection on E_G; simply pullback the connection form using the map

$$E_G \longrightarrow E_Z \times_X E_{H_1} \times_X \cdots \times_X E_{H_\ell}\,.$$

The connection on E_G given by the connections on E_Z and E_{H_i}, $[1\,,\ell]$, constructed above satisfies the condition that β is flat with respect to it. This completes the proof of the proposition. $\square$

Lemma 3.2. *Take any semisimple section $\beta_s \in H^0(X,\, \text{ad}(E_G))$ such that the element $\phi\circ\beta_s(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x, where ϕ is defined in* (2.2). *Then β_s produces a holomorphic reduction of structure group of E_G to a Levi subgroup of a parabolic subgroup of G. The conjugacy class of the Levi subgroup is determined by $\phi\circ\beta_s(x) \in \mathfrak{g}/G$.*

Proof. Fix an element

$$v_0 \in \mathfrak{g}$$

such that the image of v_0 in $\mathfrak{g}/G$ coincides with $\phi\circ\beta_s(x)$. Let $\mathbb{L} \subset G$ be the centralizer of v_0. It is known that $\mathbb{L}$ is a Levi subgroup of some parabolic subgroup of G [DM, p. 26, Proposition 1.22] (note that $\mathbb{L}$ is the centralizer of the torus in G generated by v_0). In particular, $\mathbb{L}$ is connected and reductive.
For any point $x \in X$, let $F_x \subset (E_G)_x$ be the complex submanifold consisting of all points z such that the image of $(z\,,v_0)$ in $\text{ad}(E_G)_x$ coincides with $\beta_s(x)$. (Recall that $\text{ad}(E_G)$ is a quotient of $E_G\times\mathfrak{g}$.) Let

$$F_{\mathbb{L}} \subset E_G$$

be the complex submanifold such that $F_{\mathbb{L}}\bigcap(E_G)_x = F_x$ for all $x \in X$. It is straightforward to check that $F_{\mathbb{L}}$ is a holomorphic reduction of structure group of the principal G–bundle E_G to the subgroup $\mathbb{L}$. $\square$

Remark 3.3. If $\beta_s \in H^0(X,\, \text{ad}(E_G))$ is such that $\beta_s(x)$ is semisimple for every $x \in X$, then the conjugacy class of $\beta_s(x)$ is in fact independent of x. But we do not need this here.

From the Jordan decomposition of a complex reductive Lie algebra we know that for any holomorphic section θ of $\text{ad}(E_G)$, there is a naturally associated semisimple (respectively, nilpotent) section θ_s (respectively, θ_n) such that $\theta = \theta_s+\theta_n$.
Take any $\beta \in H^0(X,\, \text{ad}(E_G))$. Let

$$\beta = \beta_s+\beta_n$$

be the Jordan decomposition. Assume that the element $\phi\circ\beta(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x, where ϕ is defined in (2.2). This implies that

$\phi \circ \beta_s(x) \in \mathfrak{g}/G$, $x \in X$, is also independent of x. Let $(\mathbb{L}\,, F_{\mathbb{L}})$ be the principal bundle constructed in Lemma 3.2 from β_s. Let H be an adjoint type simple quotient of $\mathbb{L}$. Let

$$E_H := F_{\mathbb{L}} \times^{\mathbb{L}} H \longrightarrow X$$

be the holomorphic principal H–bundle associated to $F_{\mathbb{L}}$ for the projection $\mathbb{L} \longrightarrow H$.
Since $[\beta_s\,, \beta_n] = 0$, from the construction of $F_{\mathbb{L}}$ it follows that

$$\beta_n \in H^0(X,\, \mathrm{ad}(F_{\mathbb{L}})) \subset H^0(X,\, \mathrm{ad}(E_G))\,.$$

Therefore, using the natural projection $\mathrm{ad}(F_{\mathbb{L}}) \longrightarrow \mathrm{ad}(E_H)$, given by the projection of the Lie algebra $\mathrm{Lie}(\mathbb{L}) \longrightarrow \mathrm{Lie}(H)$, the above section β_n produces a holomorphic section of $\mathrm{ad}(E_H)$. Let

$$\widetilde{\beta}_n \in H^0(X,\, \mathrm{ad}(E_H)) \tag{3.3}$$

be the section constructed from β_n.

Theorem 3.4. *Take any $\beta \in H^0(X,\, \mathrm{ad}(E_G))$. Let $\beta = \beta_s + \beta_n$ be the Jordan decomposition. Assume that*

- *E_G admits a holomorphic connection,*
- *the element $\phi \circ \beta(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x, where ϕ is defined in* (2.2)*, and*
- *for every adjoint type simple quotient H of $\mathbb{L}$, the section $\widetilde{\beta}_n$ in* (3.3) *of* $\mathrm{ad}(E_H)$ *has the property that it is either zero or it is regular nilpotent at some point of X.*

Then the principal G–bundle E_G admits a holomorphic connection for which β is flat.

Proof. Note that

$$\beta_s \in H^0(X,\, \mathrm{ad}(F_{\mathbb{L}})) \subset H^0(X,\, \mathrm{ad}(E_G))\,.$$

In fact, for each point $x \in X$, the element $\beta_s(x) \in \mathrm{ad}(F_{\mathbb{L}}))_x$ is in the center of $\mathrm{ad}(F_{\mathbb{L}}))_x$. Consider the abelian quotient

$$Z_{\mathbb{L}} = \mathbb{L}/[\mathbb{L}\,, \mathbb{L}]\,.$$

Let $F_{Z_{\mathbb{L}}}$ be the holomorphic principal $Z_{\mathbb{L}}$–bundle over X obtained by extending the structure group of the principal $\mathbb{L}$–bundle $F_{\mathbb{L}}$ using the quotient map $\mathbb{L} \longrightarrow Z_{\mathbb{L}}$. The adjoint vector bundle $\mathrm{ad}(F_{Z_{\mathbb{L}}})$ is a direct summand of $\mathrm{ad}(F_{\mathbb{L}})$. In fact, for each $x \in X$, the subspace $\mathrm{ad}(F_{Z_{\mathbb{L}}})_x \subset \mathrm{ad}(F_{\mathbb{L}})_x$ is the center of the Lie algebra $\mathrm{ad}(F_{\mathbb{L}})_x$.
A holomorphic connection on $F_{\mathbb{L}}$ induces a holomorphic connection on E_G. We can now apply Proposition 3.1 to $F_{\mathbb{L}}$ to complete the proof of the theorem. But for that we need to show that $F_{\mathbb{L}}$ admits a holomorphic connection.
Let $\mathfrak{l}$ be the Lie algebra of $\mathbb{L}$. Consider the inclusion of $\mathbb{L}$–modules $\mathfrak{l} \hookrightarrow \mathfrak{g}$ given by the inclusion of $\mathbb{L}$ in G. Since $\mathbb{L}$ is reductive, there is a sub $\mathbb{L}$–module $S \subset \mathfrak{g}$ such that the natural homomorphism

$$\mathfrak{l} \oplus S \longrightarrow \mathfrak{g}$$

is an isomorphism (so S is a complement of $\mathfrak{l}$). Let

$$p : \mathfrak{g} \longrightarrow \mathfrak{l} \tag{3.4}$$

be the projection given by the above decomposition of $\mathfrak{g}$.

Let D be a holomorphic connection on E_G. So D is a holomorphic 1–form on the total space of E_G with values in the Lie algebra $\mathfrak{g}$. Let D' be the restriction of this 1–form to the complex submanifold $F_{\mathbb{L}} \subset E_G$. Consider the $\mathfrak{l}$–valued 1–form $p \circ D'$ on $E_{\mathbb{L}}$, where p is the projection in (3.4). This $\mathfrak{l}$–valued 1–form on $F_{\mathbb{L}}$ defines a holomorphic connection of the principal $\mathbb{L}$–bundle $F_{\mathbb{L}}$. Now Proposition 3.1 completes the proof of the theorem. □

We recall that a holomorphic vector bundle W on X has a holomorphic connection if and only if each indecomposable component of W is of degree zero [We], [At, p. 203, Theorem 10]. This criterion generalizes to holomorphic principal G–bundles on X (see [AB] for details).

We now set $G = \mathrm{GL}(r, \mathbb{C})$ in Theorem 3.4. Let E be a holomorphic vector bundle of rank r on X. Take any

$$\beta \in H^0(X, End(E)).$$

Let

$$E = \bigoplus_{i=1}^{\ell} E_i \tag{3.5}$$

be the generalized eigen-bundle decomposition of E for β. Therefore,

$$\beta|_{E_i} = \lambda_i \cdot \mathrm{Id}_{E_i} + N_i\,,$$

where $\lambda \in \mathbb{C}$, and either $N_i = 0$ or N_i is nilpotent.

Then Theorem 3.4 has the following corollary:

COROLLARY 3.5. *For every $N_i \neq 0$, assume that the section N^{r_i-1} of $End(E_i)$ is nowhere vanishing, where r_i is the rank of the vector bundle E_i in* (3.5)*. If the holomorphic vector bundle E admits a holomorphic connection, then it admits a holomorphic connection D such that the section β is flat with respect to the connection on $End(E)$ induced by D.*

Consider the condition on β in Corollary 3.5 which says that N^{r_i-1} is nowhere vanishing whenever $N_i \neq 0$. This condition implies that the image of $\beta(x)$ in $\mathrm{M}(r, \mathbb{C})/\mathrm{GL}(r, \mathbb{C})$ is independent of $x \in X$ (here $\mathrm{GL}(r, \mathbb{C})$ acts on its Lie algebra $\mathrm{M}(r, \mathbb{C})$ via conjugation). Therefore, one may ask whether the above mentioned condition in Corollary 3.5 can be replaced by the weaker condition that the conjugacy class of $\beta(x)$ is independent of $x \in X$. Note that if this can be done, then the sufficient condition in Corollary 3.5 for the existence of a connection on E such that β is flat with respect to it actually becomes a necessary and sufficient condition. The following construction of the referee shows that the condition in Corollary 3.5 cannot be replaced by the weaker condition that the conjugacy class of $\beta(x)$ is independent of $x \in X$.

Example 3.6 (Referee). Let X be of sufficiently high genus. Let L and M be holomorphic line bundles on X of degree 1 and degree -2 respectively. Then there exists an indecomposable holomorphic vector bundle E of rank three on X satisfying the following condition: it admits a filtration of holomorphic subbundles

$$L = E_1 \subset E_2 \subset E$$

such that $E_2/L = M$ and $E/E_2 = L$. We omit the detailed arguments given by the referee showing that such a vector bundle E exists. Let β denote the composition

$$E \longrightarrow E/E_2 = L = E_1 \hookrightarrow E\,.$$

Clearly, the conjugacy class of $\beta(x)$ is independent of $x \in X$. The vector bundle E admits a holomorphic connection because it is indecomposable of degree zero. If D is a holomorphic connection on E such that β is flat with respect to the connection on $End(E)$ induced by D, then the subsheaf image$(\beta) \subset E$ is flat with respect to D. But image$(\beta) = L$ does not admit a holomorphic connection because it is of nonzero degree.

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