The Λ-Adic Shimura-Shintani-Waldspurger Correspondence

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Abstract. We generalize the Λ-adic Shintani lifting for $\mathrm{GL}_2(\mathbb{Q})$ to indefinite quaternion algebras over $\mathbb{Q}$.

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1. Introduction

Langlands's principle of functoriality predicts the existence of a staggering wealth of transfers (or lifts) between automorphic forms for different reductive groups. In recent years, attempts at the formulation of p-adic variants of Langlands's functoriality have been articulated in various special cases. We prove the existence of the Shimura-Shintani-Waldspurger lift for p-adic families. More precisely, Stevens, building on the work of Hida and Greenberg-Stevens, showed in [21] the existence of a Λ-adic variant of the classical Shintani lifting of [20] for $\mathrm{GL}_2(\mathbb{Q})$. This Λ-adic lifting can be seen as a formal power series with coefficients in a finite extension of the Iwasawa algebra $\Lambda := \mathbb{Z}_p[\![X]\!]$ equipped with specialization maps interpolating classical Shintani lifts of classical modular forms appearing in a given Hida family.
Shimura in [19], resp. Waldspurger in [22] generalized the classical Shimura-Shintani correspondence to quaternion algebras over $\mathbb{Q}$, resp. over any number field. In the p-adic realm, Hida ([7]) constructed a Λ-adic Shimura lifting, while Ramsey ([17]) (resp. Park [12]) extended the Shimura (resp. Shintani) lifting to the overconvergent setting.
In this paper, motivated by ulterior applications to Shimura curves over $\mathbb{Q}$, we generalize Stevens's result to any non-split rational indefinite quaternion algebra B, building on work of Shimura [19] and combining this with a result of Longo-Vigni [9]. Our main result, for which the reader is referred to Theorem 3.8 below, states the existence of a formal power series and specialization maps interpolating Shimura-Shintani-Waldspurger lifts of classical forms in a given

p-adic family of automorphic forms on the quaternion algebra B. The Λ-adic variant of Waldspurger's result appears computationally challenging (see remark in [15, Intro.]), but it seems within reach for real quadratic fields (cf. [13]).

As an example of our main result, we consider the case of families with trivial character. Fix a prime number p and a positive integer N such that $p \nmid N$. Embed the set $\mathbb{Z}^{\geq 2}$ of integers greater or equal to 2 in $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ by sending $k \in \mathbb{Z}^{\geq 2}$ to the character $x \mapsto x^{k-2}$. Let f_∞ be an Hida family of tame level N passing through a form f_0 of level $\Gamma_0(Np)$ and weight k_0. There is a neighborhood U of k_0 in $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ such that, for any $k \in \mathbb{Z}^{\geq 2} \cap U$, the weight k specialization of f_∞ gives rise to an element $f_k \in S_k(\Gamma_0(Np))$. Fix a factorization $N = MD$ with $D > 1$ a square-free product of an even number of primes and $(M, D) = 1$ (we assume that such a factorization exists). Applying the Jacquet-Langlands correspondence we get for any $k \in \mathbb{Z}^{\geq 2} \cap U$ a modular form f_k^{JL} on Γ, which is the group of norm-one elements in an Eichler order R of level Mp contained in the indefinite rational quaternion algebra B of discriminant D. One can show that these modular forms can be p-adically interpolated, up to scaling, in a neighborhood of k_0. More precisely, let $\mathcal{O}$ be the ring of integers of a finite extension F of $\mathbb{Q}_p$ and let $\mathbb{D}$ denote the $\mathcal{O}$-module of $\mathcal{O}$-valued measures on $\mathbb{Z}_p^2$ which are supported on the set of primitive elements in $\mathbb{Z}_p^2$. Let Γ_0 be the group of norm-one elements in an Eichler order $R_0 \subseteq B$ containing R. There is a canonical action of Γ_0 on $\mathbb{D}$ (see [9, §2.4] for its description). Denote by F_k the extension of F generated by the Fourier coefficients of f_k. Then there is an element $\Phi \in H^1(\Gamma_0, \mathbb{D})$ and maps $\rho_k : H^1(\Gamma_0, \mathbb{D}) \longrightarrow H^1(\Gamma, F_k)$ such that $\rho(k)(\Phi) = \phi_k$, the cohomology class associated to f_k^{JL}, with k in a neighborhood of k_0 (for this we need a suitable normalization of the cohomology class associated to f_k^{JL}, which we do not touch for simplicity in this introduction). We view Φ as a quaternionic family of modular forms. To each ϕ_k we may apply the Shimura-Shintani-Waldspurger lifting ([19]) and obtain a modular form h_k of weigh $k + 1/2$, level $4Np$ and trivial character. We show that this collection of forms can be p-adically interpolated. For clarity's sake, we present the liftings and their Λ-adic variants in a diagram, in which the horizontal maps are specialization maps of the p-adic family to weight k; JL stands for the Jacquet-Langlands correspondence; SSW stands for the Shimura-Shintani-Waldspurger lift; and the dotted arrows are constructed in this paper:

$$\begin{tikzcd}
f_\infty \arrow[r, mapsto] \arrow[d, mapsto, "\Lambda-\text{adic JL}"'] & f_k \arrow[d, mapsto, "\text{JL}"] \\
\Phi \arrow[r, mapsto, "\rho_k"] \arrow[d, mapsto, dotted, "\Lambda-\text{adic SSW}"'] & \phi_k \arrow[d, mapsto, "\text{SSW}"] \\
\Theta \arrow[r, mapsto, dotted] & h_k
\end{tikzcd}$$

More precisely, as a particular case of our main result, Theorem 3.8, we get the following

Theorem 1.1. *There exists a p-adic neighborhood U_0 of k_0 in* $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$, *$p$-adic periods Ω_k for $k \in U_0 \cap \mathbb{Z}^{\geq 2}$ and a formal expansion*

$$\Theta = \sum_{\xi \geq 1} a_\xi q^\xi$$

with coefficients a_ξ in the ring of $\mathbb{C}_p$-valued functions on U_0, such that for all $k \in U_0 \cap \mathbb{Z}^{\geq 2}$ we have

$$\Theta(k) = \Omega_k \cdot h_k.$$

Further, $\Omega_{k_0} \neq 0$.

2. Shintani integrals and Fourier coefficients of half-integral weight modular forms

We express the Fourier coefficients of half-integral weight modular forms in terms of period integrals, thus allowing a cohomological interpretation which is key to the production of the Λ-adic version of the Shimura-Shintani-Waldspurger correspondence. For the quaternionic Shimura-Shintani-Waldspurger correspondence of interest to us (see [15], [22]), the period integrals expressing the values of the Fourier coefficients have been computed generally by Prasanna in [16].

2.1. The Shimura-Shintani-Waldspurger lifting. Let $4M$ be a positive integer, $2k$ an even non-negative integer and χ a Dirichlet character modulo $4M$ such that $\chi(-1) = 1$. Recall that the space of half-integral weight modular forms $S_{k+1/2}(4M, \chi)$ consists of holomorphic cuspidal functions h on the upper-half place $\mathfrak{H}$ such that

$$h(\gamma(z)) = j^{1/2}(\gamma, z)^{2k+1}\chi(d)h(z),$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4M)$, where $j^{1/2}(\gamma, z)$ is the standard square root of the usual automorphy factor $j(\gamma, z)$ (cf. [15, 2.3]).

To any quaternionic integral weight modular form we may associate a half-integral weight modular form following Shimura's work [19], as we will describe below.

Fix an odd square free integer N and a factorization $N = M \cdot D$ into coprime integers such that $D > 1$ is a product of an even number of distinct primes. Fix a Dirichlet character ψ modulo M and a positive even integer $2k$. Suppose that

$$\psi(-1) = (-1)^k.$$

Define the Dirichlet character χ modulo $4N$ by

$$\chi(x) := \psi(x) \left(\frac{-1}{x}\right)^k.$$

Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant D. Fix a maximal order $\mathcal{O}_B$ of B. For every prime $\ell|M$, choose an isomorphism

$$i_\ell : B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \mathbb{M}_2(\mathbb{Q}_\ell)$$

such that $i_\ell(\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell) = \mathbb{M}_2(\mathbb{Z}_\ell)$. Let $R \subseteq \mathcal{O}_B$ be the Eichler order of B of level M defined by requiring that $i_\ell(R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$ is the suborder of $\mathbb{M}_2(\mathbb{Z}_\ell)$ of upper triangular matrices modulo ℓ for all $\ell|M$. Let Γ denote the subgroup of the group $R_1^\times$ of norm 1 elements in $R^\times$ consisting of those γ such that $i_\ell(\gamma) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ mod ℓ for all $\ell|M$. We denote by $S_{2k}(\Gamma)$ the $\mathbb{C}$-vector space of weight $2k$ modular forms on Γ, and by $S_{2k}(\Gamma, \psi^2)$ the subspace of $S_{2k}(\Gamma)$ consisting of forms having character ψ^2 under the action of $R_1^\times$. Fix a Hecke eigenform

$$f \in S_{2k}(\Gamma, \psi^2)$$

as in [19, Section 3].

Let V denote the $\mathbb{Q}$-subspace of B consisting of elements with trace equal to zero. For any $v \in V$, which we view as a trace zero matrix in $\mathbb{M}_2(\mathbb{R})$ (after fixing an isomorphism $i_\infty : B \otimes \mathbb{R} \simeq \mathbb{M}_2(\mathbb{R})$), set

$$G_v := \{\gamma \in \mathrm{SL}_2(\mathbb{R}) |\, \gamma^{-1} v \gamma = v\}$$

and put $\Gamma_v := G_v \cap \Gamma$. One can show that there exists an isomorphism

$$\omega : \mathbb{R}^\times \xrightarrow{\sim} G_v$$

defined by $\omega(s) = \beta^{-1}\left(\begin{smallmatrix} s & 0 \\ 0 & s^{-1} \end{smallmatrix}\right)\beta$, for some $\beta \in \mathrm{SL}_2(\mathbb{R})$. Let $\mathfrak{t}_v$ be the order of $\Gamma_v \cap \{\pm 1\}$ and let γ_v be an element of Γ_v which generates $\Gamma_v \{\pm 1\} / \{\pm 1\}$. Changing γ_v to γ_v^{-1} if necessary, we may assume $\gamma_v = \omega(t)$ with $t > 0$. Define V^* to be the $\mathbb{Q}$-subspace of V consisting of elements with strictly negative norm. For any $\alpha = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in V^*$ and $z \in \mathcal{H}$, define the quadratic form

$$Q_\alpha(z) := cz^2 - 2az - b.$$

Fix $\tau \in \mathcal{H}$ and set

$$P(f, \alpha, \Gamma) := -\big(2(-\mathrm{nr}(\alpha))^{1/2}/\mathfrak{t}_\alpha\big) \int_\tau^{\gamma_\alpha(\tau)} Q_\alpha(z)^{k-1} f(z) dz$$

where $\mathrm{nr} : B \to \mathbb{Q}$ is the norm map. By [19, Lemma 2.1], the integral is independent on the choice τ, which justifies the notation.

Remark 2.1. The definition of $P(f, \alpha, \Gamma)$ given in [19, (2.5)] looks different: the above expression can be derived as in [19, page 629] by means of [19, (2.20) and (2.22)].

Let $R(\Gamma)$ denote the set of equivalence classes of V^* under the action of Γ by conjugation. By [19, (2.6)], $P(f, \alpha, \Gamma)$ only depends on the conjugacy class of α, and thus, for $\mathcal{C} \in R(\Gamma)$, we may define $P(f, \mathcal{C}, \Gamma) := P(f, \alpha, \Gamma)$ for any choice of $\alpha \in \mathcal{C}$. Also, $q(\mathcal{C}) := -\mathrm{nr}(\alpha)$ for any $\alpha \in \mathcal{C}$.

Define $\mathcal{O}'_B$ to be the maximal order in B such that $\mathcal{O}'_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \simeq \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ for all $\ell \nmid M$ and $\mathcal{O}'_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is equal to the local order of $B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ consisting of elements γ such that $i_\ell(\gamma) = \left(\begin{smallmatrix} a & b/M \\ cM & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbb{Z}_\ell$, for all $\ell | M$. Given $\alpha \in \mathcal{O}'_B$, we can find an integer b_α such that

$$\text{(1)} \qquad i_\ell(\alpha) \equiv \begin{pmatrix} * & b_\alpha/M \\ * & * \end{pmatrix} \quad \bmod i_\ell(R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell), \quad \forall \ell | M.$$

Define a locally constant function η_ψ on V by $\eta_\psi(\alpha) = \psi(b_\alpha)$ if $\alpha \in \mathcal{O}'_B \cap V$ and $\eta(\alpha) = 0$ otherwise, with $\psi(a) = 0$ if $(a, M) \neq 1$ (for the definition of locally constant functions on V in this context, we refer to [19, p. 611]).
For any $\mathcal{C} \in R(\Gamma)$, fix $\alpha_{\mathcal{C}} \in \mathcal{C}$. For any integer $\xi \geq 1$, define

$$a_\xi(\tilde{h}) := \big(2\mu(\Gamma \backslash \mathfrak{H})\big)^{-1} \cdot \sum_{\mathcal{C} \in R(\Gamma), q(\mathcal{C}) = \xi} \eta_\psi(\alpha_{\mathcal{C}}) \xi^{-1/2} P(f, \mathcal{C}, \Gamma).$$

Then, by [19, Theorem 3.1],

$$\tilde{h} := \sum_{\xi \geq 1} a_\xi(\tilde{h}) q^\xi \in S_{k+1/2}(4N, \chi)$$

is called the Shimura-Shintani-Waldspurger lifting of f.

2.2. Cohomological interpretation. We introduce necessary notation to define the action of the Hecke action on cohomology groups; for details, see [9, §2.1]. If G is a subgroup of $B^\times$ and S a subsemigroup of $B^\times$ such that (G, S) is an Hecke pair, we let $\mathcal{H}(G, S)$ denote the Hecke algebra corresponding to (G, S), whose elements are written as $T(s) = GsG = \coprod_i Gs_i$ for $s, s_i \in S$ (finite disjoint union). For any $s \in S$, let $s^* := \mathrm{norm}(s)s^{-1}$ and denote by S^* the set of elements of the form s^* for $s \in S$. For any $\mathbb{Z}[S^*]$-module M we let $T(s)$ act on $H^1(G, M)$ at the level of cochains $c \in Z^1(G, M)$ by the formula $(c|T(s))(\gamma) = \sum_i s_i^* c(t_i(\gamma))$, where $t_i(\gamma)$ are defined by the equations $Gs_i\gamma = Gs_j$ and $s_i\gamma = t_i(\gamma)s_j$. In the following, we will consider the case of $G = \Gamma$ and

$$S = \{s \in B^\times | i_\ell(s) \text{ is congruent to } \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \bmod \ell \text{ for all } \ell | M\}.$$

For any field L and any integer $n \geq 0$, let $V_n(L)$ denote the L-dual of the L-vector space $\mathcal{P}_n(L)$ of homogeneous polynomials in 2 variables of degree n. We let $\mathbb{M}_2(L)$ act from the right on $P(x, y)$ as $P|\gamma(x, y) := P(\gamma(x, y))$, where for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we put

$$\gamma(x, y) := (ax + yb, cx + dy).$$

This also equips $V_n(L)$ with a left action by $\gamma \cdot \varphi(P) := \varphi(P|\gamma)$. To simplify the notation, we will write $P(z)$ for $P(z, 1)$.
Let F denote the finite extension of $\mathbb{Q}$ generated by the eigenvalues of the Hecke action on f. For any field K containing F, set

$$\mathbb{W}_f(K) := H^1\big(\Gamma, V_{k-2}(K)\big)^f$$

where the superscript f denotes the subspace on which the Hecke algebra acts via the character associated with f. Also, for any sign $\pm$, let $\mathbb{W}_f^\pm(K)$ denote the $\pm$-eigenspace for the action of the archimedean involution ι. Remember that ι is defined by choosing an element ω_∞ of norm -1 in $R^\times$ such that such that $i_\ell(\omega_\infty) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ mod M for all primes $\ell | M$ and then setting $\iota := T(w_\infty)$ (see [9, §2.1]). Then $\mathbb{W}_f^\pm(K)$ is one dimensional (see, e.g., [9, Proposition 2.2]); fix a generator $\phi_f^\pm$ of $\mathbb{W}_f^\pm(F)$.

To explicitly describe $\phi_f^\pm$, let us introduce some more notation. Define

$$f|\omega_\infty(z) := (Cz+D)^{-k/2}\overline{f(\omega_\infty(\bar{z}))}$$

where $i_\infty(\omega_\infty) = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Then $f|\omega_\infty \in S_{2k}(\Gamma)$ as well. If the eigenvalues of the Hecke action on f are real, then we may assume, after multiplying f by a scalar, that $f|\omega_\infty = f$ (see [19, p. 627] or [10, Lemma 4.15]). In general, let $I(f)$ denote the class in $H^1(\Gamma, V_{k-2}(\mathbb{C}))$ represented by the cocycle

$$\gamma \longmapsto \left[P \mapsto I_\gamma(f)(P) := \int_\tau^{\gamma(\tau)} f(z)P(z)dz \right]$$

for any $\tau \in \mathcal{H}$ (the corresponding class is independent on the choice of τ). With this notation,

$$P(f,\alpha,\Gamma) = -\big(2(-\mathrm{nr}(\alpha))^{1/2}/\mathfrak{t}_\alpha\big)\cdot I_{\gamma_{\alpha_{\mathcal{C}}}}(f)\big(Q_{\alpha_{\mathcal{C}}}(z)^{k-1}\big).$$

Denote by $I^\pm(f) := (1/2)\cdot I(f) \pm (1/2)\cdot I(f)|\omega_\infty$, the projection of $I(f)$ to the eigenspaces for the action of ω_∞. Then $I(f) = I^+(f) + I^-(f)$ and $I_f^\pm = \Omega_f^\pm \cdot \phi_f^\pm$, for some $\Omega_f^\pm \in \mathbb{C}^\times$.

Given $\alpha \in V^*$ of norm $-\xi$, put $\alpha' := \omega_\infty^{-1}\alpha\omega_\infty$. By [19, 4.19], we have

$$\eta(\alpha)\xi^{-1/2}P(f,\alpha,\Gamma) + \eta(\alpha')\xi^{-1/2}P(f,\alpha',\Gamma) = -\eta(\alpha)\cdot {\mathfrak{t}_\alpha}^{-1}\cdot I^+_{\gamma_\alpha}\big(Q_{\alpha_{\mathcal{C}}}(z)^{k-1}\big).$$

We then have

$$a_\xi(\tilde{h}) = \sum_{\mathcal{C}\in R_2(\Gamma), q(\mathcal{C})=\xi} \frac{-\eta_\psi(\alpha_{\mathcal{C}})}{2\mu(\Gamma\backslash\mathcal{H})\cdot \mathfrak{t}_{\alpha_{\mathcal{C}}}} \cdot I^+_{\gamma_{\alpha_{\mathcal{C}}}}\big(Q_{\alpha_{\mathcal{C}}}(z)^{k-1}\big).$$

We close this section by choosing a suitable multiple of h which will be the object of the next section. Given $Q_\alpha(z) = cz^2 - 2az - b$ as above, with α in V^*, define $\tilde{Q}_\alpha(z) := M\cdot Q_\alpha(z)$. Then, clearly, $I^\pm(f)(\tilde{Q}_{\alpha_{\mathcal{C}}}(z)^{k-1})$ is equal to $M^{k-1}I^\pm(f)(Q_{\alpha_{\mathcal{C}}}(z)^{k-1})$. We thus normalize the Fourier coefficients by setting

(2)
$$a_\xi(h) := -\frac{a_\xi(\tilde{h})\cdot M^{k-1}\cdot 2\mu(\Gamma\backslash\mathcal{H})}{\Omega_f^+} = \sum_{\mathcal{C}\in R(\Gamma), q(\mathcal{C})=\xi} \frac{\eta_\psi(\alpha_{\mathcal{C}})}{\mathfrak{t}_{\alpha_{\mathcal{C}}}}\cdot \phi_f^+\big(\tilde{Q}_{\alpha_{\mathcal{C}}}(z)^{k-1}\big).$$

So

$$h := \sum_{\xi\geq 1} a_\xi(h)q^\xi \tag{3}$$

belongs to $S_{k+1/2}(4N,\chi)$ and is a non-zero multiple of $\tilde{h}$.

3. The Λ-adic Shimura-Shintani-Waldspurger correspondence

At the heart of Stevens's proof lies the control theorem of Greenberg-Stevens, which has been worked out in the quaternionic setting by Longo–Vigni [9]. Recall that $N \geq 1$ is a square free integer and fix a decomposition $N = M \cdot D$ where D is a square free product of an even number of primes and M is coprime to D. Let $p \nmid N$ be a prime number and fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

3.1. The Hida Hecke algebra. Fix an ordinary p-stabilized newform

$$f_0 \in S_{k_0}\big(\Gamma_1(Mp^{r_0}) \cap \Gamma_0(D), \epsilon_0\big) \tag{4}$$

of level $\Gamma_1(Mp^{r_0}) \cap \Gamma_0(D)$, Dirichlet character ϵ_0 and weight k_0, and write $\mathcal{O}$ for the ring of integers of the field generated over $\mathbb{Q}_p$ by the Fourier coefficients of f_0.
Let Λ (respectively, $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$) denote the Iwasawa algebra of $W := 1 + p\mathbb{Z}_p$ (respectively, $\mathbb{Z}_p^\times$) with coefficients in $\mathcal{O}$. We denote group-like elements in Λ and $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$ as $[t]$. Let $\mathfrak{h}_\infty^{\mathrm{ord}}$ denote the p-ordinary Hida Hecke algebra with coefficients in $\mathcal{O}$ of tame level $\Gamma_1(N)$. Denote by $\mathcal{L} := \mathrm{Frac}(\Lambda)$ the fraction field of Λ. Let $\mathcal{R}$ denote the integral closure of Λ in the primitive component $\mathcal{K}$ of $\mathfrak{h}_\infty^{\mathrm{ord}} \otimes_\Lambda \mathcal{L}$ corresponding to f_0. It is well known that the Λ-algebra $\mathcal{R}$ is finitely generated as Λ-module.
Denote by $\mathcal{X}$ the $\mathcal{O}$-module $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}\text{-alg}}(\mathcal{R}, \bar{\mathbb{Q}}_p)$ of continuous homomorphisms of $\mathcal{O}$-algebras. Let $\mathcal{X}^{\mathrm{arith}}$ the set of arithmetic homomorphisms in $\mathcal{X}$, defined in [9, §2.2] by requiring that the composition

$$W \hookrightarrow \Lambda \xrightarrow{\kappa} \bar{\mathbb{Q}}_p$$

has the form $\gamma \mapsto \psi_\kappa(\gamma)\gamma^{n_\kappa}$ with $n_\kappa = k_\kappa - 2$ for an integer $k_\kappa \geq 2$ (called the weight of κ) and a finite order character $\psi_\kappa : W \to \bar{\mathbb{Q}}_p$ (called the wild character of κ). Denote by r_κ the smallest among the positive integers t such that $1 + p^t\mathbb{Z}_p \subseteq \ker(\psi_\kappa)$. For any $\kappa \in \mathcal{X}^{\mathrm{arith}}$, let P_κ denote the kernel of κ and $\mathcal{R}_{P_\kappa}$ the localization of $\mathcal{R}$ at κ. The field $F_\kappa := \mathcal{R}_{P_\kappa}/P_\kappa\mathcal{R}_{P_\kappa}$ is a finite extension of $\mathrm{Frac}(\mathcal{O})$. Further, by duality, κ corresponds to a normalized eigenform

$$f_\kappa \in S_{k_\kappa}\big(\Gamma_0(Np^{r_\kappa}), \epsilon_\kappa\big)$$

for a Dirichlet character $\epsilon_\kappa : (\mathbb{Z}/Np^{r_\kappa}\mathbb{Z})^\times \to \bar{\mathbb{Q}}_p$. More precisely, if we write $\psi_{\mathcal{R}}$ for the character of $\mathcal{R}$, defined as in [6, Terminology p. 555], and we let ω denote the Teichmüller character, we have $\epsilon_\kappa := \psi_\kappa \cdot \psi_{\mathcal{R}} \cdot \omega^{-n_\kappa}$ (see [6, Cor. 1.6]). We call $(\epsilon_\kappa, k_\kappa)$ the signature of κ. We let κ_0 denote the arithmetic character associated with f_0, so $f_0 = f_{\kappa_0}$, $k_0 = k_{\kappa_0}$, $\epsilon_0 = \epsilon_{\kappa_0}$, and $r_0 = r_{\kappa_0}$. The eigenvalues of f_κ under the action of the Hecke operators T_n ($n \geq 1$ an integer) belong to F_κ. Actually, one can show that f_κ is a p-stabilized newform on $\Gamma_1(Mp^{r_\kappa}) \cap \Gamma_0(D)$.
Let Λ_N denote the Iwasawa algebra of $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ with coefficients in $\mathcal{O}$. To simplify the notation, define $\Delta := (\mathbb{Z}/Np\mathbb{Z})^\times$. We have a canonical isomorphism of rings $\Lambda_N \simeq \Lambda[\Delta]$, which makes Λ_N a Λ-algebra, finitely generated as

Λ-module. Define the tensor product of Λ-algebras

$$\mathcal{R}_N := \mathcal{R} \otimes_\Lambda \Lambda_N,$$

which is again a Λ-algebra (resp. Λ_N-algebra) finitely generated as a Λ-module, (resp. as a Λ_N-module). One easily checks that there is a canonical isomorphism of Λ-algebras

$$\mathcal{R}_N \simeq \mathcal{R}[\Delta]$$

(where Λ acts on $\mathcal{R}$); this is also an isomorphism of Λ_N-algebras, when we let $\Lambda_N \simeq \Lambda[\Delta]$ act on $\mathcal{R}[\Delta]$ in the obvious way.
We can extend any $\kappa \in \mathcal{X}^{\mathrm{arith}}$ to a continuous $\mathcal{O}$-algebra morphism

$$\kappa_N : \mathcal{R}_N \longrightarrow \bar{\mathbb{Q}}_p$$

setting

$$\kappa_N\left(\sum_{i=1}^n r_i \cdot \delta_i\right) := \sum_{i=1}^n \kappa(r_i) \cdot \psi_{\mathcal{R}}(\delta_i)$$

for $r_i \in \mathcal{R}$ and $\delta_i \in \Delta$. Therefore, κ_N restricted to $\mathbb{Z}_p^\times$ is the character $t \mapsto \epsilon_\kappa(t)t^{n_\kappa}$. If we denote by $\mathcal{X}_N$ the $\mathcal{O}$-module of continuous $\mathcal{O}$-algebra homomorphisms from $\mathcal{R}_N$ to $\bar{\mathbb{Q}}_p$, the above correspondence sets up an injective map $\mathcal{X}^{\mathrm{arith}} \hookrightarrow \mathcal{X}_N$. Let $\mathcal{X}_N^{\mathrm{arith}}$ denote the image of $\mathcal{X}^{\mathrm{arith}}$ under this map. For $\kappa_N \in \mathcal{X}_N^{\mathrm{arith}}$, we define the signature of κ_N to be that of the corresponding κ.

3.2. The control theorem in the quaternionic setting. Recall that $B/\mathbb{Q}$ is a quaternion algebra of discriminant D. Fix an auxiliary real quadratic field F such that all primes dividing D are inert in F and all primes dividing Mp are split in F, and an isomorphism $i_F : B \otimes_{\mathbb{Q}} F \simeq \mathbb{M}_2(F)$. Let $\mathcal{O}_B$ denote the maximal order of B obtained by taking the intersection of B with $\mathbb{M}_2(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers of F. More precisely, define

$$\mathcal{O}_B := \iota^{-1}\big(i_F^{-1}\big(i_F(B \otimes 1) \cap \mathbb{M}_2(\mathcal{O}_F)\big)\big)$$

where $\iota : B \hookrightarrow B \otimes_{\mathbb{Q}} F$ is the inclusion defined by $b \mapsto b \otimes 1$. This is a maximal order in B because $i_F(B \otimes 1) \cap \mathbb{M}_2(\mathcal{O}_F)$ is a maximal order in $i_F(B \otimes 1)$. In particular, i_F and our fixed embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ induce an isomorphism

$$i_p : B \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathbb{M}_2(\mathbb{Q}_p)$$

such that $i_p(\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p) = \mathbb{M}_2(\mathbb{Z}_p)$. For any prime $\ell | M$, also choose an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$ which, composed with i_F, yields isomorphisms

$$i_\ell : B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \mathbb{M}_2(\mathbb{Q}_\ell)$$

such that $i_p(\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell) = \mathbb{M}_2(\mathbb{Z}_\ell)$. Define an Eichler order $R \subseteq \mathcal{O}_B$ of level M by requiring that for all primes $\ell | M$ the image of $R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ via i_ℓ consists of upper triangular matrices modulo ℓ. For any $r \geq 0$, let Γ_r denote the subgroup of the group $R_1^\times$ of norm-one elements in R consisting of those γ such that $i_\ell(\gamma) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \equiv 0 \mod Mp^r$ and $a \equiv d \equiv 1 \mod Mp^r$, for all primes $\ell | Mp$. To conclude this list of notation and definitions, fix an embedding $F \hookrightarrow \mathbb{R}$ and let

$$i_\infty : B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{M}_2(\mathbb{R})$$

be the induced isomorphism.
Let $\mathbb{Y} := \mathbb{Z}_p^2$ and denote by $\mathbb{X}$ the set of primitive vectors in $\mathbb{Y}$. Let $\mathbb{D}$ denote the $\mathcal{O}$-module of $\mathcal{O}$-valued measures on $\mathbb{Y}$ which are supported on $\mathbb{X}$. Note that $\mathbb{M}_2(\mathbb{Z}_p)$ acts on $\mathbb{Y}$ by left multiplication; this induces an action of $\mathbb{M}_2(\mathbb{Z}_p)$ on the $\mathcal{O}$-module of $\mathcal{O}$-valued measures on $\mathbb{Y}$, which induces an action on $\mathbb{D}$. The group $R^\times$ acts on $\mathbb{D}$ via i_p. In particular, we may define the group:

$$\mathbb{W} := H^1(\Gamma_0, \mathbb{D}).$$

Then $\mathbb{D}$ has a canonical structure of $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$-module, as well as $\mathfrak{h}_\infty^{\mathrm{ord}}$-action, as described in [9, §2.4]. In particular, let us recall that, for any $[t] \in \mathcal{O}[\![\mathbb{Z}_p^\times]\!]$, we have

$$\int_{\mathbb{X}} \varphi(x,y) d\big([t] \cdot \nu\big) = \int_{\mathbb{X}} \varphi(tx, ty) d\nu,$$

for any locally constant function φ on $\mathbb{X}$.
For any $\kappa \in \mathcal{X}^{\mathrm{arith}}$ and any sign $\pm \in \{-,+\}$, set

$$\mathbb{W}_\kappa^\pm := \mathbb{W}_{f_\kappa^{\mathrm{JL}}}^\pm(F_\kappa) = H^1\big(\Gamma_{r_\kappa}, V_{n_\kappa}(F_\kappa)\big)^{f_\kappa, \pm}$$

where f_κ^{JL} is any Jacquet-Langlands lift of f_κ to Γ_{r_κ}; recall that the superscript f_κ denotes the subspace on which the Hecke algebra acts via the character associated with f_κ, and the superscript $\pm$ denotes the $\pm$-eigenspace for the action of the archimedean involution ι. Also, recall that $\mathbb{W}_\kappa^\pm$ is one dimensional and fix a generator $\phi_\kappa^\pm$ of it.
We may define specialization maps

$$\rho_\kappa : \mathbb{D} \longrightarrow V_{n_\kappa}(F_\kappa)$$

by the formula

$$\rho_\kappa(\nu)(P) := \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon_\kappa(y) P(x,y) d\nu \tag{5}$$

which induces (see [9, §2.5]) a map:

$$\rho_\kappa : \mathbb{W}^{\mathrm{ord}} \longrightarrow \mathbb{W}_\kappa^{\mathrm{ord}}.$$

Here $\mathbb{W}^{\mathrm{ord}}$ and $\mathbb{W}_\kappa^{\mathrm{ord}}$ denote the ordinary submodules of $\mathbb{W}$ and $\mathbb{W}_\kappa$, respectively, defined as in [3, Definition 2.2] (see also [9, §3.5]). We also let $\mathbb{W}_{\mathcal{R}} := \mathbb{W} \otimes_\Lambda \mathcal{R}$, and extend the above map ρ_κ to a map

$$\rho_\kappa : \mathbb{W}_{\mathcal{R}}^{\mathrm{ord}} \longrightarrow \mathbb{W}_\kappa^{\mathrm{ord}}$$

by setting $\rho_\kappa(x \otimes r) := \rho_\kappa(x) \cdot \kappa(r)$.

Theorem 3.1. *There exists a p-adic neighborhood $\mathcal{U}_0$ of κ_0 in $\mathcal{X}$, elements $\Phi^\pm$ in $\mathbb{W}_{\mathcal{R}}^{\mathrm{ord}}$ and choices of p-adic periods $\Omega_\kappa^\pm \in F_\kappa$ for $\kappa \in \mathcal{U}_0 \cap \mathcal{X}^{\mathrm{arith}}$ such that, for all $\kappa \in \mathcal{U}_0 \cap \mathcal{X}^{\mathrm{arith}}$, we have*

$$\rho_\kappa(\Phi^\pm) = \Omega_\kappa^\pm \cdot \phi_\kappa^\pm$$

and $\Omega^{\pm}_{\kappa_0} \neq 0$.

Proof. This is an easy consequence of [9, Theorem 2.18] and follows along the lines of the proof of [21, Theorem 5.5], cf. [10, Proposition 3.2]. □

We now normalize our choices as follows. With $\mathcal{U}_0$ as above, define

$$\mathcal{U}_0^{\mathrm{arith}} := \mathcal{U}_0 \cap \mathcal{X}^{\mathrm{arith}}.$$

Fix $\kappa \in \mathcal{U}_0^{\mathrm{arith}}$ and an embedding $\bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$. Let f_κ^{JL} denote a modular form on Γ_{r_κ} corresponding to f_κ by the Jacquet-Langlands correspondence, which is well defined up to elements in $\mathbb{C}^\times$. View $\phi_\kappa^{\pm}$ as an element in $H^1(\Gamma_{r_\kappa}, V_n(\mathbb{C}))^{\pm}$. Choose a representative $\Phi_\gamma^{\pm}$ of $\Phi^{\pm}$, by which we mean that if $\Phi^{\pm} = \sum_i \Phi_i^{\pm} \otimes r_i$, then we choose a representative $\Phi_{i,\gamma}^{\pm}$ for all i. Also, we will write $\rho_\kappa(\Phi)(P)$ as

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \epsilon_\kappa(y)P(x,y)d\Phi_\gamma^{\pm} := \sum_i \kappa(r_i)\cdot \int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \epsilon_\kappa(y)P(x,y)d\Phi_{i,\gamma}^{\pm}.$$

With this notation, we see that the two cohomology classes

$$\gamma \longmapsto \int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \epsilon_\kappa(y)P(x,y)d\Phi_\gamma^{\pm}(x,y)$$

and

$$\gamma \longmapsto \Omega_\kappa^{\pm}\cdot \int_\tau^{\gamma(\tau)} P(z,1) f_\kappa^{\mathrm{JL},\pm}(z)dz$$

are cohomologous in $H^1(\Gamma_{r_\kappa}, V_{n_\kappa}(\mathbb{C}))$, for any choice of $\tau \in \mathcal{H}$.

3.3. Metaplectic Hida Hecke algebras. Let $\sigma : \Lambda_N \to \Lambda_N$ be the ring homomorphism associated to the group homomorphism $t \mapsto t^2$ on $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$, and denote by the same symbol its restriction to Λ and $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$. We let Λ_σ, $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]_\sigma$ and $\Lambda_{N,\sigma}$ denote, respectively, Λ, $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$ and Λ_N viewed as algebras over themselves via σ. The ordinary metaplectic p-adic Hida Hecke algebra we will consider is the Λ-algebra

$$\widetilde{\mathcal{R}} := \mathcal{R} \otimes_\Lambda \Lambda_\sigma.$$

Define as above

$$\widetilde{\mathcal{X}} := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}\text{-alg}}(\widetilde{\mathcal{R}}, \bar{\mathbb{Q}}_p)$$

and let the set $\widetilde{\mathcal{X}}^{\mathrm{arith}}$ of arithmetic points in $\widetilde{\mathcal{X}}$ to consist of those $\tilde{\kappa}$ such that the composition

$$W \hookrightarrow \Lambda \xhookrightarrow{\lambda\mapsto 1\otimes\lambda} \widetilde{\mathcal{R}} \xhookrightarrow{\tilde{\kappa}} \bar{\mathbb{Q}}_p$$

has the form $\gamma \mapsto \psi_{\tilde{\kappa}}(\gamma)\gamma^{n_{\tilde{\kappa}}}$ with $n_{\tilde{\kappa}} := k_{\tilde{\kappa}} - 2$ for an integer $k_{\tilde{\kappa}} \geq 2$ (called the weight of $\tilde{\kappa}$) and a finite order character $\psi_{\tilde{\kappa}} : W \to \bar{\mathbb{Q}}$ (called the wild character of $\tilde{\kappa}$). Let $r_{\tilde{\kappa}}$ the smallest among the positive integers t such that $1 + p^t\mathbb{Z}_p \subseteq \ker(\psi_{\tilde{\kappa}})$.

We have a map $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ induced by pull-back from the canonical map $\mathcal{R} \to \widetilde{\mathcal{R}}$. The map p restricts to arithmetic points.

As above, define the Λ-algebra (or Λ_N-algebra)

$$\widetilde{\mathcal{R}}_N := \mathcal{R} \otimes_\Lambda \Lambda_{N,\sigma} \tag{6}$$

via $\lambda \mapsto 1 \otimes \lambda$.

We easily see that $\widetilde{\mathcal{R}}_N \simeq \widetilde{\mathcal{R}}[\Delta]$ as Λ_N-algebras, where we enhance $\widetilde{\mathcal{R}}[\Delta]$ with the following structure of $\Lambda_N \simeq \Lambda[\Delta]$-algebra: for $\sum_i \lambda_i \cdot \delta_i \in \Lambda[\Delta]$ (with $\lambda_i \in \Lambda$ and $\delta_i \in \Delta$) and $\sum r_j \cdot \delta'_j \in \widetilde{\mathcal{R}}[\Delta]$ (with $r_j = \sum_h r_{j,h} \otimes \lambda_{j,h} \in \widetilde{\mathcal{R}}$, $r_{j,h} \in \mathcal{R}$, $\lambda_{j,h} \in \Lambda_\sigma$, and $\delta'_j \in \Delta$), we set

$$\big(\sum_i \lambda_i \cdot \delta_i\big) \cdot \big(\sum_j r_j \cdot \delta'_j\big) := \sum_{i,j,h} \big(r_{j,h} \otimes (\lambda_i \lambda_{j,h})\big) \cdot (\delta_i \delta'_j).$$

As above, extend $\tilde{\kappa} \in \widetilde{\mathcal{X}}^{\mathrm{arith}}$ to a continuous $\mathcal{O}$-algebra morphism

$$\tilde{\kappa}_N : \widetilde{\mathcal{R}}_N \longrightarrow \bar{\mathbb{Q}}_p$$

by setting

$$\tilde{\kappa}_N\left(\sum_{i=1}^n x_i \cdot \delta_i\right) := \sum_{i=1}^n \tilde{\kappa}(x_i) \cdot \psi_{\mathcal{R}}(\delta_i)$$

for $x_i \in \widetilde{\mathcal{R}}$ and $\delta_i \in \Delta$, where $\psi_{\mathcal{R}}$ is the character of $\mathcal{R}$. If we denote by $\widetilde{\mathcal{X}}_N$ the $\mathcal{O}$-module of continuous $\mathcal{O}$-linear homomorphisms from $\widetilde{\mathcal{R}}_N$ to $\bar{\mathbb{Q}}_p$, the above correspondence sets up an injective map $\widetilde{\mathcal{X}}^{\mathrm{arith}} \hookrightarrow \widetilde{\mathcal{X}}_N$ and we let $\widetilde{\mathcal{X}}_N^{\mathrm{arith}}$ denote the image of $\widetilde{\mathcal{X}}^{\mathrm{arith}}$. Put $\epsilon_{\tilde{\kappa}} := \psi_{\tilde{\kappa}} \cdot \psi_{\mathcal{R}} \cdot \omega^{-n_{\tilde{\kappa}}}$, which we view as a Dirichlet character of $(\mathbb{Z}/Np^{r_{\tilde{\kappa}}}\mathbb{Z})^\times$, and call the pair $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$ the signature of $\tilde{\kappa}_N$, where $\tilde{\kappa}$ is the arithmetic point corresponding to $\tilde{\kappa}_N$.

We also have a map $p_N : \widetilde{\mathcal{X}}_N \to \mathcal{X}_N$ induced from the map $\mathcal{R}_N \to \widetilde{\mathcal{R}}_N$ taking $r \mapsto r \otimes 1$ by pull-back. The map p_N also restricts to arithmetic points. The maps p and p_N make the following diagram commute:

$$\begin{array}{ccc} \widetilde{\mathcal{X}}^{\mathrm{arith}} & \hookrightarrow & \widetilde{\mathcal{X}}_N^{\mathrm{arith}} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_N} \\ \mathcal{X}^{\mathrm{arith}} & \hookrightarrow & \mathcal{X}_N^{\mathrm{arith}} \end{array}$$

where the projections take a signature (ϵ, k) to $(\epsilon^2, 2k)$.

3.4. The Λ-adic correspondence. In this part, we combine the explicit integral formula of Shimura and the fact that the toric integrals can be p-adically interpolated to show the existence of a Λ-adic Shimura-Shintani-Waldspurger correspondence with the expected interpolation property. This follows very closely [21, §6].

Let $\tilde{\kappa}_N \in \widetilde{\mathcal{X}}_N^{\mathrm{arith}}$ of signature $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$. Let L_r denote the order of $\mathbb{M}_2(F)$ consisting of matrices $\left(\begin{smallmatrix} a & b/Mp^r \\ Mp^r c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathcal{O}_F$. Define

$$\mathcal{O}_{B,r} := \iota^{-1}\big(i_F^{-1}\big(i_F(B \otimes 1) \cap L_r\big)\big)$$

Then $\mathcal{O}_{B,r}$ is the maximal order introduced in §2.1 (and denoted $\mathcal{O}'_B$ there) defined in terms of the maximal order $\mathcal{O}_B$ and the integer Mp^r. Also,

$$S := \mathcal{O}_B \cap \mathcal{O}_{B,r}$$

is an Eichler order of B of level Mp containing the fixed Eichler order R of level M. With $\alpha \in V^* \cap \mathcal{O}_{B,1}$, we have

$$i_F(\alpha) = \begin{pmatrix} a & b/(Mp) \\ c & -a \end{pmatrix} \tag{7}$$

in $\mathbb{M}_2(F)$ with $a, b, c \in \mathcal{O}_F$ and we can consider the quadratic forms

$$Q_\alpha(x,y) := cx^2 - 2axy - \big(b/(Mp)\big)y^2,$$

and

$$\tilde{Q}_\alpha(x,y) := Mp \cdot Q_\alpha(x,y) = Mpcx^2 - 2Mpaxy - by^2. \tag{8}$$

Then $\tilde{Q}_\alpha(x,y)$ has coefficients in $\mathcal{O}_F$ and, composing with $F \hookrightarrow \mathbb{R}$ and letting $x = z$, $y = 1$, we recover $Q_\alpha(z)$ and $\tilde{Q}_\alpha(z)$ of §2.1 (defined by means of the isomorphism i_∞). Since each prime $\ell | Mp$ is split in F, the elements a, b, c can be viewed as elements in $\mathbb{Z}_\ell$ via our fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$, for any prime $\ell | Mp$ (we will continue writing a, b, c for these elements, with a slight abuse of notation). So, letting $b_\alpha \in \mathbb{Z}$ such that $i_\ell(\alpha) \equiv \left(\begin{smallmatrix} * & b_\alpha/(Mp) \\ * & * \end{smallmatrix}\right)$ modulo $i_\ell(S \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$, for all $\ell | Mp$, we have $b \equiv b_\alpha$ modulo $Mp\mathbb{Z}_\ell$ as elements in $\mathbb{Z}_\ell$, for all $\ell | Mp$, and thus we get

$$\eta_{\epsilon_{\tilde{\kappa}}}(\alpha) = \epsilon_{\tilde{\kappa}}(b_\alpha) = \epsilon_{\tilde{\kappa}}(b) \tag{9}$$

for b as in (7).

For any $\nu \in \mathbb{D}$, we may define an $\mathcal{O}$-valued measure $j_\alpha(\nu)$ on $\mathbb{Z}_p^\times$ by the formula:

$$\int_{\mathbb{Z}_p^\times} f(t) dj_\alpha(\nu)(t) := \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} f\big(\tilde{Q}_\alpha(x,y)\big) d\nu(x,y).$$

for any continuous function $f : \mathbb{Z}_p^\times \to \mathbb{C}_p$. Recall that the group of $\mathcal{O}$-valued measures on $\mathbb{Z}_p^\times$ is isomorphic to the Iwasawa algebra $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$, and thus we may view $j_\alpha(\nu)$ as an element in $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$ (see, for example, [1, §3.2]). In particular, for any group-like element $[\lambda] \in \mathcal{O}[\![\mathbb{Z}_p^\times]\!]$ we have:

$$\int_{\mathbb{Z}_p^\times} f(t) d\big([\lambda] \cdot j_\alpha(\nu)\big)(t) = \int_{\mathbb{Z}_p^\times} \left(\int_{\mathbb{Z}_p^\times} f(ts) d[\lambda](s) \right) dj_\alpha(\nu)(t) = \int_{\mathbb{Z}_p^\times} f(\lambda t) dj_\alpha(\nu)(t).$$

On the other hand,

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} f\big(\tilde{Q}_\alpha(x,y)\big) d(\lambda \cdot \nu) = \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} f\big(\tilde{Q}_\alpha(\lambda x, \lambda y)\big) d\nu = \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} f\big(\lambda^2 \tilde{Q}_\alpha(x,y)\big) d\nu$$

and we conclude that $j_\alpha(\lambda \cdot \nu) = [\lambda^2] \cdot j_\alpha(\nu)$. In other words, j_α is a $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$-linear map

$$j_\alpha : \mathbb{D} \longrightarrow \mathcal{O}[\![\mathbb{Z}_p^\times]\!]_\sigma.$$

Before going ahead, let us introduce some notation. Let χ be a Dirichlet character modulo Mp^r, for a positive integer r, which we decompose accordingly with the isomorphism $(\mathbb{Z}/Np^r\mathbb{Z})^\times \simeq (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/p^r\mathbb{Z})^\times$ into the product $\chi = \chi_N \cdot \chi_p$ with $\chi_N : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ and $\chi_p : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \mathbb{C}^\times$. Thus, we will write $\chi(x) = \chi_N(x_N) \cdot \chi_p(x_p)$, where x_N and x_p are the projections of $x \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$ to $(\mathbb{Z}/N\mathbb{Z})^\times$ and $(\mathbb{Z}/p^r\mathbb{Z})^\times$, respectively. To simplify the notation, we will suppress the N and p from the notation for x_N and x_p, thus simply writing x for any of the two. Using the isomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \simeq (\mathbb{Z}/M\mathbb{Z})^\times \times (\mathbb{Z}/D\mathbb{Z})^\times$, decompose χ_N as $\chi_N = \chi_M \cdot \chi_D$ with χ_M and χ_D characters on $(\mathbb{Z}/M\mathbb{Z})^\times$ and $(\mathbb{Z}/D\mathbb{Z})^\times$, respectively. In the following, we only need the case when $\chi_D = 1$.

Using the above notation, we may define a $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$-linear map $J_\alpha : \mathbb{D} \to \mathcal{O}[\![\mathbb{Z}_p^\times]\!]$ by

$$J_\alpha(\nu) = \epsilon_{\tilde\kappa,M}(b) \cdot \epsilon_{\tilde\kappa,p}(-1) \cdot j_\alpha(\nu)$$

with b as in (7). Set $\mathbb{D}_N := \mathbb{D} \otimes_{\mathcal{O}[\![\mathbb{Z}_p^\times]\!]} \Lambda_N$, where the map $\mathcal{O}[\![\mathbb{Z}_p^\times]\!] \to \Lambda_N$ is induced from the map $\mathbb{Z}_p^\times \to \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ on group-like elements given by $x \mapsto x \otimes 1$. Then J_α can be extended to a Λ_N-linear map $J_\alpha : \mathbb{D}_N \to \Lambda_{N,\sigma}$. Setting $\mathbb{D}_{\mathcal{R}_N} := \mathcal{R}_N \otimes_{\Lambda_N} \mathbb{D}_N$ and extending by $\mathcal{R}_N$-linearity over Λ_N we finally obtain a $\mathcal{R}_N$-linear map, again denoted by the same symbol,

$$J_\alpha : \mathbb{D}_{\mathcal{R}_N} \longrightarrow \widetilde{\mathcal{R}}_N.$$

For $\nu \in \mathbb{D}_N$ and $r \in \mathcal{R}_N$ we thus have

$$J_\alpha(r \otimes \nu) = \epsilon_{\tilde\kappa,M}(b) \cdot \epsilon_{\tilde\kappa,p}(-1) \cdot r \otimes j_\alpha(\nu).$$

For the next result, for any arithmetic point $\kappa_N \in \mathcal{X}_N^{\mathrm{arith}}$ coming from $\kappa \in \mathcal{X}^{\mathrm{arith}}$, extend ρ_κ in (5) by $\mathcal{R}_N$-linearity over $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$, to get a map

$$\rho_{\kappa_N} : \mathbb{D}_{\mathcal{R}_N} \longrightarrow V_{n_\kappa}$$

defined by $\rho_{\kappa_N}(r \otimes \nu) := \rho_\kappa(\nu) \cdot \kappa_N(r)$, for $\nu \in \mathbb{D}$ and $r \in \mathcal{R}_N$. To simplify the notation, set

$$\langle \nu, \alpha \rangle_{\kappa_N} := \rho_{\kappa_N}(\nu)(\tilde Q_\alpha^{n_{\tilde\kappa}/2}). \tag{10}$$

The following is essentially [21, Lemma (6.1)].

Lemma 3.2. *Let $\tilde\kappa_N \in \widetilde{\mathcal{X}}_N^{\mathrm{arith}}$ with signature $(\epsilon_{\tilde\kappa}, k_{\tilde\kappa})$ and define $\kappa_N := p_N(\tilde\kappa_N)$. Then for any $\nu \in \mathbb{D}_{\mathcal{R}_N}$ we have:*

$$\tilde\kappa_N\big(J_\alpha(\nu)\big) = \eta_{\epsilon_{\tilde\kappa}}(\alpha) \cdot \langle \nu, \alpha \rangle_{\kappa_N}.$$

Proof. For $\nu \in \mathbb{D}_N$ and $r \in \mathcal{R}_N$ we have

$$\begin{aligned}\tilde\kappa_N\big(J_\alpha(r \otimes \nu)\big) &= \tilde\kappa_N\big(\epsilon_{\tilde\kappa,M}(b) \cdot \epsilon_{\tilde\kappa,p}(-1) \cdot r \otimes j_\alpha(\nu)\big)\\ &= \epsilon_{\tilde\kappa,M}(b) \cdot \epsilon_{\tilde\kappa,p}(-1) \cdot \tilde\kappa_N(r \otimes 1) \cdot \tilde\kappa_N\big(1 \otimes j_\alpha(\nu)\big)\\ &= \epsilon_{\tilde\kappa,M}(b) \cdot \epsilon_{\tilde\kappa,p}(-1) \cdot \kappa_N(r) \cdot \int_{\mathbb{Z}_p^\times} \tilde\kappa_N(t) dj_\alpha(\nu)\end{aligned}$$

and thus, noticing that $\tilde\kappa_N$ restricted to $\mathbb{Z}_p^\times$ is $\tilde\kappa_N(t) = \epsilon_{\tilde\kappa,p}(t)t^{n_{\tilde\kappa}}$, we have

$$\tilde\kappa_N\big(J_\alpha(r \otimes \nu)\big) = \epsilon_{\tilde\kappa,M}(b) \cdot \epsilon_{\tilde\kappa,p}(-1) \cdot \kappa_N(r) \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon_{\tilde\kappa,p}(\tilde Q_\alpha(x,y)) \tilde Q_\alpha(x,y)^{n_{\tilde\kappa}/2} d\nu.$$

Recalling (8), and viewing a, b, c as elements in $\mathbb{Z}_p$, we have, for $(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p^\times$,

$$\epsilon_{\tilde{\kappa},p}\big(\tilde{Q}_\alpha(x,y)\big) = \epsilon_{\tilde{\kappa},p}(-by^2) = \epsilon_{\tilde{\kappa},p}(-b)\epsilon_{\tilde{\kappa},p}(y^2) = \epsilon_{\tilde{\kappa},p}(-b)\epsilon^2_{\tilde{\kappa},p}(y) = \epsilon_{\tilde{\kappa},p}(-b)\epsilon_{\kappa,p}(y).$$

Thus, since $\epsilon_{\tilde{\kappa}}(-1)^2 = 1$, we get:

$$\tilde{\kappa}_N\big(J_\alpha(r \otimes \nu)\big) = \kappa_N(r) \cdot \epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(b) \cdot \rho_\kappa(\nu)(\tilde{Q}_\alpha^{n_{\tilde{\kappa}}/2}) = \eta_{\epsilon_\kappa}(\alpha) \cdot \langle \nu, \alpha \rangle_{\kappa_N}$$

where for the last equality use (9) and (10). □

Define

$$\mathbb{W}_{\mathcal{R}_N} := \mathbb{W} \otimes_{\mathcal{O}[\![\mathbb{Z}_p^\times]\!]} \mathcal{R}_N,$$

the structure of $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$-module of $\mathcal{R}_N$ being that induced by the composition of the two maps $\mathcal{O}[\![\mathbb{Z}_p^\times]\!] \to \Lambda_N \to \mathcal{R}_N$ described above. There is a canonical map

$$\vartheta : \mathbb{W}_{\mathcal{R}_N} \longrightarrow H^1(\Gamma_0, \mathbb{D}_{\mathcal{R}_N})$$

described as follows: if ν_γ is a representative of an element ν in $\mathbb{W}$ and $r \in \mathcal{R}_N$, then $\vartheta(\nu \otimes r)$ is represented by the cocycle $\nu_\gamma \otimes r$.
For $\nu \in \mathbb{W}_{\mathcal{R}_N}$ represented by ν_γ and $\xi \geq 1$ an integer, define

$$\theta_\xi(\nu) := \sum_{\mathcal{C} \in R(\Gamma_1), q(\mathcal{C}) = \xi} \frac{J_{\alpha_\mathcal{C}}(\nu_{\gamma_{\alpha_\mathcal{C}}})}{\mathfrak{t}_{\alpha_\mathcal{C}}}.$$

Definition 3.3. For $\nu \in \mathbb{W}_{\mathcal{R}_N}$, the formal Fourier expansion

$$\Theta(\nu) := \sum_{\xi \geq 1} \theta_\xi(\nu) q^\xi$$

in $\mathcal{R}_N[\![q]\!]$ is called the Λ-adic Shimura-Shintani-Waldspurger lift of ν. For any $\tilde{\kappa} \in \widetilde{\mathcal{X}}^{\mathrm{arith}}$, the formal power series expansion

$$\Theta(\nu)(\tilde{\kappa}_N) := \sum_{\xi \geq 1} \tilde{\kappa}_N\big(\theta_\xi(\nu)\big) q^\xi$$

is called the $\tilde{\kappa}$-specialization of $\Theta(\nu)$.

There is a natural map

$$\mathbb{W}_\mathcal{R} \longrightarrow \mathbb{W}_{\mathcal{R}_N}$$

taking $\nu \otimes r$ to itself (use that $\mathcal{R}$ has a canonical map to $\mathcal{R}_N \simeq \mathcal{R}[\Delta]$, as described above). So, for any choice of sign, $\Phi^\pm \in \mathbb{W}_\mathcal{R}$ will be viewed as an element in $\mathbb{W}_{\mathcal{R}_N}$.
¿From now on we will use the following notation. Fix $\tilde{\kappa}_0 \in \widetilde{\mathcal{X}}^{\mathrm{arith}}$ and put $\kappa_0 := p(\tilde{\kappa}_0) \in \mathcal{X}^{\mathrm{arith}}$. Recall the neighborhood $\mathcal{U}_0$ of κ_0 in Theorem 3.1. Define $\widetilde{\mathcal{U}}_0 := p^{-1}(\mathcal{U}_0)$ and

$$\widetilde{\mathcal{U}}_0^{\mathrm{arith}} := \widetilde{\mathcal{U}}_0 \cap \widetilde{\mathcal{X}}^{\mathrm{arith}}.$$

For each $\tilde{\kappa} \in \widetilde{\mathcal{U}}_0^{\mathrm{arith}}$ put $\kappa = p(\tilde{\kappa}) \in \mathcal{U}_0^{\mathrm{arith}}$. Recall that if $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$ is the signature of $\tilde{\kappa}$, then $(\epsilon_\kappa, k_\kappa) := (\epsilon_{\tilde{\kappa}}^2, 2k_{\tilde{\kappa}})$ is that of κ_0. For any $\kappa := p(\tilde{\kappa})$ as above, we may consider the modular form

$$f_\kappa^{\mathrm{JL}} \in S_{k_\kappa}(\Gamma_{r_\kappa}, \epsilon_\kappa)$$

and its Shimura-Shintani-Waldspurger lift

$$h_\kappa = \sum_\xi a_\xi(h_\kappa) q^\xi \in S_{k_\kappa+1/2}(4Np^{r_\kappa}, \chi_\kappa), \quad \text{where } \chi_\kappa(x) := \epsilon_{\tilde{\kappa}}(x)\left(\frac{-1}{x}\right)^{k_\kappa},$$

normalized as in (2) and (3). For our fixed κ_0, recall the elements $\Phi := \Phi^+$ chosen as in Theorem 3.1 and define $\phi_\kappa := \phi_\kappa^+$ and $\Omega_\kappa := \Omega_\kappa^+$ for $\kappa \in \mathcal{U}_0^{\mathrm{arith}}$.

Proposition 3.4. *For all $\tilde{\kappa} \in \widetilde{\mathcal{U}}_0^{\mathrm{arith}}$ such that $r_\kappa = 1$ we have*

$$\tilde{\kappa}_N\big(\theta_\xi(\Phi)\big) = \Omega_\kappa \cdot a_\xi(h_\kappa) \quad \textit{and} \quad \Theta(\Phi)(\tilde{\kappa}_N) = \Omega_\kappa \cdot h_\kappa.$$

Proof. By Lemma 3.2 we have

$$\tilde{\kappa}_N\big(\theta_\xi(\Phi)\big) = \sum_{\mathcal{C}\in R(\Gamma_1), q(\mathcal{C})=\xi} \frac{\eta_{\epsilon_{\tilde{\kappa}}}(\alpha_{\mathcal{C}})}{\mathfrak{t}_{\alpha_{\mathcal{C}}}} \rho_{\kappa_N}(\Phi)(\tilde{Q}_{\alpha_{\mathcal{C}}}^{n_{\tilde{\kappa}}/2}).$$

Using Theorem 3.1, we get

$$\tilde{\kappa}_N\big(\theta_\xi(\Phi)\big) = \sum_{\mathcal{C}\in R(\Gamma_1), q(\mathcal{C})=\xi} \frac{\eta_{\epsilon_{\tilde{\kappa}}}(\alpha_{\mathcal{C}}) \cdot \Omega_\kappa}{\mathfrak{t}_{\alpha_{\mathcal{C}}}} \phi_\kappa(\tilde{Q}_{\alpha_{\mathcal{C}}}^{k_\kappa-1}).$$

Now (2) shows the statement on $\tilde{\kappa}_N(\theta_\xi(\Phi))$, while that on $\Theta(\Phi)(\tilde{\kappa}_N)$ is a formal consequence of the previous one. □

Corollary 3.5. *Let a_p denote the image of the Hecke operator T_p in $\mathcal{R}$. Then $\Theta(\Phi)|T_p^2 = a_p \cdot \Theta(\Phi)$.*

Proof. For any $\kappa \in \mathcal{X}^{\mathrm{arith}}$, let $a_p(\kappa) := \kappa(T_p)$, which is a p-adic unit by the ordinarity assumption. For all $\tilde{\kappa} \in \widetilde{\mathcal{U}}_0^{\mathrm{arith}}$ with $r_\kappa = 1$, we have

$$\Theta(\Phi)(\tilde{\kappa}_N)|T_p^2 = \Omega_\kappa \cdot h_\kappa|T_p^2 = a_p(\kappa)\cdot\Omega_\kappa\cdot h_\kappa = a_p(\kappa)\cdot\Theta(\Phi)(\tilde{\kappa}_N).$$

Consequently,

$$\tilde{\kappa}_N\big(\theta_{\xi p^2}(\Phi)\big) = a_p(\kappa)\cdot\tilde{\kappa}_N\big(\theta_\xi(\Phi)\big)$$

for all $\tilde{\kappa}$ such that $r_\kappa = 1$. Since this subset is dense in $\widetilde{\mathcal{X}}_N$, we conclude that $\theta_{\xi p^2}(\Phi) = a_p \cdot \theta_\xi(\Phi)$ and so $\Theta(\Phi)|T_p^2 = a_p \cdot \Theta(\Phi)$. □

For any integer $n \geq 1$ and any quadratic form Q with coefficients in F, write $[Q]_n$ for the class of Q modulo the action of $i_F(\Gamma_n)$. Define $\mathcal{F}_{n,\xi}$ to be the subset of the F-vector space of quadratic forms with coefficients in F consisting of quadratic forms $\tilde{Q}_\alpha$ such that $\alpha \in V^* \cap \mathcal{O}_{B,n}$ and $-\mathrm{nr}(\alpha) = \xi$. Writing $\delta_{\tilde{Q}_\alpha}$ for the discriminant of Q_α, the above set can be equivalently described as

$$\mathcal{F}_{n,\xi} := \{\tilde{Q}_\alpha |\, \alpha \in V^* \cap \mathcal{O}_{B,n},\, \delta_{\tilde{Q}_\alpha} = Np^n\xi\}.$$

Define $\mathcal{F}_{n,\xi}/\Gamma_n$ to be the set $\{[\tilde{Q}_\alpha]_n |\, \tilde{Q}_\alpha \in \mathcal{F}_{n,\xi}\}$ of equivalence classes of $\mathcal{F}_{n,\xi}$ under the action of $i_F(\Gamma_n)$. A simple computation shows that $Q_{g^{-1}\alpha g} = Q_\alpha|g$ for all $\alpha \in V^*$ and all $g \in \Gamma_n$, and thus we find

$$\mathcal{F}_{n,\xi}/\Gamma_n = \{[\tilde{Q}_{\mathcal{C}_\alpha}]_n |\, \mathcal{C} \in R(\Gamma_n),\, \delta_{\tilde{Q}_\alpha} = Np^n\xi\}.$$

We also note that, in the notation of §2.1, if f has weight character ψ, defined modulo Np^n, and level Γ_n, the Fourier coefficients $a_\xi(h)$ of the Shimura-Shintani-Waldspurger lift h of f are given by

$$a_\xi(h) = \sum_{[Q]\in\mathcal{F}_{n,\xi}/\Gamma_n} \frac{\psi(Q)}{\mathfrak{t}_Q}\phi_f^+\big(Q(z)^{k-1}\big) \tag{11}$$

and, if $Q = \tilde{Q}_\alpha$, we put $\psi(Q) := \eta_\psi(b_\alpha)$ and $\mathfrak{t}_Q := \mathfrak{t}_\alpha$. Also, if we let

$$\mathcal{F}_n/\Gamma_n := \coprod_\xi \mathcal{F}_{n,\xi}/\Gamma_n$$

we can write

$$h = \sum_{[Q]\in\mathcal{F}_n/\Gamma_n} \frac{\psi(Q)}{\mathfrak{t}_Q}\phi_f^+\big(Q(z)^{k-1}\big)q^{\delta_Q/(Np^n)}. \tag{12}$$

Fix now an integer $m \geq 1$ and let $n \in \{1, m\}$. For any $t \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ and any integer $\xi \geq 1$, define $\mathcal{F}_{n,\xi,t}$ to be the subset of $\mathcal{F}_{n,\xi}$ consisting of forms such that $Np^n b_\alpha \equiv t \mod Np^m$. Also, define $\mathcal{F}_{n,\xi,t}/\Gamma_n$ to be the set of equivalence classes of $\mathcal{F}_{n,\xi,t}$ under the action of $i_F(\Gamma_n)$. If $\alpha \in V^* \cap \mathcal{O}_{B,m}$ and

$$i_F(\alpha) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

then

$$\tilde{Q}_\alpha(x,y) = Np^n cx^2 - 2Np^n axy - Np^n by^2 \tag{13}$$

from which we see that there is an inclusion $\mathcal{F}_{m,\xi,t} \subseteq \mathcal{F}_{1,\xi p^{m-1},t}$. If $\tilde{Q}_\alpha$ and $\tilde{Q}_{\alpha'}$ belong to $\mathcal{F}_{m,\xi,t}$, and $\alpha' = g\alpha g^{-1}$ for some $g \in \Gamma_m$, then, since $\Gamma_m \subseteq \Gamma_1$, we see that $\tilde{Q}_\alpha$ and $\tilde{Q}_{\alpha'}$ represent the same class in $\mathcal{F}_{1,\xi p^{m-1},t}/\Gamma_1$. This shows that $[\tilde{Q}_\alpha]_m \mapsto [\tilde{Q}_\alpha]_1$ gives a well-defined map

$$\pi_{m,\xi,t} : \mathcal{F}_{m,\xi,t}/\Gamma_m \longrightarrow \mathcal{F}_{1,\xi p^{m-1},t}/\Gamma_1.$$

Lemma 3.6. *The map $\pi_{m,\xi,t}$ is bijective.*

Proof. We first show the injectivity. For this, suppose $\tilde{Q}_\alpha$ and $\tilde{Q}_{\alpha'}$ are in $\mathcal{F}_{m,\xi,t}$ and $[\tilde{Q}_\alpha]_1 = [\tilde{Q}_{\alpha'}]_1$. So there exists $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ in $i_F(\Gamma_1)$ such that such that $\tilde{Q}_\alpha = \tilde{Q}_{\alpha'}|g$. If $\tilde{Q}_\alpha = cx^2 - 2axy - by^2$, and easy computation shows that $\tilde{Q}_{\alpha'} = c'x^2 - 2a'xy - b'y^2$ with

$$c' = c\alpha^2 - 2a\alpha\gamma - b\gamma^2$$

$$a' = -c\alpha\beta + a\beta\gamma + a\alpha\delta + b\gamma\delta$$

$$b' = -c\beta^2 + 2a\beta\delta + b\delta^2.$$

The first condition shows that $\gamma \equiv 0 \mod Np^m$. We have $b \equiv b' \equiv t \mod Np^m$, so $\delta^2 \equiv 1 \mod Np^m$. Since $\delta \equiv 1 \mod Np$, we see that $\delta \equiv 1 \mod Np^m$ too.

We now show the surjectivity. For this, fix $[\tilde{Q}_{\alpha_{\mathcal{C}}}]_1$ in the target of π, and choose a representative

$$\tilde{Q}_{\alpha_{\mathcal{C}}} = cx^2 - 2axy - by^2$$

(recall $Mp^m\xi|\delta_{\tilde{Q}_{\alpha_{\mathcal{C}}}}$, $Np|c$, $Np|a$, and $b \in \mathcal{O}_F^\times$, the last condition due to $\eta_\psi(\alpha_{\mathcal{C}}) \neq 0$). By the Strong Approximation Theorem, we can find $\tilde{g} \in \Gamma_1$ such that

$$i_\ell(\tilde{g}) \equiv \begin{pmatrix} 1 & 0 \\ ab^{-1} & 1 \end{pmatrix} \mod Np^m$$

for all $\ell|Np$. Take $g := i_F(\tilde{g})$, and put $\alpha := g^{-1}\alpha_{\mathcal{C}}g$. An easy computation, using the expressions for a', b', c' in terms of a, b, c and $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ as above, shows that $\alpha \in V^* \cap \mathcal{O}_{B,m}$, $\eta_\psi(\alpha) = t$ and $\delta_{\tilde{Q}_\alpha} = Np^m\xi$, and it follows that $\tilde{Q}_\alpha \in \mathcal{F}_{m,\xi,t}$. Now

$$\pi\big([\tilde{Q}_\alpha]_m\big) = [\tilde{Q}_\alpha]_1 = [\tilde{Q}_{g^{-1}\alpha_{\mathcal{C}}g}]_1 = [\tilde{Q}_{\alpha_{\mathcal{C}}}]_1$$

where the last equality follows because $g \in \Gamma_1$. □

Proposition 3.7. *For all $\tilde{\kappa} \in \widetilde{\mathcal{U}}_0^{\mathrm{arith}}$ we have*

$$\Theta(\Phi)(\tilde{\kappa}_N)|T_p^{r_\kappa - 1} = \Omega_\kappa \cdot h_\kappa.$$

Proof. For $r_\kappa = 1$, this is Proposition 3.4 above, so we may assume $r_\kappa \geq 2$. As in the proof of Proposition 3.4, combining Lemma 3.2 and Theorem 3.1 we get

$$\Theta(\Phi)(\tilde{\kappa}_N) = \sum_{\xi \geq 1} \left(\sum_{\mathcal{C} \in R(\Gamma_1), q(\mathcal{C}) = \xi} \frac{\eta_{\epsilon_{\tilde{\kappa}}}(\alpha_{\mathcal{C}}) \cdot \Omega_\kappa}{\mathsf{t}_{\alpha_{\mathcal{C}}}} \phi_\kappa(\tilde{Q}_{\alpha_{\mathcal{C}}}^{k_\kappa - 1}) \right) q^\xi$$

which, by (11) and (12) above we may rewrite as

$$\Theta(\Phi)(\tilde{\kappa}_N) = \sum_{[Q] \in \mathcal{F}_1/\Gamma_1} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{\mathsf{t}_Q} \phi_\kappa(Q^{k_\kappa - 1}) q^{\delta_Q/(Np)}$$

By definition of the action of T_p on power series, we have

$$\Theta(\Phi)(\tilde{\kappa}_N)|T_p^{r_\kappa - 1} = \sum_{[Q] \in \mathcal{F}_1/\Gamma_1,\, p^{r_\kappa}|\delta_Q} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{\mathsf{t}_Q} \phi_\kappa(Q^{k_\kappa - 1}) q^{\delta_Q/(Np^{r_\kappa})}.$$

Setting $\mathcal{F}_{n,t}/\Gamma_n := \coprod_{\xi \geq 1} \mathcal{F}_{n,t,\xi}/\Gamma_n$ for $n \in \{1, r_\kappa\}$, Lemma 3.6 shows that

$$\mathcal{F}_{1,t}^* := \{[Q] \in \mathcal{F}_{1,t}/\Gamma_{1,t} \text{ such that } p^{r_\kappa}|\delta_Q\} \text{ is equal to } \mathcal{F}_{r_\kappa,t}.$$

Therefore, splitting the above sum over $t \in (\mathbb{Z}/Np^{r_\kappa}\mathbb{Z})^\times$, we get

$$\begin{aligned}
\Theta(\Phi)(\tilde{\kappa}_N)|T_p^{r_\kappa - 1} &= \sum_{t \in (\mathbb{Z}/p^{r_\kappa - 1}\mathbb{Z})^\times} \sum_{[Q] \in \mathcal{F}_{1,t}^*} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{\mathsf{t}_Q} \phi_\kappa(Q^{k_\kappa - 1}) q^{\delta_Q/(Np^{r_\kappa})} \\
&= \sum_{t \in (\mathbb{Z}/p^{r_\kappa - 1}\mathbb{Z})^\times} \sum_{[Q] \in \mathcal{F}_{m,t}/\Gamma_m} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{\mathsf{t}_Q} \phi_\kappa(Q^{k_\kappa - 1}) q^{\delta_Q/(Np^{r_\kappa})} \\
&= \sum_{[Q] \in \mathcal{F}_m/\Gamma_m} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{\mathsf{t}_Q} \phi_\kappa(Q^{k_\kappa - 1}) q^{\delta_Q/(Np^{r_\kappa})}.
\end{aligned}$$

Comparing this expression with (12) gives the result. □

We are now ready to state the analogue of [21, Thm. 3.3], which is our main result. For the reader's convenience, we briefly recall the notation appearing below. We denote by $\mathcal{X}$ the points of the ordinary Hida Hecke algebra, and by $\mathcal{X}^{\mathrm{arith}}$ its arithmetic points. For $\kappa_0 \in \mathcal{X}^{\mathrm{arith}}$, we denote by $\mathcal{U}_0$ the p-adic neighborhood of κ_0 appearing in the statement of Theorem 3.1 and put $\mathcal{U}_0^{\mathrm{arith}} := \mathcal{U}_0 \cap \mathcal{X}^{\mathrm{arith}}$. We also denote by $\Phi = \Phi^+ \in \mathbb{W}_{\mathcal{R}}^{\mathrm{ord}}$ the cohomology class appearing in Theorem 3.1. The points $\widetilde{\mathcal{X}}$ of the metaplectic Hida Hecke algebra defined in §3.3 are equipped with a canonical map $p : \widetilde{\mathcal{X}}^{\mathrm{arith}} \to \mathcal{X}^{\mathrm{arith}}$ on arithmetic points. Let $\widetilde{\mathcal{U}}_0^{\mathrm{arith}} := \widetilde{\mathcal{U}}_0 \cap \widetilde{\mathcal{X}}^{\mathrm{arith}}$. For each $\tilde\kappa \in \widetilde{\mathcal{U}}_0^{\mathrm{arith}}$, put $\kappa = p(\tilde\kappa) \in \mathcal{U}_0^{\mathrm{arith}}$. Recall that if $(\epsilon_{\tilde\kappa}, k_{\tilde\kappa})$ is the signature of $\tilde\kappa$, then the signature of κ is $(\epsilon_\kappa, k_\kappa) := (\epsilon_{\tilde\kappa}^2, 2k_{\tilde\kappa})$. For any $\kappa := p(\tilde\kappa)$ as above, we may consider the modular form

$$f_\kappa^{\mathrm{JL}} \in S_{k_\kappa}(\Gamma_{r_\kappa}, \epsilon_\kappa)$$

and its Shimura-Shintani-Waldspurger lift

$$h_\kappa = \sum_\xi a_\xi(h_\kappa) q^\xi \in S_{k_\kappa+1/2}(4Np^{r_\kappa}, \chi_\kappa), \quad \text{where } \chi_\kappa(x) := \epsilon_{\tilde\kappa}(x) \left(\frac{-1}{x}\right)^{k_\kappa},$$

normalized as in (2) and (3). Finally, for $\tilde\kappa \in \widetilde{\mathcal{X}}^{\mathrm{arith}}$, we denote by $\tilde\kappa_N$ its extension to the metaplectic Hecke algebra $\widetilde{\mathcal{R}}_N$ defined in §3.3.

Theorem 3.8. *Let $\kappa_0 \in \mathcal{X}^{\mathrm{arith}}$. Then there exists a choice of p-adic periods Ω_κ for $\kappa \in \mathcal{U}_0$ such that the Λ-adic Shimura-Shintani-Waldspurger lift of Φ*

$$\Theta(\Phi) := \sum_{\xi \geq 1} \theta_\xi(\Phi) q^\xi$$

in $\mathcal{R}_N[\![q]\!]$ has the following properties:

(1) $\Omega_{\kappa_0} \neq 0$.

(2) *For any $\tilde\kappa \in \widetilde{\mathcal{U}}_0^{\mathrm{arith}}$, the $\tilde\kappa$-specialization of $\Theta(\Phi)$*

$$\Theta(\nu)(\tilde\kappa_N) := \sum_{\xi \geq 1} \tilde\kappa\big(\theta_\xi(\Phi)\big) q^\xi \text{ belongs to } S_{k_\kappa+1/2}(4Np^{r_\kappa}, \chi'_\kappa),$$

where $\chi'_\kappa(x) := \chi_\kappa(x) \cdot \left(\frac{p}{x}\right)^{k_\kappa - 1}$, and satisfies

$$\Theta(\Phi)(\tilde\kappa_N) = \Omega_\kappa \cdot h_\kappa | T_p^{1-r_\kappa}.$$

Proof. The elements Ω_κ are those Ω_κ^+ appearing in Theorem 3.1, which we used in Propositions 3.4 and 3.7 above, so (1) is clear. Applying $T_p^{r_\kappa-1}$ to the formula of Proposition 3.7, using Corollary 3.5 and applying $a_p(\kappa)^{1-r_\kappa}$ on both sides gives

$$\Theta(\Phi)(\tilde\kappa_N) = a_p(\kappa)^{1-r_\kappa} \Omega_\kappa \cdot h_\kappa | T_p^{r_\kappa - 1}.$$

By [18, Prop. 1.9], each application of T_p has the effect of multiplying the character by $\left(\frac{p}{\cdot}\right)$, hence

$$h'_\kappa := h_\kappa | T_p^{r_\kappa - 1} \in S_{k_\kappa+1/2}(4Np^{r_\kappa}, \chi'_\kappa)$$

with χ'_κ as in the statement. This gives the first part of (2), while the last formula follows immediately from Proposition 3.7. $\square$

Remark 3.9. Theorem 1.1 is a direct consequence of Theorem 3.8, as we briefly show below.

Recall the embedding $\mathbb{Z}^{\geq 2} \hookrightarrow \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ which sends $k \in \mathbb{Z}^{\geq 2}$ to the character $x \mapsto x^{k-2}$. Extending characters by $\mathcal{O}$-linearity gives a map

$$\mathbb{Z}^{\geq 2} \hookrightarrow \mathcal{X}(\Lambda) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}\text{-alg}}(\Lambda, \bar{\mathbb{Q}}_p).$$

We denote by $k^{(\Lambda)}$ the image of $k \in \mathbb{Z}^{\geq 2}$ in $\mathcal{X}(\Lambda)$ via this embedding. We also denote by $\varpi : \mathcal{X} \to \mathcal{X}(\Lambda)$ the finite-to-one map obtained by restriction of homomorphisms to Λ. Let $k^{(\mathcal{R})}$ be a point in $\mathcal{X}$ of signature $(k, 1)$ such that $\varpi(k^{(\mathcal{R})}) = k^{(\Lambda)}$. A well-known result by Hida (see [6, Cor. 1.4]) shows that $\mathcal{R}/\Lambda$ is unramified at $k^{(\Lambda)}$. As shown in [21, §1], this implies that there is a section $s_{k^{(\Lambda)}}$ of ϖ which is defined on a neighborhood $\mathcal{U}_{k^{(\Lambda)}}$ of $k^{(\Lambda)}$ in $\mathcal{X}(\Lambda)$ and sends $k^{(\Lambda)}$ to $k^{(\mathcal{R})}$.

Fix now k_0 as in the statement of Theorem 1.1, corresponding to a cuspform f_0 of weight k_0 with trivial character. The form f_0 corresponds to an arithmetic character $k_0^{(\mathcal{R})}$ of signature $(1, k_0)$ belonging to $\mathcal{X}$. Let $\mathcal{U}'_0$ be the inverse image of $\mathcal{U}_0$ under the section $s^{-1}_{k_0^{(\Lambda)}}$, where $\mathcal{U}_0 \subseteq \mathcal{X}$ is the neighborhood of $k_0^{(\mathcal{R})}$ in Theorem 3.8. Extending scalars by $\mathcal{O}$ gives, as above, an injective continuos map $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \hookrightarrow \mathcal{X}(\Lambda)$, and we let U_0 be any neighborhood of the character $x \mapsto x^{k_0-2}$ which maps to $\mathcal{U}'_0$ and is contained in the residue class of k_0 modulo $p-1$. Composing this map with the section $\mathcal{U}'_0 \hookrightarrow \mathcal{U}_0$ gives a continuous injective map

$$\varsigma : U_0 \hookrightarrow \mathcal{U}'_0 \hookrightarrow \mathcal{U}_0$$

which takes k_0 to $k_0^{(\mathcal{R})}$, since by construction the image of k_0 by the first map is $k_0^{(\Lambda)}$. We also note that, more generally, $\varsigma(k) = k^{(\mathcal{R})}$ because by construction $\varsigma(k)$ restricts to $k^{(\Lambda)}$ and its signature is $(1, k)$, since the character of $\varsigma(k)$ is trivial. To show the last assertion, recall that the character of $\varsigma(k)$ is $\psi_k \cdot \psi_{\mathcal{R}} \cdot \omega^{-k}$, and note that ψ_k is trivial because $k^{(\Lambda)}(x) = x^{k-1}$, and $\psi_{\mathcal{R}} \cdot \omega^{-k}$ is trivial because the same is true for k_0 and $k \equiv k_0$ modulo $p-1$. In other words, arithmetic points in $\varsigma(U_0)$ correspond to cuspforms with trivial character. This is the Hida family of forms with trivial character that we considered in the Introduction.

We can now prove Theorem 1.1. For all $k \in U_0 \cap \mathbb{Z}^{\geq 2}$, put $\Omega_k := \Omega_{k^{(\mathcal{R})}}$ and define $\Theta := \Theta(\Phi) \circ \varsigma$ with Φ as in Theorem 3.8 for $\kappa_0 = k_0^{(\mathcal{R})}$. Applying Theorem 3.8 to $k_0^{(\mathcal{R})}$, and restricting to $\varsigma(U_0)$, shows that U_0, Ω_k and Θ satisfy the conclusion of Theorem 1.1.

Remark 3.10. For $\tilde{\kappa} \in \widetilde{\mathcal{U}}_0^{\mathrm{arith}}$ of signature $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$ with $r_{\tilde{\kappa}} = 1$ as in the above theorem, h_κ is trivial if $(-1)^{k_{\tilde{\kappa}}} \neq \epsilon_{\tilde{\kappa}}(-1)$. However, since $\phi_{\kappa_0} \neq 0$, it follows that h_{κ_0} is not trivial as long as the necessary condition $(-1)^{k_0} = \epsilon_0(-1)$ is verified.

Remark 3.11. This result can be used to produce a quaternionic Λ-adic version of the Saito-Kurokawa lifting, following closely the arguments in [8, Cor. 1].

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