Birational Motivic Homotopy Theories and the Slice Filtration

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Abstract. We show that there is an equivalence of categories between the orthogonal components for the slice filtration and the birational motivic stable homotopy categories which are constructed in this paper. Relying on this equivalence, we are able to describe the slices for projective spaces (including $\mathbb{P}^\infty$), Thom spaces and blow ups.

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1 Definitions and Notation

Our main result, theorem 3.6, shows that there is an equivalence of categories between the orthogonal components for the slice filtration (see definition 1.1) and the weakly birational motivic stable homotopy categories which are constructed in this paper (see definition 2.9). Relying on this equivalence; we are able to describe over an arbitrary base scheme (see theorems 4.2, 4.4 and 4.6) the slices for projective spaces (including $\mathbb{P}^\infty$), Thom spaces and blow ups. We also construct the birational motivic stable homotopy categories (see definition 2.4), which are a natural generalization of the weakly birational motivic stable homotopy categories, and show (see proposition 2.12) that there exists a Quillen equivalence between them when the base scheme is a perfect field. Our approach was inspired by the work of Kahn-Sujatha [1] on birational motives, where the existence of a connection between the layers of the slice filtration and birational invariants is explicitly suggested. Furthermore, this approach allows to obtain analogues for the slice filtration in the unstable setting (see remark 3.8).

In this paper X will denote a Noetherian separated base scheme of finite Krull dimension, Sch_X the category of schemes of finite type over X and Sm_X the full subcategory of Sch_X consisting of smooth schemes over X regarded as a site with the Nisnevich topology. All the maps between schemes will be considered over the base X. Given $Y \in Sch_X$, all the closed subsets Z of Y will be considered as closed subschemes with the reduced structure.

Let $\mathcal{M}$ be the category of pointed simplicial presheaves in Sm_X equipped with the motivic Quillen model structure [14] constructed by Morel-Voevodsky [8, p. 86 Thm. 3.2], taking the affine line $\mathbb{A}^1_X$ as interval. Given a map $f : Y \to W$ in Sm_X, we will abuse notation and denote by f the induced map $f : Y_+ \to W_+$ in $\mathcal{M}$ between the corresponding pointed simplicial presheaves represented by Y and W respectively.

We define T in $\mathcal{M}$ to be the pointed simplicial presheaf represented by $S^1 \wedge \mathbb{G}_m$, where $\mathbb{G}_m$ is the multiplicative group $\mathbb{A}^1_X - \{0\}$ pointed by 1, and S^1 denotes the simplicial circle. Given an arbitrary integer $r \geq 1$, S^r (respectively $\mathbb{G}_m^r$) will denote the iterated smash product $S^1 \wedge \cdots \wedge S^1$ (respectively $\mathbb{G}_m \wedge \cdots \wedge \mathbb{G}_m$) with r-factors; $S^0 = \mathbb{G}_m^0$ will be by definition equal to the pointed simplicial presheaf X_+ represented by the base scheme X.

Let $Spt(\mathcal{M})$ denote Jardine's category of symmetric T-spectra on $\mathcal{M}$ equipped with the motivic model structure defined in [6, Thm. 4.15] and let $\mathcal{SH}$ denote its homotopy category, which is triangulated. We will follow Jardine's notation [6, p. 506-507] where F_n denotes the left adjoint to the n-evaluation functor

$$\begin{array}{rcl} Spt(\mathcal{M}) & \xrightarrow{ev_n} & \mathcal{M} \\ (X^m)_{m\geq 0} & \longmapsto & X^n \end{array}$$

Notice that $F_0(A)$ is just the usual infinite suspension spectrum $\Sigma^\infty_T A$.

For every integer $q \in \mathbb{Z}$, we consider the following family of symmetric T-spectra

$$C^q_{\mathit{eff}} = \{F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \mid n, r, s \geq 0; s - n \geq q; U \in Sm_X\} \tag{1.1}$$

where U_+ denotes the simplicial presheaf represented by U with a disjoint base point. Let $\Sigma^q_T \mathcal{SH}^{\mathit{eff}}$ denote the smallest full triangulated subcategory of $\mathcal{SH}$ which contains C^q_{eff} and is closed under arbitrary coproducts. Voevodsky [16] defines the slice filtration in $\mathcal{SH}$ to be the following family of triangulated subcategories

$$\cdots \subseteq \Sigma^{q+1}_T \mathcal{SH}^{\mathit{eff}} \subseteq \Sigma^q_T \mathcal{SH}^{\mathit{eff}} \subseteq \Sigma^{q-1}_T \mathcal{SH}^{\mathit{eff}} \subseteq \cdots$$

It follows from the work of Neeman [9], [10] that the inclusion

$$i_q : \Sigma^q_T \mathcal{SH}^{\mathit{eff}} \to \mathcal{SH}$$

has a right adjoint $r_q : \mathcal{SH} \to \Sigma_T^q \mathcal{SH}^{\mathit{eff}}$, and that the following functors

$$\begin{aligned} f_q :& \mathcal{SH} \to \mathcal{SH} \\ s_{<q} :& \mathcal{SH} \to \mathcal{SH} \\ s_q :& \mathcal{SH} \to \mathcal{SH} \end{aligned}$$

are triangulated, where f_q is defined as the composition $i_q \circ r_q$; and $s_{<q}, s_q$ are characterized by the fact that for every $E \in \mathcal{SH}$, we have distinguished triangles in $\mathcal{SH}$:

$$f_qE \xrightarrow{\theta_q^E} E \xrightarrow{\pi_{<q}^E} s_{<q}E \longrightarrow S^1 \wedge f_qE$$

$$f_{q+1}E \xrightarrow{\rho_q^E} f_qE \xrightarrow{\pi_q^E} s_qE \longrightarrow S^1 \wedge f_{q+1}E$$

We will refer to f_qE as the $(q-1)$-*connective cover* of E, to $s_{<q}E$ as the q-*orthogonal component* of E, and to s_qE as the q-*slice* of E. It follows directly from the definition that $s_{<q+1}E, s_qE$ satisfy that for every symmetric T-spectrum K in $\Sigma_T^{q+1}\mathcal{SH}^{\mathit{eff}}$:

$$\mathrm{Hom}_{\mathcal{SH}}(K, s_{<q+1}E) = \mathrm{Hom}_{\mathcal{SH}}(K, s_qE) = 0$$

Definition 1.1. Let $E \in \mathit{Spt}(\mathcal{M})$ be a symmetric T-spectrum. We will say that E is n-*orthogonal*, if for all $K \in \Sigma_T^n \mathcal{SH}^{\mathit{eff}}$

$$\mathrm{Hom}_{\mathcal{SH}}(K, E) = 0$$

Let $\mathcal{SH}^{\perp}(n)$ denote the full subcategory of $\mathcal{SH}$ consisting of the n-orthogonal objects.

The slice filtration admits an alternative definition in terms of (left and right) Bousfield localization of $\mathit{Spt}(\mathcal{M})$ [11, 12]. The Bousfield localizations are constructed following Hirschhorn's approach [2]. In order to be able to apply Hirschhorn's techniques, it is necessary to know that $\mathit{Spt}(\mathcal{M})$ is *cellular* [2, Def. 12.1.1] and *proper* [2, Def. 13.1.1].

Theorem 1.2. *The Quillen model category $\mathit{Spt}(\mathcal{M})$ is:*

1. cellular *(see [5], [3, Cor. 1.6] or [12, Thm. 2.7.4]).*
2. proper *(see [6, Thm. 4.15]).*

For details and definitions about Bousfield localization we refer the reader to Hirschhorn's book [2]. Let us just mention the following theorem of Hirschhorn, which guarantees the existence of left and right Bousfield localizations.

Theorem 1.3 (see [2, Thms. 4.1.1 and 5.1.1]). *Let $\mathcal{A}$ be a Quillen model category which is cellular and proper. Let L be a set of maps in $\mathcal{A}$ and let K be a set of objects in $\mathcal{A}$. Then:*

1. *The left Bousfield localization of $\mathcal{A}$ with respect to L exists.*
2. *The right Bousfield localization of $\mathcal{A}$ with respect to the class of K-colocal equivalences exists.*

Now, we can describe the slice filtration in terms of suitable Bousfield localizations of $Spt(\mathcal{M})$.

Theorem 1.4 (see [12]).

1. *Let $R_{C^q_{eff}} Spt(\mathcal{M})$ be the right Bousfield localization of $Spt(\mathcal{M})$ with respect to the set of objects C^q_{eff} (see Eqn. (1.1)). Then its homotopy category $R_{C^q_{eff}}\mathcal{SH}$ is triangulated and naturally equivalent to $\Sigma^q_T \mathcal{SH}^{eff}$. Moreover, the functor f_q is canonically isomorphic to the following composition of triangulated functors:*

$$\mathcal{SH} \xrightarrow{R} R_{C^q_{eff}}\mathcal{SH} \xrightarrow{C_q} \mathcal{SH}$$

where R is a fibrant replacement functor in $Spt(\mathcal{M})$, and C_q a cofibrant replacement functor in $R_{C^q_{eff}} Spt(\mathcal{M})$.

2. *Let $L_{<q} Spt(\mathcal{M})$ be the left Bousfield localization of $Spt(\mathcal{M})$ with respect to the set of maps*

$$\{F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \to * \mid F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C^q_{eff}\}$$

Then its homotopy category $L_{<q}\mathcal{SH}$ is triangulated and naturally equivalent to $\mathcal{SH}^{\perp}(q)$. Moreover, the functor $s_{<q}$ is canonically isomorphic to the following composition of triangulated functors:

$$\mathcal{SH} \xrightarrow{Q} L_{<q}\mathcal{SH} \xrightarrow{W_q} \mathcal{SH}$$

where Q is a cofibrant replacement functor in $Spt(\mathcal{M})$, and W_q a fibrant replacement functor in $L_{<q} Spt(\mathcal{M})$.

3. *Let $S^q Spt(\mathcal{M})$ be the right Bousfield localization of $L_{<q+1} Spt(\mathcal{M})$ with respect to the set of objects*

$$\{F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \mid n, r, s \geq 0; s - n = q; U \in Sm_X\}$$

Then its homotopy category $S^q\mathcal{SH}$ is triangulated and the identity functor

$$id : R_{C^q_{eff}} Spt(\mathcal{M}) \to S^q Spt(\mathcal{M})$$

is a left Quillen functor. Moreover, the functor s_q is canonically isomorphic to the following composition of triangulated functors:

$$\mathcal{SH} \xrightarrow{R} R_{C^q_{eff}}\mathcal{SH} \xrightarrow{C_q} S^q\mathcal{SH} \xrightarrow{W_{q+1}} R_{C^q_{eff}}\mathcal{SH} \xrightarrow{C_q} \mathcal{SH}$$

Proof. (1) and (3) follow directly from [12, Thms. 3.3.9, 3.3.25, 3.3.50, 3.3.68]. On the other hand, (2) follows from proposition 3.2.27(3) together with theorem 3.3.26; proposition 3.3.30 and theorem 3.3.45 in [12] □

2 BIRATIONAL AND WEAKLY BIRATIONAL COHOMOLOGY THEORIES

In this section, we construct the birational and weakly birational motivic stable homotopy categories. These are defined as left Bousfield localizations of $Spt(\mathcal{M})$ with respect to maps which are induced by open immersions with a numerical condition in the codimension of the closed complement (which is assumed to be smooth in the weakly birational case). The existence of the left Bousfield localizations considered in this section follows immediately from theorems 1.2 and 1.3.

LEMMA 2.1. *Let $a, a', b, b', p, p' \geq 0$ be integers such that $a - p = a' - p'$ and $b - p = b' - p'$. Assume that $p \geq p'$, then for every $Y \in Sm_X$, there is a weak equivalence in $Spt(\mathcal{M})$, which is natural with respect to Y*

$$g^{a,b}_{p,p'}(Y) : F_p(S^a \wedge \mathbb{G}^b_m \wedge Y_+) \to F_{p'}(S^{a'} \wedge \mathbb{G}^{b'}_m \wedge Y_+)$$

Proof. We have the following adjunction (see [12, Def. 2.6.8])

$$(F_p, ev_p, \varphi) : \mathcal{M} \to Spt(\mathcal{M})$$

Using this adjunction, we define $g^{a,b}_{p,p'}(Y)$ as adjoint to the identity map:

$$\begin{aligned} S^a \wedge \mathbb{G}^b_m \wedge Y_+ \xrightarrow{id} ev_p(F_{p'}(S^{a'} \wedge \mathbb{G}^{b'}_m \wedge Y_+)) &\cong S^{p-p'} \wedge \mathbb{G}^{p-p'}_m \wedge S^{a'} \wedge \mathbb{G}^{b'}_m \wedge Y_+ \\ &\cong S^a \wedge \mathbb{G}^b_m \wedge Y_+ \end{aligned}$$

Thus, it is clear that $g^{a,b}_{p,p'}(Y)$ is natural in Y, and it follows from [12, Prop. 2.4.26] that it is a weak equivalence in $Spt(\mathcal{M})$. □

DEFINITION 2.2 (see [13, section 7.5]). Let $Y \in Sch_X$, and Z a closed subscheme of Y. The *codimension* of Z in Y, $codim_Y Z$ is the infimum (over the generic points z_i of Z) of the dimensions of the local rings $\mathcal{O}_{Y,z_i}$.

Since X is Noetherian of finite Krull dimension and Y is of finite type over X, $codim_Y Z$ is always finite.

DEFINITION 2.3. We fix an arbitrary integer $n \geq 0$, and consider the following set of open immersions which have a closed complement of codimension at least $n + 1$

$$\begin{aligned} B_n = \{\iota_{U,Y} : & U \to Y \text{ open immersion} \mid \\ & Y \in Sm_X; Y \text{ irreducible}; (codim_Y Y \backslash U) \geq n + 1\} \end{aligned}$$

The letter B stands for birational.

Now we consider the left Bousfield localization of $Spt(\mathcal{M})$ with respect to a suitable set of maps induced by the families of open immersions B_n described above.

Definition 2.4. Let $n \in \mathbb{Z}$ be an arbitrary integer.

1. Let $B_nSpt(\mathcal{M})$ denote the left Bousfield localization of $Spt(\mathcal{M})$ with respect to the set of maps
$$sB_n = \{F_p(\mathbb{G}_m^b \wedge \iota_{U,Y}) : b,p,r \geq 0, b-p \geq n-r; \iota_{U,Y} \in B_r\}.$$
2. Let $b^{(n)}$ denote its fibrant replacement functor and $\mathcal{SH}(B_n)$ its associated homotopy category.

For $n \neq 0$ we will call $\mathcal{SH}(B_n)$ the *codimension* $n+1$*-birational motivic stable homotopy category,* and for $n = 0$ we will call it the *birational motivic stable homotopy category.*

Lemma 2.5. *Let $n \in \mathbb{Z}$ be an arbitrary integer. Then for every $a \geq 0$, the maps*
$$S^a \wedge sB_n = \{F_p(S^a \wedge \mathbb{G}_m^b \wedge \iota_{U,Y}) : b,p,r \geq 0, b-p \geq n-r; \iota_{U,Y} \in B_r\}$$
are weak equivalences in $B_nSpt(\mathcal{M})$.

Proof. Let $F_p(\mathbb{G}_m^b \wedge \iota_{U,Y}) \in sB_n$ with $\iota_{U,Y} \in B_r$. Both $F_p(\mathbb{G}_m^b \wedge U_+)$ and $F_p(\mathbb{G}_m^b \wedge Y_+)$ are cofibrant in $Spt(\mathcal{M})$ (see [12, Props. 2.4.17, 2.6.18 and Thm. 2.6.30]) and hence also in $B_nSpt(\mathcal{M})$. By construction, $F_p(\mathbb{G}_m^b \wedge \iota_{U,Y})$ is a weak equivalence in $B_nSpt(\mathcal{M})$; and [2, Thm. 4.1.1.(4)] implies that $B_nSpt(\mathcal{M})$ is a simplicial model category. Thus, it follows from Ken Brown's lemma (see [4, lemma 1.1.12]) that $F_p(S^a \wedge \mathbb{G}_m^b \wedge \iota_{U,Y})$ is also a weak equivalence in $B_nSpt(\mathcal{M})$ for every $a \geq 0$. $\square$

Proposition 2.6. *Let E be an arbitrary symmetric T-spectrum. Then E is fibrant in $B_nSpt(\mathcal{M})$ if and only if the following conditions hold:*

1. *E is fibrant in $Spt(\mathcal{M})$.*
2. *For every $a,b,p,r \geq 0$ such that $b-p \geq n-r$; and every $\iota_{U,Y} \in B_r$, the induced map*
$$\mathrm{Hom}_{\mathcal{SH}}(F_p(S^a \wedge \mathbb{G}_m^b \wedge Y_+), E) \xrightarrow[\cong]{\iota_{U,Y}^*} \mathrm{Hom}_{\mathcal{SH}}(F_p(S^a \wedge \mathbb{G}_m^b \wedge U_+), E)$$
is an isomorphism.

Proof. ($\Rightarrow$): Since the identity functor
$$id : Spt(\mathcal{M}) \to B_nSpt(\mathcal{M})$$

is a left Quillen functor, the conclusion follows from the derived adjunction

$$(Q, b^{(n)}, \varphi) : \mathcal{SH} \to \mathcal{SH}(B_n)$$

together with lemma 2.5.
($\Leftarrow$): Assume that E satisfies (1) and (2). Let ω_0, η_0 denote the base points of the pointed simplicial sets $\mathrm{Map}_*(F_p(\mathbb{G}_m^b \wedge Y_+), E)$ and $\mathrm{Map}_*(F_p(\mathbb{G}_m^b \wedge U_+), E)$ respectively. Since $F_p(\mathbb{G}_m^b \wedge Y_+)$ and $F_p(\mathbb{G}_m^b \wedge U_+)$ are always cofibrant, by [2, Def. 3.1.4(1)(a) and Thm. 4.1.1(2)] it is enough to show that every map in sB_n induces a weak equivalence of simplicial sets:

$$\mathrm{Map}_*(F_p(\mathbb{G}_m^b \wedge Y_+), E) \xrightarrow{\iota^*_{U,Y}} \mathrm{Map}_*(F_p(\mathbb{G}_m^b \wedge U_+), E)$$

Since $Spt(\mathcal{M})$ is a pointed simplicial model category, we observe that lemma 6.1.2 in [4] and remark 2.4.3(2) in [12] imply that the following diagram is commutative for $a \geq 0$ and all the vertical arrows are isomorphisms

$$\begin{tikzcd}
\pi_{a,\omega_0}\mathrm{Map}_*(F_p(\mathbb{G}_m^b \wedge Y_+), E) \arrow[r, "\iota^*_{U,Y}"] \arrow[d, "\cong"'] & \pi_{a,\eta_0}\mathrm{Map}_*(F_p(\mathbb{G}_m^b \wedge U_+), E) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{SH}}(F_p(S^a \wedge \mathbb{G}_m^b \wedge Y_+), E) \arrow[r, "\iota^*_{U,Y}"] & \mathrm{Hom}_{\mathcal{SH}}(F_p(S^a \wedge \mathbb{G}_m^b \wedge U_+), E)
\end{tikzcd}$$

by hypothesis, the bottom row is an isomorphism, hence the top row is also an isomorphism. This implies that for every map in sB_n, the induced map

$$\mathrm{Map}_*(F_p(\mathbb{G}_m^b \wedge Y_+), E) \xrightarrow{\iota^*_{U,Y}} \mathrm{Map}_*(F_p(\mathbb{G}_m^b \wedge U_+), E)$$

is a weak equivalence when it is restricted to the path component of $\mathrm{Map}_*(F_p(\mathbb{G}_m^b \wedge Y_+), E)$ containing ω_0. This holds in particular for

$$\mathrm{Map}_*(F_{p+1}(\mathbb{G}_m^{b+1} \wedge Y_+), E) \xrightarrow{\iota^*_{U,Y}} \mathrm{Map}_*(F_{p+1}(\mathbb{G}_m^{b+1} \wedge U_+), E)$$

Therefore, the following map is a weak equivalence of pointed simplicial sets, since taking S^1-loops kills the path components that do not contain the base point

$$\begin{tikzcd}
\mathrm{Map}_*(S^1, \mathrm{Map}_*(F_{p+1}(\mathbb{G}_m^{b+1} \wedge Y_+), E)) \arrow[d] \\
\mathrm{Map}_*(S^1, \mathrm{Map}_*(F_{p+1}(\mathbb{G}_m^{b+1} \wedge U_+), E))
\end{tikzcd}$$

Now, since $Spt(\mathcal{M})$ is a simplicial model category we deduce that the rows in the following commutative diagram are isomorphisms

$$\begin{tikzcd}
\mathrm{Map}_*(S^1, \mathrm{Map}_*(F_{p+1}(\mathbb{G}_m^{b+1} \wedge Y_+), E)) \arrow[d, "\iota_{U,Y}^*"'] \arrow[dr, "\cong"] & \\
& \mathrm{Map}_*(F_{p+1}(S^1 \wedge \mathbb{G}_m^{b+1} \wedge Y_+), E) \arrow[d, "\iota_{U,Y}^*"] \\
\mathrm{Map}_*(S^1, \mathrm{Map}_*(F_{p+1}(\mathbb{G}_m^{b+1} \wedge U_+), E)) \arrow[dr, "\cong"] & \\
& \mathrm{Map}_*(F_{p+1}(S^1 \wedge \mathbb{G}_m^{b+1} \wedge U_+), E)
\end{tikzcd}$$

Thus, by the three out of two property for weak equivalences, we conclude that

$$\mathrm{Map}_*(F_{p+1}(S^1 \wedge \mathbb{G}_m^{b+1} \wedge Y_+), E) \xrightarrow{\iota_{U,Y}^*} \mathrm{Map}_*(F_{p+1}(S^1 \wedge \mathbb{G}_m^{b+1} \wedge U_+), E)$$

is also a weak equivalence of pointed simplicial sets. Finally, lemma 2.1 implies that the following diagram is commutative and the vertical arrows are weak equivalences in $Spt(\mathcal{M})$

$$\begin{tikzcd}
\mathrm{Map}_*(F_{p+1}(S^1 \wedge \mathbb{G}_m^{b+1} \wedge Y_+), E) \arrow[r, "\iota_{U,Y}^*"] & \mathrm{Map}_*(F_{p+1}(S^1 \wedge \mathbb{G}_m^{b+1} \wedge U_+), E) \\
\mathrm{Map}_*(F_p(\mathbb{G}_m^{b} \wedge Y_+), E) \arrow[u, "g_{p+1,p}^{1,b+1}(Y)^*"] \arrow[r, "\iota_{U,Y}^*"] & \mathrm{Map}_*(F_p(\mathbb{G}_m^{b} \wedge U_+), E) \arrow[u, "g_{p+1,p}^{1,b+1}(U)^*"']
\end{tikzcd}$$

Thus, we conclude by the two out of three property for weak equivalences that the bottom arrow is also a weak equivalence in $Spt(\mathcal{M})$. $\square$

PROPOSITION 2.7. *The homotopy category $\mathcal{SH}(B_n)$ is a compactly generated triangulated category in the sense of Neeman [9, Def. 1.7].*

Proof. We will prove first that $\mathcal{SH}(B_n)$ is a triangulated category. For this, it is enough to show that the smash product with the simplicial circle induces a Quillen equivalence (see [14, sections I.2, I.3])

$$(- \wedge S^1, \Omega_{S^1}-, \varphi) : B_n Spt(\mathcal{M}) \to B_n Spt(\mathcal{M})$$

It follows from [2, Thm. 4.1.1.(4)] that this adjunction is a Quillen adjunction, and the same argument as in [12, Cor. 3.2.38] (replacing [12, Prop. 3.2.32] with proposition 2.6) allows us to conclude that it is a Quillen equivalence. Finally, since $\mathcal{SH}$ is a compactly generated triangulated category (see [12, Prop. 3.1.5]) and the identity functor is a left Quillen functor

$$id : Spt(\mathcal{M}) \to B_n Spt(\mathcal{M})$$

it follows from the derived adjunction

$$(Q, b^{(n)}, \varphi) : \mathcal{SH} \to \mathcal{SH}(B_n)$$

that $\mathcal{SH}(B_n)$ is also compactly generated, having exactly the same set of generators as $\mathcal{SH}$. □

Definition 2.8. We fix an arbitrary integer $n \geq 0$, and consider the following set of open immersions with smooth closed complement of codimension at least $n+1$

$$\begin{aligned} WB_n = \{\iota_{U,Y} : & U \to Y \text{ open immersion} \mid \\ & Y, Z = Y \backslash U \in Sm_X; Y \text{ irreducible}; (codim_Y Z) \geq n+1\} \end{aligned}$$

Notice that every map in WB_n is also in B_n, but the converse doesn't hold. The reason to consider maps $\iota_{U,Y}$ in WB_n is that if the closed complement is smooth, then the Morel-Voevodsky homotopy purity theorem (see [8, Thm. 2.23]) characterizes the homotopy cofibre of $\iota_{U,Y}$ in terms of the Thom space of the normal bundle for the closed immersion $Y \backslash U \to Y$.

Definition 2.9. Let $n \in \mathbb{Z}$ be an arbitrary integer.

1. Let $WB_n Spt(\mathcal{M})$ denote the left Bousfield localization of $Spt(\mathcal{M})$ with respect to the set of maps

$$sWB_n = \{F_p(\mathbb{G}_m^b \wedge \iota_{U,Y}) : b, p, r \geq 0, b - p \geq n - r; \iota_{U,Y} \in WB_r\}.$$

2. Let $wb^{(n)}$ denote its fibrant replacement functor and $\mathcal{SH}(WB_n)$ its associated homotopy category.

For $n \neq 0$ we will call $\mathcal{SH}(WB_n)$ the *codimension* $n+1$*-weakly birational motivic stable homotopy category*, and for $n = 0$ we will call it the *weakly birational motivic stable homotopy category*.

Proposition 2.10. *Let E be an arbitrary symmetric T-spectrum. Then E is fibrant in $WB_n Spt(\mathcal{M})$ if and only if the following conditions hold:*

1. *E is fibrant in $Spt(\mathcal{M})$.*
2. *For every $a, b, p, r \geq 0$ such that $b - p \geq n - r$; and every $\iota_{U,Y} \in WB_r$, the induced map*

$$\mathrm{Hom}_{\mathcal{SH}}(F_p(S^a \wedge \mathbb{G}_m^b \wedge Y_+), E) \xrightarrow[\cong]{\iota_{U,Y}^*} \mathrm{Hom}_{\mathcal{SH}}(F_p(S^a \wedge \mathbb{G}_m^b \wedge U_+), E)$$

is an isomorphism.

Proof. The proof is exactly the same as in proposition 2.6. □

PROPOSITION 2.11. *The homotopy category $\mathcal{SH}(WB_n)$ is a compactly generated triangulated category in the sense of Neeman.*

Proof. The proof is exactly the same as in proposition 2.7. $\square$

PROPOSITION 2.12. *Assume that the base scheme $X = \mathrm{Spec}\, k$, with k a perfect field, then the Quillen adjunction:*

$$(id, id, \varphi) : WB_nSpt(\mathcal{M}) \to B_nSpt(\mathcal{M})$$

is a Quillen equivalence.

Proof. Consider the following commutative diagram

$$\begin{tikzcd} & Spt(\mathcal{M}) \arrow[dl, "id"'] \arrow[dr, "id"] & \\ WB_nSpt(\mathcal{M}) \arrow[rr, dashed, "id"'] & & B_nSpt(\mathcal{M}) \end{tikzcd}$$

where the solid arrows are left Quillen functors. Clearly, $WB_r \subseteq B_r$ for every $r \geq 0$, so $sWB_n \subseteq sB_n$, and we conclude that every sWB_n-local equivalence is a sB_n-local equivalence. Therefore, the universal property of left Bousfield localizations implies that the horizontal arrow is also a left Quillen functor.
The universal property for left Bousfield localizations also implies that it is enough to show that all the maps in

$$sB_n = \{F_p(\mathbb{G}_m^b \wedge \iota_{U,Y}) : b, p, r \geq 0, b - p \geq n - r; \iota_{U,Y} \in B_r\}$$

become weak equivalences in $WB_nSpt(\mathcal{M})$. Given $F_p(\mathbb{G}_m^b \wedge \iota_{U,Y}) \in sB_n$ with $\iota_{U,Y} \in B_r$, we proceed by induction on the dimension of $Z = Y\backslash U$. If $\dim Z = 0$, then $Z \in Sm_X$ since k is a perfect field (and we are considering Z with the reduced scheme structure), hence $F_p(\mathbb{G}_m^b \wedge \iota_{U,Y}) \in sWB_n$ and then a weak equivalence in $WB_nSpt(\mathcal{M})$.
If $\dim Z > 0$, then we consider the singular locus Z_s of Z over X. We have that $\dim Z_s < \dim Z$ since k is a perfect field. Therefore, by induction on the dimension $F_p(\mathbb{G}_m^b \wedge \iota_{V,Y})$ is a weak equivalence in $WB_nSpt(\mathcal{M})$, where $V = Y\backslash Z_s$. On the other hand, $F_p(\mathbb{G}_m^b \wedge \iota_{U,V})$ is also a weak equivalence in $WB_nSpt(\mathcal{M})$ since $\iota_{U,V}$ is also in B_r and its closed complement $V\backslash U = Z\backslash Z_s$ is smooth over X, by construction of Z_s.
But $F_p(\mathbb{G}_m^b \wedge \iota_{U,Y}) = F_p(\mathbb{G}_m^b \wedge \iota_{V,Y}) \circ F_p(\mathbb{G}_m^b \wedge \iota_{U,V})$, so by the two out of three property for weak equivalences we conclude that $F_p(\mathbb{G}_m^b \wedge \iota_{U,Y})$ is a weak equivalence in $WB_nSpt(\mathcal{M})$. $\square$

3 A CHARACTERIZATION OF THE SLICES

This section contains our main results. We give a characterization of the slices in terms of effectivity and birational conditions (in the sense of definition 3.1), and we also show that there is an equivalence between the notion of orthogonality (see definition 1.1) and weak birationality (see definition 3.1).

Definition 3.1. Let $E \in Spt(\mathcal{M})$ be a symmetric T-spectrum and $n \in \mathbb{Z}$.

1. We will say that E is *n+1-birational* (respectively *weakly n+1-birational*), if E is fibrant in $B_nSpt(\mathcal{M})$ (respectively $WB_nSpt(\mathcal{M})$). If $n = 0$, we will simply say that E is *birational* (respectively *weakly birational*).

2. We will say that E is an *n-slice* if E is isomorphic in $\mathcal{SH}$ to $s_n(E')$ for some symmetric T-spectrum E'.

Definition 3.2. 1. Let $\iota_{U,Y}$ be an open immersion in Sm_X. Let Y/U denote the pushout of the following diagram in $\mathcal{M}$ (i.e. the homotopy cofibre of $\iota_{U,Y}$ in $\mathcal{M}$)

$$\begin{tikzcd} U_+ \arrow[r, "\iota_{U,Y}"] \arrow[d] & Y_+ \arrow[d] \\ X \arrow[r] & Y/U \end{tikzcd}$$

2. Given a vector bundle $\pi : V \to Y$ with $Y \in Sm_X$, let $Th(V)$ denote the Thom space of V, i.e. $V/(V \backslash \sigma_0(Y))$, where $\sigma_0 : Y \to V$ denotes the zero section of V.

Lemma 3.3. *Let $\iota_{U,Y} \in WB_r$ for some $r \geq 0$, and let $a, b, p \geq 0$ be arbitrary integers such that $b - p \geq n - r$. Then*

$$F_p(S^a \wedge \mathbb{G}_m^b \wedge Y/U) \in \Sigma_T^{n+1}\mathcal{SH}^{\textit{eff}}$$

Proof. Since $\Sigma_T^{n+1}\mathcal{SH}^{\textit{eff}}$ is a triangulated category, it is enough to consider the case $a = 0$. It is also clear that it suffices to show that $F_0(Y/U) \in \Sigma_T^{r+1}\mathcal{SH}^{\textit{eff}}$. Now, it follows from the Morel-Voevodsky homotopy purity theorem (see [8, Thm. 2.23]) that there is an isomorphism in $\mathcal{SH}$

$$F_0(Y/U) \to F_0(Th(N))$$

where Th(N) is the Thom space of the normal bundle N of the (smooth) complement Z of U in Y:

$$e : Y \backslash U = Z \to Y$$

But, $\iota_{U,Y} \in WB_r$; so e is a regular embedding of codimension c at least $r + 1$, hence N is a vector bundle of rank at least $r + 1$. Therefore, if N is a trivial vector bundle we conclude from [8, Prop. 2.17(2)] that

$$F_0(Th(N)) \cong F_0(S^c \wedge \mathbb{G}_m^c \wedge Z_+) \in \Sigma_T^c\mathcal{SH}^{\textit{eff}} \subseteq \Sigma_T^{r+1}\mathcal{SH}^{\textit{eff}}$$

Finally, we conclude in the general case by choosing a Zariski cover of Z which trivializes N and using the Mayer-Vietoris property for Zariski covers. □

Lemma 3.4. *Let $U \in Sm_X$. Consider the open immersion in Sm_X*

$$m_U : \mathbb{A}^1_U \backslash U \to \mathbb{A}^1_U$$

given by the complement of the zero section. Then $m_U \in WB_0$, and there exists a weak equivalence in $Spt(\mathcal{M})$ between its homotopy cofibre in $\mathcal{M}$, $\mathbb{A}^1_U/(\mathbb{A}^1_U\backslash U)$ and $S^1 \wedge \mathbb{G}_m \wedge U_+$

$$t_U : \mathbb{A}^1_U/(\mathbb{A}^1_U\backslash U) \to S^1 \wedge \mathbb{G}_m \wedge U_+$$

Proof. Since the zero section $i_0 : U \to \mathbb{A}^1_U$ is a closed embedding of codimension 1 between smooth schemes over X, it follows from the definition of WB_0 that $m_U \in WB_0$. Finally, [8, Prop. 2.17(2)] implies the existence of the weak equivalence t_U. □

Proposition 3.5. *Let $E \in Spt(\mathcal{M})$ be a symmetric T-spectrum and $n \in \mathbb{Z}$. Consider the following conditions:*

1. *E is fibrant in $L_{<n+1}Spt(\mathcal{M})$.*
2. *E is weakly $n+1$-birational (see definition 3.1(1)).*
3. *E is $n+1$-birational (see definition 3.1(1)).*

Then (1) and (2) are equivalent. In addition, if the base scheme $X = \mathrm{Spec}\ k$, with k a perfect field, then (1), (2) and (3) are equivalent.

Proof. (1)⇒(2): Assume that E is fibrant in $L_{<n+1}Spt(\mathcal{M})$. By proposition 2.10 it suffices to show that for every $a, b, p, r \geq 0$ with $b - p \geq n - r$, and every $\iota_{U,Y} \in WB_r$; the induced map

$$\mathrm{Hom}_{\mathcal{SH}}(F_p(S^a \wedge \mathbb{G}_m^b \wedge Y_+), E) \xrightarrow[\cong]{\iota^*_{U,Y}} \mathrm{Hom}_{\mathcal{SH}}(F_p(S^a \wedge \mathbb{G}_m^b \wedge U_+), E)$$

is an isomorphism. We observe that

$$F_p(S^a \wedge \mathbb{G}_m^b \wedge -) : \mathcal{M} \to Spt(\mathcal{M})$$

is a left Quillen functor, therefore the following

$$F_p(S^a \wedge \mathbb{G}_m^b \wedge U_+) \xrightarrow{F_p(S^a \wedge \mathbb{G}_m^b \wedge \iota_{U,Y})} F_p(S^a \wedge \mathbb{G}_m^b \wedge Y_+) \twoheadrightarrow F_p(S^a \wedge \mathbb{G}_m^b \wedge Y/U)$$

is a cofibre sequence in $Spt(\mathcal{M})$. However, $\mathcal{SH}$ is a triangulated category and lemma 2.1 implies that

$$F_{p+1}(S^a \wedge \mathbb{G}_m^{b+1} \wedge Y/U) \cong \Omega_{S^1} \circ R \circ F_p(S^a \wedge \mathbb{G}_m^b \wedge Y/U)$$

are isomorphic in $\mathcal{SH}$, where R denotes a fibrant replacement functor in $Spt(\mathcal{M})$. Hence it suffices to show that

$$\mathrm{Hom}_{\mathcal{SH}}(F_{p+1}(S^a \wedge \mathbb{G}_m^{b+1} \wedge Y/U), E) = \mathrm{Hom}_{\mathcal{SH}}(F_p(S^a \wedge \mathbb{G}_m^b \wedge Y/U), E) = 0$$

But this follows from lemma 3.3 together with [12, Prop. 3.3.30], since we are assuming that E is fibrant in $L_{<n+1}Spt(\mathcal{M})$.
(2)⇒(1) Assume that E is $n+1$-weakly birational. Then, proposition 3.3.30 in [12] implies that it suffices to show that

$$\mathrm{Hom}_{\mathcal{SH}}(F_p(S^a \wedge \mathbb{G}_m^b \wedge U_+), E) = 0$$

for every $F_p(S^a \wedge \mathbb{G}_m^b \wedge U_+) \in C^{n+1}_{\mathit{eff}}$.
The same argument as in lemma 2.5 implies that it is enough to consider the case when $F_p(\mathbb{G}_m^b \wedge U_+) \in C^{n+1}_{\mathit{eff}}$. Moreover, we can further reduce to the case where $b, p \geq 1$ and $F_p(S^1 \wedge \mathbb{G}_m^b \wedge U_+) \in C^{n+1}_{\mathit{eff}}$. In effect, if $F_p(\mathbb{G}_m^b \wedge U_+) \in C^{n+1}_{\mathit{eff}}$, then lemma 2.1 implies that the natural map

$$g^{1,b+1}_{p+1,p}(U) : F_{p+1}(S^1 \wedge \mathbb{G}_m^{b+1} \wedge U_+) \to F_p(\mathbb{G}_m^b \wedge U_+)$$

is a weak equivalence in $Spt(\mathcal{M})$.
Now, it follows from lemma 3.4, that if $b \geq 1$, and $0 - p + (b-1) \geq n$ (i.e. $b - p \geq n+1$); then $F_p(\mathbb{G}_m^{b-1} \wedge m_U) \in sWB_n$, i.e. a weak equivalence in $WB_nSpt(\mathcal{M})$.
Since $\mathcal{SH}(WB_n)$ is a triangulated category, $id : Spt(\mathcal{M}) \to WB_nSpt(\mathcal{M})$ is a left Quillen functor, and $F_p(\mathbb{G}_m^{b-1} \wedge (\mathbb{A}^1_U/(\mathbb{A}^1_U \backslash U_+)))$ is the homotopy cofibre of $F_p(\mathbb{G}_m^{b-1} \wedge m_U)$; we deduce that E being $n+1$-weakly birational implies that

$$\mathrm{Hom}_{\mathcal{SH}}(F_p(\mathbb{G}_m^{b-1} \wedge (\mathbb{A}^1_U/(\mathbb{A}^1_U \backslash U_+))), E) = 0$$

Finally, it follows from lemma 3.4 that the following groups are isomorphic

$$\begin{aligned} 0 &= \mathrm{Hom}_{\mathcal{SH}}(F_p(\mathbb{G}_m^{b-1} \wedge (\mathbb{A}^1_U/(\mathbb{A}^1_U \backslash U_+))), E) \\ &\cong \mathrm{Hom}_{\mathcal{SH}}(F_p(S^1 \wedge \mathbb{G}_m^b \wedge U_+), E) \end{aligned}$$

(2)⇔(3): This follows directly from proposition 2.12. □

Theorem 3.6. *The Quillen adjunction*

$$(id, id, \varphi) : WB_nSpt(\mathcal{M}) \to L_{<n+1}Spt(\mathcal{M})$$

is a Quillen equivalence. In addition, if the base scheme $X = \mathrm{Spec}\, k$, with k a perfect field, then the Quillen adjunction

$$(id, id, \varphi) : B_nSpt(\mathcal{M}) \to L_{<n+1}Spt(\mathcal{M})$$

is also a Quillen equivalence.

Proof. We show first that $WB_nSpt(\mathcal{M})$ and $L_{<n+1}Spt(\mathcal{M})$ are Quillen equivalent. Since $WB_nSpt(\mathcal{M})$, $L_{<n+1}Spt(\mathcal{M})$ are both left Bousfield localizations of $Spt(\mathcal{M})$, we deduce that they are simplicial model categories with the same cofibrant replacement functor Q. Thus, it suffices to show that they have the same class of weak equivalences.

However, proposition 3.5 implies that $WB_nSpt(\mathcal{M})$, and $L_{<n+1}Spt(\mathcal{M})$ also have the same class of fibrant objects. Therefore, it follows from [2, Thm. 9.7.4] that they have exactly the same class of weak equivalences.
Finally, if the base scheme is a perfect field, by proposition 2.12 we conclude that $WB_nSpt(\mathcal{M})$ and $B_nSpt(\mathcal{M})$ are Quillen equivalent. □

THEOREM 3.7. *Let E be fibrant in $Spt(\mathcal{M})$. Then E is an n-slice (see definition 3.1(2)) if and only if the following conditions hold:*

S1 *E is n-effective, i.e. $E \in \Sigma_T^n \mathcal{SH}^{eff}$.*

S2 *E is $n+1$-weakly birational.*

In addition, if the base scheme $X = \operatorname{Spec} k$, with k a perfect field, then E is an n-slice if and only if the following conditions hold:

GSS1 *E is n-effective, i.e. $E \in \Sigma_T^n \mathcal{SH}^{eff}$.*

GSS2 *E is $n+1$-birational.*

Proof. Assume that E is an n-slice. Then theorems 1.4(1) and 1.4(3) imply that E is n-effective and fibrant in $L_{<n+1}Spt(\mathcal{M})$. Hence, proposition 3.5 implies that E is also $n+1$-weakly birational.
Now we assume that E satisfies the conditions S1 and S2 above. Then, proposition 3.5 implies that E is fibrant in $L_{<n+1}Spt(\mathcal{M})$. Therefore, theorem 1.4(3) implies that E is isomorphic in $\mathcal{SH}$ to its own n-slice $s_n(E)$.
Finally, if the base scheme is a perfect field, then by proposition 3.5 the conditions S2 and GSS2 are equivalent; hence we can conclude applying the same argument as above. □

Remark 3.8. Notice that theorem 3.6 implies that it is possible to construct the slice filtration directly from the Quillen model categories $WB_nSpt(\mathcal{M})$ described in definition 2.9 without making any reference to the effective categories $\Sigma_T^q \mathcal{SH}^{eff}$. One of the interesting consequences of this fact is that it is possible to obtain analogues of the slice filtration in the unstable setting, since the suspension with respect to T or S^1 does not play an essential role in the construction of $WB_nSpt(\mathcal{M})$, i.e. we could consider the left Bousfield localization of the motivic unstable homotopy category $\mathcal{M}$ with respect to the maps in definition 2.8. We will study the details of this construction in a future work.

4 SOME COMPUTATIONS

In this section we use the characterization of the slices obtained in theorem 3.7 to describe the slices of projective spaces, Thom spaces and blow ups.
To simplify the notation, given a simplicial presheaf $K \in \mathcal{M}$ or a map $f \in \mathcal{M}$; let $s_j(K)$, $s_j(f)$ (respectively $s_{<j}(K)$, $s_{<j}(f)$) denote $s_j(F_0(K))$, $s_j(F_0(f))$ (respectively $s_{<j}(F_0(K))$, $s_{<j}(F_0(f))$).

Lemma 4.1. *Let $g : E \to F$ be a map in $\mathcal{SH}$ such that $s_{<n}(g)$ and $s_{<n+1}(g)$ are both isomorphisms in $\mathcal{SH}$. Then the n-slice of g, $s_n(g)$ is also an isomorphism in $\mathcal{SH}$.*

Proof. It follows from [12, Prop. 3.1.19] that the rows in the following commutative diagram are distinguished triangles in $\mathcal{SH}$

$$\begin{tikzcd}
s_n(E) \arrow[r] \arrow[d, "s_n(g)"'] & s_{<n+1}(E) \arrow[r] \arrow[d, "s_{<n+1}(g)"] & s_{<n}(E) \arrow[r] \arrow[d, "s_{<n}(g)"] & S^1 \wedge s_n(E) \arrow[d, "S^1 \wedge s_n(g)"] \\
s_n(F) \arrow[r] & s_{<n+1}(F) \arrow[r] & s_{<n}(F) \arrow[r] & S^1 \wedge s_n(F)
\end{tikzcd}$$

Thus, we conclude that $s_n(g)$ is also an isomorphism in $\mathcal{SH}$. $\square$

Consider $Y \in Sm_X$. Let $\mathbb{P}^n(Y)$ denote the trivial projective bundle of rank n over Y, and let $\mathbb{P}^\infty(Y)$ denote the colimit in $\mathcal{M}$ of the following filtered diagram

$$\mathbb{P}^0(Y) \to \mathbb{P}^1(Y) \to \cdots \to \mathbb{P}^n(Y) \to \cdots \tag{4.1}$$

given by the inclusions of the respective hyperplanes at infinity.

Theorem 4.2. *Let $Y \in Sm_X$. Then for any integer $j \leq n$, the diagram 4.1 induces the following isomorphisms in $\mathcal{SH}$*

$$s_j(\mathbb{P}^n(Y)_+) \xrightarrow{\cong} s_j(\mathbb{P}^{n+1}(Y)_+) \xrightarrow{\cong} \cdots \xrightarrow{\cong} s_j(\mathbb{P}^\infty(Y)_+)$$

Proof. Let $k > n$, and consider the closed embedding induced by the diagram (4.1) $\lambda_n^k : \mathbb{P}^n(Y) \to \mathbb{P}^k(Y)$. It is possible to choose a linear embedding $\mathbb{P}^{k-n-1}(Y) \to \mathbb{P}^k(Y)$ such that its open complement $U_{k,n}$ contains $\mathbb{P}^n(Y)$ and has the structure of a vector bundle over $\mathbb{P}^n(Y)$, with zero section σ_n^k:

$$\begin{tikzcd}
U_{k,n} \arrow[r, "v_n^k"] \arrow[d, shift right] & \mathbb{P}^k(Y) & \mathbb{P}^{k-n-1}(Y) \arrow[l] \\
\mathbb{P}^n(Y) \arrow[u, shift right, "\sigma_n^k"'] \arrow[ur, bend right, "\lambda_n^k"'] & &
\end{tikzcd}$$

By homotopy invariance $s_{<j}(\sigma_n^k)$ is an isomorphism in $\mathcal{SH}$ for every integer j. On the other hand, if $j \leq n$, then $F_0(v_n^k)$ is a weak equivalence in $WB_jSpt(\mathcal{M})$ since the codimension of its closed complement is $n+1$. Thus, theorems 1.4(2) and 3.6 imply that if $j \leq n+1$, then $s_{<j}(v_n^k)$ is also an isomorphism in $\mathcal{SH}$. Therefore, $s_{<j}(\lambda_n^k) = s_{<j}(v_n^k) \circ s_{<j}(\sigma_n^k)$ is an isomorphism in $\mathcal{SH}$ for $j \leq n+1$; and using lemma 4.1 we conclude that the induced map on the slices $s_j(\lambda_n^k)$ is also an isomorphism for $j \leq n$.
Finally, the result for $\mathbb{P}^\infty(Y)$ follows directly from the fact that the slices commute with filtered homotopy colimits. $\square$

Let $H\mathbb{Z}$ denote Voevodsky's Eilenberg-MacLane spectrum (see [15, section 6.1]) representing motivic cohomology in $\mathcal{SH}$.

Corollary 4.3. *Assume that the base scheme $X = \operatorname{Spec} k$, with k a perfect field. Then, in the following diagram all the symmetric T-spectra are isomorphic to $H\mathbb{Z}$:*

$$H\mathbb{Z} \xrightarrow{\cong} s_0(\mathbb{P}^0(k)_+) \xrightarrow{\cong} s_0(\mathbb{P}^1(k)_+) \xrightarrow{\cong} \cdots$$
$$\cdots \xrightarrow{\cong} s_0(\mathbb{P}^n(k)_+) \xrightarrow{\cong} \cdots \xrightarrow{\cong} s_0(\mathbb{P}^\infty(k)_+)$$

Proof. This follows immediately from theorem 4.2 together with the computation of Levine [7, Thm. 10.5.1] and Voevodsky [17] for the zero slice of the sphere spectrum. □

Theorem 4.4. *Let $\iota_{U,Y} \in WB_n$, $\pi : V \to Y$ a vector bundle of rank r together with a trivialization $t : \pi^{-1}(U) \to \mathbb{A}^r_U$ of its restriction to U. Then for every integer $j \leq n$, there exists an isomorphism in $\mathcal{SH}$ (see definition 3.2(2))*

$$s_j(Th(V)) \cong S^r \wedge \mathbb{G}_m^r \wedge s_{j-r}(Y_+)$$

Proof. Let $Z \in Sm_X$ be the closed complement of $\iota_{U,Y}$. Consider the following diagram in Sm_X, where all the squares are cartesian

$$\begin{tikzcd}
\pi^{-1}(Z) \cap (V \backslash \sigma_0(Y)) \arrow[r] \arrow[d] & V \backslash \sigma_0(Y) \arrow[d] & \pi^{-1}(U) \cap (V \backslash \sigma_0(Y)) \arrow[l, "\beta"'] \arrow[d] \\
\pi^{-1}(Z) \arrow[r] \arrow[d] & V \arrow[d, "\pi"] & \pi^{-1}(U) \arrow[l, "\alpha"'] \arrow[d] \\
Z \arrow[r] & Y & U \arrow[l, "\iota_{U,Y}"]
\end{tikzcd}$$

and let $\gamma : Th(\pi^{-1}(U)) \to Th(V)$ be the induced map between the corresponding Thom spaces. We observe that α, β also belong to WB_n; thus, if $j \leq n$ we conclude that $F_0(\iota_{U,Y}), F_0(\alpha), F_0(\beta)$ are all weak equivalences in $WB_jSpt(\mathcal{M})$. Therefore, theorems 1.4(2) and 3.6 imply that if $j \leq n+1$, then $s_{<j}(\iota_{U,Y})$, $s_{<j}(\alpha)$, $s_{<j}(\beta)$ are isomorphisms in $\mathcal{SH}$. We claim that if $j \leq n+1$, then

$$s_{<j}(\gamma) : s_{<j}(Th(\pi^{-1}(U))) \to s_{<j}(Th(V))$$

is also an isomorphism in $\mathcal{SH}$. In effect, by construction of the Thom spaces, we deduce that for any integer $j \in \mathbb{Z}$, the rows in the following commutative diagram in $\mathcal{SH}$ are in fact distinguished triangles

$$\begin{tikzcd}
s_{<j}((\pi^{-1}(U) \cap (V \backslash \sigma_0(Y)))_+) \arrow[r] \arrow[d, "s_{<j}(\beta)"'] & s_{<j}(\pi^{-1}(U)_+) \arrow[r] \arrow[d, "s_{<j}(\alpha)"'] & s_{<j}(Th(\pi^{-1}(U))) \arrow[d, "s_{<j}(\gamma)"'] \\
s_{<j}((V \backslash \sigma_0(Y))_+) \arrow[r] & s_{<j}(V_+) \arrow[r] & s_{<j}(Th(V))
\end{tikzcd}$$

Since $s_{<j}(\alpha), s_{<j}(\beta)$ are isomorphisms in $\mathcal{SH}$ for $j \leq n+1$, we conclude that for $j \leq n+1$, $s_{<j}(\gamma)$ is also an isomorphism in $\mathcal{SH}$.
Thus, lemma 4.1 implies that for $j \leq n$, $s_j(\iota_{U,Y}), s_j(\gamma)$ are isomorphisms in $\mathcal{SH}$.
Now, we use the trivialization t to obtain the following commutative diagram in Sm_X where the rows are isomorphisms

$$\begin{tikzcd}
\mathbb{A}^r_U \backslash U \arrow[d] & & \pi^{-1}(U) \cap (V \backslash \sigma_0(Y)) \arrow[ll, "\tilde{t}", "\cong"'] \arrow[d] \\
\mathbb{A}^r_U \arrow[dr] & & \pi^{-1}(U) \arrow[ll, "t", "\cong"'] \arrow[dl, "\pi_U"] \\
& U &
\end{tikzcd}$$

The same argument as above, shows that for every integer $j \in \mathbb{Z}$, there is an isomorphism in $\mathcal{SH}$

$$s_j(\bar{t}) : s_j(Th(\pi^{-1}(U))) \to s_j(Th(\mathbb{A}^r_U))$$

On the other hand, [8, Prop. 2.17(2)] implies that there is a weak equivalence $w : F_0(Th(\mathbb{A}^r_U)) \to S^r \wedge \mathbb{G}^r_m \wedge F_0(U_+)$ in $Spt(\mathcal{M})$. Thus, for $j \leq n$ there exist isomorphisms in $\mathcal{SH}$

$$\begin{tikzcd}
s_j(Th(\pi^{-1}(U))) \arrow[r, "s_j(\bar{t})"] \arrow[d, "s_j(\gamma)"'] & s_j(Th(\mathbb{A}^r_U)) \arrow[d, "s_j(w)"] \\
s_j(Th(V)) & s_j(S^r \wedge \mathbb{G}^r_m \wedge U_+)
\end{tikzcd}$$

However, there exists a canonical isomorphism in $\mathcal{SH}$

$$s_j(S^r \wedge \mathbb{G}^r_m \wedge U_+) \to S^r \wedge \mathbb{G}^r_m \wedge s_{j-r}(U_+)$$

Finally, we conclude by using the isomorphism $s_{j-r}(\iota_{U,Y})$ (notice that if $j \leq n$ then certainly $j - r \leq n$, since $r \geq 0$). □

Corollary 4.5. *Assume that the base scheme $X = \operatorname{Spec} k$, with k a perfect field. Let $\iota_{U,Y} \in B_n$, $\pi : V \to Y$ a vector bundle of rank r together with a trivialization $t : \pi^{-1}(U) \to \mathbb{A}^r_U$ of its restriction to U. Then for every integer $j \leq n$, there exists an isomorphism in $\mathcal{SH}$*

$$s_j(Th(V)) \cong S^r \wedge \mathbb{G}^r_m \wedge s_{j-r}(Y_+)$$

Proof. Proposition 2.12 implies that $F_0(\iota_{U,Y})$ is a weak equivalence in $WB_jSpt(\mathcal{M})$ for $j \leq n$. Hence, the result follows using exactly the same argument as in theorem 4.4. □

Given a closed embedding $Z \to Y$ of smooth schemes over X, let $\mathcal{B}\ell_Z Y$ denote the blowup of Y with center in Z.

Theorem 4.6. *Let $\iota_{U,Y} \in WB_n$ with closed complement Z, and $j \in \mathbb{Z}$ an arbitrary integer. Consider the following cartesian square in Sm_X*

$$\begin{tikzcd} D \arrow[r, "d"] \arrow[d, "q"'] & \mathcal{B}\ell_Z Y \arrow[d, "p"] & U \arrow[l, "u"'] \arrow[d, equal] \\ Z \arrow[r, "i"'] & Y & U \arrow[l, "\iota_{U,Y}"] \end{tikzcd} \tag{4.2}$$

and let q_j, d_j, p_j, i_j denote $s_j(q), s_j(d), s_j(p), s_j(i)$ respectively. Then the cartesian square (4.2) induces the following distinguished triangle in $\mathcal{SH}$

$$s_j(D_+) \xrightarrow{\binom{-d_j}{q_j}} s_j(\mathcal{B}\ell_Z Y_+) \oplus s_j(Z_+) \xrightarrow{(p_j, i_j)} s_j(Y_+) \tag{4.3}$$

If $j \leq n$, then $s_j(\iota_{U,Y})$ is an isomorphism in $\mathcal{SH}$, and the following distinguished triangles in $\mathcal{SH}$ split

$$\begin{tikzcd} s_j(D_+) \arrow[r, shift left, "\binom{-d_j}{q_j}"] & s_j(\mathcal{B}\ell_Z Y_+) \oplus s_j(Z_+) \arrow[l, shift left] \arrow[r, shift left, "(p_j, i_j)"] & s_j(Y_+) \arrow[l, shift left, "\binom{r_j}{0}"] \end{tikzcd} \tag{4.4}$$

$$\begin{tikzcd} s_j(Y_+) \arrow[r, shift left, "r_j"] & s_j(\mathcal{B}\ell_Z Y_+) \arrow[l, shift left, "p_j"] \arrow[r, shift left] & s_j(Th(\mathcal{O}_D(1))) \arrow[l, shift left] \end{tikzcd} \tag{4.5}$$

where $r_j = s_j(u) \circ (s_j(\iota_{U,Y}))^{-1}$, and $\mathcal{O}_D(1)$ denotes the canonical line bundle of the projective bundle $q : D \to Z$.

Proof. It follows from [8, Prop. 2.29 and Rmk. 2.30] that the following square is homotopy cocartesian in $\mathcal{M}$

$$\begin{tikzcd} S^1 \wedge D_+ \arrow[r, "id \wedge d"] \arrow[d, "id \wedge q"'] & S^1 \wedge \mathcal{B}\ell_Z Y_+ \arrow[d, "id \wedge p"] \\ S^1 \wedge Z_+ \arrow[r, "id \wedge i"'] & S^1 \wedge Y_+ \end{tikzcd}$$

Thus, we deduce that the following diagram is a distinguished triangle in $\mathcal{SH}$

$$F_0(D_+) \xrightarrow{\binom{-F_0(d)}{F_0(q)}} F_0(\mathcal{B}\ell_Z Y_+) \oplus F_0(Z_+) \xrightarrow{(F_0(p), F_0(i))} F_0(Y_+)$$

Since the slices s_j are triangulated functors, it follows that diagram (4.3) is a distinguished triangle in $\mathcal{SH}$.

Now, we prove that $s_j(\iota_{U,Y})$ is an isomorphism for $j \leq n$. By lemma 4.1, it suffices to show that $s_{<j}(\iota_{U,Y})$ is an isomorphism in $\mathcal{SH}$ for $j \leq n+1$. But this follows directly from theorems 3.6 and 1.4(2) since $F_0(\iota_{U,Y})$ is clearly a weak equivalence in $WB_jSpt(\mathcal{M})$ for $j \leq n$.
Thus, r_j is well defined for $j \leq n$, and the following diagram shows that it gives a splitting for the distinguished triangle (4.4)

$$\begin{tikzcd} s_j(U_+) \arrow[r, "s_j(u)"] \arrow[d, equal] & s_j(\mathcal{B}\ell_Z Y_+) \arrow[d, "p_j"] \\ s_j(U_+) \arrow[r, "s_j(\iota_{U,Y})"'] & s_j(Y_+) \end{tikzcd} \tag{4.6}$$

Finally, since the normal bundle of the closed embedding $d : D \to \mathcal{B}\ell_Z Y$ is given by $\mathcal{O}_D(1)$, we deduce from the Morel-Voevodsky homotopy purity theorem (see [8, Thm. 2.23]) that the following diagram is a distinguished triangle in $\mathcal{SH}$

$$s_j(U_+) \xrightarrow{s_j(u)} s_j(\mathcal{B}\ell_Z Y_+) \longrightarrow s_j(Th(\mathcal{O}_D(1)))$$

Combining this distinguished triangle with diagram (4.6) above, we conclude that diagram (4.5) is a split distinguished triangle in $\mathcal{SH}$ for $j \leq n$. $\square$

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