Documenta Mathematica

Journal der
Deutschen Mathematiker-Vereinigung
Gegründet 1996

Band 17 · 2012

Documenta Mathematica, Journal der Deutschen Mathematiker-Vereinigung, veröffentlicht Forschungsarbeiten aus allen mathematischen Gebieten und wird in traditioneller Weise referiert. Es wird indiziert durch Mathematical Reviews, Science Citation Index Expanded, Zentralblatt für Mathematik.

Artikel können als TEX-Dateien per E-Mail bei einem der Herausgeber eingereicht werden. Hinweise für die Vorbereitung der Artikel können unter der unten angegebenen WWW-Adresse gefunden werden.

Documenta Mathematica, Journal der Deutschen Mathematiker-Vereinigung, publishes research manuscripts out of all mathematical fields and is refereed in the traditional manner. It is indexed in Mathematical Reviews, Science Citation Index Expanded, Zentralblatt für Mathematik.

Manuscripts should be submitted as TEX -files by e-mail to one of the editors. Hints for manuscript preparation can be found under the following web address.

`http://www.math.uni-bielefeld.de/documenta`

ISSN 1431-0635 (Print), ISSN 1431-0643 (Internet)

SPARC
Leading Edge

Documenta Mathematica is a Leading Edge Partner of SPARC, the Scholarly Publishing and Academic Resource Coalition of the Association of Research Libraries (ARL), Washington DC, USA.

Address of Technical Managing Editor: Ulf Rehmann, Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Typesetting in TEX.

Documenta Mathematica

Band 17, 2012

Degenerate Cohomological Hall Algebra and Quantized Donaldson-Thomas Invariants for m-Loop Quivers

Markus Reineke

Received: June 7, 2011

Communicated by Peter Schneider

Abstract. We derive a combinatorial formula for quantized Donaldson-Thomas invariants of the m-loop quiver. Our main tools are the combinatorics of noncommutative Hilbert schemes and a degenerate version of the Cohomological Hall algebra of this quiver.

2010 Mathematics Subject Classification: Primary 16G20, secondary 05E05, 14N35
Keywords and Phrases: Quantized Donaldson-Thomas invariant, Cohomological Hall algebra, Noncommutative Hilbert scheme

1 Introduction

Generalized Donaldson-Thomas invariants of (noncommutative) varieties arise from factorizations of generating series of motivic invariants of Hilbert scheme-type varieties into Euler products. For 3-Calabi-Yau manifolds, this principle is developed extensively in [10].

In [13], the author showed that the wall-crossing formulae of [10] can be modelled using Hilbert schemes of path algebras of quivers; explicit calculations for these varieties in [14] allowed to derive relative integrality (that is, preservation of integrality under wall-crossing) of generalized Donaldson-Thomas invariants.

In [9] a general framework for the study of such integrality properties is proposed, the central tools being Cohomological Hall algebras and the geometric concept of factorization systems.

The purpose of the present paper is to develop an explicit, in most parts purely combinatorial, setup for the study of the quantized Donaldson-Thomas invariants of [9] in the very special, but typical, case of the m-loop quiver. The relevant concepts of [9] are discussed in sections 3, 4. Our approach is based on the explicit description of Hilbert schemes attached to this quiver of [12], which is reviewed in Section 2. It allows us to give a combinatorial description of a degenerate version of the Cohomological Hall algebra, whose structure is easily described (see Section 5). Using number-theoretic arguments similar to [14], we obtain explicit formulas for these quantized Donaldson-Thomas invariants (see Theorem 6.8) in terms of cyclic classes of certain integer sequences in Section 6. We also relate this combinatorics to a similar one appearing in the study of Higgs moduli in [7], thereby proving a conjecture of [6] on the Hitchin nullcone; see Section 7.

Roughly speaking, our approach uses (the combinatorics of) noncommutative Hilbert schemes as a transitional tool between the geometric problem of determination of Donaldson-Thomas invariants and the combinatorial object of cyclic configurations. However, the present approach is not strong enough to yield the positivity properties conjectured in [9]. A proof of these is announced in [3]. The quantized Donaldson-Thomas invariants considered here also appear in [16].

Acknowledgments: The author would like to thank S. Mozgovoy for several discussions concerning this work, and in particular for pointing out the potential relation to Higgs moduli. After this work was first submitted to the arXiv, the author learned about the work of A. Efimov [3] and of F. Rodriguez-Villegas [16]. This work was started while the author participated in the workshop "Wall-crossing in Mathematics and Physics" at Urbana-Champaign, and finished during stays at the Issac Newton Institute Cambridge and the Hausdorff Institute Bonn. The author would like to thank the organizers and participants of these programmes for the inspiring atmosphere.

2 Noncommutative Hilbert schemes

In this section, we recall the definition of noncommutative Hilbert schemes and their main properties following [12]. We also relate the relevant combinatorics of trees to a combinatorics of partitions which will play a major role in the following.

Fix an integer $m \geq 1$. For $n \geq 0$, we call a pair consisting of a tuple $(\varphi_1, \ldots, \varphi_m)$ of linear operators on $\mathbf{C}^n$ and a vector $v \in \mathbf{C}^n$ *stable* if v is cyclic for the representation of the free algebra $F^{(m)} = \mathbf{C}\langle x_1, \ldots, x_m\rangle$ on $\mathbf{C}^n$ defined by the operators φ_i, that is, if $\mathbf{C}\langle\varphi_1, \ldots, \varphi_m\rangle v = \mathbf{C}^n$. This defines an open subset of the affine space $\mathrm{End}(\mathbf{C}^n)^m \oplus \mathbf{C}^n$, for which a geometric quotient by the action of $\mathrm{GL}_n(\mathbf{C})$ via $g(\varphi_1, \ldots, \varphi_m, v) = (g\varphi_1 g^{-1}, \ldots, g\varphi_m g^{-1}, gv)$ exists. This quotient is denoted by $\mathrm{Hilb}_n^{(m)}$ and is called a *noncommutative Hilbert* *scheme* for $F^{(m)}$: in analogy with the Hilbert scheme of n points of an affine variety X parametrizing codimension n ideals in the coordinate ring of X, the variety $\mathrm{Hilb}_n^{(m)}$ parametrizes left ideals I in $F^{(m)}$ of codimension n, that is, ideals such that $\dim_{\mathbf{C}} F^{(m)}/I = n$. Namely, to a tuple as above we associate the left ideal of polynomials $P \in F^{(m)}$ such that $P(\varphi_1, \ldots, \varphi_m)v = 0$. Conversely, given a left ideal $I \subset F^{(m)}$, we choose an isomorphism between $F^{(m)}/I$ and $\mathbf{C}^n$. The operators φ_i are induced by left multiplication by x_i on $F^{(m)}/I$ via this isomorphism, whereas v is induced by the coset of the unit $1 \in F^{(m)}$. This tuple is stable by definition, and well defined up to the choice of the isomorphism $F^{(m)}/I \simeq \mathbf{C}^n$, that is, up to the $\mathrm{GL}_n(\mathbf{C})$-action.

Consider the set $\Omega^{(m)}$ of words $\omega = i_1 \ldots i_k$ in the alphabet $\{1, ...m\}$. Composition of words defines a monoid structure on $\Omega^{(m)}$; define ω' to be a *left subword* of ω if $\omega = \omega'\omega''$ for a word ω''. The set $\Omega^{(m)}$ carries a lexicographic ordering $\leq_{\mathrm{lex}}$ induced by the canonical total ordering on the alphabet $\{1, \ldots, m\}$. An *m-ary tree* is a finite subset $T \subset \Omega^{(m)}$ which is closed under left subwords. This terminology is explained as follows: a subset T is visualized as the tree with nodes ω for $\omega \in T$ and an edge of colour i from ω to ωi if $\omega, \omega i \in T$; the empty word corresponds to the root of the tree.

For a tree T, define its *corona* $C(T)$ as the set of all $\omega \in \Omega^{(m)}$ such that $\omega \notin T$, but $\omega' \in T$ for $\omega = \omega' i$ for some i. We have $|C(T)| = (m-1)|T| + 1$.

Given a word $\omega = i_1 \ldots i_k$ and a tuple of operators $(\varphi_1, \ldots, \varphi_m)$ as above, we define $\varphi_\omega = \varphi_{i_k} \circ \ldots \circ \varphi_{i_1}$. For a tree T of cardinality n, define $Z_T \subset \mathrm{Hilb}_n^{(m)}$ as the set of classes of tuples $(\varphi_1, \ldots, \varphi_m, v)$ such that:

1. the elements $\varphi_\omega v$ for $\omega \in T$ form a basis of $\mathbf{C}^n$,
2. if $\omega \in C(T)$, then $\varphi_\omega v = \sum_{\omega'} \lambda_{\omega,\omega'} \varphi_{\omega'} v$, where the sum ranges over all words $\omega' \in T$ such that $\omega' <_{\mathrm{lex}} \omega$.

Denote by $d(T)$ the number of pairs (ω, ω') such that $\omega \in C(T)$, $\omega' \in T$ and $\omega' <_{\mathrm{lex}} \omega$.

Theorem 2.1 *[12, Theorem 1.3] The following holds:*

1. *Z_T is a locally closed subset of $\mathrm{Hilb}_n^{(m)}$, which is isomorphic to an affine space of dimension $d(T)$.*
2. *The subsets Z_T, for T ranging over all trees of cardinality n, define a cell decomposition of $\mathrm{Hilb}_n^{(m)}$, that is, there exists a decreasing filtration of $\mathrm{Hilb}_n^{(m)}$ by closed subvarieties, such that the successive complements are the subsets Z_T.*

As a corollary to this geometric description, we can derive precise information on the cohomology (singular cohomology with rational coefficients) of $\mathrm{Hilb}_n^{(m)}$. The existence of a cell decomposition implies vanishing of odd cohomology (and algebraicity of even cohomology), thus we can consider the following generating series of Poincaré polynomials

$$F(q,t)=\sum_{n\geq 0}q^{(m-1)\binom{n}{2}}\sum_k \dim H^k(\mathrm{Hilb}_n^{(m)})q^{-k/2}t^n\in\mathbf{Z}[q,q^{-1}][[t]],$$

as well as its specialization

$$F(t)=F(1,t)=\sum_{n\geq 0}\chi(\mathrm{Hilb}_n^{(m)})t^n\in\mathbf{Z}[[t]].$$

We also define

$$H(q,t)=\sum_{n\geq 0}\frac{q^{(m-1)\binom{n}{2}}}{(1-q^{-1})\cdot\ldots\cdot(1-q^{-n})}t^n\in\mathbf{Q}(q)[[t]],$$

which is a q-hypergeometric series whose major role for the following will be explained in the next section.

COROLLARY 2.2 *We have the following explicit descriptions of the series $F(q,t)$ and $F(t)$:*

1. *The series $F(q,t)$ is uniquely determined as the solution in $\mathbf{Q}(q)[[t]]$ of the algebraic q-difference equation*

$$F(q,t)=1+t\prod_{k=0}^{m-1}F(q,q^kt).$$

2. *The series $F(t)$ is uniquely determined as the solution in $\mathbf{Q}[[t]]$ of the algebraic equation*

$$F(t)=1+tF(t)^m.$$

3. *The Euler characteristic of $\mathrm{Hilb}_n^{(m)}$ equals the number of m-ary trees with n nodes, which is $\frac{1}{(m-1)n+1}\binom{mn}{n}$.*

4. *We have $F(q,t)=\frac{H(q,t)}{H(q,q^{-1}t)}$.*

PROOF: In the notation of [12], the series $F(q,t)$ equals the series $\overline{\zeta}_1^{(m)}(q,t)$ of [12, Section 5] by [12, Corollary 4.4.]. The first statement translates the operation of grafting of trees; see [12, Theorem 5.5.]. Specialization of the functional equation to $q=1$ yields the second statement. The third statement follows from an explicit formula for the number of m-ary trees; see [12, Corollary 4.5.]. The fourth statement is a special case of [4, Theorem 5.2.]; in the present case, it is easily derived from the identity

$$H(q,t)=H(q,q^{-1}t)+tH(q,q^{m-1}t)$$

(which follows by a direct calculation from the definition of $H(q,t)$), together with the first statement.

$\square$

Remark: These explicit descriptions of the series $F(q,t)$ are also derived in [1, 2]; these references are already used in [12] to derive asymptotic properties of the cohomology of $\mathrm{Hilb}_n^{(m)}$.

Denote by T_n the set of partitions (see the beginning of Section 5 for a discussion of this non-standard definition of partitions) $\lambda = (0 \le \lambda_1 \le \ldots \le \lambda_n)$ such that $\lambda_i \le (m-1)(i-1)$ for all $i = 1, \ldots, n$. Define the *weight* of $\lambda \in T_n$ as $\mathrm{wt}(\lambda) = (m-1)\binom{n}{2} - |\lambda|$. We also define a weight function $\mathrm{wt}(T)$ on trees T as above by

$$\mathrm{wt}(T) = (m-1)\binom{|T|}{2} - |\{(\omega', \omega) \in C(T) \times T \,:\, \omega' <_{\mathrm{lex}} \omega\}|,$$

thus $\mathrm{wt}(T) = d(T) - (m-1)\binom{n+1}{2} - n$ by definition of $d(T)$.

Given an m-ary tree $T \subset \Omega^{(m)}$ with n vertices as above, write $T = \{\omega_1, \ldots, \omega_n\}$ with $\omega_1 <_{\mathrm{lex}} \ldots <_{\mathrm{lex}} \omega_n$. We define a partition $\lambda(T)$ by

$$\lambda(T)_i = |\{\omega \in C(T) \,:\, \omega <_{\mathrm{lex}} \omega_i\}|.$$

Proposition 2.3 *The map associating $\lambda(T)$ to T defines a weight-preserving bijection between m-ary trees with n nodes and T_n.*

Proof: To prove that $\lambda(T)$ belongs to T_n, we observe that an element $\omega \in C(T)$ such that $\omega <_{\mathrm{lex}} \omega_k$ belongs to $C(T_k) \setminus \{\omega_k\}$ for the subtree $T_k = \{\omega_1, \ldots, \omega_{k-1}\}$ of T; this is a set of cardinality $(m-1)(k-1)$. We reconstruct the tree from the partition $\lambda \in T_n$ inductively as follows: we start with the empty tree T_0. In the k-th step, we list the elements of the corona of T_{k-1} in ascending lexicographic order as $C(T_{k-1}) = \{\omega_i^k, \ldots, \omega_{(m-1)(k-1)-1}^k\}$ and define $T_k = T_{k-1} \cup \{\omega_{\lambda_k+1}^k\}$. We then have $\{\omega \in C(T),\ \omega <_{\mathrm{lex}} \omega_k\} = \{\omega_1^k, \ldots, \omega_{\lambda_k}^k\}$, proving that T is reconstructed from $\lambda(T)$. The equality of the weights of T and $\lambda(T)$ follows from the definitions.

$\square$

3 Donaldson-Thomas type invariants

The following definition of *Donaldson-Thomas type invariants* for the m-loop quiver (recall that $m \ge 1$) is motivated by [10].

Definition 3.1 *Define $\mathrm{DT}_n^{(m)} \in \mathbf{Q}$ for $n \ge 1$ by writing*

$$F((-1)^{m-1}t) = \prod_{n \ge 1} (1 - t^n)^{-(-1)^{(m-1)n} n \mathrm{DT}_n^{(m)}}.$$

These numbers are well-defined since $F(t)$ is an integral power series with constant term 1. A priori, we have $n\mathrm{DT}_n^{(m)} \in \mathbf{Z}$.

Theorem 3.2 *[14] We have $\mathrm{DT}_n^{(m)} \in \mathbf{N}$; explicitly, these numbers are given by the following formula:*

$$\mathrm{DT}_n^{(m)} = \frac{1}{n^2} \sum_{d|n} \mu(\frac{n}{d})(-1)^{(m-1)(n-d)} \binom{mn-1}{n-1}.$$

We make this formula more explicit by giving some examples, in which we observe that $\mathrm{DT}_n^{(m)}$ is a polynomial in m except if $n \equiv 2 \bmod 4$ (this phenomenon will become more transparent in the following sections).

$$\mathrm{DT}_1^{(m)} = 1, \quad \mathrm{DT}_2^{(m)} = \left\lfloor \frac{m}{2} \right\rfloor, \quad \mathrm{DT}_3^{(m)} = \frac{m(m-1)}{2},$$

$$\mathrm{DT}_4^{(m)} = \frac{m(m-1)(2m-1)}{3}, \quad \mathrm{DT}_5^{(m)} = \frac{5m(m-1)(5m^2-5m+2)}{24},$$

$$\mathrm{DT}_6^{(m)} = \frac{m(m-1)(36m^3-54m^2+31m-\frac{13+(-1)^{m-1}5}{2})}{20},$$

$$\mathrm{DT}_7^{(m)} = \frac{7m(m-1)(343m^4-686m^3+539m^2-196m+36)}{720}.$$

Remark: In general, $\mathrm{DT}_n^{(m)}$ has leading term $\frac{n^{n-2}}{n!}m^{n-1}$ considered as a function of m. It would be interesting to give a graph-theoretic explanation of this, in the spirit of the graph-theoretic explanation for the leading term of the polynomial counting isomorphism classes of absolutely indecomposable representations of dimension n of the m-loop quiver in [8].

In [9], a conjecture is formulated which implies the above theorem; we formulate a slight variant of this conjecture.

Conjecture 3.3 *[9, Section 2.6] There exists a product expansion*

$$H(q,(-1)^{m-1}t) = \prod_{n\geq 1} \prod_{k\geq 0} \prod_{l\geq 0} (1-q^{k-l}t^n)^{-(-1)^{(m-1)n}c_{n,k}}$$

for nonnegative integers $c_{n,k}$, such that only finitely many $c_{n,k}$ are nonzero for any fixed n.

Assuming this conjecture, we have

$$F(q,(-1)^{m-1}t) = \prod_{n\geq 1} \prod_{k\geq 0} \prod_{l=0}^{n-1} (1-q^{k-l}t^n)^{-(-1)^{(m-1)n}c_{n,k}}$$

and thus

$$F((-1)^{m-1}t) = \prod_{n\geq 1} (1-t^n)^{-(-1)^{(m-1)n} n \sum_k c_{n,k}}.$$

Thus, setting $\mathrm{DT_n}^{(m)}(q)=\sum_{k\geq 0}c_{n,k}q^k$, the conjecture implies that $\mathrm{DT}_n^{(m)}(q)$ is a polynomial with nonnegative coefficients, such that $\mathrm{DT}_n^{(m)}(1)=\mathrm{DT}_n^{(m)}$.

In the following, we will use a simplified notation for product expansions as in the conjecture, using the λ-ring exponential Exp, see Section 8. Using Lemma 8.3, the product of the conjecture can be rewritten as

$$\mathrm{Exp}(\frac{1}{1-q^{-1}}\sum_{n\geq 1}\mathrm{DT}_n^{(m)}(q)((-1)^{m-1}t)).$$

4 The Cohomological Hall algebra

In this section, we review the definition and the main properties of the Cohomological Hall algebra of [9] for the m-loop quiver. In particular, we formulate the main conjecture of [9] on these algebras and relate it to the conjecture of the previous section.

For a vector space V, we denote by $E_V=\mathrm{End}(V)^m$ the space of m-tuples of endomorphisms of V. The group $G_V=\mathrm{GL}(V)$ acts on E_V by simultaneous conjugation. For complex vector spaces V and W of dimension n_1 and n_2, respectively, we consider the subspace $E_{V,W}$ of $E_{V\oplus W}$ of m-tuples of endomorphisms $(\varphi_1,\ldots,\varphi_m)$ respecting the subspace V of $V\oplus W$, that is, such that $\varphi_i(V)\subset V$ for all $i=1,\ldots,m$. We have an obvious projection map $p:E_{V,W}\to E_V\times E_W$ mapping $(\varphi_1,\ldots,\varphi_m)$ to $((\varphi_1|_V,\ldots,\varphi_m|_V),(\overline{\varphi_1},\ldots,\overline{\varphi_m}))$, where $\overline{\varphi_i}$ denotes the endomorphism of W induced by φ_i. The action of $G_{V\oplus W}$ on $E_{V\oplus W}$ restricts to an action of the parabolic subgroup $P_{V,W}$ of $G_{V\oplus W}$, consisting of automorphisms respecting the subspace V, on $E_{V,W}$. The projection p is equivariant, if the action of $P_{V,W}$ on $E_V\times E_W$ is defined through the Levi subgroup $G_V\times G_W$ of $P_{V,W}$. Moreover, the closed embedding of $E_{V,W}$ into $E_{V\oplus W}$ is $P_{V,W}$-equivariant.

Using these maps $E_V\times E_W\leftarrow E_{V,W}\to E_{V\oplus W}$ and their $P_{V,W}$-equivariance, we can define the following map in equivariant cohomology with rational coefficients:

$$\begin{array}{rcl} H^*_{G_V}(E_V)\otimes H^*_{G_W}(E_W) & \simeq & H^*_{G_V\times G_W}(E_V\times E_W)\\ & \simeq & H^*_{P_{V,W}}(E_{V,W})\\ & \to & H^{*+2s_1}_{P_{V,W}}(E_{V\oplus W})\\ & \to & H^{*+2s_1+2s_2}_{G_{V\oplus W}}(E_{V\oplus W}),\end{array}$$

where the shifts in cohomological degree are $s_1=\dim E_{V\oplus W}-\dim E_{V,W}=m\dim V\dim W$ and $s_2=-\dim G_{V\oplus W}/P_{V,W}=-\dim V\dim W$ (see [9, Section 2.2.] for the details). Then the following holds:

Theorem 4.1 *[9, Theorem 1] The above maps induce an associative unital*

$\mathbf{Q}$-algebra structure on $\mathcal{H} = \bigoplus_{n\geq 0} H^*_{G_{\mathbf{C}^n}}(E_{\mathbf{C}^n})$, which is $\mathbf{N}\times\mathbf{Z}$-bigraded if $H^k_{G_{\mathbf{C}^n}}(E_{\mathbf{C}^n})$ is placed in bidegree $(n,(m-1)\binom{n}{2}-k/2)$.

The algebra $\mathcal{H}$ is called the *Cohomological Hall algebra* of the m-loop quiver in [9]. Since all spaces E_V are contractible, the vector space underlying $\mathcal{H}$ is independent of m, whereas the algebra structure depends on m.

The above bigrading differs slightly from the one in [9]; it is more suited to our purposes of studying the series $H(q,t)$ in relation to the generating series $F(q,t)$ of Poincaré polynomials of $\mathrm{Hilb}_n^{(m)}$.

We consider the Poincaré-Hilbert series of $\mathcal{H}$:

$$P_{\mathcal{H}}(q,t) = \sum_{n\geq 0}\sum_{k\in\mathbf{Z}} \dim_{\mathbf{Q}} \mathcal{H}_{n,k} q^k t^n.$$

Lemma 4.2 *The series $P_{\mathcal{H}}(q,t)$ equals $H(q,t)$.*

Proof: The homogeneous component of $\mathcal{H}$ with respect to the first component of the bidegree equals $H^*_{G_{\mathbf{C}^n}}(E_{\mathbf{C}^n}) \simeq H^*_{G_{\mathbf{C}^n}}(\mathrm{pt})$, which is isomorphic to a polynomial ring in n generators placed in bidegree $(n,(m-1)\binom{n}{2}-i)$ for $i=1,\ldots,n$. Thus, this component has Poincaré-Hilbert series

$$\frac{q^{(m-1)\binom{n}{2}}t^n}{(1-q^{-1})\cdot\ldots\cdot(1-q^{-n})}.$$

□

Using torus fixed point localization, one obtains the following algebraic description of $\mathcal{H}$:

Theorem 4.3 *[9, Theorem 2] The algebra $\mathcal{H}$ is isomorphic to the following shuffle-type algebra structure on $\bigoplus_{n\geq 0}\mathbf{Q}[x_1,\ldots,x_n]^{S_n}$, the space of symmetric polynomials in all possible numbers of variables:*

$$(f_1 * f_2)(x_1,\ldots,x_{n_1+n_2}) =$$

$$\sum f_1(x_{i_1},\ldots,x_{i_{n_1}})f_2(x_{j_1},\ldots,x_{j_{n_2}})(\prod_{k=1}^{n_1}\prod_{l=1}^{n_2}(x_{j_l}-x_{i_k}))^{m-1},$$

the sum ranging over all shuffles $\{i_1<\ldots<i_{n_1}\}\cup\{j_1<\ldots<j_{n_2}\} = \{1,\ldots,n_1+n_2\}$. A homogeneous symmetric function of degree k in n variables is placed in bidegree $(n,(m-1)\binom{n}{2}-k)$.

From this description we see that $\mathcal{H}$ is commutative in case m is odd, and supercommutative in case m is even.

Conjecture 4.4 *[9, Conjecture 1] The bigraded algebra $\mathcal{H}$ is isomorphic to* $\mathrm{Sym}(C \otimes \mathbf{Q}[z])$*, the (graded) symmetric algebra over a bigraded super vector space. For any fixed $n \geq 1$, only finitely many homogeneous components $C_{n,k}$ are nonvanishing and $k \geq 0$ in this case, and z is a homogeneous element of bidegree* $(0, -1)$.

This conjecture immediately implies Conjecture 3.3 for $c_{n,k} = \dim_{\mathbf{Q}} C_{n,k}$, since the Poincaré-Hilbert series of a symmetric algebra has a natural product expansion, namely $P_{\mathrm{Sym}(V)} = \mathrm{Exp}(P_V)$. A proof of Conjecture 4.4 is announced in [3].

5 The degenerate Cohomological Hall Algebra

We introduce a degenerate form of the Cohomological Hall algebra $\mathcal{H}$ and show that it is of purely combinatorial nature. We analyze its structure using the combinatorics of partitions in the set T_n introduced in Section 2.

In the following, we will adopt a non-standard notation for partitions: a partition of length n is a non-decreasing sequence $\lambda = (0 \leq \lambda_1 \leq \ldots \leq \lambda_n)$ of (not necessarily non-zero) integers. Denote by Λ_n the set of partitions of length $l(\lambda) = n$, and denote the disjoint union of all Λ_n (for $n \geq 0$) by Λ. For $N \in \mathbf{N}$, define $S^N\lambda = (\lambda_1+N, \ldots, \lambda_n+N)$. Define the union $\mu \cup \nu$ of partitions $\mu, \nu \in \Lambda$ as the partition with parts $\mu_1, \ldots, \mu_{l(\mu)}, \nu_1, \ldots, \nu_{l(\nu)}$, resorted in ascending order.

Generalizing the definition in Section 2, the weight of a partition λ is defined as $\mathrm{wt}(\lambda) = (m-1)\binom{n}{2} - |\lambda|$, where $|\lambda| = \lambda_1 + \ldots + \lambda_n$. Comparison with the argument in the proof of Lemma 4.2 immediately shows:

Lemma 5.1 *The generating function $\sum_{\lambda \in \Lambda} q^{\mathrm{wt}(\lambda} t^{l(\lambda)}$ of Λ by weight and length equals $H(q, t)$.*

Definition 5.2 *Define an algebra structure $*$ on the vector space A with basis elements $\lambda \in \Lambda$ by*

$$\mu * \nu = \mu \cup S^{(m-1)l(\mu)}\nu$$

for $\mu, \nu \in \Lambda$.

This multiplication is obviously associative, but non-commutative unless $m = 1$. It is easy to verify that this algebra is bigraded by weight and length of partitions, and thus has $H(q, t)$ as its Poincaré series.

The explicit description of the Cohomological Hall algebra in Theorem 4.3 allows us to define the following (naive) quantization.

Definition 5.3 *Define the quantized Cohomological Hall algebra $\mathcal{H}_q$ as the bigraded $\mathbf{Q}[q]$-module $\bigoplus_{n \geq 0} \mathbf{Q}[q][x_1, \ldots, x_n]^{S_n}$ with the product*

$$(f_1 * f_2)(x_1, \ldots, x_{n_1+n_2}) =$$

$$\sum f_1(x_{i_1},\ldots,x_{i_{n_1}})f_2(x_{j_1},\ldots,x_{j_{n_2}})(\prod_{k=1}^{n_1}\prod_{l=1}^{n_2}(x_{j_l}-qx_{i_k}))^{m-1}.$$

REMARK: It would be interesting to realize this algebra geometrically, as the convolution algebra in some appropriate cohomology theory on the G_V-spaces E_V of the previous section.

We can specialize the algebra $\mathcal{H}_q$ to any $q\in\mathbf{Q}$, in particular to $q=0$, yielding an algebra $\mathcal{H}_0$.

PROPOSITION 5.4 *We have an isomorphism of bigraded algebras $A\simeq\mathcal{H}_0$ by mapping a partition λ to the symmetric polynomial*

$$P_\lambda(x_1,\ldots,x_n)=\sum_{\sigma\in S_n}x_{\sigma(1)}^{\lambda_1}\cdot\ldots\cdot x_{\sigma(n)}^{\lambda_n}.$$

PROOF: The polynomial P_λ is a suitable multiple of the monomial symmetric polynomial $m_\lambda(x_1,\ldots,x_n)$. The multiplication in $\mathcal{H}_0$ reduces to

$$(f_1*f_2)(x_1,\ldots,x_{n_1+n_2})=$$

$$\sum f_1(x_{i_1},\ldots,x_{i_{n_1}})f_2(x_{j_1},\ldots,x_{j_{n_2}})(\prod_{l=1}^{n_2}x_{j_l})^{(m-1)n_1}.$$

Identification of shuffles with cosets $S_{n_1+n_2}/(S_{n_1}\times S_{n_2})$ immediately shows that $P_\lambda*P_\mu=P_{\lambda*\mu}$.

□

Recall from section 2 the subset $T_n\subset\Lambda_n$ of partitions $\lambda\in\Lambda_n$ such that $\lambda_i\leq(m-1)(i-1)$ for all $i=1,\ldots,n$, and define T as the disjoint union of all T_n.

LEMMA 5.5 *The subspace B of A generated by the basis elements indexed by T is stable under the multiplication $*$, thus B is a subalgebra of A. In computing a product $\lambda*\mu$ for $\lambda,\mu\in T$, it suffices to append $S^{(m-1)l(\lambda)}\mu$ to λ (without resorting parts).*

PROOF: Using the definition of T and of $*$, this is immediately verified.

□

Denote by S the linear operator on A induced by the operation S on partitions.

LEMMA 5.6 *Multiplication induces an isomorphism of bigraded vector spaces $B\otimes SA\simeq A$.*

PROOF: On the level of partitions, this reduces to the statement that multiplication induces a bijection between $\bigcup_{k+l=n}T_k\times S(\Lambda_l)$ and Λ_n preserving weights. Suppose λ is given. If $\lambda\in T_n$, we map λ to $(\lambda,())\in T_n\times\Lambda_0$. Otherwise, let i be maximal such that $\lambda_i\leq(m-1)(i-1)$ (thus $i<n$). We

define $\mu = (\lambda_1, \ldots, \lambda_i)$. We have $\lambda_j > (m-1)(j-1)$ for all $j > i$, thus $(\lambda_{i+1}, \ldots, \lambda_n) = S^{(m-1)i+1}\nu$ for the partition ν of length $n-i$ with parts $\nu_k = \lambda_{i+k} - (m-1)i - 1 \geq 0$. Then λ is mapped to (μ, ν). By a simple calculation, compatibility of this bijection with the weight is verified.

□

We can iterate this lemma to get (the infinite tensor product meaning finite combinations of finite pure tensors):

Corollary 5.7 *Multiplication induces an isomorphism*

$$\bigotimes_{i\geq 0} S^i B = B \otimes SB \otimes S^2 B \otimes \ldots \simeq A.$$

Proof: Iteration of the previous lemma shows that any λ admits a finite decomposition $\lambda = \lambda^1 * \ldots * \lambda^s$ such that $\lambda^k \in S^k B$ for degree reasons.

□

Next, we analyze the structure of the algebra B. Denote by $T_n^0 \subset T_n$ the subset of all $\lambda \in T_n$ such that $\lambda_i < (m-1)(i-1)$ for $i = 2, \ldots, n$, by T^0 the disjoint union of all T_n^0, and by B^0 the subspace of B linearly generated by T^0.

Lemma 5.8 *B is isomorphic to the tensor algebra $T(B^0)$.*

Proof: In a product $\lambda = \lambda^1 * \ldots * \lambda^k$ of partitions $\lambda^i \in T^0$, the set of indices $l = 2, \ldots, n$ such that $\lambda_l = (m-1)(l-1)$ is precisely the set $\{l(\lambda^1)+1, l(\lambda^1) + l(\lambda^2)+1, \ldots, l(\lambda^1)+\ldots+l(\lambda^{k-1})+1\}$. This observation shows that any $\lambda \in T$ admits a unique such decomposition.

□

We define a total ordering on T^0 by the lexicographic ordering, viewing partitions as words in the alphabet $\mathbf{N}$. This induces a total ordering, the lexicographic ordering in the alphabet T^0, on words in T^0. Call a word in the alphabet T^0 *Lyndon* if it is strictly bigger than all of its proper cyclic shifts. Denote by T^L the set of all $\lambda^1 * \ldots * \lambda^k$ for $\lambda^1 \ldots \lambda^k$ a Lyndon word in T^0, thus T^L is the union of all $T_n^L = T^L \cap T_n$, and by B^L the subspace of B generated by T^L.

Lemma 5.9 *Multiplication induces an isomorphism of bigraded vector spaces* $\mathrm{Sym}(B^L) \simeq B$.

Proof: By the previous lemma, we have $B \simeq T(B^0)$, thus $B \simeq \mathrm{Sym}(L(B^0))$ as vector spaces by Poincaré-Birkhoff-Witt, where $L(B^0)$ is the free Lie algebra in B^0 (since the free algebra of a vector space is the enveloping algebra of its free Lie algebra). By general results on free Lie algebras [17], the Lyndon words parametrize a basis of the free Lie algebra, since every word can be written uniquely as a product of Lyndon words, weakly increasing with respect to lexicographic ordering on words. This construction provides an isomorphism between B and $\mathrm{Sym}(B^L)$. The latter inherits its bigrading from B^L, and the construction obviously respects the bigradings.

□

Combining the above lemmas, we arrive at the following description of the algebra A:

THEOREM 5.10 *We have an isomorphism of bigraded vector spaces*

$$A \simeq \mathrm{Sym}(\bigoplus_{i\geq 0} S^i B^L).$$

PROOF: The result follows from the following chain of isomorphisms:

$$A \simeq \bigotimes_{i\geq 0} S^i B \simeq \bigotimes_{i\geq 0} S^i \mathrm{Sym}(B^L) \simeq \bigotimes_{i\geq 0} \mathrm{Sym}(S^i B^L) \simeq \mathrm{Sym}(\bigoplus_{i\geq 0} S^i B^L).$$

□

REMARK: This result is not a direct analogue of Conjecture 4.4 for the algebra $A \simeq \mathcal{H}_0$, since the operator S induces a shift of $(0, -n)$ in bidegree on a homogeneous component $B^L_{(n,k)}$ of B^L.

Comparing Poincaré-Hilbert series of both sides in the formula of Theorem 5.10, we get:

COROLLARY 5.11 *We have the following product expansion:*

$$H(q,t) = \mathrm{Exp}(\sum_{n\geq 1} \frac{1}{1-q^{-n}} \sum_{\lambda\in T^L_n} q^{\mathrm{wt}(\lambda)} t^n).$$

For application to (quantized) Donaldson-Thomas invariants, we have to describe $H(q, (-1)^{m-1}t)$, thus it is necessary to derive a signed analogue of the previous corollary. Define $T^{L,+}$ as T^L if m is odd, and as

$$T^{L,+} = T^L \cup \{\lambda * \lambda \,|\, \lambda \in T^L,\, l(\lambda) \text{ odd}\}$$

if m is even. Define $T^{L,+}_n = T^{L,+} \cap T_n$.

THEOREM 5.12 *We have a product expansion*

$$H(q, (-1)^{m-1}t) = \mathrm{Exp}(\sum_{n\geq 1} \frac{1}{1-q^{-n}} \sum_{\lambda\in T^{L,+}_n} q^{\mathrm{wt}(\lambda)} ((-1)^{m-1}t)^n).$$

PROOF: If m is odd, there is nothing to prove, so suppose that m is even. From the identity

$$(1 + q^a t^b)^{-1} = \mathrm{Exp}(q^{2a} t^{2b} - q^a t^b)$$

it follows that $H(q, (-1)^{m-1}t)$ equals

$$\mathrm{Exp}(\sum_{n\geq 1} \frac{1}{1-q^{-n}} \sum_{\lambda\in T^L_n} q^{\mathrm{wt}(\lambda)} ((-1)^{m-1}t)^n + \sum_{\substack{n\geq 1\\ \text{odd}}} \frac{1}{1-q^{-2n}} \sum_{\lambda\in T^L_n} q^{2\mathrm{wt}(\lambda)} t^{2n}).$$

Now it remains to recall that length and weight double when passing from λ to $\lambda * \lambda$, and the claim follows.

□

Arguing as in Section 3, this implies the following combinatorial interpretation of Donaldson-Thomas invariants.

Corollary 5.13 *We have* $\mathrm{DT}_n^{(m)} = \frac{1}{n}|T_n^{L,+}|$.

Define the polynomials $\overline{Q}_n(q) = \sum_{\lambda \in T_n^L} q^{\mathrm{wt}(\lambda)}$ and $Q_n(q) = \sum_{\lambda \in T_n^{L,+}} q^{\mathrm{wt}(\lambda)}$; we thus have $Q_n(q) = \overline{Q}_n(q)$ except in case m is even and $n = 2\overline{n}$ for odd $\overline{n}$, where $Q_n(q) = \overline{Q}_n(q) + \overline{Q}_{\overline{n}}(q^2)$. We can reformulate Theorem 5.12 as

$$H(q, (-1)^{m-1}t) = \mathrm{Exp}(\sum_{n \geq 1} \frac{1}{1-q^{-n}} Q_n(q)((-1)^{m-1}t)^n).$$

Example: To illustrate the classes of partitions $T^0 \subset T^L \subset T \subset \Lambda$ defined above, we consider the case $m = 2$, $n = 4$. The set T_4 consists of the 14 partitions

$$\underline{(0000)}, \underline{(0001)}, \underline{(0002)}, (0003), \underline{(0011)}, \underline{(0012)}, (0013),$$

$$(0022), (0023), (0111), (0112), (0113), (0122), (0123).$$

The five underlined partitions belong to T_4^0; for the other ones, we have the following decompositions:

$$(0003) = (000) * (0), (0013) = (001) * (0), (0022) = (00) * (00),$$

$$(0023) = (00) * (0) * (0), (0111) = (0) * (000), (0112) = (0) * (001),$$

$$(0113) = (0) * (00) * (0), (0122) = (0) * (0) * (00), (0123) = (0) * (0) * (0) * (0).$$

The lexicographic ordering on T^0 gives $(0) <_{\mathrm{lex}} (00) <_{\mathrm{lex}} (000) <_{\mathrm{lex}} (001)$, thus we have the following eight elements in T_4^L:

$$(0000), (0001), (0002), (0003), (0011), (0012), (0013), (0023).$$

6 Explicit formulas and integrality

Denote by U_n the set of all sequences $(a_1, \ldots, a_n)$ of nonnegative integers which sum up to $(m-1)n$. We consider the natural action of the n-element cyclic group C_n on U_n by cyclic shift; call a sequence *primitive* if it is different from all its proper cyclic shifts. Every non-primitive sequence can be written as the (n/d)-fold repetition of a primitive sequence in U_d for d a proper divisor of n; we denote the corresponding subset of U_n by $U_n^{d-\mathrm{prim}}$, and in particular by $U_n^{\mathrm{prim}} = U_n^{n-\mathrm{prim}}$ the subset of primitive sequences. We relate U_n^{prim}/C_n, the set of C_n-orbits of primitive sequences, to the set T_n^L of the previous section.

LEMMA 6.1 *We have an injective map φ from T_n to U_n given by*

$$(\lambda_1,\ldots,\lambda_n)\mapsto(\lambda_2-\lambda_1,\lambda_3-\lambda_2,\ldots,\lambda_n-\lambda_{n-1},(m-1)n-\lambda_n).$$

Its inverse is given by

$$(a_1,\ldots,a_n)\mapsto(0,a_1,a_1+a_2,\ldots,a_1+\ldots+a_{n-1}).$$

The image of φ consists of the sequences $(a_1,\ldots,a_n)$ such that $a_1+\ldots+a_i\leq (m-1)i$ for all $i=1,\ldots,n$.

PROOF: This is immediately verified using the definitions.

□

Call a sequence $(a_1,\ldots,a_n)$ as above *admissible* if the condition of the previous lemma is satisfied, that is, if it belongs to the image of φ.

LEMMA 6.2 *Every cyclic class in U_n contains at least one admissible element.*

PROOF: Define an auxilliary sequence $(b_1,\ldots,b_n)$ of integers by $b_i=a_i-(m-1)$; then $\sum_i b_i=0$, and the admissibility condition translates into $\sum_{j=1}^i b_j\leq 0$ for all $i\leq n$. Choose an index i_0 such that $b_1+\ldots+b_{i_0}$ is maximal among these partial sums. Then $(a_{i_0+1},\ldots,a_n,a_1,\ldots,a_{i_0})$ is admissible: for $i_0\leq i\leq n$ we have

$$b_{i_0+1}+\ldots+b_i=(b_1+\ldots+b_i)-(b_1+\ldots+b_{i_0})\leq 0.$$

For $i\leq i_0$, we have (since the b_i sum up to 0):

$$b_{i_0+1}+\ldots+b_n+b_1+\ldots+b_i=(b_1+\ldots+b_i)-(b_1+\ldots+b_{i_0})\leq 0.$$

□

PROPOSITION 6.3 *The map φ induces a bijection between T_n^L and $U_n^{\rm prim}/C_n$.*

PROOF: If $\mu\in T_k$ and $\nu\in T_l$ for $k+l=n$, then $\varphi(\mu*\nu)$ is just the concatenation of the sequences $\varphi(\mu)$ and $\varphi(\nu)$. Thus, $\varphi(\mu*\nu)$ and $\varphi(\nu*\mu)$ are cyclic shifts of each other. Conversely, if a sequence $a\in U_n$ and a proper cyclic shift $a'=(a_{i+1},\ldots,a_n,a_1,\ldots,a_i)$ of a are both admissible, both subsequences $(a_1,\ldots,a_i)$ and $(a_{i+1},\ldots,a_n)$ are admissible. It follows that $a=\varphi(\mu*\nu)$ and $a'=\varphi(\nu*\mu)$ for some μ,ν.

We conclude that the restriction of φ to T_n^L only maps to primitive classes, and that each such cyclic class is hit precisely once.

□

Define $U_n^{\rm prim,+}$ as $U_n^{\rm prim}\cup U_n^{\overline{n}-\rm prim}$ if m is even and $n=2\overline{n}\equiv 2 \bmod 4$, and as $U_n^{\rm prim}$ otherwise. We have the following variant of the previous proposition:

COROLLARY 6.4 *The map φ induces a bijection between $T_n^{L,+}$ and $U_n^{\rm prim,+}/C_n$.*

Under the above map φ, the weight $\mathrm{wt}(\lambda)$ of a partition translates into the function

$$\mathrm{wt}(a_1,\ldots,a_n)=\sum_{i=1}^n(n-i)(m-1-a_i).$$

Lemma 6.5 *Considered modulo n, the function* wt *on U_n is invariant under cyclic shift. In each cyclic class, it assumes its maximum at an admissible element. If $a\in U_n$ is the $\frac{n}{d}$-fold repetition of a sequence $b\in U_d$, then* $\mathrm{wt}(a)=\frac{n}{d}\mathrm{wt}(b)$.

Proof: We have

$$\mathrm{wt}(a_{i+1},\ldots,a_n,a_1,\ldots,a_i)=\mathrm{wt}(a_1,\ldots,a_n)-n((m-1)i-a_1-\ldots-a_i),$$

proving the first two claims. It follows from a direct calculation that the function wt is additive with respect to concatenation of sequences as above, proving the third claim.

□

Defining $\mathrm{wt}(C)$ for a cyclic class $C\in U_n^{\mathrm{prim},+}/C_n$ as the maximal weight of sequences in class C, we can thus rewrite the polynomial $Q_n(q)$ of the previous section as $Q_n(q)=\sum_{C\in U_n^{\mathrm{prim},+}/C_n}q^{\mathrm{wt}(C)}$. We also derive the identity

$$nQ_n(q)\equiv\sum_{a\in U_n^{\mathrm{prim},+}}q^{\mathrm{wt}(a)}\bmod (q^n-1).$$

Define $P_n(q)=\sum_{a\in U_n}q^{\mathrm{wt}(a)}$. Using again the previous lemma, we have

$$P_n(q)=\sum_{d|n}\sum_{a\in U_n^{d-\mathrm{prim}}}q^{\mathrm{wt}(a)}=\sum_{d|n}\sum_{b\in U_d^{\mathrm{prim}}}q^{\frac{n}{d}\mathrm{wt}(b)}$$

and thus

$$P_n(q)\equiv\sum_{d|n}d\overline{Q}_d(q^{\frac{n}{d}})\bmod (q^n-1).$$

By Moebius inversion, this gives

Lemma 6.6 *We have*

$$\overline{Q}_n(q)\equiv\frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})P_d(q^{\frac{n}{d}})\bmod (q^n-1).$$

Remark: Arguments like the above also appear in the context of the "cyclic sieving phenomenon" for Gaussian binomial coefficients, see [15].

Theorem 6.7 *The polynomial $Q_n(q)$ is divisible by $[n]=1+q+\ldots+q^{n-1}$, and the quotient $\frac{1}{[n]}Q_n(q)$ is a polynomial in $\mathbf{Z}[q]$.*

Proof: Unwinding the definitions of the polynomial $P_n(q)$, of the set U_n and of its weight statistics wt, we see that $P_n(q)$ equals the $t^{(m-1)n}$-term in

$$\sum_{a_1,\ldots,a_n\geq 0} q^{\sum_i (n-i)(m-1-a_i)}t^{\sum_i a_i} = \frac{q^{(m-1)\binom{n}{2}}}{\prod_{i=0}^{n-1}(1-q^{-i}t)}.$$

Let ζ_n be a primitive n-th root of unity. Specializing q at an arbitrary n-th root of unity ζ_n^s for $s=1,\ldots,n$, we see that $P_n(\zeta_n^s)$ equals the $t^{(m-1)n}$-term in

$$\frac{\zeta_n^{(m-1)\binom{n}{2}s}}{\prod_{i=0}^{n-1}(1-\zeta_n^{si}t)} = \frac{\zeta_n^{(m-1)\binom{n}{2}s}}{(\prod_{i=0}^{\frac{n}{g}-1}(1-\zeta_n^{si}t))^g} = \frac{\zeta_n^{(m-1)\binom{n}{2}s}}{(1-t^{\frac{n}{g}})^g} =$$

$$= \zeta_n^{(m-1)\binom{n}{2}s}\sum_{k\geq 0}\binom{k+g-1}{g-1}t^{\frac{n}{g}k},$$

where $g=\gcd(s,n)$. The term $\zeta_n^{(m-1)\binom{n}{2}s}$ is easily seen to equal the sign $(-1)^{(m-1)(n-1)s}$, thus

$$P_n(\zeta_n^s) = (-1)^{(m-1)(n-1)s}\binom{m\gcd(s,n)-1}{\gcd(s,n)-1}.$$

Substituting this into the Moebius inversion formula of the previous lemma, we arrive at

$$\overline{Q}_n(\zeta_n^s) = \frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})(-1)^{(m-1)(n-1)s}\binom{m\gcd(s,n)-1}{\gcd(s,n)-1}.$$

In particular, we have $\overline{Q}_n(1) = \sum_{d|n}\mu(\frac{n}{d})\binom{mn-1}{n-1}$. Applying Lemma 8.4, we see that $Q_n(\zeta_n^s) = \overline{Q}_n(\zeta_n^s) = 0$ except in case m even, n even, $s=\overline{n}=\frac{n}{2}$ odd, where $\overline{Q}_n(-1) = -\frac{1}{\overline{n}}\sum_{d|\overline{n}}\mu(\frac{\overline{n}}{d})\binom{md-1}{d-1}$. Using the above formula for $\overline{Q}_n(1)$, in this case we thus get $Q_n(-1) = \overline{Q}_n(-1)+\overline{Q}_{\overline{n}}(1) = 0$.

We have proved that $Q_n(\zeta_n^s) = 0$ for $s=1,\ldots,n-1$, thus $Q_n(q)\in\mathbf{Z}[q]$ is divisible in $\mathbf{Z}[q]$ by all nontrivial cyclotomic polynomials $\Phi_d(q)$ for $1\neq d|n$, and thus $Q_n(q)$ is divisible in $\mathbf{Z}[q]$ by their product, which equals the polynomial $[n]$.

$\square$

We thus arrive at the following explicit formulas:

Theorem 6.8 *The following holds for all $m\geq 1$ and all $m\geq 1$:*

1. *The quantized Donaldson-Thomas invariant $\mathrm{DT}_n^{(m)}(q)$ is given by*

$$\mathrm{DT}_n^{(m)}(q) = q^{1-n}\frac{1}{[n]}\sum_{C\in U_n^{\mathrm{prim},+}} q^{\mathrm{wt}(C)}$$

and is a polynomial with integer coefficients.

2. *For every $i = 0, \ldots, n-1$, the (unquantized) Donaldson-Thomas invariant $\mathrm{DT}_n^{(m)}$ equals the number of classes $C \in U_n^{\mathrm{prim},+}$ of weight $\mathrm{wt}(C) \equiv i \bmod n$.*

Proof: Using Corollary 5.11 and the definition of $\mathrm{DT}_n^{(m)}(q)$ of Section 3, the first part follows from Theorem 6.7. The second part follows by comparing coefficients of the polynomials $Q_n(q)$ and $\mathrm{DT}_n^{(m)}(q)$.

□

Remark: There seems to be no natural weight function s on classes $C \in U_n^{\mathrm{prim},+}$ of weight $\mathrm{wt}(C) \equiv i \bmod n$ such that $\sum_C q^{s(C)} = \mathrm{DT}_n^{(m)}(q)$.

7 Relation to Higgs moduli

For $n \in \mathbf{N}$ and $d \in \mathbf{Z}$, define $H_{n,d}$ as the set of all sequences $(l_1, \ldots, l_n) \in \mathbf{Z}^n$ with the following properties:

1. $l_{k+1} - l_k + (m-1) \geq 0$ for all $k = 1, \ldots, n-1$,
2. $\sum_{i=1}^n l_i = d$,
3. $\frac{\sum_{i=1}^k l_i}{k} \geq \frac{d}{n}$ for all $k < n$.

These sequences arise as certain fixed points (so-called type $(1, \ldots, 1)$-fixed points) in the moduli space of SL_n-Higgs bundles, for the action of $\mathbf{C}^*$ scaling the Higgs field; see [7, Proposition 10.1]. A relation to Conjecture 3.3 is provided by [6, Remark 4.4.6].

Remark: Shifting every entry of such a sequence by 1 defines a bijection $H_{n,d} \simeq H_{n,d+n}$. We also have a duality $H_{n,d} \simeq H_{n,-d}$ by mapping $(l_1, \ldots, l_n)$ to $(-l_n, \ldots, -l_1)$. The elements of $H_{n,0}$ appear in combinatorics as "score sequences of complete tournaments" [11].

The aim of this section is to prove a conjecture of T. Hausel and F. Rodriguez-Villegas originating in [6, Remark 4.4.6] (see also [16]):

Theorem 7.1 *If d is coprime to n, the cardinality of $H_{n,d}$ equals $\mathrm{DT}_n^{(m)}$.*

Proof: We continue to work with the sets U_n, U_n^{prim}, $U_n^{\mathrm{prim},+}$ and $U_n^{\mathrm{prim}(,+)}/C_n$ of the previous section. We define a map $\Phi : H_{n,d} \to U_n$ by associating to $l_* = (l_1, \ldots, l_n)$ the sequence $(a_1, \ldots, a_n)$ defined by

$$a_k = l_{k+1} - l_k + (m-1) \text{ for } k = 1, \ldots, n,$$

where we formally set $l_{n+1} = l_1$ (the second and third of the defining conditions of $H_{n,d}$ have to be used to ensure $a_n \geq 0$). Obviously this map is injective. It is also compatible with cyclic shifts, from which it follows easily that the image of Φ consists only of primitive sequences, and that each cyclic class in

U_n^{prim} is hit at most once by the image of Φ (compare the proof of Proposition 6.3). In other words, Φ induces an embedding of $H_{n,d}$ into U_n^{prim}/C_n. The weight of $\Phi(l_*)$ is easily computed as $nl_1 - d$, thus it is congruent to $-d \bmod n$. We want to prove that, conversely, every primitive cyclic class a_* of weight $\mathrm{wt}(a_*) \equiv -d \bmod n$ belongs to the image of Φ. We first choose an arbitrary element a_* in such a cyclic class and associate to it the integers

$$l_k = l_1 + \sum_{i=1}^{k-1} a_i - (m-1)(k-1) \text{ where } l_1 = \frac{\mathrm{wt}(a_*) + d}{n}.$$

This sequence does not necessarily belong to $H_{n,d}$; only the first two defining conditions are fulfilled a priori, together with the condition $l_1 - l_n + (m-1) \geq 0$. We define k_0 as the maximal index $k \in \{0, \ldots, n\}$ where $l_1 + \ldots + l_k - \frac{d}{n}k$ reaches its minimum. Then the cyclic shift $(l_{k_0+1}, \ldots, l_n, l_1, \ldots, l_{k_0})$ fulfills the third defining condition by definition of k_0 (compare the proof of Lemma 6.2), and the first two conditions are still valid. From the definition, it also follows that this defines an inverse map to Φ.

We have thus proved that $H_{n,d}$ is in bijection to $(U_n^{\mathrm{prim}}/C_n)_{-d}$, the subset of U_n^{prim}/C_n of sequences of weight $\equiv -d \bmod n$. For parity reasons, sequences of such a weight cannot be twice a shorter sequence, thus we can replace U_n^{prim}/C_n by $U_n^{\mathrm{prim},+}/C_n$. By Theorem 6.8, the cardinality of the latter equals $\mathrm{DT}_n^{(m)}$. □

Remark: As in the case of $(U_n^{\mathrm{prim}}/C_n)_{-d}$, there seems to be no natural weight function on sequences $(l_1, \ldots, l_n)$ as above which gives $\mathrm{DT}_n^{(m)}(q)$.

8 Appendix: λ-ring exponential and Moebius inversion

Let R be the ring $\mathbf{Z}[q, q^{-1}][[t]]$ of formal power series in t with coefficients being integral Laurent series in q, and denote by R^+ the ideal of formal series without constant term. Then exp and log define mutually inverse isomorphisms between the additive group of R^+ and the multiplicative group $1 + R^+$ of formal series with constant term 1.

We can define (see [5]) a λ-ring structure on R with Adams operations ψ_i for $i \geq 1$ given by $\psi_i(q) = q^i$, $\psi(t) = t^i$; thus, in particular, we have $\psi_1 = \mathrm{id}$ and $\psi_i\psi_j = \psi_{ij}$. We define $\Psi = \sum_{i\geq 1} \frac{1}{i}\psi_i$.

Lemma 8.1 *The operator Ψ is invertible with inverse $\Psi^{-1} = \sum_{i\geq 1} \frac{\mu(i)}{i}\psi_i$, where μ denotes the number-theoretic Möbius function.*

Proof: Composition of Ψ with the operator defined on the right hand side of the claimed formula yields $\sum_{n\geq 1}\sum_{i|n} \mu(i)\frac{\psi_n}{n}$, which equals $\psi_1 = \mathrm{id}$ by properties of the Moebius function.

□

Using this explicit form of the operators Ψ and Ψ^{-1}, we can derive the following q-Moebius inversion formula for polynomials $f_n(q), g_n(q)$ in q (viewing them as coefficients of formal series):

Lemma 8.2 *We have*

$$ng_n(q) = \sum_{d|n} f_d(q^{n/d}) \iff f_n(q) = \sum_{d|n} \mu(\frac{n}{d}) d g_d(q^{n/d}).$$

We define the λ-ring exponential $\mathrm{Exp} : R^+ \to 1 + R^+$ by $\mathrm{Exp} = \exp \circ \Psi$. Its inverse is the λ-ring logarithm $\mathrm{Log} = \Psi^{-1} \circ \log$. We have the following explicit formula:

Lemma 8.3 *For coefficients $c_{i,k} \in \mathbf{Z}$ such that, for fixed $i \in \mathbf{N}$, we have $c_{i,k} \neq 0$ for only finitely many $k \in \mathbf{Z}$, the following formula holds:*

$$\mathrm{Exp}(\sum_{i\geq 1}\sum_{k\in\mathbf{Z}} c_{i,k} q^k t^i) = \prod_{i\geq 1}\prod_{k\in\mathbf{Z}} (1 - q^k t^i)^{-c_{i,k}}.$$

Proof: It suffices to compute $\mathrm{Exp}(q^k t^i)$, which is

$$\exp(\sum_{j\geq 1} \frac{1}{j}(q^k t^i)^j) = \exp(-\log(1 - q^k t^i)) = (1 - q^k t^i)^{-1}.$$

The lemma follows.

□

In Section 6, we make use of the following Moebius inversion type result.

Lemma 8.4 *Let $f : \mathbf{N} \to \mathbf{Z}$ be function on non-negative integers. For $n \geq 1$ and a proper divisor s of n, the sum*

$$\frac{1}{n}\sum_{d|n} \mu(\frac{n}{d})(-1)^{(m-1)(d-1)s} f(\gcd(d,s))$$

equals 0*, except when m is even, n is even, $s = \overline{n} = \frac{n}{2}$ is odd, where it equals $-\frac{1}{\overline{n}}\sum_{d|\overline{n}} \mu(\frac{\overline{n}}{d}) f(d)$.*

Proof: Suppose first that m is even, n is even, and $s = \overline{n}$ is odd. Every divisor of n is a divisor d of $\overline{n}$ or twice such a d. We can then split the sum in question into

$$\frac{1}{n}(\sum_{d|\overline{n}} \mu(\frac{n}{d})(-1)^{d-1} f(d) + \sum_{d|\overline{n}} \mu(\frac{\overline{n}}{d})(-1)^{2d-1} f(d)).$$

Since all divisors d are odd, we have $\mu(\frac{n}{d}) = -\mu(\frac{\overline{d}}{n})$, and the sum simplifies to

$$-\frac{2}{n}\sum_{d|\overline{n}} \mu(\frac{\overline{n}}{d}) f(d) = -\frac{1}{\overline{n}}\sum_{d|\overline{n}} \mu(\frac{\overline{n}}{d}) f(d),$$

as claimed.

Now suppose that s is an arbitrary proper divisor of n, but $s \neq \frac{n}{2}$ in case m is even and $n \equiv 2 \bmod 4$. We write

$$\frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})(-1)^{(m-1)(d-1)s}f(\gcd(d,s)) =$$

$$\frac{1}{n}\sum_{g|s}\sum_{\substack{d|\frac{n}{g}\\ \gcd(d,\frac{s}{g})=1}}\mu(\frac{n}{gd})(-1)^{(m-1)(gd-1)s}f(g).$$

We can uniquely decompose $\frac{n}{g}$ as n_1n_2, where n_1 collects all prime factors of $\frac{n}{g}$ dividing $\frac{s}{g}$; we then have $\gcd(n_1,n_2)=1$, and the divisors d of $\frac{n}{g}$ such that $\gcd(d,\frac{s}{g})=1$ are precisely the divisors of n_2. Thus, we can rewrite the above sum as

$$\frac{1}{n}\sum_{g|s}\sum_{d|n_2}\mu(n_1)\mu(\frac{n_2}{d})(-1)^{(m-1)(gd-1)s}f(g) =$$

$$\frac{1}{n}(-1)^{(m-1)s}\sum_{g|s}\mu(n_1)f(g)\sum_{d|n_2}\mu(\frac{n_2}{d})(-1)^{(m-1)gds}.$$

By Moebius inversion, the inner sum, temporarily called $\rho(g)$, equals zero except in the case $n_2=1$, or $n_2=2$ and $(m-1)gs$ is even.

Now suppose that in the above sum, the summand corresponding to a divisor g of s is non-zero, that is, both $\mu(n_1)$ and $\rho(g)$ are non-zero. First consider the case $n_2=1$, thus $n_1=\frac{n}{g}$ is squarefree. Since $s\neq n$, there exists a prime p dividing $\frac{n}{s}$, and thus also $\frac{n}{g}$. Since $n_2=1$, the prime p also divides $\frac{s}{g}$, thus p^2 divides $\frac{n}{g}$, a contradiction. Now consider the case $n_2=2$ and $(m-1)gs$ even, thus $n_1=\frac{n}{2g}$ is squarefree. Again, a prime p dividing $\frac{n}{2g}$ also divides $\frac{s}{g}$. If p is odd, the argument of the first case again yields a contradiction. So suppose that 2 is the only prime divisor of $\frac{n}{g}$, that $n=2^kn'$ for odd n', and $g=2^ln'$. Then $s=2^{l'}n'$ for some $l'\leq l$, and $\frac{n}{2g}=2^{k-l}$. Since $\frac{n}{2g}$ is squarefree, we have $k=l$ or $k=l+1$. If $k=l$, then $s\geq g=n$, a contradiction. If $k=l+1$, then $g=\frac{n}{2}$, both s and g are odd, and thus m is even. But then $n\equiv 2 \bmod 4$, and by assumption $s\neq\frac{n}{2}$, a contradiction.

Thus we see that no summand above can be non-zero, proving the claim.

□

References

[1] P. Duchon: *On the enumeration and generation of generalized Dyck words.* Formal power series and algebraic combinatorics (Toronto, ON, 1998). Discrete Math. 225 (2000), no. 1-3, 121-135.

[2] P. Duchon: *q-grammars and wall polyominoes.* Ann. Comb. 3 (1999), no. 2-4, 311-321.

[3] A. I. Efimov: *Cohomological Hall algebra of a symmetric quiver.* arXiv:1103.2736

[4] J. Engel, M. Reineke: *Smooth models of quiver moduli.* Math. Z. 262 (2009), 4, 817-848.

[5] E. Getzler: *Mixed Hodge structures of configuration spaces.* arXiv:alg-geom/9510018.

[6] T. Hausel and F. Rodriguez-Villegas: *Mixed Hodge polynomials of character varieties (with an appendix by Nicholas Katz).* Inv. Math. 174 (2008), 3, 555-624.

[7] T. Hausel, M. Thaddeus: *Mirror symmetry, Langlands duality and Hitchin systems.* Invent. Math., 153 (2003), 1, 197-229.

[8] G. Helleloid, F. Rodriguez-Villegas: *Counting Quiver Representations over Finite Fields Via Graph Enumeration.* arXiv:0810.2127

[9] M. Kontsevich, Y. Soibelman: *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants.* arXiv:1006.2706

[10] M. Kontsevich, Y. Soibelman: *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations.* arXiv:0811.2435

[11] H. G. Landau: *On Dominance Relations and the Structure of Animal Societies, III. The Condition for a Score Structure.* Bull. Math. Biophys. 15 (1953), 143-148.

[12] M. Reineke: *Cohomology of noncommutative Hilbert schemes.* Algebr. Represent. Theory. 8 (2005), 541-561.

[13] M. Reineke: *Poisson automorphisms and quiver moduli.* J. Inst. Math. Jussieu 9 (2010), 3, 653-667.

[14] M. Reineke: *Cohomology of quiver moduli, functional equations, and integrality of Donaldson-Thomas type invariants.* Compositio Math. 147 (2011), 943-964.

[15] V. Reiner, D. Stanton, and D. White: *The cyclic sieving phenomenon.* J. Combin. Theory Ser. A (2004), 17-50.

[16] F. Rodriguez-Villegas: *A refinement of the A-polynomial of quivers.* arXiv:1102.5308

[17] C. Reutenauer: *Free Lie algebras.* Oxford University Press, 1993.

[18] R. P. Stanley: *Enumerative Combinatorics, vol. 2.* Cambridge University Press, New York/Cambridge, 1999.

Markus Reineke
Fachbereich C
Mathematik
und Naturwissenschaften
Bergische Universität Wuppertal
Gaußstr. 20
D 42097 Wuppertal
Deutschland
reineke@math.uni-wuppertal.de

Smooth Representations of $\mathrm{GL}_m(\mathrm{D})$
V: Endo-Classes

In memory of Martin Grabitz

P. Broussous, V. Sécherre, and S. Stevens[(1)]

Abstract. Let F be a locally compact nonarchimedean local field. In this article, we extend to any inner form of GL_n over F, with $n \geqslant 1$, the notion of endo-class introduced by Bushnell and Henniart for $\mathrm{GL}_n(\mathrm{F})$. We investigate the intertwining relations of simple characters of these groups, in particular their preservation properties under transfer. This allows us to associate to any discrete series representation of an inner form of $\mathrm{GL}_n(\mathrm{F})$ an endo-class over F. We conjecture that this endo-class is invariant under the local Jacquet-Langlands correspondence.

2010 Mathematics Subject Classification: 22E50
Keywords and Phrases: representations of p-adic groups, simple characters, type theory, Shintani lift, Jacquet-Langlands correspondence

Contents

(1) This work was supported by: EPSRC (grants GR/T21714/01, EP/G001480/1), the British Council and the Partenariat Hubert Curien in the framework of the Alliance programme (number 19418YK), the Agence Nationale de la Recherche (ANR-08-BLAN-0259-01) and the Université de la Méditerranée Aix-Marseille 2.

Introduction

This is the fifth in a series of articles whose objective is a complete description of the category of smooth complex representations of $\mathrm{GL}_r(\mathrm{D})$, with r a positive integer and D a division algebra over a locally compact nonarchimedean local field. The longer term aim is an explicit description, in terms of types, of the local Jacquet-Langlands correspondence [14, 1], as begun by Bushnell and Henniart [18, 7, 9], and by Silberger and Zink [26, 27].

The main object of study in this paper is the notion of endo-equivalence class, or *endo-class*, of simple characters. This notion has been introduced by Bushnell and Henniart [6] for the group $\mathrm{GL}_n(\mathrm{F})$, with n a positive integer and F a locally compact nonarchimedean local field: an endo-class is an invariant associated to an irreducible cuspidal representation of $\mathrm{GL}_n(\mathrm{F})$, constructed by explicit methods related to the description of this representation as compactly induced from an irreducible representation of a compact-mod-centre subgroup of $\mathrm{GL}_n(\mathrm{F})$ (see [10, 6]). The arithmetic significance of this invariant has been described in [8], in the case where F is of characteristic zero: if we denote by $\mathscr{W}_{\mathrm{F}}$ the Weil group of F (relative to an algebraic closure) and by $\mathscr{P}_{\mathrm{F}}$ its wild inertia subgroup, there is a bijection between the set $\mathcal{E}(\mathrm{F})$ of endo-classes over F and the set of $\mathscr{W}_{\mathrm{F}}$-conjugacy classes of irreducible representations of $\mathscr{P}_{\mathrm{F}}$, which is compatible with the local Langlands correspondence.

In this article, we extend the notion of endo-class to any inner form of $\mathrm{GL}_n(\mathrm{F})$, $n \geqslant 1$, that is, to any group of the form $\mathrm{GL}_r(\mathrm{D})$, with r a positive integer and D an F-central division algebra of dimension d^2 over F, with $n = rd$. For this we develop a *Shintani lift*, or *base change*, for simple characters, which is also of independent interest (see below). If G is an inner form of $\mathrm{H} = \mathrm{GL}_n(\mathrm{F})$, and if $\mathcal{D}(\mathrm{G})$ denotes the discrete series of G (that is, the set of isomorphism classes of essentially square-integrable irreducible representations of G), we define a map:

$$\boldsymbol{\Theta}_{\mathrm{G}} : \mathcal{D}(\mathrm{G}) \to \mathcal{E}(\mathrm{F})$$

(see paragraph 9.2) which associates an endo-class over F to any discrete series representation of G. This map should play an important role in an explicit description of the local Jacquet–Langlands correspondence:

$$\mathbf{JL} : \mathcal{D}(\mathrm{G}) \to \mathcal{D}(\mathrm{H}).$$

In particular, we expect that $\mathbf{JL}$ preserves the endo-class (see Conjecture 9.5), that is:

$$\boldsymbol{\Theta}_{\mathrm{H}} \circ \mathbf{JL} = \boldsymbol{\Theta}_{\mathrm{G}}.$$

This conjectural property can be seen as a generalization of the fact that the correspondence **JL** preserves the representations of level zero (see [26]). The notion of endo-class also plays a central role in:

– the construction of semisimple types, which leads to a complete description of the structure of the category of smooth complex representations of G (see [25]);

– the study of smooth representations of G with coefficients in a field of non-zero characteristic different from the residue characteristic of F (see [19]).

Before giving more details, let us mention that there are roughly speaking two main obstacles to overcome: First, one has to compare simple characters in $\mathrm{GL}_r(\mathrm{D})$ with simple characters in $\mathrm{GL}_{r'}(\mathrm{D}')$ where $\mathrm{GL}_r(\mathrm{D})$ and $\mathrm{GL}_{r'}(\mathrm{D}')$ are two inner forms of $\mathrm{GL}_n(\mathrm{F})$ with D and D' not necessarily isomorphic. It is to overcome this that we need to develop a Shintani lift, or base change, for simple characters. This process is of independent interest and may be used to define a Shintani lift for irreducible representations of $\mathrm{GL}_r(\mathrm{D})$. The second problem is due to the notion of *embedding type*, a phenomenon first discovered by Fröhlich [15]; this problem, and its resolution, will be discussed in more detail below.

One of the objectives of [20], completed in [24], is the construction of simple characters, which are certain special characters of particular compact open subgroups of G. These simple characters are attached to data called *simple strata*, and are a fundamental part of the construction of more elaborate objects called *simple types* (see [21, 22]). One knows from [22, 24] that every irreducible discrete series representation π of G contains a simple character θ attached to a simple stratum. Neither the simple stratum nor the simple character are unique, but every other simple character θ' contained in π intertwines θ, that is, there is an element $g \in \mathrm{G}$ such that θ' and the conjugate character θ^g coincide on the intersection of the compact open subgroups where they are defined. It is this observation which leads to the notion of endo-class.

An endo-class is an equivalence class of objects called *potential simple characters* (or *ps-characters* for short), for a relation called *endo-equivalence.* A ps-character Θ is characterized by giving a simple stratum $[\Lambda, n, m, \beta]$ in an F-central simple algebra A and a simple character θ attached to this simple stratum. The pair $([\Lambda, n, m, \beta], \theta)$ is called a *realization* of Θ. Another simple stratum $[\Lambda', n', m', \beta]$ in another F-central simple algebra A' (note that β is unchanged) and a simple character θ' for this stratum define the same ps-character precisely when θ and θ' are linked by the transfer map defined in [20] (see paragraph 1.2 below). Two ps-characters Θ_1 and Θ_2 are said to be endo-equivalent (see Definition 1.10) if they can be characterized by giving realizations $([\Lambda, n_i, m_i, \beta_i], \theta_i)$ in an F-central simple algebra A, for $i = 1, 2$ (note that A and Λ do not depend on i), of the same degree and normalized level, and such that the simple characters θ_1 and θ_2 intertwine in $\mathrm{A}^\times$.

The properties of endo-equivalence depend on important intertwining properties of simple characters, notably the preservation of these properties under the transfer map. This article centres on two important technical results: the property of "preservation of intertwining" (Theorem 1.11) and the "intertwining implies conjugacy" property (Theorem 1.12). Partial results on these questions were already given by Grabitz [17], notably a proof of "intertwining implies conjugacy", but these results are proved under unnecessarily restrictive hypotheses: that the simple strata underlying the construction are *sound* in the sense of Definition 1.14. We have sought to develop the notion of endo-class in as general a situation as possible, emphasizing the functorial properties of the objects involved. However, rather than starting again from scratch, we decided to use the work of Grabitz as much as possible. We note that, as well as [17], our proofs rely heavily on the results of Bushnell, Henniart and Kutzko [10, 6] in the split case.

Let us now describe in more detail the results, and the techniques used, in this article. For $i = 1, 2$, let Θ_i be a ps-character defined by a simple stratum $[\Lambda, n_i, m_i, \beta_i]$ in an F-central simple algebra A and a simple character θ_i in $\mathcal{C}(\Lambda, m_i, \beta_i)$ attached to this stratum (see paragraph 1.1 for the notation). Suppose from now on that the ps-characters Θ_1 and Θ_2 are endo-equivalent so that, in particular, we may assume the characters θ_1 and θ_2 intertwine in $\mathrm{A}^\times$. The "preservation of intertwining" property can be stated as follows:

Theorem (see Theorem 1.11). — *For $i = 1, 2$, let $[\Lambda', n'_i, m'_i, \beta_i]$ be a simple stratum in a simple central* F*-algebra* A′ *and $\theta'_i \in \mathcal{C}(\Lambda', m'_i, \beta_i)$ defining the ps-character Θ_i, that is, θ'_i is the transfer of θ_i. Then the characters θ'_1 and θ'_2 intertwine in $\mathrm{A}'^\times$.*

This means that the property that two simple characters intertwine is invariant under transfer. The statement above is the same as its analogue [6, Theorem 8.7] in the case that A is split and Λ is strict. However, we will see that the proof requires new ideas.

One of the important results in [10] is the "intertwining implies conjugacy" property for simple characters, which expresses the fact that intertwining of simple characters is a very stringent relation. It is this property which allows a classification "up to conjugacy" of the irreducible cuspidal representations of $\mathrm{GL}_n(\mathrm{F})$. This property no longer holds in the general case, as was already observed in [5] for simple strata. To remedy the situation, we introduce the notion of *embedding type* of a simple stratum (see Definition 1.8): two simple strata $[\Lambda, n_i, m_i, \beta_i]$ have the same embedding type if the maximal unramified subextensions of $\mathrm{F}(\beta_i)/\mathrm{F}$ are conjugate under the normalizer of Λ in $\mathrm{A}^\times$. With the same notation and hypotheses as above, we prove the following:

Theorem (see Theorem 1.12). — *Suppose that $n_1 = n_2$, $m_1 = m_2$, and the simple strata $[\Lambda, n_i, m_i, \beta_i]$ have the same embedding type. Write K_i for the maximal unramified extension of* F *contained in $\mathrm{F}(\beta_i) \subseteq \mathrm{A}$. Then there is* *an element of the normalizer of* Λ *in* $\mathrm{A}^\times$ *which simultaneously conjugates* K_1 *to* K_2 *and* θ_1 *to* θ_2*.*

This result was proved by Grabitz [17, Corollary 10.15] with the additional assumption that the simple strata $[\Lambda, n, m, \beta_i]$ are sound. We prove it here without this hypothesis.

Once one has proved that endo-equivalence preserves certain numerical invariants (see Lemma 4.7), it is not hard to see that the proofs of these two Theorems can be reduced to the following:

Theorem (see Theorem 1.13). — *For* $i = 1, 2$*, let* $[\Lambda', n', m', \beta_i]$ *be a simple stratum in a simple central* F*-algebra* A' *and* $\theta'_i \in \mathcal{C}(\Lambda', m', \beta_i)$ *defining the ps-character* Θ_i*, that is,* θ'_i *is the transfer of* θ_i*. Assume the simple strata have the same embedding type and write* K_i *for the maximal unramified extension of* F *contained in* $\mathrm{F}(\beta_i) \subseteq \mathrm{A}'$*. Then there is an element of the normalizer of* Λ' *in* $\mathrm{A}'^\times$ *which simultaneously conjugates* K_1 *to* K_2 *and* θ'_1 *to* θ'_2*.*

Now let us describe the scheme of the proof. We begin with our endo-equivalent ps-characters Θ_1 and Θ_2, together with realizations $([\Lambda, n_i, m_i, \beta_i], \theta_i)$ in A such that the simple characters θ_1 and θ_2 intertwine in $\mathrm{A}^\times$. In order to use the results of Grabitz, we need first to produce sound realizations of the ps-characters Θ_i with the same embedding type, which intertwine. For sound strata, the embedding type is determined by a single integer, the *Fröhlich invariant*, which can also be defined for arbitrary strata (see Definition 4.1). One can then realize Θ_i on the lattice sequence $\Lambda \oplus \Lambda$ in such a way that the Fröhlich invariant is 1 and the simple characters still intertwine (see Lemma 4.4). In particular, replacing our original realizations of Θ_1 and Θ_2 with these new ones, we can assume the simple strata $[\Lambda, n_i, m_i, \beta_i]$ have the same Fröhlich invariant. Now we define a process:

$$([\Lambda, n, m, \beta], \theta) \mapsto ([\Lambda^\ddagger, n, m, \beta], \theta^\ddagger)$$

from arbitrary realizations to sound realizations, with $\theta^\ddagger$ the transfer of θ, which preserves intertwining and the Fröhlich invariant (see paragraph 2.7). In particular, from θ_1 and θ_2 one obtains simple characters $\theta_1^\ddagger$ and $\theta_2^\ddagger$ on sound simple strata with the same Fröhlich invariant (so same embedding type) which intertwine. Thus we can apply Grabitz's results, together with a reduction to the case $m_1 = m_2$, to deduce that $\theta_1^\ddagger$ and $\theta_2^\ddagger$ are conjugate under $\mathrm{A}^{\ddagger\times}$ (where $\mathrm{A}^\ddagger$ is the simple central F-algebra with respect to which the stratum $[\Lambda^\ddagger, n, m, \beta]$ is defined). Changing again our realizations of Θ_1 and Θ_2 we can suppose we have an equality $\theta_1 = \theta_2$ of simple characters. This is given in Proposition 4.9, the culmination of the first stage of the proof.

To show that other realizations θ'_1 and θ'_2 on simple strata in A' with the same embedding type are conjugate, we would like to reduce to the split case so that we can use results from [10, 6]. For this we define an *interior lifting* (see section 5):

$$([\Lambda, n, m, \beta], \theta) \mapsto ([\Gamma, n, m, \beta], \theta^{\mathrm{K}})$$

relative to the extension K/F, the maximal unramified subextension of $\mathrm{F}(\beta)/\mathrm{F}$, where $[\Gamma, n, m, \beta]$ is a simple stratum in the centralizer C of K in the simple central F-algebra A with respect to which $[\Lambda, n, m, \beta]$ is defined. Then we make a *base change* (see section 7):

$$([\Gamma, n, m, \beta], \theta^{\mathrm{K}}) \mapsto ([\overline{\Gamma}, n, m, \beta], \overline{\theta}^{\mathrm{K}})$$

relative to L/K, a finite unramified extension which is sufficiently large so that the algebra $\mathrm{C} \otimes_{\mathrm{K}} \mathrm{L}$ is split. The definition of the base change used here is somewhat subtle: indeed, it is not clear how to make a good definition which will preserve intertwining and, when applied to our characters θ_i, will be independent of i. Moreover, it is necessary to begin with the interior lift or else the base change process would produce *quasi-simple* characters (see [20]), rather than simple characters.

In order to apply these processes, we note that the maximal unramified subextension K of $\mathrm{F}(\beta_i)/\mathrm{F}$ in A can be assumed to be independent of i since the simple strata have the same embedding type. Combining now interior lifting and base change, we get a process:

$$([\Lambda, n, m, \beta], \theta) \mapsto ([\overline{\Gamma}, n, m, \beta], \overline{\theta}^{\mathrm{K}})$$

denoted here $\theta \mapsto \widetilde{\theta}$ for simplicity, which is both injective and equivariant, so it is enough to show that $\widetilde{\theta}'_1$ and $\widetilde{\theta}'_2$ are conjugate under $\mathrm{A}'^{\times}$. Now the hypothesis $\theta_1 = \theta_2$ implies $\widetilde{\theta}_1 = \widetilde{\theta}_2$ (see Propositions 6.11 and 7.5), so that the ps-characters $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_2$ defined by $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ are endo-equivalent. Moreover, for each i, the simple character $\widetilde{\theta}'_i$ is the transfer of $\widetilde{\theta}_i$ (see Theorem 6.7), so it is another realization of the ps-character $\widetilde{\Theta}_i$. We are now in the split case so, modulo a finesse in the case that we do not have strict lattice sequences, we deduce from endo-equivalence [6] that the characters $\widetilde{\theta}'_i$ intertwine. Thus, from the "intertwining implies conjugacy" property [10], the characters $\widetilde{\theta}'_1$ and $\widetilde{\theta}'_2$ are conjugate under $(\mathrm{C}' \otimes_{\mathrm{K}} \mathrm{L})^{\times}$, where C′ denotes the centralizer of K in A′. Thanks to the invariance property of the base change under the action of the Galois group Gal(L/K) (see Proposition 7.7), a cohomological argument (see Lemma 8.1) allows us to show that they are actually conjugate under $\mathrm{C}'^{\times}$. This completes the proof.

Notation

Let F be a nonarchimedean locally compact field. All F-algebras are supposed to be finite-dimensional with a unit. By an F*-division algebra* we mean a central F-algebra which is a division algebra.

For K a finite extension of F, or more generally a division algebra over a finite extension of F, we denote by $\mathcal{O}_{\mathrm{K}}$ its ring of integers, by $\mathfrak{p}_{\mathrm{K}}$ the maximal ideal of $\mathcal{O}_{\mathrm{K}}$ and by $\mathfrak{k}_{\mathrm{K}}$ its residue field.

For A a simple central algebra over a finite extension K of F, we denote by $\mathrm{N}_{\mathrm{A}/\mathrm{K}}$ and $\mathrm{tr}_{\mathrm{A}/\mathrm{K}}$ respectively the reduced norm and trace of A over K.

For u a real number, we denote by $\lceil u \rceil$ the smallest integer which is greater than or equal to u, and by $\lfloor u \rfloor$ the greatest integer which is smaller than or equal to u, that is, its integer part.
A *character* of a topological group G is a continuous homomorphism from G to the group $\mathbb{C}^\times$ of non-zero complex numbers.
All representations are supposed to be smooth with complex coefficients.

1. Statement of the main results

In this section, we recall some well known facts about lattice sequences, simple strata and simple characters in a simple central F-algebra (see [4, 10, 12, 20, 24] for more details), and we state the main results of this article.

1.1. Let A be a simple central F-algebra, and let V be a simple left A-module. The algebra $\mathrm{End}_{\mathrm{A}}(\mathrm{V})$ is an F-division algebra, the opposite of which we denote by D. Considering V as a right D-vector space, we have a canonical isomorphism of F-algebras between A and $\mathrm{End}_{\mathrm{D}}(\mathrm{V})$.

Definition 1.1. — An $\mathcal{O}_{\mathrm{D}}$*-lattice sequence* on V is a sequence $\Lambda = (\Lambda_k)_{k\in\mathbb{Z}}$ of $\mathcal{O}_{\mathrm{D}}$-lattices of V such that $\Lambda_k \supseteq \Lambda_{k+1}$ for all $k \in \mathbb{Z}$, and such that there exists a positive integer e satisfying $\Lambda_{k+e} = \Lambda_k \mathfrak{p}_{\mathrm{D}}$ for all $k \in \mathbb{Z}$. This integer is called the *period* of Λ over $\mathcal{O}_{\mathrm{D}}$.

If $\Lambda_k \supsetneq \Lambda_{k+1}$ for all $k \in \mathbb{Z}$, then the lattice sequence Λ is said to be *strict*.

Associated with an $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ on V, we have an $\mathcal{O}_{\mathrm{F}}$-lattice sequence on A defined by:

$$\mathfrak{P}_k(\Lambda) = \{a \in \mathrm{A} \mid a\Lambda_i \subseteq \Lambda_{i+k},\ i \in \mathbb{Z}\}, \quad k \in \mathbb{Z}.$$

The lattice $\mathfrak{A}(\Lambda) = \mathfrak{P}_0(\Lambda)$ is a hereditary $\mathcal{O}_{\mathrm{F}}$-order in A, and $\mathfrak{P}(\Lambda) = \mathfrak{P}_1(\Lambda)$ is its Jacobson radical. They depend only on the set $\{\Lambda_k \mid k \in \mathbb{Z}\}$.

We denote by $\mathfrak{K}(\Lambda)$ the $\mathrm{A}^\times$-*normalizer* of Λ, that is the subgroup of $\mathrm{A}^\times$ made of all elements $g \in \mathrm{A}^\times$ for which there is an integer $n \in \mathbb{Z}$ such that $g(\Lambda_k) = \Lambda_{k+n}$ for all $k \in \mathbb{Z}$. Given $g \in \mathfrak{K}(\Lambda)$, such an integer is unique: it is denoted $\upsilon_\Lambda(g)$ and called the Λ-*valuation* of g. This defines a group homomorphism υ_Λ from $\mathfrak{K}(\Lambda)$ to $\mathbb{Z}$. Its kernel, denoted $\mathrm{U}(\Lambda)$, is the group of invertible elements of $\mathfrak{A}(\Lambda)$. We set $\mathrm{U}_0(\Lambda) = \mathrm{U}(\Lambda)$ and, for $k \geqslant 1$, we set $\mathrm{U}_k(\Lambda) = 1 + \mathfrak{P}_k(\Lambda)$.

Let F' be a finite extension of F contained in A. An $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ on V is said to be F'-*pure* if it is normalized by $\mathrm{F}'^\times$. The centralizer of F' in A, denoted A', is a simple central F'-algebra. We fix a simple left A'-module V' and write D' for the algebra opposite to $\mathrm{End}_{\mathrm{A}'}(\mathrm{V}')$. By [24, Théorème 1.4] (see also [4, Theorem 1.3]), given an F'-pure $\mathcal{O}_{\mathrm{D}}$-lattice sequence on V, there is an $\mathcal{O}_{\mathrm{D}'}$-lattice sequence Λ' on V' such that:

$$\mathfrak{P}_k(\Lambda) \cap \mathrm{A}' = \mathfrak{P}_k(\Lambda'), \quad k \in \mathbb{Z}. \tag{1.1}$$

It is unique up to translation of indices, and its $\mathrm{A}'^\times$-normalizer is $\mathfrak{K}(\Lambda) \cap \mathrm{A}'^\times$.

Definition 1.2. — A *stratum* in A is a quadruple $[\Lambda, n, m, \beta]$ made of an $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ on V, two integers m, n such that $0 \leqslant m \leqslant n-1$ and an element $\beta \in \mathfrak{P}_{-m}(\Lambda)$.

For $i = 1, 2$, let $[\Lambda, n, m, \beta_i]$ be a stratum in A. We say these two strata are *equivalent* if $\beta_2 - \beta_1 \in \mathfrak{P}_{-m}(\Lambda)$.

Given a stratum $[\Lambda, n, m, \beta]$ in A, we denote by E the F-algebra generated by β. This stratum is said to be *pure* if E is a field, if Λ is E-pure and if $\upsilon_\Lambda(\beta) = -n$. In this situation, we denote by:

$$e_\beta(\Lambda)$$

the period of Λ as an $\mathcal{O}_{\mathrm{E}}$-lattice sequence. Given a pure stratum $[\Lambda, n, m, \beta]$, we denote by B the centralizer of E in A. For $k \in \mathbb{Z}$, we set:

$$\mathfrak{n}_k(\beta, \Lambda) = \{x \in \mathfrak{A}(\Lambda) \mid \beta x - x\beta \in \mathfrak{P}_k(\Lambda)\}.$$

The smallest integer $k \geqslant \upsilon_\Lambda(\beta)$ such that $\mathfrak{n}_{k+1}(\beta, \Lambda)$ is contained in $\mathfrak{A}(\Lambda) \cap \mathrm{B} + \mathfrak{P}(\Lambda)$ is called the *critical exponent* of the stratum $[\Lambda, n, m, \beta]$, denoted $k_0(\beta, \Lambda)$.

Definition 1.3. — The stratum $[\Lambda, n, m, \beta]$ is said to be *simple* if it is pure and if we have $m \leqslant -k_0(\beta, \Lambda) - 1$.

Let $[\Lambda, n, m, \beta]$ be a simple stratum in A. In [24] (see paragraph 2.4), one attaches to this simple stratum a compact open subgroup $\mathrm{H}^{m+1}(\beta, \Lambda)$ of $\mathrm{A}^\times$ and a finite set $\mathcal{C}(\Lambda, m, \beta)$ of characters of $\mathrm{H}^{m+1}(\beta, \Lambda)$, called simple characters of level m, depending on the choice of an additive character:

$$\Psi : \mathrm{F} \to \mathbb{C}^\times \tag{1.2}$$

which is trivial on $\mathfrak{p}_{\mathrm{F}}$ but not on $\mathcal{O}_{\mathrm{F}}$, and which will be fixed once and for all throughout this paper. If $\lfloor n/2 \rfloor \leqslant m$, then $\mathrm{H}^{m+1}(\beta, \Lambda) = \mathrm{U}_{m+1}(\Lambda)$, and the set $\mathcal{C}(\Lambda, m, \beta)$ reduces to a single character Ψ_β^{A} of $\mathrm{U}_{m+1}(\Lambda)$ defined by:

$$\Psi_\beta^{\mathrm{A}} : x \mapsto \Psi \circ \mathrm{tr}_{\mathrm{A}/\mathrm{F}}(\beta(x-1)), \tag{1.3}$$

which depends only on the equivalence class of $[\Lambda, n, m, \beta]$. More generally, for any possible value of m, the subgroup $\mathrm{H}^{m+1}(\beta, \Lambda)$ and the set $\mathcal{C}(\Lambda, m, \beta)$ depend only on the equivalence class of $[\Lambda, n, m, \beta]$.

1.2. Let β be a non-zero element of some finite extension of F. We set $\mathrm{E} = \mathrm{F}(\beta)$ and:

$$n_{\mathrm{F}}(\beta) = -\upsilon_{\mathrm{E}}(\beta),$$
$$e_{\mathrm{F}}(\beta) = e(\mathrm{E} : \mathrm{F}),$$
$$f_{\mathrm{F}}(\beta) = f(\mathrm{E} : \mathrm{F}),$$

where $e(\mathrm{E} : \mathrm{F})$ and $f(\mathrm{E} : \mathrm{F})$ stand for the ramification index and the residue class degree of E over F respectively, and υ_{E} for the valuation map of the field E giving the value 1 to any uniformizer of E. The lattice sequence $i \mapsto \mathfrak{p}_{\mathrm{E}}^i$, denoted $\Lambda(\mathrm{E})$, is the unique (up to translation) E-pure strict $\mathcal{O}_{\mathrm{F}}$-lattice sequence on the

F-vector space E, and its valuation map coincide with υ_{E} on $\mathrm{E}^{\times}$. To any integer $0 \leqslant k \leqslant n_{\mathrm{F}}(\beta)-1$ we can attach the pure stratum $[\Lambda(\mathrm{E}), n_{\mathrm{F}}(\beta), k, \beta]$ of the split F-algebra $\mathrm{A}(\mathrm{E}) = \mathrm{End}_{\mathrm{F}}(\mathrm{E})$, the critical exponent of which we denote by:

$$k_{\mathrm{F}}(\beta) = k_0(\beta, \Lambda(\mathrm{E})).$$

This is an integer greater than or equal to $-n_{\mathrm{F}}(\beta)$. In the case where this integer is equal to $-n_{\mathrm{F}}(\beta)$, the element β is said to be *minimal* over F. Let us recall the definition of a simple pair over F (see [6, Definition 1.5]).

Definition 1.4. — A *simple pair* over F is a pair (k, β) consisting of a non-zero element β of some finite extension of F and an integer $0 \leqslant k \leqslant -k_{\mathrm{F}}(\beta)-1$.

Given a simple pair (k, β) over F, there is the simple stratum $[\Lambda(\mathrm{E}), n_{\mathrm{F}}(\beta), k, \beta]$ in $\mathrm{A}(\mathrm{E})$ together with a compact open subgroup of $\mathrm{A}(\mathrm{E})^{\times}$ and a set of simple characters:

$$\mathrm{H}^{k+1}_{\mathrm{F}}(\beta) = \mathrm{H}^{k+1}(\beta, \Lambda(\mathrm{E})), \quad \mathcal{C}_{\mathrm{F}}(k, \beta) = \mathcal{C}(\Lambda(\mathrm{E}), k, \beta).$$

Now let A be a simple central F-algebra and V be a simple left A-module. A *realization* of (k, β) in A is a stratum in A of the form $[\Lambda, n, m, \varphi(\beta)]$ made of:

(1) a homomorphism φ of F-algebra from $\mathrm{F}(\beta)$ to A;

(2) an $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ on V normalized by the image of $\mathrm{F}(\beta)^{\times}$ under φ;

(3) an integer m such that $\lfloor m/e_{\varphi(\beta)}(\Lambda) \rfloor = k$.

The integer $-n$ is then the Λ-valuation of $\varphi(\beta)$. By [20, Proposition 2.25] we have:

$$k_0(\varphi(\beta), \Lambda) = e_{\varphi(\beta)}(\Lambda) k_{\mathrm{F}}(\beta), \tag{1.4}$$

which implies that any realization of a simple pair is a simple stratum. According to [20] again (*ibid.*, paragraph 3.3), for such a realization there is a canonical bijective map:

$$\boldsymbol{\tau}_{\Lambda,m,\varphi} : \mathcal{C}_{\mathrm{F}}(k, \beta) \to \mathcal{C}(\Lambda, m, \varphi(\beta)) \tag{1.5}$$

called the *transfer* map. Some of its properties have been studied in [24] and some further properties will be given in sections 6 and 7 of the present article. Given another realization $[\Lambda', n', m', \varphi'(\beta)]$ of the pair (k, β) in some simple central F-algebra A', we have a transfer map from $\mathcal{C}(\Lambda, m, \varphi(\beta))$ to $\mathcal{C}(\Lambda', m', \varphi'(\beta))$ by composing $\boldsymbol{\tau}^{-1}_{\Lambda,m,\varphi}$ with $\boldsymbol{\tau}_{\Lambda',m',\varphi'}$.

Associated with (k, β) is the set $\mathfrak{C}_{(k,\beta)}$ of all pairs $([\Lambda, n, m, \varphi(\beta)], \theta)$ made of a realization $[\Lambda, n, m, \varphi(\beta)]$ of (k, β) in a simple central F-algebra and a simple character $\theta \in \mathcal{C}(\Lambda, m, \varphi(\beta))$. Hence the surjective map:

$$([\Lambda, n, m, \varphi(\beta)], \theta) \mapsto \boldsymbol{\tau}^{-1}_{\Lambda,m,\varphi}(\theta) \in \mathcal{C}_{\mathrm{F}}(k, \beta)$$

is well defined on $\mathfrak{C}_{(k,\beta)}$ and induces, by its fibers, an equivalence relation on it.

Definition 1.5. — A *potential simple character* over F (or *ps-character* for short) is a triple (Θ, k, β) made of a simple pair (k, β) over F and an equivalence class Θ in $\mathfrak{C}_{(k,\beta)}$.

When the context is clear, we will often denote by Θ the ps-character (Θ, k, β). Given a realization $[\Lambda, n, m, \varphi(\beta)]$ of (k, β), we will denote by $\Theta(\Lambda, m, \varphi)$ the simple character θ such that the pair $([\Lambda, n, m, \varphi(\beta)], \theta)$ belongs to Θ.

1.3. We now state the main results which are proved in this article. Our first task is to extend the notion of endo-equivalence of simple pairs developed by Bushnell and Henniart in [6]. More precisely, we extend it to realizations in non-necessarily split simple central F-algebras with non-necessarily strict lattice sequences.

Definition 1.6. — For $i = 1, 2$, let (k_i, β_i) be a simple pair over F. We say that these pairs are *endo-equivalent*, denoted:

$$(k_1, \beta_1) \approx (k_2, \beta_2),$$

if $k_1 = k_2$ and $[\mathrm{F}(\beta_1) : \mathrm{F}] = [\mathrm{F}(\beta_2) : \mathrm{F}]$, and if there exists a simple central F-algebra A together with realizations $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ of (k_i, β_i) in A, with $i = 1, 2$, which intertwine in A.

Recall that two strata $[\Lambda, n_i, m_i, \beta_i]$ in A, with $i \in \{1, 2\}$, *intertwine* in A if there exists $g \in \mathrm{A}^\times$ such that:

$$(\beta_1 + \mathfrak{P}_{-m_1}(\Lambda)) \cap g(\beta_2 + \mathfrak{P}_{-m_2}(\Lambda))g^{-1} \neq \varnothing. \tag{1.6}$$

As we will see in paragraph 2.5 (see Corollary 2.9), this definition of endo-equivalence of simple pairs is equivalent to [6, Definition 1.14], although more general in appearance.

We now investigate the intertwining relations among various realizations of given simple pairs, and in particular their preservation properties. Our first result is the following proposition, which generalizes [6, Proposition 1.10] and is proved in paragraph 2.6.

Proposition 1.7. — *For $i = 1, 2$, let (k, β_i) be a simple pair over* F*, and suppose these pairs are endo-equivalent. Let* A *be a simple central* F*-algebra and let $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ be a realization of (k, β_i) in* A*, for $i = 1, 2$. These strata then intertwine in* A.

Broussous and Grabitz remarked in [5] that two simple strata $[\Lambda, n, m, \beta_i]$, $i = 1, 2$, in A which intertwine in A may be not conjugate under $\mathrm{A}^\times$, unlike the case where A is split (see [10, Theorem 2.6.1] for the case where A is split and Λ is strict). In order to remedy this, they introduced the notion of an embedding type (see also Fröhlich [15]). Here we extend this notion to non-necessarily strict lattice sequences.

We fix a simple central F-algebra A and a simple left A-module V as in paragraph 1.1. Associated with it, we have an F-division algebra D. An *embedding* in A is a pair (E, Λ) made of a finite extension E of F contained in A and an E-pure $\mathcal{O}_\mathrm{D}$-lattice sequence Λ on V. Given such a pair, we denote by $\mathrm{E}^\diamond$ the maximal finite unramified extension of F which is contained in E and whose degree divides the reduced degree of D over F.

Two embeddings $(\mathrm{E}_i, \Lambda_i)$, $i = 1, 2$, in A are said to be *equivalent* in A if there exists an element $g \in \mathrm{A}^\times$ such that Λ_1 is in the translation class of $g\Lambda_2$ and $\mathrm{E}_1^\diamond = g\mathrm{E}_2^\diamond g^{-1}$. This defines an equivalence relation on the set of embeddings in A, and an equivalence class for this relation is called an *embedding type* in A.

Definition 1.8. — The *embedding type* of a pure stratum $[\Lambda, n, m, \beta]$ is the embedding type of the pair $(\mathrm{F}(\beta), \Lambda)$ in A.

This allows us to state the following "intertwining implies conjugacy" theorem, which generalizes [10, Theorem 2.6.1] and [5, Proposition 4.1.2] and is proved in paragraph 3.3.

Proposition 1.9. — For $i = 1, 2$, let $[\Lambda, n, m, \beta_i]$ be a simple stratum in A. *Assume that they intertwine in* A *and have the same embedding type. Write* K_i *for the maximal unramified extension of* F *contained in* $\mathrm{F}(\beta_i)$. *Then there is $u \in \mathfrak{K}(\Lambda)$ such that* $\mathrm{K}_1 = u\mathrm{K}_2u^{-1}$ *and* $\beta_1 - u\beta_2u^{-1} \in \mathfrak{P}_{-m}(\Lambda)$.

1.4. We now extend the notion of endo-equivalence of simple characters developed by Bushnell and Henniart in [6]. As for simple pairs, we extend it to realizations in non-necessarily split simple central F-algebras with non-necessarily strict lattice sequences.

Definition 1.10. — For $i = 1, 2$, let (Θ_i, k_i, β_i) be a ps-character over F. We say that these ps-characters are *endo-equivalent*, denoted:

$$\Theta_1 \approx \Theta_2,$$

if $k_1 = k_2$ and $[\mathrm{F}(\beta_1) : \mathrm{F}] = [\mathrm{F}(\beta_2) : \mathrm{F}]$, and if there exists a simple central F-algebra A together with realizations $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ of (k_i, β_i) in A, with $i = 1, 2$, such that the simple characters $\Theta_1(\Lambda, m_1, \varphi_1)$ and $\Theta_2(\Lambda, m_2, \varphi_2)$ intertwine in $\mathrm{A}^\times$.

Recall that two simple characters $\theta_i \in \mathcal{C}(\Lambda, m_i, \beta_i)$, $i = 1, 2$, *intertwine* in $\mathrm{A}^\times$ if there exists $g \in \mathrm{A}^\times$ such that:

$$\theta_2(x) = \theta_1(gxg^{-1}), \quad x \in \mathrm{H}^{m_2+1}(\beta_2, \Lambda) \cap g^{-1}\mathrm{H}^{m_1+1}(\beta_1, \Lambda)g. \tag{1.7}$$

As we will see at the end of this article (see Corollary 8.2), this definition of endo-equivalence of simple characters is equivalent to [6, Definition 8.6].

We now state the main results of this article concerning properties of simple characters with respect to intertwining and conjugacy. The following generalizes [6, Theorem 8.7].

Theorem 1.11. — *For $i = 1, 2$, let (Θ_i, k_i, β_i) be a ps-character over* F*, and suppose that $\Theta_1 \approx \Theta_2$. Let* A *be a simple central* F*-algebra and, for $i = 1, 2$, let $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ be realizations of (k_i, β_i) in* A*. Then $\Theta_1(\Lambda, m_1, \varphi_1)$ and $\Theta_2(\Lambda, m_2, \varphi_2)$ intertwine in $\mathrm{A}^\times$.*

The following "intertwining implies conjugacy" theorem for simple characters generalizes [10, Theorem 3.5.11] and [17, Corollary 10.15] to simple characters in non-necessarily split simple central F-algebras with non-necessarily strict lattice sequences.

Theorem 1.12. — *Let* A *be a simple central* F*-algebra. For $i = 1, 2$, let $[\Lambda, n, m, \beta_i]$ be a simple stratum in* A*, and let $\theta_i \in \mathcal{C}(\Lambda, m, \beta_i)$ be a simple character. Write K_i for the maximal unramified extension of* F *contained in $\mathrm{F}(\beta_i)$. Assume that θ_1 and θ_2 intertwine in $\mathrm{A}^\times$ and that the strata $[\Lambda, n, m, \beta_i]$ have the same embedding type. Then there is an element $u \in \mathfrak{K}(\Lambda)$ such that:*

(1) $\mathrm{K}_1 = u\mathrm{K}_2u^{-1}$;
(2) $\mathcal{C}(\Lambda, m, \beta_1) = \mathcal{C}(\Lambda, m, u\beta_2u^{-1})$;
(3) $\theta_2(x) = \theta_1(uxu^{-1})$, *for all* $x \in \mathrm{H}^{m+1}(\beta_2, \Lambda) = u^{-1}\mathrm{H}^{m+1}(\beta_1, \Lambda)u$.

We will see in section 4 (see Corollary 4.8) that the proofs of these two theorems can be reduced to that of the following statement, which will be proved in section 8.

Theorem 1.13. — *For $i = 1, 2$, let (Θ_i, k_i, β_i) be a ps-character over* F*, and suppose that we have $\Theta_1 \approx \Theta_2$. Let* A *be a simple central* F*-algebra, and let $[\Lambda, n, m, \varphi_i(\beta_i)]$ be realizations of (k_i, β_i) in* A*, for $i = 1, 2$. Write K_i for the maximal unramified extension of* F *contained in $\mathrm{F}(\beta_i)$ and θ_i for the simple character $\Theta_i(\Lambda, m, \varphi_i)$. Assume these strata have the same embedding type. Then there is an element $u \in \mathfrak{K}(\Lambda)$ such that:*

(1) $\varphi_1(\mathrm{K}_1) = u\varphi_2(\mathrm{K}_2)u^{-1}$;
(2) $\mathcal{C}(\Lambda, m, \varphi_1(\beta_1)) = \mathcal{C}(\Lambda, m, u\varphi_2(\beta_2)u^{-1})$;
(3) $\mathrm{H}^{m+1}(\varphi_1(\beta_1), \Lambda) = u\mathrm{H}^{m+1}(\varphi_2(\beta_2), \Lambda)u^{-1}$;
(4) $\theta_2(x) = \theta_1(uxu^{-1})$ *for all* $x \in \mathrm{H}^{m+1}(\varphi_2(\beta_2), \Lambda)$.

The main ingredient in this reduction step is Lemma 4.7, which states that the endo-equivalence relation preserves certain numerical invariants attached to a ps-character.

1.5. As has been explained in the introduction, this article makes a large use of the results of Bushnell, Henniart and Kutzko in the split case [6, 10] (see paragraphs 1.3 and 1.4), as well as results of Grabitz [17] which are based on the following definition.

Definition 1.14. — A simple stratum $[\Lambda, n, m, \beta]$ in A is *sound* if Λ is strict, $\mathfrak{A} \cap \mathrm{B}$ is principal and $\mathfrak{K}(\mathfrak{A}) \cap \mathrm{B}^\times = \mathfrak{K}(\mathfrak{A} \cap \mathrm{B})$, where $\mathfrak{A}$ is the hereditary $\mathcal{O}_\mathrm{F}$-order defined by Λ.

More generally, an embedding (E,Λ) in A is *sound* if the conditions of Definition 1.14 are fulfilled with B the centralizer of E in A.

Remark 1.15. — Note that the condition on $\mathfrak{A}\cap\mathrm{B}$ forces $\mathfrak{A}$ to be a principal $\mathcal{O}_{\mathrm{F}}$-order. In the split case, a simple stratum $[\Lambda,n,m,\beta]$ is sound if and only if Λ is strict and $\mathfrak{A}$ is principal.

When Λ is strict, its translation class is entirely determined by the hereditary $\mathcal{O}_{\mathrm{F}}$-order $\mathfrak{A}=\mathfrak{A}(\Lambda)$. In this case, we will sometimes write $(\mathrm{E},\mathfrak{A})$ and $[\mathfrak{A},n,m,\beta]$ rather than (E,Λ) and $[\Lambda,n,m,\beta]$.

In the case where the simple strata $[\Lambda,n,m,\beta_i]$, $i=1,2$, are sound, Grabitz has proved in [17] the "intertwining implies conjugacy" theorem for simple characters (see *ibid.*, Theorem 10.3 and Corollary 10.15). More precisely, he has proved the following result.

Given K/F an unramified extension contained in A, a sound simple stratum $[\Lambda,n,m,\beta]$ in A is K-*special* (see [17, Definition 3.1]) if it is K-pure in the sense of Definition 5.1 and if $(\mathrm{K}(\beta),\mathfrak{A}(\Lambda)\cap\mathrm{C})$ is a sound embedding in C, where C is the centralizer of K in A.

Theorem 1.16 ([17], Theorem 10.3). — *For $i=1,2$, let $[\Lambda,n,m,\beta_i]$ be a sound simple stratum in a simple central* F*-algebra* A *and let $\theta_i\in\mathcal{C}(\Lambda,m,\beta_i)$ be a simple character. Let f be a multiple of the greatest common divisor of $f_{\mathrm{F}}(\beta_1)$ and $f_{\mathrm{F}}(\beta_2)$, and let K_i be an unramified extension of* F *of degree f contained in* A *such that $[\Lambda,n,m,\beta_i]$ is K_i-special. Assume (K_1,Λ) and (K_2,Λ) are equivalent embeddings in* A*, and that θ_1 and θ_2 intertwine in $\mathrm{A}^\times$. Then:*

(1) $e_{\mathrm{F}}(\beta_1)=e_{\mathrm{F}}(\beta_2)$ *and* $f_{\mathrm{F}}(\beta_1)=f_{\mathrm{F}}(\beta_2)$*;*
(2) K_i *contains the maximal unramified extension of* F *contained in* $\mathrm{F}[\beta_i]$.

Moreover, there exists $u\in\mathfrak{K}(\Lambda)$ such that:

(3) $\mathrm{K}_1=u\mathrm{K}_2u^{-1}$*;*
(4) $\mathcal{C}(\Lambda,m,\beta_1)=\mathcal{C}(\Lambda,m,u\beta_2u^{-1})$*;*
(5) $\theta_2(x)=\theta_1(uxu^{-1})$*, for all* $x\in\mathrm{H}^{m+1}(\beta_2,\Lambda)=u^{-1}\mathrm{H}^{m+1}(\beta_1,\Lambda)u$.

We will also need the following result.

Proposition 1.17 ([17], Propositions 9.1, 9.9). — *For $i=1,2$, let $[\Lambda,n,m,\beta_i]$ be a sound simple stratum in* A*. Assume that $\mathcal{C}(\Lambda,m,\beta_1)\cap\mathcal{C}(\Lambda,m,\beta_2)\neq\varnothing$. Then $e_{\mathrm{F}}(\beta_1)=e_{\mathrm{F}}(\beta_2)$, $f_{\mathrm{F}}(\beta_1)=f_{\mathrm{F}}(\beta_2)$, $k_{\mathrm{F}}(\beta_1)=k_{\mathrm{F}}(\beta_2)$ and $\mathcal{C}(\Lambda,m,\beta_1)=\mathcal{C}(\Lambda,m,\beta_2)$.*

Note that [17, Proposition 9.1] gives us an equality between $[\mathrm{F}(\beta_1):\mathrm{F}]$ and $[\mathrm{F}(\beta_2):\mathrm{F}]$, but the two finer equalities between the ramification indexes and residue class degrees come from Theorem 1.16.

Our proof of Theorem 1.13 in section 8 is decomposed into two steps. The first step consists of treating the case where the extensions $\mathrm{F}(\beta_i)/\mathrm{F}$ are totally ramified, and the second step consists of reducing to the totally ramified case. In section 5, we develop an interior lifting process for simple strata and simple characters with respect to a finite unramified extension K of F, in a way similar to [6] and [17]. Its compatibility with transfer is explored in section 6. This interior lifting is enough to reduce to the totally ramified case. The totally ramified case is more subtle. For this we develop an 'exterior lifting' or unramified base change in section 7.

2. Realizations and intertwining for simple strata

In this section, we introduce various constructions which will be used throughout the paper. More precisely, we describe various processes, preserving intertwining, which associate to a realization of a simple pair in some simple central F-algebra a realization in a (possibly different) simple central F-algebra, with additional properties. This allows us to prove that Definition 1.6 and the definition of endo-equivalence of simple pairs given in [6] are equivalent (see Corollary 2.9), and to prove Proposition 1.7.

2.1. We fix a simple central F-algebra A and a simple left A-module V. We set:

$$\widetilde{\mathrm{A}} = \mathrm{End}_{\mathrm{F}}(\mathrm{V}),$$

which is a split simple central F-algebra in which the algebra A embeds naturally. To any stratum $[\Lambda, n, m, \beta]$ in A we can attach a stratum $[\widetilde{\Lambda}, n, m, \beta]$ in $\widetilde{\mathrm{A}}$, where $\widetilde{\Lambda}$ denotes the $\mathcal{O}_{\mathrm{F}}$-lattice sequence defined by Λ. By [20, Théorème 2.23], this latter stratum is simple if and only the first one is, and in this case they are realizations of the same simple pair over F. Moreover, we have the following result.

Proposition 2.1. — For $i = 1, 2$, let $[\Lambda, n_i, m_i, \beta_i]$ be a simple stratum in A. *Assume they intertwine in* A. *Then the strata $[\widetilde{\Lambda}, n_i, m_i, \beta_i]$ intertwine in* $\widetilde{\mathrm{A}}$.

Proof. — This follows immediately from the definition of intertwining and the fact that the $\mathcal{O}_{\mathrm{F}}$-module $\mathfrak{P}_k(\Lambda)$ is contained in $\mathfrak{P}_k(\widetilde{\Lambda})$ for all $k \in \mathbb{Z}$. □

2.2. Let $[\Lambda, n, m, \beta]$ be a simple stratum in A, which is a realization of a simple pair (k, β) over F. The *affine class* of Λ is the set of all $\mathcal{O}_{\mathrm{D}}$-lattice sequences on V of the form:

$$a\Lambda + b : k \mapsto \Lambda_{\lceil (k-b)/a \rceil}, \tag{2.1}$$

with $a, b \in \mathbb{Z}$ and $a \geqslant 1$. The period of (2.1) is a times the period $e(\Lambda)$ of Λ. Given an integer $l \geqslant 1$, we set $\mathrm{V}' = \mathrm{V} \oplus \cdots \oplus \mathrm{V}$ (l times) and $\mathrm{A}' = \mathrm{End}_{\mathrm{D}}(\mathrm{V}')$, and embed A in A′ diagonally. For each $j \in \{1, \ldots, l\}$, we choose a lattice sequence Λ^j in the affine class of Λ, and assume the periods of the Λ^j's are all equal to a common integer $ae(\Lambda)$ with $a \geqslant 1$. We now form the $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ' on V′ defined by:

$$\Lambda' = \Lambda^1 \oplus \Lambda^2 \oplus \cdots \oplus \Lambda^l, \tag{2.2}$$

and fix a non-negative integer m' such that:

$$\lfloor m'/a \rfloor = m.$$

If we set $n' = an$, this gives us a simple stratum $[\Lambda', n', m', \beta]$ in A', which is a realization of the simple pair (k, β). In the particular case where $l = 1$, we have the following result.

Lemma 2.2. — *Assume that $l = 1$, so that Λ' is in the affine class of Λ. Then we have $\mathrm{H}^{m'+1}(\beta, \Lambda') = \mathrm{H}^{m+1}(\beta, \Lambda)$ and $\mathcal{C}(\Lambda', m', \beta) = \mathcal{C}(\Lambda, m, \beta)$. Moreover, the transfer map from $\mathcal{C}(\Lambda', m', \beta)$ to $\mathcal{C}(\Lambda, m, \beta)$ is the identity map.*

Proof. — The first assertion is straightforward by induction on β (that is, on the integer $k_{\mathrm{F}}(\beta)$ defined in paragraph 1.2). For the second one, see [24, Théorème 2.13]. $\square$

2.3. Assume now we are given two simple strata $[\Lambda, n_i, m_i, \beta_i]$, $i = 1, 2$, in A. For each i, we set $n'_i = an_i$ and fix a non-negative integer m'_i such that we have $\lfloor m'_i/a \rfloor = m_i$, so that we have a simple stratum $[\Lambda', n'_i, m'_i, \beta_i]$ in A'.

Proposition 2.3. — *Assume that the strata $[\Lambda, n_i, m_i, \beta_i]$, $i \in \{1, 2\}$, intertwine in* A. *Then the strata $[\Lambda', n'_i, m'_i, \beta_i]$, $i \in \{1, 2\}$, intertwine in* A'.

Proof. — We start with an element $g \in \mathrm{A}^\times$ which intertwines the two strata $[\Lambda, n_i, m_i, \beta_i]$, that is, which satisfies the condition (1.6), and we let ι denote the diagonal embedding of A in A' (which we omit from the notation when the context is clear). For $j \in \{1, \ldots, l\}$, write V^j for the jth copy of V in $\mathrm{V}' = \mathrm{V} \oplus \cdots \oplus \mathrm{V}$. Then for each i, we have:

$$\mathfrak{P}_{-m'_i}(\Lambda') \cap \mathrm{End}_{\mathrm{D}}(\mathrm{V}^j) = \mathfrak{P}_{-m'_i}(\Lambda^j), \quad j \in \{1, \ldots, l\},$$

which is equal to $\mathfrak{P}_{-m_i}(\Lambda)$ as can be seen by a direct computation in the case $l = 1$. This implies that ι induces an $\mathcal{O}_{\mathrm{F}}$-module embedding of $\mathfrak{P}_{-m_i}(\Lambda)$ in $\mathfrak{P}_{-m'_i}(\Lambda')$, from which we deduce that $g' = \iota(g) \in \mathrm{A}'^\times$ intertwines the strata $[\Lambda', n'_i, m'_i, \beta_i]$. $\square$

Remark 2.4. — Note that ι induces a group homomorphism of $\mathfrak{K}(\Lambda)$ into $\mathfrak{K}(\Lambda')$. Therefore, if $g \in \mathfrak{K}(\Lambda)$ intertwines two simple strata $[\Lambda, n, m, \beta_i]$, $i = 1, 2$, that is, if we have:

$$\beta_2 - g\beta_1 g^{-1} \in \mathfrak{P}_{-m}(\Lambda),$$

and if we set $n' = an$ and fix a non-negative integer m' such that $\lfloor m'/a \rfloor = m$, then the element $\iota(g) \in \mathfrak{K}(\Lambda')$ intertwines the strata $[\Lambda', n', m', \beta_i]$.

Proposition 2.5. — *Assume that the strata $[\Lambda, n_i, m_i, \beta_i]$, for $i \in \{1, 2\}$, have the same embedding type. Then the strata $[\Lambda', n'_i, m'_i, \beta_i]$, $i \in \{1, 2\}$, have the same embedding type.*

Proof. — Given $g \in \mathfrak{K}(\Lambda)$ which conjugates the unramified extensions $\mathrm{F}(\beta_i)^\diamond$, $i \in \{1, 2\}$, then $\iota(g) \in \mathfrak{K}(\Lambda')$ conjugates the extensions $\mathrm{F}(\iota(\beta_i))^\diamond$, $i \in \{1, 2\}$. $\square$

2.4. For $i=1,2$, let θ_i be a simple character in $\mathcal{C}(\Lambda, m_i, \beta_i)$, and let θ'_i be its transfer in $\mathcal{C}(\Lambda', m'_i, \beta_i)$. The following result is an analogue of Proposition 2.3 for simple characters.

Proposition 2.6. — Assume that θ_1 and θ_2 intertwine in $\mathrm{A}^\times$. Then θ'_1 and θ'_2 intertwine in $\mathrm{A}'^\times$.

Proof. — The decomposition of V' into a sum of copies of V defines a Levi subgroup:

$$\mathrm{M} = \mathrm{A}^\times \times \cdots \times \mathrm{A}^\times \tag{2.3}$$

of $\mathrm{A}'^\times$. We fix a parabolic subgroup P of $\mathrm{A}^\times$ with Levi factor M and unipotent radical N, and we write N^- for the unipotent radical of the parabolic subgroup of $\mathrm{A}^\times$ opposite to P with respect to M. According to [24, Théorème 2.17], we have an Iwahori decomposition:

$$\mathrm{H}^{m'_i+1}(\beta_i,\Lambda') = \big(\mathrm{H}^{m'_i+1}(\beta_i,\Lambda')\cap\mathrm{N}^-\big)\big(\mathrm{H}^{m'_i+1}(\beta_i,\Lambda')\cap\mathrm{M}\big)\big(\mathrm{H}^{m'_i+1}(\beta_i,\Lambda')\cap\mathrm{N}\big),$$
$$\mathrm{H}^{m'_i+1}(\beta_i,\Lambda')\cap\mathrm{M} = \mathrm{H}^{m_i+1}(\beta_i,\Lambda)\times\cdots\times\mathrm{H}^{m_i+1}(\beta_i,\Lambda)$$

for each integer $i=1,2$. We have the following result.

Lemma 2.7. — The simple character θ'_i is trivial on $\mathrm{H}^{m'_i+1}(\beta_i,\Lambda')\cap\mathrm{N}$ and $\mathrm{H}^{m'_i+1}(\beta_i,\Lambda')\cap\mathrm{N}^-$, and we have:

$$\theta'_i \mid \mathrm{H}^{m'_i+1}(\beta_i,\Lambda')\cap\mathrm{M} = \theta_i\otimes\cdots\otimes\theta_i.$$

Proof. — This derives from [24, Théorème 2.17]. Indeed, for $j\in\{1,\dots,l\}$, the restriction of θ'_i to $\mathrm{H}^{m'_i+1}(\beta_i,\Lambda')\cap\mathrm{Aut}_{\mathrm{D}}(\mathrm{V}^j)$ is the transfer of θ'_i to $\mathcal{C}(\Lambda^j, m'_i, \beta_i)$, which is equal to θ_i by Lemma 2.2. ☐

Now let $g\in\mathrm{A}^\times$ intertwine θ_1 and θ_2 as in (1.7), and set $g'=\iota(g)\in\mathrm{M}$. If we write $\mathrm{H}_i=\mathrm{H}^{m_i+1}(\beta_i,\Lambda)$ and $\mathrm{H}'_i=\mathrm{H}^{m'_i+1}(\beta_i,\Lambda')$ for each integer $i\in\{1,2\}$, we get an Iwahori decomposition:

$$\mathrm{H}'_2\cap g'^{-1}\mathrm{H}'_1g' = \big(\mathrm{H}'_2\cap g'^{-1}\mathrm{H}'_1g'\cap\mathrm{N}^-\big)\big(\mathrm{H}'_2\cap g'^{-1}\mathrm{H}'_1g'\cap\mathrm{M}\big)\big(\mathrm{H}'_2\cap g'^{-1}\mathrm{H}'_1g'\cap\mathrm{N}\big),$$
$$\mathrm{H}'_2\cap g'^{-1}\mathrm{H}'_1g'\cap\mathrm{M} = \big(\mathrm{H}_2\cap g^{-1}\mathrm{H}_1g\big)\times\cdots\times\big(\mathrm{H}_2\cap g^{-1}\mathrm{H}_1g\big).$$

According to Lemma 2.7, the simple characters θ'_1 and θ'_2 are trivial on the two subgroups $\mathrm{H}'_2\cap g'^{-1}\mathrm{H}'_1g'\cap\mathrm{N}$ and $\mathrm{H}'_2\cap g'^{-1}\mathrm{H}'_1g'\cap\mathrm{N}^-$, and we have:

$$\theta'_i \mid \mathrm{H}'_2\cap g'^{-1}\mathrm{H}'_1g'\cap\mathrm{M} = \big(\theta_i \mid \mathrm{H}_2\cap g^{-1}\mathrm{H}_1g\big)\otimes\cdots\otimes\big(\theta_i \mid \mathrm{H}_2\cap g^{-1}\mathrm{H}_1g\big)$$

for each $i\in\{1,2\}$. This ensures that g' intertwines the simple characters θ'_1 and θ'_2. ☐

2.5. We give here an example which will be of particular interest for us. Let $[\Lambda, n, m, \beta]$ be a simple stratum in A, which is a realization of a simple pair (k, β) over F, and let e denote the period of Λ over $\mathcal{O}_\mathrm{D}$. We set:

$$\Lambda^\dagger : k \mapsto \Lambda_k \oplus \Lambda_{k+1} \oplus \cdots \oplus \Lambda_{k+e-1}, \tag{2.4}$$

which is a strict $\mathcal{O}_\mathrm{D}$-lattice sequence on $\mathrm{V}^\dagger = \mathrm{V} \oplus \cdots \oplus \mathrm{V}$ (e times) of the form (2.2). Thus we can form the simple stratum $[\Lambda^\dagger, n, m, \beta]$ in $\mathrm{A}^\dagger = \mathrm{End}_\mathrm{D}(\mathrm{V}^\dagger)$, which is a realization of (k, β). Moreover, the hereditary $\mathcal{O}_\mathrm{F}$-order $\mathfrak{A}^\dagger$ defined by $\Lambda^\dagger$ is principal, and we have the following result, which derives from Propositions 2.3 and 2.5.

Proposition 2.8. — For $i = 1, 2$, let $[\Lambda, n_i, m_i, \beta_i]$ be a simple stratum in A*. Assume they intertwine in* A *(resp. have the same embedding type). Then the strata $[\Lambda^\dagger, n_i, m_i, \beta_i]$ intertwine in $\mathrm{A}^\dagger$ (resp. have the same embedding type).*

Note that the operations $\Lambda \mapsto \widetilde{\Lambda}$ (see paragraph 2.1) and $\Lambda \mapsto \Lambda^\dagger$ commute, so that there is no ambiguity in writing $\widetilde{\Lambda}^\dagger$ for the strict $\mathcal{O}_\mathrm{F}$-lattice sequence defined by $\Lambda^\dagger$.

Corollary 2.9. — Definition 1.6 is equivalent to Definition [6, 1.14].

Proof. — Assume we are given two simple pairs (k, β_i), $i = 1, 2$, which are endo-equivalent in the sense of Definition 1.6. Then we have $[\mathrm{F}(\beta_1) : \mathrm{F}] = [\mathrm{F}(\beta_2) : \mathrm{F}]$, and there exists a simple central F-algebra A together with two realizations $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ of (k, β_i) in A, with $i = 1, 2$, which intertwine in A. By replacing A and Λ by $\widetilde{\mathrm{A}}^\dagger$ and $\widetilde{\Lambda}^\dagger$, we have realizations $[\widetilde{\Lambda}^\dagger, n_i, m_i, \varphi_i(\beta_i)]$ of (k, β_i) in $\widetilde{\mathrm{A}}^\dagger$, with $i = 1, 2$, and these realizations intertwine in $\widetilde{\mathrm{A}}^\dagger$ according to Propositions 2.1 and 2.8. Thus the simple pairs (k, β_1) and (k, β_2) are endo-equivalent in the sense of [6, Definition 1.14]. Conversely, two simple pairs which are endo-equivalent in this sense are clearly endo-equivalent in the sense of Definition 1.6. □

2.6. We now prove the preservation property of intertwining for simple strata, that is, Proposition 1.7. We first prove that the endo-equivalence relation preserves certain numerical invariants attached to simple pairs. Compare the following proposition with [6], Property (1.15). See paragraph 1.2 for the notation.

Proposition 2.10. — For $i = 1, 2$, let (k, β_i) be a simple pair over F*, and suppose that (k, β_1) and (k, β_2) are endo-equivalent. Then we have $n_\mathrm{F}(\beta_1) = n_\mathrm{F}(\beta_2)$, $e_\mathrm{F}(\beta_1) = e_\mathrm{F}(\beta_2)$, $f_\mathrm{F}(\beta_1) = f_\mathrm{F}(\beta_2)$ and $k_\mathrm{F}(\beta_1) = k_\mathrm{F}(\beta_2)$.*

Proof. — By Corollary 2.9, we may assume that the pairs (k, β_i) are endo-equivalent in the sense of [6]. The result follows from [6, Proposition 1.10]. □

For $i = 1, 2$, let (k, β_i) be a simple pair over F, and suppose that $(k, \beta_1) \approx (k, \beta_2)$.

Let A be a simple central F-algebra and, for $i = 1, 2$, let $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ be a realization of (k, β_i) in A. Let V denote the simple left A-module on which Λ is a lattice sequence and write D for the F-algebra opposite to $\mathrm{End}_{\mathrm{A}}(\mathrm{V})$. For $i = 1, 2$, let E_i denote the F-algebra $\mathrm{F}(\beta_i)$. We fix a simple right $\mathrm{E}_1 \otimes_{\mathrm{F}} \mathrm{D}$-module S and set $\mathrm{A(S)} = \mathrm{End}_{\mathrm{D}}(\mathrm{S})$, and we denote by ρ_1 the natural F-algebra homomorphism $\mathrm{E}_1 \to \mathrm{A(S)}$. Let $\mathfrak{S}$ denote the unique (up to translation) E_1-pure strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence on S.

Lemma 2.11. — There is a homomorphism of F*-algebras* $\rho_2 : \mathrm{E}_2 \to \mathrm{A(S)}$ *such that* $\mathfrak{S}$ *is* $\rho_2(\mathrm{E}_2)$*-pure, and such that the pairs* $(\rho_1(\mathrm{E}_1), \mathfrak{S})$ *and* $(\rho_2(\mathrm{E}_2), \mathfrak{S})$ *have the same embedding type in* A(S) *(see paragraph 1.3).*

Proof. — As (k, β_1) and (k, β_2) are endo-equivalent, Proposition 2.10 gives us the equalities $e_{\mathrm{F}}(\beta_1) = e_{\mathrm{F}}(\beta_2)$ and $f_{\mathrm{F}}(\beta_1) = f_{\mathrm{F}}(\beta_2)$. The result follows from [5, Corollary 3.16]. □

Remark 2.12. — We actually have a stronger result: for any F-algebra homomorphism ρ_2 such that $\mathfrak{S}$ is $\rho_2(\mathrm{E}_2)$-pure, the pairs $(\rho_1(\mathrm{E}_1), \mathfrak{S})$ and $(\rho_2(\mathrm{E}_2), \mathfrak{S})$ have the same embedding type in A(S). Indeed, if ρ_2 is such a homomorphism and if η_2 is an F-algebra homomorphism as in Lemma 2.11, the Skolem-Noether theorem gives us $g \in \mathrm{A(S)}^{\times}$ which conjugates these F-algebra homomorphisms ρ_2 and η_2. As E_1 and E_2 have the same degree over F, the lattice sequence $\mathfrak{S}$ is the unique (up to translation) $\rho_2(\mathrm{E}_2)$-pure strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence — and also the unique (up to translation) $\eta_2(\mathrm{E}_2)$-pure strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence — on S. It follows that g normalizes the lattice sequence $\mathfrak{S}$ and that the pairs $(\rho_2(\mathrm{E}_2), \mathfrak{S})$ and $(\eta_2(\mathrm{E}_2), \mathfrak{S})$ have the same embedding type in A(S).

Let us fix an F-algebra homomorphism ρ_2 as in Lemma 2.11. As (k, β_1) and (k, β_2) are endo-equivalent, we have $n_{\mathrm{F}}(\beta_1) = n_{\mathrm{F}}(\beta_2)$ and $e_{\mathrm{F}}(\beta_1) = e_{\mathrm{F}}(\beta_2)$, so that the $\mathfrak{S}$-valuation of $\rho_i(\beta_i)$, denoted n_0, and the period $e_{\rho_i(\beta_i)}(\mathfrak{S})$ do not depend on $i \in \{1, 2\}$. We set:

$$m_0 = e_{\rho_i(\beta_i)}(\mathfrak{S})k.$$

For each $i \in \{1, 2\}$, we have a stratum $[\mathfrak{S}, n_0, m_0, \rho_i(\beta_i)]$, which is a realization of (k, β_i) in A(S). By paragraph 2.1, we have a realization $[\widetilde{\mathfrak{S}}, n_0, m_0, \rho_i(\beta_i)]$ of (k, β_i) in the split simple central F-algebra $\mathrm{End}_{\mathrm{F}}(\mathrm{S})$, and the $\mathcal{O}_{\mathrm{F}}$-lattice sequence $\widetilde{\mathfrak{S}}$ is strict. Hence we can apply [6, Proposition 1.10], which implies that these realizations, for $i = 1, 2$, intertwine in $\mathrm{End}_{\mathrm{F}}(\mathrm{S})$. By our assumption (see Lemma 2.11), the strata $[\mathfrak{S}, n_0, m_0, \rho_i(\beta_i)]$, for $i = 1, 2$, have the same embedding type. Here we need to recall the following statement, due to Broussous and Grabitz.

Proposition 2.13 ([5], Proposition 4.1.3). — *For* $i = 1, 2$, *let* $[\Sigma, n, m, \gamma_i]$ *be a simple stratum in a simple central* F*-algebra* U*, where* Σ *is strict. Assume that they have the same embedding type, and that the strata* $[\widetilde{\Sigma}, n, m, \gamma_i]$ *intertwine in* $\widetilde{\mathrm{U}}$*. Then there exists an element* $u \in \mathfrak{K}(\Sigma)$ *such that* $\gamma_1 - u\gamma_2 u^{-1} \in \mathfrak{P}_{-m}(\Sigma)$*.*

Moreover, u can be chosen such that the maximal unramified extension of F *contained in* $\mathrm{F}(\gamma_1)$ *is equal to that of* $\mathrm{F}(u\gamma_2 u^{-1})$.

We deduce from Proposition 2.13 that there exists an element $g \in \mathfrak{K}(\mathfrak{S})$ such that:

$$\rho_1(\beta_1) - g\rho_2(\beta_2)g^{-1} \in \mathfrak{P}_{-m_0}(\mathfrak{S}). \tag{2.5}$$

We now fix a decomposition:

$$\mathrm{V} = \mathrm{V}^1 \oplus \cdots \oplus \mathrm{V}^l$$

of V into simple right $\mathrm{E}_1 \otimes_{\mathrm{F}} \mathrm{D}$-modules (which all are copies of S) such that the lattice sequence Λ decomposes into the direct sum of the $\Lambda^j = \Lambda \cap \mathrm{V}^j$, for $j \in \{1, \dots, l\}$.

Lemma 2.14. — There are isomorphisms of $\mathrm{E}_1 \otimes_{\mathrm{F}} \mathrm{D}$*-modules* $\mathrm{V}^j \to \mathrm{S}$*,* $j \in \{1, \dots, l\}$*, such that the resulting* F*-algebra homomorphism* $\iota : \mathrm{A(S)} \to \mathrm{A}$ *satisfies* $\iota \circ \rho_1 = \varphi_1$.

Proof. — Since each V^j, for $j \in \{1, \dots, l\}$, is an E_1-vector subspace of V, the F-algebra homomorphism φ_1 has the form $x \mapsto (\omega_1(x), \dots, \omega_l(x))$, where ω_j is an F-algebra homomorphism from E_1 to $\mathrm{End}_{\mathrm{D}}(\mathrm{V}^j)$. By the Skolem-Noether theorem, one can choose, for each integer j, a suitable $\mathrm{E}_1 \otimes_{\mathrm{F}} \mathrm{D}$-module isomorphism between V^j and S such that the resulting F-algebra homomorphism π_j between $\mathrm{End}_{\mathrm{D}}(\mathrm{V}^j)$ and A(S) satisfies the condition $\pi_j \circ \omega_j = \rho_1$. Then the F-algebra homomorphism ι defined by $\iota(x) = (\pi_1^{-1}(x), \dots, \pi_l^{-1}(x))$ for $x \in \mathrm{A(S)}$ satisfies the required condition. □

We now fix isomorphisms of $\mathrm{E}_1 \otimes_{\mathrm{F}} \mathrm{D}$-modules $\mathrm{V}^j \to \mathrm{S}$, $j \in \{1, \dots, l\}$, as in Lemma 2.14. Then each Λ^j is in the affine class of $\mathfrak{S}$ (see (2.1) and also [22, §1.4.8]), and these lattice sequences all have the same period, equal to that of Λ. Therefore, we are in the situation of paragraph 2.2. We set:

$$n = n_i, \quad m = e_{\varphi_i(\beta_i)}(\Lambda)k,$$

which both do not depend on $i \in \{1, 2\}$. By (2.5) and Remark 2.4, the element $\iota(g)$ normalizes Λ and conjugates $[\Lambda, n, m, \iota(\rho_2(\beta_2))]$ into a simple stratum in A which is equivalent to $[\Lambda, n, m, \varphi_1(\beta_1)]$. By the Skolem-Noether theorem, there is an element $x \in \mathrm{A}^\times$ which conjugates the F-algebra homomorphisms $\iota \circ \rho_2$ and φ_2, and thus intertwines the simple strata $[\Lambda, n, m, \iota(\rho_2(\beta_2))]$ and $[\Lambda, n, m, \varphi_2(\beta_2)]$. Therefore the strata $[\Lambda, n, m, \varphi_i(\beta_i)]$ intertwine. As $m \leqslant m_1, m_2$, the strata $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ intertwine, which ends the proof of Proposition 1.7.

Remark 2.15. — There is a gap in the proof of the existence of the transfer map given in [20, Théorème 3.53], in the case where Λ is a strict lattice sequence. To complete this proof, one has to prove that, given a non-minimal simple pair (k, β) over F together with a realization $[\Lambda, n, m, \varphi(\beta)]$ of this pair in a simple central F-algebra A, there is a simple pair (k', γ) over F having realizations in A(E) and A which are approximations of β and $\varphi(\beta)$, respectively. More precisely, set $q = -k_0(\beta, \Lambda)$, and start with a stratum $[\Lambda, n, q, \gamma]$ in A which is simple and equivalent to $[\Lambda, n, q, \varphi(\beta)]$. If we denote by (k', γ) the simple pair of which this stratum is a realization, and if we set $n_0 = n_{\mathrm{F}}(\beta)$ and $q_0 = -k_{\mathrm{F}}(\beta)$, then we search for a realization $[\Lambda(\mathrm{E}), n_0, q_0, \varphi_0(\gamma)]$ of (k', γ) in $\mathrm{A}(\mathrm{E})$ which is equivalent to the pure stratum $[\Lambda(\mathrm{E}), n_0, q_0, \beta]$ (see paragraph 1.2). Let us remark that, when passing to $\widetilde{\mathrm{A}}$ (see paragraph 2.1), we get a stratum $[\widetilde{\Lambda}, n, q, \gamma]$ which is simple and equivalent to $[\widetilde{\Lambda}, n, q, \varphi(\beta)]$. Now let $[\Lambda(\mathrm{E}), n_0, q_0, \delta]$ be a stratum in $\mathrm{A}(\mathrm{E})$ which is simple and equivalent to $[\Lambda(\mathrm{E}), n_0, q_0, \beta]$. By choosing a suitable decomposition of the F-vector space V into a direct sum of copies of E, we get an F-embedding:

$$\iota : \mathrm{A}(\mathrm{E}) \to \widetilde{\mathrm{A}},$$

thus a stratum $[\widetilde{\Lambda}, n, q, \iota(\delta)]$ in $\widetilde{\mathrm{A}}$ which is simple and equivalent to $[\widetilde{\Lambda}, n, q, \iota(\beta)]$. By the Skolem-Noether theorem, there is an element $g \in \widetilde{\mathrm{A}}^{\times}$ which conjugates $\iota(\beta)$ and $\varphi(\beta)$, thus intertwines the strata $[\widetilde{\Lambda}, n, q, \gamma]$ and $[\widetilde{\Lambda}, n, q, \iota(\delta)]$. The simple pairs (k', γ) and (k', δ) are thus endo-equivalent. Now let $[\Lambda(\mathrm{E}), n_0, q_0, \jmath(\gamma)]$ be a realization of (k', γ) in $\mathrm{A}(\mathrm{E})$ which intertwines with $[\Lambda(\mathrm{E}), n_0, q_0, \delta]$. By the "intertwining implies conjugacy" theorem [10, Theorem 3.5.11] in the split simple central F-algebra $\mathrm{A}(\mathrm{E})$, there is $g \in \mathrm{U}(\Lambda(\mathrm{E}))$ such that $g\jmath(\gamma)g^{-1} - \delta \in \mathfrak{P}(\Lambda(\mathrm{E}))^{-q_0}$. The homomorphism of F-algebras $\varphi_0 : x \mapsto g\jmath(x)g^{-1}$ has the required property.

2.7. Before closing this section, we give a more elaborate example than that of paragraph 2.5, which will be very useful in the sequel. As in paragraph 2.5, let $[\Lambda, n, m, \beta]$ be a simple stratum in A, which is a realization of a simple pair (k, β) over F, and let e denote the period of Λ over $\mathcal{O}_{\mathrm{D}}$. Write B for the centralizer of the field $\mathrm{E} = \mathrm{F}(\beta)$ in A, fix a simple left B-module V_β and write D_β for the E-algebra opposite to the algebra of B-endomorphisms of V_β. Let Σ denote an $\mathcal{O}_{\mathrm{D}_\beta}$-lattice sequence on V_β corresponding to Λ by (1.1), and let e' denote its period over $\mathcal{O}_{\mathrm{D}_\beta}$. We fix an integer l which is a multiple of e and e' and set:

$$\Lambda^{\ddagger} : k \mapsto \Lambda_k \oplus \Lambda_{k+1} \oplus \cdots \oplus \Lambda_{k+l-1}, \tag{2.6}$$

which is a strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence on $\mathrm{V}^{\ddagger} = \mathrm{V} \oplus \cdots \oplus \mathrm{V}$ (l times) of the form (2.2). Thus we can form the simple stratum $[\Lambda^{\ddagger}, n, m, \beta]$ in $\mathrm{A}^{\ddagger} = \mathrm{End}_{\mathrm{D}}(\mathrm{V}^{\ddagger})$, which is a realization of (k, β). Moreover, the hereditary $\mathcal{O}_{\mathrm{F}}$-order $\mathfrak{A}^{\ddagger}$ defined by $\Lambda^{\ddagger}$ is principal, and we have the following result.

Lemma 2.16. — The stratum $[\Lambda^{\ddagger}, n, m, \beta]$ is sound (see Definition 1.14).

Proof. — Write $\mathrm{B}^{\ddagger}$ for the centralizer of E in $\mathrm{A}^{\ddagger}$ and $\Sigma^{\ddagger}$ for the $\mathcal{O}_{\mathrm{D}_\beta}$-lattice sequence on $\mathrm{V}_\beta \times \cdots \times \mathrm{V}_\beta$ (l times) defined by:

$$\Sigma^{\ddagger} : k \mapsto \Sigma_k \oplus \Sigma_{k+1} \oplus \cdots \oplus \Sigma_{k+l-1}.$$

This is a strict lattice sequence, which defines a principal order of $\mathrm{B}^\ddagger$. By direct computation of each block, we get for all $k \in \mathbb{Z}$:

$$\mathfrak{P}_k(\Lambda^\ddagger) \cap \mathrm{B}^\ddagger = \mathfrak{P}_k(\Sigma^\ddagger), \tag{2.7}$$

which amounts to saying that $\Sigma^\ddagger$ is an $\mathcal{O}_{\mathrm{D}_\beta}$-lattice sequence which corresponds to $\Lambda^\ddagger$ by (1.1). In particular, its $\mathrm{B}^{\ddagger\times}$-normalizer is $\mathfrak{K}(\Lambda^\ddagger) \cap \mathrm{B}^{\ddagger\times}$. As $\Lambda^\ddagger$ is strict, its normalizer is equal to $\mathfrak{K}(\mathfrak{A}^\ddagger)$, and a similar statement holds for the lattice sequence $\Sigma^\ddagger$, so that we have $\mathfrak{K}(\mathfrak{A}^\ddagger) \cap \mathrm{B}^{\ddagger\times} = \mathfrak{K}(\mathfrak{A}^\ddagger \cap \mathrm{B}^\ddagger)$. Finally, if we choose $k = 0$ in (2.7), we deduce that $\mathfrak{A}^\ddagger \cap \mathrm{B}^\ddagger$ is principal. □

Note that, unlike (2.4), the process defined by (2.6) depends on E and l, and not only on the lattice sequence Λ.

Now let $[\Lambda, n_i, m_i, \beta_i]$, for $i = 1, 2$, be simple strata in A. Let e denote the period of Λ over $\mathcal{O}_\mathrm{D}$, and write e'_i for the period of the $\mathcal{O}_{\mathrm{D}_{\beta_i}}$-lattice sequence associated with Λ as above.

Proposition 2.17. — Let $l \geqslant 1$ be a multiple of e'_1, e'_2 and e, and assume that the simple strata $[\Lambda, n_i, m_i, \beta_i]$, $i = 1, 2$, intertwine in A *(resp. have the same embedding type). Then the simple strata $[\Lambda^\ddagger, n_i, m_i, \beta_i]$, $i = 1, 2$, are sound, and intertwine in $\mathrm{A}^\ddagger$ (resp. have the same embedding type).*

Proof. — This derives from Propositions 2.3 and 2.5, and Lemma 2.16. □

3. Intertwining implies conjugacy for simple strata

In this section, we prove the "intertwining implies conjugacy" property for simple strata, that is Proposition 1.9. We fix a simple central F-algebra A and a simple left A-module V as in paragraph 1.1. Associated with it, we have an F-division algebra D.

3.1. We will need the following general lemma on embedding types. Let $\mathscr{B}$ be a D-basis of V, and let L be a maximal unramified extension of F contained in D. The choice of $\mathscr{B}$ defines an isomorphism of F-algebras between A and $\mathrm{M}_r(\mathrm{D})$ for some integer $r \geqslant 1$, which allows us to identify these F-algebras. In particular, we will consider L as an extension of F contained in A. We write I_r for the identity matrix.

An embedding (K, Λ) in A is said to be *standard* with respect to the pair $(\mathscr{B}, \mathrm{L})$ if K is a subfield of L and if Λ is split by the basis $\mathscr{B}$ in the sense of [3].

Lemma 3.1. — Let $(\mathscr{B}, \mathrm{L})$ be a pair as above.

(1) *Any embedding in* A *is equivalent to an embedding which is standard with respect to the pair $(\mathscr{B}, \mathrm{L})$.*

(2) *Let (K, Λ) be standard with respect to $(\mathscr{B}, \mathrm{L})$, and let ϖ be a uniformizer of* D *normalizing* L. *Then conjugation by the diagonal matrix $\varpi \cdot \mathrm{I}_r$ normalizes* K *and Λ, and any element of* $\mathrm{Gal}(\mathrm{K}/\mathrm{F})$ *is induced by conjugation by a power of $\varpi \cdot \mathrm{I}_r$.*

Proof. — Assertion (2) follows from the fact that the map $x \mapsto \varpi x \varpi^{-1}$, for $x \in \mathrm{L}$, is a generator of the group $\mathrm{Gal(L/F)}$. To prove (1), let (E, Λ) be an embedding in A, and set $\mathrm{K} = \mathrm{E}^{\diamond}$ (see paragraph 1.3 for the notation). One first notices that one can conjugate the pair (K, Λ) so that $\mathrm{K} \subseteq \mathrm{L}$, which we will assume. Let $\mathscr{I}$ be the non-enlarged Bruhat-Tits building of $\mathrm{A}^{\times}$ and $\mathscr{I}'$ be that of the centralizer $\mathrm{C}^{\times}$ of $\mathrm{K}^{\times}$ in $\mathrm{A}^{\times}$. Since the group $\mathrm{C}^{\times}$ identifies with $\mathrm{GL}_r(\mathrm{D}')$, where D' is the centralizer of K in D, the two buildings $\mathscr{I}$ and $\mathscr{I}'$ have same dimension $r-1$. Recall (see [3, Théorème II.1.1]) that there exists a unique mapping:

$$\boldsymbol{j} = \boldsymbol{j}_{\mathrm{K/F}} : \mathscr{I}' \to \mathscr{I}$$

which is affine and $\mathrm{C}^{\times}$-equivariant. Its image is the set of $\mathrm{K}^{\times}$-fixed points in $\mathscr{I}$. The basis $\mathscr{B}$ gives rise to an apartment $\mathscr{A}$ of $\mathscr{I}$ (see e.g. [3, §0]), and points in that apartment are fixed by diagonal matrices of $\mathrm{A}^{\times}$ of the form $x \cdot \mathrm{I}_r$, with $x \in \mathrm{D}^{\times}$. In particular, they are fixed by $\mathrm{K}^{\times}$. It easily follows that there is some apartment $\mathscr{A}'$ in $\mathscr{I}'$ such that we have $\mathscr{A} = \boldsymbol{j}(\mathscr{A}')$.
The affine class of Λ determines a point y of the building $\mathscr{I}$ (see [3, I.7]). Since $\mathrm{K}^{\times}$ normalizes Λ, this point writes $\boldsymbol{j}(x)$, for some $x \in \mathscr{I}'$. Since $\mathrm{C}^{\times}$ acts transitively on the set of all apartments of $\mathscr{I}'$, and since any point of $\mathscr{I}'$ is contained in some apartment, there is an element $h \in \mathrm{C}^{\times}$ such that $h \cdot x \in \mathscr{A}'$. Its follows that $h \cdot y = \boldsymbol{j}(h \cdot x)$ lies in $\mathscr{A}$. By [3, Proposition I.2.7], this means that the lattice sequence $h\Lambda$ is split by the basis $\mathscr{B}$, i.e. that $(h\mathrm{K}h^{-1}, h\Lambda) = (\mathrm{K}, h\Lambda)$ is standard with respect to the pair $(\mathscr{B}, \mathrm{L})$, as required. □

Remark 3.2. — We can rephrase Assertion (1) of the above lemma by saying that, for any embedding (E, Λ) in A, there is $g \in \mathrm{A}^{\times}$ such that $(\mathrm{E}^{\diamond}, \Lambda)$ is standard with respect to the pair $(g\mathscr{B}, g\mathrm{L}g^{-1})$.
If one writes $\mathrm{N}_{\mathrm{A}^{\times}}(\mathrm{K})$ for the normalizer of K in $\mathrm{A}^{\times}$, Assertion (2) can also be rephrased by saying that conjugation induces a surjective group homomorphism from the intersection $\mathfrak{K}(\Lambda) \cap \mathrm{N}_{\mathrm{A}^{\times}}(\mathrm{K})$ onto $\mathrm{Gal(K/F)}$. With the notation of the proof of Lemma 3.1, the kernel of this homomorphism is $\mathfrak{K}(\Lambda) \cap \mathrm{C}^{\times}$.

3.2. We will also need the following result, which generalizes [6, Lemma 1.6].

Proposition 3.3. — *Let* Λ *be an* $\mathcal{O}_{\mathrm{D}}$*-lattice sequence on* V *and* E/F *a finite extension. Suppose that there are two homomorphisms* $\varphi_i : \mathrm{E} \to \mathrm{A}$ *of* F*-algebras,* $i = 1, 2$*, such that the pairs* $(\varphi_1(\mathrm{E}), \Lambda)$ *and* $(\varphi_2(\mathrm{E}), \Lambda)$ *are two equivalent embeddings in* A*. Then there is an element* $u \in \mathfrak{K}(\Lambda)$ *such that:*

$$\varphi_1(x) = u\varphi_2(x)u^{-1}, \quad x \in \mathrm{E}. \tag{3.1}$$

Remark 3.4. — In particular, if K denotes the maximal unramified extension of F contained in E, then u conjugates $\varphi_2(\mathrm{K})$ to $\varphi_1(\mathrm{K})$.

Proof. — Since the embeddings $(\varphi_1(\mathrm{E}), \Lambda)$ and $(\varphi_2(\mathrm{E}), \Lambda)$ are equivalent, there exists an element $g \in \mathfrak{K}(\Lambda)$ such that $\varphi_1(\mathrm{E}^{\diamond}) = g\varphi_2(\mathrm{E}^{\diamond})g^{-1}$. Then the mapping:

$$x \mapsto g\varphi_2(\varphi_1^{-1}(x))g^{-1} \tag{3.2}$$

is an F-automorphism of $\varphi_1(\mathrm{E}^\diamond)$. By Lemma 3.1(2), there is $h \in \mathfrak{K}(\Lambda)$ such that this F-automorphism is $x \mapsto hxh^{-1}$. We thus have $\varphi_1(x) = w\varphi_2(x)w^{-1}$, for all $x \in \mathrm{E}^\diamond$, where $w = h^{-1}g$. So replacing φ_2 by a $\mathfrak{K}(\Lambda)$-conjugate, one may reduce to the case where φ_1 and φ_2 coincide on $\mathrm{E}^\diamond$. Assume now that we are is this case, and put $\mathrm{K} = \varphi_2(\mathrm{E}^\diamond)$. Let C be the centralizer of K in A, and write $\mathfrak{C}$ for the intersection of $\mathfrak{A} = \mathfrak{A}(\Lambda)$ with C.

Lemma 3.5. — There is $u \in \mathrm{U}(\mathfrak{C})$ such that (3.1) *holds.*

Proof. — We fix an unramified extension L of K such that the degree of L/F is equal to the reduced degree of D over F, denoted d. The L-algebra $\overline{\mathrm{C}} = \mathrm{C} \otimes_{\mathrm{K}} \mathrm{L}$ is thus split and, as E/K has residue class degree prime to d, the L-algebra $\mathrm{E} \otimes_{\mathrm{K}} \mathrm{L}$ is an extension of L, denoted $\overline{\mathrm{E}}$. For each i, the K-algebra homomorphism φ_i extends to a homomorphism of L-algebras $\overline{\mathrm{E}} \to \overline{\mathrm{C}}$, still denoted φ_i. By applying [6, Lemma 1.6] with the $\mathcal{O}_{\mathrm{L}}$-order $\overline{\mathfrak{C}} = \mathfrak{C} \otimes_{\mathcal{O}_{\mathrm{K}}} \mathcal{O}_{\mathrm{L}}$ and the homomorphisms of L-algebras φ_1 and φ_2, we get $u \in \mathrm{U}(\overline{\mathfrak{C}})$ satisfying (3.1). If we write $\overline{\mathrm{B}}$ for the centralizer of $\varphi_2(\mathrm{E})$ in $\overline{\mathrm{C}}$, then the 1-cocycle $\sigma \mapsto u^{-1}\sigma(u)$ defines a class in the Galois cohomology set:

$$\mathrm{H}^1(\mathrm{Gal}(\mathrm{L}/\mathrm{K}), \mathrm{U}(\overline{\mathfrak{C}}) \cap \overline{\mathrm{B}}^\times).$$

This cohomology set is trivial by a standard filtration argument. (For more detail, see e.g. [5, §6].) Hence we actually may choose u in $\mathrm{U}(\mathfrak{C})$, which ends the proof of the lemma. □

Proposition 3.3 follows immediately from Lemma 3.5. □

Remark 3.6. — The conclusion of Proposition 3.3 does not hold if the pairs $(\varphi_1(\mathrm{E}), \Lambda)$ and $(\varphi_2(\mathrm{E}), \Lambda)$ are not assumed to be equivalent in A. For instance, take $\mathrm{A} = \mathrm{M}_2(\mathrm{D})$ where D is a quaternionic algebra over F, and let E/F be an unramified quadratic extension. One may embed E in $\mathrm{M}_2(\mathrm{F})$ so that the multiplicative group of the image normalizes the order $\mathrm{M}_2(\mathcal{O}_{\mathrm{F}})$. This gives an embedding φ_1 of E in $\mathrm{A} = \mathrm{M}_2(\mathrm{D}) = \mathrm{M}_2(\mathrm{F}) \otimes_{\mathrm{F}} \mathrm{D}$, such that $\varphi_1(\mathrm{E}^\times)$ normalizes $\mathrm{M}_2(\mathcal{O}_{\mathrm{D}}) = \mathrm{M}_2(\mathcal{O}_{\mathrm{F}}) \otimes_{\mathcal{O}_{\mathrm{F}}} \mathcal{O}_{\mathrm{D}}$. One also may embed E in D. The diagonal embedding of D in A gives rise to a second embedding φ_2 such that $\varphi_2(\mathrm{E}^\times)$ normalizes $\mathrm{M}_2(\mathcal{O}_{\mathrm{D}})$. Take Λ to be a strict lattice sequence in $\mathrm{D} \times \mathrm{D}$ defining the order $\mathfrak{A} = \mathrm{M}_2(\mathcal{O}_{\mathrm{D}})$, so that:

$$\mathfrak{K}(\Lambda) = \mathfrak{K}(\mathfrak{A}) = \langle \varpi \rangle \cdot \mathrm{U}(\mathfrak{A}),$$

where ϖ denotes a uniformizer of D and $\langle \varpi \rangle$ the subgroup generated by ϖ. One can check that the pairs $(\varphi_i(\mathrm{E}), \Lambda)$, $i = 1, 2$, are inequivalent. Assume for a contradiction that there is an element $u \in \mathfrak{K}(\mathfrak{A})$ such that $\varphi_1(\mathrm{E}) = u\varphi_2(\mathrm{E})u^{-1}$, and write $\mathfrak{P}$ for the radical of $\mathfrak{A}$. For $i = 1, 2$, the map φ_i induces an embedding of the residue field $\mathfrak{k}_{\mathrm{E}}$ in the $\mathfrak{k}_{\mathrm{F}}$-algebra $\mathfrak{A}/\mathfrak{P}$, which is isomorphic to $\mathrm{M}_2(\mathfrak{k}_{\mathrm{D}})$, and the images $\varphi_i(\mathfrak{k}_{\mathrm{E}})$, $i = 1, 2$, are conjugate under the action of u on the quotient $\mathfrak{A}/\mathfrak{P}$. But this action stabilizes the centre of $\mathrm{M}_2(\mathfrak{k}_{\mathrm{D}})$ and $\varphi_2(\mathfrak{k}_{\mathrm{E}})$ lies in this centre. This implies that $\varphi_1(\mathfrak{k}_{\mathrm{E}})$ is central: a contradiction.

3.3. We now prove the "intertwining implies conjugacy" property for simple strata, that is, Proposition 1.9. For $i = 1, 2$, let $[\Lambda, n, m, \beta_i]$ a simple stratum in A. Assume that they intertwine in A and have the same embedding type, and write K_i for the maximal unramified extension of F contained in $\mathrm{E}_i = \mathrm{F}(\beta_i)$. By Remark 3.4, we may replace β_2 by some $\mathfrak{K}(\Lambda)$-conjugate and assume that K_1 and K_2 are equal to a common extension K of F. We write $\mathrm{N}_{\mathrm{A}^\times}(\mathrm{K})$ for the normalizer of K in $\mathrm{A}^\times$. Therefore, we are reduced to proving that there is an element $u \in \mathfrak{K}(\Lambda) \cap \mathrm{N}_{\mathrm{A}^\times}(\mathrm{K})$ such that we have $\beta_1 - u\beta_2 u^{-1} \in \mathfrak{P}_{-m}(\Lambda)$.

We proceed as in the proof of proposition 1.7 (see paragraph 2.6). Let us fix a simple right $\mathrm{E}_1 \otimes_{\mathrm{F}} \mathrm{D}$-module S and set $\mathrm{A}(\mathrm{S}) = \mathrm{End}_{\mathrm{D}}(\mathrm{S})$. Let us denote by ρ_1 the natural F-algebra homomorphism $\mathrm{E}_1 \to \mathrm{A}(\mathrm{S})$. We write $\mathfrak{S}$ for the unique (up to translation) E_1-pure strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence on S and fix an F-algebra homomorphism $\rho_2 : \mathrm{E}_2 \to \mathrm{A}(\mathrm{S})$ such that $\mathfrak{S}$ is $\rho_2(\mathrm{E}_2)$-pure, and such that $(\rho_1(\mathrm{E}_1), \mathfrak{S})$ and $(\rho_2(\mathrm{E}_2), \mathfrak{S})$ have the same embedding type in A(S) (see Lemma 2.11). We also fix a decomposition:

$$\mathrm{V} = \mathrm{V}^1 \oplus \cdots \oplus \mathrm{V}^l \tag{3.3}$$

of V into simple right $\mathrm{K}(\beta) \otimes_{\mathrm{F}} \mathrm{D}$-modules (which all are copies of S) such that Λ is decomposed by (3.3) in the sense of [22, Définition 1.13], that is, Λ is the direct sum of the lattice sequences $\Lambda^j = \Lambda \cap \mathrm{V}^j$, for $j \in \{1, \dots, l\}$. By choosing, for each j, an isomorphism of $\mathrm{K}(\beta) \otimes_{\mathrm{F}} \mathrm{D}$-modules between S and V^j, this decomposition gives us an F-algebra homomorphism:

$$\iota : \mathrm{A}(\mathrm{S}) \to \mathrm{A}.$$

Using Lemma 2.14, we may assume that it satisfies $\iota(\rho_1(\beta_1)) = \beta_1$.

For $i \in \{1, 2\}$, let (k, β_i) be the simple pair of which the stratum $[\Lambda, n, m, \beta_i]$ is a realization. By putting $n_0 = n_{\mathrm{F}}(\beta_i)$ and $m_0 = e_{\rho_i(\beta_i)}(\mathfrak{S})k$, which do not depend on i by Proposition 2.10, we get a simple stratum $[\mathfrak{S}, n_0, m_0, \rho_i(\beta_i)]$ which is a realization of (k, β_i) in A(S). The proof of [5, Theorem 4.1.2] (see also [17, Lemma 10.5]) gives us an element $v \in \mathfrak{K}(\mathfrak{S})$ such that $\rho_1(\mathrm{K}) = v\rho_2(\mathrm{K})v^{-1}$ and $\beta_1 - v\beta_2 v^{-1} \in \mathfrak{P}_{-m_0}(\mathfrak{S})$. By Proposition 3.3, there is $w \in \mathfrak{K}(\Lambda)$ such that $\iota(\rho_2(x)) = wxw^{-1}$ for all $x \in \mathrm{E}_2$, and, by Remark 3.4, this element satisfies $w\mathrm{K}w^{-1} = \iota(\rho_2(\mathrm{K}))$. Thus the element $u = \iota(v)w$ normalizes K and Λ and satisfies the required condition:

$$\beta_1 - u\beta_2 u^{-1} \in \mathfrak{P}_{-e_{\beta_i}(\Lambda)k}(\Lambda) \subseteq \mathfrak{P}_{-m}(\Lambda),$$

which ends the proof of Proposition 1.9.

4. Realizations and intertwining for simple characters

The two main results of this section are Propositions 4.9 and 4.11. The first one asserts that two endo-equivalent ps-characters have realizations with very special properties, allowing us to use the results of [17]. The second one leads to the rigidity theorem 4.16, and will also give us an important property of the base change map in paragraph 7.2.

4.1. In this paragraph, we generalize the construction given in paragraph 2.7 by incorporating the notion of embedding type. For this, we will need the following definition.

Let $[\Lambda, n, m, \beta]$ be a simple stratum in A, which is a realization of a simple pair (k, β) over F, and set $E = F(\beta)$. The containment of $\mathcal{O}_E$ in $\mathfrak{A}(\Lambda)$ allows us to identify the residue field $\mathfrak{k} = \mathfrak{k}_{E^\diamond}$ with its canonical image in the $\mathfrak{k}_F$-algebra $\overline{\mathfrak{A}} = \mathfrak{A}(\Lambda)/\mathfrak{P}(\Lambda)$.

Definition 4.1. — The *Fröhlich invariant* of $[\Lambda, n, m, \beta]$ is the degree over $\mathfrak{k}_F$ of the intersection of $\mathfrak{k}$ with the centre of $\overline{\mathfrak{A}}$.

Recall that this invariant has been introduced by Fröhlich (see [15]) for sound strata. In this case, we have the following important property.

Theorem 4.2 ([15], Theorem 2). — *For $i = 1, 2$, let (K_i, Λ) be a sound embedding in A where K_i/F is an unramified extension contained in A. These embeddings are equivalent if and only if $[K_1^\diamond : F] = [K_2^\diamond : F]$ and they have the same Fröhlich invariant.*

We will need the two following lemmas.

Lemma 4.3. — *Let us fix an integer $l \geqslant 1$, an $\mathcal{O}_D$-lattice sequence Λ' and an integer m' as in paragraph 2.2, and let us form the simple stratum $[\Lambda', n', m', \beta]$ in A'. The simple strata $[\Lambda, n, m, \beta]$ and $[\Lambda', n', m', \beta]$ have the same Fröhlich invariant.*

Proof. — Let us identify A' with the matrix algebra $M_l(A)$, and write j for the $\mathfrak{k}_F$-algebra homomorphism $\mathfrak{k} \to \overline{\mathfrak{A}}' = \mathfrak{A}(\Lambda')/\mathfrak{P}(\Lambda')$ induced by the embedding of $\mathcal{O}_E$ in $\mathfrak{A}(\Lambda')$ (which is the restriction to $\mathcal{O}_E$ of the diagonal embedding of E in A'). By a direct computation, we see that the diagonal blocks of $\mathfrak{A}(\Lambda')$ are equal to $\mathfrak{A}(\Lambda)$, and that of its radical $\mathfrak{P}(\Lambda')$ are equal to $\mathfrak{P}(\Lambda)$. This is enough to prove that $j(x)$ is central in $\overline{\mathfrak{A}}'$ if and only if x is central in $\overline{\mathfrak{A}}$. Thus the strata $[\Lambda, n, m, \beta]$ and $[\Lambda', n', m', \beta]$ have the same Fröhlich invariant. □

Lemma 4.4. — *We set $\Lambda' = \Lambda \oplus \Lambda$ and $m' = m$ (thus $l = 2$). There exists an element $u \in A'^{\times}$ such that Λ' is $uF(\beta)u^{-1}$-pure and the simple stratum $[\Lambda', n, m, u\beta u^{-1}]$ in A' has Fröhlich invariant* 1.

Proof. — We fix a D-basis $\mathscr{B}$ of V, a maximal unramified extension L of F contained in D and a uniformizer ϖ of D normalizing L (see paragraph 3.1). According to Lemma 3.1, we may identify A with $M_r(D)$ and assume that the embedding $(E^\diamond, \Lambda)$ is in standard form with respect to $(\mathscr{B}, L)$. The map $\varphi : x \mapsto \varpi x \varpi^{-1}$ defines a generator of $\mathrm{Gal}(E^\diamond/F)$, and thus induces on the residue field $\mathfrak{k} = \mathfrak{k}_{E^\diamond}$ a generator of $\mathrm{Gal}(\mathfrak{k}/\mathfrak{k}_F)$, denoted σ. We write j for the $\mathfrak{k}_F$-algebra homomorphism from $\mathfrak{k}$ to $\overline{\mathfrak{A}}$ induced by φ, which is the composite of σ with the canonical embedding of $\mathfrak{k}$ in $\overline{\mathfrak{A}}$. Thus, one has $j(x) = x$ if and only if $x \in \mathfrak{k}_{\mathrm{F}}$. We now set:

$$u = \begin{pmatrix} \mathrm{I}_r & 0 \\ 0 & \varpi \cdot \mathrm{I}_r \end{pmatrix} \in \mathrm{M}_2(\mathrm{A}) = \mathrm{A}'.$$

If one identifies the $\mathfrak{k}_{\mathrm{F}}$-algebra $\overline{\mathfrak{A}}' = \mathfrak{A}(\Lambda')/\mathfrak{P}(\Lambda')$ with $\mathrm{M}_2(\overline{\mathfrak{A}})$, then the $\mathfrak{k}_{\mathrm{F}}$-algebra homomorphism j' from $\mathfrak{k}$ to $\overline{\mathfrak{A}}'$ induced by $x \mapsto uxu^{-1}$ is given by:

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & j(x) \end{pmatrix}.$$

Therefore, $j'(x)$ is central in $\overline{\mathfrak{A}}'$ if and only if $x = j(x)$ is central in $\overline{\mathfrak{A}}$, that is, if and only if $x \in \mathfrak{k}_{\mathrm{F}}$. □

This leads us to the following result. For $i = 1, 2$, let (k, β_i) be a simple pair over F, let $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ be a realization of (k, β_i) in A and let θ_i be a simple character in $\mathcal{C}(\Lambda, m_i, \varphi_i(\beta_i))$.

Proposition 4.5. — *Assume θ_1 and θ_2 intertwine in $\mathrm{A}^\times$. Then there is a simple central* F*-algebra* A′ *together with realizations $[\Lambda', n_i, m_i, \varphi'_i(\beta_i)]$ of (k, β_i) in* A′ *(with the same n_i and m_i), with $i = 1, 2$, which are sound and have the same embedding type, and such that θ'_1 and θ'_2 intertwine in $\mathrm{A}'^\times$, where $\theta'_i \in \mathcal{C}(\Lambda', m_i, \varphi'_i(\beta_i))$ denotes the transfer of θ_i.*

Proof. — First, we reduce to the case where the strata $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ have Fröhlich invariant 1. Let $g \in \mathrm{A}^\times$ intertwine the characters θ_1 and θ_2 as in (1.7). We set $\Lambda' = \Lambda \oplus \Lambda$ and $\mathrm{A}' = \mathrm{M}_2(\mathrm{A})$ and, for each i, we fix an element $u_i \in \mathrm{A}'^\times$ as in Lemma 4.4 so that the simple stratum $[\Lambda', n_i, m_i, u_i\varphi_i(\beta_i)u_i^{-1}]$ has Fröhlich invariant 1. For each i, let θ'_i be the transfer of θ_i in $\mathcal{C}(\Lambda', m_i, \varphi_i(\beta_i))$, and let θ''_i be that of θ_i in $\mathcal{C}(\Lambda', m_i, u_i\varphi_i(\beta_i)u_i^{-1})$, which is equal to the conjugate character $x \mapsto \theta'_i(u_i^{-1}xu_i)$. By the proof of Proposition 2.6, the element $g' = \iota(g) \in \mathrm{A}'^\times$ intertwines θ'_1 and θ'_2, where ι denotes the diagonal embedding of A in A′, and it follows that $g'' = u_1^{-1}g'u_2$ intertwines θ''_1 and θ''_2. Thus we can assume that the strata $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ have Fröhlich invariant 1. Using Proposition 2.17 (with some suitable integer $l \geqslant 1$) and Lemma 4.3 together, we see that the simple strata $[\Lambda^\ddagger, n_i, m_i, \varphi_i(\beta_i)]$ are sound with Fröhlich invariant 1. By Theorem 4.2, they have the same embedding type. Let $\theta_i^\ddagger$ be the transfer of θ_i in $\mathcal{C}(\Lambda^\ddagger, m_i, \varphi_i(\beta_i))$. The fact that $\theta_1^\ddagger$ and $\theta_2^\ddagger$ intertwine in $\mathrm{A}^{\ddagger\times}$ follows from Proposition 2.6. □

Remark 4.6. — The assumption $[\mathrm{F}(\beta_1) : \mathrm{F}] = [\mathrm{F}(\beta_2) : \mathrm{F}]$ is not needed in the proof.

4.2. Before proving the first main result of this section, that is Proposition 4.9, we will need the following lemmas. Compare the first one with Proposition 2.10.

Lemma 4.7. — *For $i = 1, 2$, let (Θ_i, k, β_i) be a ps-character over* F*, and suppose that Θ_1 and Θ_2 are endo-equivalent. Then $n_{\mathrm{F}}(\beta_1) = n_{\mathrm{F}}(\beta_2)$, $e_{\mathrm{F}}(\beta_1) = e_{\mathrm{F}}(\beta_2)$, $f_{\mathrm{F}}(\beta_1) = f_{\mathrm{F}}(\beta_2)$ and $k_{\mathrm{F}}(\beta_1) = k_{\mathrm{F}}(\beta_2)$.*

Proof. — By assumption, we have $[\mathrm{F}(\beta_1) : \mathrm{F}] = [\mathrm{F}(\beta_2) : \mathrm{F}]$ and there is a simple central F-algebra A together with realizations $[\Lambda, n_i, m_i, \beta_i]$ of (k, β_i), for $i = 1, 2$, such that the corresponding simple characters θ_1 and θ_2 intertwine in $\mathrm{A}^\times$. By Proposition 4.5, we can assume that these realizations are sound and have the same embedding type. We now follow the proof of [6, Proposition 8.4]. An argument similar to the first part of this proof (which we do not reproduce) gives us $n_1 = n_2$, denoted n. Now consider the integers m_1, m_2. By symmetry, we can assume that $m_1 \geqslant m_2$. Let us choose a simple stratum $[\Lambda, n, m_1, \gamma]$ in A which is equivalent to $[\Lambda, n, m_1, \beta_2]$ and let θ_0 denote the restriction of θ_2 to $\mathrm{H}^{m_1+1}(\gamma, \Lambda)$. The characters θ_0 and θ_1 still intertwine, which implies, by the "intertwining implies conjugacy" theorem [17, Corollary 10.15], the existence of $u \in \mathfrak{K}(\Lambda)$ such that $\mathcal{C}(\Lambda, m_1, \beta_1) = \mathcal{C}(\Lambda, m_1, u\gamma u^{-1})$. By Proposition 1.17, we get:

$$k_{\mathrm{F}}(\beta_1) = k_{\mathrm{F}}(\gamma), \quad [\mathrm{F}(\beta_1) : \mathrm{F}] = [\mathrm{F}(\gamma) : \mathrm{F}]. \tag{4.1}$$

By [5, Theorem 5.1(ii)], the equality $[\mathrm{F}(\beta_2) : \mathrm{F}] = [\mathrm{F}(\gamma) : \mathrm{F}]$ implies that $[\Lambda, n, m_1, \beta_2]$ is a simple stratum in A. By Theorem 1.16, we get $e_{\mathrm{F}}(\beta_1) = e_{\mathrm{F}}(\beta_2)$ and $f_{\mathrm{F}}(\beta_1) = f_{\mathrm{F}}(\beta_2)$, and (4.1) gives us $k_{\mathrm{F}}(\beta_1) = k_{\mathrm{F}}(\beta_2)$. The remaining equality is a consequence of the identity $n_i = e_{\beta_i}(\Lambda) n_{\mathrm{F}}(\beta_i)$. □

Corollary 4.8. — *Theorem 1.13 implies Theorems 1.11 and 1.12.*

Proof. — For $i = 1, 2$, let (Θ_i, k, β_i) be a ps-character over F, and suppose that Θ_1 and Θ_2 are endo-equivalent. Let A be a simple central F-algebra. For each i, let $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ be a realization of (k, β_i) in A, and put $\theta_i = \Theta_i(\Lambda, m_i, \varphi_i)$. Write $n = n_i$ and:

$$m = e_{\varphi_i(\beta_i)}(\Lambda)k,$$

which do not depend on i by Lemma 4.7. As $m_1, m_2 \geqslant m$, we may assume without loss of generality that $m_1 = m_2 = m$. Let us fix an F-algebra homomorphism $\varphi_3 : \mathrm{F}(\beta_2) \to \mathrm{A}$ such that the simple strata $[\Lambda, n, m, \varphi_1(\beta_1)]$ and $[\Lambda, n, m, \varphi_3(\beta_2)]$ have the same embedding type, and let θ_3 denote the transfer of θ_2 in $\mathcal{C}(\Lambda, m, \varphi_3(\beta_2))$. According to Theorem 1.13, there is an element $u \in \mathfrak{K}(\Lambda)$ such that $\theta_3(x) = \theta_1(uxu^{-1})$ for all $x \in \mathrm{H}^{m+1}(\varphi_3(\beta_2), \Lambda)$ and, by the Skolem-Noether theorem, there is an element $g \in \mathrm{A}^\times$ such that $\varphi_3(x) = g\varphi_2(x)g^{-1}$. Thus g intertwines θ_3 and θ_2, which proves that θ_1 and θ_2 intertwine in $\mathrm{A}^\times$ and ends the proof of Theorem 1.11.
Assume now that the strata $[\Lambda, n, m, \varphi_i(\beta_i)]$, $i = 1, 2$ have the same embedding type. Then applying Theorem 1.13 gives immediately Theorem 1.12. □

We are thus reduced to proving Theorem 1.13, which will be done in section 8. For this we will have to develop base change methods (see sections 5, 6 and 7). We now state and prove the first main result of this section.

Proposition 4.9. — *For $i = 1, 2$, let (Θ_i, k, β_i) be a ps-character over* F*, and suppose that Θ_1 and Θ_2 are endo-equivalent. Write K_i for the maximal unramified extension of* F *contained in $\mathrm{F}(\beta_i)$. Then there exists a simple central* F*-algebra* A *together with realizations $[\Lambda, n, m, \varphi_i(\beta_i)]$ of (k, β_i), for $i = 1, 2$, which are sound and have the same embedding type, and such that:*

(1) *m is a multiple of k;*
(2) *$\varphi_1(\mathrm{K}_1) = \varphi_2(\mathrm{K}_2)$;*
(3) *$\Theta_1(\Lambda, m, \varphi_1) = \Theta_2(\Lambda, m, \varphi_2)$.*

Proof. — By Proposition 4.5, there is a simple central F-algebra A together with realizations $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ of (k, β_i), for $i = 1, 2$, sound and having the same embedding type, such that $\theta_1 = \Theta_1(\Lambda, m_1, \varphi_1)$ and $\theta_2 = \Theta_2(\Lambda, m_2, \varphi_2)$ intertwine in $\mathrm{A}^\times$. By Lemma 4.7, we have $n_1 = n_2$, and the integer $m = e_{\varphi_i(\beta_i)}(\Lambda)k$ does not depend on i.

Lemma 4.10. — *For each i, there exists a unique $\vartheta_i \in \mathcal{C}(\Lambda, m, \varphi_i(\beta_i))$ extending θ_i, and the characters ϑ_1 and ϑ_2 intertwine in $\mathrm{A}^\times$.*

Proof. — The proof is similar to that of [10, Lemma 3.6.7] and [6, Lemma 8.5] together. One just has to replace Corollary 3.3.21 of [10] by Proposition 2.16 of [24], and Theorems 3.5.8, 3.5.9 and 3.5.11 of [10] by Corollary 10.15 and Propositions 9.9 and 9.10 of [17]. □

Therefore we can assume that m_1, m_2 are both equal to m. The result follows from the "intertwining implies conjugacy" theorem [17, Corollary 10.15]. □

4.3. We now assume that we are in the situation of paragraph 2.4. Let us fix two simple strata $[\Lambda, n, m, \beta_i]$, $i = 1, 2$, in A. We set $n' = an$ and fix a non-negative integer m' such that $\lfloor m'/a \rfloor = m$, so that we have simple strata $[\Lambda', n', m', \beta_i]$, $i = 1, 2$, in A′, where Λ' is defined by (2.2). We fix a simple character θ_i in $\mathcal{C}(\Lambda, m, \beta_i)$ and write θ'_i for its transfer in $\mathcal{C}(\Lambda', m', \beta_i)$. The aim of this paragraph is to prove the following proposition, which is the second main result of this section.

Proposition 4.11. — *Assume that θ_1 and θ_2 are equal. Then θ'_1 and θ'_2 are equal.*

Proof. — We first prove the following lemma, which generalizes [10, Theorem 3.5.9] and [17, Proposition 9.10] (see also [13, Lemme 7.9], which gives a similar result in the split case for *semisimple characters* and whose proof we follow).

Lemma 4.12. — *Assume that $m \geqslant 1$, and that $\mathcal{C}(\Lambda, m, \beta_1) \cap \mathcal{C}(\Lambda, m, \beta_2)$ is not empty. Then we have $\mathrm{H}^m(\beta_1, \Lambda) = \mathrm{H}^m(\beta_2, \Lambda)$.*

Proof. — We put $\nu = 2m - 1$ and, for $i = 1, 2$, we choose a simple stratum $[\Lambda, n, \nu, \gamma_i]$ equivalent to $[\Lambda, n, \nu, \beta_i]$ in A. Then, for each $i = 1, 2$, we have the equality $\mathcal{C}(\Lambda, \nu, \beta_i) = \mathcal{C}(\Lambda, \nu, \gamma_i)$ and, from [24, Proposition 2.15], we have $\mathrm{H}^m(\beta_i, \Lambda) = \mathrm{H}^m(\gamma_i, \Lambda)$. Since the restriction of a simple character to the subgroup $\mathrm{H}^{\nu+1}(\beta_1,\Lambda)=\mathrm{H}^{\nu+1}(\beta_2,\Lambda)$ is still a simple character, the intersection $\mathcal{C}(\Lambda,\nu,\gamma_1)\cap\mathcal{C}(\Lambda,\nu,\gamma_2)$ is not empty. By computing the intertwining of an element of this intersection via the formula of [24, Théorème 2.23], we get:

$$\Omega_{q_1-\nu}(\gamma_1,\Lambda)\mathrm{B}_{\gamma_1}^{\times}\Omega_{q_1-\nu}(\gamma_1,\Lambda)=\Omega_{q_2-\nu}(\gamma_2,\Lambda)\mathrm{B}_{\gamma_2}^{\times}\Omega_{q_2-\nu}(\gamma_2,\Lambda)$$

with the notations of *loc. cit.* and where, for each $i=1,2$, we write B_{γ_i} for the centralizer of $\mathrm{F}(\gamma_i)$ in A and $q_i=-k_0(\gamma_i,\Lambda)$. Taking the intersection with $\mathfrak{P}_m(\Lambda)$ and then its additive closure, we find that the following set:

$$\mathfrak{Q}_m^i+(\mathfrak{P}_{q_i-\nu}(\Lambda)\cap\mathfrak{n}_{-\nu}(\gamma_i,\Lambda))\,\mathfrak{Q}_m^i+\mathfrak{Q}_m^i\mathfrak{J}^{\lceil q_i/2\rceil}(\gamma_i,\Lambda), \tag{4.2}$$

is independent of i, where we have put $\mathfrak{Q}_m^i=\mathfrak{P}_m(\Lambda)\cap\mathrm{B}_{\gamma_i}$ and where the notations $\mathfrak{J}^k$ and $\mathfrak{H}^k$, for $k\geqslant 0$, are defined in [24, §2.4]. We claim that the set in (4.2) is contained in $\mathfrak{H}^m(\gamma_i,\Lambda)=\mathfrak{H}^m(\beta_i,\Lambda)$. For then, adding $\mathfrak{H}^{m+1}(\gamma_i,\Lambda)=\mathfrak{H}^{m+1}(\beta_i,\Lambda)$, which is also independent of i, we see that:

$$\mathfrak{H}^m(\beta_i,\Lambda)=\mathfrak{H}^m(\gamma_i,\Lambda)=\mathfrak{Q}_m^i+\mathfrak{H}^{m+1}(\gamma_i,\Lambda)$$

is independent of i, as required. We now need the following lemma (see [28, Lemma 3.11(i)]).

Lemma 4.13. — Let $[\Lambda,n,m,\beta]$ *be a simple stratum in* A *with* $q=-k_0(\beta,\Lambda)$. *For each integer* $1\leqslant k\leqslant q-1$*, we have:*

$$(\mathfrak{n}_{-k}(\beta,\Lambda)\cap\mathfrak{P}_{q-k}(\Lambda))\,\mathfrak{J}^{\lceil k/2\rceil}(\beta,\Lambda)\subseteq\mathfrak{H}^{\lfloor k/2\rfloor+1}(\beta,\Lambda).$$

Proof. — We write $[\widetilde{\Lambda},n,m,\beta]$ for the simple stratum in $\widetilde{\mathrm{A}}=\mathrm{End}_{\mathrm{F}}(\mathrm{V})$ associated with $[\Lambda,n,m,\beta]$ (see paragraph 2.1). Then we have:

$$\left(\mathfrak{n}_{-k}(\beta,\widetilde{\Lambda})\cap\mathfrak{P}_{q-k}(\widetilde{\Lambda})\right)\mathfrak{J}^{\lceil k/2\rceil}(\beta,\widetilde{\Lambda})\subseteq\mathfrak{H}^{\lfloor k/2\rfloor+1}(\beta,\widetilde{\Lambda})$$

by [28, Lemma 3.11(i)]. By taking the intersection with A, we get the expected result. □

We now see that:

$$(\mathfrak{P}_{q_i-\nu}(\Lambda)\cap\mathfrak{n}_{-\nu}(\gamma_i,\Lambda))\,\mathfrak{Q}_m^i\subseteq(\mathfrak{P}_{q_i-\nu}(\Lambda)\cap\mathfrak{n}_{-\nu}(\gamma_i,\Lambda))\,\mathfrak{J}^{\lceil\nu/2\rceil}(\gamma_i,\Lambda),$$

which is contained in $\mathfrak{H}^m(\gamma_i,\Lambda)$. Similarly, we have:

$$\mathfrak{Q}_m^i\mathfrak{J}^{\lceil q_i/2\rceil}(\gamma_i,\Lambda)\subseteq\left(\mathfrak{P}_{q_i-(q_i-m)}(\Lambda)\cap\mathfrak{n}_{m-q_i}(\gamma_i,\Lambda)\right)\mathfrak{J}^{\lceil(q_i-m)/2\rceil}(\gamma_i,\Lambda),$$

which is contained in $\mathfrak{H}^{\lfloor(q_i-m)/2\rfloor+1}(\gamma_i,\Lambda)$. Since the left hand side here is clearly also contained in $\mathfrak{P}_m(\Lambda)$, we see that it is contained in $\mathfrak{H}^m(\gamma_i,\Lambda)$ as required. This also completes the proof of Lemma 4.12. □

For each i, write Θ_i for the ps-character defined by the pair $([\Lambda,n,m,\beta_i],\theta_i)$, and recall that θ_1 and θ_2 are equal.

Lemma 4.14. — We have $e_{\mathrm{F}}(\beta_1)=e_{\mathrm{F}}(\beta_2)$ *and* $f_{\mathrm{F}}(\beta_1)=f_{\mathrm{F}}(\beta_2)$.

Proof. — By Proposition 4.5, there is a simple central F-algebra A together with realizations $[\Lambda^0, n, m, \varphi_i^0(\beta_i)]$ of (k, β_i), with $i = 1, 2$, which are sound and have the same embedding type, and such that $\Theta_1(\Lambda^0, m, \varphi_1^0)$ and $\Theta_2(\Lambda^0, m, \varphi_2^0)$ intertwine in $A^{0\times}$. Let us write f for the greatest common divisor of $f_F(\beta_1)$ and $f_F(\beta_2)$ and K_i for the maximal unramified extension of F contained in $F(\varphi_i^0(\beta_i))$. Then Theorem 1.16 gives us the expected equality. □

Thus the ps-characters Θ_1 and Θ_2 are endo-equivalent, which allows us to use Lemma 4.7.

Lemma 4.15. — The characters θ'_1 and θ'_2 are equal if and only if we have:

$$H^{m'+1}(\beta_1, \Lambda') = H^{m'+1}(\beta_2, \Lambda'). \tag{4.3}$$

Proof. — This follows immediately from Lemma 2.7. □

Thus we are reduced to proving equality (4.3), and for this, we claim that it is enough to prove that:

$$H^{q'}(\beta_1, \Lambda') = H^{q'}(\beta_2, \Lambda'), \tag{4.4}$$

where $q' = -k_0(\beta_i, \Lambda')$ is independent of i by Lemma 4.7. Indeed, assume that (4.4) holds, and let t' be the smallest integer in $\{m', \dots, q' - 1\}$ such that:

$$H^{t'+1}(\beta_1, \Lambda') = H^{t'+1}(\beta_2, \Lambda'). \tag{4.5}$$

Suppose that $t' \neq m'$. By Lemma 4.15, the characters θ'_1 and θ'_2 agree on (4.5), that is, the intersection $\mathcal{C}(\Lambda', t', \beta_1) \cap \mathcal{C}(\Lambda', t', \beta_2)$ is not empty. By Lemma 4.12, we get an equality which contradicts the minimality of t'. Hence $t' = m'$ and we are thus reduced to proving (4.4), which we do by induction on β_1. Assume first that β_1 is minimal over F. Then so is β_2 by Lemma 4.7, so that we have:

$$H^{q'}(\beta_1, \Lambda') = U_{q'}(\Lambda') = H^{q'}(\beta_2, \Lambda').$$

Assume now that β_1 is not minimal over F, set $q = -k_0(\beta_i, \Lambda)$, which is independent of i by Lemma 4.7, and choose a simple stratum $[\Lambda, n, q, \gamma_i]$ in A equivalent to the stratum $[\Lambda, n, q, \beta_i]$, for each $i \in \{1, 2\}$. We then have:

$$H^{q'}(\beta_i, \Lambda') = H^{q'}(\gamma_i, \Lambda'),$$

and the restriction $\vartheta_i = \theta_i \mid H^{q+1}(\gamma_i, \Lambda)$ belongs to $\mathcal{C}(\Lambda, q, \gamma_i)$. As $\beta_i - \gamma_i \in \mathfrak{P}_{-q}(\Lambda)$, the simple characters ϑ_1 and ϑ_2 are equal. If we write ϑ'_i for the transfer of ϑ_i to the set $\mathcal{C}(\Lambda', q', \gamma_i)$, then the inductive hypothesis implies that $\vartheta'_1 = \vartheta'_2$. Therefore, the intersection $\mathcal{C}(\Lambda', q', \gamma_1) \cap \mathcal{C}(\Lambda', q', \gamma_2)$ is not empty, and Lemma 4.12 gives us the required equality (4.4). This ends the proof of Proposition 4.11. □

4.4. Before closing this section, we prove the following rigidity theorem for simple characters, which generalizes [10, Theorem 3.5.8] and [17, Proposition 9.9] to simple characters in non-necessarily split simple central F-algebras with non-necessarily strict lattice sequences.

Theorem 4.16. — *For $i = 1, 2$, let $[\Lambda, n, m, \beta_i]$ be a simple stratum in a simple central* F*-algebra* A*. Assume that the intersection $\mathcal{C}(\Lambda, m, \beta_1) \cap \mathcal{C}(\Lambda, m, \beta_2)$ is not empty. Then we have $\mathcal{C}(\Lambda, m, \beta_1) = \mathcal{C}(\Lambda, m, \beta_2)$.*

Proof. — For each $i \in \{1, 2\}$, we fix a simple character $\theta_i \in \mathcal{C}(\Lambda, m, \beta_i)$ and assume that θ_1 and θ_2 are equal. In particular, we have:

$$\mathrm{H}^{m+1}(\beta_1, \Lambda) = \mathrm{H}^{m+1}(\beta_2, \Lambda). \tag{4.6}$$

By choosing an integer l as in Proposition 2.17, we have sound simple strata $[\Lambda^{\ddagger}, n, m, \beta_i]$, $i = 1, 2$, in $\mathrm{A}^{\ddagger}$. If we write $\theta_i^{\ddagger}$ for the transfer of θ_i to $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_i)$, then it follows from Proposition 4.11 that the simple characters $\theta_1^{\ddagger}$ and $\theta_2^{\ddagger}$ are equal, hence that the intersection $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_1) \cap \mathcal{C}(\Lambda^{\ddagger}, m, \beta_2)$ is not empty. By Proposition 1.17, the sets $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_i)$, $i = 1, 2$, are equal. As the transfer map from $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_i)$ to $\mathcal{C}(\Lambda, m, \beta_i)$ is the restriction map from $\mathrm{H}^{m+1}(\beta_i, \Lambda^{\ddagger})$ to $\mathrm{H}^{m+1}(\beta_i, \Lambda)$, the equality (4.6) implies that $\mathcal{C}(\Lambda, m, \beta_1) = \mathcal{C}(\Lambda, m, \beta_2)$. □

It is natural to ask whether the simple strata $[\Lambda, n, m, \beta_i]$ in Theorem 4.16 have the same embedding type. We have the following conjecture(2).

Conjecture 4.17. — *For $i = 1, 2$, let $[\Lambda, n, m, \beta_i]$ be a simple stratum in a simple central* F*-algebra* A*. Assume that the intersection $\mathcal{C}(\Lambda, m, \beta_1) \cap \mathcal{C}(\Lambda, m, \beta_2)$ is not empty, and that Λ is strict. Then these simple strata have the same embedding type.*

Note that we know from [5, Lemma 5.2] that two equivalent simple strata (with respect to a strict lattice sequence) have the same embedding type.

In the case where the strata are sound, we will prove below that this conjecture is true. First we need a series of lemmas.

Lemma 4.18. — *Let* E/F *be a finite extension with ramification index e, contained in a simple central* F*-algebra* A*, and let $\mathfrak{B}$ be a principal $\mathcal{O}_{\mathrm{E}}$-order of period r in the centralizer* B *of* E *in* A*. Write $\mathrm{A} \simeq \mathrm{M}_k(\mathrm{D})$ for some $k \geqslant 1$ and some* F*-division algebra* D*, and write d for the reduced degree of* D *over* F*.*

(1) *There exists a unique* E*-pure hereditary $\mathcal{O}_{\mathrm{F}}$-order $\mathfrak{A}$ in* A *such that $\mathfrak{B} = \mathfrak{A} \cap \mathrm{B}$ and $\mathfrak{K}(\mathfrak{B}) = \mathfrak{K}(\mathfrak{A}) \cap \mathrm{B}^{\times}$, and such an order is principal.*

(2) *The period of $\mathfrak{A}$ is equal to $re/(re, d)$, where (re, d) denotes the greatest common divisor of re and d.*

Proof. — The first part is given by [16, Corollary 1.4(ii)]. Part (2) follows for instance from the formula given in the proof of [24, Théorème 1.7]. □

(2) This conjecture — and an even more general statement — is proven in [25, Lemma 3.5].

In other words, there exists a unique hereditary $\mathcal{O}_F$-order $\mathfrak{A}$ in A such that $\mathfrak{A} \cap B = \mathfrak{B}$ and that $(E, \mathfrak{A})$ is a sound embedding in A.

Lemma 4.19. — *For $i = 1, 2$, let E_i be an extension of F contained in A and let $\mathfrak{A}$ be a hereditary $\mathcal{O}_F$-order in A such that $(E_i, \mathfrak{A})$ is a sound embedding in A. Write $\mathfrak{B}_i$ for the intersection of $\mathfrak{A}$ with the centralizer of E_i in A. Let f be the greatest common divisor of $f(E_1 : F)$ and $f(E_2 : F)$, and for each i, let K_i be the unramified extension of F of degree f contained in E_i. Assume E_1 and E_2 have the same ramification order e and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have the same period r. Then the embeddings $(K_1, \mathfrak{A})$ and $(K_2, \mathfrak{A})$ are equivalent in A.*

Proof. — Let $\mathfrak{C}_i$ denote the intersection of $\mathfrak{A}$ with the centralizer C_i of K_i in A. If we write B_i for the centralizer of E_i in A, then we have $\mathfrak{B}_i = \mathfrak{C}_i \cap B_i$ and $\mathfrak{K}(\mathfrak{B}_i) = \mathfrak{K}(\mathfrak{C}_i) \cap B_i^\times$. Using Lemma 4.18, the period of $\mathfrak{C}_i$ is equal to $re/(re, d_i)$, where d_i is the reduced degree of the K_i-division algebra D_i such that C_i is isomorphic to $M_{k_i}(D_i)$ for some $k_i \geqslant 1$. Using for instance [29], we have $d_i = d/(d, f)$, which does not depend on i. By the Skolem-Noether theorem, there is $g \in A^\times$ such that $gK_1g^{-1} = K_2$. Thus $g\mathfrak{C}_1g^{-1}$ and $\mathfrak{C}_2$ are two principal $\mathcal{O}_{K_2}$-orders in C_2 with the same period, which implies that there exists $h \in C_2^\times$ such that $g\mathfrak{C}_1g^{-1} = h\mathfrak{C}_2h^{-1}$. Let us write $u = h^{-1}g$. Using the unicity property (1) of Lemma 4.18, we get $\mathfrak{A} = u^{-1}\mathfrak{A}u$, that is $u \in \mathfrak{K}(\mathfrak{A})$. □

We now prove Conjecture 4.17 in the case where the strata are sound.

Proposition 4.20. — *For $i = 1, 2$, let $[\Lambda, n, m, \beta_i]$ be a sound simple stratum in a simple central F-algebra A. Assume that $\mathcal{C}(\Lambda, m, \beta_1) \cap \mathcal{C}(\Lambda, m, \beta_2)$ is not empty. Then these simple strata have the same embedding type.*

Proof. — For each i, we fix a simple character $\theta_i \in \mathcal{C}(\Lambda, m, \beta_i)$ and assume θ_1 and θ_2 are equal. Thus we have $[F(\beta_1) : F] = [F(\beta_2) : F]$ by Proposition 1.17. By Lemma 4.7, we also have $e_F(\beta_1) = e_F(\beta_2)$ and $f_F(\beta_1) = f_F(\beta_2)$. If we write $I_{U(\Lambda)}(\theta_i)$ for the intertwining of θ_i in $U(\Lambda)$, then [20, Théorème 3.50] gives us:

$$I_{U(\Lambda)}(\theta_i)U^1(\Lambda)/U^1(\Lambda) \simeq U(\mathfrak{B}_i)/U^1(\mathfrak{B}_i),$$

where $\mathfrak{B}_i$ is the intersection of $\mathfrak{A} = \mathfrak{A}(\Lambda)$ with the centralizer of β_i in A. As the stratum $[\Lambda, n, m, \beta_i]$ is sound, $\mathfrak{B}_i$ is a principal $\mathcal{O}_{F(\beta_i)}$-order. Thus there are a finite extension $\mathfrak{k}_i$ of $\mathfrak{k}_F$ and two positive integers $r_i, s_i \geqslant 1$ such that:

$$U(\mathfrak{B}_i)/U^1(\mathfrak{B}_i) \simeq GL_{s_i}(\mathfrak{k}_i)^{r_i}.$$

Since it does not depend on i, we have $r_1 = r_2$. Now write K_i for the maximal unramified extension of F contained in $F(\beta_i)$. Using Lemma 4.19 with $E_i = F(\beta_i)$, we deduce that the embeddings $(K_1, \mathfrak{A})$ and $(K_2, \mathfrak{A})$ are equivalent in A. □

5. The interior lifting

In this section, we develop an interior lifting process for simple strata and characters with respect to a finite unramified extension K of F, in a way similar to [6] and [17]. The situation in [6] is somewhat more general than ours, since the authors only assume K/F to be tamely ramified, but is only concerned with simple strata and characters in split simple central F-algebras attached to strict lattice sequences. The situation in [17] deals with any simple central F-algebra, but puts an unnecessarily restrictive condition on the simple strata (they are supposed to be sound).

5.1. Let A be a simple central F-algebra and K/F be a finite unramified extension contained in A. Let C denote the centralizer of K in A. We fix a simple left A-module V and a simple left C-module W. The following definition extends [6, Definition 2.2] to strata with non-necessarily strict lattice sequences.

Definition 5.1. — A stratum $[\Lambda, n, m, \beta]$ in A is said to be K-*pure* if it is pure, if β centralizes K and if the algebra $\mathrm{K}[\beta]$ is a field such that $\mathrm{K}[\beta]^\times$ normalizes Λ.

Given a K-pure stratum $[\Lambda, n, m, \beta]$ in A, we can form the pure stratum $[\Gamma, n, m, \beta]$, where Γ is the unique (up to translation) lattice sequence on W defined by:

$$\mathfrak{P}_k(\Lambda) \cap \mathrm{C} = \mathfrak{P}_k(\Gamma), \quad k \in \mathbb{Z}. \tag{5.1}$$

Note that the $\mathrm{C}^\times$-normalizer of Γ is equal to $\mathfrak{K}(\Lambda) \cap \mathrm{C}^\times$. We then get a process:

$$[\Lambda, n, m, \beta] \mapsto [\Gamma, n, m, \beta] \tag{5.2}$$

giving an injection, respecting equivalence, between the set of K-pure strata of A and the set of pure strata of C. The fact that Γ is defined only up to translation makes (5.2) not well defined, but this will be of no importance in the sequel. We now discuss the image of simple K-pure strata of A by (5.2).

Proposition 5.2. — (1) *Let* $[\Lambda, n, m, \beta]$ *be a* K*-pure stratum in* A*. Then:*

$$k_0(\beta, \Gamma) \leqslant k_0(\beta, \Lambda). \tag{5.3}$$

(2) *Suppose moreover that* $[\Lambda, n, m, \beta]$ *is a simple stratum. Then the stratum* $[\Gamma, n, m, \beta]$ *given by the map (5.2) is simple.*

Proof. — As K is unramified over F, the lattice sequences Λ and Γ have the same period over $\mathcal{O}_\mathrm{F}$. By (1.4) it is then enough to prove that $k_\mathrm{K}(\beta) \leqslant k_\mathrm{F}(\beta)$. Let $\mathfrak{L}$ denote the strict $\mathcal{O}_\mathrm{F}$-lattice sequence on $\mathrm{K}(\beta)$ defined by $i \mapsto \mathfrak{p}^i_{\mathrm{K}(\beta)}$. By [6, Theorem 2.4], we have:

$$k_\mathrm{K}(\beta) \leqslant k_0(\beta, \mathfrak{L}).$$

On the other hand, we have $e_\beta(\mathfrak{L}) = 1$ as K is unramified over F. By (1.4) again, we get the expected result. Suppose now that the stratum $[\Lambda, n, m, \beta]$ is simple. Then the fact that $[\Gamma, n, m, \beta]$ is simple derives immediately from (5.3). □

Remark 5.3. — For a case where the map (5.2) is not surjective, see [24, Exemple 1.6]. Compare with the split case [6, (2.3)].

5.2. Given a simple stratum $[\Gamma, n, m, \beta]$ in C in the image of (5.2), the K-pure stratum of A corresponding to it may not be simple. However, we have the following result, which generalizes [6, Corollary 3.8].

Proposition 5.4. — *Let* $[\Lambda, n, m, \beta]$ *be a* K*-pure stratum in* A *such that* $[\Gamma, n, m, \beta]$ *is simple. Then there exists a simple stratum* $[\Gamma, n, m, \beta']$ *in* C *equivalent to* $[\Gamma, n, m, \beta]$ *such that the stratum* $[\Lambda, n, m, \beta']$ *is simple.*
Moreover, β' *can be chosen such that the maximal unramified extension of* F *contained in* $\mathrm{F}(\beta')$ *is contained in that of* $\mathrm{F}(\beta)$*.*

Proof. — Let (k, β) denote the simple pair over K of which $[\Gamma, n, m, \beta]$ is a realization, fix a simple right $\mathrm{K}(\beta) \otimes_{\mathrm{F}} \mathrm{D}$-module S and set $\mathrm{A(S)} = \mathrm{End}_{\mathrm{D}}(\mathrm{S})$. Write ρ for the natural K-algebra homomorphism from $\mathrm{K}(\beta)$ to A(S). Let $\mathfrak{S}$ denote the unique (up to translation) $\rho(\mathrm{K}(\beta))$-pure strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence on S and n_0 the $\mathfrak{S}$-valuation of $\rho(\beta)$, and set:

$$m_0 = e_{\rho(\beta)}(\mathfrak{S})k,$$

so that $[\mathfrak{S}, n_0, m_0, \rho(\beta)]$ is a K-pure stratum in A(S). Write C(S) for the centralizer of K in A(S), fix a simple left C(S)-module T and let $[\mathfrak{T}, n_0, m_0, \rho(\beta)]$ be the stratum in C(S) attached to $[\mathfrak{S}, n_0, m_0, \rho(\beta)]$ by (5.2). This stratum is a realization of (k, β) in C(S), hence this is a simple stratum. According to [6, Theorem 3.7], the simple pair (k, β) is endo-equivalent to a simple pair (k, α) over K which is a K/F-lift of some simple pair over F in the sense of [6] (see paragraph 3). By [6, Proposition 1.10], the extensions $\mathrm{K}(\alpha)$ and $\mathrm{K}(\beta)$ have the same ramification index and residue class degree over K, which implies by [5, Corollary 3.16] that there is a realization $[\mathfrak{T}, n_0, m_0, \varphi(\alpha)]$ of (k, α) in C(S), having the same embedding type as $[\mathfrak{T}, n_0, m_0, \rho(\beta)]$.

We now pass to the strata $[\widetilde{\mathfrak{T}}, n_0, m_0, \varphi(\alpha)]$ and $[\widetilde{\mathfrak{T}}, n_0, m_0, \rho(\beta)]$ in the K-algebra $\mathrm{End}_{\mathrm{K}}(\mathrm{T})$ (see paragraph 2.1). By [24] (see Théorème 1.7 and Remarque 1.8), the lattice sequence $\mathfrak{T}$ (and thus $\widetilde{\mathfrak{T}}$) is in the affine class of a strict lattice sequence, so that, up to renormalization, one may consider it as being strict (see Lemma 2.2). By [6, Proposition 1.10], these strata thus intertwine. Hence, using Proposition 2.13, we can replace φ by some $\mathfrak{K}(\mathfrak{T})$-conjugate and assume that the strata $[\mathfrak{T}, n_0, m_0, \varphi(\alpha)]$ and $[\mathfrak{T}, n_0, m_0, \rho(\beta)]$ are equivalent, and that the maximal unramified extension of K contained in $\mathrm{K}(\varphi(\alpha))$ is equal to that of $\mathrm{K}(\rho(\beta))$. We check that the stratum $[\mathfrak{S}, n_0, m_0, \varphi(\alpha)]$ is simple as in the proof of [6, Proposition 4.3]. We now fix a decomposition:

$$\mathrm{V} = \mathrm{V}^1 \oplus \cdots \oplus \mathrm{V}^l \tag{5.4}$$

of V into simple right $\mathrm{K}(\beta) \otimes_{\mathrm{F}} \mathrm{D}$-modules (which all are copies of S) such that Λ is decomposed by (5.4) in the sense of [22, Définition 1.13], that is, Λ is the direct sum of the lattice sequences $\Lambda^j = \Lambda \cap \mathrm{V}^j$, for $j \in \{1, \ldots, l\}$. By choosing, for each j, an isomorphism of $\mathrm{K}(\beta)\otimes_{\mathrm{F}}\mathrm{D}$-modules between S and V^j, this decomposition gives us an F-algebra homomorphism:

$$\iota : \mathrm{A}(\mathrm{S}) \to \mathrm{A}.$$

Using Lemma 2.14, we may assume that this homomorphism satisfies $\iota(\rho(\beta)) = \beta$. If we set $\beta' = \iota(\varphi(\alpha))$, then the stratum $[\Gamma, n, m, \beta']$ is simple and satisfies the conditions of the first part of Proposition 5.4.

In particular, the pure strata $[\Lambda, n, m, \beta]$ and $[\Lambda, n, m, \beta']$ are equivalent, and the second one is simple. By replacing the lattice sequence Λ by $\Lambda^\dagger$ (see paragraph 2.5), we can apply [5, Theorem 5.1(ii)] and thus get that $f_{\mathrm{F}}(\beta')$ divides $f_{\mathrm{F}}(\beta)$. Moreover, the maximal unramified extension of K contained in $\mathrm{K}(\beta')$ is equal to that of $\mathrm{K}(\beta)$, denoted L. As K/F is unramified, the extension L/F is unramified. Thus the maximal unramified extension of F contained in $\mathrm{F}(\beta')$ and that of $\mathrm{F}(\beta)$ are two finite unramified extensions of F contained in L. According to the condition on their degrees, it follows that the maximal unramified extension of F contained in $\mathrm{F}(\beta')$ in contained in that of $\mathrm{F}(\beta)$. □

5.3. Let $[\Lambda, n, m, \beta]$ be a K-pure simple stratum in A, and let $[\Gamma, n, m, \beta]$ be the stratum in C given by the map (5.2), which is simple by Proposition 5.2. Recall that one attaches to these simple strata compact open subgroups $\mathrm{H}^{m+1}(\beta,\Lambda)$ of $\mathrm{A}^\times$ and $\mathrm{H}^{m+1}(\beta,\Gamma)$ of $\mathrm{C}^\times$, respectively.

Proposition 5.5. — Let $[\Lambda, n, m, \beta]$ be a K*-pure simple stratum in* A*, and let $[\Gamma, n, m, \beta]$ correspond to it by (5.2). Then we have:*

$$\mathrm{H}^{m+1}(\beta,\Lambda)\cap\mathrm{C}^\times = \mathrm{H}^{m+1}(\beta,\Gamma).$$

Proof. — It is enough to prove it when $m = 0$. The proof is by induction on β. Let R denote the centralizer of $\mathrm{K}(\beta)$ in A. Assume first that β is minimal over F, so that:

$$\mathrm{H}^1(\beta,\Lambda) = (\mathrm{U}_1(\Lambda)\cap\mathrm{B}^\times)\mathrm{U}_{\lfloor n/2\rfloor+1}(\Lambda).$$

According to (5.1), we get:

$$\mathrm{H}^1(\beta,\Lambda)\cap\mathrm{C}^\times = (\mathrm{U}_1(\Gamma)\cap\mathrm{R}^\times)\mathrm{U}_{\lfloor n/2\rfloor+1}(\Gamma),$$

which is equal to $\mathrm{H}^1(\beta,\Gamma)$ as β is minimal over K by Proposition 5.2. Now assume that β is not minimal over F, set $q = -k_0(\beta,\Lambda)$ and $r = \lfloor q/2\rfloor + 1$, and choose a simple stratum $[\Gamma, n, q, \gamma]$ equivalent to $[\Gamma, n, q, \beta]$ such that $[\Lambda, n, q, \gamma]$ is simple and K-pure, which is possible thanks to Proposition 5.4. We then have:

$$\mathrm{H}^1(\beta,\Lambda) = (\mathrm{U}_1(\Lambda)\cap\mathrm{B}^\times)\mathrm{H}^r(\gamma,\Lambda)$$

and, if we set $q_1 = -k_0(\beta,\Gamma)$ and $r_1 = \lfloor q_1/2\rfloor + 1$, we have:

$$\mathrm{H}^1(\beta,\Gamma) = (\mathrm{U}_1(\Gamma)\cap\mathrm{R}^\times)\mathrm{H}^{r_1}(\gamma,\Gamma).$$

As $-k_0(\gamma,\Gamma) \geqslant q_1 \geqslant q$, the group $\mathrm{H}^r(\gamma,\Gamma)$ is equal to $(\mathrm{U}_r(\Gamma)\cap\mathrm{R}^\times)\mathrm{H}^{r_1}(\gamma,\Gamma)$. It follows from (5.1) that the group $\mathrm{H}^1(\beta,\Gamma)$ is equal to $\mathrm{H}^1(\beta,\Lambda)\cap\mathrm{C}^\times$. This ends the proof of Proposition 5.5. □

5.4. We now want to lift simple characters. For this, given a simple stratum $[\Lambda, n, m, \beta]$ in A, we will need a characterization of the set $\mathcal{C}(\Lambda, m, \beta)$ by induction on β, generalizing [20, Proposition 3.47] to the case where Λ is non-necessarily strict.

Lemma 5.6. — Let $[\Lambda, n, m, \beta]$ *be a simple stratum in* A *and* θ *be a character of the group* $\mathrm{H}^{m+1}(\beta, \Lambda)$*, and set* $q = -k_0(\beta, \Lambda)$ *and* $m' = \max\{m, \lfloor q/2 \rfloor\}$*. Then* $\theta \in \mathcal{C}(\Lambda, m, \beta)$ *if and only if it is normalized by* $\mathfrak{K}(\Lambda) \cap \mathrm{B}^{\times}$ *and satisfies the following conditions:*

(1) *if* β *is minimal over* F*, then* $\theta \mid \mathrm{U}_{m'+1}(\Lambda) = \Psi_{\beta}^{\mathrm{A}}$ *and* $\theta \mid \mathrm{U}_{m+1}(\Lambda) \cap \mathrm{B}^{\times} = \chi \circ \mathrm{N}_{\mathrm{B}/\mathrm{E}}$ *for some character* χ *of* $1 + \mathfrak{p}_{\mathrm{E}}$ *(see (1.3) for the definition of* $\Psi_{\beta}^{\mathrm{A}}$*);*

(2) *if* β *is not minimal over* F*, and if* $[\Lambda, n, q, \gamma]$ *is simple and equivalent to* $[\Lambda, n, q, \beta]$ *in* A*, then* $\theta \mid \mathrm{H}^{m'+1}(\beta, \Lambda) = \theta_0 \Psi_{\beta-\gamma}^{\mathrm{A}}$ *and* $\theta \mid \mathrm{H}^{m+1}(\beta, \Lambda) \cap \mathrm{B}^{\times} = \chi \circ \mathrm{N}_{\mathrm{B}/\mathrm{E}}$ *for some simple character* $\theta_0 \in \mathcal{C}(\Lambda, m', \gamma)$ *and some character* χ *of* $1 + \mathfrak{p}_{\mathrm{E}}$*.*

Proof. — The proof is similar to that of [20, Proposition 3.11], and we do not repeat it. Note that [17, Lemma 1.9] is actually not needed in the proof, and that [12, Corollary 5.3] has to be replaced by [24, Proposition 1.20] and [10, Proposition 3.3.9] by [20, Proposition 3.30]. □

Let $[\Lambda, n, m, \beta]$ be a simple K-pure stratum in A, and let $[\Gamma, n, m, \beta]$ correspond to it by (5.2). We write $\mathcal{C}(\Gamma, m, \beta)$ for the set of simple characters attached to $[\Gamma, n, m, \beta]$ with respect to the additive character:

$$\Psi_{\mathrm{K}} = \Psi \circ \mathrm{tr}_{\mathrm{K}/\mathrm{F}}, \tag{5.5}$$

which is trivial on $\mathfrak{p}_{\mathrm{K}}$ but not on $\mathcal{O}_{\mathrm{K}}$, as K is unramified over F. Compare the following theorem with [6, Theorem 7.7] and [17, Proposition 7.1].

Theorem 5.7. — Let $[\Lambda, n, m, \beta]$ *be a simple* K*-pure stratum in* A*, and let* $[\Gamma, n, m, \beta]$ *correspond to it by (5.2). Then, for any* $\theta \in \mathcal{C}(\Lambda, m, \beta)$*, we have:*

$$\theta \mid \mathrm{H}^{m+1}(\beta, \Gamma) \in \mathcal{C}(\Gamma, m, \beta).$$

Proof. — The proof is by induction on β. Let θ^{K} denote the restriction of θ to the group $\mathrm{H}^{m+1}(\beta, \Gamma)$ and R the centralizer of $\mathrm{K}(\beta)$ in A. Assume first that β is minimal over F. By Proposition 5.2, it is also minimal over K. If $m \geqslant \lfloor n/2 \rfloor$, we have $\mathcal{C}(\Lambda, m, \beta) = \{\Psi_{\beta}^{\mathrm{A}}\}$ and $\mathcal{C}(\Gamma, m, \beta) = \{\Psi_{\beta}^{\mathrm{C}}\}$, where $\Psi_{\beta}^{\mathrm{C}}$ denotes the character of $\mathrm{U}_{m+1}(\Gamma)$ defined by:

$$\Psi_{\beta}^{\mathrm{C}} : x \mapsto \Psi_{\mathrm{K}} \circ \mathrm{tr}_{\mathrm{C}/\mathrm{K}}(\beta(x-1)).$$

So we just need to prove that:

$$\Psi_{\beta}^{\mathrm{A}} \mid \mathrm{U}_{m+1}(\Gamma) = \Psi_{\beta}^{\mathrm{C}}, \tag{5.6}$$

which is given by [6, Property (7.6)]. If $m \leqslant \lfloor n/2 \rfloor$, then any $\theta \in \mathcal{C}(\Lambda, m, \beta)$ extends the character $\Psi_{\beta}^{\mathrm{A}} \mid \mathrm{U}_{\lfloor n/2 \rfloor+1}(\Lambda)$ and its restriction to $\mathrm{U}_{m+1}(\Lambda) \cap \mathrm{B}^{\times}$ has the form:

$$\theta \mid \mathrm{U}_{m+1}(\Lambda) \cap \mathrm{B}^{\times} = \chi \circ \mathrm{N}_{\mathrm{B}/\mathrm{E}}$$

for some character χ of $1+\mathfrak{p}_{\mathrm{E}}$. Therefore the character θ^{K} extends the character $\Psi_\beta^{\mathrm{C}} \mid \mathrm{U}_{\lfloor n/2\rfloor+1}(\Gamma)$, and its restriction to $\mathrm{U}_{m+1}(\Gamma)\cap \mathrm{R}^\times$ has the form:

$$\theta^{\mathrm{K}} \mid \mathrm{U}_{m+1}(\Gamma)\cap \mathrm{R}^\times = \chi\circ \mathrm{N}_{\mathrm{K}(\beta)/\mathrm{E}}\circ \mathrm{N}_{\mathrm{R}/\mathrm{K}(\beta)}.$$

Finally, the group $\mathfrak{K}(\Gamma)\cap \mathrm{R}^\times$, which normalizes both θ and $\mathrm{H}^{m+1}(\beta,\Gamma)$, normalizes θ^{K}. It follows from Lemma 5.6 that $\theta^{\mathrm{K}}\in \mathcal{C}(\Gamma,m,\beta)$.
Now assume that β is not minimal over F. We set $q=-k_0(\beta,\Lambda)$ and $r=\lfloor q/2\rfloor+1$, and choose a simple stratum $[\Gamma,n,q,\gamma]$ equivalent to $[\Gamma,n,q,\beta]$ such that $[\Lambda,n,q,\gamma]$ is simple and K-pure. If $m\geqslant \lfloor q/2\rfloor$, then any $\theta\in\mathcal{C}(\Lambda,m,\beta)$ can be written as $\theta=\theta_0\Psi^{\mathrm{A}}_{\beta-\gamma}$ for some simple character $\theta_0\in\mathcal{C}(\Lambda,m,\gamma)$. Now we claim that:

$$\mathrm{H}^{m+1}(\beta,\Gamma)=\mathrm{H}^{m+1}(\gamma,\Gamma). \tag{5.7}$$

We write $q_1=-k_0(\beta,\Gamma)$. If $q_1=q$, then the equality (5.7) follows by definition. Otherwise, we have $q_1>q$ by Proposition 5.2. The strata $[\Gamma,n,q,\beta]$ and $[\Gamma,n,q,\gamma]$ are thus both simple, and (5.7) follows. We now restrict the character θ to the group given by (5.7) and get $\theta^{\mathrm{K}}=\theta_0^{\mathrm{K}}\Psi^{\mathrm{C}}_{\beta-\gamma}$, where θ_0^{K} denotes the restriction $\theta_0\mid \mathrm{H}^{m+1}(\gamma,\Gamma)$, and this restriction is in $\mathcal{C}(\Gamma,m,\gamma)$ by the inductive hypothesis. If $q_1=q$, then θ^{K} is in $\mathcal{C}(\Gamma,m,\beta)$ by definition. Otherwise, $[\Gamma,n,q,\beta]$ is simple and the result follows from [24, Proposition 2.15]. The case $m\leqslant\lfloor q/2\rfloor$ reduces to the previous one exactly as in the minimal case. ☐

6. Interior lifting and transfer

In this section, we define the interior lift of a ps-character. This amounts to studying the behaviour of the interior lifting process with respect to transfer.

6.1. As in section 5, we are given in this section a simple central F-algebra A and a finite unramified extension K/F contained in A. We fix a finite unramified extension L of K such that the L-algebra:

$$\overline{\mathrm{A}}=\mathrm{A}\otimes_{\mathrm{F}}\mathrm{L}$$

is split. This L-algebra inherits an action of the Galois group of L/F in the obvious way, and we consider A as being naturally embedded in $\overline{\mathrm{A}}$ by $j_{\mathrm{A}}: a\mapsto a\otimes_{\mathrm{F}}1$. We have a decomposition:

$$\mathrm{K}\otimes_{\mathrm{F}}\mathrm{L}=\mathrm{K}^1\oplus\cdots\oplus\mathrm{K}^f \tag{6.1}$$

into simple $\mathrm{K}\otimes_{\mathrm{F}}\mathrm{L}$-modules, where f denotes the degree of K/F. For each $i\in\{1,\ldots,f\}$, we write e^i for the minimal idempotent in $\mathrm{K}\otimes_{\mathrm{F}}\mathrm{L}$ corresponding to K^i. The centralizer of $\mathrm{K}\otimes_{\mathrm{F}}\mathrm{L}$ in $\overline{\mathrm{A}}$, denoted $\overline{\mathrm{U}}$, is equal to $\mathrm{C}\otimes_{\mathrm{F}}\mathrm{L}$. By identifying it with $\mathrm{C}\otimes_{\mathrm{K}}(\mathrm{K}\otimes_{\mathrm{F}}\mathrm{L})$ and using (6.1), we get a decomposition:

$$\overline{\mathrm{U}}=\overline{\mathrm{U}}^1\oplus\cdots\oplus\overline{\mathrm{U}}^f,$$

where the K^i-algebra $\overline{\mathrm{U}}^i=\mathrm{e}^i\overline{\mathrm{A}}\mathrm{e}^i$ identifies with $\mathrm{C}\otimes_{\mathrm{K}}\mathrm{K}^i$ for each $i\in\{1,\ldots,f\}$.

In a similar way, we may consider the centralizer C of K in A as being embedded in the split L-algebra $\overline{\mathrm{C}} = \mathrm{C} \otimes_{\mathrm{K}} \mathrm{L}$ by the K-algebra homomorphism $j_{\mathrm{C}} : c \mapsto c \otimes_{\mathrm{K}} 1$.

Similarly to the case of simple characters (see paragraph 5.4), we will define the interior lift of a quasi-simple character by restriction from $\overline{\mathrm{A}}$ to $\overline{\mathrm{C}}$. For this we need an embedding of $\overline{\mathrm{C}}$ in $\overline{\mathrm{A}}$ satisfying some conditions with respect to j_{A} and j_{C} (see below), but there is no canonical such embedding. We choose a set:

$$\mathrm{S} = \{\sigma_1, \dots, \sigma_f\} \subseteq \mathrm{Gal}(\mathrm{L}/\mathrm{F}) \tag{6.2}$$

of representatives of $\mathrm{Hom}_{\mathrm{F}}(\mathrm{K}, \mathrm{L})$ in $\mathrm{Gal}(\mathrm{L}/\mathrm{F})$, that is a subset of $\mathrm{Gal}(\mathrm{L}/\mathrm{F})$ such that the restriction map from L to K induces a bijection from S to $\mathrm{Hom}_{\mathrm{F}}(\mathrm{K}, \mathrm{L})$. For simplicity, we assume that we have ordered the e^i's so that:

$$\begin{gathered} \mathrm{K}^1 \textit{ and } \mathrm{L} \textit{ are isomorphic } \mathrm{K} \otimes \mathrm{L}\textit{-modules} \\ \textit{and } \sigma_i(\mathrm{e}^1) = \mathrm{e}^i \textit{ for any } i \in \{1, \dots, f\}. \end{gathered} \tag{6.3}$$

This gives us an F-algebra homomorphism:

$$\varkappa : \overline{\mathrm{C}} \xrightarrow{\simeq} \overline{\mathrm{U}}^1 \subseteq \overline{\mathrm{U}}, \tag{6.4}$$

and $\sigma_i \circ \varkappa$ is an F-algebra homomorphism from $\overline{\mathrm{C}}$ to $\overline{\mathrm{U}}^i$ for each integer $i \in \{1, \dots, f\}$. The following lemma gives us a relationship between (6.4) and the embeddings j_{A} and j_{C}.

Lemma 6.1. — Let $j_{\mathrm{A,C}}$ denote the restriction of j_{A} to C*, with values in $\overline{\mathrm{U}}$. Then the* F*-algebra homomorphism from $\overline{\mathrm{C}}$ to $\overline{\mathrm{U}}$ defined by:*

$$\iota = \iota_{\mathrm{S}} : x \mapsto \sigma_1 \circ \varkappa(x) + \dots + \sigma_f \circ \varkappa(x) \tag{6.5}$$

satisfies the equality $\iota \circ j_{\mathrm{C}} = j_{\mathrm{A,C}}$.

Proof. — We have $\sigma_i(\mathrm{e}^1 j_{\mathrm{A}}(x)) = \mathrm{e}^i j_{\mathrm{A}}(x)$ for all $i \in \{1, \dots, f\}$ and $x \in \mathrm{C}$, which implies that $\iota \circ \mathrm{e}^1 j_{\mathrm{A,C}} = j_{\mathrm{A,C}}$. Note that $\mathrm{e}^1 j_{\mathrm{A,C}} = j_{\mathrm{C}}$, so that we get the expected equality. □

6.2. Let $[\Lambda, n, m, \beta]$ be a simple stratum in A, which is a realization of a simple pair (k, β) over F. *In this paragraph, we assume that* Λ *is a strict lattice sequence.* If we fix a simple left $\overline{\mathrm{A}}$-module $\overline{\mathrm{V}}$, then there is a unique (up to translation) $\mathcal{O}_{\mathrm{L}}$-lattice sequence $\overline{\Lambda}$ on $\overline{\mathrm{V}}$ such that:

$$\mathfrak{P}_k(\overline{\Lambda}) = \mathfrak{P}_k(\Lambda) \otimes_{\mathcal{O}_{\mathrm{F}}} \mathcal{O}_{\mathrm{L}}, \quad k \in \mathbb{Z} \tag{6.6}$$

(see [20, §2.2]). This provides us with a stratum $[\overline{\Lambda}, n, m, \beta]$ in $\overline{\mathrm{A}}$, called the *quasi-simple* L/F*-lift* of the simple stratum $[\Lambda, n, m, \beta]$. This quasi-simple lift is pure if and only if the residue class degree of E over F is prime to the degree of L over F, and in this case it is a simple stratum (*ibid.*).

In [20] (see paragraph 3.2.3), one attaches to the stratum $[\overline{\Lambda}, n, m, \beta]$ a compact open subgroup $\mathrm{H}^{m+1}(\beta, \overline{\Lambda})$ of $\overline{\mathrm{A}}^{\times}$ and a set $\mathcal{Q}(\overline{\Lambda}, m, \beta)$ of characters of the group $\mathrm{H}^{m+1}(\beta, \overline{\Lambda})$, called *quasi-simple characters* of level m and depending on an additive character:

$$\boldsymbol{\Psi} : \mathrm{L} \to \mathbb{C}^\times \tag{6.7}$$

extending the additive character (1.2), being trivial on $\mathfrak{p}_\mathrm{L}$ but not on $\mathcal{O}_\mathrm{L}$. Recall that the restriction map from $\mathrm{H}^{m+1}(\beta, \overline{\Lambda})$ to $\mathrm{H}^{m+1}(\beta, \Lambda)$ induces a surjective map from $\mathcal{Q}(\overline{\Lambda}, m, \beta)$ to $\mathcal{C}(\Lambda, m, \beta)$.

Let $[\Lambda', n', m', \beta]$ be another realization of (k, β) in a simple central F-algebra A', *with* Λ' *strict. We assume that* Λ *and* Λ' *have the same period and that* $m = m'$ *is a multiple of* k. We assume that the extension L/F is chosen such that the L-algebras $\overline{\mathrm{A}}$ and $\overline{\mathrm{A}}'$ are both split, and we set:

$$\mathrm{V}^0 = \overline{\mathrm{V}} \oplus \overline{\mathrm{V}}', \quad \Lambda^0 = \overline{\Lambda} \oplus \overline{\Lambda}'. \tag{6.8}$$

Then Λ^0 is a strict $\mathcal{O}_\mathrm{L}$-lattice sequence on the L-vector space V^0. Moreover $\mathrm{A}^0 = \mathrm{End}_\mathrm{L}(\mathrm{V}^0)$ is a split simple central L-algebra, in which $\mathrm{E} = \mathrm{F}(\beta)$ is naturally embedded. We write M for $\overline{\mathrm{A}}^\times \times \overline{\mathrm{A}}'^\times$ considered as a Levi subgroup of $\mathrm{A}^{0\times}$. We have the decomposition:

$$\mathrm{H}^{m+1}(\beta, \Lambda^0) \cap \mathrm{M} = \mathrm{H}^{m+1}(\beta, \overline{\Lambda}) \times \mathrm{H}^{m+1}(\beta, \overline{\Lambda}'). \tag{6.9}$$

We will need the following characterization of the transfer map.

Proposition 6.2. — Let $\theta \in \mathcal{C}(\Lambda, m, \beta)$ *and* $\theta' \in \mathcal{C}(\Lambda', m', \beta)$ *be two simple characters. Assume* Λ *and* Λ' *are strict, have the same period and* $m = m'$ *is a multiple of* k. *Then* θ' *is the transfer of* θ *if and only if there exists* $\boldsymbol{\theta}^0 \in \mathcal{Q}(\Lambda^0, m, \beta)$ *such that:*

$$\boldsymbol{\theta}^0 \mid \mathrm{H}^{m+1}(\beta, \Lambda) \times \mathrm{H}^{m+1}(\beta, \Lambda') = \theta \otimes \theta'. \tag{6.10}$$

Proof. — Recall (see [20, §3.3]) that θ and θ' are transfers of each other if and only if there exist two quasi-simple characters $\boldsymbol{\theta} \in \mathcal{Q}(\overline{\Lambda}, m, \beta)$ and $\boldsymbol{\theta}' \in \mathcal{Q}(\overline{\Lambda}', m, \beta)$, extending θ and θ' respectively, which are transfers of each other.

Lemma 6.3. — The map from $\mathcal{Q}(\Lambda^0, m, \beta)$ *to* $\mathcal{Q}(\overline{\Lambda}, m, \beta)$ *induced by the restriction from* $\mathrm{H}^{m+1}(\beta, \Lambda^0)$ *to* $\mathrm{H}^{m+1}(\beta, \overline{\Lambda})$ *is the transfer.*

Proof. — We have a decomposition of the L-algebra $\mathrm{E} \otimes_\mathrm{F} \mathrm{L}$ into simple $\mathrm{E} \otimes_\mathrm{F} \mathrm{L}$-modules E^j, for $j \in \{1, \ldots, s\}$, where s denotes the greatest common divisor of the degree of L/F and the residue class degree of E/F. For each j, we write $\mathbf{1}^j$ for the minimal idempotent in $\mathrm{E} \otimes_\mathrm{F} \mathrm{L}$ corresponding to E^j, as well as $\Lambda^{0,j}$ for the projection of Λ^0 onto $\mathrm{V}^{0,j} = \mathbf{1}^j \mathrm{V}^0$ and β^j for $\mathbf{1}^j \beta$. Thus we get a simple stratum $[\Lambda^{0,j}, n, m, \beta^j]$ in the F-algebra $\mathrm{A}^{0,j} = \mathbf{1}^j \mathrm{A}^0 \mathbf{1}^j$ and, similarly, we get a simple stratum $[\overline{\Lambda}^j, n, m, \beta^j]$ in $\overline{\mathrm{A}}^j$. By [20, Corollaire 3.34], there are bijections:

$$\mathcal{Q}(\Lambda^0, m, \beta) \to \prod_{j=1}^{s} \mathcal{C}(\Lambda^{0,j}, m, \beta^j), \quad \mathcal{Q}(\overline{\Lambda}, m, \beta) \to \prod_{j=1}^{s} \mathcal{C}(\overline{\Lambda}^j, m, \beta^j),$$

which are compatible with transfer. Therefore, it is enough to prove that, for each j, the map from $\mathcal{C}(\Lambda^{0,j}, m, \beta^j)$ to $\mathcal{C}(\overline{\Lambda}^j, m, \beta^j)$ induced by the restriction from $\mathrm{H}^{m+1}(\beta^j, \Lambda^{0,j})$ to $\mathrm{H}^{m+1}(\beta^j, \overline{\Lambda}^j)$ is the transfer. This is [24, Théorème 2.17]. □

Assume first that there is a quasi-simple character $\boldsymbol{\theta}^0 \in \mathcal{Q}(\Lambda^0, m, \beta)$ such that (6.10) is satisfied, and write $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ for the restrictions of $\boldsymbol{\theta}^0$ to $\mathrm{H}^{m+1}(\beta, \overline{\Lambda})$ and $\mathrm{H}^{m+1}(\beta, \overline{\Lambda}')$, respectively. By Lemma 6.3, these are quasi-simple characters which are transfers of each other. By (6.10), they extend the simple characters θ and θ'. It follows that θ and θ' are transfers of each other.
Conversely, assume θ and θ' are transfers of each other. Let $\boldsymbol{\theta}$ be a quasi-simple character in $\mathcal{Q}(\overline{\Lambda}, m, \beta)$ extending θ, and let $\boldsymbol{\theta}^0$ be its transfer to $\mathcal{Q}(\Lambda^0, m, \beta)$. By Lemma 6.3, the restriction of $\boldsymbol{\theta}^0$ to $\mathrm{H}^{m+1}(\beta, \overline{\Lambda}')$ is the transfer of $\boldsymbol{\theta}$, and thus extends θ'. Therefore, the identity (6.10) is satisfied. □

6.3. Let $[\Lambda, n, m, \beta]$ be a K-pure simple stratum in A, and let $[\Gamma, n, m, \beta]$ denote the simple stratum in C associated with $[\Lambda, n, m, \beta]$ by (5.2). *In this paragraph, we assume that* Λ *and* Γ *are strict lattice sequences.*

If we fix a simple left $\overline{\mathrm{C}}$-module $\overline{\mathrm{W}}$, we can form the quasi-simple lift $[\overline{\Gamma}, n, m, \beta]$ of the simple stratum $[\Gamma, n, m, \beta]$ with respect to L/K. One attaches to this quasi-simple lift a compact open subgroup $\mathrm{H}^{m+1}(\beta, \overline{\Gamma})$ of $\overline{\mathrm{C}}^\times$ and a set $\mathcal{Q}(\overline{\Gamma}, m, \beta)$ of characters of $\mathrm{H}^{m+1}(\beta, \overline{\Gamma})$ with respect to the additive character:

$$\boldsymbol{\Psi}_{\mathrm{K}} = \boldsymbol{\Psi} \circ (\sigma_1 + \cdots + \sigma_f) \tag{6.11}$$

of L, depending on the choice of the set S fixed in (6.2). It is trivial on $\mathfrak{p}_{\mathrm{L}}$ and, thanks to the condition on S, it extends the character Ψ_{K} defined by (5.5); hence it is not trivial on $\mathcal{O}_{\mathrm{L}}$. This comes with a surjective restriction map from $\mathcal{Q}(\overline{\Gamma}, m, \beta)$ to $\mathcal{C}(\Gamma, m, \beta)$.

Lemma 6.4. — The image of $\mathrm{H}^{m+1}(\beta, \overline{\Gamma})$ *by* ι *is contained in* $\mathrm{H}^{m+1}(\beta, \overline{\Lambda})$.

Proof. — First we have to prove that:

$$\varkappa(\mathrm{H}^{m+1}(\beta, \overline{\Gamma})) = \mathrm{H}^{m+1}(\beta, \overline{\Lambda}) \cap \overline{\mathrm{U}}^1 = \mathrm{e}^1 \mathrm{H}^{m+1}(\beta, \overline{\Lambda}) \mathrm{e}^1.$$

This follows from the definition of the groups $\mathrm{H}^{m+1}(\beta, \overline{\Gamma})$ and $\mathrm{H}^{m+1}(\beta, \overline{\Lambda})$ by induction on β, and from the fact that e^1 commutes to β. According to (6.3), we get:

$$\sigma_i \circ \varkappa(\mathrm{H}^{m+1}(\beta, \overline{\Gamma})) = \mathrm{H}^{m+1}(\beta, \overline{\Lambda}) \cap \overline{\mathrm{U}}^i = \mathrm{e}^i \mathrm{H}^{m+1}(\beta, \overline{\Lambda}) \mathrm{e}^i$$

for each $i \in \{1, \ldots, f\}$, and the result follows. □

This gives rise to the following result.

Proposition 6.5. — Let $\theta \in \mathcal{C}(\Lambda, m, \beta)$ *be a simple character, let* $\boldsymbol{\theta} \in \mathcal{Q}(\overline{\Lambda}, m, \beta)$ *be a quasi-simple character extending* θ*, and set:*

$$\boldsymbol{\theta}^{\mathrm{K}}(x) = \boldsymbol{\theta}(\iota(x)), \quad x \in \mathrm{H}^{m+1}(\beta, \overline{\Gamma}). \tag{6.12}$$

Then $\boldsymbol{\theta}^{\mathrm{K}}$ *is a quasi-simple character in* $\mathcal{Q}(\overline{\Gamma}, m, \beta)$ *extending the character* $\theta^{\mathrm{K}} = \theta \mid \mathrm{H}^{m+1}(\beta, \Gamma)$.

Proof. — By Lemmas 6.1 and 6.4, the character $\boldsymbol{\theta}^{\mathrm{K}}$ is well defined and extends the simple character θ^{K}. It thus remains to prove that it is in $\mathcal{Q}(\overline{\Gamma}, m, \beta)$. The proof is by induction on β (see [20, Définition 3.22]). Assume first that β is minimal over F. Then it is minimal over K by Proposition 5.2. If $m \geqslant \lfloor n/2 \rfloor$, the set $\mathcal{Q}(\overline{\Lambda}, m, \beta)$ consists of a single element $\boldsymbol{\Psi}_{\beta}^{\overline{\mathrm{A}}}$, which is the character of $\mathrm{U}_{m+1}(\overline{\Lambda})$ defined by:

$$\boldsymbol{\Psi}_{\beta}^{\overline{\mathrm{A}}}(x) = \boldsymbol{\Psi} \circ \mathrm{tr}_{\overline{\mathrm{A}}/\mathrm{L}}(\beta(x-1)), \quad x \in \mathrm{U}_{m+1}(\overline{\Lambda}),$$

and the set $\mathcal{Q}(\overline{\Gamma}, m, \beta)$ consists of a single element $\boldsymbol{\Psi}_{\beta}^{\overline{\mathrm{C}}}$, which is the character of $\mathrm{U}_{m+1}(\overline{\Gamma})$ defined by:

$$\boldsymbol{\Psi}_{\beta}^{\overline{\mathrm{C}}}(x) = \boldsymbol{\Psi} \circ \mathrm{tr}_{\overline{\mathrm{C}}/\mathrm{L}}(\beta(x-1)), \quad x \in \mathrm{U}_{m+1}(\overline{\Gamma}).$$

So we just need to prove that:

$$\boldsymbol{\Psi}_{\beta}^{\overline{\mathrm{A}}} \circ \iota(x) = \boldsymbol{\Psi}_{\beta}^{\overline{\mathrm{C}}}(x), \quad x \in \mathrm{U}_{m+1}(\overline{\Gamma}), \tag{6.13}$$

which follows from the fact that:

$$\begin{aligned} \mathrm{tr}_{\overline{\mathrm{A}}/\mathrm{L}} \circ \iota &= \sum_{i=1}^{f} \mathrm{tr}_{\overline{\mathrm{A}}/\mathrm{L}} \circ \sigma_i \circ \varkappa \\ &= (\sigma_1 + \cdots + \sigma_f) \circ \mathrm{tr}_{\overline{\mathrm{A}}/\mathrm{L}} \circ \varkappa = (\sigma_1 + \cdots + \sigma_f) \circ \mathrm{tr}_{\overline{\mathrm{C}}/\mathrm{L}}. \end{aligned}$$

If $m \leqslant \lfloor n/2 \rfloor$, then $\boldsymbol{\theta}$ extends the character $\boldsymbol{\Psi}_{\beta}^{\overline{\mathrm{A}}} \mid \mathrm{U}_{\lfloor n/2 \rfloor + 1}(\overline{\Lambda})$ and its restriction to $\mathrm{U}_{m+1}(\overline{\Lambda}) \cap \overline{\mathrm{B}}^{\times}$ has the form:

$$\boldsymbol{\theta} \mid \mathrm{U}_{m+1}(\overline{\Lambda}) \cap \overline{\mathrm{B}}^{\times} = \boldsymbol{\chi} \circ \mathrm{N}_{\overline{\mathrm{B}}/\mathrm{E} \otimes_{\mathrm{F}} \mathrm{L}}, \tag{6.14}$$

where we write $\overline{\mathrm{B}}$ for the centralizer of E in $\overline{\mathrm{A}}$ and where $\boldsymbol{\chi}$ denotes some character of the subgroup $1 + \mathfrak{p}_{\mathrm{E}} \otimes \mathcal{O}_{\mathrm{L}}$ of $(\mathrm{E} \otimes_{\mathrm{F}} \mathrm{L})^{\times}$. Then, if we write $\overline{\mathrm{R}}$ for the centralizer of $\mathrm{K}(\beta)$ in $\overline{\mathrm{A}}$, the character $\boldsymbol{\theta}^{\mathrm{K}}$ extends $\boldsymbol{\Psi}_{\beta}^{\overline{\mathrm{C}}} \mid \mathrm{U}_{\lfloor n/2 \rfloor + 1}(\overline{\Gamma})$, and its restriction to $\mathrm{U}_{m+1}(\overline{\Gamma}) \cap \overline{\mathrm{R}}^{\times}$ has the form:

$$\boldsymbol{\theta}^{\mathrm{K}} \mid \mathrm{U}_{m+1}(\overline{\Gamma}) \cap \overline{\mathrm{R}}^{\times} = \boldsymbol{\chi}^{\mathrm{S}} \circ \mathrm{N}_{\overline{\mathrm{R}}/\mathrm{K}(\beta) \otimes_{\mathrm{K}} \mathrm{L}} \tag{6.15}$$

where $\boldsymbol{\chi}^{\mathrm{S}}$ is the product of all the $\boldsymbol{\chi} \circ \sigma_i$'s for all $i \in \{1, \ldots, f\}$, as required.
Assume now that β is not minimal over F. We set $q = -k_0(\beta, \Lambda)$ and $r = \lfloor q/2 \rfloor + 1$, and choose a simple stratum $[\Gamma, n, q, \gamma]$ equivalent to $[\Gamma, n, q, \beta]$ such that $[\Lambda, n, q, \gamma]$ is simple and K-pure. By [5, Theorem 5.1] and Proposition 5.4 together, one may assume that the maximal unramified extension of F contained in $\mathrm{F}(\gamma)$ is contained in that of $\mathrm{F}(\beta)$, which implies that the L-canonical decomposition of γ is finer than that of β (see paragraph 2.3.4 and the proof of Lemme 3.16 in [20]). If $m \geqslant \lfloor q/2 \rfloor$, then any $\boldsymbol{\theta} \in \mathcal{Q}(\overline{\Lambda}, m, \beta)$ can be written as

$\boldsymbol{\theta} = \boldsymbol{\theta}_0 \boldsymbol{\Psi}^{\overline{\mathrm{A}}}_{\beta-\gamma}$ for some quasi-simple character $\boldsymbol{\theta}_0 \in \mathcal{Q}(\overline{\Lambda}, m, \gamma)$. Now we claim that:

$$\mathrm{H}^{m+1}(\beta, \overline{\Gamma}) = \mathrm{H}^{m+1}(\gamma, \overline{\Gamma}). \tag{6.16}$$

We write $q_1 = -k_0(\beta, \Gamma)$. If $q_1 = q$, then the equality (6.16) follows by definition. Otherwise, we have $q_1 > q$ by Proposition 5.2. The strata $[\Gamma, n, q, \beta]$ and $[\Gamma, n, q, \gamma]$ are thus simple, and (6.16) follows. We now form the character $\boldsymbol{\theta}^{\mathrm{K}} = \boldsymbol{\theta} \circ \iota \mid \mathrm{H}^{m+1}(\beta, \overline{\Gamma})$ and get the equality $\boldsymbol{\theta}^{\mathrm{K}} = \boldsymbol{\theta}_0^{\mathrm{K}} \boldsymbol{\Psi}^{\overline{\mathrm{C}}}_{\beta-\gamma}$, where $\boldsymbol{\theta}_0^{\mathrm{K}}$ denotes the character $\boldsymbol{\theta}_0 \circ \iota \mid \mathrm{H}^{m+1}(\gamma, \overline{\Gamma})$, and this character is in $\mathcal{Q}(\overline{\Gamma}, m, \gamma)$ by the inductive hypothesis. If $q_1 = q$, then $\boldsymbol{\theta}^{\mathrm{K}}$ is in $\mathcal{Q}(\overline{\Gamma}, m, \beta)$ by definition. Otherwise, the strata $[\Gamma, n, q, \beta]$ and $[\Gamma, n, q, \gamma]$ are simple and the result follows from [24, Proposition 2.15]. The case $m \leqslant \lfloor q/2 \rfloor$ reduces to the previous one as in the minimal case.

It remains to prove that the subgroup $\mathfrak{K}(\overline{\Gamma}) \cap \overline{\mathrm{R}}^{\times}$ normalizes $\boldsymbol{\theta}^{\mathrm{K}}$. If $g \in \mathfrak{K}(\overline{\Gamma}) \cap \overline{\mathrm{R}}^{\times}$, then we have:

$$\iota(g) \cdot \overline{\Lambda}_k = \bigoplus_{i=1}^{f} \sigma_i(\varkappa(g)) \cdot \mathrm{e}^i \overline{\Lambda}_k = \bigoplus_{i=1}^{f} \mathrm{e}^i \overline{\Lambda}_{k+\upsilon(\sigma_i(\varkappa(g)))} \tag{6.17}$$

where υ denotes the valuation map associated with $\overline{\Lambda}$. As all the $\sigma_i(\varkappa(g))$'s have the same valuation, the equality (6.17) gives us $\iota(g) \in \mathfrak{K}(\overline{\Lambda}) \cap \overline{\mathrm{B}}^{\times}$. Proposition 6.5 now follows from the fact that $\mathfrak{K}(\overline{\Lambda}) \cap \overline{\mathrm{B}}^{\times}$ normalizes $\boldsymbol{\theta}$. □

Remark 6.6. — Note that the interior lifting map from $\mathcal{Q}(\overline{\Lambda}, m, \beta)$ to $\mathcal{Q}(\overline{\Gamma}, m, \beta)$ defined by Proposition 6.5 depends on the choice of the set S chosen in (6.2).

6.4. Let $[\Lambda, n, m, \beta]$ and $[\Lambda', n', m', \beta]$ be realizations of a simple pair (k, β) over F in simple central F-algebras A and A′, respectively. Assume further that A and A′ contain K, that the strata $[\Lambda, n, m, \beta]$ and $[\Lambda', n', m', \beta]$ are K-pure and that the strata $[\Gamma, n, m, \beta]$ and $[\Gamma', n', m', \beta]$ associated with them by (5.2) are realizations of the same simple pair over K. (This is equivalent to saying that the extensions of K generated by β in A and A′ are K-isomorphic.) We have the following relation between the transfer maps and the interior lifting maps.

Theorem 6.7. — Let $\theta \in \mathcal{C}(\Lambda, m, \beta)$ and $\theta' \in \mathcal{C}(\Lambda', m', \beta)$ be transfers of each other. Then the simple characters:

$$\theta \mid \mathrm{H}^{m+1}(\beta, \Gamma), \quad \theta' \mid \mathrm{H}^{m'+1}(\beta, \Gamma')$$

are transfers of each other.

Proof. — One can assume without loss of generality that m and m' are multiples of k. By rescaling the lattice sequences Λ and Λ', one can also assume that they have the same period thanks to Lemma 2.2. Thus $m = m'$ and $n = n'$. The proof decomposes into two parts.

(1) First we prove the theorem in the case where all the lattice sequences are strict, so that we can apply the results of paragraphs 6.2 and 6.3. We fix a quasi-simple character $\boldsymbol{\theta}$ in $\mathcal{Q}(\overline{\Lambda}, m, \beta)$ extending θ and write $\boldsymbol{\theta}'$ for its transfer in $\mathcal{Q}(\overline{\Lambda}', m, \beta)$. The restriction of $\boldsymbol{\theta}'$ to $\mathrm{H}^{m+1}(\beta, \Lambda')$ is thus equal to θ'. By Proposition 6.2, there exists a quasi-simple character $\boldsymbol{\theta}^0$ in $\mathcal{Q}(\Lambda^0, m, \beta)$ extending $\boldsymbol{\theta} \otimes \boldsymbol{\theta}'$. We write $\overline{\mathrm{C}}$ and $\overline{\mathrm{U}}$ as in paragraph 6.1, and use similar notations $\overline{\mathrm{C}}'$ and $\overline{\mathrm{U}}'$. We have:

$$\mathrm{H}^{m+1}(\beta, \Lambda^0) \cap \big(\overline{\mathrm{C}}^{\times} \times \overline{\mathrm{C}}'^{\times}\big) = \mathrm{H}^{m+1}(\beta, \overline{\Gamma}) \times \mathrm{H}^{m+1}(\beta, \overline{\Gamma}'). \tag{6.18}$$

We define ι by (6.5) and write $\boldsymbol{\theta}^{\mathrm{K}}$ for the quasi-simple character defined by (6.12). We also define ι' and $\boldsymbol{\theta}'^{\mathrm{K}}$ in a similar way. If we restrict the map $x \mapsto (\iota(x), \iota'(x))$ to the subgroup (6.18) and then compose it with $\boldsymbol{\theta}^0$, then we get the character $\boldsymbol{\theta}^{\mathrm{K}} \otimes \boldsymbol{\theta}'^{\mathrm{K}}$. This implies that $\boldsymbol{\theta}^{\mathrm{K}}$ and $\boldsymbol{\theta}'^{\mathrm{K}}$ are transfers of each other. By Propositions 6.5 and 6.2 together, their restrictions $\boldsymbol{\theta}^{\mathrm{K}} \mid \mathrm{H}^{m+1}(\beta, \Gamma) = \theta^{\mathrm{K}}$ and $\boldsymbol{\theta}'^{\mathrm{K}} \mid \mathrm{H}^{m+1}(\beta, \Gamma') = \theta'^{\mathrm{K}}$ are transfers of each other.

(2) We now reduce the general case to Case (1). For this we fix a positive integer l as in Lemma 2.16, and form the sound simple strata $[\Lambda^{\ddagger}, n, m, \beta]$ and $[\Lambda'^{\ddagger}, n, m, \beta]$. Write $\mathrm{C}^{\ddagger}$ for the centralizer of K in $\mathrm{A}^{\ddagger}$ and $[\Gamma^{\ddagger}, n, m, \beta]$ for the simple stratum in $\mathrm{C}^{\ddagger}$ associated with $[\Lambda^{\ddagger}, n, m, \beta]$ by (5.2). In a similar way, we have a K-algebra $\mathrm{C}'^{\ddagger}$ and a simple stratum $[\Gamma'^{\ddagger}, n, m, \beta]$. Then the simple strata $[\Gamma^{\ddagger}, n, m, \beta]$ and $[\Gamma'^{\ddagger}, n, m, \beta]$ are realizations of the same simple pair over K. Write $\theta^{\ddagger}$ for the transfer of θ in $\mathcal{C}(\Lambda^{\ddagger}, m, \beta)$. In a similar way, we have a simple character $\theta'^{\ddagger}$. By Case (1), the simple characters:

$$\theta^{\ddagger} \mid \mathrm{H}^{m+1}(\beta, \Gamma^{\ddagger}), \quad \theta'^{\ddagger} \mid \mathrm{H}^{m+1}(\beta, \Gamma'^{\ddagger})$$

are transfers of each other. Thus it remains to prove the following lemma.

Lemma 6.8. — The characters $\theta \mid \mathrm{H}^{m+1}(\beta, \Gamma)$ and $\theta^{\ddagger} \mid \mathrm{H}^{m+1}(\beta, \Gamma^{\ddagger})$ are transfers of each other.

Proof. — Write M for the Levi subgroup of $\mathrm{A}^{\ddagger\times}$ defined by the decomposition of $\mathrm{V}^{\ddagger}$ into copies of V. According to Lemma 2.7, the character $\theta^{\ddagger}$ is characterized by the identity:

$$\theta^{\ddagger} \mid \mathrm{H}^{m+1}(\beta, \Lambda^{\ddagger}) \cap \mathrm{M} = \theta \otimes \cdots \otimes \theta.$$

Thus its restriction to $\mathrm{H}^{m+1}(\beta, \Gamma^{\ddagger}) \cap \mathrm{M} = \mathrm{H}^{m+1}(\beta, \Gamma) \times \cdots \times \mathrm{H}^{m+1}(\beta, \Gamma)$ is equal to the tensor product of l copies of θ^{K}. ☐

This ends the proof of Theorem 6.7. ☐

Remark 6.9. — In the case where $[\Lambda, n, m, \beta]$ and $[\Lambda', n', m', \beta]$ are sound, this theorem implies that Grabitz's transfer [17] is the same as the transfer defined in [20].

6.5. Before closing this section, we prove the following result. Let $[\Lambda, n, m, \beta]$ be a simple K-pure stratum in A, and write $[\Gamma, n, m, \beta]$ for the simple stratum in C which corresponds to it by (5.2). Theorem 5.7 gives us a map from $\mathcal{C}(\Lambda, m, \beta)$ to $\mathcal{C}(\Gamma, m, \beta)$, called the interior lifting map, and denoted $\boldsymbol{l}_{\mathrm{K/F}} : \theta \mapsto \theta^{\mathrm{K}}$. It has the following properties.

Proposition 6.10. — The map $\boldsymbol{l}_{\mathrm{K/F}}$ is injective and $\mathfrak{K}(\Gamma)$-equivariant.

Proof. — Note that the second assertion is immediate. Let us fix a positive integer $l \geqslant 1$ as in Lemma 2.16, and form the sound simple stratum $[\Lambda^{\ddagger}, n, m, \beta]$. Write $\mathrm{C}^{\ddagger}$ for the centralizer of K in $\mathrm{A}^{\ddagger}$ and $[\Gamma^{\ddagger}, n, m, \beta]$ for the simple stratum in $\mathrm{C}^{\ddagger}$ associated with the stratum $[\Lambda^{\ddagger}, n, m, \beta]$ by (5.2). Now let $\theta \in \mathcal{C}(\Lambda, m, \beta)$ be a simple character and write $\theta^{\ddagger}$ for its transfer in $\mathcal{C}(\Lambda^{\ddagger}, m, \beta)$. Then, by Lemma 6.8, the transfer of θ^{K} to $\mathcal{C}(\Gamma^{\ddagger}, m, \beta)$ is equal to $\theta^{\ddagger} \mid \mathrm{H}^{m+1}(\beta, \Gamma^{\ddagger})$. As the transfer map from $\mathcal{C}(\Lambda, m, \beta)$ to $\mathcal{C}(\Lambda^{\ddagger}, m, \beta)$ is bijective, we may replace Λ by $\Lambda^{\ddagger}$ and assume that the stratum $[\Lambda, n, m, \beta]$ is sound. In this case, the injectivity of the map $\boldsymbol{l}_{\mathrm{K/F}}$ follows from [17, Proposition 7.1]. $\square$

Assume we are given two K-pure simple strata $[\Lambda, n, m, \beta_i]$, $i = 1, 2$, in A. For each i, let θ_i be a simple character in $\mathcal{C}(\Lambda, m, \beta_i)$.

Proposition 6.11. — Assume θ_1 and θ_2 are equal. Then $\boldsymbol{l}_{\mathrm{K/F}}(\theta_1)$ and $\boldsymbol{l}_{\mathrm{K/F}}(\theta_2)$ are equal.

Proof. — It suffices to verify that the groups $\mathrm{H}^{m+1}(\beta_i, \Gamma)$, $i = 1, 2$, are equal. This follows from Proposition 5.5 and the fact that the groups $\mathrm{H}^{m+1}(\beta_i, \Lambda)$, $i = 1, 2$, are equal. $\square$

7. The base change

In this section, we develop a base change process for simple strata and characters with respect to a finite unramified extension K of F, in a way similar to [6].

7.1. Let K/F be an unramified extension of degree f. Given a simple central F-algebra A, we set:

$$\widehat{\mathrm{A}} = \mathrm{A} \otimes_{\mathrm{F}} \mathrm{End}_{\mathrm{F}}(\mathrm{K}).$$

Then K embeds naturally in $\widehat{\mathrm{A}}$, and its centralizer, denoted A_{K}, is canonically isomorphic to $\mathrm{A} \otimes_{\mathrm{F}} \mathrm{K}$ as a K-algebra. Let V be a simple left A-module. Then $\widehat{\mathrm{V}} = \mathrm{V} \otimes_{\mathrm{F}} \mathrm{K}$ is a simple left $\widehat{\mathrm{A}}$-module and, if we fix an F-basis of K, we have a decomposition:

$$\widehat{\mathrm{V}} = \mathrm{V} \oplus \cdots \oplus \mathrm{V} \tag{7.1}$$

of $\widehat{\mathrm{V}}$ into a sum of f copies of V, so that we are in the situation of paragraph 2.2.

Let $[\Lambda, n, m, \beta]$ be a simple stratum in A and set $\mathrm{E} = \mathrm{F}(\beta)$. Let us form the simple stratum $[\widehat{\Lambda}, n, m, \beta]$ in $\widehat{\mathrm{A}}$, where $\widehat{\Lambda} = \Lambda \oplus \cdots \oplus \Lambda$ is the direct sum of f copies of Λ. This simple stratum is not K-pure in general. We have a decomposition:

$$\mathrm{E} \otimes_{\mathrm{F}} \mathrm{K} = \mathrm{E}^1 \oplus \cdots \oplus \mathrm{E}^s$$

into simple $\mathrm{E} \otimes_{\mathrm{F}} \mathrm{K}$-modules, where s denotes the greatest common divisor of f and the residue class degree of E over F. For each $j \in \{1, \ldots, s\}$, we write c^j for the minimal idempotent in $\mathrm{E} \otimes_{\mathrm{F}} \mathrm{K}$ corresponding to E^j, and we set:

$$\beta^j = \mathrm{c}^j \beta, \quad j \in \{1, \ldots, s\}.$$

These are the various K/F-lifts of β. If we write $\widehat{\Lambda}^j$ for the projection of $\widehat{\Lambda}$ onto the space $\widehat{\mathrm{V}}^j = \mathrm{c}^j \widehat{\mathrm{V}}$ for each j, we get a simple stratum $[\widehat{\Lambda}^j, n, m, \beta^j]$ in the F-algebra $\widehat{\mathrm{A}}^j = \mathrm{c}^j \widehat{\mathrm{A}} \mathrm{c}^j$, which is K-pure for the natural embedding of K in $\widehat{\mathrm{A}}^j$. Thus one can form the interior lift $[\widehat{\Gamma}^j, n, m, \beta^j]$ in the centralizer of K in $\widehat{\mathrm{A}}^j$ (see paragraph 5.1).

Given a simple character $\theta \in \mathcal{C}(\Lambda, m, \beta)$, let $\widehat{\theta}$ denote its transfer to $\mathcal{C}(\widehat{\Lambda}, m, \beta)$ and write $\widehat{\theta}^j$ for the transfer of $\widehat{\theta}$ to $\mathcal{C}(\widehat{\Lambda}^j, m, \beta^j)$, that is the restriction of $\widehat{\theta}$ to $\mathrm{H}^{m+1}(\beta^j, \widehat{\Lambda}^j)$. Let us denote by θ^j_{K} the restriction of $\widehat{\theta}^j$ to $\mathrm{H}^{m+1}(\beta^j, \widehat{\Gamma}^j)$, which belongs to $\mathcal{C}(\widehat{\Gamma}^j, m, \beta^j)$ by Theorem 5.7. We have the following definition.

Definition 7.1. — The process:

$$\boldsymbol{b}_{\mathrm{K/F}} : \theta \mapsto \{\theta^j_{\mathrm{K}}, \ j = 1, \ldots, s\}$$

is the K/F-*base change* for simple characters. For each j, the simple character θ^j_{K} is called the K/F-lift of θ corresponding to the K/F-lift β^j of β.

Now let (Θ, k, β) be a ps-character over F. Let $[\Lambda, n, m, \varphi(\beta)]$ be a realization of the pair (k, β) in a simple central F-algebra A, and let θ denote the simple character $\Theta(\Lambda, m, \varphi)$. Let (k, β^j), for $j \in \{1, \ldots, s\}$, be the various K/F-lifts of the pair (k, β), and let φ^j denote the homomorphism of K-algebras from $\mathrm{K}(\beta^j)$ to the centralizer of K in $\widehat{\mathrm{A}}^j$ induced by φ. Thus the sum of the φ^j's is the K-algebra homomorphism $\varphi \otimes \mathrm{id}_{\mathrm{K}}$ from $\mathrm{E} \otimes_{\mathrm{F}} \mathrm{K}$ to A_{K}. For each j, let us denote by $(\Theta^j_{\mathrm{K}}, k, \beta^j)$ the ps-character defined by $([\widehat{\Gamma}^j, n, m, \beta^j], \theta^j_{\mathrm{K}})$.

Definition 7.2. — The process:

$$\boldsymbol{b}_{\mathrm{K/F}} : (\Theta, k, \beta) \mapsto \{(\Theta^j_{\mathrm{K}}, k, \beta^j), \ j = 1, \ldots, s\}$$

is the K/F-*base change* for ps-characters, and Θ^j_{K} is called the K/F-lift of Θ corresponding to the K/F-lift β^j of β.

This definition does not depend on the choice of the realization $[\Lambda, n, m, \varphi(\beta)]$. Indeed, let $[\Lambda', n', m', \varphi'(\beta)]$ be another realization of (k, β) in a simple central F-algebra A', and let us write θ' for the transfer of θ to $\mathcal{C}(\Lambda', m', \varphi'(\beta))$. Then it follows from Theorem 6.7 that, for each j, the K/F-lifts θ^j_{K} and θ'^j_{K} are transfers of each other.

7.2. In this paragraph, we study in more details the case where $s = 1$, that is the case where the residue class degree of $\mathrm{F}(\beta)/\mathrm{F}$ is prime to f. In this case, the simple pair (k, β) has exactly one K/F-lift. If we write Λ_{K} for the $\mathcal{O}_{\mathrm{K}}$-lattice sequence defined by $\widehat{\Lambda}$, then the base change process gives rise to a map:

$$\boldsymbol{b}_{\mathrm{K}/\mathrm{F}} : \mathcal{C}(\Lambda, m, \beta) \to \mathcal{C}(\Lambda_{\mathrm{K}}, m, \beta) \tag{7.2}$$

having the following properties.

Proposition 7.3. — The map $\boldsymbol{b}_{\mathrm{K}/\mathrm{F}}$ is injective and $\mathfrak{K}(\Lambda)$-equivariant.

Proof. — As $\boldsymbol{b}_{\mathrm{K}/\mathrm{F}}$ is the composite of the transfer map from $\mathcal{C}(\Lambda, m, \beta)$ to $\mathcal{C}(\widehat{\Lambda}, m, \beta)$ and the interior lifting from $\mathcal{C}(\widehat{\Lambda}, m, \beta)$ to $\mathcal{C}(\Lambda_{\mathrm{K}}, m, \beta)$, this follows from Proposition 6.10. ☐

Assume we are given two simple strata $[\Lambda, n_i, m_i, \beta_i]$, $i = 1, 2$, in A, such that $f_{\mathrm{F}}(\beta_1)$ and $f_{\mathrm{F}}(\beta_2)$ are prime to f. For each i, let θ_i be a simple character in $\mathcal{C}(\Lambda, m_i, \beta_i)$.

Proposition 7.4. — Assume θ_1 and θ_2 intertwine in $\mathrm{A}^\times$. Then $\boldsymbol{b}_{\mathrm{K}/\mathrm{F}}(\theta_1)$ and $\boldsymbol{b}_{\mathrm{K}/\mathrm{F}}(\theta_2)$ intertwine in $\mathrm{A}_{\mathrm{K}}^\times$.

Proof. — Assume θ_1 and θ_2 are intertwined by $g \in \mathrm{A}^\times$. By the proof of Proposition 2.6, the characters $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are intertwined by $\iota(g)$, where ι denotes the diagonal embedding of A in $\widehat{\mathrm{A}} = \mathrm{M}_f(\mathrm{A})$. As $\iota(g)$ is actually in $\mathrm{A}_{\mathrm{K}}^\times$, we deduce that the characters $\boldsymbol{b}_{\mathrm{K}/\mathrm{F}}(\theta_1)$ and $\boldsymbol{b}_{\mathrm{K}/\mathrm{F}}(\theta_2)$ intertwine in $\mathrm{A}_{\mathrm{K}}^\times$. ☐

We now suppose that $n_1 = n_2$ and $m_1 = m_2$.

Proposition 7.5. — Assume θ_1 and θ_2 are equal. Then $\boldsymbol{b}_{\mathrm{K}/\mathrm{F}}(\theta_1)$ and $\boldsymbol{b}_{\mathrm{K}/\mathrm{F}}(\theta_2)$ are equal.

Proof. — If θ_1 and θ_2 are equal, then Proposition 4.11 gives us $\widehat{\theta}_1 = \widehat{\theta}_2$ and Proposition 6.11 gives us the expected equality. ☐

Let $[\Lambda, n, m, \beta]$ and $[\Lambda', n', m', \beta]$ be two realizations of the simple pair (k, β), let θ be a simple character in $\mathcal{C}(\Lambda, m, \beta)$ and let θ' be its transfer in $\mathcal{C}(\Lambda', m', \beta)$. The following proposition is a special case of Theorem 6.7.

Proposition 7.6. — The character $\boldsymbol{b}_{\mathrm{K}/\mathrm{F}}(\theta')$ is the transfer of $\boldsymbol{b}_{\mathrm{K}/\mathrm{F}}(\theta)$ in $\mathcal{C}(\Lambda'_{\mathrm{K}}, m', \beta)$.

Finally, we will need the following result. Note that Gal(K/F) acts naturally on A_{K}.

Proposition 7.7. — Let $\theta \in \mathcal{C}(\Lambda_{\mathrm{K}}, m, \beta)$ be a simple character. Then for any $\sigma \in \mathrm{Gal}(\mathrm{K}/\mathrm{F})$, we have $\theta \circ \sigma \in \mathcal{C}(\Lambda_{\mathrm{K}}, m, \beta)$.

Proof. — One checks by induction on β that the image of $\mathcal{C}(\Lambda_{\mathrm{K}}, m, \beta)$ by $\theta \mapsto \theta \circ \sigma$ is the set of simple characters attached to the image of $[\Lambda_{\mathrm{K}}, n, m, \beta]$ by σ^{-1} with respect to the additive character $\Psi_{\mathrm{K}} \circ \sigma$. The result follows from the fact that this stratum and the additive character Ψ_{K} are invariant by σ. $\square$

7.3. We prove the following theorem, which generalizes [10, Corollary 3.6.3].

Theorem 7.8. — *For $i = 1, 2$, let (k_i, β_i) be a simple pair over* F. *Let us fix two realizations $[\Lambda, n, m, \beta_i]$ and $[\Lambda', n', m', \beta_i]$ of (k_i, β_i). Assume $\mathcal{C}(\Lambda, m, \beta_i)$ and $\mathcal{C}(\Lambda', m', \beta_i)$ do not depend on i. Then the transfer map $\boldsymbol{\tau}_i : \mathcal{C}(\Lambda, m, \beta_i) \to \mathcal{C}(\Lambda', m', \beta_i)$ does not depend on i.*

Proof. — The proof decomposes into three steps.

(1) In the first step, we reduce to the case where the strata are all sound. For this, we fix an integer l as in Proposition 2.17 which is large enough for Λ and Λ'. Write $\boldsymbol{a}_i$ for the transfer map from $\mathcal{C}(\Lambda, m, \beta_i)$ to $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_i)$. There is also a map $\boldsymbol{a}'_i$ for Λ'. Thus we have a commutative diagram:

$$\begin{tikzcd} \mathcal{C}(\Lambda^{\ddagger}, m, \beta_i) \arrow[r, "\boldsymbol{\tau}_i^{\ddagger}"] & \mathcal{C}(\Lambda'^{\ddagger}, m', \beta_i) \\ \mathcal{C}(\Lambda, m, \beta_i) \arrow[u, "\boldsymbol{a}_i"] \arrow[r, "\boldsymbol{\tau}_i"'] & \mathcal{C}(\Lambda', m', \beta_i) \arrow[u, "\boldsymbol{a}'_i"'] \end{tikzcd}$$

where $\tau_i^{\ddagger}$ denotes the transfer map from $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_i)$ to $\mathcal{C}(\Lambda'^{\ddagger}, m', \beta_i)$. By Proposition 4.11, the vertical maps $\boldsymbol{a}_i$ and $\boldsymbol{a}'_i$ do not depend on i, and Proposition 1.17 implies that the sets $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_i)$ and $\mathcal{C}(\Lambda'^{\ddagger}, m', \beta_i)$ do not depend on i. Since $\boldsymbol{a}'_i$ is bijective, the equality $\boldsymbol{\tau}_1^{\ddagger} = \boldsymbol{\tau}_2^{\ddagger}$ implies that $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$. We thus may replace Λ by $\Lambda^{\ddagger}$ and Λ' by $\Lambda'^{\ddagger}$ and assume that all the strata are sound.

(2) We now assume that all the strata are sound, and we reduce to the case where the extensions $\mathrm{F}(\beta_i)/\mathrm{F}$ are totally ramified. By Proposition 4.20, for each i, the simple strata $[\Lambda, n, m, \beta_i]$ and $[\Lambda', n', m', \beta_i]$ have the same embedding type. Write K_i for the maximal unramified extension of F contained in $\mathrm{F}(\beta_i)$, and fix $\theta_i \in \mathcal{C}(\Lambda, m, \beta_i)$. Assume that the characters θ_1 and θ_2 are equal. Using the "intertwining implies conjugacy" theorem [17, Corollary 10.15], one may assume that $\mathrm{K}_1 = \mathrm{K}_2$, denoted K. Write $\boldsymbol{l}_i$ for the interior lifting map from $\mathcal{C}(\Lambda, m, \beta_i)$ to $\mathcal{C}(\Gamma, m, \beta_i)$. There is also a map $\boldsymbol{l}'_i$ for Λ'. By Theorem 6.7, we have a commutative diagram:

$$\begin{tikzcd} \mathcal{C}(\Gamma, m, \beta_i) \arrow[r, "\boldsymbol{\tau}_i^{\mathrm{K}}"] & \mathcal{C}(\Gamma', m', \beta_i) \\ \mathcal{C}(\Lambda, m, \beta_i) \arrow[u, "\boldsymbol{l}_i"] \arrow[r, "\boldsymbol{\tau}_i"'] & \mathcal{C}(\Lambda', m', \beta_i) \arrow[u, "\boldsymbol{l}'_i"'] \end{tikzcd}$$

where $\boldsymbol{\tau}_i^{\mathrm{K}}$ denotes the transfer map from $\mathcal{C}(\Gamma, m, \beta_i)$ to $\mathcal{C}(\Gamma', m', \beta_i)$. By Proposition 6.11, the vertical maps $\boldsymbol{l}_i$ and $\boldsymbol{l}'_i$ do not depend on i, and Theorem 4.16

implies that the sets $\mathcal{C}(\Gamma, m, \beta_i)$ and $\mathcal{C}(\Gamma', m', \beta_i)$ do not depend on i. By the same argument as above, using that the map $\boldsymbol{l}'_i$ is injective (see Proposition 6.10), we may assume that $\mathrm{F}(\beta_i)$ is totally ramified over F.

(3) We now assume that $f_{\mathrm{F}}(\beta_1) = f_{\mathrm{F}}(\beta_2) = 1$, and reduce to the split case. Let us fix a finite unramified extension L/F such that the L-algebras $\overline{\mathrm{A}}$ and $\overline{\mathrm{A}}'$ are split. Write $\boldsymbol{b}_i$ for the base change map from $\mathcal{C}(\Lambda, m, \beta_i)$ to $\mathcal{C}(\overline{\Lambda}, m, \beta_i)$. There is also a map $\boldsymbol{b}'_i$ for Λ'. By Proposition 7.6, we have a commutative diagram:

$$\begin{tikzcd}
\mathcal{C}(\overline{\Lambda}, m, \beta_i) \arrow[r, "\overline{\boldsymbol{\tau}}_i"] & \mathcal{C}(\overline{\Lambda}', m', \beta_i) \\
\mathcal{C}(\Lambda, m, \beta_i) \arrow[u, "\boldsymbol{b}_i"] \arrow[r, "\boldsymbol{\tau}_i"'] & \mathcal{C}(\Lambda', m', \beta_i) \arrow[u, "\boldsymbol{b}'_i"']
\end{tikzcd}$$

where $\overline{\boldsymbol{\tau}}_i$ denotes the transfer map from $\mathcal{C}(\overline{\Lambda}, m, \beta_i)$ to $\mathcal{C}(\overline{\Lambda}', m', \beta_i)$. By Proposition 7.4, the maps $\boldsymbol{b}_i$ and $\boldsymbol{b}'_i$ do not depend on i. Thus [10, Theorem 3.5.8] (the rigidity theorem for simple characters in the split case) implies that the sets of simple characters $\mathcal{C}(\overline{\Lambda}, m, \beta_i)$ and $\mathcal{C}(\overline{\Lambda}', m', \beta_i)$ do not depend on i. By the same argument as above, using that the map $\boldsymbol{b}'_i$ is injective (see Proposition 7.3), we may assume that A is split and Λ is strict.

The result then follows from [10, Corollary 3.6.3]. $\square$

8. Endo-equivalence of simple characters

8.1. In this paragraph, we prove Theorem 1.13 in the totally ramified case. For $i = 1, 2$, let (Θ_i, k, β_i) be a ps-character over F with $f_{\mathrm{F}}(\beta_i) = 1$, and suppose that Θ_1 and Θ_2 are endo-equivalent. Let A be a simple central F-algebra and let $[\Lambda, n, m, \varphi_i(\beta_i)]$ be realizations of (k, β_i) in A, with $i = 1, 2$. Write θ_i for the simple character $\Theta_i(\Lambda, m, \varphi_i)$. We have to prove that θ_1 and θ_2 are conjugate under $\mathfrak{K}(\Lambda)$.

For each i, we write E_i for the F-algebra $\mathrm{F}(\beta_i)$, which is a totally ramified finite extension of F. By assumption, we have $[\mathrm{E}_1 : \mathrm{F}] = [\mathrm{E}_2 : \mathrm{F}]$. Using Proposition 4.9, there exists a simple central F-algebra A' together with sound realizations $[\Lambda', n', m', \varphi'_i(\beta_i)]$ of (k, β_i), with $i = 1, 2$, such that k divides m' and $\theta'_1 = \theta'_2$, where we write $\theta'_i = \Theta_i(\Lambda', m'_i, \varphi'_i)$.

Now let A be a simple central F-algebra and $[\Lambda, n, m, \varphi_i(\beta_i)]$ be realizations of (k, β_i) in A, for $i = 1, 2$. Let V denote the simple left A-module on which Λ is a lattice sequence and write D for the F-algebra opposite to $\mathrm{End}_{\mathrm{A}}(\mathrm{V})$. Let us fix a finite unramified extension L of F such that the L-algebra $\overline{\mathrm{A}} = \mathrm{A} \otimes_{\mathrm{F}} \mathrm{L}$ is split and a simple left $\overline{\mathrm{A}}$-module $\overline{\mathrm{V}}$. As E_i is totally ramified over F, the quasi-simple lift $[\overline{\Lambda}, n, m, \beta_i]$ is a simple stratum in $\overline{\mathrm{A}}$ (see [20, Théorème 2.30] and [24, Remarque 2.9]). We denote by $\mathcal{C}(\overline{\Lambda}, m, \beta_i)$ the set of simple characters attached to this quasi-simple lift with respect to the character $\Psi \circ \mathrm{tr}_{\mathrm{L/F}}$. The base change process developed in paragraph 7.2 gives rise to an injective and $\mathfrak{K}(\Lambda)$-equivariant map:

$$\boldsymbol{b}_{\mathrm{L/F}} : \mathcal{C}(\Lambda, m, \beta_i) \to \mathcal{C}(\overline{\Lambda}, m, \beta_i),$$

simply denoted $\boldsymbol{b}$. We use similar notations for A'. For each i, we write θ_i for the simple character $\Theta_i(\Lambda, m, \varphi_i)$. By Proposition 7.4, we have $\boldsymbol{b}(\theta'_1) = \boldsymbol{b}(\theta'_2)$. By Proposition 7.6, for each i, the lifts $\boldsymbol{b}(\theta_i)$ and $\boldsymbol{b}(\theta'_i)$ are transfers of each other. At this point, we cannot apply [6, 10] to deduce that $\boldsymbol{b}(\theta_1)$ and $\boldsymbol{b}(\theta_2)$ are $\mathfrak{K}(\overline{\Lambda})$-conjugate, because the lattice sequence Λ is not necessarily strict.

Let us fix a simple right $\mathrm{E}_1 \otimes_{\mathrm{F}} \mathrm{D}$-module S. We set $\mathrm{A(S)} = \mathrm{End}_{\mathrm{D}}(\mathrm{S})$, and denote by ρ_1 the natural F-algebra homomorphism $\mathrm{E}_1 \to \mathrm{A(S)}$. Let $\mathfrak{S}$ denote the unique (up to translation) E_1-pure strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence on S, and let us fix an F-algebra homomorphism $\rho_2 : \mathrm{E}_2 \to \mathrm{A(S)}$ such that $\mathfrak{S}$ is $\rho_2(\mathrm{E}_2)$-pure. Write n_0 for the $\mathfrak{S}$-valuation of $\rho_i(\beta_i)$ and:

$$m_0 = e_{\rho_i(\beta_i)}(\mathfrak{S})k,$$

which do not depend on i. We thus can form the stratum $[\mathfrak{S}, n_0, m_0, \rho_i(\beta_i)]$, which is a realization of (k, β_i) in A(S). Write ϑ_i for the simple character $\Theta_i(\mathfrak{S}, m_0, \rho_i)$. We now form the simple stratum $[\overline{\mathfrak{S}}, n_0, m_0, \rho_i(\beta_i)]$ in the split simple central L-algebra $\mathrm{A(S)} \otimes_{\mathrm{F}} \mathrm{L}$. It is a realization of (k, β_i) over L, and the $\mathcal{O}_{\mathrm{L}}$-lattice sequence $\overline{\mathfrak{S}}$ is strict. We thus can apply [6, Theorem 8.7] and [10, Theorem 3.5.11], which imply together that there exists $u \in \mathfrak{K}(\overline{\mathfrak{S}})$ such that:

$$\boldsymbol{b}(\vartheta_2)(x) = \boldsymbol{b}(\vartheta_1)(uxu^{-1}), \quad x \in \mathrm{H}^{m+1}(\rho_2(\beta_2), \overline{\mathfrak{S}}) = u^{-1}\mathrm{H}^{m+1}(\rho_1(\beta_1), \overline{\mathfrak{S}})u.$$

We need the following lemma.

Lemma 8.1. — We may assume that $u \in \mathfrak{K}(\mathfrak{S})$.

Proof. — By Proposition 7.7, the map $\sigma \mapsto u^{-1}\sigma(u)$ is a 1-cocycle on Gal(L/F) with values in the $\mathrm{U}(\overline{\mathfrak{S}})$-normalizer of $\boldsymbol{b}(\vartheta_2)$, which is equal to $\mathrm{J}(\rho_2(\beta_2), \overline{\mathfrak{S}})$ according to [10]. This cocycle defines a class in the cohomology set:

$$\mathrm{H}^1(\mathrm{Gal(L/F)}, \mathrm{J}(\rho_2(\beta_2), \overline{\mathfrak{S}})).$$

We claim this cohomology set is trivial. According to [20, Proposition 2.39], it is enough to prove that:

$$\mathrm{H}^1(\mathrm{Gal(L/F)}, \mathrm{J}(\rho_2(\beta_2), \overline{\mathfrak{S}})/\mathrm{J}^1(\rho_2(\beta_2), \overline{\mathfrak{S}}))$$

is trivial, which is given by a standard filtration argument (see [5, §6]). □

Using Proposition 7.3, we thus may replace ρ_2 by a $\mathfrak{K}(\mathfrak{S})$-conjugate and assume that the characters ϑ_1 and ϑ_2 are equal. We now fix a decomposition:

$$\mathrm{V} = \mathrm{V}^1 \oplus \cdots \oplus \mathrm{V}^l$$

of V into simple right $\mathrm{E}_1 \otimes_{\mathrm{F}} \mathrm{D}$-modules (which all are copies of S) such that the lattice sequence Λ decomposes into the direct sum of the $\Lambda^j = \Lambda \cap \mathrm{V}^j$, for

$j \in \{1, \dots, l\}$. By choosing, for each j, an isomorphism of $\mathrm{K}(\beta) \otimes_{\mathrm{F}} \mathrm{D}$-modules between S and V^j, this gives us an F-algebra homomorphism:

$$\iota : \mathrm{A(S)} \to \mathrm{A}.$$

Using Lemma 2.14, we may assume that $\iota \circ \rho_1 = \varphi_1$, and, by Lemma 3.5, on may replace φ_2 by a $\mathfrak{K}(\Lambda)$-conjugate and assume that $\iota \circ \rho_2 = \varphi_2$. We now remark that, for each i, the map $\vartheta_i \mapsto \theta_i$ corresponds to the process described in paragraph 2.4. The equality $\theta_1 = \theta_2$ thus follows from Proposition 4.11.

8.2. In this paragraph, we reduce the proof of Theorem 1.13 to the totally ramified case, which has been treated in paragraph 8.1. For $i = 1, 2$, let (Θ_i, k, β_i) be a ps-character over F, set $\mathrm{E}_i = \mathrm{F}(\beta_i)$ and write K_i for the maximal unramified extension of F contained in E_i, and suppose that $\Theta_1 \approx \Theta_2$. Then we have $[\mathrm{E}_1 : \mathrm{F}] = [\mathrm{E}_2 : \mathrm{F}]$ and, using Proposition 4.9, there is a simple central F-algebra A together with realizations $[\Lambda, n, m, \varphi_i(\beta_i)]$ of (k, β_i), with $i = 1, 2$, which are sound and have the same embedding type, with k dividing m and such that $\varphi_1(\mathrm{K}_1) = \varphi_2(\mathrm{K}_2)$, denoted K, and $\theta_1 = \theta_2$, where $\theta_i = \Theta_i(\Lambda, m_i, \varphi_i)$. Let C denote the centralizer of K in A and write $[\Gamma, n, m, \beta_i]$ for the stratum in C associated with $[\Lambda, n, m, \beta_i]$ by (5.2). By Proposition 6.11, the K/F-lifts θ_1^{K} and θ_2^{K} are equal.

Now let A' be a simple central F-algebra and $[\Lambda', n', m', \varphi'_i(\beta_i)]$ be realizations of (k, β_i) in A, with $i = 1, 2$, having the same embedding type. By Remark 3.4, we may conjugate φ'_2 by $\mathfrak{K}(\Lambda')$ and assume that the maximal unramified extensions of F contained in $\varphi'_1(\mathrm{E}_1)$ and $\varphi'_2(\mathrm{E}_2)$ are equal to a common extension K' of F, say. Moreover, by Lemma 3.1, we may conjugate again φ'_2 by $\mathfrak{K}(\Lambda')$ and assume that the F-algebra isomorphisms $\varphi'_1 \circ \varphi_1^{-1}$ and $\varphi'_2 \circ \varphi_2^{-1}$ agree on K (and thus identify K and K'). Let C' denote the centralizer of K' in A' and write $[\Gamma', n', m', \varphi'_i(\beta_i)]$ for the stratum in C associated with $[\Lambda', n', m', \varphi'_i(\beta_i)]$ by (5.2). Thus the simple strata $[\Gamma, n, m, \varphi_i(\beta_i)]$ and $[\Gamma', n', m', \varphi'_i(\beta_i)]$ are realizations of the same simple pair over K. For each i, we write θ'_i for the character $\Theta_i(\Lambda', m', \varphi'_i)$. By Theorem 6.7, for each i, the K/F-lifts θ_i^{K} and $\theta_i'^{\mathrm{K}}$ are transfers of each other. Therefore, by paragraph 8.1, there exists $u \in \mathfrak{K}(\Gamma')$ such that:

$$\theta_2'^{\mathrm{K}}(x) = \theta_1'^{\mathrm{K}}(uxu^{-1}), \quad x \in \mathrm{H}^{m+1}(\varphi'_2(\beta_2), \Gamma') = u^{-1}\mathrm{H}^{m+1}(\varphi'_1(\beta_1), \Gamma')u.$$

The equality ${\theta'}_1^u = \theta'_2$ follows from Proposition 6.10.

Corollary 8.2. — Definition 1.10 is equivalent to [6, Definition 8.6].

Proof. — Assume we are given two ps-characters (Θ_i, k, β_i), $i = 1, 2$, which are endo-equivalent in the sense of Definition 1.10, and let A be a simple central split F-algebra together with realizations $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ of (k, β_i) in A, with $i = 1, 2$, such that Λ is strict. By Theorem 1.11, the simple characters $\Theta_i(\Lambda, m_i, \varphi_i)$ intertwine in $\mathrm{A}^\times$, that is, the ps-characters (Θ_i, k, β_i) are endo-equivalent in the sense of [6, Definition 8.6]. Conversely, two simple pairs which are endo-equivalent in this sense are clearly endo-equivalent in the sense of Definition 1.10. □

Corollary 8.3. — The relation $\approx$ on ps-characters is an equivalence relation.

Proof. — This comes from [6, Corollary 8.10] together with Corollary 8.2. □

9. The endo-class of a discrete series representation

9.1. Let A be a simple central F-algebra, and let V be a simple left A-module. Associated with it, there is an F-division algebra D. We write d for the reduced degree of D over F and m for the dimension of V as a right D-vector space. We set $\mathrm{G} = \mathrm{A}^\times$, identified with $\mathrm{GL}_m(\mathrm{D})$.
Let π be an irreducible smooth representation of G, and assume that its inertial class (in the sense of Bushnell and Kutzko's theory of types [11]), denoted $\mathfrak{s}(\pi)$, is homogeneous. Thus there is a positive integer r dividing m, an irreducible cuspidal representation ρ of the group $\mathrm{G}_0 = \mathrm{GL}_{m/r}(\mathrm{D})$ and unramified characters χ_i of G_0, with $i \in \{1, \dots, r\}$, such that π is isomorphic to a quotient of the normalized parabolically induced representation $\rho\chi_1 \times \dots \times \rho\chi_r$ (see for instance [2] for the notation).
In this section, we associate with π an endo-class $\boldsymbol{\Theta}(\pi)$ over F, and show that it depends only on the inertial class $\mathfrak{s} = \mathfrak{s}(\pi)$.

9.2. Let π be a representation of G as above, and write $\mathfrak{s} = \mathfrak{s}(\pi)$ for its inertial class. According to [24, Théorème 5.23], this inertial class possesses a type in the sense of [11]. Such a type is a pair (J, λ) formed of a compact open subgroup J of G and of an irreducible smooth representation λ of J such that an irreducible smooth representation of G has inertial class $\mathfrak{s}$ if and only if λ occurs in its restriction to J. More precisely, (J, λ) can be chosen to be a *simple type* in the sense of [22]. We won't give a precise description of simple types; the only property of interest for us is the following fact, which is a weak form of [24, Théorème 5.23].

Fact 9.1. — There is a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A *together with a simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ such that the order $\mathfrak{A} \cap \mathrm{B}$ (with* B *the centralizer of* $\mathrm{F}(\beta)$ *in* A*) is principal of period r and the character θ occurs in the restriction of π to $\mathrm{H}^1(\beta, \mathfrak{A})$.*

Neither $[\mathfrak{A}, n, 0, \beta]$ nor the character θ are uniquely determined. We let $(\Theta, 0, \beta)$ be the ps-character defined by the pair $([\mathfrak{A}, n, 0, \beta], \theta)$ and we denote by $\boldsymbol{\Theta}$ its endo-class.

Theorem 9.2. — The endo-class $\boldsymbol{\Theta}$ depends only on the inertial class $\mathfrak{s}$.

Proof. — We have to prove that $\boldsymbol{\Theta}$ does not depend on the choice of the simple stratum $[\mathfrak{A}, n, 0, \beta]$ and the simple character θ satisfying the conditions of Fact 9.1. For $i = 1, 2$, let $[\mathfrak{A}_i, n_i, 0, \beta_i]$ be a simple stratum and θ_i be a simple character satisfying the conditions of Fact 9.1, and let $(\Theta_i, 0, \beta_i)$ denote the ps-character that it defines. Let $\mathfrak{A}'_i$ denote the unique principal $\mathcal{O}_{\mathrm{F}}$-order in A such that the pair $(\mathrm{E}_i, \mathfrak{A}'_i)$ is a sound embedding in A (see Lemma 4.18) and let θ'_i denote the transfer of θ_i in $\mathcal{C}(\mathfrak{A}'_i, 0, \beta_i)$. By a standard argument using [24, Théorème 2.13], the character θ'_i occurs in the restriction of π to $\mathrm{H}^1(\beta_i, \mathfrak{A}'_i)$. Therefore, we can assume without changing Θ_i that $(\mathrm{E}_i, \mathfrak{A}_i)$ is sound.

Lemma 9.3. — The extensions E_1/F *and* E_2/F *have the same ramification index.*

Proof. — We are going to prove that this ramification index is determined by the irreducible cuspidal representation ρ of paragraph 9.1. Let $n(\rho)$ denote the number of unramified characters χ of G_0 such that $\rho\chi$ is equivalent to ρ. Write q for the cardinality of the residue field of F and $|\cdot|_{\mathrm{F}}$ for the absolute value on F giving the value q^{-1} to any uniformizer. Let $s(\rho)$ denote the unique positive real number such that $\rho \times \rho\nu_\rho$ is reducible, where ν_ρ is the unramified character $g \mapsto |\mathrm{N}_{\mathrm{A}/\mathrm{F}}(g)|_{\mathrm{F}}^{s(\rho)}$ (see section 4 of [23] for more details). By using [23, Theorem 4.6], the product $n(\rho)s(\rho)$ is equal to the quotient of md by the ramification index of E_i/F, for any $i = 1, 2$. □

By Lemma 4.18, the principal orders $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have the same period (as $\mathfrak{A}_i \cap \mathrm{B}_i$ has period r). Thus one may conjugate $([\mathfrak{A}_1, n_1, 0, \beta_1], \theta_1)$ by an element of G and assume that $\mathfrak{A}_1 = \mathfrak{A}_2$, denoted $\mathfrak{A}$. For each i, we have $\theta_i \in \mathcal{C}(\mathfrak{A}, 0, \beta_i)$ and θ_i occurs in the restriction of π to the subgroup $\mathrm{H}^1(\beta_i, \mathfrak{A})$. Thus the characters θ_1 and θ_2 intertwine in $\mathrm{A}^\times$. To prove that Θ_1 and Θ_2 are endo-equivalent, it remains to prove that $\mathrm{F}(\beta_1)$ and $\mathrm{F}(\beta_2)$ have the same degree over F. By copying the beginning of the proof of Lemma 4.7, we get $n_1 = n_2$. We now write f for the greatest common divisor of $f_{\mathrm{F}}(\beta_1)$ and $f_{\mathrm{F}}(\beta_2)$ and K_i for the maximal unramified extension of F contained in $\mathrm{F}(\beta_i)$. Then Theorem 1.16 gives us the expected equality. □

We call the class $\boldsymbol{\Theta}$ *the endo-class* of π (or of $\mathfrak{s}$). We have actually obtained more.

Theorem 9.4. — Let π be an irreducible representation with inertial class $\mathfrak{s}$ as above, and let $[\mathfrak{A}, n, 0, \beta]$ and θ satisfy the conditions of Fact 9.1. Assume moreover $[\mathfrak{A}, n, 0, \beta]$ is sound. The following objects are invariants of the inertial class $\mathfrak{s}$:

1. *the ramification index* $e_{\mathrm{F}}(\beta)$ *and the residue class degree* $f_F(\beta)$*;*
2. *the* G*-conjugacy class of the order* $\mathfrak{A}$*;*
3. *the embedding type of* $(\mathrm{F}(\beta), \mathfrak{A})$.

Proof. — Assertions (1) and (2) have already been proved. Assertion (3) follows immediately from Lemma 4.19. □

9.3. Recall that an irreducible smooth representation π of G is *essentially square integrable* if there is a character χ of G such that $\pi\chi$ is unitary and has a non-zero coefficient which is square integrable on G/Z, where Z denotes the centre of G. We write $\mathcal{D}(\mathrm{G})$ for the set of isomorphism classes of essentially square integrable representation of G. According to [2, §2.2], any essentially square integrable representation of G has an inertial class which is homogeneous in the sense of paragraph 9.1. Thus the construction of paragraph 9.2 gives us a map:

$$\boldsymbol{\Theta}_{\mathrm{G}} : \mathcal{D}(\mathrm{G}) \to \mathcal{E}(\mathrm{F}) \tag{9.1}$$

from $\mathcal{D}(\mathrm{G})$ to the set of endo-classes of ps-characters over F.

We now write $\mathrm{H} = \mathrm{GL}_{md}(\mathrm{F})$, and let **JL** denote the Jacquet-Langlands correspondence (see [1, 14]) from $\mathcal{D}(\mathrm{G})$ to $\mathcal{D}(\mathrm{H})$. We have the following conjecture.

Conjecture 9.5. — For any π in $\mathcal{D}(G)$, we have:

$$\Theta_{\mathrm{H}}(\mathbf{JL}(\pi)) = \Theta_{\mathrm{G}}(\pi). \tag{9.2}$$

This conjecture generalizes the fact that, for any level zero representation π in $\mathcal{D}(\mathrm{G})$, the representation $\mathbf{JL}(\pi)$ has level zero. It allows one to refine the correspondence **JL** by fixing the endo-class: given $\boldsymbol{\Theta}$ an endo-class over F, Conjecture 9.5 implies that we have a bijective map:

$$\mathbf{JL}_{\boldsymbol{\Theta}} : \mathcal{D}(\mathrm{G}, \boldsymbol{\Theta}) \to \mathcal{D}(\mathrm{H}, \boldsymbol{\Theta})$$

where we write $\mathcal{D}(\mathrm{G}, \boldsymbol{\Theta})$ for the set of isomorphism classes of essentially square integrable representations of G of endo-class $\boldsymbol{\Theta}$.

References

[1] A. I. Badulescu – "Correspondance de Jacquet-Langlands pour les corps locaux de caractéristique non nulle", *Ann. Sci. École Norm. Sup. (4)* 35 (2002), no. 5, p. 695–747.

[2] A. I. Badulescu, G. Henniart, B. Lemaire & V. Sécherre – "Sur le dual unitaire de $\mathrm{GL}(r, D)$", *Amer. J. Math.* 132 (2010), no. 5, p. 1365–1396.

[3] P. Broussous & B. Lemaire – "Building of $\mathrm{GL}(m, D)$ and centralizers", *Transform. Groups* 7 (2002), no. 1, p. 15–50.

[4] P. Broussous – "Hereditary orders and embeddings of local fields in simple algebras", *J. Algebra* 204 (1998), no. 1, p. 324–336.

[5] P. Broussous & M. Grabitz – "Pure elements and intertwining classes of simple strata in local central simple algebras", *Comm. Algebra* 28 (2000), no. 11, p. 5405–5442.

[6] C. J. Bushnell & G. Henniart – "Local tame lifting for $\mathrm{GL}(N)$. I. Simple characters", *Inst. Hautes Études Sci. Publ. Math.* (1996), no. 83, p. 105–233.

[7] ______, "Correspondance de Jacquet-Langlands explicite. II : Le cas de degré égal à la caractéristique résiduelle", *Manuscripta Math.* 102 (2000), no. 2, p. 211–225.

[8] ______, "Local tame lifting for $\mathrm{GL}(n)$. IV. Simple characters and base change", *Proc. London Math. Soc. (3)* 87 (2003), no. 2, p. 337–362.

[9] ______, "Local tame lifting for GL(n). III. Explicit base change and Jacquet-Langlands correspondence", *J. Reine Angew. Math.* 580 (2005), p. 39–100.

[10] C. J. Bushnell & P. C. Kutzko – *The admissible dual of* GL(N) *via compact open subgroups*, Princeton University Press, Princeton, NJ, 1993.

[11] ______, "Smooth representations of reductive p-adic groups: structure theory via types", *Proc. London Math. Soc. (3)* 77 (1998), no. 3, p. 582–634.

[12] ______, "Semisimple types in GL_n", *Compositio Math.* 119 (1999), no. 1, p. 53–97.

[13] J.-F. Dat – "Finitude pour les représentations lisses de groupes p-adiques", *J. Inst. Math. Jussieu* 8 (2009), no. 2, p. 261–333.

[14] P. Deligne, D. Kazhdan & M.-F. Vignéras – "Représentations des algèbres centrales simples p-adiques", in *Representations of reductive groups over a local field*, Travaux en Cours, Hermann, Paris, 1984, p. 33–117.

[15] A. Fröhlich – "Principal orders and embedding of local fields in algebras", *Proc. London Math. Soc. (3)* 54 (1987), no. 2, p. 247–266.

[16] M. Grabitz – "Continuation of hereditary orders in local central simple algebras", *J. Number Theory* 77 (1999), no. 1, p. 1–26.

[17] ______, "Simple characters for principal orders in $\mathrm{M}_m(D)$", *J. Number Theory* 126 (2007), no. 1, p. 1–51.

[18] G. Henniart – "Correspondance de Jacquet-Langlands explicite. I : Le cas modéré de degré premier", in *Séminaire de Théorie des Nombres, Paris, 1990–91*, Progr. Math., vol. 108, Birkhäuser, Boston, MA, 1993, p. 85–114.

[19] A. Minguez & V. Sécherre – "Représentations lisses ℓ-modulaires de GL(m, D)", preprint, `http://arxiv.org/abs/1110.1467`.

[20] V. Sécherre – "Représentations lisses de GL(m, D), I : caractères simples", *Bull. Soc. math. France* 132 (2004), no. 3, p. 327–396.

[21] ______, "Représentations lisses de GL(m, D), II : β-extensions", *Compos. Math.* 141 (2005), p. 1531–1550.

[22] ______, "Représentations lisses de GL(m, D), III : types simples", *Ann. Scient. Éc. Norm. Sup.* 38 (2005), p. 951–977.

[23] ______, "Proof of the Tadic Conjecture U0 on the unitary dual of GL(m, D)", *J. Reine Angew. Math.* 626 (2009), p. 187-204.

[24] V. Sécherre & S. Stevens – "Représentations lisses de GL(m, D), IV : représentations supercuspidales", *J. Inst. Math. Jussieu* 7 (2008), no. 3, p. 527–574.

[25] ______, "Smooth representations of GL(m, D), VI : semi-simple types", *Int. Math. Res. Not.* (2011).

[26] A. J. Silberger & E.-W. Zink – "Weak explicit matching for level zero discrete series of unit groups of $\mathfrak{p}$-adic simple algebras", *Canad. J. Math.* 55 (2003), no. 2, p. 353–378.

[27] ______, "An explicit matching theorem for level zero discrete series of unit groups of $\mathfrak{p}$-adic simple algebras", *J. Reine Angew. Math.* 585 (2005), p. 173–235.

[28] S. Stevens – "Semisimple characters for p-adic classical groups", *Duke Math. J.* 127 (2005), no. 1, p. 123–173.

[29] E.-W. Zink – "More on embeddings of local fields in simple algebras", *J. Number Theory* 77 (1999), no. 1, p. 51–61.

P. Broussous
Université de Poitiers
Laboratoire de Mathématiques
Téléport 2
Boulevard Marie et Pierre Curie
BP 30179
F-86962 Futuroscope-Chasseneuil cedex
France
Paul.Broussous@math.univ-poitiers.fr

V. Sécherre
Université de Versailles Saint-Quentin
Laboratoire de Mathématiques de Versailles
45 avenue des Etats-Unis
78035 Versailles cedex
France
vincent.secherre@math.uvsq.fr

S. Stevens
School of Mathematics
University of East Anglia
Norwich NR4 7TJ
United Kingdom
Shaun.Stevens@uea.ac.uk

The Equivariant Cohomology of Isotropy Actions on Symmetric Spaces

Oliver Goertsches

Received: September 9, 2011

Communicated by Eckhard Meinrenken

Abstract. We show that for every symmetric space G/K of compact type with K connected, the K-action on G/K by left translations is equivariantly formal.

2010 Mathematics Subject Classification: 53C35, 55N25, 57S15

1 Introduction

Given compact connected Lie groups $K \subset G$ of equal rank, it is well-known that the K-action on the homogeneous space G/K is equivariantly formal because the odd de Rham cohomology groups of G/K vanish. (See for example [7] for an investigation of the equivariant cohomology of such spaces.) If however the rank of K is strictly smaller than the rank of G, then the isotropy action is not necessarily equivariantly formal, and in general it is unclear when this is the case.[1] Restricting our attention to symmetric spaces of compact type, we will prove the following theorem.

Theorem. *Let (G, K) be a symmetric pair of compact type, where G and K are compact connected Lie groups. Then the K-action on the symmetric space $M = G/K$ by left translations is equivariantly formal.*

For symmetric spaces of type II, i.e., compact Lie groups, this result is already known, see Section 4.3. More generally, in the case of symmetric spaces of split rank ($\operatorname{rank} G = \operatorname{rank} K + \operatorname{rank} G/K$), the fact that all K-isotropy groups have maximal rank implies equivariant formality, see Section 4.5. However, for the general case we have to rely on an explicit calculation of the dimension of the

[1] A sufficient condition for equivariant formality of the isotropy action was introduced in [17], see Remark 4.2 below. If K belongs to a certain class of subtori of G this condition is in fact an equivalence, see [18].

cohomology of the T-fixed point set M^T, where $T \subset K$ is a maximal torus, in order to use the characterization of equivariant formality via the condition $\dim H^*(M^T) = \dim H^*(M)$. With the help of the notion of compartments introduced in [1] and several results proven therein we will find in Section 4.1 a calculable expression for this dimension, and after reducing to the case of an irreducible simply-connected symmetic space in Section 4.2 we can invoke the classification of such spaces to show equivariant formality in each of the remaining cases by hand. On the way we obtain a formula for the number of compartments in a fixed K-Weyl chamber, see Proposition 4.14.

Acknowledgements. The author wishes to express his gratitude to Augustin-Liviu Mare for interesting discussions on a previous version of the paper.

2 Symmetric spaces

Let G be a connected Lie group and $K \subset G$ a closed subgroup. Then K is said to be a symmetric subgroup of G if there is an involutive automorphism $\sigma : G \to G$ such that K is an open subgroup of the fixed point subgroup G^σ. We will refer to the pair (G, K) as a symmetric pair, and G/K is a symmetric space.

Given a symmetric pair (G, K) with corresponding involution $\sigma : G \to G$, then the Lie algebra $\mathfrak{g}$ decomposes into the (± 1)-eigenspaces of σ: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and the usual commutation relations hold: $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The rank of G/K is by definition the maximal dimension of an abelian subalgebra of $\mathfrak{p}$. Then clearly $\operatorname{rank} G - \operatorname{rank} K \leq \operatorname{rank} G/K$, and if equality holds, then we say that G/K is of split rank.

A symmetric pair (G, K) is called (almost) effective if G acts (almost) effectively on G/K. Given a symmetric pair (G, K), then the kernel $N \subset G$ of the G-action on G/K is contained in K, and $(G/N, K/N)$ is an effective symmetric pair with $(G/N)/(K/N) = G/K$. An almost effective symmetric pair (G, K) (and the corresponding symmetric space G/K) will be called of compact type if G is a compact semisimple Lie group. In this paper only symmetric spaces of compact type will occur. If (G, K) is effective, then G can be regarded as a subgroup of the isometry group of G/K with respect to any G-invariant Riemannian metric on G/K. If (G, K) is additionally of compact type, then this inclusion is in fact an isomorphism between G and the identity component of the isometry group.

3 Equivariant formality

The equivariant cohomology of an action of a compact connected Lie group K on a compact manifold M is by definition the cohomology of the Borel construction

$$H^*_K(M) = H^*(EK \times_K M);$$

we use real coefficients throughout the paper. The projection $EK \times_K M \to EK/K = BK$ to the classifying space BK of K induces on $H^*_K(M)$ the structure of an $H^*(BK)$-algebra.

An action of a compact connected Lie group K on a compact manifold M is called equivariantly formal in the sense of [3] if $H^*_K(M)$ is a free $H^*(BK)$-module. If the K-action on M is equivariantly formal then automatically

$$H^*_K(M) = H^*(M) \otimes H^*(BK) \tag{1}$$

as graded $H^*(BK)$-modules, see [2, Proposition 2.3]. In the following proposition we collect some known equivalent characterizations of equivariant formality.

Proposition 3.1. *Consider an action of a compact connected Lie group K on a compact manifold M, and let $T \subset K$ be a maximal torus. Then the following conditions are equivalent:*

1. *The K-action on M is equivariantly formal.*
2. *The T-action on M is equivariantly formal.*
3. *The cohomology spectral sequence associated to the fibration $ET \times_T M \to BT$ collapses at the E_2-term.*
4. *We have $\dim H^*(M) = \dim H^*(M^T)$.*
5. *The natural map $H^*_T(M) \to H^*(M)$ is surjective.*

Proof. For the equivalence of (1) and (2) see [5, Proposition C.26]. The Borel localization theorem implies that the rank of $H^*_T(M)$ as an $H^*(BT)$-module always equals $\dim H^*(M^T)$. Then [5, Lemma C.24] implies the equivalence of (2), (3), and (4); see also [10, p. 46]. For the equivalence to (5), see [13, p. 148]. □

Note that by [10, p. 46] the inequality $\dim H^*(M^T) \leq \dim H^*(M)$ holds for any T-action on M. Condition (5) in the proposition shows that

Corollary 3.2. *If a compact connected Lie group K acts equivariantly formally on a compact manifold M, then so does every connected closed subgroup of K.*

Applying the gap method to the spectral sequence in Item (3) of Proposition 3.1 we obtain the following well-known sufficient condition for equivariant formality.

Proposition 3.3. *Any action of a compact Lie group K on a compact manifold M with $H^{odd}(M) = 0$ is equivariantly formal.*

4 Isotropy actions on symmetric spaces of compact type

Let G be a compact connected Lie group and $K \subset G$ a compact connected subgroup. Because an equivariantly formal torus action always has fixed points, the only tori $T \subset G$ that can act equivariantly formally on G/K by left translations are those that are conjugate to a subtorus of K. On the other hand, if a maximal torus T of K acts equivariantly formally on G/K, then we know by Corollary 3.2 that all these tori do in fact act equivariantly formally. In the following, we will prove that this indeed happens for symmetric spaces of compact type. More precisely:

Theorem 4.1. *Let (G, K) be a symmetric pair of compact type, where G and K are compact connected Lie groups. Then the K-action on the symmetric space G/K by left translations is equivariantly formal.*

Remark 4.2. The pair (G, K) is a Cartan pair in the sense of [4], see [4, p. 448]. Therefore, [17, Theorem A] shows that a sufficient condition for the K-action on G/K to be equivariantly formal is that the map $H^*(G/K)^{N_G(K)} \to H^*(G)$ induced by the projection $G \to G/K$, where $N_G(K)$ acts on G/K from the right, is injective. It would be interesting to know whether a symmetric pair always satisfies this condition.

4.1 The fixed point set of a maximal torus in K

Let (G, K) be a symmetric pair of compact type, where G and K are compact connected Lie groups. Denote by $\sigma : G \to G$ the corresponding involutive automorphism. Then $M = G/K$ is a symmetric space of compact type. We fix maximal tori $T_K \subset K$ and $T_G \subset G$ such that $T_K \subset T_G$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of the Lie algebra $\mathfrak{g}$ into eigenspaces of σ.
In order to prove Theorem 4.1 we can without loss of generality assume that the symmetric pair (G, K) is effective: if $N \subset K$ is the kernel of the G-action on G/K, then clearly the K-action on $G/K = (G/N)/(K/N)$ is equivariantly formal if and only if the K/N-action is equivariantly formal. (This follows for example from Proposition 3.1 because the fixed point sets of appropriately chosen maximal tori in K and K/N coincide.)

Lemma 4.3. *The T_K-fixed point set in M is $N_G(T_K)/N_K(T_K)$.*

Proof. An element $gK \in M$ is fixed by T_K if and only if $g^{-1}T_Kg \subset K$ (i.e., $g^{-1}T_Kg$ is a maximal torus in the compact Lie group K), which is the case if and only if there is some $k \in K$ with $k^{-1}g^{-1}T_Kgk = T_K$. Thus, $(G/K)^{T_K} = N_G(T_K)/N_G(T_K) \cap K = N_G(T_K)/N_K(T_K)$. □

Lemma 4.4 ([15, Proposition VII.3.2]). *T_G is the unique maximal torus in G containing T_K.*

Lemma 4.4 implies that the Lie algebra $\mathfrak{t}_\mathfrak{g}$ of T_G decomposes according to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as $\mathfrak{t}_\mathfrak{g} = \mathfrak{t}_\mathfrak{k} \oplus \mathfrak{t}_\mathfrak{p}$. (In fact, this statement is the first part of the proof of [15, Proposition VII.3.2].)

Proposition 4.5. *Each connected component of M^{T_K} is a torus of dimension* $\operatorname{rank} G - \operatorname{rank} K$.

Proof. Because of Lemma 4.4, the abelian subalgebra $\mathfrak{t}_\mathfrak{p} \subset \mathfrak{p}$ is the space of elements in $\mathfrak{p}$ that commute with $\mathfrak{t}_\mathfrak{k}$. Thus, Lemma 4.3 implies that the component of M^{T_K} containing eK is $T_G/(T_G \cap K) = T_G/T_K$ (note that the centralizer of T_K in K is exactly T_K), i.e., a $\operatorname{rank} G - \operatorname{rank} K$-dimensional torus. Because the fixed set M^{T_K} is a homogeneous space, all components are diffeomorphic. $\square$

We therefore understand the structure of the T_K-fixed point set M^{T_K} if we know its number of connected components, which we denote by r. In view of condition (4) in Proposition 3.1, we are mostly interested in the dimension of its cohomology. Proposition 4.5 implies immediately:

Proposition 4.6. *We have* $\dim H^*(M^{T_K}) = 2^{\operatorname{rank} G - \operatorname{rank} K} \cdot r$.

In order to get a calculable expression for r we will use several results from [1, Sections 5 and 6] which we now collect. Denote by $\Delta_G = \Delta_\mathfrak{g}$ the root system of G with respect to the maximal torus T_G, i.e., the set of nonzero elements $\alpha \in \mathfrak{t}_\mathfrak{g}^*$ such that the corresponding eigenspace $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^\mathbb{C} \mid [W,X] = i\alpha(W)X \text{ for all } W \in \mathfrak{t}_\mathfrak{g}\}$ is nonzero. Then we have the root space decomposition

$$\mathfrak{g}^\mathbb{C} = \mathfrak{t}_\mathfrak{g}^\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta_\mathfrak{g}} \mathfrak{g}_\alpha. \tag{2}$$

The $\mathfrak{g}$-Weyl chambers are the connected components of the set $\mathfrak{t}_\mathfrak{g} \setminus \bigcup_{\alpha \in \Delta_\mathfrak{g}} \ker \alpha$. Because of Lemma 4.4, $\mathfrak{t}_\mathfrak{k}$ contains $\mathfrak{g}$-regular elements, hence no root in $\Delta_\mathfrak{g}$ vanishes on $\mathfrak{t}_\mathfrak{k}$. Therefore some of the $\mathfrak{g}$-Weyl chambers intersect $\mathfrak{t}_\mathfrak{k}$ nontrivially, and following [1] we will refer to these intersections as *compartments*. Considering as in [1] the decomposition of $\Delta_\mathfrak{g}$ into complementary subsets $\Delta_\mathfrak{g} = \Delta' \cup \Delta''$, where

$$\Delta' = \{\alpha \in \Delta_\mathfrak{g} \mid \mathfrak{g}_\alpha \not\subset \mathfrak{p}^\mathbb{C}\}, \quad \Delta'' = \{\alpha \in \Delta_\mathfrak{g} \mid \mathfrak{g}_\alpha \subset \mathfrak{p}^\mathbb{C}\}, \tag{3}$$

we have by [1, Lemma 9] that the root system $\Delta_K = \Delta_\mathfrak{k}$ of K with respect to T_K is given by

$$\Delta_\mathfrak{k} = \{\alpha|_{\mathfrak{t}_\mathfrak{k}} \mid \alpha \in \Delta'\}. \tag{4}$$

In particular, $\mathfrak{g}$-regular elements in $\mathfrak{t}_\mathfrak{k}$ are also $\mathfrak{k}$-regular, and hence each compartment is contained in a $\mathfrak{k}$-Weyl chamber.

Because of Lemma 4.4, the group $N_G(T_K)$ is a subgroup of $N_G(T_G)$. Both groups have the same identity component T_G, so we may regard the quotient group $N_G(T_K)/T_G$ as a subgroup of the Weyl group $W(G)$ of G. The free action of $W(G)$ on the $\mathfrak{g}$-Weyl chambers induces an action of $N_G(T_K)/T_G$ on the set of compartments. Because any two compartments are G-conjugate [1, Theorem 10], this action is simply transitive on the set of compartments, and it follows that the number of connected components of $N_G(T_K)$ equals the total number of compartments in $\mathfrak{t}_\mathfrak{k}$. On the other hand no connected component of $N_G(T_K)$ contains more than one connected component of $N_K(T_K)$.

(It is sufficient to check this for the identity component T_G of $N_G(T_K)$. An element in $N_K(T_K) \cap T_G$ is an element in K centralizing T_K, hence already contained in T_K.) Because the number of connected components of $N_K(T_K)$ equals the number of $\mathfrak{k}$-Weyl chambers, and each $\mathfrak{k}$-Weyl chamber contains the same number of compartments [1, Theorem 10], we have shown the following lemma.

Lemma 4.7. *The number r of connected components of $M^{T_K} = N_G(T_K)/N_K(T_K)$ is the number of compartments in a fixed $\mathfrak{k}$-Weyl chamber. In particular it only depends on the Lie algebra pair $(\mathfrak{g}, \mathfrak{k})$.*

Let C be a $\mathfrak{g}$-Weyl chamber that intersects $\mathfrak{t}_\mathfrak{k}$ nontrivially. By [1, Lemma 8] the compartment $C \cap \mathfrak{t}_\mathfrak{k}$ can be described explicitly: The involution $\sigma : G \to G$ permutes the $\mathfrak{g}$-Weyl chambers and fixes $\mathfrak{t}_\mathfrak{k}$, hence it fixes C. Let $B = \{\alpha_1, \ldots, \alpha_{\mathrm{rank}\, G}\}$ be the corresponding simple roots such that C is exactly the set of points where the elements of B take positive values. The involution σ acts as a permutation group on B because for any i the linear form $\alpha_i \circ \sigma$ is again positive on C. Note that for every root $\alpha \in \Delta_\mathfrak{g}$ the linear form $\frac{1}{2}(\alpha + \alpha \circ \sigma)$ vanishes on $\mathfrak{t}_\mathfrak{p}$ and coincides with $\alpha|_{\mathfrak{t}_\mathfrak{k}}$ on $\mathfrak{t}_\mathfrak{k}$. The set $B|_{\mathfrak{t}_\mathfrak{k}} = \{\alpha_i|_{\mathfrak{t}_\mathfrak{k}} \mid i = 1, \ldots, \mathrm{rank}\, G\}$ is a basis of $\mathfrak{t}_\mathfrak{k}^*$ (in particular it consists of $\dim \mathfrak{t}_\mathfrak{k}$ elements) and the compartment $C \cap \mathfrak{t}_\mathfrak{k}$ is exactly the set of points in $\mathfrak{t}_\mathfrak{k}$ where all $\alpha_i|_{\mathfrak{t}_\mathfrak{k}}$ take positive values. It is a simplicial cone bounded by the hyperplanes $\ker \alpha_i|_{\mathfrak{t}_\mathfrak{k}}$. Any such hyperplane is either a wall of a $\mathfrak{k}$-Weyl chamber or the kernel of a $\mathfrak{g}$-root α_i with $\alpha_i \circ \sigma = \alpha_i$, see (4). In any case, reflection along the hyperplane defines an element of $N_G(T_K)/T_G$ and takes $C \cap \mathfrak{t}_\mathfrak{k}$ to an adjacent compartment. (This argument is taken from the proof of [1, Theorem 10].)
It follows that the action of $N_G(T_K)/T_G$ on the set of compartments described above is generated by the reflections along all hyperplanes $\ker \alpha|_{\mathfrak{t}_\mathfrak{k}}$, where $\alpha \in \Delta_\mathfrak{g}$. Let $\langle \cdot, \cdot \rangle$ be the Killing form on $\mathfrak{g}$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is orthogonal with respect to $\langle \cdot, \cdot \rangle$. We identify $\mathfrak{t}_\mathfrak{g}^*$ with $\mathfrak{t}_\mathfrak{g}$ and $\mathfrak{t}_\mathfrak{k}^*$ with $\mathfrak{t}_\mathfrak{k}$ via $\langle \cdot, \cdot \rangle$. For $\alpha \in \Delta_\mathfrak{g}$, let $H_\alpha \in \mathfrak{t}_\mathfrak{g}$ be the element such that $\alpha(H) = \langle H, H_\alpha \rangle$ for all $H \in \mathfrak{t}_\mathfrak{g}$. Given $X \in \mathfrak{t}_\mathfrak{g}$, we write $X^\mathfrak{k}$ and $X^\mathfrak{p}$ for the $\mathfrak{k}$- and $\mathfrak{p}$-parts of X respectively. Then $H_\alpha^\mathfrak{k}$ corresponds to $\alpha|_{\mathfrak{t}_\mathfrak{k}}$ under the isomorphism $\mathfrak{t}_\mathfrak{k} \cong \mathfrak{t}_\mathfrak{k}^*$.

Lemma 4.8. *Let $\alpha \in \Delta_\mathfrak{g}$ be a root with $\alpha \circ \sigma \neq \alpha$. Then either*

1. *$\langle H_\alpha, H_{\alpha\circ\sigma} \rangle = 0$ and $|H_\alpha^\mathfrak{p}|^2 = |H_\alpha^\mathfrak{k}|^2$ or*
2. *$2 \cdot \frac{\langle H_\alpha, H_{\alpha\circ\sigma}\rangle}{|H_\alpha|^2} = -1$, $|H_\alpha^\mathfrak{p}|^2 = 3|H_\alpha^\mathfrak{k}|^2$ and $\alpha + \alpha \circ \sigma \in \Delta_\mathfrak{g}$.*

Proof. We have $H_{\alpha\circ\sigma} = H_\alpha^\mathfrak{k} - H_\alpha^\mathfrak{p}$, and because $\Delta_\mathfrak{g}$ is a root system it follows that

$$2 \cdot \frac{\langle H_\alpha, H_{\alpha\circ\sigma}\rangle}{|H_\alpha|^2} = 2 \cdot \frac{|H_\alpha^\mathfrak{k}|^2 - |H_\alpha^\mathfrak{p}|^2}{|H_\alpha^\mathfrak{k}|^2 + |H_\alpha^\mathfrak{p}|^2} \in \mathbb{Z}.$$

Because α and $\alpha \circ \sigma$ are roots of equal length, this integer can only equal 0 or ± 1 [12, Proposition 2.48.(d)]. Further, because $\alpha - \alpha \circ \sigma$ is not a root (by Lemma 4.4 no root vanishes on $\mathfrak{t}_\mathfrak{k}$) and not 0, only the possibilities 0 and -1

remain, and in the latter case we also have that $\alpha+\alpha\circ\sigma\in\Delta_{\mathfrak{g}}$ [12, Proposition 2.48.(e)]. ☐

Proposition 4.9. *The set* $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}}=\{\alpha|_{\mathfrak{t}_{\mathfrak{k}}}\mid\alpha\in\Delta_{\mathfrak{g}}\}$ *is a root system in* $\mathfrak{t}_{\mathfrak{k}}^*$.

Proof. It is clear that $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}}$ spans $\mathfrak{t}_{\mathfrak{k}}^*$. We have to check that for all $\alpha,\beta\in\Delta_{\mathfrak{g}}$, the quantity

$$2\cdot\frac{\langle H_\alpha^{\mathfrak{k}},H_\beta^{\mathfrak{k}}\rangle}{|H_\alpha^{\mathfrak{k}}|^2}\tag{5}$$

is an integer. With respect to the decomposition $\Delta_{\mathfrak{g}}=\Delta'\cup\Delta''$ (see (3)) there are four cases:
If both α and β are elements of Δ', then (5) is an integer because $\alpha|_{\mathfrak{t}_{\mathfrak{k}}}$ and $\beta|_{\mathfrak{t}_{\mathfrak{k}}}$ are $\mathfrak{k}$-roots, see (4). In case α and β are elements of Δ'', then the corresponding vectors H_α and H_β are already elements of $\mathfrak{t}_{\mathfrak{k}}$, so $H_\alpha^{\mathfrak{k}}=H_\alpha$ and $H_\beta^{\mathfrak{k}}=H_\beta$, hence (5) is an integer.
Consider the case that $\alpha\in\Delta''$ and $\beta\in\Delta'$. Then $H_\alpha=H_\alpha^{\mathfrak{k}}\in\mathfrak{t}_{\mathfrak{k}}$, hence

$$2\cdot\frac{\langle H_\alpha^{\mathfrak{k}},H_\beta^{\mathfrak{k}}\rangle}{|H_\alpha^{\mathfrak{k}}|^2}=2\cdot\frac{\langle H_\alpha,H_\beta\rangle}{|H_\alpha|^2}\in\mathbb{Z}.$$

The last case to be considered is that $\alpha\in\Delta'$ and $\beta\in\Delta''$. In this case $H_\beta=H_\beta^{\mathfrak{k}}\in\mathfrak{t}_{\mathfrak{k}}$. It may happen that $H_\alpha\in\mathfrak{t}_{\mathfrak{k}}$, but then the claim would follow as before, so we may assume that $H_\alpha\notin\mathfrak{t}_{\mathfrak{k}}$. It follows that $\alpha\circ\sigma$ is a root different from α. By Lemma 4.8 we have $|H_\alpha^{\mathfrak{p}}|^2=c|H_\alpha^{\mathfrak{k}}|^2$ with $c=1$ or $c=3$. We know that

$$2\cdot\frac{\langle H_\alpha,H_\beta\rangle}{|H_\alpha|^2}=2\cdot\frac{\langle H_\alpha^{\mathfrak{k}},H_\beta^{\mathfrak{k}}\rangle}{|H_\alpha^{\mathfrak{k}}|^2+|H_\alpha^{\mathfrak{p}}|^2}=\frac{2}{1+c}\cdot\frac{\langle H_\alpha^{\mathfrak{k}},H_\beta^{\mathfrak{k}}\rangle}{|H_\alpha^{\mathfrak{k}}|^2}$$

is an integer, hence multiplying with the integer $1+c$ shows that (5) is an integer in this case as well.
Next we have to check that for each $\alpha\in\Delta_{\mathfrak{g}}$ the reflection $s_{\alpha|_{\mathfrak{t}_{\mathfrak{k}}}}:\mathfrak{t}_{\mathfrak{k}}\to\mathfrak{t}_{\mathfrak{k}}$ along $\ker\alpha|_{\mathfrak{t}_{\mathfrak{k}}}$ defined by

$$X\mapsto X-2\cdot\frac{\langle H_\alpha^{\mathfrak{k}},X\rangle}{|H_\alpha^{\mathfrak{k}}|^2}H_\alpha^{\mathfrak{k}}\tag{6}$$

sends $\{H_\beta^{\mathfrak{k}}\mid\beta\in\Delta_{\mathfrak{g}}\}$ to itself. If $H_\alpha\in\mathfrak{t}_{\mathfrak{k}}$ (this includes the case $\alpha\in\Delta''$), then the reflection $s_\alpha:\mathfrak{t}_{\mathfrak{g}}\to\mathfrak{t}_{\mathfrak{g}}$ along $\ker\alpha$ leaves invariant $\mathfrak{t}_{\mathfrak{k}}$, and (6) is nothing but the restriction of this reflection to $\mathfrak{t}_{\mathfrak{k}}$. Thus, $\{H_\beta^{\mathfrak{k}}\mid\beta\in\Delta_{\mathfrak{g}}\}$ is sent to itself.
Let $\alpha\in\Delta'$ with $H_\alpha\notin\mathfrak{t}_{\mathfrak{k}}$. We treat the two cases that can arise by Lemma 4.8 separately: assume first that $\langle H_\alpha,H_{\alpha\circ\sigma}\rangle=0$. In this case the two reflections

s_α and $s_{\alpha\circ\sigma}$ commute and we have, recalling that $H_{\alpha\circ\sigma} = H_\alpha^{\mathfrak{k}} - H_\alpha^{\mathfrak{p}}$,

$$\begin{aligned} s_{\alpha\circ\sigma}\circ s_\alpha(X) &= X - 2\cdot\frac{\langle H_\alpha, X\rangle}{|H_\alpha|^2}H_\alpha - 2\cdot\frac{\langle H_{\alpha\circ\sigma}, X\rangle}{|H_{\alpha\circ\sigma}|^2}H_{\alpha\circ\sigma} \\ &= X - 2\cdot\frac{\langle H_\alpha, X\rangle + \langle H_{\alpha\circ\sigma}, X\rangle}{2|H_\alpha^{\mathfrak{k}}|^2}H_\alpha^{\mathfrak{k}} - 2\cdot\frac{\langle H_\alpha, X\rangle - \langle H_{\alpha\circ\sigma}, X\rangle}{2|H_\alpha^{\mathfrak{p}}|^2}H_\alpha^{\mathfrak{p}} \\ &= X - 2\cdot\frac{\langle H_\alpha^{\mathfrak{k}}, X\rangle}{|H_\alpha^{\mathfrak{k}}|^2}H_\alpha^{\mathfrak{k}} + 2\cdot\frac{\langle H_\alpha^{\mathfrak{p}}, X\rangle}{|H_\alpha^{\mathfrak{p}}|^2}H_\alpha^{\mathfrak{p}}. \end{aligned}$$

In particular for each $\beta \in \Delta_{\mathfrak{g}}$ the vector $H_\beta^{\mathfrak{k}} - 2\cdot\frac{\langle H_\alpha^{\mathfrak{k}}, H_\beta^{\mathfrak{k}}\rangle}{|H_\alpha^{\mathfrak{k}}|^2}H_\alpha^{\mathfrak{k}}$ is the $\mathfrak{k}$-part of some vector H_γ, which shows that (6) sends $\{H_\beta^{\mathfrak{k}} \mid \beta \in \Delta_{\mathfrak{g}}\}$ to itself.
In the second case of Lemma 4.8 we have that $\alpha + \alpha\circ\sigma \in \Delta_{\mathfrak{g}}$, with $\ker(\alpha + \alpha\circ\sigma) = \ker \alpha|_{\mathfrak{t}_{\mathfrak{k}}} \oplus \mathfrak{t}_{\mathfrak{p}}$. Thus, the reflection $s_{\alpha|_{\mathfrak{t}_{\mathfrak{k}}}}$ is nothing but the restriction of $s_{\alpha+\alpha\circ\sigma}$ to $\mathfrak{t}_{\mathfrak{k}}$; in particular it sends $\{H_\beta^{\mathfrak{k}} \mid \beta \in \Delta_{\mathfrak{g}}\}$ to itself. □

Remark 4.10. The root system $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}}$ is not necessarily reduced: if there exists a root $\alpha \in \Delta_{\mathfrak{g}}$ with $\alpha\circ\sigma \neq \alpha$ for which the second case of Lemma 4.8 holds, then it contains $\alpha|_{\mathfrak{t}_{\mathfrak{k}}}$ as well as $2\cdot\alpha|_{\mathfrak{t}_{\mathfrak{k}}}$. This happens for instance for $\mathrm{SU}(2m+1)/\mathrm{SO}(2m+1)$.

Because B is the set of simple roots of $\Delta_{\mathfrak{g}}$ every root $\alpha \in \Delta_{\mathfrak{g}}$ can be written as a linear combination of elements in B with integer coefficients of the same sign. It follows that every restriction $\alpha|_{\mathfrak{t}_{\mathfrak{k}}} \in \Delta|_{\mathfrak{t}_{\mathfrak{k}}}$ is a linear combination of elements in $B|_{\mathfrak{t}_{\mathfrak{k}}}$ of the same kind. We thus have proven the following lemma.

Lemma 4.11. *The $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}}$-Weyl chambers are exactly the compartments. If C is a $\mathfrak{g}$-Weyl chamber that intersects $\mathfrak{t}_{\mathfrak{k}}$ nontrivially, with corresponding set of simple roots $B \subset \Delta_{\mathfrak{g}}$, then $B|_{\mathfrak{t}_{\mathfrak{k}}}$ is the set of simple roots of the root system $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}}$ corresponding to $C \cap \mathfrak{t}_{\mathfrak{k}}$.*

Recall that the $N_G(T_K)/T_G$-action on the set of compartments was shown to be generated by the reflections along all hyperplanes $\ker \alpha|_{\mathfrak{t}_{\mathfrak{k}}}$, where $\alpha \in \Delta_{\mathfrak{g}}$. Thus, we obtain

Corollary 4.12. *The $N_G(T_K)/T_G$-action on the set of compartments is the same as the action of the Weyl group $W(\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}})$. In particular, it is generated by the reflections along the hyperplanes $\ker \alpha_i|_{\mathfrak{t}_{\mathfrak{k}}}$. Furthermore, $r = \frac{|W(\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}})|}{|W(\mathfrak{k})|}$.*

Recall that whereas a reduced root system is determined by its simple roots [12, Proposition 2.66], this is no longer true for nonreduced root systems such as $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}}$, see [12, II.8]. However, the reduced elements in a nonreduced root system always form a reduced root system [12, Lemma 2.91] with the same simple roots and the same Weyl group. Using the following proposition taken from [15] we will identify this reduced root system contained in $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}}$ with the root system of a second symmetric subalgebra $\mathfrak{k}' \subset \mathfrak{g}$.

PROPOSITION 4.13 ([15, Proposition VII.3.4]). *There is an extension of* $\sigma : \mathfrak{t}_\mathfrak{g} \to \mathfrak{t}_\mathfrak{g}$ *to an involutive automorphism* $\sigma' : \mathfrak{g} \to \mathfrak{g}$ *such that its* $\mathbb{C}$*-linear extension* $\sigma' : \mathfrak{g}^\mathbb{C} \to \mathfrak{g}^\mathbb{C}$ *satisfies* $\sigma'|_{\mathfrak{g}_\alpha} = \mathrm{id}$ *for every root* $\alpha \in B$ *with* $\alpha = \alpha \circ \sigma$. *The root system of the fixed point algebra* $\mathfrak{k}' = \mathfrak{g}^{\sigma'}$ *relative to the maximal abelian subalgebra* $\mathfrak{t}_\mathfrak{k}$ *has* $B|_{\mathfrak{t}_\mathfrak{k}}$ *as simple roots.*

The roots of $\mathfrak{k}'$ relative to $\mathfrak{t}_\mathfrak{k}$ are restrictions of certain (not necessarily all) elements in $\Delta_\mathfrak{g}$ to $\mathfrak{t}_\mathfrak{k}$; the restrictions of all elements in B occur. See [15, p. 129] for the root space decomposition of $\mathfrak{k}'$ with respect to $\mathfrak{k}_\mathfrak{k}$. Because the sub-root system of reduced elements in $\Delta_\mathfrak{g}|_{\mathfrak{t}_\mathfrak{k}}$ and the root system of $\mathfrak{k}'$ have the same simple roots, these reduced root systems coincide. In particular we obtain the following formula for r:

PROPOSITION 4.14. *We have* $r = \frac{|W(\mathfrak{k}')|}{|W(\mathfrak{k})|}$.

EXAMPLE 4.15. If $\operatorname{rank} G = \operatorname{rank} K$, i.e., if T_K is also a maximal torus of G, then the identity on $\mathfrak{g}$ satisfies the conditions of Proposition 4.13. Hence $\mathfrak{k}' = \mathfrak{g}$ and the proposition says $r = \frac{|W(G)|}{|W(K)|}$. This however follows already from Lemma 4.3.

EXAMPLE 4.16. If G/K is a symmetric space of split rank, i.e., $\operatorname{rank} G = \operatorname{rank} K + \operatorname{rank} G/K$, then σ itself satisfies the conditions of Proposition 4.13. In fact, let $\alpha \in B$ with $\alpha = \alpha \circ \sigma$. In this case α vanishes on $\mathfrak{t}_\mathfrak{p}$, which implies that $\mathfrak{g}_\alpha$ is contained either in $\mathfrak{k}^\mathbb{C}$ or in $\mathfrak{p}^\mathbb{C}$. But if it was contained in $\mathfrak{p}^\mathbb{C}$, then $[\mathfrak{t}_\mathfrak{p}, \mathfrak{g}_\alpha] = 0$ and $[\mathfrak{t}_\mathfrak{p}, \mathfrak{g}_{-\alpha}] = 0$, which would contradict the fact that $\mathfrak{t}_\mathfrak{p}$ is maximal abelian in $\mathfrak{p}$. Thus, we have $r = 1$ in the split rank case. Note that $r = 1$ also follows from [1, Lemma 13], combined with Lemma 4.7.

EXAMPLE 4.17. The symmetric space G/K', where K' is the connected subgroup of G with Lie algebra $\mathfrak{k}'$, is not always of split rank. Assume as in Remark 4.10 that there exists a root $\alpha \in \Delta_\mathfrak{g}$ with $\alpha \circ \sigma \neq \alpha$ such that $\alpha + \alpha \circ \sigma \in \Delta_\mathfrak{g}$. Let $X \in \mathfrak{g}_\alpha$ be nonzero. Then $[X, \sigma'(X)]$ is a nonzero element in $\mathfrak{g}_{\alpha + \alpha\circ\sigma}$. We have $\sigma'([X, \sigma'(X)]) = -[X, \sigma'(X)]$, thus $[X, \sigma'(X)] \in \mathfrak{p}'$, where $\mathfrak{p}'$ is the -1-eigenspace of σ'. By definition of σ' we have $\mathfrak{t}_\mathfrak{p} \subset \mathfrak{p}'$, but $\mathfrak{t}_\mathfrak{p}$ is not a maximal abelian subspace of $\mathfrak{p}'$ because it commutes with $[X, \sigma'(X)]$. For example, in the case $\mathrm{SU}(2m+1)/\mathrm{SO}(2m+1)$ we have $K' = K$ although the space is not of split rank, see Subsection 4.6.2 below.

We will use below that the symmetric subalgebra $\mathfrak{k}'$ can be determined via the Dynkin diagram of G: σ defines an automorphism of the Dynkin diagram of G (because it is a permutation group of B), which is nontrivial if and only if $\operatorname{rank} \mathfrak{g} > \operatorname{rank} \mathfrak{k}$. One can calculate the root system of $\mathfrak{k}'$ via the fact that by Proposition 4.13 the simple roots of $\mathfrak{k}'$ are given by $B|_{\mathfrak{t}_\mathfrak{k}} = \{\frac{1}{2}(\alpha_i + \alpha_i \circ \sigma) \mid i = 1, \ldots, \operatorname{rank} G\}$.

4.2 Reduction to the irreducible case

Lemma 4.18. *If (G,K) and (G',K') are two effective symmetric pairs of connected compact semisimple Lie groups associated to the same pair of Lie algebras $(\mathfrak{g},\mathfrak{k})$, then the K-action on G/K is equivariantly formal if and only if the K'-action on G'/K' is equivariantly formal.*

Proof. Because K and K' are connected, both $H^*(G/K)$ and $H^*(G'/K')$ are given as the $\mathbb{R}$-algebra of $\mathfrak{k}$-invariant elements in $\Lambda^*\mathfrak{p}$, see [20, Theorem 8.5.8]. In particular $\dim H^*(G/K) = \dim H^*(G'/K')$. Choosing maximal tori $T \subset K$ and $T' \subset K'$, we furthermore know from Propositions 4.6 and 4.7 that $\dim H^*((G/K)^T) = \dim H^*((G'/K')^{T'})$ because (G,K) and (G',K') correspond to the same Lie algebra pair. The statement then follows from Proposition 3.1. □

Lemma 4.19. *Given actions of compact connected Lie groups K_i on compact manifolds M_i $(i = 1\ldots n)$, then the $K_1\times\ldots\times K_n$-action on $M_1\times\ldots\times M_n$ is equivariantly formal if and only if all the K_i-actions on M_i are equivariantly formal.*

Proof. Choose maximal tori $T_i \subset K_i$. Then $T_1\times\ldots\times T_n$ is a maximal torus in $K_1\times\ldots\times K_n$. The claim follows from Proposition 3.1 because the $T_1\times\ldots\times T_n$-fixed point set is exactly the product of the T_i-fixed point sets. □

Lemmas 4.18 and 4.19 imply that for proving Theorem 4.1 it suffices to check it for effective symmetric pairs (G,K) of compact connected Lie groups such that G/K is an irreducible simply-connected symmetric space of compact type. Below we will make use of the classification of such spaces, see [8].

4.3 Lie groups

Given a compact connected Lie group G, the product $G\times G$ acts on G via $(g_1,g_2)\cdot g = g_1 g g_2^{-1}$. The isotropy group of the identity element is the diagonal $D(G) \subset G\times G$. In the language of Helgason [8], we obtain an irreducible symmetric pair $(G\times G, D(G))$ of type II. The $D(G)$-action on $(G\times G)/D(G)$ is nothing but the action of G on itself by conjugation. But for any compact connected Lie group, the action on itself by conjugation is equivariantly formal. In fact, if $T \subset G$ is a maximal torus, then the fixed point set of the T-action, G^T, is T itself, and thus $\dim H^*(G^T) = \dim H^*(T) = 2^{\operatorname{rank} G} = \dim H^*(G)$. For other ways to prove that this action is equivariantly formal see [2, Example 4.6]. For instance, equivariant formality would also follow from Proposition 4.23 below as $(G\times G, D(G))$ is of split rank.

4.4 Inner symmetric spaces

Consider the case that the symmetric space G/K of compact type is inner, i.e., that the involution σ is inner. By [8, Theorem IX.5.6] this is the case if and only if $\operatorname{rank} G = \operatorname{rank} K$. Hence, a maximal torus $T_K \subset K$ is also a maximal torus in G, and the T_K-fixed point set is by Lemma 4.3 a finite set of cardinality $\frac{|W(G)|}{|W(K)|}$. Because of the following classical result (see for example [4, Chapter XI, Theorem VII]), the case of inner symmetric spaces is easy to deal with.

Proposition 4.20. *Given any compact connected Lie groups $K \subset G$, the following conditions are equivalent:*

1. $\operatorname{rank} G = \operatorname{rank} K$.
2. $\chi(G/K) > 0$.
3. $H^{odd}(G/K) = 0$.

It follows from Proposition 3.3 that the K-action on a homogeneous space G/K with $\operatorname{rank} G = \operatorname{rank} K$ is always equivariantly formal. Alternatively, [2, Corollary 4.5] implies that the G-action on G/K is equivariantly formal because all its isotropy groups have rank equal to the rank of G. Then by Corollary 3.2 any closed subgroup of G acts equivariantly formally on G/K.

Proposition 4.21. *If $\operatorname{rank} G = \operatorname{rank} K$, then the K-action on G/K is equivariantly formal. If $T_K \subset K$ is a maximal torus, then the fixed point set of the induced T_K-action consists of exactly $\dim H^*(G/K) = \frac{|W(G)|}{|W(K)|}$ points.*

Remark 4.22. This is not a new result. For an investigation of the (algebra structure of the) equivariant cohomology of homogeneous spaces G/K with $\operatorname{rank} G = \operatorname{rank} K$ see [7], or [9, Section 5] for an emphasis on other coefficient rings.

4.5 Spaces of split rank

Also when G/K is of split rank, i.e., $\operatorname{rank} G = \operatorname{rank} K + \operatorname{rank} G/K$, there is a general argument that implies equivariant formality of the K-action on G/K.

Proposition 4.23. *If G/K is of split rank, then the natural K-action on G/K is equivariantly formal.*

Proof. We will show that every K-isotropy algebra has maximal rank, i.e., rank equal to $\operatorname{rank} \mathfrak{k}$. Then equivariant formality follows from [2, Corollary 3.5].
Consider the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and choose any Ad_K-invariant scalar product on $\mathfrak{p}$ that turns G/K into a Riemannian symmetric space. Then we have an exponential map $\exp : \mathfrak{p} \to G/K$, and it is known that every orbit of the K-action on G/K meets $\exp(\mathfrak{a})$, where $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$. Because G/K is of split rank, there is a maximal torus $T_K \subset K$ such that $\mathfrak{t}_\mathfrak{k} \oplus \mathfrak{a}$ is abelian. The torus T_K acts trivially on $\exp(\mathfrak{a})$. Thus, the K-isotropy algebra of any point in $\exp(\mathfrak{a})$ (and hence of any point in M) has maximal rank. □

In the split-rank case we have $r = 1$ by Example 4.16. We thus have

Proposition 4.24. *If G/K is of split rank then* $\dim H^*(G/K) = 2^{\operatorname{rank} G/K}$. *If $T_K \subset K$ is a maximal torus, then the fixed point set of the induced T_K-action on G/K is a* $\operatorname{rank} G/K$*-dimensional torus (in particular connected).*

4.6 Outer symmetric spaces which are not of split rank

For the remaining cases that are not covered by any of the arguments above, i.e., irreducible simply-connected symmetric spaces of type I that are neither of equal nor of split rank, we do not have a general argument for equivariant formality of the isotropy action. Using the classification of symmetric spaces [8, p. 518], we calculate for each of these spaces the dimension of the cohomology of the T_K-fixed point set and show that it coincides with the dimension of the cohomology of G/K (which we take from the literature), upon which we conclude equivariant formality via Proposition 3.1. Fortunately, there are only three (series of) such symmetric spaces, namely

$$\mathrm{SU}(n)/\mathrm{SO}(n), \quad \mathrm{SO}(2p+2q+2)/\mathrm{SO}(2p+1)\times\mathrm{SO}(2q+1), \text{ and } E_6/\mathrm{PSp}(4),$$

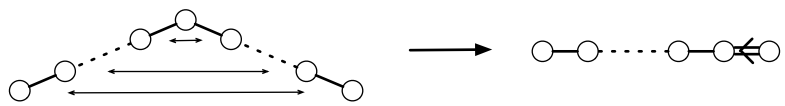
where $n \geq 4$ and $p, q \geq 1$. We have shown with Propositions 4.6 and 4.14 that

$$\dim H^*((G/K)^{T_K}) = 2^{\operatorname{rank}\mathfrak{g}-\operatorname{rank}\mathfrak{k}} \cdot \frac{|W(\mathfrak{k}')|}{|W(\mathfrak{k})|},$$

where the symmetric subalgebra $\mathfrak{k}' \subset \mathfrak{g}$ was introduced in Proposition 4.13. Because in this section we are dealing with outer symmetric spaces, we have $\operatorname{rank}\mathfrak{g} > \operatorname{rank}\mathfrak{k}$, so $\mathfrak{k}' \neq \mathfrak{g}$ is a symmetric subgroup of $\mathfrak{g}$. The orders of the appearing Weyl groups are listed in [11, p. 66].

4.6.1 $\mathrm{SU}(2m)/\mathrm{SO}(2m)$

Let $M = \mathrm{SU}(2m)/\mathrm{SO}(2m)$, where $m \geq 2$, and $T \subset \mathrm{SO}(2m)$ be a maximal torus. The only connected symmetric subgroup of $\mathrm{SU}(2m)$ of rank m different from $\mathrm{SO}(2m)$ is $\mathrm{Sp}(m)$. The fact that $\mathfrak{k}' = \mathfrak{sp}(m)$ can be visualized via the Dynkin diagrams: the involution σ fixes only the middle root of the Dynkin diagram A_{2m-1} of $\mathrm{SU}(2m)$. Hence, after restricting, the middle root becomes a root which is longer than the other roots, and only in C_m there exists a root longer than the others, not in D_m.



We thus may calculate

$$r = \frac{|W(C_m)|}{|W(D_m)|} = \frac{2^m \cdot m!}{2^{m-1}\cdot m!} = 2;$$

note that for this example the number of compartments was also calculated in [1, p. 11]. It is known that $\dim H^*(M) = 2^m$ (see for example [4, p. 493] or [14, Theorem III.6.7.(2)]), hence

$$\dim H^*(M^T) = 2^{2m-1-m} \cdot r = 2^m = \dim H^*(M).$$

Thus, the action is equivariantly formal.

4.6.2 $\mathrm{SU}(2m+1)/\,\mathrm{SO}(2m+1)$

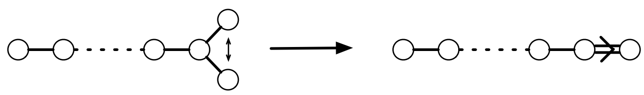
Let $M = \mathrm{SU}(2m+1)/\,\mathrm{SO}(2m+1)$, where $m \geq 2$, and $T \subset \mathrm{SO}(2m+1)$ be a maximal torus. It is known that $\dim H^*(M) = 2^m$ (see for example [4, p. 493] or [14, Theorem III.6.7.(2)]), hence

$$2^m \cdot r = \dim H^*(M^T) \leq \dim H^*(M) = 2^m$$

for some natural number r. Thus necessarily $r = 1$ (in fact $\mathfrak{k}' = \mathfrak{so}(2m+1)$) and the action is equivariantly formal. Note that this space is also listed as an exception in [1] as it is the only outer symmetric space which is not of split rank such that the corresponding involution fixes no root in the Dynkin diagram (and hence every compartment is a K-Weyl chamber).

4.6.3 $\mathrm{SO}(2p+2q+2)/\,\mathrm{SO}(2p+1) \times \mathrm{SO}(2q+1)$

Let $M = \mathrm{SO}(2p+2q+2)/\,\mathrm{SO}(2p+1) \times \mathrm{SO}(2q+1)$, where $p, q \geq 1$, and $T \subset \mathrm{SO}(2p+1) \times \mathrm{SO}(2q+1)$ be a maximal torus. The only connected symmetric subgroups of $\mathrm{SO}(2p+2q+2)$ of rank $p+q$ are $\mathrm{SO}(2p'+1) \times \mathrm{SO}(2q'+1)$, where $p'+q' = p+q$. The involution σ fixes all roots of the Dynkin diagram D_{p+q+1} of $\mathrm{SO}(2p+2q+2)$ but two; after restricting, these two become a single root which is shorter than the others. Because $A_{p+q-1} \oplus A_1$ and D_{p+q} do not appear as the Dynkin diagram of any of the possible symmetric subgroups, the Dynkin diagram of $\mathfrak{k}'$ is forced to be B_{p+q}, which means that $\mathfrak{k}' = \mathfrak{so}(2p+2q+1)$.



We thus have

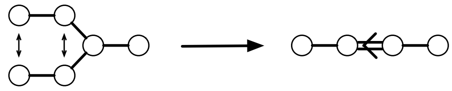
$$r = \frac{|W(B_{p+q})|}{|W(B_p)| \cdot |W(B_q)|} = \frac{2^{p+q} \cdot (p+q)!}{2^p \cdot p! \cdot 2^q \cdot q!} = \binom{p+q}{p}.$$

By [4, p. 496] we have $\dim H^*(M) = 2 \cdot \binom{p+q}{p}$, and it follows that the action is equivariantly formal because of

$$\dim H^*(M^T) = 2^{p+q+1-p-q} \cdot r = 2 \cdot \binom{p+q}{p} = \dim H^*(M).$$

4.6.4 $E_6/\operatorname{PSp}(4)$

Let $M = E_6/\operatorname{PSp}(4)$ and $T \subset \operatorname{PSp}(4)$ be a maximal torus. The only symmetric subalgebra of $\mathfrak{e}_6$ of rank 4 different from $\mathfrak{sp}(4)$ is $\mathfrak{f}_4$.



We obtain

$$r = \frac{|W(F_4)|}{|W(C_4)|} = \frac{2^7 \cdot 3^2}{2^4 \cdot 4!} = 3.$$

It is shown in [19] that $\dim H^*(M) = 12$. Thus,

$$\dim H^*(M^T) = 2^{6-4} \cdot r = 2^2 \cdot 3 = 12 = \dim H^*(M)$$

shows that the action is equivariantly formal.

References

[1] J.-H. Eschenburg, A.-L. Mare and P. Quast, *Pluriharmonic maps into outer symmetric spaces and a subdivision of Weyl chambers*, Bull. Lond. Math. Soc. (2010), doi: 10.1112/blms/bdq070.

[2] O. Goertsches and S. Rollenske, *Torsion in equivariant cohomology and Cohen-Macaulay G-actions*, Transform. Groups. 16 (2011), no. 4, 1063-1080.

[3] M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. 131 (1998), no. 1, 25-83.

[4] W. Greub, S. Halperin, and R. Vanstone, *Connections, curvature, and cohomology. Volume III*, Pure and Applied Mathematics, Vol. 47-III. Academic Press, New York-London, 1976.

[5] V. W. Guillemin, V. L. Ginzburg, and Y. Karshon, *Moment maps, cobordisms, and Hamiltonian group actions*, Mathematical Surveys and Monographs, 96. AMS, Providence, RI, 2002.

[6] V. W. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory*, Springer-Verlag, Berlin 1999.

[7] V. W. Guillemin, T. Holm, and C. Zara, *A GKM description of the equivariant cohomology ring of a homogeneous space*, J. Algebraic Combin. 23 (2006), no. 1, 21–41.

[8] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, 80. Academic Press, Inc., New York-London, 1978.

[9] T. Holm and R. Sjamaar, *Torsion and abelianization in equivariant cohomology*, Transform. Groups 13 (2008), no. 3-4, 585615.

[10] W. Y. Hsiang, *Cohomology theory of topological transformation groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 85. Springer-Verlag, New York-Heidelberg, 1975.

[11] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, New York-Berlin, 1972.

[12] A. W. Knapp, *Lie groups beyond an introduction*, Second edition. Progress in Mathematics, 140. Birkhuser Boston, Inc., Boston, MA, 2002.

[13] J. McCleary, *A user's guide to spectral sequences*, Second edition. Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001.

[14] M. Mimura and H. Toda, *Topology of Lie groups, I and II*, Translations of Mathematical Monographs, 91. American Mathematical Society, Providence, RI, 1991.

[15] O. Loos, *Symmetric spaces. II: Compact spaces and classification*, W. A. Benjamin, Inc., New York-Amsterdam 1969.

[16] T. Sakai, *Riemannian geometry*, Translations of Mathematical Monographs, 149. American Mathematical Society, Providence, RI, 1996.

[17] H. Shiga, *Equivariant de Rham cohomology of homogeneous spaces*, J. Pure Appl. Algebra 106 (1996), no. 2, 173–183.

[18] H. Shiga and H. Takahashi, *Remarks on equivariant cohomology of homogeneous spaces*, Technical report of the Technological University of Nagaoka, 17 (1995), 7-9.

[19] M. Takeuchi, *On Pontrjagin classes of compact symmetric spaces*, J. Fac. Sci. Univ. Tokyo Sect. I 9 (1962), 313–328.

[20] J. A. Wolf, *Spaces of constant curvature*, McGraw-Hill Book Co., New York-London-Sydney, 1967.

Oliver Goertsches
Universität Hamburg
Fachbereich Mathematik
Bundesstraße 55
20146 Hamburg
Germany

Equivariant Cobordism of Schemes

Amalendu Krishna

Received: October 22, 2010
Revised: October 21, 2011

Communicated by Alexander Merkurjev

Abstract. Let k be a field of characteristic zero. For a linear algebraic group G over k acting on a scheme X, we define the equivariant algebraic cobordism of X and establish its basic properties. We explicitly describe the relation of equivariant cobordism with equivariant Chow groups, K-groups and complex cobordism.

We show that the rational equivariant cobordism of a G-scheme can be expressed as the Weyl group invariants of the equivariant cobordism for the action of a maximal torus of G. As applications, we show that the rational algebraic cobordism of the classifying space of a complex linear algebraic group is isomorphic to its complex cobordism.

2010 Mathematics Subject Classification: Primary 14C25; Secondary 19E15
Keywords and Phrases: Algebraic cobordism, group actions

1. Introduction

Let k be a field of characteristic zero. Based on the construction of the motivic algebraic cobordism spectrum MGL by Voevodsky, Levine and Morel [31] gave a geometric construction of the algebraic cobordism and showed that this is a universal oriented Borel-Moore homology theory in the category of varieties over the field k. Their definition was extended by Deshpande [9] in the equivariant set-up that led to the notion of the equivariant cobordism of smooth varieties acted upon by linear algebraic groups. This in particular allowed one to define the algebraic cobordism of the classifying spaces analogous to their complex cobordism.

Apart from its many applications in the equivariant set up which are parallel to the ones in the non-equivariant world, an equivariant cohomology theory often leads to the description of the corresponding non-equivariant cohomology by mixing the geometry of the variety with the representation theory of the underlying groups.

Our aim in this first part of a series of papers is to develop the theory of equivariant cobordism in the category of all k-schemes with action of a linear algebraic group. We establish the fundamental properties of this theory and give applications. In the second part [20] of this series, we shall give many important applications of the results of this paper. Some further applications of the results of this paper to the computation of the non-equivariant cobordism rings appear in [21], [24] and [15]. We now describe some of the main results in of this paper.

Let G be a linear algebraic group over k. In this paper, a scheme will mean a quasi-projective k-scheme and all G-actions will be assumed to be linear. If X is a smooth scheme with a G-action, Deshpande defined the equivariant cobordism $\Omega^G_*(X)$ using the coniveau filtration on the Levine-Morel cobordism of certain smooth mixed spaces. This was based on the construction of the Chow groups of classifying spaces in [39] and the equivariant Chow groups in [10].

Using a *niveau* filtration on the algebraic cobordism, which is based on the analogous filtration on any Borel-Moore homology theory as described in [2, Section 3], we define the equivariant algebraic cobordism of any k-scheme with G-action in Section 4. This is defined by taking a projective limit over the quotients of the Levine-Morel cobordism of certain mixed spaces by various levels of the niveau filtration. In order to make sense of this construction, one needs to prove various properties of the above niveau filtration which is done in Section 3. These equivariant cobordism groups coincide with the one in [9] for smooth schemes. We also show in Section 5 how one can recover the formula for the cobordism group of certain classifying spaces directly from the above definition, by choosing suitable models for the underlying mixed spaces.

In Section 5, we establish the basic properties such as functoriality, homotopy invariance, exterior product, projection formula and existence of Chern classes for equivariant vector bundles in Theorem 5.2. Although we do not have the equivariant version of the localization sequence for the algebraic cobordism, we shall show that the restriction map induced by a G-equivariant open immersion is indeed surjective.

In Section 7, we show how the equivariant cobordism is related to other equivariant cohomology theories such as equivariant Chow groups, equivariant K-groups and equivariant complex cobordism. Using some properties of the niveau filtration and known relation between the non-equivariant cobordism and Chow groups, we deduce an explicit formula (*cf.* Proposition 7.2) which relates the equivariant cobordism and the equivariant Chow groups of k-schemes. Using this and the main results of [17], we give a formula in Theorem 7.4 which relates the equivariant cobordism with the equivariant K-theory of smooth schemes. We also construct a natural transformation from the algebraic to the equivariant version of the complex cobordism for schemes over the field of complex numbers.

Our next main result of this paper is Theorem 8.6, where we show that for a connected linear algebraic group G acting on a scheme X, there is a canonical isomorphism $\Omega^G_*(X) \xrightarrow{\cong} \left(\Omega^T_*(X)\right)^W$ with rational coefficients, where T is a split maximal torus of a Levi subgroup of G with Weyl group W. This is mainly achieved by the Morita isomorphism of Proposition 5.4 and a detour to the motivic cobordism MGL and its extension MGL' to singular schemes by Levine [28]. The use of MGL'-theory in our context is motivated by the recent comparison result of Levine [29] which shows that the Levine-Morel cobordism theory is a piece of the more general MGL'-theory.

As an easy consequence of Proposition 7.2, we recover Totaro's cycle class map (*cf.* [39])

$$CH^*(BG) \to MU^*(BG) \otimes_{\mathbb{L}} \mathbb{Z} \to H^*(BG, \mathbb{Z})$$

for a complex linear algebraic group G. It is conjectured that this map is an isomorphism of rings. This conjecture has been shown to be true by Totaro for some classical groups such as BGL_n, O_n, Sp_{2n} and SO_{2n+1}. Although, we can not say anything about this conjecture here, we do show as a consequence of Theorem 8.6 that the map $CH^*(BG) \to MU^*(BG) \otimes_{\mathbb{L}} \mathbb{Z}$ is indeed an isomorphism of rings with the rational coefficients (see Theorem 8.9 for the full statement). We do this by first showing that there is a natural ring homomorphism $\Omega^*(BG) \to MU^*(BG)$ (with integer coefficients) which lifts Totaro's map. We then show that this map is in fact an isomorphism with rational coefficients using Theorem 8.6.

We now make a remark on our definition and the notation for the equivariant cobordism groups. In many of the topology texts, the cobordism rings of classifying spaces are expressed as rings which are complete. For example, one often writes $MU^*(\mathbb{C}\mathrm{P}^\infty)$ as the formal power series $\mathbb{L}[[t]]$ instead of the graded power series ring. This does not allow one to write an expression of the cobordism in each degree. Since our interest is to give an expression of the cobordism groups in each component, we shall express the equivariant cobordism of a smooth scheme as a graded ring. This notation has been used earlier by other authors in the topological context (see [25, Section 2], [27]). We refer the reader to Subsection 6.1 for more about the comparison between the two notations.

We end this introduction with the following comment. One of the initial motivations for this article was to find a definition of the equivariant algebraic cobordism which has all the expected properties of an equivariant cohomology theory and which is simultaneously, simple enough to compute. Although much of this objective is achieved, the equivariant cobordism as considered here has two drawbacks. The first one is that there is not always a natural map from the algebraic to the complex equivariant cobordism for complex varieties with group action (*cf.* Proposition 7.5). A more serious problem is the lack of the localization sequence (*cf.* Proposition 5.3). One way to take care of these two problems is to consider Voevodsky's motivic cobordism MGL from the equivariant point of view. It turns out that although this approach does have certain advantages, it becomes computationally much harder. So the challenge is to study and analyze the situations when these two approaches yield the same answer so that one can use either of the two, depending on what one would like to prove. These questions will be studied in further detail in [23].

2. Recollection of algebraic cobordism

In this section, we briefly recall the definition of algebraic cobordism of Levine-Morel. We also recall the other definition of this object as given by Levine-Pandharipande. Since we shall be concerned with the study of schemes with group actions and the associated quotient schemes, and since such quotients often require the original scheme to be quasi-projective, we shall assume throughout this paper that all schemes over k are quasi-projective.

Notations. We shall denote the category of quasi-projective k-schemes by $\mathcal{V}_k$. By a scheme, we shall mean an object of $\mathcal{V}_k$. The category of smooth quasi-projective schemes will be denoted by $\mathcal{V}^S_k$. If G is a linear algebraic group over k, we shall denote the category of quasi-projective k-schemes with a G-action and G-equivariant maps by $\mathcal{V}_G$. The associated category of smooth G-schemes will be denoted by $\mathcal{V}^S_G$. All G-actions in this paper will be assumed to be linear. Recall that this means that all G-schemes are assumed to admit G-equivariant ample line bundles. This assumption is always satisfied for normal schemes (*cf.* [36, Theorem 2.5], [37, 5.7]).

2.1. Algebraic cobordism. Before we define the algebraic cobordism, we recall the Lazard ring $\mathbb{L}$. It is a polynomial ring over $\mathbb{Z}$ on infinite but countably many variables and is given by the quotient of the polynomial ring $\mathbb{Z}[A_{ij}|(i,j) \in \mathbb{N}^2]$ by the relations, which uniquely define the universal formal group law $F_{\mathbb{L}}$ of rank one on $\mathbb{L}$. This formal group law is given by the power series

$$F_{\mathbb{L}}(u,v) = u + v + \sum_{i,j \ge 1} a_{ij} u^i v^j,$$

where a_{ij} is the equivalence class of A_{ij} in the ring $\mathbb{L}$. The Lazard ring is graded by setting the degree of a_{ij} to be $1-i-j$. In particular, one has $\mathbb{L}_0 = \mathbb{Z}, \mathbb{L}_{-1} = \mathbb{Z}a_{11}$ and $\mathbb{L}_i = 0$ for $i \ge 1$, that is, $\mathbb{L}$ is non-positively graded. We shall write $\mathbb{L}_*$ for the graded ring such that $\mathbb{L}_{*,i} = \mathbb{L}_{-i}$ for $i \in \mathbb{Z}$. We now define the algebraic cobordism of Levine and Morel [31].

Let X be an equi-dimensional k-scheme. A cobordism cycle over X is a family $\alpha = [Y \xrightarrow{f} X, L_1, \cdots, L_r]$, where Y is a smooth scheme, the map f is projective, and L_i's are line bundles on Y. Here, one allows the set of line bundles to be empty. The degree of such a cobordism cycle is defined to be $\deg(\alpha) = \dim_k(Y) - r$ and its codimension is defined to be $\dim(X) - \deg(\alpha)$. Let $\mathcal{Z}^*(X)$ be the free abelian group generated by the cobordism cycles of the above type. Note that this group is graded by the codimension of the cycles. In particular, for $j \in \mathbb{Z}$, $\mathcal{Z}^j(X)$ is the free abelian group on cobordism cycles $\alpha = [Y \xrightarrow{f} X, L_1, \cdots, L_r]$, where Y is smooth and irreducible and codimension of α is j. We impose several relations on $\mathcal{Z}^*(X)$ in order to define the algebraic cobordism group. The first among these is the so called *dimension axiom*: let $\mathcal{R}^*_{\dim}(X)$ be the graded subgroup of $\mathcal{Z}^*(X)$ generated by the cobordism cycles $\alpha = [Y \xrightarrow{f} X, L_1, \cdots, L_r]$ such that $\dim_k Y < r$. Let

$$\mathcal{Z}^*_{\dim}(X) = \frac{\mathcal{Z}^*(X)}{\mathcal{R}^*_{\dim}(X)}.$$

For a line bundle L on X and cobordism cycle α as above, we define the Chern class operator on $\mathcal{Z}^*_{\dim}(X)$ by letting $c_1(L)(\alpha) = [Y \xrightarrow{f} X, L_1, \cdots, L_r, f^*(L)]$.
Next, we impose the so called *section axiom*. Let $\mathcal{R}^*_{\rm sec}(X)$ be the graded subgroup of $\mathcal{Z}^*_{\dim}(X)$ generated by cobordism cycles of the form $[Y \to X, L] - [Z \to X]$, where $Y \xrightarrow{s} L$ is a section of the line bundle L on Y which is transverse to the zero-section, and $Z \hookrightarrow Y$ is the closed subvariety of Y defined by the zeros of s. The transversality of s ensures that Z is a smooth variety. In particular, $[Z \to X]$ is a well-defined cobordism cycle on X. Define

$$\underline{\Omega}^*(X) = \frac{\mathcal{Z}^*_{\dim}(X)}{\mathcal{R}^*_{\rm sec}(X)}.$$

The assignment $X \mapsto \underline{\Omega}^*(X)$ is called the pre-cobordism theory.
Finally, we impose the *formal group law* on the cobordism using the following relation. For X as above, let $\mathcal{R}^*_{\rm FGL}(X) \subset \mathbb{L}\otimes_{\mathbb{Z}}\underline{\Omega}^*(X)$ be the graded $\mathbb{L}$-submodule generated by elements of the form

$$\left\{F_{\mathbb{L}}\left(c_1(L), c_1(M)\right)(x) - c_1(L \otimes M)(x) | x \in \underline{\Omega}^*(X), L, M \in \mathrm{Pic}(X)\right\}.$$

We define the *algebraic cobordism* group of X by

$$\Omega^*(X) = \frac{\mathbb{L}\otimes_{\mathbb{Z}}\underline{\Omega}^*(X)}{\mathcal{R}^*_{\rm FGL}(X)}. \tag{2.1}$$

If X is not necessarily equi-dimensional, we define $\mathcal{Z}_*(X)$ to be same as $\mathcal{Z}^*(X)$ except that $\mathcal{Z}_*(X)$ is now graded by the degree of the cobordism cycles. In particular, $\mathcal{Z}_i(X)$ is the free abelian group on cobordism cycles $[Y \xrightarrow{f} X, L_1, \cdots, L_r]$ such that f is projective and Y is smooth and irreducible such that $\dim(Y) - r = i$. One then defines $\Omega_*(X)$ to be the quotient of $\mathbb{L}_* \otimes_{\mathbb{Z}} \underline{\Omega}_*(X)$ in the same way as above. Note that for X equi-dimensional of dimension d and $i \in \mathbb{Z}$, one has $\Omega^i(X) \cong \Omega_{d-i}(X)$.
Observe that $\Omega^*(X)$ is a graded $\mathbb{L}$-module such that $\Omega^j(X) = 0$ for $j > \dim(X)$ and $\Omega^j(X)$ can be non-zero for any given $-\infty < j \le \dim(X)$. Similarly, $\Omega_*(X)$ is a graded $\mathbb{L}_*$-module which has no component in the negative degrees and it can be non-zero in arbitrarily large positive degree.
The following is the main result of Levine and Morel from which most of their other results on algebraic cobordism are deduced. We refer to *loc. cit.* for more properties.

Theorem 2.1. *The functor $X \mapsto \Omega_*(X)$ is the universal Borel-Moore homology on the category $\mathcal{V}_k$. In other words, it is universal among the homology theories on $\mathcal{V}_k$ which have functorial push-forward for projective morphism, pull-back for smooth morphism (any morphism of smooth schemes), Chern classes for line bundles, and which satisfy Projective bundle formula, homotopy invariance,* *the above dimension, section and formal group law axioms. Moreover, for a k-scheme X and closed subscheme Z of X with open complement U, there is a localization exact sequence*

$$\Omega_*(Z) \to \Omega_*(X) \to \Omega_*(U) \to 0.$$

It was also shown in *loc. cit.* that the natural composite map

$$\Phi : \mathbb{L} \to \mathbb{L} \otimes_{\mathbb{Z}} \underline{\Omega}^*(k) \twoheadrightarrow \Omega^*(k)$$
$$a \mapsto [a]$$

is an isomorphism of commutative graded rings.
As an immediate corollary of Theorem 2.1, we see that for a smooth k-scheme X and an embedding $\sigma : k \to \mathbb{C}$, there is a natural morphism of graded rings

$$\Phi_X^{top} : \Omega^*(X) \to MU^{2*}(X_\sigma(\mathbb{C})), \tag{2.2}$$

where $MU^*(X_\sigma(\mathbb{C}))$ is the complex cobordism ring of the complex manifold $X_\sigma(\mathbb{C})$ given by the complex points of $X \times_k \mathbb{C}$. This map is an isomorphism for $X = \mathrm{Spec}(k)$. In particular, there are isomorphisms of graded rings

$$\mathbb{L} \xrightarrow{\cong} \Omega^*(k) \xrightarrow{\cong} MU^{2*} \xrightarrow{\cong} MU^*, \tag{2.3}$$

where MU^* is the complex cobordism ring of a point. As a corollary, we see that for any field extension $k \hookrightarrow K$, the natural map $\Omega^*(k) \to \Omega^*(K)$ is an isomorphism.

2.2. COBORDISM VIA DOUBLE POINT DEGENERATION. To enforce the formal group law on the algebraic cobordism in order to make it an oriented cohomology theory on the category of smooth varieties, Levine and Morel artificially imposed this condition by tensoring their pre-cobordism theory with the Lazard ring. Although they were able to show that the resulting map $\mathcal{Z}_*(X) \to \Omega_*(X)$ is still surjective, they were unable to describe the explicit geometric relations in $\mathcal{Z}_*(X)$ that define $\Omega_*(X)$. This was subsequently accomplished by Levine-Pandharipande [32]. We conclude our introduction to the algebraic cobordism by briefly discussing the construction of Levine-Pandharipande. For $n \geq 1$, let $\square^n$ denote the space $(\mathbb{P}^1_k - \{1\})^n$.

DEFINITION 2.2. A morphism $Y \xrightarrow{\pi} \square^1$ is called a *double point degeneration*, if Y is a smooth scheme and $\pi^{-1}(0)$ is scheme-theoretically given as the union $A \cup B$, where A and B are smooth divisors on Y which intersect transversely. The intersection $D = A \cap B$ is called the double point locus of π. Here, A, B and D are allowed to be disconnected or, even empty.

For a double point degeneration as above, notice that the scheme D is also smooth and $\mathcal{O}_D(A+B)$ is trivial. In particular, one sees that $N_{A/D} \otimes_D N_{B/D} \cong \mathcal{O}_D$. This is turn implies that the projective bundles $\mathbb{P}(\mathcal{O}_D \oplus N_{A/D}) \to D$ and $\mathbb{P}(\mathcal{O}_D \oplus N_{B/D}) \to D$ are isomorphic, where $N_{A/D}$ and $N_{B/D}$ are the normal bundles of D in A and B respectively. Let $\mathbb{P}(\pi) \to D$ denote any of these two projective bundles.

Let X be a k-scheme and let $Y \xrightarrow{f} X \times \square^1$ be a projective morphism from a smooth scheme Y. Assume that the composite map $\pi : Y \to X \times \square^1 \to \square^1$ is a double point degeneration such that $Y_\infty = \pi^{-1}(\infty)$ is smooth. We define the cobordism cycle on X associated to the morphism f to be the cycle

$$C(f) = [Y_\infty \to X] - [A \to X] - [B \to X] + [\mathbb{P}(\pi) \to X]. \tag{2.4}$$

Let $\mathcal{M}_*(X)$ be the free abelian group on the isomorphism classes of the morphisms $[Y \xrightarrow{f} X]$, where Y is smooth and irreducible and f is projective. Then $\mathcal{M}_*(X)$ is a graded abelian group, where the grading is by the dimension of Y. Let $\mathcal{R}_*(X)$ be the subgroup of $\mathcal{M}_*(X)$ generated by all cobordism cycles $C(f)$, where $C(f)$ is as in (2.4). Note that $\mathcal{R}_*(X)$ is a graded subgroup of $\mathcal{M}_*(X)$. Define

$$\omega_*(X) = \frac{\mathcal{M}_*(X)}{\mathcal{R}_*(X)}. \tag{2.5}$$

Theorem 2.3 ([32]). *There is a canonical isomorphism*

$$\omega_*(X) \xrightarrow{\cong} \Omega_*(X) \tag{2.6}$$

of oriented Borel-Moore homology theories on $\mathcal{V}$.

3. Niveau filtration on algebraic cobordism

In this section, we introduce the niveau filtration on the algebraic cobordism which plays an important role in the definition of the equivariant algebraic cobordism. Our main result here is a refined localization sequence for the cobordism which preserves the niveau filtration. This new localization sequence will have interesting consequences in the study of the equivariant cobordism.

Let X be a k-scheme of dimension d. For $j \in \mathbb{Z}$, let Z_j be the set of all closed subschemes $Z \subset X$ such that $\dim_k(Z) \le j$ (we assume $\dim(\emptyset) = -\infty$). The set Z_j is then ordered by the inclusion. For $i \ge 0$, we define

$$\Omega_i(Z_j) = \varinjlim_{Z \in Z_j} \Omega_i(Z) \quad \text{and} \quad \text{put}$$

$$\Omega_*(Z_j) = \bigoplus_{i \ge 0} \Omega_i(Z_j).$$

It is immediate that $\Omega_*(Z_j)$ is a graded $\mathbb{L}_*$-module and there is a graded $\mathbb{L}_*$-linear map $\Omega_*(Z_j) \to \Omega_*(X)$.

Following [2, Section 3], we let Z_j/Z_{j-1} denote the ordered set of pairs $(Z, Z') \in Z_j \times Z_{j-1}$ such that $Z' \subset Z$ with the ordering

$$(Z, Z') \ge (Z_1, Z_1') \quad \text{if} \quad Z_1 \subseteq Z \quad \text{and} \quad Z_1' \subseteq Z'.$$

If $(Z, Z') \ge (Z_1, Z_1')$, then the functoriality of the push-forward maps and the localization sequence yield a map $\Omega_i(Z_1 - Z_1') \to \Omega_i(Z - Z')$ (*cf.* (3.1)). We let

$$\Omega_i\left(Z_j/Z_{j-1}(X)\right) := \varinjlim_{(Z,Z') \in Z_j/Z_{j-1}} \Omega_i(Z - Z').$$

Lemma 3.1. *For $f : X' \to X$ projective, the push-forward map $\Omega_*(X') \xrightarrow{f_*} \Omega_*(X)$ induces a push-forward map $\Omega_*\left(Z_j/Z_{j-1}(X')\right) \to \Omega_*\left(Z_j/Z_{j-1}(X)\right)$.*

Proof. Let $(Z, Z') \in Z_j/Z_{j-1}(X')$. Then $(W, W') = (\mathrm{Im}(Z), \mathrm{Im}(Z')) \in Z_j/Z_{j-1}(X)$. It suffices now to show that f_* induces a natural map $\Omega_*(Z - Z') \to \Omega_*(W - W')$. However, this follows directly from the localization exact sequences

$$\begin{tikzcd} \Omega_*(Z') \arrow[r] \arrow[d, "f_*"'] & \Omega_*(Z) \arrow[r] \arrow[d, "f_*"] & \Omega_*(Z - Z') \arrow[r] \arrow[d, dotted] & 0 \\ \Omega_*(W') \arrow[r] & \Omega_*(W) \arrow[r] & \Omega_*(W - W') \arrow[r] & 0 \end{tikzcd} \tag{3.1}$$

and the fact that the square on the left is commutative. □

For $x \in Z_j$, let

$$\widetilde{\Omega_*(k(x))} = \varinjlim_{U \subseteq \overline{\{x\}}} \Omega_*(U), \tag{3.2}$$

where the limit is taken over all non-empty open subsets of $\overline{\{x\}}$. Taking the limit over the localization sequences

$$\Omega_*(Z') \to \Omega_*(Z) \to \Omega_*(Z - Z') \to 0$$

for $(Z, Z') \in Z_j/Z_{j-1}$, one now gets an exact sequence

$$\Omega_*(Z_{j-1}) \to \Omega_*(Z_j) \to \bigoplus_{x \in (Z_j - Z_{j-1})} \widetilde{\Omega_*(k(x))} \to 0. \tag{3.3}$$

Definition 3.2. We define $F_j\Omega_*(X)$ to be the image of the natural $\mathbb{L}_*$-linear map $\Omega_*(Z_j) \to \Omega_*(X)$. In other words, $F_j\Omega_*(X)$ is the image of all $\Omega_*(W) \to \Omega_*(X)$, where $W \to X$ is a projective map such that $\dim(\mathrm{Image}(W)) \le j$. Using the localization sequence, this is same as saying that $F_j\Omega_*(X)$ is the set of all elements $s \in \Omega_*(X)$ such that $i^*(s) = 0$ for some open subset $i : U \hookrightarrow X$, whose complement has dimension at most j.

One checks at once that there is a canonical *niveau filtration*

$$0 = F_{-1}\Omega_*(X) \subseteq F_0\Omega_*(X) \subseteq \cdots \subseteq F_{d-1}\Omega_*(X) \subseteq F_d\Omega_*(X) = \Omega_*(X). \tag{3.4}$$

Lemma 3.3. *If $f : X' \to X$ is a projective morphism, then $f_*\left(F_j\Omega_*(X')\right) \subseteq F_j\Omega_*(X)$. If $g : X' \to X$ is a smooth morphism of relative dimension r, then $g^*\left(F_j\Omega_*(X)\right) \subseteq F_{j+r}\Omega_*(X')$.*

Proof. The first assertion is obvious from the definition. In fact, the push-forward map preserves the niveau filtration at the level of the free abelian groups of cobordism cycles. The second assertion also follows immediately using the fact that for a cobordism cycle $[Y \to X]$, one has $g^*([Y \to X]) =$

$[Y \times_X X' \to X']$. This in turn implies that $g^* \circ f_* = f'_* \circ g'^*$ for a Cartesian square

$$\begin{tikzcd} W' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ W \arrow[r, "f"'] & X \end{tikzcd}$$

such that f is projective and g is smooth. □

Proposition 3.4. *Let X be a k-scheme and let Z be a closed subscheme of X with the complement U. Then for every $j \geq \dim(Z)$, there is an exact sequence*

$$\Omega_*(Z) \to F_j\Omega_*(X) \to F_j\Omega_*(U) \to 0.$$

Proof. Since $j \geq \dim(Z)$, we see that the image of the map $\Omega_*(Z) \to \Omega_*(X)$ lies in $F_j\Omega_*(X)$. Using the localization sequence of the algebraic cobordism (*cf.* Theorem 2.1), we only need to show that the map $F_j\Omega_*(X) \to F_j\Omega_*(U)$ is surjective.

Let $F_j\mathcal{Z}_*(X)$ be the free abelian group on cobordism cycles $[Y \xrightarrow{f} X]$ such that Y is irreducible and $\dim(f(Y)) \leq j$. Note that $f(Y)$ is a closed and irreducible subscheme of X since Y is irreducible and f is projective. It is then clear that $F_j\mathcal{Z}_*(X) \subset \mathcal{Z}_*(X)$ and $F_j\mathcal{Z}_*(X) \twoheadrightarrow F_j\Omega_*(X)$.

Let $[Y \xrightarrow{f} U]$ be a cobordism cycle on U such that Y is smooth and irreducible, f is projective and $\dim(f(Y)) \leq j$. We have a factorization $Y \hookrightarrow \mathbb{P}^n_k \times U \to U$ where the first inclusion is a closed immersion. Let $\overline{Y}$ denote a resolution of singularities of the closure of Y in $\mathbb{P}^n_k \times X$ and let $\overline{Y} \xrightarrow{\bar{f}} X$ be the projection map. It is then easy to verify that $[\overline{Y} \xrightarrow{\bar{f}} X]$ is a cobordism cycle on X which restricts to $[Y \xrightarrow{f} U]$ in $\Omega_*(U)$ and $\dim(\bar{f}(\overline{Y})) = \dim(f(Y)) \leq j$. This proves the required surjection. □

Theorem 3.5. *Let X be a k-scheme and let Z be a closed subscheme of X with the complement U. Then for every $j \in \mathbb{Z}$, there is an exact sequence*

$$\frac{\Omega_*(Z)}{F_j\Omega_*(Z)} \to \frac{\Omega_*(X)}{F_j\Omega_*(X)} \to \frac{\Omega_*(U)}{F_j\Omega_*(U)} \to 0. \tag{3.5}$$

Proof. Let $f : U \to X$ and $g : Z \to X$ be the inclusion maps. The surjectivity of the second map in (3.5) follows from the localization sequence of algebraic cobordism (*cf.* Theorem 2.1). It suffices thus to show that

$$f^{*(-1)}\left(F_j\Omega_*(U)\right) \subseteq \mathrm{Image}\left(\Omega_*(Z) \oplus F_j\Omega_*(X) \to \Omega_*(X)\right) \tag{3.6}$$

in order to prove the theorem.

So let $\alpha \in \Omega_*(X)$ be such that $\beta = f^*(\alpha) \in F_j\Omega_*(U)$. We can find a closed subscheme $q : W \hookrightarrow U$ of dimension at most j and a cobordism cycle $\beta' \in \Omega_*(W)$ such that $\beta = q_*(\beta')$. Let Y be the closure of W in X and let $W \xrightarrow{f'} Y \xrightarrow{p} X$ be the open and the closed immersions.

Using Theorem 2.1 (the localization sequence), we can find $\alpha' \in \Omega_*(Y)$ such that $\beta' = f'^*(\alpha')$. We conclude from this that $f^*(\alpha - p_*(\alpha')) = 0$ in $\Omega_*(U)$. Using Theorem 2.1 (the localization sequence) again, we see that $\alpha = g_*(\gamma) + p_*(\alpha')$ for some $\gamma \in \Omega_*(Z)$. Since $\dim(Y) = \dim(W) \le j$, it also follows that $p_*(\alpha') \in F_j\Omega_*(X)$. This proves (3.6) and hence the theorem. □

The following is an immediate consequence of Theorem 3.5.

Corollary 3.6. *Let X be a k-scheme. Then for any $j \ge 0$ and any closed subscheme $Z \subset X$ of dimension at most j, the natural map $\Omega_*(X) \to \Omega_*(X-Z)$ induces an isomorphism*

$$\frac{\Omega_*(X)}{F_j\Omega_*(X)} \xrightarrow{\cong} \frac{\Omega_*(X-Z)}{F_j\Omega_*(X-Z)}.$$

Lemma 3.7. *For a k-scheme X and $i \ge 0$, the natural map $\Omega_i(X) \to \mathrm{CH}_i(X)$ has the factorization*

$$\Omega_i(X) \to \frac{\Omega_i(X)}{F_{i-1}\Omega_i(X)} \to \mathrm{CH}_i(X).$$

Proof. By Theorem 2.3, $\Omega_*(X)$ is generated by the cobordism cycles $[Y \to X]$, where Y is smooth and f is projective. It follows from the definition of the niveau filtration that $F_j\Omega_*(X)$ is generated by the cobordism cycles of the form $i_*([Y \to Z])$, where $Z \overset{\phi}{\hookrightarrow} X$ is a closed subscheme of X of dimension at most j. Since $\Omega_* \to \mathrm{CH}_*$ is a natural transformation of oriented Borel-Moore homology theories, we get a commutative diagram

$$\begin{array}{ccc} \Omega_i(Z) & \longrightarrow & \mathrm{CH}_i(Z) \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} \\ \Omega_i(X) & \longrightarrow & \mathrm{CH}_i(X). \end{array}$$

The lemma now follows from the fact that $\mathrm{CH}_i(Z) = 0$ if $j \le i-1$. □

Lemma 3.8. *For any $s \in F_j\Omega_*(X)$, there are elements $a_i \in \mathbb{L}_*$ and $s_i \in \Omega_{\le j}(X)$ such that $s = \sum_i a_i s_i$.*

Proof. It is a simple variant of the generalized degree formula [30, Theorem 4.7]. We can assume that s is a homogeneous element of $\Omega_*(X)$. We have seen in the proof of Proposition 3.4 that $F_j\mathcal{Z}_*(X)$ a free abelian subgroup of $\mathcal{Z}_*(X)$ such that $F_j\mathcal{Z}_*(X) \twoheadrightarrow F_j\Omega_*(X)$. We can thus assume that $s = [Y \xrightarrow{f} X]$, where Y is smooth and irreducible and f is projective such that $\dim(f(Y)) \le j$.
Let $\iota : f(Y) = W \hookrightarrow X$ denote the inclusion of the closed subset and let U be the complement of W in X. Then the image of s dies in $\Omega_*(U)$ under the restriction map. It follows from the localization sequence of the algebraic cobordism (*cf.* Theorem 2.1) and [30, Theorem 4.7] that we can write $s = \iota_*\left(a[\widetilde{W} \to W] + \sum_i a_i[\widetilde{Z_i} \to Z_i]\right)$, where $\widetilde{W} \to W$ is a resolution of singularities of W, $\widetilde{Z_i} \to Z_i$ is a resolution of singularities of a closed subscheme Z_i

of W of dimension strictly less than that of W and $a, a_i \in \mathbb{L}_*$. It follows from this expression that $s = \sum_i a_i s_i$ such that $s_i \in \Omega_{\le j}(X)$ and $a_i \in \mathbb{L}_*$ $\square$

Proposition 3.9. *Let $E \xrightarrow{f} X$ be a vector bundle of rank r. Then the pull-back map $f^* : \Omega_*(X) \to \Omega_*(E)$ induces an isomorphism*

$$F_j\Omega_*(X) \xrightarrow{\cong} F_{j+r}\Omega_*(E)$$

for all $j \in \mathbb{Z}$. In particular, $F_{<r}\Omega_*(E) = 0$.

Remark 3.10. The reader should be warned that the map f^* shifts the degree of the grading by r.

Proof. Using Lemma 3.8, this can be proved in the same way as [9, Lemma 3.3], where a similar result is proven for smooth varieties and coniveau filtration. We sketch the proof in the singular case.
The homotopy invariance of the algebraic cobordism tells us that the natural map $\Omega_*(X) \xrightarrow{f^*} \Omega_*(E)$ is an isomorphism. So we only need to show that this map is surjective at each level of the niveau filtration. So let $e \in F_{j+r}\Omega_*(E)$. We can assume that $e \in \Omega_i(E)$ is a homogeneous element.
By Lemma 3.8, we can write $e = \sum_p a_p s_p$, where each s_p is a homogeneous element of $\Omega_{\le j+r}(E)$ and $a_p \in \mathbb{L}_*$. Since f^* is an isomorphism of graded abelian groups which shifts the degree by r, we can write $s_p = f^*(x_p)$ such that $x_p \in \Omega_{\le j}(X)$. Letting $x = \sum_p a_p x_p$, we see that $x \in F_j\Omega_*(X)$ and $s = f^*(x)$. This proves the proposition. $\square$

4. Equivariant algebraic cobordism

In this text, G will denote a linear algebraic group of dimension g over k. All representations of G will be finite dimensional. The definition of equivariant cobordism needs one to consider certain kind of mixed spaces which in general may not be a scheme even if the original space is a scheme. The following well known (*cf.* [10, Proposition 23]) lemma shows that this problem does not occur in our context and all the mixed spaces in this paper are schemes with ample line bundles.

Lemma 4.1. *Let H be a linear algebraic group acting freely and linearly on a k-scheme U such that the quotient U/H exists as a quasi-projective variety. Let X be a k-scheme with a linear action of H. Then the mixed quotient $X \overset{H}{\times} U$ exists for the diagonal action of H on $X \times U$ and is quasi-projective. Moreover, this quotient is smooth if both U and X are so. In particular, if H is a closed subgroup of a linear algebraic group G and X is a k-scheme with a linear action of H, then the quotient $G \overset{H}{\times} X$ is a quasi-projective scheme.*

Proof. It is already shown in [10, Proposition 23] using [12, Proposition 7.1] that the quotient $X \overset{H}{\times} U$ is a scheme. Moreover, as U/H is quasi-projective, [12, Proposition 7.1] in fact shows that $X \overset{H}{\times} U$ is also quasi-projective. The similar conclusion about $G \overset{H}{\times} X$ follows from the first case by taking $U = G$ and by observing that G/H is a smooth quasi-projective scheme (*cf.* [3, Theorem 6.8]). The assertion about the smoothness is clear since $X \times U \to X \overset{H}{\times} U$ is a principal H-bundle. $\square$

For any integer $j \geq 0$, let V_j be an l-dimensional representation of G and let U_j be a G-invariant open subset of V_j such that the codimension of the complement $(V_j - U_j)$ in V_j is at least j and G acts freely on U_j such that the quotient U_j/G is a quasi-projective scheme. Such a pair (V_j, U_j) will be called a *good* pair for the G-action corresponding to j (*cf.* [17, Section 2]). It is easy to see that a good pair always exists (*cf.* [10, Lemma 9]). Let X_G denote the mixed quotient $X \overset{G}{\times} U_j$ of the product $X \times U_j$ by the diagonal action of G, which is free.
Let X be a k-scheme of dimension d with a G-action. Fix $j \geq 0$ and let (V_j, U_j) be an l-dimensional good pair corresponding to j. For $i \in \mathbb{Z}$, set

$$\Omega_i^G(X)_j = \frac{\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)}. \tag{4.1}$$

Lemma 4.2. *For a fixed $j \geq 0$, the group $\Omega_i^G(X)_j$ is independent (in a canonical way) of the choice of the good pair (V_j, U_j).*

Proof. Let (V_j, U_j) and (V'_j, U'_j) be two good pairs of dimensions and l and l' respectively corresponding to j. Using the results of Section 3, one can follow the proof of the similar result for the equivariant Chow groups in [10, Proposition 1] to construct a canonical isomorphism

$$\alpha_{vv'} : \frac{\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)} \xrightarrow{\cong} \frac{\Omega_{i+l'-g}\left(X \overset{G}{\times} U'_j\right)}{F_{d+l'-g-j}\Omega_{i+l'-g}\left(X \overset{G}{\times} U'_j\right)}$$

as follows.
We let $V = V_j \oplus V'_j$ and $U = (U_j \oplus V'_j) \cup (V_j \oplus U'_j)$. Let G act diagonally on V. Then it is easy to see that the complement of the open subset $X \overset{G}{\times} (U_j \oplus V'_j)$ in $X \overset{G}{\times} U$ has dimension at most $d + l + l' - g - j$. Hence by Corollary 3.6, the map
(4.2)

$$\frac{\Omega_{i+l+l'-g}\left(X \overset{G}{\times} U\right)}{F_{d+l+l'-g-j}\Omega_{i+l+l'-g}\left(X \overset{G}{\times} U\right)} \xrightarrow{\iota_v^*} \frac{\Omega_{i+l+l'-g}\left(X \overset{G}{\times} (U_j \oplus V'_j)\right)}{F_{d+l+l'-g-j}\Omega_{i+l+l'-g}\left(X \overset{G}{\times} (U_j \oplus V'_j)\right)}$$

is an isomorphism. On the other hand, the map $X \stackrel{G}{\times} (U_j \oplus V'_j) \xrightarrow{\phi_v} X \stackrel{G}{\times} U_j$ is a vector bundle of rank l' and hence by Proposition 3.9, the map

(4.3)
$$\frac{\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_j\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_j\right)} \xrightarrow{\phi_v^*} \frac{\Omega_{i+l+l'-g}\left(X \stackrel{G}{\times} (U_j \oplus V'_j)\right)}{F_{d+l+l'-g-j}\Omega_{i+l+l'-g}\left(X \stackrel{G}{\times} (U_j \oplus V'_j)\right)}$$

is also an isomorphism. Combining the above two isomorphisms, we get the canonical isomorphism

$$(\iota_v^*)^{-1} \circ \phi_v^* : \frac{\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_j\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_j\right)} \xrightarrow{\cong} \frac{\Omega_{i+l+l'-g}\left(X \stackrel{G}{\times} U\right)}{F_{d+l+l'-g-j}\Omega_{i+l+l'-g}\left(X \stackrel{G}{\times} U\right)}.$$

In the same way, we also get an isomorphism

$$(\phi_{v'}^*)^{-1} \circ \iota_{v'}^* : \frac{\Omega_{i+l+l'-g}\left(X \stackrel{G}{\times} U\right)}{F_{d+l+l'-g-j}\Omega_{i+l+l'-g}\left(X \stackrel{G}{\times} U\right)} \xrightarrow{\cong}$$

$$\xrightarrow{\cong} \frac{\Omega_{i+l'-g}\left(X \stackrel{G}{\times} U'_j\right)}{F_{d+l'-g-j}\Omega_{i+l'-g}\left(X \stackrel{G}{\times} U'_j\right)}.$$

The composite $\alpha_{vv'} = ((\phi_{v'}^*)^{-1} \circ \iota_{v'}^*) \circ ((\iota_v^*)^{-1} \circ \phi_v^*)$ is the desired canonical isomorphism (see the proof of [39, Theorem 1.1]). $\square$

Lemma 4.3. *For $j' \geq j \geq 0$, there is a natural surjective map $\Omega_i^G(X)_{j'} \twoheadrightarrow \Omega_i^G(X)_j$.*

Proof. Choose a good pair $(V_{j'}, U_{j'})$ for j'. Then it is clearly a good pair for j too. Moreover, there is a natural surjection

$$\frac{\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_{j'}\right)}{F_{d+l-g-j'}\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_{j'}\right)} \twoheadrightarrow \frac{\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_{j'}\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_{j'}\right)}.$$

On the other hand, the left and the right terms are $\Omega_i^G(X)_{j'}$ and $\Omega_i^G(X)_j$ respectively by Lemma 4.2. $\square$

Definition 4.4. Let X be a k-scheme of dimension d with a G-action. For any $i \in \mathbb{Z}$, we define the *equivariant algebraic cobordism* of X to be

$$\Omega_i^G(X) = \varprojlim_j \Omega_i^G(X)_j.$$

The reader should note from the above definition that unlike the ordinary cobordism, the equivariant algebraic cobordism $\Omega^G_i(X)$ can be non-zero for any $i \in \mathbb{Z}$. We set

$$\Omega^G_*(X) = \bigoplus_{i\in\mathbb{Z}} \Omega^G_i(X).$$

If X is an equi-dimensional k-scheme with G-action, we let $\Omega^i_G(X) = \Omega^G_{d-i}(X)$ and $\Omega^*_G(X) = \underset{i\in\mathbb{Z}}{\oplus} \Omega^i_G(X)$. We shall denote the equivariant cobordism $\Omega^*_G(k)$ of the ground field by $S(G)$. This is also called the algebraic cobordism of the classifying space of G and often written as $\Omega^*(BG)$.

Remark 4.5. If G is the trivial group, we can take the good pair (V_j, V_j) for every j where V_j is any l-dimensional k-vector space. In that case, we get $\Omega_{i+l}(X \overset{G}{\times} V_j) \cong \Omega_{i+l}(X \times V_j)$ which is isomorphic to $\Omega_i(X)$ by the homotopy invariance of the non-equivariant cobordism. Moreover, $F_{d+l-j}\Omega_{i+l}(X \times V_j)$ is isomorphic to $F_{d-j}\Omega_i(X)$ by Proposition 3.9 and this last term is zero for all large j. In particular, we see from (4.1) and the definition of the equivariant cobordism that there is a canonical isomorphism $\Omega^G_*(X) \xrightarrow{\cong} \Omega_*(X)$.

Remark 4.6. Let X be a G-scheme and let H be a closed normal subgroup of G with quotient W. If (V_j, U_j) is a good pair for the G-action for any $j \geq 0$, then W naturally acts on the mixed quotient $X_j = X \overset{H}{\times} U_j$ and hence it acts on $\Omega_*(X_j)$. Since W acts on X_j by automorphisms, it keeps the niveau filtration invariant. In particular, it acts on $\Omega^H_*(X)_j$. It is clear that if (V'_j, U'_j) is another good pair, then the isomorphisms in (4.2) and (4.3) are W-equivariant. In other words, the W-action on $\Omega_*(X_j)$ does not depend on the choice of good pairs. Furthermore, for $j' \geq j$, we can choose a good pair for j' and that makes the maps in the inverse system $\{\Omega_*(X_j)\}_{j\geq 0}$ W-equivariant. We conclude that W acts on the equivariant cobordism $\Omega^H_*(X)$. One example of such a situation is where H is a maximal torus in a linear algebraic group and G is its normalizer. The quotient W is then the Weyl group. In that case, $\Omega^H_*(X)$ becomes a $\mathbb{Z}[W]$-module.

Remark 4.7. It is easy to check from the above definition of the niveau filtration that if X is a smooth and irreducible k-scheme of dimension d, then $F_j\Omega_i(X) = F^{d-j}\Omega^{d-i}(X)$, where $F^\bullet\Omega^*(X)$ is the coniveau filtration used in [9]. Furthermore, one also checks in this case that if G acts on X, then

$$\Omega^i_G(X) = \varprojlim_j \frac{\Omega^i\left(X \overset{G}{\times} U_j\right)}{F^j\Omega^i\left(X \overset{G}{\times} U_j\right)}, \tag{4.4}$$

where (V_j, U_j) is a good pair corresponding to any $j \geq 0$. Thus the above definition 4.4 of the equivariant cobordism coincides with that of [9] for smooth schemes.

Remark 4.8. As is evident from the above definition (see Example 6.6), the equivariant cobordism $\Omega^G_i(X)$ can not in general be computed in terms of the algebraic cobordism of one single mixed space. This makes these groups more complicated to compute than the equivariant Chow groups, which can be computed in terms of a single mixed space. This also motivates one to ask if the equivariant cobordism can be defined in such a way that they can be calculated using one single mixed space in a given degree. It follows however from Lemmas 4.2 and 4.3 that for a given i and j, each component of the projective system $\left\{\Omega^G_i(X)_j\right\}_{j\ge 0}$ can be computed using a single mixed space.

4.1. Change of groups. If $H \subset G$ is a closed subgroup of dimension h, then any l-dimensional good pair (V_j, U_j) for G-action is also a good pair for the induced H-action. Moreover, for any $X \in \mathcal{V}_G$ of dimension d, $X \stackrel{H}{\times} U_j \to X \stackrel{G}{\times} U_j$ is an étale locally trivial G/H-fibration and hence a smooth map (*cf.* [3, Theorem 6.8]) of relative dimension $g-h$. This induces the inverse system of pull-back maps

$$\Omega^G_i(X)_j = \frac{\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_j\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_j\right)} \to h$$

$$\to \frac{\Omega_{i+l-h}\left(X \stackrel{H}{\times} U_j\right)}{F_{d+l-h-j}\Omega_{i+l-h}\left(X \stackrel{H}{\times} U_j\right)} = \Omega^H_i(X)_j$$

and hence a natural restriction map

$$r^G_{H,X} : \Omega^G_*(X) \to \Omega^H_*(X). \tag{4.5}$$

Taking $H = \{1\}$ and using Remark 4.5, we get the *forgetful* map

$$r^G_X : \Omega^G_*(X) \to \Omega_*(X) \tag{4.6}$$

from the equivariant to the non-equivariant cobordism. Since $r^G_{H,X}$ is obtained as a pull-back under the smooth map, it commutes with any projective push-forward and smooth pull-back (*cf.* Theorem 5.2). We remark here that although the definition of $r^G_{H,X}$ uses a good pair, it is easy to see as in Lemma 4.2 that it is independent of the choice of such good pairs.

4.2. Fundamental class of cobordism cycles. Let $X \in \mathcal{V}_G$ and let $Y \xrightarrow{f} X$ be a morphism in $\mathcal{V}_G$ such that Y is smooth of dimension d and f is projective. For any $j \ge 0$ and any l-dimensional good pair (V_j, U_j), $[Y_G \xrightarrow{f_G} X_G]$ is an ordinary cobordism cycle of dimension $d+l-g$ by Lemma 5.1 and hence defines an element $\alpha_j \in \Omega^G_d(X)_j$. Moreover, it is evident that the image of $\alpha_{j'}$ is α_j for $j' \ge j$. Hence we get a unique element $\alpha \in \Omega^G_d(X)$, called the *G-equivariant fundamental class* of the cobordism cycle $[Y \xrightarrow{f} X]$. We also see from this more generally that if $[Y \xrightarrow{f} X, L_1, \cdots, L_r]$ is as above with each L_i a G-equivariant line bundle on Y, then this defines a unique class in $\Omega^G_{d-r}(X)$. It is interesting question to ask under what conditions on the group G, the equivariant cobordism group $\Omega^G_*(X)$ is generated by the fundamental classes of G-equivariant cobordism cycles on X. It turns out that this question indeed has a positive answer if G is a split torus by [20, Theorem 4.11].

5. Some properties of equivariant cobordism

In this section, we establish some basic properties of equivariant algebraic cobordism that are analogous to the non-equivariant case. We begin with the following elementary result. This will be used in the sequel for the morphisms between mixed quotients.

LEMMA 5.1. *Let $f : X \to Y$ be a projective G-equivariant map in $\mathcal{V}_G$ with free G-actions such that Y/G is quasi-projective. Then $X/G \in \mathcal{V}_k$ and the induced map $\overline{f} : X/G \to Y/G$ of quotients is projective.*

Proof. It follows from our assumption and [12, Proposition 7.1] that X/G exists and that $\overline{f} : X' = X/G \to Y/G = Y'$ is a morphism in $\mathcal{V}_k$. Furthermore, the square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "\overline{f}"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

is Cartesian. Since both the horizontal maps are the principal G-bundles, they are smooth and surjective. Since proper maps have smooth descent (in fact fpqc descent), we see that $\overline{f} : X' \to Y'$ is proper. Since these schemes are quasi-projective, we leave it as an exercise to show that $\overline{f}$ is also quasi-projective and hence must be projective. $\square$

THEOREM 5.2. *The equivariant algebraic cobordism satisfies the following properties.*
(i) Functoriality : The assignment $X \mapsto \Omega^G_*(X)$ is covariant for projective maps and contravariant for smooth maps in $\mathcal{V}_G$. It is also contravariant for l.c.i. morphisms in $\mathcal{V}_G$. Moreover, for a fiber diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

in $\mathcal{V}_G$ with f projective and g smooth, one has $g^* \circ f_* = {f'}_* \circ {g'}^* : \Omega^G_*(X) \to \Omega^G_*(Y')$.
(ii) Homotopy : If $f : E \to X$ is a G-equivariant vector bundle, then $f^* : \Omega^G_*(X) \xrightarrow{\cong} \Omega^G_*(E)$.

(iii) Chern classes : For any G-equivariant vector bundle $E \xrightarrow{f} X$ of rank r, there are equivariant Chern class operators $c^G_m(E) : \Omega^G_*(X) \to \Omega^G_{*-m}(X)$ for $0 \le m \le r$ with $c^G_0(E) = 1$. These Chern classes have same functoriality properties as in the non-equivariant case. Moreover, they satisfy the Whitney sum formula.
(iv) Free action : If G acts freely on X with quotient Y, then $\Omega^G_*(X) \xrightarrow{\cong} \Omega_*(Y)$.
(v) Exterior Product : There is a natural product map

$$\Omega^G_i(X) \otimes_{\mathbb{Z}} \Omega^G_{i'}(X') \to \Omega^G_{i+i'}(X \times X').$$

In particular, $\Omega^G_*(k)$ is a graded algebra and $\Omega^G_*(X)$ is a graded $\Omega^G_*(k)$-module for every $X \in \mathcal{V}_G$.
(vi) Projection formula : For a projective map $f : X' \to X$ in $\mathcal{V}^S_G$, one has for $x \in \Omega^G_*(X)$ and $x' \in \Omega^G_*(X')$, the formula : $f_*(x' \cdot f^*(x)) = f_*(x') \cdot x$.

Proof. Assume that the dimensions of X and Y are m and n respectively and let $d = m - n$ be the relative dimension of a projective G-equivariant morphism $f : X \to Y$. For a fixed $j \ge 0$, let (V_j, U_j) be an l-dimensional good pair for j. Since f is projective, Lemma 5.1 implies that $\overline{f} : X_G \to Y_G$ is projective and hence by Theorem 2.1 and Lemma 3.3, there is a push-forward map

$$\frac{\Omega_{i+l-g}(X_G)}{F_{m+l-g-j}\Omega_{i+l-g}(X_G)} \to \frac{\Omega_{i+l-g}(Y_G)}{F_{m+l-g-j}\Omega_{i+l-g}(Y_G)} = \frac{\Omega_{i+l-g}(Y_G)}{F_{n+l-g-(j-d)}\Omega_{i+l-g}(Y_G)}.$$

In particular, we get a compatible system of maps

$$\Omega^G_i(X)_{j+d} \to \Omega^G_i(Y)_j.$$

Taking the inverse limits, one gets the desired push-forward map $\Omega^G_i(X) \xrightarrow{f_*} \Omega^G_i(Y)$.
If f is smooth of relative dimension d, then $\overline{f} : X_G \to Y_G$ is also smooth of same relative dimension. Hence, we get a compatible system of pull-back maps $\Omega^G_i(Y)_j \xrightarrow{\overline{f}^*} \Omega^G_{i+d}(X)_j$. Taking the inverse limit, we get the desired pull-back map of the equivariant algebraic cobordism groups. If f is a l.c.i. morphism of G-schemes, the same proof applies using the existence of similar map in the non-equivariant case. The required commutativity of the pull-back and push-forward maps follows exactly in the same way from the corresponding result for the non-equivariant cobordism groups.
To prove the homotopy property, let $E \xrightarrow{f} X$ be a G-equivariant vector bundle of rank r. For any $j \ge 0$, let (V_j, U_j) be a good pair for j. Then the map of mixed quotients $E_G \to X_G$ is a vector bundle of rank r (cf. [10, Lemma 1]). Hence by Proposition 3.9, the pull-back map $\Omega^G_i(X)_j \to \Omega^G_{i+r}(E)_j$ is an isomorphism. If $j' \ge j$, then we can choose a common good pair for both j and j'. Hence, we have a pull-back map of the inverse systems

$\{\Omega^G_i(X)_j\} \to \{\Omega^G_{i+r}(E)_j\}$ which is an isomorphism at each level. Hence $f^* : \Omega^G_i(X) \to \Omega^G_{i+r}(E)$ is an isomorphism.

To define the Chern classes of an equivariant vector bundle E of rank r, we choose an l-dimensional good pair (V_j, U_j) and consider the vector bundle $E_G \to X_G$ as above and let $c^G_{m,j} : \Omega_{i+l-g}(X_G) \to \Omega_{i+l-g-m}(X_G)$ be the non-equivariant Chern class as in [31, 4.1.7]. For a closed subscheme $Z \overset{\iota}{\hookrightarrow} X_G$, the projection formula for the non-equivariant cobordism

$$c^G_{m,j}(E_G) \circ \iota_* = \iota_* \circ \left(c^G_{m,j}\left(\iota^*(E_G)\right)\right) \tag{5.1}$$

implies that $c^G_{m,j}(E_G)$ descends to maps $c^G_{m,j} : \Omega^G_i(X)_j \to \Omega^G_{i-m}(X)_j$.

One shows as in Lemma 4.2 that this is independent of the choice of the good pairs. Furthermore, choosing a common good pair for $j' \geq j$, we see that $c^G_{m,j}$ actually defines a map of the inverse systems. Taking the inverse limit, we get the Chern classes $c^G_m(E) : \Omega^G_i(X) \to \Omega^G_{i-m}(X)$ for $0 \leq m \leq r$ with $c^G_0(E) = 1$. The functoriality and the Whitney sum formula for the equivariant Chern classes are easily proved along the above lines using the analogous properties of the non-equivariant Chern classes.

The statement about the free action follows from [9, Lemma 7.2] and Remark 4.5.

We now show the existence of the exterior product of the equivariant cobordism which requires some work. Let d and d' be the dimensions of X and X' respectively. We first define maps

$$\Omega^G_i(X)_j \otimes \Omega^G_{i'}(X')_j \to \Omega^G_{i+i'}(X \times X')_j \quad \text{for } j \geq 0. \tag{5.2}$$

Let (V_j, U_j) be an l-dimensional good pair for j and let $\alpha = [Y \xrightarrow{f} X_G]$ and $\alpha' = [Y' \xrightarrow{f'} X'_G]$ be the cobordism cycles on X_G and X'_G respectively. Using the fact that $X \times U_j \to X_G$ and $X' \times U_j \to X'_G$ are principal G-bundles, we get the unique cobordism cycles $[\widetilde{Y} \to X \times U_j]$ and $[\widetilde{Y'} \to X' \times U_j]$ whose G-quotients are the above chosen cycles. We define $\alpha \star \alpha' = [\widetilde{Y} \overset{G}{\times} \widetilde{Y'} \to (X \times X')_G]$. Note that $(V_j \times V_j, U_j \times U_j)$ is a good pair for j of dimension $2l$ and $(X \times X')_G$ is the quotient of $X \times X' \times U_j \times U_j$ for the free diagonal action of G and $\alpha \star \alpha'$ is a well defined cobordism cycle by Lemma 5.1.

Suppose now that $W \xrightarrow{p} X_G \times \square^1$ is a projective morphism from a smooth scheme W such that the composite map $\pi : W \to X_G \times \square^1 \to \square^1$ is a double point degeneration with $W_\infty = \pi^{-1}(\infty)$ smooth. Letting G act trivially on $\square^1$, this gives a unique G-equivariant double point degeneration $\widetilde{W} \xrightarrow{\widetilde{p}} X \times U_j \times \square^1$ of G-schemes. This implies in particular that $\widetilde{W} \times \widetilde{Y'} \xrightarrow{\widetilde{p} \times \widetilde{f'}} X \times X' \times U_j \times U_j \times \square^1$ is also a G-equivariant double point degeneration whose quotient for the free G-action gives a double point degeneration $\widetilde{W} \overset{G}{\times} \widetilde{Y'} \xrightarrow{q} (X \times X')_G \times \square^1$. Moreover, it is easy to see from this that $C(p) \star \alpha' = C(q)$ (cf. (2.4)). Reversing the roles of X and X' and using (2.4) and Theorem 2.3, we get the maps

$$\Omega_{i+l-g}(X_G) \otimes \Omega_{i'+l-g}(X'_G) \to \Omega_{i+i'-2l-g}\left((X \times X')_G\right).$$

It is also clear from the definition of $\alpha \star \alpha'$ and the niveau filtration that

$$\begin{aligned}&\{F_{d+l-g-j}\Omega_{i+l-g}(X_G) \otimes \Omega_{i'+l-g}(X'_G)\} + \\ &\qquad + \{\Omega_{i+l-g}(X_G) \otimes F_{d'+l-g-j}\Omega_{i'+l-g}(X'_G)\} \to \\ &\qquad\qquad \to F_{d+d'+2l-g-j}\Omega_{i+i'-2l-g-j}\left((X \times X')_G\right).\end{aligned}$$

This defines the maps as in (5.2). One can now show as in Lemma 4.2 that these maps are independent of the choice of the good pairs. We get the desired exterior product as the composite map

$$\begin{aligned}\Omega^G_i(X) \otimes_{\mathbb{Z}} \Omega^G_{i'}(X') &= \varprojlim_j \Omega^G_i(X)_j \otimes_{\mathbb{Z}} \varprojlim_j \Omega^G_{i'}(X')_j \\ &\to \varprojlim_j \left(\Omega^G_i(X)_j \otimes_{\mathbb{Z}} \Omega^G_{i'}(X')_j\right) &(5.3)\\ &\to \varprojlim_j \Omega^G_{i+i'}(X \times X')_j = \Omega^G_{i+i'}(X \times X'). &(5.4)\end{aligned}$$

Finally for X smooth, we get the product structure on $\Omega^*_G(X)$ via the composite $\Omega^*_G(X) \otimes_{\mathbb{Z}} \Omega^*_G(X) \to \Omega^*_G(X \times X) \xrightarrow{\Delta^*_X} \Omega^*_G(X)$. The projection formula can now be proven by using the non-equivariant version of such a formula (*cf.* [31, 5.1.4]) at each level of the projective system $\{\Omega^i_G(X)_j\}$ and then taking the inverse limit. $\square$

We now turn our attention to the question of the localization sequence in equivariant cobordism. In the topological context, Buhstaber-Miscenko [6, 7] defined the topological K-theory of an infinite CW-complex as the projective limit of the K-theory of finite skeleta. They suggested that this theory might not have the Gysin exact sequence. In [26], Landweber showed that such a phenomenon for the K-theory is also reflected in the complex cobordism. This makes us believe that one should not expect the full localization sequence for the equivariant algebraic cobordism considered here. On the positive side however, we can prove the following weaker result.

Proposition 5.3. *Let X be a G-scheme of dimension d and let $f : U \hookrightarrow X$ be a G-invariant open subscheme. Then the restriction map $f^* : \Omega^G_*(X) \to \Omega^G_*(U)$ is surjective.*

Proof. Let Z be the complement of U in X with the reduced induced closed subscheme structure and let $g : Z \hookrightarrow X$ be the inclusion map.
We fix integers $i \in \mathbb{Z}$ and $j \geq 0$ and choose a good pair (V_j, U_j) of dimension l for j. Then we see that Z is a G-invariant closed subscheme of X and $Z_G \subseteq X_G$ is a closed subscheme with the complement U_G. Hence by applying Theorem 3.5

at the appropriate levels of the niveau filtration and taking the quotients, we get an exact sequence

$$\frac{\Omega_{i+l-g}(Z_G)}{F_{d+l-g-j}\Omega_{i+l-g}(Z_G)} \to \frac{\Omega_{i+l-g}(X_G)}{F_{d+l-g-j}\Omega_{i+l-g}(X_G)} \to \frac{\Omega_{i+l-g}(U_G)}{F_{d+l-g-j}\Omega_{i+l-g}(U_G)} \to 0.$$

If $d' = \dim(Z)$, then $F_{d'+l-g-j}\Omega_{i+l-g}(Z_G) \subseteq F_{d+l-g-j}\Omega_{i+l-g}(Z_G)$ and hence by Lemma 4.2, we get an exact sequence of inverse systems

$$\Omega_i^G(Z)_j \xrightarrow{\phi_j^i} \Omega_i^G(X)_j \to \Omega_i^G(U)_j \to 0. \tag{5.5}$$

Setting M_j^i and N_j^i to be the kernel and the image of the map ϕ_j^i respectively, (5.5) can be split into the short exact sequences of inverse systems

$$0 \to M_j^i \to \Omega_i^G(Z)_j \to N_j^i \to 0; \tag{5.6}$$

$$0 \to N_j^i \to \Omega_i^G(X)_j \to \Omega_i^G(U)_j \to 0. \tag{5.7}$$

It follows from Lemma 4.3 that $\{\Omega_i^G(Z)_j\}_{j\ge 0}$ is an inverse system of surjective maps and hence so is $\{N_j^i\}_{j\ge 0}$. In particular, it satisfies the Mittag-Leffler (ML) condition. As a consequence, we get an exact sequence of inverse limits

$$\varprojlim_j N_j^i \to \Omega_i^G(X) \to \Omega_i^G(U) \to 0$$

and this proves the proposition. □

Proposition 5.4 (Morita Isomorphism). *Let $H \subset G$ be a closed subgroup and let $X \in \mathcal{V}_H$. Then there is a canonical isomorphism*

$$\Omega_*^G\left(G \overset{H}{\times} X\right) \xrightarrow{\cong} \Omega_*^H(X). \tag{5.8}$$

Proof. Define an action of $H \times G$ on $G \times X$ by

$$(h, g) \cdot (g', x) = \left(gg'h^{-1}, hx\right),$$

and an action of $H \times G$ on X by $(h, g) \cdot x = hx$. Then the projection map $G \times X \xrightarrow{p} X$ is $(H \times G)$-equivariant which is a G-torsor. Hence by [9, Lemma 7.2], the natural map $\Omega_*^H(X) \xrightarrow{p^*} \Omega_*^{H\times G}(G \times X)$ is an isomorphism. On the other hand, the projection map $G \times X \to G \overset{H}{\times} X$ is $(H \times G)$-equivariant which is an H-torsor. Hence we get an isomorphism $\Omega_*^G\left(G \overset{H}{\times} X\right) \xrightarrow{\cong} \Omega_*^{H\times G}(G \times X)$. The proposition follows by combining these two isomorphisms. □

6. Computations

Let X be a k-scheme of dimension d with a G-action. We have seen above that unlike the situation of Chow groups, the cobordism group $\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)$ is not independent of the choice of the l-dimensional good pair (V_j, U_j) even if j is large enough. This anomaly is rectified by considering the quotients of the cobordism groups of the good pairs by the niveau filtration. Our main result in this section is to show that if we suitably choose a sequence of good pairs $\{(V_j, U_j)\}_{j \geq 0}$, then the above equivariant cobordism group can be computed without taking quotients by the niveau filtration. This reduction is often very helpful in computing the equivariant cobordism groups.

Theorem 6.1. *Let $\{(V_j, U_j)\}_{j \geq 0}$ be a sequence of l_j-dimensional good pairs such that*
(i) $V_{j+1} = V_j \oplus W_j$ as representations of G with $\dim(W_j) > 0$ and
(ii) $U_j \oplus W_j \subsetneq U_{j+1}$ as G-invariant open subsets.
(iii) $\operatorname{codim}_{V_{j+1}} (V_{j+1} \setminus U_{j+1}) > \operatorname{codim}_{V_j} (V_j \setminus U_j)$.
Then for any scheme X as above and any $i \in \mathbb{Z}$, we have

$$\varprojlim_j \Omega_{i+l_j-g}\left(X \stackrel{G}{\times} U_j\right) \xrightarrow{\cong} \Omega_i^G(X).$$

Moreover, such a sequence $\{(V_j, U_j)\}_{j \geq 0}$ of good pairs always exists.

Proof. Let $\{(V_j, U_j)\}_{j \geq 0}$ be a sequence of good pairs as in the theorem. We have natural maps

$$\begin{aligned} (6.1) \qquad \Omega_{i+l_{j+1}-g}\left(X \stackrel{G}{\times} U_{j+1}\right) &\twoheadrightarrow \\ \twoheadrightarrow \Omega_{i+l_{j+1}-g}\left(X \stackrel{G}{\times} (U_j \oplus W_j)\right) &\xleftarrow{\cong} \Omega_{i+l_j-g}\left(X \stackrel{G}{\times} U_j\right), \end{aligned}$$

where the first map is the restriction to an open subset and the second is the pull-back via a vector bundle. Taking the quotients by the niveau filtrations, we get natural maps (*cf.* proof of Lemma 4.2)
(6.2)

$$\begin{tikzcd}
\frac{\Omega_{i+l_{j+1}-g}\left(X \stackrel{G}{\times} U_{j+1}\right)}{F_{d+l_{j+1}-g-j-1}\Omega_{i+l_{j+1}-g}\left(X \stackrel{G}{\times} U_{j+1}\right)} \arrow[r] & \frac{\Omega_{i+l_{j+1}-g}\left(X \stackrel{G}{\times} (U_j \oplus W_j)\right)}{F_{d+l_{j+1}-g-j-1}\Omega_{i+l_{j+1}-g}\left(X \stackrel{G}{\times} (U_j \oplus W_j)\right)} \\
\frac{\Omega_{i+l_j-g}\left(X \stackrel{G}{\times} U_j\right)}{F_{d+l_j-g-j}\Omega_{i+l_j-g}\left(X \stackrel{G}{\times} U_j\right)} & \frac{\Omega_{i+l_j-g}\left(X \stackrel{G}{\times} U_j\right)}{F_{d+l_j-g-j-1}\Omega_{i+l_j-g}\left(X \stackrel{G}{\times} U_j\right)} \arrow[l, two heads] \arrow[u, "\cong"']
\end{tikzcd}$$

where the right vertical arrow is an isomorphism by Proposition 3.9. Setting $X_j = X \stackrel{G}{\times} U_j$, we get natural maps

$$(6.3) \qquad \begin{tikzcd}
\Omega_{i+l_{j+1}-g}\left(X_{j+1}\right) \arrow[r, "\nu_j^{j+1}"] \arrow[d, two heads] & \Omega_{i+l_j-g}\left(X_j\right) \arrow[d, two heads] \\
\frac{\Omega_{i+l_{j+1}-g}(X_{j+1})}{F_{d+l_{j+1}-g-j-1}\Omega_{i+l_{j+1}-g}(X_{j+1})} \arrow[r] & \frac{\Omega_{i+l_j-g}(X_j)}{F_{d+l_j-g-j}\Omega_{i+l_j-g}(X_j)}.
\end{tikzcd}$$

Since (V_j, U_j) is a good pair for each j, we see that $\frac{\Omega_{i+l_j-g}(X_j)}{F_{d+l_j-g-j}\Omega_{i+l_j-g}(X_j)} \cong \Omega_i^G(X)_j$. Hence, we only have to show that the map

$$\varprojlim_j \Omega_{i+l_j-g}(X_j) \to \varprojlim_j \frac{\Omega_{i+l_j-g}(X_j)}{F_{d+l_j-g-j}\Omega_{i+l_j-g}(X_j)} \tag{6.4}$$

is an isomorphism in order to prove the theorem.
To prove (6.4), we only need to show that for any given $j \geq 0$, the map $\Omega_{i+l_{j'}-g}(X_{j'}) \xrightarrow{\nu_j^{j'}} \Omega_{i+l_j-g}(X_j)$ factors through

$$\frac{\Omega_{i+l_{j'}-g}(X_{j'})}{F_{d+l_{j'}-g-j'}\Omega_{i+l_{j'}-g}(X_{j'})} \to \Omega_{i+l_j-g}(X_j) \quad \text{for all } j' \gg j. \tag{6.5}$$

However, it follows from (6.2) that $\nu_j^{j'}$ induces the map

$$\frac{\Omega_{i+l_{j'}-g}(X_{j'})}{F_{d+l_{j'}-g-j'}\Omega_{i+l_{j'}-g}(X_{j'})} \to \frac{\Omega_{i+l_j-g}(X_j)}{F_{d+l_j-g-j'}\Omega_{i+l_j-g}(X_j)}.$$

On the other hand $F_{d+l_j-g-j'}\Omega_{i+l_j-g}(X_j)$ vanishes for $j' \gg j$. This proves (6.5) and hence (6.4).
Finally, it follows easily from the proof of Lemma 4.2 (see also [39, Remark 1.4]) that a sequence of good pairs as in Theorem 6.1 always exists. $\square$

As a simple corollary of Theorem 6.1, we get the following localization sequence for the equivariant cobordism in a special case.

Corollary 6.2. *Let X be a G-scheme and let $Z \subseteq X$ be a G-invariant closed subscheme with the complement U. Assume that there is a G-equivariant projective morphism $p : X \to Y$ whose restriction to Z is an isomorphism. Then there is a short exact sequence*

$$0 \to \Omega_*^G(Z) \to \Omega_*^G(X) \to \Omega_*^G(U) \to 0. \tag{6.6}$$

Proof. Let $\{(V_j, U_j)\}_{j \geq 0}$ be a sequence of good pairs as in Theorem 6.1. The localization sequence for the ordinary algebraic cobordism yields for any $i \in \mathbb{Z}$, an exact sequence of inverse systems

$$\Omega_{i+l_j-g}\left(Z \overset{G}{\times} U_j\right) \to \Omega_{i+l_j-g}\left(X \overset{G}{\times} U_j\right) \to \Omega_{i+l_j-g}\left(U \overset{G}{\times} U_j\right) \to 0. \tag{6.7}$$

It follows from Lemma 5.1 that there are projective morphisms $Z \overset{G}{\times} U_j \xrightarrow{f_j} X \overset{G}{\times} U_j \xrightarrow{p_j} Y \overset{G}{\times} U_j$ such that $p_j \circ f_j$ is an isomorphism. This implies that the map ${p_j}_* \circ {f_j}_*$ is an isomorphism. In other words, (6.8) is in fact a short exact sequence of inverse systems

$$0 \to \Omega_{i+l_j-g}\left(Z \overset{G}{\times} U_j\right) \to \Omega_{i+l_j-g}\left(X \overset{G}{\times} U_j\right) \to \Omega_{i+l_j-g}\left(U \overset{G}{\times} U_j\right) \to 0. \tag{6.8}$$

We have moreover seen in (6.1) that $\{\Omega_{i+l_j-g}(Z \overset{G}{\times} U_j)\}_{j\ge 0}$ is an inverse system of surjective maps. It follows that (6.8) remains short exact after taking limit, which proves (6.6). $\square$

Another consequence of Theorem 6.1 is that for a linear algebraic group G acting on a scheme X of dimension d, the forgetful map $r^G_X : \Omega^G_*(X) \to \Omega_*(X)$ (*cf.* (4.6)) can be easily shown to be analogous to the one used in [10, Subsection 2.2] for the Chow groups. This interpretation of the forgetful map has some interesting applications in the computation of the non-equivariant cobordism using the equivariant techniques (*cf.* [20], [21]).
So let $\{(V_j, U_j)\}_{j\ge 0}$ be a sequence of good pairs as in Theorem 6.1. We choose a k-rational point $x \in U_0$ and let x_j be its image in U_j/G under the natural map $U_0 \to U_0/G \to U_j/G$. Setting $X_j = X \overset{G}{\times} U_j$, this yields a commutative diagram

$$\begin{tikzcd} X \times U_j \arrow[r, "p_j"] \arrow[d, "\pi_j"'] & X_j \arrow[r, "\phi_j"] \arrow[d, "\psi_j"'] & X_{j+1} \arrow[d, "\psi_{j+1}"] \\ U_j \arrow[r] & U_j/G \arrow[r] & U_{j+1}/G \end{tikzcd} \tag{6.9}$$

such that the left square is Cartesian and

$$X \cong \pi_j^{-1}(x) \xrightarrow{\cong} \psi_j^{-1}(x_j) \xrightarrow{\cong} \psi_{j+1}^{-1}(x_{j+1}). \tag{6.10}$$

Let $\nu_j : \psi_j^{-1}(x_j) \hookrightarrow X_j$ be the closed embedding. Notice that since U_j/G is smooth and ψ_j is flat, it follows that ν_j is a regular closed embedding (hence an l.c.i. morphism). Using the identification in (6.10), we get maps $\nu_j^* : \Omega_*(X_j) \to \Omega_*(X)$ such that $\nu_j^* \circ \phi_j^* = \nu_{j+1}^*$. Taking the limit over $j \ge 0$, this yields for any $i \in \mathbb{Z}$, a restriction map

$$\widetilde{r}^G_X : \Omega^G_i(X) = \varprojlim_{j\ge 0} \Omega^G_i(X)_j \to \Omega_i(X). \tag{6.11}$$

Corollary 6.3. *The maps r^G_X, $\widetilde{r}^G_X : \Omega^G_i(X) \to \Omega_i(X)$ coincide.*

Proof. Using the construction of the map r^G_X in (4.6) and the diagram (6.9), it suffices to show that for any $i \in \mathbb{Z}$, the natural maps

$$\frac{\Omega_i(X)}{F_{d-j}\Omega_i(X)} \leftarrow \frac{\Omega_i(X \times V_j)}{F_{d+l_j-j}\Omega_i(X \times V_j)} \to \frac{\Omega_i(X \times U_j)}{F_{d+l_j-j}\Omega_i(X \times U_j)}$$

are isomorphisms for all $j \gg 0$. Here, the first map is the restriction induced by the section corresponding to the rational point $x_j \in U_j/G$, and the second map is the restriction to an open subset. The existence of the first map follows from Proposition 3.9. The assertion that these two maps are isomorphisms follows immediately from Corollary 3.6 and Proposition 3.9. $\square$

Remark 6.4. It follows from Corollary 6.3 that the map $\widetilde{r}^G_X$ does not depend on the choice of the k-rational point $x \in U_0$.

6.1. Graded vs. completed cobordism rings. Another consequence of Theorem 6.1 is that it allows us to explain the relation between the graded (or noncomplete) and the nongraded (or complete) versions of the equivariant cobordism rings of smooth G-schemes. If $\{(V_j, U_j)\}_{j\ge 0}$ is a sequence of l_j-dimensional good pairs as in Theorem 6.1, then it can be easily checked from the proof of this theorem that the expression

$$\widehat{\Omega^*_G(X)} = \varprojlim_j \Omega^*\left(X \overset{G}{\times} U_j\right) \tag{6.12}$$

is well-defined and there is a natural map $\iota_X : \Omega^*_G(X) \to \widehat{\Omega^*_G(X)}$.
Furthermore, the surjectivity of the map $\Omega^*\left(X \overset{G}{\times} U_{j+1}\right) \to \Omega^*\left(X \overset{G}{\times} U_j\right)$ (*cf.* (6.1)) implies that ι_X identifies $\widehat{\Omega^*_G(X)}$ as the completion of $\Omega^*_G(X)$ with respect to the linear topology given by the decreasing filtration

$$F^j\Omega^*_G(X) = \mathrm{Ker}\left(\Omega^*_G(X) \twoheadrightarrow \Omega^*\left(X \overset{G}{\times} U_j\right)\right).$$

6.2. Formal group law in equivariant cobordism. Let G be a linear algebraic group over k acting on a scheme X of dimension d. We have seen before that the equivariant line bundles on X give rise to the equivariant Chern class operators on $\Omega^G_*(X)$. Below, we write down an expression for the equivariant Chern class of the tensor product of two such line bundles.
Let $\{(V_j, U_j)\}_{j\ge 0}$ be a sequence of l_j-dimensional good pairs as in Theorem 6.1. Letting $X_j = X \overset{G}{\times} U_j$, we see that for every $j \ge 0$, $\Omega_*(X_j) = \bigoplus_{i\in\mathbb{Z}} \Omega_{i+l_j-g}(X_j)$ is an $\mathbb{L}$-module and for $j' \ge j$, there is a natural surjection $\Omega_*(X_{j'}) \twoheadrightarrow \Omega_*(X_j)$ of $\mathbb{L}$-modules.
Given G-equivariant line bundles L, M on X, we get line bundles L_j, M_j on X_j, where $L_j = L \overset{G}{\times} U_j$ for $j \ge 0$. The formal group law of the non-equivariant cobordism yields

$$c_1\left((L\otimes M)_j\right) = \\ = c_1\left(L_j \otimes M_j\right) = c_1(L_j) + c_1(M_j) + \sum_{i,i'\ge 1} a_{i,i'}\left(c_1(L_j)\right)^i \circ \left(c_1(M_j)\right)^{i'}.$$

Note that if $(x_j) \in \Omega^G_i(X)$, then the evaluation of the operator $c^G_1(L)(x_j)$ at any level $j \ge 0$ is a finite sum above.
Taking the limit over $j \ge 0$ and noting that the sum (and the product) in the equivariant cobordism groups are obtained by taking the limit of the sums (and the products) at each level of the inverse system, we get the same formal group law for the equivariant Chern classes:

$$c^G_1(L\otimes M) = c^G_1(L) + c^G_1(M) + \sum_{i,i'\ge 1} a_{i,i'}\left(c^G_1(L)\right)^i \circ \left(c^G_1(M)\right)^{i'}. \tag{6.13}$$

Note that the coefficients $a_{i,i'}$ are homogeneous elements of $\mathbb{L}$ and can be considered as elements of $S(G)$ under the natural inclusion of graded rings $\mathbb{L} \hookrightarrow S(G)$. One should also observe that unlike the case of ordinary cobordism, the evaluation of the above sum on any given equivariant cobordism cycle may no longer be finite. In other words, the equivariant Chern classes are not in general locally nilpotent.

6.3. Cobordism ring of classifying spaces. Let R be a Noetherian ring and let $A = \underset{j \in \mathbb{Z}}{\oplus} A_j$ be a $\mathbb{Z}$-graded R-algebra with $R \subseteq A_0$. Recall that the *graded power series ring* $S^{(n)} = \underset{i \in \mathbb{Z}}{\oplus} S_i$ is a graded ring such that S_i is the set of formal power series of the form $f(\mathbf{t}) = \sum\limits_{m(\mathbf{t}) \in \mathcal{C}} a_{m(\mathbf{t})} m(\mathbf{t})$ such that $a_{m(\mathbf{t})}$ is a homogeneous element of A of degree $|a_{m(\mathbf{t})}|$and $|a_{m(\mathbf{t})}| + |m(\mathbf{t})| = i$. Here, $\mathcal{C}$ is the set of all monomials in $\mathbf{t} = (t_1, \cdots, t_n)$ and $|m(\mathbf{t})| = i_1 + \cdots + i_n$ if $m(\mathbf{t}) = t_1^{i_1} \cdots t_n^{i_n}$. We call $|m(\mathbf{t})|$ to be the degree of the monomial $m(\mathbf{t})$.
We shall write the above graded power series ring as $A[[\mathbf{t}]]_{\mathrm{gr}}$ to distinguish it from the usual formal power series ring $A[[\mathbf{t}]]$. Notice that if A is only non-negatively graded, then $S^{(n)}$ is nothing but the standard polynomial ring $A[t_1, \cdots, t_n]$ over A. It is also easy to see that $S^{(n)}$ is indeed a graded ring which is a subring of the formal power series ring $A[[t_1, \cdots, t_n]]$. The following result summarizes some basic properties of these rings. The proof is straightforward and is left as an exercise.

Lemma 6.5. *(i) There are inclusions of rings $A[t_1, \cdots, t_n] \subset S^{(n)} \subset A[[t_1, \cdots, t_n]]$, where the first is an inclusion of graded rings.*
(ii) These inclusions are analytic isomorphisms with respect to the $\mathbf{t}$-adic topology. In particular, the induced maps of the associated graded rings

$$A[t_1, \cdots, t_n] \to \mathrm{Gr}_{(\mathbf{t})} S^n \to \mathrm{Gr}_{(\mathbf{t})} A[[t_1, \cdots, t_n]]$$

are isomorphisms.
(iii) $S^{(n-1)}[[t_n]]_{\mathrm{gr}} \xrightarrow{\cong} S^{(n)}$.
(iv) $\frac{S^{(n)}}{(t_{i_1}, \cdots, t_{i_r})} \xrightarrow{\cong} S^{(n-r)}$ *for any $n \geq r \geq 1$, where $S^{(0)} = A$.*
(v) The sequence $\{t_1, \cdots, t_n\}$ is a regular sequence in $S^{(n)}$.
(vi) If $A = R[x_1, x_2, \cdots]$ is a polynomial ring with $|x_i| < 0$ and $\lim\limits_{i \to \infty} |x_i| = -\infty$, then $S^{(n)} \xrightarrow{\cong} \varprojlim\limits_{i} R[x_1, \cdots, x_i][[\mathbf{t}]]_{\mathrm{gr}}$.

Examples 6.6. In the following examples, we compute $\Omega^*(BG) = \Omega^*_G(k)$ for some classical groups G over k. These computations follow directly from the definition of equivariant cobordism and suitable choices of good pairs.
We first consider the case when $G = \mathbb{G}_m$ is the multiplicative group. For any $j \geq 1$, we choose the good pair (V_j, U_j), where V_j is the j-dimensional representation of $\mathbb{G}_m$ with all weights -1 and U_j is the complement of the origin. We see then that $U_j/\mathbb{G}_m \cong \mathbb{P}^{j-1}_k$. Let ζ be the class of $c_1(\mathcal{O}(-1))(1) \in \Omega^1(\mathbb{P}^{j-1}_k)$. The projective bundle formula for the ordinary algebraic cobordism

implies that $(\Omega^i_G)_j = \bigoplus_{0 \le p \le j-1} \mathbb{L}^{i-p}\zeta^p$. Taking the inverse limit over $j \ge 1$, we find from this that for $i \in \mathbb{Z}$,

$$\Omega^i_{\mathbb{G}_m}(k) = \prod_{p \ge 0} \mathbb{L}^{i-p}\zeta^p.$$

It particular, if $x = \sum_{j=1}^{n} x_{i_j}$ is a sum of homogeneous elements of $\Omega^*(B\mathbb{G}_m)$, then we get a natural map

$$\Omega^*(B\mathbb{G}_m) \to \mathbb{L}[[t]]_{\mathrm{gr}} \tag{6.14}$$

$$x = \left(x_{i_1} = \prod a_p^{i_1}\zeta^p, \cdots, x_{i_n} = \prod a_p^{i_n}\zeta^p\right) \mapsto \sum_{p \ge 0} \left(\sum_{1 \le j \le n} a_p^{i_j}\right) t^p,$$

which is an isomorphism of graded $\mathbb{L}$-algebras. Observe that $\Omega^*\widehat{(B\mathbb{G}_m)}$ (*cf.* (6.12)) is the formal power series ring $\mathbb{L}[[t]]$.

For a general split torus T of rank n, we choose a basis $\{\chi_1, \cdots, \chi_n\}$ of the character group $\widehat{T}$. This is equivalent to a decomposition $T = T_1 \times \cdots \times T_n$ with each T_i isomorphic to $\mathbb{G}_m$ and χ_i is a generator of $\widehat{T_i}$. Let L_χ be the one-dimensional representation of T, where T acts via χ. For any $j \ge 1$, we take the good pair (V_j, U_j) such that $V_j = \prod_{i=1}^{n} L^{\oplus j}_{\chi_i}$, $U_j = \prod_{i=1}^{n} \left(L^{\oplus j}_{\chi_i} \setminus \{0\}\right)$ and T acts on V_j by $(t_1, \cdots, t_n)(x_1, \cdots, x_n) = (\chi_1(t_1)(x_1), \cdots, \chi_n(t_n)(x_n))$. It is then easy to see that $U_j/T \cong X_1 \times \cdots \times X_n$ with each X_i isomorphic to $\mathbb{P}^{j-1}_k$. Moreover, the T-line bundle L_{χ_i} gives the line bundle $L_{\chi_i} \overset{T_i}{\times} \left(L^{\oplus j}_{\chi_i} \setminus \{0\}\right) \to X_i$ which is $\mathcal{O}(\pm 1)$. Letting ζ_i be the first Chern class of this line bundle, the projective bundle formula for the non-equivariant cobordism shows that

$$\Omega^i_T(k) = \prod_{p_1, \cdots, p_n \ge 0} \mathbb{L}^{i - (\sum_{i=1}^{n} p_i)} \zeta_1^{p_1} \cdots \zeta_n^{p_n},$$

which is isomorphic to the set of formal power series in $\{\zeta_1, \cdots, \zeta_n\}$ of degree i with coefficients in $\mathbb{L}$. It particular, one concludes as in the rank one case above that

Proposition 6.7. *Let $\{\chi_1, \cdots, \chi_n\}$ be a chosen basis of the character group of a split torus T of rank n. The assignment $t_i \mapsto c_1^T(L_{\chi_i})$ yields a graded $\mathbb{L}$-algebra isomorphism*

$$\mathbb{L}[[t_1, \cdots, t_n]]_{\mathrm{gr}} \to \Omega^*(BT).$$

For $G = GL_n$, we can take a good pair for j to be (V_j, U_j), where V_j is the vector space of $n \times p$ matrices with $p > n$ with GL_n acting by left multiplication, and U_j is the open subset of matrices of maximal rank. Then the mixed quotient is the Grassmannian $Gr(n, p)$. We can now calculate the cobordism ring of $Gr(n,p)$ using the projective bundle formula (by standard stratification technique) and then we can use the similar calculations as above to get a natural isomorphism

$$\Omega^*(BGL_n) \to \mathbb{L}[[\gamma_1, \cdots, \gamma_n]]_{\mathrm{gr}} \tag{6.15}$$

of graded $\mathbb{L}$-algebras, where γ_i's are the elementary symmetric polynomials in $t_1, \cdots, t_n$ that occur in (6.14).
Another way to obtain the isomorphism (6.15) is to observe that the Weyl group of GL_n is the permutation group S_n and $t_{GL_n} = 1$, where t_G denotes the torsion index of a connected reductive group G. It follows from [15, Theorem 3.7] that we can assume the base field to be the field of complex numbers. Subsequently, it follows from [15, Proposition 4.8] that the natural map $\Omega^*(BGL_n) \to (\Omega^*(BT))^{S_n} = \mathbb{L}[[\gamma_1, \cdots, \gamma_n]]_{\mathrm{gr}}$ is an isomorphism. Using the same argument, on obtains an isomorphism $\Omega^*(BSL_n) \xrightarrow{\cong} \mathbb{L}[[\gamma_2, \cdots, \gamma_n]]_{\mathrm{gr}}$.

Remark 6.8. The cobordism rings of BGL_n and BSL_n have also been written down by Deshpande in [9, Section 4]. His expressions depend on the assumption that these groups are isomorphic to the complex cobordism. As the reader will find in Subsection 7.3, there may not in general exist a map from the algebraic to the complex equivariant cobordism although such a map does exist for a classifying space BG (*cf.* Corollary 7.7). Moreover, it is not clear when such a map is an isomorphism. We refer the reader to [15, Theorem 3.7] for a result in this direction.

7. Comparison with other equivariant cohomology theories

In this paper, we fix the following notation for the tensor product while dealing with inverse systems of modules over a commutative ring. Let A be a commutative ring with unit and let $\{L_n\}$ and $\{M_n\}$ be two inverse systems of A-modules with inverse limits L and M respectively. Following [38], one defines the *topological tensor product* of L and M by

$$L\widehat{\otimes}_A M := \varprojlim_n (L_n \otimes_A M_n). \tag{7.1}$$

In particular, if D is an integral domain with quotient field F and if $\{A_n\}$ is an inverse system of D-modules with inverse limit A, one has $A\widehat{\otimes}_D F = \varprojlim_n (A_n \otimes_D F)$. The examples $\widehat{\mathbb{Z}_{(p)}} = \varprojlim_n \mathbb{Z}/p^n$ and $\mathbb{Z}[[x]] \otimes_{\mathbb{Z}} \mathbb{Q} \to \varprojlim_n \frac{\mathbb{Z}[x]}{(x^n)} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[[x]]$ show that the map $A \otimes_D F \to A\widehat{\otimes}_D F$ is in general neither injective nor surjective. We shall denote $A\widehat{\otimes}_D F$ in the sequel by A_F to simplify the notations.
If R is a $\mathbb{Z}$-graded ring and if M and N are two R-graded modules, then recall that $M \otimes_R N$ is also a graded R-module given by the quotient of $M \otimes_{R_0} N$ by the graded submodule generated by the homogeneous elements of the type $ax \otimes y - x \otimes ay$ where a, x and y are the homogeneous elements of R, M and

N respectively. If all the graded pieces M_i and N_i are the limits of inverse systems $\{M_i^{\lambda}\}$ and $\{N_i^{\lambda}\}$ of R_0-modules, we define the *graded topological tensor product* as $M\widehat{\otimes}_R N = \bigoplus_{i\in\mathbb{Z}} \left(M\widehat{\otimes}_R N\right)_i$, where

$$\left(M\widehat{\otimes}_R N\right)_i = \varprojlim_{\lambda} \left(\bigoplus_{j+j'=i} \frac{M_j^{\lambda} \otimes_{R_0} N_{j'}^{\lambda}}{(ax \otimes y - x \otimes ay)} \right). \tag{7.2}$$

Notice that this reduces to the ordinary tensor product of graded R-modules if the underlying inverse systems are trivial.

7.1. Comparison with equivariant Chow groups. Let X be a k-scheme of dimension d with a G-action. It was shown by Levine and Morel [31] that there is a natural map $\Omega_*(X) \to \mathrm{CH}_*(X)$ of graded abelian groups which is a ring homomorphism if X is smooth. Moreover, this map induces a graded isomorphism

$$\Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z} \xrightarrow{\cong} \mathrm{CH}_*(X). \tag{7.3}$$

Recall from [39] and [10] that the equivariant Chow groups of X are defined as $\mathrm{CH}_i^G(X) = \mathrm{CH}_{i+l-g}\left(X \overset{G}{\times} U\right)$, where (V,U) is an l-dimensional good pair corresponding to $d-i+1$. It is known that $\mathrm{CH}_i^G(X)$ is well-defined and can be non-zero for any $-\infty < i \le d$. We set $\mathrm{CH}_*^G(X) = \bigoplus_i \mathrm{CH}_i^G(X)$. If X is equi-dimensional, we let $\mathrm{CH}^i_G(X) = \mathrm{CH}^G_{d-i}(X)$ and set $\mathrm{CH}^*_G(X) = \bigoplus_{i\ge 0} \mathrm{CH}^i_G(X)$. Notice that in this case, $\mathrm{CH}^i_G(X)$ is same as $\mathrm{CH}^i\left(X \overset{G}{\times} U\right)$, where (V,U) is an l-dimensional good pair corresponding to $i+1$.

If we fix $i \in \mathbb{Z}$ and choose an l-dimensional good pair (V_j, U_j) corresponding to $j \ge \max(0, d-i+1)$, the universality of the algebraic cobordism gives a unique map $\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right) \to \mathrm{CH}_{i+l-g}\left(X \overset{G}{\times} U_j\right)$. By Lemma 3.7, this map factors through

$$\frac{\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)}{F_{i+l-g-1}\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)} \to \mathrm{CH}_{i+l-g}\left(X \overset{G}{\times} U_j\right). \tag{7.4}$$

Since $j \ge d-i+1$ by the choice, we have $d+l-g-j \le i+l-g-1$ and hence we get the map

(7.5)

$$\Omega_i^G(X)_j = \frac{\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)} \to \mathrm{CH}_{i+l-g}\left(X \overset{G}{\times} U_j\right) = \mathrm{CH}_i^G(X).$$

It is easily shown using the proof of Lemma 4.2 that this map is independent of the choice of the good pair (V_j, U_j). Taking the inverse limit over $j \geq 0$, we get a natural map $\Omega_i^G(X) \to \mathrm{CH}_i^G(X)$ and hence a map of graded abelian groups

$$\Phi_X : \Omega_*^G(X) \to \mathrm{CH}_*^G(X) \tag{7.6}$$

which is in fact a map of graded $\mathbb{L}$-modules. Notice that the right side of (7.5) does not depend on j as long as $j \gg 0$. If X is equi-dimensional, we write the above map cohomologically as $\Omega_G^*(X) \to \mathrm{CH}_G^*(X)$.

Example 7.1. Let T be a split torus of rank n over k. It follows from Proposition 6.7 that $\Omega^*(BT)$ is isomorphic to the *graded* power series ring $\mathbb{L}[[t_1, \cdots, t_n]]_{\mathrm{gr}}$. One knows that $\mathrm{CH}^*(BT)$ is isomorphic to the polynomial ring $\mathbb{Z}[t_1, \cdots, t_n]$ (*cf.* [10, 3.2]). And the map $\Phi_k : \Omega^*(BT) \to \mathrm{CH}^*(BT)$ in this case is the obvious map $\mathbb{L}[[t_1, \cdots, t_n]]_{\mathrm{gr}} \to \mathbb{Z}[t_1, \cdots, t_n]$ obtained by killing the ideal $\mathbb{L}^{<0}$.

Proposition 7.2. *The map Φ_X induces an isomorphism of graded $\mathbb{L}$-modules*

$$\Phi_X : \Omega_*^G(X) \widehat{\otimes}_{\mathbb{L}} \mathbb{Z} \xrightarrow{\cong} \mathrm{CH}_*^G(X).$$

Proof. Let $\{(V_j, U_j)\}_{j \geq 0}$ be a sequence of l_j-dimensional good pairs as in Theorem 6.1. It follows from (7.3) that for any $i \in \mathbb{Z}$, there is a short exact sequence

$$0 \to \left(\mathbb{L}^{<0}\Omega_*(X_j) \cap \Omega_{i+l_j-g}(X_j)\right) \to \Omega_{i+l_j-g}(X_j) \to \mathrm{CH}_{i+l_j-g}(X_j) \to 0. \tag{7.7}$$

By comparing this exact sequence for $j' \geq j \geq 0$, using the surjection $\Omega_{i+l_{j+1}-g}(X_{j+1}) \to \Omega_{i+l_j-g}(X_j)$ as in (6.1) and using the localization sequences for the cobordism and Chow groups, we find that the map

$$\left(\mathbb{L}^{<0}\Omega_*(X_{j+1}) \cap \Omega_{i+l_{j+1}-g}(X_{j+1})\right) \to \left(\mathbb{L}^{<0}\Omega_*(X_j) \cap \Omega_{i+l_j-g}(X_j)\right)$$

is surjective for each $j \geq 0$. Taking the limit in (7.7) and using Theorem 6.1, we get a short exact sequence

$$0 \to \varprojlim_{j \geq 0} \left(\mathbb{L}^{<0}\Omega_*(X_j) \cap \Omega_{i+l_j-g}(X_j)\right) \to \Omega_i^G(X) \to \mathrm{CH}_i^G(X) \to 0.$$

Observe here that the inverse system $\{\mathrm{CH}_{i+l-g}(X_j)\}_{j \geq 0}$ is eventually constant with $\mathrm{CH}_i^G(X)$ as its limit. We now take the direct sum over $i \in \mathbb{Z}$ to get the desired isomorphism $\Omega_*^G(X) \widehat{\otimes}_{\mathbb{L}} \mathbb{Z} \xrightarrow{\cong} \mathrm{CH}_*^G(X)$. ☐

Let $C(G) = \mathrm{CH}_G^*(k)$ denote the equivariant Chow ring of the field k. The following is the equivariant analogue of (7.3).

Corollary 7.3. *For a k-scheme X with a G-action, the natural map*

$$\Omega_*^G(X) \otimes_{S(G)} C(G) \to \mathrm{CH}_*^G(X)$$

is an isomorphism of $C(G)$-modules. This is a ring isomorphism if X is smooth.

Proof. It is clear that the above map is a ring homomorphism if X is smooth. So we only need to prove the first assertion. But this follows directly from the isomorphisms $\Omega^G_*(X) \otimes_{S(G)} C(G) \cong \Omega^G_*(X) \otimes_{S(G)} \left(S(G) \widehat{\otimes}_{\mathbb{L}} \mathbb{Z}\right) \cong \Omega^G_*(X) \widehat{\otimes}_{\mathbb{L}} \mathbb{Z}$ using Proposition 7.2. $\square$

7.2. Comparison with equivariant K-theory. It was shown by Levine and Morel in [30, Corollary 11.11] that the universal property of the algebraic cobordism implies that there is a canonical isomorphism of oriented cohomology theories

$$\Omega^*(X) \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \xrightarrow{\cong} K_0(X)[\beta, \beta^{-1}] \tag{7.8}$$

in the category of smooth k-schemes. This was later generalized to a complete algebraic analogue of the Conner-Floyd isomorphism

$$MGL^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \xrightarrow{\cong} K_*(X)[\beta, \beta^{-1}]$$

between the motivic cobordism and algebraic K-theory by Panin, Pimenov and Röndigs [33]. Since the equivariant cobordism is a Borel style cohomology theory, one can not expect an equivariant version of the isomorphism (7.8) even with the rational coefficients. However, we show here that the equivariant Conner-Floyd isomorphism holds after we base change the above by the completion of the representation ring of G with respect to the ideal of virtual representations of rank zero. In fact, it can be shown easily that such a base change is the minimal requirement. In Theorem 7.4, all cohomology groups are considered with rational coefficients (*cf.* Section 8).
For a linear algebraic group G, let $R(G)$ denote the representation ring of G. Let I denote the ideal of of virtual representations of rank zero in $R(G)$ and let $\widehat{R(G)}$ denote the associated completion of $R(G)$. Let $\widehat{C(G)}$ denote the completion of $C(G)$ with respect to the augmentation ideal of algebraic cycles of positive codimensions. For a scheme X with G-action, let $K^G_0(X)$ denote the Grothendieck group of G-equivariant vector bundles on X. By [11, Theorem 4.1], there is a natural ring isomorphism $\widehat{R(G)} \xrightarrow{\cong} \widehat{C(G)}$ given by the equivariant Chern character. We identify these two rings via this isomorphism. In particular, the maps $S(G) \twoheadrightarrow C(G) \to \widehat{C(G)}$ yield a ring homomorphism $S(G) \to \widehat{R(G)}$.

Theorem 7.4. *Let X be a smooth scheme with a G-action. Then, with rational coefficients, there is a natural isomorphism of rings*

$$\Psi_X : \Omega^*_G(X) \otimes_{S(G)} \widehat{R(G)} \xrightarrow{\cong} K^G_0(X) \otimes_{R(G)} \widehat{R(G)}.$$

Proof. By [17, Theorem 1.2], there is a Chern character isomorphism $K^G_0(X) \otimes_{R(G)} \widehat{R(G)} \xrightarrow{\cong} \mathrm{CH}^*(X) \otimes_{C(G)} \widehat{C(G)}$ of cohomology rings. Thus, we only need to show that the map $\Omega^*_G(X) \otimes_{S(G)} \widehat{C(G)} \to \mathrm{CH}^*(X) \otimes_{C(G)} \widehat{C(G)}$ is an isomorphism. However, we have

$$\begin{aligned} \Omega^*_G(X) \otimes_{S(G)} \widehat{C(G)} &\cong \left(\Omega^*_G(X) \otimes_{S(G)} C(G)\right) \otimes_{C(G)} \widehat{C(G)} \\ &\cong \mathrm{CH}^*_G(X) \otimes_{C(G)} \widehat{C(G)}, \end{aligned}$$

where the last isomorphism follows from Corollary 7.3. This finishes the proof. $\square$

7.3. Comparison with complex cobordism. Let G be a complex Lie group acting on a finite CW-complex X. We define the *equivariant complex cobordism* ring of X as

$$MU^*_G(X) := MU^*\left(X \overset{G}{\times} EG\right) \tag{7.9}$$

where $EG \to BG$ is universal principal G-bundle over the classifying space BG of G. If $E'G \to B'G$ is another such bundle, then the projection $(X \times EG \times E'G)/G \to X \overset{G}{\times} EG$ is a fibration with contractible fiber. In particular, $MU^*_G(X)$ is well-defined. Moreover, if G acts freely on X with quotient X/G, then the map $X \overset{G}{\times} EG \to X/G$ is a fibration with contractible fiber EG and hence we get $MU^*_G(X) \cong MU^*(X/G)$.

For a linear algebraic group G over $\mathbb{C}$ acting on a $\mathbb{C}$-scheme X, let $H^*_G(X, A)$ denote the (equivariant) cohomology of the complex analytic space $X(\mathbb{C})$ with coefficients in the ring A.

Proposition 7.5. *Assume that $X \in \mathcal{V}^S_G$ is such that $H^*_G(X, \mathbb{Z})$ is torsion-free. Then there is a natural homomorphism of graded rings*

$$\rho^G_X : \ \Omega^*_G(X) \to MU^{2*}_G(X).$$

Proof. If $\{(V_j, U_j)\}$ is a sequence of good pairs as in Theorem 6.1, then the universality of the Levine-Morel cobordism gives a natural $\mathbb{L}$-algebra map of inverse systems

$$\Omega^i\left(X \overset{G}{\times} U_j\right) \to MU^{2i}\left(X \overset{G}{\times} U_j\right)$$

which after taking limits yields the map

$$\Omega^i_G(X) = \varprojlim_{j \ge 0} \Omega^i\left(X \overset{G}{\times} U_j\right) \to \varprojlim_{j \ge 0} MU^{2i}\left(X \overset{G}{\times} U_j\right). \tag{7.10}$$

On the other hand, it follows from [15, Lemma 3.2] (see also [39, Theorem 2.1]) that there is a Milnor exact sequence

(7.11)

$$0 \to {\varprojlim_{j \ge 0}}^1 MU^{2i-1}\left(X \overset{G}{\times} U_j\right) \to MU^{2i}\left(X \overset{G}{\times} EG\right) \to \varprojlim_{j \ge 0} MU^{2i}\left(X \overset{G}{\times} U_j\right) \to 0.$$

Moreover, it follows from our assumption and [26, Corollary 1] that the first term of this exact sequence vanishes. This yields the natural map $\rho^G_X : \Omega^i_G(X) \to MU^{2i}_G(X)$. $\square$

It follows from the proof of [26, Corollary 1] that the first term in (7.11) always vanishes if we work over the rationals. We can thus imitate the proof of Proposition 7.5 to see that there is a natural map $\Omega^*_G(X)_{\mathbb{Q}} \to MU^{2*}_G(X)_{\mathbb{Q}}$. Combining this with Proposition 7.2 (with rational coefficients), one concludes the following.

Corollary 7.6. *For any $X \in \mathcal{V}^S_G$, there is a natural map of graded $\mathbb{L}_{\mathbb{Q}}$-algebras*

$$\rho^G_X : \ \Omega^*_G(X)_{\mathbb{Q}} \to MU^{2*}_G(X)_{\mathbb{Q}}.$$

In particular, there is a natural ring homomorphism

$$\overline{\rho}^G_X : \ \mathrm{CH}^*_G(X)_{\mathbb{Q}} \to MU^{2*}_G(X)_{\mathbb{Q}} \widehat{\otimes}_{\mathbb{L}_{\mathbb{Q}}} \mathbb{Q}$$

which factors the cycle class map $\mathrm{CH}^*_G(X) \to H^{2*}_G(X, \mathbb{Q})$.

Corollary 7.7. *There is a natural morphism $\Omega^*(BG) \to MU^{2*}(BG)$ of graded $\mathbb{L}$-algebras. In particular, there is a natural ring homomorphism $\mathrm{CH}^*(BG) \to MU^{2*}(BG) \widehat{\otimes}_{\mathbb{L}} \mathbb{Z}$ which factors the cycle class map $\mathrm{CH}^*(BG) \to H^{2*}(BG, \mathbb{Z})$.*

Proof. The first assertion follows immediately from (7.10), (7.11) and [26, Theorem 1] using the fact that BG is homotopy equivalent to the classifying space of its maximal compact subgroup. The second assertion follows from the first and Proposition 7.2 using the identification $\mathbb{L} \xrightarrow{\cong} MU^*$. $\square$

Remark 7.8. The map $\mathrm{CH}^*(BG) \to MU^{2*}(BG) \widehat{\otimes}_{\mathbb{L}} \mathbb{Z}$ has also been constructed by Totaro [39] by a different method.

We shall study the above realization maps in more detail in the next section.

8. Reduction of arbitrary groups to tori

The main result of this section is to show that with the rational coefficients, the equivariant cobordism of schemes with an action of a connected linear algebraic group can be written in terms of the Weyl group invariants of the equivariant cobordism for the action of the maximal torus. This reduces the problems about the equivariant cobordism to the case where the underlying group is a torus. We draw some consequences of this for the cycle class map from the rational Chow groups to the complex cobordism groups of classifying spaces. We first prove some reduction results about the equivariant cobordism which reflect the relations between the G-equivariant cobordism and the equivariant cobordism for actions of subgroups of G. The results of this section are used in [19] and [21] to compute the non-equivariant cobordism ring of flag varieties and flag bundles.

Proposition 8.1. *Let G be a connected reductive group over k. Let B be a Borel subgroup of G containing a maximal torus T over k. Then for any $X \in \mathcal{V}_G$, the restriction map*

$$\Omega^B_*(X) \xrightarrow{r^B_{T,X}} \Omega^T_*(X) \tag{8.1}$$

is an isomorphism.

Proof. By Proposition 5.4, we only need to show that

$$\Omega_*^B\left(B \overset{T}{\times} X\right) \cong \Omega_*^B(X). \tag{8.2}$$

By [8, XXII, 5.9.5], there exists a characteristic filtration $B^u = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n = \{1\}$ of the unipotent radical B^u of B such that U_{i-1}/U_i is a vector group, each U_i is normal in B and $TU_i = T \ltimes U_i$. Moreover, this filtration also implies that for each i, the natural map $B/TU_i \to B/TU_{i-1}$ is a torsor under the vector bundle $U_{i-1}/U_i \times B/TU_{i-1}$ on B/TU_{i-1}. Hence, the homotopy invariance (*cf.* Theorem 5.2) gives an isomorphism

$$\Omega_*^B(B/TU_{i-1} \times X) \xrightarrow{\cong} \Omega_*^B(B/TU_i \times X).$$

Composing these isomorphisms successively for $i = 1, \cdots, n$, we get

$$\Omega_*^B(X) \xrightarrow{\cong} \Omega_*^B(B/T \times X).$$

The canonical isomorphism of B-varieties $B \overset{T}{\times} X \cong B/T \times X$ and Proposition 5.4 together now prove (8.2) and hence (8.1). □

Proposition 8.2. *Let H be a possibly non-reductive group over k. Let $H = L \ltimes H^u$ be the Levi decomposition of H (which exists since k is of characteristic zero). Then the restriction map*

$$\Omega_*^H(X) \xrightarrow{r^H_{L,X}} \Omega_*^L(X) \tag{8.3}$$

is an isomorphism.

Proof. Since the ground field is of characteristic zero, the unipotent radical H^u of H is split over k. Now the proof is exactly same as the proof of Proposition 8.1, where we just have to replace B and T by H and L respectively. □

Notation: All results in the rest of this section will be proven with the rational coefficients. In order to simplify our notations, an abelian group A from now on will actually mean the $\mathbb{Q}$-vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$, and an inverse limit of abelian groups will mean the limit of the associated $\mathbb{Q}$-vector spaces. In particular, all cohomology groups will be considered with the rational coefficients and $\Omega_i^G(X)$ will mean

$$\Omega_i^G(X) := \varprojlim_j \left(\Omega_i^G(X)_j \otimes_{\mathbb{Z}} \mathbb{Q}\right).$$

Notice that this is same as $\Omega_i^G(X)\widehat{\otimes}_{\mathbb{Z}}\mathbb{Q}$ in our earlier notation.

8.1. The motivic cobordism theory. Before we prove our main results of this section, we recall the theory of motivic algebraic cobordism $MGL_{*,*}$ introduced by Voevodsky in [41]. This is a bi-graded ring cohomology theory in the category of smooth schemes over k. Levine has recently shown in [28] that $MGL_{*,*}$ extends uniquely to a bi-graded oriented Borel-Moore homology theory $MGL'_{*,*}$ on the category of all schemes over k. This homology theory has exterior products, homotopy invariance, localization exact sequence and Mayer-Vietoris among other properties (*cf.* [*loc. cit.*, Section 3]). Moreover, the universality of Levine-Morel cobordism theory implies that there is a unique map

$$\vartheta : \Omega_* \to MGL'_{2*,*}$$

of oriented Borel-Moore homology theories. Our motivation for studying the motivic cobordism theory in this text comes from the following result of Levine.

Theorem 8.3 ([29]). *For any $X \in \mathcal{V}_k$, the map ϑ_X is an isomorphism.*

We recall from [28] that for a smooth k-scheme X, there is a Hopkins-Morel spectral sequence

$$E_2^{p,q}(n) = \mathrm{CH}^{n-q}(X, 2n-p-q) \otimes \mathbb{L}^q \Rightarrow MGL^{p+q,n}(X) \tag{8.4}$$

which is an algebraic analogue of the Atiyah-Hirzebruch spectral sequence in complex cobordism.

If X is possibly singular, we embed it as a closed subscheme of a smooth scheme M. Then, the functoriality of the above spectral sequence with respect to an open immersion yields a spectral sequence

$$E_2^{p,q}(n) = \mathrm{CH}_X^{n-q}(M, 2n-p-q) \otimes \mathbb{L}^q \Rightarrow MGL_X^{p+q,n}(M)$$

of cohomology with support. Since the higher Chow groups and the motivic cobordism groups of M with support in X are canonically isomorphic to the higher Chow groups and the Borel-Moore motivic cobordism groups of X (*cf.* [1], [29, Section 3]), the above spectral sequence is identified with

$$E^2_{p,q}(n) = \mathrm{CH}_n(X, p) \otimes \mathbb{L}^q \Rightarrow MGL'_{2n+2q-p,n+q}(X). \tag{8.5}$$

Now, suppose that a finite group G acts on X. By embedding X equivariantly in a smooth G-scheme M, the formula

$$MGL'_{p,q}(X) := \mathrm{Hom}_{\mathcal{SH}(k)}\left(\Sigma_T^{\infty} M/(M-X), \Sigma^{p',q'} MGL\right)$$

(where $p' = 2\dim(M) - p, q' = \dim(M) - q$) shows that G acts naturally on $MGL'_{p,q}(X)$. It also acts on the higher Chow groups $\mathrm{CH}_p(X, q)$ likewise, where we just have to replace MGL in (8.5) by the Eilenberg-MacLane spectrum $H\mathbb{Z}$. Recall furthermore that MGL and $H\mathbb{Z}$ are ring spectra and the spectral sequence (8.4) (and hence (8.5)) is obtained by first showing that there is a natural quotient morphism of ring spectra $MGL \to H\mathbb{Z}$. In particular, for any G-scheme X, (8.5) is a spectral sequence of $\mathbb{Z}[G]$-modules. The reader will notice that the rational coefficients have not been used so far and Theorem 8.3 as well as the spectral sequence (8.5) hold integrally.

We shall now use rational coefficients everywhere and draw some consequences of Theorem 8.3 and (8.5). Since the functor of taking "G-invariants" is exact on the category of $\mathbb{Q}[G]$-modules, the spectral sequence (8.5) over the rationals, yields the spectral sequence of G-invariants

$$E'^{2}_{p,q}(n) = \left(\mathrm{CH}_n(X,p)\right)^G \otimes \mathbb{L}^q \Rightarrow \left(MGL'_{2n+2q-p,n+q}(X)\right)^G. \tag{8.6}$$

Recall that a connected and reductive group G over k is said to be *split*, if it contains a split maximal torus T over k such that G is given by a root datum relative to T. One knows that every connected and reductive group containing a split maximal torus is split (*cf.* [8, Chapter XXII, Proposition 2.1]). In such a case, the normalizer N of T in G and all its connected components are defined over k and the quotient N/T is the Weyl group W of the corresponding root datum. As an application of the spectral sequences (8.5) and (8.6), we get the following.

Lemma 8.4. *Let G be a connected reductive group with split maximal torus T and the associated Weyl group W. Let G act freely on a scheme X. Then, with rational coefficients, the pull-back map $\Omega_*(X/G) \to \left(\Omega_*(X/T)\right)^W$ (up to a shift) is an isomorphism.*

Proof. It follows from [17, Corollary 3.9] (see also [22, Corollary 8.9]) that in the case under consideration, the natural map $\mathrm{CH}_*(X/G,p) \to \left(\mathrm{CH}_*(X/T,p)\right)^W$ is an isomorphism for all $p \ge 0$. We can thus apply the spectral sequences (8.5) and (8.6) to conclude that the map $MGL'_{*,*}(X/G) \to \left(MGL'_{*,*}(X/T)\right)^W$ is an isomorphism. The lemma now follows from this isomorphism and Theorem 8.3. $\square$

Remark 8.5. The proof of Lemma 8.4 is based on the existence of the Atiyah-Hirzebruch spectral sequence in the motivic cobordism. There is no published proof of the existence of this spectral sequence, though it has been presented by the authors during various seminars. We can give another proof of the above lemma without using the spectral sequence as follows.

It was proven by Levine and Morel in [31, Theorems 4.1.28, 4.5.1] that there is a morphism of oriented Borel-Moore homology theories $\Omega_* \to \mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}$ which is an isomorphism with rational coefficients. Here, $\mathrm{CH}_*[\mathbf{t}]$ is the polynomial module $\mathrm{CH}_*[t_1, t_2, \cdots]$ in infinitely many variables with $\deg(t_i) = i$. Recall also that $\mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}$ is same as $\mathrm{CH}_*[\mathbf{t}]$ as $\mathrm{CH}_*(k)$-module.

Applying the above isomorphism to X/T and X/G and using the isomorphism $\mathrm{CH}_*(X/G) \xrightarrow{\cong} \left(\mathrm{CH}_*(X/T)\right)^W$ (*cf.* [10, Proposition 9]), we immediately get that the map $\Omega_*(X/G) \to \left(\Omega_*(X/T)\right)^W$ is an isomorphism.

Recall from Remark 4.6 that if a connected reductive group G acts on a scheme X, then the Weyl group W acts on $\Omega^T_*(X)$ where T is a maximal torus of G.

Theorem 8.6. [1] *Let G be a connected linear algebraic group and let L be a Levi subgroup of G with a split maximal torus T. Let W denote the Weyl group of L with respect to T. Then for any $X \in \mathcal{V}_G$, the natural map*

$$\Omega^G_*(X) \to \left(\Omega^T_*(X)\right)^W \tag{8.7}$$

is an isomorphism with rational coefficients.

Proof. By Proposition 8.2, we can assume that $G = L$ and hence G is a connected reductive group with split maximal torus T.
We choose a sequence of l_j-dimensional good pairs $\{(V_j, U_j)\}$ as in Theorem 6.1 for the G-action. Then, this is also a sequence of good pairs for the action of T. Setting $X^j_H = \{X \stackrel{H}{\times} U_j\}$ for any closed subgroup $H \subseteq G$, we see that $\{X^j_T\}$ is a sequence of W-schemes, each term of which has a free W-action. It follows from Lemma 8.4 (or Remark 8.5) that the smooth pull-back map

$$\Omega_{i+l_j-g}\left(X^j_G\right) \to \left(\Omega_{i+l_j-n}\left(X^j_T\right)\right)^W \tag{8.8}$$

is an isomorphism, where $\dim(G) = g$ and $\dim(T) = n$.
Since the action of W on the inverse system $\left\{\Omega_{i+l_j-n}\left(X^j_T\right)\right\}_j$ induces the similar action on the inverse limit and since the inverse limit commutes with taking the W-invariants, we get

$$\varprojlim_j \Omega_{i+l_j-g}\left(X^j_G\right) \xrightarrow{\cong} \left(\varprojlim_j \Omega_{i+l_j-n}\left(X^j_T\right)\right)^W. \tag{8.9}$$

Since the left and the right terms are same as $\Omega^G_i(X)$ and $\left(\Omega^T_i(X)\right)^W$ respectively by choice of our good pairs and Theorem 6.1, we conclude that $\Omega^G_i(X) \xrightarrow{\cong} \left(\Omega^T_i(X)\right)^W$. This completes the proof of the theorem. □

Corollary 8.7. *Let $X \in \mathcal{V}_G$ be as in Theorem 8.6. Then the restriction map*

$$\Omega^G_*(X)_{\mathbb{Q}} \xrightarrow{r^G_{T,X}} \Omega^T_*(X)_{\mathbb{Q}} \tag{8.10}$$

is a split monomorphism which is natural for the morphisms in $\mathcal{V}_G$. In particular, if H is any closed subgroup of G, then there is a split injective map

$$\Omega^H_*(X)_{\mathbb{Q}} \xrightarrow{r^G_{T,X}} \Omega^T_*\left(G \stackrel{H}{\times} X\right)_{\mathbb{Q}}. \tag{8.11}$$

Proof. The first statement follows directly from Theorem 8.6, where the splitting is given by the trace map. The second statement follows from the first and Proposition 5.4. □

[1] It has been shown recently in [15, Proposition 4.8] that the map $S(G) \to S(T)^W$ is an isomorphism over $\mathbb{Z}[t_G^{-1}]$, where t_G is the torsion index of G.

Before we apply Theorem 8.6 to study the rational cobordism rings of classifying spaces, we need the following topological analogue, which is much simpler to prove. Recall from (7.9) that if G is a complex Lie group and X is a finite CW-complex with a G-action, then its *equivariant complex cobordism* is defined as

$$MU^*_G(X) := MU^*\left(X \overset{G}{\times} EG\right). \tag{8.12}$$

Theorem 8.8. *Let G be a complex Lie group with a maximal torus T and Weyl group W. Then for any X as above, the natural map*

$$MU^*_G(X) \to \left(MU^*_T(X)\right)^W \tag{8.13}$$

is an isomorphism with rational coefficients.

Proof. As in the proof of Theorem 8.6, we can reduce to the case when G is reductive. It follows from the above definition of the equivariant complex cobordism and the similar definition of the equivariant singular cohomology of X, plus the Atiyah-Hirzebruch spectral sequence in topology that there is a spectral sequence

$$E_2^{p,q} = H^p_G(X, \mathbb{Q}) \otimes_{\mathbb{Q}} MU^q \Rightarrow MU^{p+q}_G(X). \tag{8.14}$$

Since the Atiyah-Hirzebruch spectral sequence degenerates rationally, we see that the above spectral sequence degenerates too. Since one knows that $H^*_G(X) \cong \left(H^*_T(X)\right)^W$ (*cf.* [4, Proposition 1]), the corresponding result for the cobordism follows. □

Theorem 8.9. *For a connected linear algebraic group G over $\mathbb{C}$, the degree doubling map $\rho^G : \Omega^*(BG) \to MU^*(BG)$ (cf. Corollary 7.7) of $\mathbb{L}$-algebras, is an isomorphism with rational coefficients. In particular, the natural map of $\mathbb{Q}$-algebras*

$$\mathrm{CH}^*(BG)_{\mathbb{Q}} \xrightarrow{\overline{\rho}^G} MU^*(BG)\widehat{\otimes}_{\mathbb{L}}\mathbb{Q}$$

is an isomorphism.

Proof. To prove the first isomorphism, we can use Theorems 8.6 and 8.8 to reduce to the case of a torus. But this case is already known even with the integer coefficients (*cf.* (6.14) and [39]). The second isomorphism follows from the first and Proposition 7.2. □

Remark 8.10. The map $\overline{\rho}^G : \mathrm{CH}^*(BG) \to MU^*(BG)\widehat{\otimes}_{\mathbb{L}}\mathbb{Z}$ was found by Totaro in [39] even before Levine and Morel discovered their algebraic cobordism. It was conjectured that the map $\overline{\rho}^G$ should be an isomorphism with the integer coefficients for a connected complex algebraic group G. Totaro modified this conjecture to an expectation that $\overline{\rho}^G$ should be an isomorphism after localization at a prime p such that $MU^*(BG)_{(p)}$ is concentrated in even degree. The above theorem proves the isomorphism in general with the rational coefficients. We also remark that the map $MU^*(BG)\widehat{\otimes}_{\mathbb{L}}\mathbb{Q} \to H^*(BG, \mathbb{Q})$ is an isomorphism (*cf.* [38]). The above result then shows that the cycle class map for the classifying space is an isomorphism with the rational coefficients. One wonders if the techniques of this paper could be applied to the algebraic version of the Brown-Peterson cobordism theory to prove the Totaro's modified conjecture. We do not know the answer yet.

Acknowledgments. The author would like to thank the referee for his/her comments and suggestions, which were immensely helpful in improving the contents and the presentation of this paper.

References

[1] S. Bloch, *Algebraic cycles and higher K-theory*, Advances in Math., 61, (1986), 267-304.
[2] S. Bloch, A. Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup., (4), 7, (1975), 181-201
[3] A. Borel, *Linear Algebraic groups*, Second edition, GTM 8, Springer-Verlag, (1991).
[4] M. Brion, *Equivariant cohomology and equivariant intersection theory*, Representation theory and algebraic geometry, NATO ASI series, C514, Kluwer Academic Publishers, (1997).
[5] M. Brion, *Equivariant Chow groups for torus actions*, Transform. Groups, 2, (1997), no. 3, 225-267.
[6] V. Buhstaber, A. Miscenko, *K-theory on the category of infinite cell complexes*, (Russian), Izv. Akad. Nauk SSSR Ser. Mat., 32, (1968), 560V604.
[7] V. Buhstaber, A. Miscenko, *Elements of infinite filtration in KK-theory*, (Russian), Dokl. Akad. Nauk SSSR, 178, (1968), 1234V1237.
[8] M. Demazure, A. Grothendieck, *Schémas en Groupes*, Lecture Notes in Math., 153, (1970), Springer-Verlag.
[9] D. Deshpande, *Algebraic Cobordism of Classifying Spaces*, MathArxiv, mathAG/0907.4437, (2009).
[10] D. Edidin, W. Graham, *Equivariant intersection theory*, Invent. Math., 131, (1998), 595-634.
[11] D. Edidin, W. Graham, *Riemann-Roch for equivariant Chow groups*, Duke Math. J., 102, (2000), no. 3, 567-594.
[12] J. Fogarty, F. Kirwan, D. Mumford, *Geometric Invariant Theory*, 3rd Edition, Springer-Verlag, (1994).
[13] W. Fulton, *Intersection Theory*, 2nd edition, Springer-Verlag, (1998).
[14] R. Joshua, *Algebraic K-theory and higher Chow groups of linear varieties*, Math. Proc. Cambridge Philos. Soc., 130, (2001), no. 1, 37-60.
[15] V. Kiritchenko, A. Krishna, *Equivariant cobordism of flag varieties and of symmetric varieties*, MathArxiv, mathAG/1104.1089, (2011).
[16] B. Kock, *Chow motifs and higher Chow theory of G/P*, Manuscripta Math., 70, (1991), no. 4, 363-372.
[17] A. Krishna, *Riemann-Roch for equivariant K-theory*, MathArxiv, mathAG/0906.1696, (2009).
[18] A. Krishna, *Equivariant K-theory and higher Chow groups of smooth varieties*, MathArxiv, mathAG/0906.3109, (2009).

[19] A. Krishna, *Equivariant cobordism of schemes*, MathArxiv, mathAG/1006.3176, (2010).

[20] A. Krishna, *Equivariant cobordism of schemes with torus action*, MathArxiv, mathAG/1010.6182, (2010).

[21] A. Krishna, *Cobordism of flag bundles*, MathArxiv, mathAG/1007.1083, (2010).

[22] A. Krishna, *Higher Chow groups of varieties with group actions*, Submitted to journal, (2011).

[23] A. Krishna, *The equivariant motivic cobordism*, In preparation, (2011).

[24] A. Krishna, V. Uma, *Cobordism rings of toric varieties*, MathArxiv, mathAG/1011.0573, (2010).

[25] P. Landweber, *Coherence, flatness and cobordism of classifying spaces*, Proc. Adv. study Inst. Alg. top., II, Aarhus, (1970), 256-269.

[26] P. Landweber, *Elements of infinite filtration in complex cobordism*, Math. Scand., 30, (1972), 223-226.

[27] P. Landweber, *Unique factorization in graded power series rings*, Proc. of the AMS., 42, (1974), 73-76.

[28] M. Levine, *Oriented cohomology, Borel-Moore homology, and algebraic cobordism*, Special volume in honor of Melvin Hochster, Michigan Math. J., 57, (2008).

[29] M. Levine, *Comparison of cobordism theories*, J. Algebra, 322, (2009), no. 9, 3291-3317.

[30] M. Levine, F. Morel, *Algebraic cobordism I*, Preprint, www.math.uiuc.edu/K-theory/0547, (2002).

[31] M. Levine, F. Morel, *Algebraic cobordism*, Springer Monographs in Mathematics, Springer, Berlin, (2007).

[32] M. Levine, R. Pandharipande, *Algebraic cobordism revisited*, Invent. Math., 176, (2009), no. 1, 63-130.

[33] I. Panin, K. Pimenov, O. Röndings, *On the relation of Voevodsky's algebraic cobordism to Quillen's K-theory*, Invent. Math., 175, (2009), no. 2, 435-451.

[34] D. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math., 7, (1971), 29-56.

[35] T. Springer, *Linear algebraic groups*, Second edition, Progress in Math., 9, (1998), Birkhauser.

[36] H. Sumihiro, *Equivariant completion II*, J. Math. Kyoto, 15, (1975), 573-605.

[37] R. Thomason, *Equivariant algebraic vs. topological K-homology Atiyah-Segal-style*, Duke Math. J., 56, (1988), 589-636.

[38] B. Totaro, *Torsion algebraic cycles and complex cobordism*, J. Amer. Math. Soc., 10, (1997), no. 2, 467-493.

[39] B. Totaro, *The Chow ring of a classifying space*, Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math., Amer. Math. Soc., 67, (1999), 249-281.

[40] G. Vezzosi, A. Vistoli, *Higher algebraic K-theory for actions of diagonalizable groups*, Invent. Math., 153, (2003), no. 1, 1-44.
[41] V. Voevodsky, *$\mathbb{A}^1$-homotopy theory*, Proceedings of the International Congress of Mathematicians, 1, (Berlin, 1998), Doc. Math., (1998), 579-604.

Amalendu Krishna
School of Mathematics
Tata Institute
of Fundamental Research
Homi Bhabha Road
Colaba
Mumbai
India
amal@math.tifr.res.in

Basic Polynomial Invariants, Fundamental Representations and the Chern Class Map

S. Baek, E. Neher, K. Zainoulline

Received: September 19, 2012
Revised: January 3, 2012

Communicated by Alexander Merkurjev

Abstract. Consider a crystallographic root system together with its Weyl group W acting on the weight lattice Λ. Let $\mathbb{Z}[\Lambda]^W$ and $S(\Lambda)^W$ be the W-invariant subrings of the integral group ring $\mathbb{Z}[\Lambda]$ and the symmetric algebra $S(\Lambda)$ respectively. A celebrated result by Chevalley says that $\mathbb{Z}[\Lambda]^W$ is a polynomial ring in classes of fundamental representations $\rho_1, ..., \rho_n$ and $S(\Lambda)^W \otimes \mathbb{Q}$ is a polynomial ring in basic polynomial invariants $q_1, ..., q_n$. In the present paper we establish and investigate the relationship between ρ_i's and q_i's over the integers. As an application we provide estimates for the torsion of the Grothendieck γ-filtration and the Chow groups of some twisted flag varieties up to codimension 4.

2010 Mathematics Subject Classification: Primary 13A50; Secondary 14L24
Keywords and Phrases: Dynkin index, polynomial invariant, fundamental representation, Chow group, gamma-filtration, twisted flag variety, torsion

Introduction

Consider a crystallographic root system Φ together with its Weyl group W acting on the weight lattice Λ of Φ. Let $\mathbb{Z}[\Lambda]^W$ and $S^*(\Lambda)^W$ be the W-invariant subrings of the integral group ring $\mathbb{Z}[\Lambda]$ and the symmetric algebra $S^*(\Lambda)$. A celebrated theorem of Chevalley says that $\mathbb{Z}[\Lambda]^W$ is a polynomial ring over $\mathbb{Z}$ in classes of fundamental representations $\rho_1, \ldots, \rho_n$ and $S^*(\Lambda)^W \otimes \mathbb{Q}$ is a polynomial ring over $\mathbb{Q}$ in basic polynomial invariants $q_1, \ldots, q_n$, where $n = \mathrm{rank}(\Phi)$. Another classical result due to Demazure says that the kernels of characteristic maps $\mathbb{Z}[\Lambda] \to K_0(X)$ and $S^*(\Lambda) \to \mathrm{CH}^*(X)$, where X is the variety of

Borel subgroups of the associated linear algebraic group, are generated by non-constant W-invariants. This fact provides a link between combinatorics of the W-action on $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$ and the respective cohomology rings.

In the present paper we establish and investigate the relationship between ρ_i's and q_i's. To do this we introduce an equivariant analogue of the Chern class map ϕ_i that provides an isomorphism between the truncated rings $\mathbb{Z}[\Lambda]/I_m^j$ and $S^*(\Lambda)/I_a^j$ modulo powers of the respective augmentation ideals. This allows us to express basic polynomial invariants in terms of fundamental representations and vice versa, hence, relating the representation theory of the respective Lie algebra $\mathfrak{g}$ with the geometry of the variety of Borel subgroups X.

A multiple of ϕ_i restricted to the respective cohomology (K_0 and CH^*) of X gives the classical Chern class map $c_i\colon K_0(X) \to \mathrm{CH}^i(X)$. This geomeric interpretation provides a powerful tool to compute the annihilators of the torsion of the Grothendieck γ-filtration on K_0 of twisted forms of X as well as a tool to estimate the torsion part of its Chow groups in small codimensions.

The paper is organized as follows. In the first section we introduce the I-adic filtrations on $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$ together with an isomorphism ϕ_i on their truncations. Then we study the subrings of invariants and introduce the key notion of an exponent τ_i of a W-action on a free abelian group Λ. Roughly speaking, the integers τ_i measure how far is the ring $S^*(\Lambda)^W$ from being a polynomial ring in q_i's. In section 5 we estimate all the exponents up to degree 4 and show that they all divide the Dynkin index of the Lie algebra $\mathfrak{g}$. We would like to stress that the procedure of estimating τ_i-s has an algorithmic nature, i.e. given a group and an integer i one can estimate τ_i for this group just using the explicit formulas for W-invariant polynomials. Finally, we apply the obtained results to estimate the torsion in Grothendieck γ-filtration of some twisted flag varieties.

Acknowledgments. The first author has been partially supported from the NSERC grants of the other two authors and from the Fields Institute. The second author gratefully acknowledges support through NSERC Discovery grant 8836-20121. The last author has been supported by the NSERC Discovery grant 385795-2010, Accelerator Supplement 396100-2010 and an Early Researcher Award (Ontario).

1. Two filtrations

Consider the two covariant functors $S^*(-)$ and $\mathbb{Z}[-]$ from the category of abelian groups to the category of commutative rings

$$S^*(-)\colon \Lambda \mapsto S^*(\Lambda) \text{ and } \mathbb{Z}[-]\colon \Lambda \mapsto \mathbb{Z}[\Lambda]$$

given by taking the symmetric algebra of an abelian group Λ and the integral group ring of Λ respectively. The ith graded component $S^i(\Lambda)$ is additively generated by monomials $\lambda_1\lambda_2\ldots\lambda_i$ with $\lambda_j \in \Lambda$ and the ring $\mathbb{Z}[\Lambda]$ is additively generated by exponents e^λ, $\lambda \in \Lambda$.

The trivial group homomorphism induces the ring homomorphisms

$$\epsilon_a\colon S^*(\Lambda) \to \mathbb{Z} \text{ and } \epsilon_m\colon \mathbb{Z}[\Lambda] \to \mathbb{Z}$$

called the augmentation maps. By definition ϵ_a sends every element of positive degree to 0 and ϵ_m sends every e^{λ} to 1. Let I_a and I_m denote the kernels of ϵ_a and ϵ_m respectively. Observe that $I_a = S^{>0}(\Lambda)$ consists of elements of positive degree and I_m is generated by differences $(1-e^{-\lambda})$, $\lambda \in \Lambda$. Consider the respective I-adic filtrations:

$$S^*(\Lambda) = I_a^0 \supseteq I_a \supseteq I_a^2 \supseteq \ldots \text{ and } \mathbb{Z}[\Lambda] = I_m^0 \supseteq I_m \supseteq I_m^2 \supseteq \ldots$$

and let

$$gr_a^*(\Lambda) = \bigoplus_{i\geq 0} I_a^i/I_a^{i+1} \text{ and } gr_m^*(\Lambda) = \bigoplus_{i\geq 0} I_m^i/I_m^{i+1}$$

denote the associated graded rings. Observe that $gr_a^*(\Lambda) = S^*(\Lambda)$.

1.1. Example. If $\Lambda \simeq \mathbb{Z}$, then the ring $S^*(\Lambda)$ can be identified with the polynomial ring in one variable $\mathbb{Z}[\omega]$, where ω is a generator of Λ and the ring $\mathbb{Z}[\Lambda]$ can be identified with the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}]$ where $x = e^{\omega}$. The augmentations ϵ_a and ϵ_m are given by

$$\epsilon_a\colon \omega \mapsto 0 \text{ and } \epsilon_m\colon x \mapsto 1.$$

We have $I_a = (\omega)$ and I_m is additively generated by differences $(1-x^n)$, $n \in \mathbb{Z}$. Note that the rings $\mathbb{Z}[\omega]$ and $\mathbb{Z}[x,x^{-1}]$ are not isomorphic, however they become isomorphic after the truncation. Namely for every $i \geq 0$ there is ring isomorphism

$$\phi_i\colon \mathbb{Z}[x,x^{-1}]/I_m^{i+1} \xrightarrow{\simeq} \mathbb{Z}[\omega]/I_a^{i+1}$$

defined by $\phi_i\colon x \mapsto (1-\omega)^{-1} = 1 + \omega + \ldots + \omega^i$ with the inverse defined by $\phi_i^{-1}\colon \omega \mapsto 1 - x^{-1}$. It is useful to keep the following picture in mind

$$\begin{tikzcd}
\mathbb{Z}[x,x^{-1}] \arrow[d] \arrow[dr] & \mathbb{Z}[\omega] \arrow[l] \arrow[d] \\
\mathbb{Z}[x,x^{-1}]/I_m^{i+1} \arrow[r, "\phi_i", "\simeq"'] & \mathbb{Z}[\omega]/I_a^{i+1}
\end{tikzcd}$$

observing that the inverse ϕ_i^{-1} can be lifted to the map $\mathbb{Z}[\omega] \to \mathbb{Z}[x,x^{-1}]$ but ϕ_i can't.

The example can be generalized as follows:

1.2. Lemma. [GZ10, 2.1] *Assume that Λ is a free abelian group of finite rank n. The rings $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$ become isomorphic after truncation. Namely, if $\{\omega_1, \ldots, \omega_n\}$ is a $\mathbb{Z}$-basis of Λ, then for every $i \geq 0$ there is a ring isomorphism*

$$\phi_i\colon \mathbb{Z}[\Lambda]/I_m^{i+1} \xrightarrow{\simeq} S^*(\Lambda)/I_a^{i+1}$$

defined by $\phi_i(1) = 1$ and

$$\phi_i(e^{\sum_{j=1}^n a_j\omega_j}) = \prod_{j=1}^n (1-\omega_j)^{-a_j}$$

with the inverse defined by $\phi_i^{-1}(\omega_j) = 1 - e^{-\omega_j}$.

Note that the map ϕ_i preserves the I-adic filtrations. Indeed, by definition $\phi_i(I_m^j) \subseteq I_a^j$ for every $0 \le j \le i$. Moreover, we have the following

1.3. Lemma. *(cf.* [CPZ, 4.2]*) The isomorphism* ϕ_i *restricted to the subsequent quotients* I_m^i/I_m^{i+1} *doesn't depend on the choice of a basis of* Λ. *Hence, there is an induced canonical isomorphism of graded rings*

$$\phi_* = \oplus_{i\ge 0}\phi_i : gr_m^*(\Lambda) \xrightarrow{\simeq} gr_a^*(\Lambda) = S^*(\Lambda).$$

Proof. Indeed, in this case we can define the inverse $\phi_i^{-1}\colon I_a^i/I_a^{i+1} \to I_m^i/I_m^{i+1}$ by

$$\phi_i^{-1}(\lambda_1\lambda_2\ldots\lambda_i) = (1-e^{-\lambda_1})(1-e^{-\lambda_2})\ldots(1-e^{-\lambda_i}).$$

It is well-defined since $(1-e^{-\lambda-\lambda'}) = (1-e^{-\lambda}) + (1-e^{-\lambda'})$ modulo I_m^2. □

Consider the composite of the map ϕ_i with the projections

$$\phi^{(i)}\colon \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda]/I_m^{i+1} \xrightarrow{\phi_i} S^*(\Lambda)/I_a^{i+1} \to S^i(\Lambda).$$

The map $\phi^{(i)}$, and therefore ϕ_i, can be computed on generators e^λ, $\lambda \in \Lambda$ as follows:
Let $f(z) = \prod_j (1-\omega_j z)^{-a_j}$, where $\lambda = \sum_j a_j\omega_j$. Then

$$\phi^{(i)}(e^{\sum_j a_j\omega_j}) = \frac{1}{i!}\frac{d^i f(z)}{dz^i}\Big|_{z=0}$$

To compute the derivatives of $f(z)$ we observe that $f'(z) = f(z)g(z)$, where $g(z) = \sum_j a_j\omega_j(1-\omega_j z)^{-1}$ and $\frac{d^i\, g(z)}{d\, z^i} = \sum_j \frac{i!\, a_j\omega_j^{i+1}}{(1-\omega_j z)^{i+1}}$. Hence, starting with $g_0 = 1$ we obtain the following recursive formulas

$$\frac{d^i\, f(z)}{d\, z^i} = f(z)\cdot g_i(z), \text{ where } g_i(z) = g(z)g_{i-1}(z) + g_{i-1}'(z).$$

1.4. Example. For small values of i we obtain

i	$i!\cdot\phi^{(i)}(e^\lambda) =$
1	λ
2	$\lambda^2 + \lambda(2)$
3	$\lambda^3 + 3\lambda(2)\lambda + 2\lambda(3)$
4	$\lambda^4 + 6\lambda(4) + 6\lambda(2)\lambda^2 + 8\lambda(3)\lambda + 3\lambda(2)^2$

where given a presentation $\lambda = \sum_{j=1}^n a_{j,\lambda}\omega_j,\ a_{j,\lambda} \in \mathbb{Z}$ in terms of the basis $\{\omega_1, \omega_2, \ldots \omega_n\}$ we set $\lambda(m) = \sum_{j=1}^n a_{j,\lambda}\omega_j^m$ for $m \ge 1$.

2. Invariants and exponents

Let W be a finite group which acts on a free abelian group Λ of finite rank by $\mathbb{Z}$-linear automorphisms. Consider the induced action of W on $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$. Observe that it is compatible with the I-adic filtrations, i.e. $W(I_m^i) \subseteq I_m^i$ and $W(I_a^i) \subseteq I_a^i$ for every $i \ge 0$.

Note that the isomorphisms ϕ_i and ϕ_i^{-1} are not necessarily W-equivariant. However, by Lemma 1.3 their restrictions to the subsequent quotients I_m^i/I_m^{i+1} and $I_a^i/I_a^{i+1} = S^i(\Lambda)$ are W-equivariant and we have

$$(I_m^i/I_m^{i+1})^W \simeq (I_a^i/I_a^{i+1})^W.$$

Let I_m^W denote the ideal of $\mathbb{Z}[\Lambda]$ generated by W-invariant elements from the augmentation ideal I_m, i.e., by elements from $\mathbb{Z}[\Lambda]^W \cap I_m$. Similarly, let I_a^W denote the ideal of $S^*(\Lambda)$ generated by W-invariant elements from I_a, i.e., by elements from $S^*(\Lambda)^W \cap I_a$.

For each $\chi \in \Lambda$ let $\rho(\chi) = \sum_{\lambda\in W(\chi)} e^\lambda$ denote the sum over all elements of the W-orbit of χ. Every element in I_m^W can be written as a finite linear combination with integer coefficients of the elements $\hat\rho(\chi) = \rho(\chi) - \epsilon_m(\rho(\chi))$, $\chi \in \Lambda$. Therefore, the ideal I_m^W is generated by the elements $\hat\rho(\chi)$, i.e.,

$$I_m^W = \langle \hat\rho(\chi) \mid \chi \in \Lambda\rangle.$$

The image of I_m^W by means of the composite

$$\mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda]/I_m^{i+1} \xrightarrow{\phi_i} S^*(\Lambda)/I_a^{i+1}\,.$$

is an ideal in $S^*(\Lambda)/I_a^{i+1}$ generated by the elements $\phi_i(\hat\rho(\chi))$, $\chi \in \Lambda$. Therefore, the image of I_m^W in $S^i(\Lambda)$ is the ith homogeneous component of the ideal generated by $\phi^{(j)}(\hat\rho(\chi))$, where $1 \le j \le i$, $\chi \in \Lambda$, i.e.

$$\phi^{(i)}(I_m^W) = \langle f\cdot \phi^{(j)}(\hat\rho(\chi)) \mid 1 \le j \le i,\ f \in S^{i-j}(\Lambda),\ \chi\in\Lambda\rangle_{\mathbb{Z}}\,.$$

We are ready now to introduce the central notion of the present paper:

2.1. Definition. We say that an action of W on Λ *has finite exponent in degree* i if there exists a non-zero integer N_i such that

$$N_i \cdot (I_a^W)^{(i)} \subseteq \phi^{(i)}(I_m^W),$$

where $(I_a^W)^{(i)} = I_a^W \cap S^i(\Lambda)$. In this case the g.c.d. of all such N_is will be called the *i-th exponent* of the W-action and will be denoted by τ_i.

Observe that if $\phi^{(i)}(I_m^W)$ is a subgroup of finite index in $(I_a^W)^{(i)}$, then τ_i is simply the exponent of $\phi^{(i)}(I_m^W)$ in $(I_a^W)^{(i)}$. Note also that by the very definition $\tau_0 = 1$.

2.2. Example. Consider $\Lambda = \mathbb{Z}\cdot\omega$ with the action $\omega \mapsto -\omega$ of $W = \mathbb{Z}/2\mathbb{Z}$. Then (I_a^W) is generated by $\omega^2, \omega^4, \cdots$, hence $(I_a^W)^{(i)} = \mathbb{Z}\cdot\omega^i$ if i is even, 0 otherwise. On the other hand, $\phi^{(i)}(I_m^W)$ is generated by $\phi^{(i)}(\hat\rho(\omega)) = \phi^{(i)}(e^\omega + e^{-\omega} - 2) = \omega^i$ if $i \ge 2$, 0 otherwise. Therefore, we have $\tau_i = 1$ for every $i \ge 0$.

3. Essential actions

In the present section we study W-actions that have no W-invariant linear forms, i.e. we assume that $\Lambda^W = 0$. In the theory of reflection groups such actions are called *essential* (see [B4-6, V, §3.7] or [Hu]). Note that this immediately implies that $\tau_1 = 1$.

3.1. Lemma. *For every $\chi \in \Lambda$ and $m \in \mathbb{N}_+$ we have $\sum_{\lambda \in W(\chi)} \lambda(m) = 0$.*

Proof. Let $\omega_1, \omega_2, \ldots . \omega_n$ be a $\mathbb{Z}$-basis of Λ. For $m \in \mathbb{N}_+$ we have

$$\sum_{\lambda \in W(\chi)} \lambda(m) = \sum_{\lambda \in W(\chi)} \Big(\sum_{j=1}^{n} a_{j,\lambda} \omega_j^m\Big) = \sum_{j=1}^{n} \Big(\sum_{\lambda \in W(\chi)} a_{j,\lambda}\Big) \omega_j^m.$$

In particular, for $m = 1$ we obtain

$$\sum_{\lambda \in W(\chi)} \lambda = \sum_{j=1}^{n} \Big(\sum_{\lambda \in W(\chi)} a_{j,\lambda}\Big) \omega_i.$$

Since $\Lambda^W = 0$, we have $\sum_{\lambda \in W(\chi)} \lambda = 0$. Since ω_j, $1 \leq j \leq n$ are $\mathbb{Z}$-free, we have $\sum_{\lambda \in W(\chi)} a_{j,\lambda} = 0$ for all $1 \leq j \leq n$. □

3.2. Corollary. *For every $\chi \in \Lambda$ we have*

$$\phi^{(2)}(\rho(\chi)) = \tfrac{1}{2} \sum_{\lambda \in W(\chi)} \lambda^2.$$

In particular, the quadratic form $\phi^{(2)}(\rho(\chi))$ is W-invariant, i.e.

$$\phi^{(2)}(\rho(\chi)) \in S^2(\Lambda)^W.$$

Proof. By the formula for $\phi^{(2)}$ in Example 1.4 and by Lemma 3.1 we obtain that

$$\phi^{(2)}\Big(\sum_{\lambda \in W(\chi)} e^{\lambda}\Big) = \tfrac{1}{2} \sum_{\lambda \in W(\chi)} (\lambda^2 + \lambda(2)) = \tfrac{1}{2} \sum_{\lambda \in W(\chi)} \lambda^2. \quad \square$$

3.3. Corollary. *If $S^2(\Lambda)^W = \langle q \rangle$ for some q, then $\phi^{(2)}(I_m^W)$ is a subgroup of finite index in $(I_a^W)^{(2)}$.*

Proof. The image of the ideal I_m^W is generated by $\phi^{(1)}(\rho(\chi))$ and $\phi^{(2)}(\rho(\chi))$. Since $\Lambda^W = 0$, $\phi^{(1)}(\rho(\chi)) = \sum_{\lambda \in W(\chi)} \lambda = 0$ and by Corollary 3.2, $\phi^{(2)}(I_m^W)$ is generated only by the W-invariant quadratic forms $\phi^{(2)}(\rho(\chi))$. For every $\chi \in \Lambda$ let

$$\phi^{(2)}(\rho(\chi)) = N_\chi \cdot q, \; N_\chi \in \mathbb{N}. \tag{1}$$

Then the subgroup $\phi^{(2)}(I_m^W)$ is a subgroup of $(I_a^W)^{(2)}$ of exponent

$$\tau_2 = \gcd_{\chi \in \Lambda} N_\chi. \quad \square$$

We now investigate the invariants of degree 3 and 4.

3.4. Lemma. *For every $\chi \in \Lambda$ we have*

$$\phi^{(3)}(\rho(\chi)) = \tfrac{1}{6} \sum_{\lambda \in W(\chi)} (\lambda^3 + 3\lambda(2)\lambda).$$

Proof. By the formula for $\phi^{(3)}$ in Example 1.4 and by Lemma 3.1 we obtain that

$$\phi^{(3)}(\rho(\chi)) = \tfrac{1}{6}\sum_{\lambda\in W(\chi)}(\lambda^3+3\lambda(2)\lambda+2\lambda(3)) = \tfrac{1}{6}\sum_{\lambda\in W(\chi)}(\lambda^3+3\lambda(2)\lambda). \quad \square$$

3.5. Lemma. *For every $\chi\in\Lambda$ we have*

$$\phi^{(4)}(\rho(\chi)) = \tfrac{1}{24}\sum_{\lambda\in W(\chi)}[\lambda^4+6\lambda(2)\lambda^2+8\lambda(3)\lambda+3\lambda(2)^2].$$

Proof. It follows from Example 1.4 and Lemma 3.1. $\square$

4. The Dynkin index

In the present section we show that the action of the Weyl group W of a crystallographic root system Φ on the weight lattice Λ has finite exponent in degree 2 which coincides with the Dynkin index of the respective Lie algebra.

Let W be the Weyl group of a crystallographic root system Φ and let Λ be its weight lattice as defined in [Hu, §2.9]. Let $\{\omega_1,\ldots,\omega_n\}$ be a basis of Λ consisting of fundamental weights (here n is the rank of Φ).

The Weyl group W acts on $\lambda\in\Lambda$ by means of simple reflections

$$s_j(\lambda) = \lambda - \langle\alpha_j^\vee,\lambda\rangle\cdot\alpha_j, \quad j=1\ldots n$$

where $\alpha_j^\vee$ is the j-th simple coroot and $\langle -,-\rangle$ is the usual pairing. Note that $\langle\alpha_j^\vee,\omega_i\rangle=\delta_{ij}$, where δ_{ij} is the Kronecker symbol.

The subring of invariants $\mathbb{Z}[\Lambda]^W$ is the representation ring of the respective Lie algebra $\mathfrak{g}$. By a theorem of Chevalley it is the polynomial ring in classes of fundamental representations $\mathrm{ch}(V_j)\in\mathbb{Z}[\Lambda]^W$, i.e.

$$\mathbb{Z}[\Lambda]^W \simeq \mathbb{Z}[\mathrm{ch}(V_1),\ldots,\mathrm{ch}(V_n)].$$

Note that every $\mathrm{ch}(V_l)$ is a sum of W-orbits $\rho(\chi)$ with some multiplicities.

Therefore, the image $\phi^{(i)}(I_m^W)$ is the i-th homogeneous component of the ideal generated by $\phi^{(j)}(\mathrm{ch}(V_l))$, $1\le j\le i$, $l=1\ldots n$.

4.1. Lemma. *We have $\Lambda^W=0$ and hence also*

$$\phi^{(1)}(\mathbb{Z}[\Lambda]^W)=\phi^{(1)}(I_m^W)=0.$$

Proof. Let $\eta\in\Lambda^W$. Since $\eta=s_{\alpha_j}(\eta)=\eta-\langle\eta,\alpha_j^\vee\rangle\alpha_j$ we have $\langle\eta,\alpha_j^\vee\rangle = \frac{2(\alpha_j,\eta)}{(\alpha_j,\alpha_j)}=0$ for all simple roots α_j which implies that $\eta=0$. $\square$

4.2. Lemma. *We have $S^2(\Lambda)^W=\langle q\rangle$.*

Proof. By [GN04, Prop. 4] there exists an integer valued W-invariant quadratic form on Λ which has value 1 on short coroots. As the group $S^2(\Lambda)^W$ is identical to the group of all integral W-invariant quadratic forms on $T_*\otimes\mathbb{R}$, the result follows. $\square$

4.3. Corollary. *The image $\phi^{(2)}(I_m^W)$ is a subgroup of $(I_a^W)^{(2)}$ of finite index.*

Proof. This follows from Corollary 3.3 and Lemma 4.1. □

We recall briefly the notion of indices of representations introduced by Dynkin [Dy57, §2] (See also [Br91]).

Let $f : \mathfrak{g} \to \mathfrak{g}'$ be a morphism between simple Lie algebras. Then there exists a unique number $j_f \in \mathbb{C}$, called the *Dynkin index of* f, satisfying

$$(f(x), f(y)) = j_f(x, y),$$

for all $x, y \in \mathfrak{g}$, where $(–,–)$ is the Killing form on $\mathfrak{g}$ and $\mathfrak{g}'$ normalized such that $(\alpha, \alpha) = 2$ for any long root α. In particular, if $f : \mathfrak{g} \to \mathfrak{sl}(V)$ is a linear representation, j_f is a positive integer, called the *Dynkin index of the linear representation* f, defined by

$$\mathrm{tr}(f(x), f(y)) = j_f(x, y).$$

The *Dynkin index of* $\mathfrak{g}$ is defined to be the greatest common divisor of all the Dynkin indices of all linear representations of $\mathfrak{g}$. By [Dy57, (2.24) and (2.25)], the Dynkin index of $\mathfrak{g}$ is the greatest common divisor of the Dynkin indexes j_l of its fundamental representations V_l, $l = 1 \ldots m$. All the Dynkin indexes j_l were calculated in [Dy57, Table 5]. We provide below the list of Dynkin indexes taken from [LS97, Prop. 2.6]:

type of $\mathfrak{g}$	A or C	B_n $(n \geq 3)$, D_n $(n \geq 4)$, G_2	F_4 or E_6	E_7	E_8
Dynkin index	1	2	6	12	60

Using the $\mathfrak{sl}_2$-representation theory, the Dynkin index of a linear representation $f : \mathfrak{g} \to \mathfrak{sl}(V)$ can be described as follows. Let α be a long root. For the formal character $\mathrm{ch}(V) = \sum_\lambda n_\lambda e^\lambda$, one has (see [LS97, Lemma 2.4] or [KNR, 5.1 and Lemma 5.2])

$$j_f = \frac{1}{2} \sum_\lambda n_\lambda \langle \lambda, \alpha^\vee \rangle^2.$$

4.4. Theorem. *The second exponent equals the Dynkin index of* $\mathfrak{g}$.

Proof. As explained at the beginning of this section, the image $\phi^{(2)}(I_m^W)$ is spanned by $\phi^{(2)}(\mathrm{ch}(V_l))$, where V_l is the l-th fundamental representation. It follows that τ_2 is the greatest common divisor of the integers N_l defined by $\phi^{(2)}(\mathrm{ch}(V_l)) = N_l \cdot q$ as in Corollary 3.3.

To find the precise value of τ_2 we use the explicit formula for $\phi^{(2)}(\rho(\chi))$ given in Corollary 3.2, that is

$$\phi^{(2)}(\rho(\chi)) = \tfrac{1}{2} \sum_{\lambda \in W(\chi)} \lambda^2.$$

Recall that $\mathrm{ch}(V_l)$ is a sum of W-orbits $\rho(\chi)$ of some $\chi \in \Lambda$ with some multiplicities. Evaluating $\phi^{(2)}(\mathrm{ch}(V_l))$ (considered as a linear combination of $\phi^{(2)}(\rho(\chi))$) at $\alpha^\vee$, where α is long, we obtain that $j_l = N_l q(\alpha^\vee) = N_l$. Therefore, $gcd(j_1, \ldots, j_n) = gcd(N_1, \ldots, N_n) = \tau_2$. □

We note that Theorem 4.4 was shown in [GZ10, §2] with a different proof.

5. Exponents of degrees 3 and 4

In the present section we show that $\tau_2 = N_3 = N_4$ for all crystallographic root systems, i.e. that the exponents τ_3 and τ_4 divide the Dynkin index of G.

Let $S = \{\lambda_1, \ldots, \lambda_r\}$ be a finite set of weights. We denote by $-S$ the set of opposite weights $\{-\lambda_1, \ldots, -\lambda_r\}$, by S_+ the set of sums $\{\lambda_i + \lambda_j\}_{i<j}$, by S_- the set of differences $\{\lambda_i - \lambda_j\}_{i<j}$ and by $S_\pm$ the disjoint union $S_+ \amalg S_-$. By definition we have $|S_+| = |S_-| = \binom{r}{2}$.

Using the fact that $(\lambda+\lambda')(m) = \lambda(m)+\lambda'(m)$ for every $\lambda, \lambda' \in \Lambda$ and $m \geq 0$ we obtain the following lemma which will be extensively used in the computations

5.1. Lemma. *(i) For every integer $m_1, m_2, x, y \geq 0$ and a finite subset $S \subset \Lambda$ we have*

$$\sum_{\lambda \in S \amalg -S} \lambda(m_1)^x \lambda(m_2)^y = (1 + (-1)^{x+y}) \sum_{\lambda \in S} \lambda(m_1)^x \lambda(m_2)^y.$$

In particular, $\sum_{\lambda \in S \amalg -S} \lambda(2)\lambda^2 = 0$.

(ii) For every subset $S \subset \Lambda$ with $|S| = r$ and for every $m_1, m_2 \geq 0$ we have

$$\sum_{\lambda \in S_+} \lambda(m_1)\lambda(m_2) = (r-1) \sum_{\lambda \in S} \lambda(m_1)\lambda(m_2) + \sum_{i \neq j} \lambda_i(m_1)\lambda_j(m_2) \text{ and}$$

$$\sum_{\lambda \in S_-} \lambda(m_1)\lambda(m_2) = (r-1) \sum_{\lambda \in S} \lambda(m_1)\lambda(m_2) - \sum_{i \neq j} \lambda_i(m_1)\lambda_j(m_2).$$

In particular, this implies that

$$\sum_{\lambda \in S_\pm} \lambda(m_1)\lambda(m_2) = 2(r-1) \sum_{\lambda \in S} \lambda(m_1)\lambda(m_2).$$

A_n-case. Let Φ be of type A_n for $n \geq 3$. We denote the canonical basis of $\mathbb{R}^{n+1}$ by e_i with $1 \leq i \leq n+1$. According to [Hu, §3.5 and §3.12] the basic polynomial invariants of the W-action on Λ (algebraically independent homogeneous generators of $S^*(\Lambda)^W$ as a $\mathbb{Q}$-algebra) are given by the symmetric power sums

$$q_i := e_1^i + \cdots + e_{n+1}^i, \quad 2 \leq i \leq n+1.$$

Let s_i denote the ith elementary symmetric function in $e_1, \ldots, e_{n+1}$. Using the classical identities

$$q_1 = s_1, \quad q_i = s_1 q_{i-1} - s_2 q_{i-2} + \ldots + (-1)^i s_{i-1} q_1 + (-1)^{i+1} i \cdot s_i, \quad 1 < i < n+1$$

and the fact that $s_1 = 0$, we obtain that

$$q_2/2 = -s_2, \; q_3/3 = s_3, \text{ and } q_4/2 = s_2^2 - 2s_4.$$

generate (with integral coefficients) the ideal I_a^W up to degree 4.

The fundamental weights of Φ can be expressed as follows

$$\omega_1 = e_1, \; \omega_2 = e_1 + e_2, \; \ldots \; , \omega_{n-1} = e_1 + \ldots + e_{n-1}, \; \omega_n = -e_{n+1},$$

where $e_1 + e_2 + \ldots + e_{n+1} = 0$. The orbits of ω_1, $\omega_1 + \omega_n$, ω_n and ω_2, ω_{n-1} under the action of the Weyl group $W = S_{n+1}$ are given by

$$W(\omega_1) = \{e_1, \ldots, e_{n+1}\} = -W(\omega_n),\ W(\omega_1 + \omega_n) = \{e_i - e_j\}_{i \neq j} \text{ and}$$

$$W(\omega_2) = \{e_i + e_j\}_{i<j} = -W(\omega_{n-1}).$$

Therefore, $W(\omega_1 + \omega_n) = S_- \amalg -S_-$ and $W(\omega_2) = S_+$, where $S = W(\omega_1)$.

Applying Lemma 3.5 and Lemma 5.1 we obtain that

$$\phi^{(4)}(\rho(\omega_1) + \rho(\omega_n)) = \tfrac{1}{12} \sum_{\lambda \in S} (\lambda^4 + 8\lambda(3)\lambda + 3\lambda(2)^2) \text{ and}$$

$$\phi^{(4)}(\rho(\omega_1 + \omega_n) + \rho(\omega_2) + \rho(\omega_{n-1})) = \tfrac{1}{24} \sum_{\lambda \in S_\pm \amalg -S_\pm} (\lambda^4 + 8\lambda(3)\lambda + 3\lambda(2)^2) =$$

$$= \tfrac{1}{24} \sum_{\lambda \in S_\pm \amalg -S_\pm} \lambda^4 + \tfrac{n}{6} \sum_{\lambda \in S} (8\lambda(3)\lambda + 3\lambda(2)^2).$$

Then the difference

$$\phi^{(4)}(\rho(\omega_1 + \omega_n) + \rho(\omega_2) + \rho(\omega_{n-1})) - 2n \cdot \phi^{(4)}(\rho(\omega_1) + \rho(\omega_n)) =$$

$$= \tfrac{1}{24} \sum_{\lambda \in S_\pm \amalg -S_\pm} \lambda^4 - \tfrac{n}{6} \sum_{\lambda \in S} \lambda^4 = \tag{2}$$

is a symmetric function in $e_1, \ldots, e_{n+1}$ and, therefore, it can be always written as a polynomial in q_is. Indeed, since

$$\sum_{\lambda \in S_\pm \amalg -S_\pm} \lambda^4 = 2 \sum_{i<j} ((e_i + e_j)^4 + (e_i - e_j)^4) = 4n \sum_{\lambda \in S} \lambda^4 + 24 \sum_{i<j} e_i^2 e_j^2,$$

the difference (2) equals

$$= \sum_{i<j} e_i^2 e_j^2 = (q_2^2 - q_4)/2.$$

5.2. Lemma. *For a root system of type A_n, $n \geq 2$, we have $\tau_2 = \tau_3 = \tau_4 = 1$.*

Proof. It is enough to show that the generators $q_2/2$, $q_3/3$ and $q_4/2$ are in the ideal generated by the image of $\phi^{(i)}$, $i \leq 4$.

By Corollary 3.2 we have $\phi^{(2)}(\rho(\omega_1)) = \frac{1}{2} \sum_{\lambda \in S} \lambda^2 = q_2/2$. By Lemma 3.4 we have $q_3/3 = \phi^{(3)}(\rho(\omega_1)) - \phi^{(3)}(\rho(\omega_n))$ (see also [GZ10, §1C]). If Φ is of type A_2, then $s_4 = 0$ and, hence, $q_4 = q_2^2/2$. If Φ is of type A_n, $n \geq 3$, then by (2) the generator $q_4/2$ belongs to the ideal generated by the images of $\phi^{(2)}$ and $\phi^{(4)}$. $\square$

B_n, C_n and D_n cases. Let Φ be of type B_n or C_n for $n \geq 2$ or of type D_n for $n \geq 4$. We denote the canonical basis of $\mathbb{R}^n$ by e_i with $1 \leq i \leq n$. By [Hu, §3.5 and §3.12] the basic polynomial invariants of the W-action on Λ are given by even power sums

$$q_{2i} := e_1^{2i} + \cdots + e_n^{2i}, \quad 1 \leq i \leq n,$$

together with $p_n := e_1 \cdots e_n$ if Φ is of type D_n.

The first two fundamental weights of Φ are given by $\omega_1 = e_1$, $\omega_2 = e_1 + e_2$ and their W-orbits are

$$W(\omega_1) = \{\pm e_1, \dots, \pm e_n\} \text{ and } W(\omega_2) = \{\pm e_i \pm e_j\}_{i<j}.$$

Hence $W(\omega_1) = S \amalg -S$ and $W(\omega_2) = S_\pm \amalg -S_\pm$, where $S = \{e_1, \dots, e_n\}$.

Applying Lemma 3.5 and Lemma 5.1 we obtain that

$$\phi^{(4)}(\rho(\omega_1)) = \tfrac{1}{12} \sum_{\lambda \in S} \lambda^4 + \tfrac{1}{12} \sum_{\lambda \in S} (8\lambda(3)\lambda + 3\lambda(2)^2) \text{ and}$$

$$\phi^{(4)}(\rho(\omega_2)) = \tfrac{1}{24} \sum_{\lambda \in S_\pm \amalg -S_\pm} \lambda^4 + \tfrac{n-1}{6} \sum_{\lambda \in S} (8\lambda(3)\lambda + 3\lambda(2)^2).$$

Then similar to the A_n-case we obtain

$$\phi^{(4)}(\rho(\omega_2)) - 2(n-1)\phi^{(4)}(\rho(\omega_1)) = (q_2^2 - q_4)/2, \tag{3}$$

where $q_i = e_1^i + \ldots + e_n^i$ and

$$-\phi^{(4)}(\rho(\omega_3)) + \phi^{(4)}(\rho(\omega_4)) = p_4, \tag{4}$$

if Φ is of type D_4.

5.3. Lemma. *For a root system of type B_n or C_n, $n \geq 2$ or D_n, $n \geq 4$ the exponents τ_3 and τ_4 divide the Dynkin index τ_2.*

Proof. Since there are no basic polynomial invariants in degree 3 [Hu, §3.7 Table 1] we have $\tau_3 \mid \tau_2 = 2$. For D_4, by (4) the invariant p_4 is in the ideal generated by the image of $\phi^{(4)}$. Hence, to show that $\tau_4 \mid \tau_2$ it is enough to show that $q_4/2$ is in the ideal generated by the image of $\phi^{(2)}$ and $\phi^{(4)}$. Indeed, by Corollary 3.2 we have $\phi^{(2)}(\rho(\omega_1)) = \sum_{\lambda \in S} \lambda^2 = q_2$. Therefore, by (3)

$$q_4/2 = (q_2/2) \cdot \phi^{(2)}(\rho(\omega_1)) - \phi^{(4)}(\rho(\omega_2)) + 2(n-1)\phi^{(4)}(\rho(\omega_1)). \quad \square$$

5.4. Theorem. *For every crystallographic root system Φ the exponents τ_3 and τ_4 divide the Dynkin index τ_2.*

Proof. If Φ is of type A_n, this follows from Lemma 5.2. If Φ is of type B_n, C_n or D_n this follows from Lemma 5.3; for all other types τ_3 and τ_4 divide τ_2 since there are no basic polynomial invariants of degree 3 and 4 (see [Hu, §3.7 Table 1]). $\square$

6. Torsion in the Grothendieck γ-filtration

The goal of the present section is to provide geometric interpretation (see (6)) of the map ϕ_i and the exponents τ_i.

Let G be a split simple simply-connected group over a field k. We fix a maximal split torus T of G and a Borel subgroup $B \supset T$. Let Λ be the group of characters of T. Since G is simply-connected, Λ coincides with the weight lattice of G.

Let X denote the variety of Borel subgroups of G (conjugate to B). Consider the Chow ring $\mathrm{CH}^*(X)$ of algebraic cycles modulo rational equivalence and the Grothendieck ring $K_0(X)$. Following [De74, §1] to every character $\lambda \in \Lambda$ we may associate the line bundle $\mathcal{L}(\lambda)$ over X. It induces the ring homomorphisms (called the characteristic maps)

$$\mathfrak{c}_a \colon S^*(\Lambda) \to \mathrm{CH}^*(X) \text{ and } \mathfrak{c}_m \colon \mathbb{Z}[\Lambda] \twoheadrightarrow K_0(X)$$

by sending $\lambda \mapsto c_1(\mathcal{L}(\lambda))$ and $e^\lambda \mapsto [\mathcal{L}(\lambda)]$ respectively. Note that the map $\mathfrak{c}_a$ is an isomorphism in codimension one, hence, giving

$$\mathfrak{c}_a \colon S^1(\Lambda) = \Lambda \xrightarrow{\simeq} Pic(X) = \mathrm{CH}^1(X)$$

and the map $\mathfrak{c}_m$ is surjective. Let W be the Weyl group and let I_a^W and I_m^W denote the respective W-invariant ideals. Then according to [De73, §4 Cor.2,§9] and [CPZ, §6]

$$\ker \mathfrak{c}_m = I_m^W \tag{5}$$

and $\ker \mathfrak{c}_a$ is generated by elements of $S^*(\Lambda)$ such that their multiples are in I_a^W.

Consider the Grothendieck γ-filtration on $K_0(X)$ (see [GZ10, §1]). Its ith term is an ideal generated by products

$$\gamma^i(X) := \langle (1-[\mathcal{L}_1^\vee])(1-[\mathcal{L}_2^\vee]) \cdot \ldots \cdot (1-[\mathcal{L}_i^\vee])\rangle,$$

where $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_i$ are line bundles over X. Consider the ith subsequent quotient $\gamma^i(X)/\gamma^{i+1}(X)$. The usual Chern class c_i induces a group homomorphism $c_i \colon \gamma^i(X)/\gamma^{i+1}(X) \to \mathrm{CH}^i(X)$.

6.1. Proposition. *For every $i \geq 0$ there is a commutative diagram of group homomorphisms*

$$\begin{tikzcd} I_m^i/I_m^{i+1} \arrow[r, "(-1)^{i-1}(i-1)!\cdot\phi_i"] \arrow[d, "\mathfrak{c}_m"] & S^i(\Lambda) \arrow[d, "\mathfrak{c}_a"] \\ \gamma^i(X)/\gamma^{i+1}(X) \arrow[r, "c_i"] & \mathrm{CH}^i(X) \end{tikzcd} \tag{6}$$

Proof. Indeed, the γ-filtration on $K_0(X)$ is the image of the I_m-adic filtration on $\mathbb{Z}[\Lambda]$, i.e. $\gamma^i(X) = \mathfrak{c}_m(I_m^i)$ for every $i \geq 0$. The Proposition then follows from the identity

$$c_i\Big((1-[\mathcal{L}_1^\vee])(1-[\mathcal{L}_2^\vee])\ldots(1-[\mathcal{L}_i^\vee])\Big) = (-1)^{i-1}(i-1)!\cdot c_1(\mathcal{L}_1)c_1(\mathcal{L}_2)\ldots c_1(\mathcal{L}_i),$$

where $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_i$ are line bundles over X and $\mathcal{L}_i^\vee$ denotes the dual of $\mathcal{L}_i$. $\square$

6.2. Remark. Note that $\mathbb{Z}[\Lambda]$ can be identitfied with the T-equivariant K_0 of a point $pt = Spec\, k$ and $S^*(\Lambda)$ with the T-equivariant CH of a point (see [GZ11]). The maps $\mathfrak{c}_a$ and $\mathfrak{c}_m$ then can be identified with the pull-backs $K_0^T(pt) \to K_0^T(G)$ and $\mathrm{CH}_T(pt) \to \mathrm{CH}_T(G)$ induced by the structure map $G \to pt$.
In view of these identifications the map ϕ_i can be viewed as an equivariant analogue of the Chern class map c_i.

Consider the diagram (6) with $\mathbb{Q}$-coefficients. In this case the Chern class map c_i will become an isomorphism (by the Riemann-Roch theorem), the characteristic map $\mathfrak{c}_a$ will turn into a surjection and the map $(-1)^{i-1}(i-1)! \cdot \phi_i$ will be an isomorphism as well. In view of (5) we obtain an isomorphism

$$\phi^{(i)} \otimes \mathbb{Q} \colon I_m^W \cap I_m^i / I_m^W \cap I_m^{i+1} \otimes \mathbb{Q} \longrightarrow (I_a^W)^{(i)} \otimes \mathbb{Q}$$

on the kernels of $\mathfrak{c}_m$ and $\mathfrak{c}_a$. By the very definition of the exponents τ_i this implies that

6.3. Corollary. *The action of the Weyl group of a crystallograhic root system has finite exponent τ_i for every i.*

6.4. Lemma. *We have $(\ker \mathfrak{c}_a)^{(i)} = (I_a^W)^{(i)}$ for each $i \leq 4$ except the case $i = 4$ and G is of type B_n $(n \geq 3)$ or D_n $(n \geq 5)$ where we have $2(\ker \mathfrak{c}_a)^{(4)} \subseteq (I_a^W)^{(4)}$.*

Proof. The statement follows by the same analysis as in [GZ10, §1B]. For the exception it is enough to show that the polynomial $P = q \cdot f_2 + d \cdot (q_4/2)$ in ω_i-s is not divisible by 4, where $d \in \mathbb{Z}$, f_2 is a polynomial of degree 2, $q_4/2$ is the basic polynomial invariant of degree 4 and $g.c.d.(f_2, d) = 1$.
Assume that $4 \mid P$, we claim that in this case $g.c.d.(f_2, d) = 2$. Indeed, let $f_2 = \sum_{i=1}^n a_i\omega_i^2 + \sum_{i<j} a_{ij}\omega_i\omega_j$, $a_i, a_{ij} \in \mathbb{Z}$. Take ω_i and ω_j corresponding to adjacent long roots. Set $\omega_k = 0$ for $k \neq i, j$. Then the congruence $P \equiv 0\,(mod\, 4)$ turns into

$$(\omega_i^2 - \omega_i\omega_j + \omega_j^2)(a_i\omega_i^2 + a_{ij}\omega_i\omega_j + a_j\omega_j^2) + d(\omega_i^4 - 2\omega_i^3\omega_j + 3\omega_i^2\omega_j^2 - 2\omega_i\omega_j^3 + \omega_j^4) \equiv 0$$

which gives $a_i \equiv a_j \equiv -d$, $a_{ij} - a_i \equiv a_{ij} - a_j \equiv -2d$ and $a_i - a_{ij} + a_j \equiv 3d$. This implies that $2d \equiv 0$, therefore, $2 \mid d$. Finally, since q is indivisible, $2 \mid f_2$.
In the D_4-case let $Q = q \cdot f_2 + d \cdot (q_4/2) + e \cdot p_4$ with $g.c.d.(f_2, d, e) = 1$. If $4 \mid Q$, then we have $d \equiv a_i \equiv 0\,(mod\, 2)$ by the same argument. Hence, $2 \mid q \cdot f_2 + e \cdot p_4$. Set $\omega_2 = 0$. Then we have

$$(\omega_1^2 + \omega_3^2 + \omega_4^2) f_2 \mid_{\omega_2 = 0} + e(\omega_1^2\omega_3^2 - \omega_1^2\omega_4^2) \equiv 0\,(mod\, 2).$$

In particular, $2 \mid a_1 + a_3 + e$. As $2 \mid a_i$, we have $2 \mid e$, which implies that $2 \mid f_2$. $\square$

We are now ready to prove the main result of this section

6.5. Theorem. *The integer $\tau_i \cdot (i-1)!$ annihilates the torsion of the ith subsequent quotient $\gamma^i(X)/\gamma^{i+1}(X)$ of the γ-filtration on $K_0(X)$ for $i = 2,3,4$ except the case $i = 4$ and G is of type B_n $(n \geq 3)$ or D_n $(n \geq 5)$ where the torsion of $\gamma^4(X)/\gamma^5(X)$ is annihilated by 24.*

6.6. Remark. Note that by [SGA6, Exposé XIV, 4.5] for groups of types A_n and C_n the quotients $\gamma^i(X)/\gamma^{i+1}(X)$ have no torsion.

Proof. Assume that α is a torsion element in $\gamma^i(X)/\gamma^{i+1}(X)$. Then $c_i(\alpha) = 0$ since $\mathrm{CH}^i(G/B)$ has no torsion. Let $\tilde{\alpha}$ be a preimage of α via $\mathfrak{c}_m$ in $I_m^i/I_m^{i+1} \subseteq \mathbb{Z}[\Lambda]/I_m^{i+1}$. By (6) we obtain that

$$(i-1)!\,\phi_i(\tilde{\alpha}) \in (\ker \mathfrak{c}_a)^{(i)}$$

where $(\ker \mathfrak{c}_a)^{(i)}$ coincides with $(I_a^W)^{(i)}$ up to a multiple (see Lemma 6.4). By definition of the index τ_i we have

$$\tau_i \cdot (i-1)!\,\phi_i(\tilde{\alpha}) = \phi_i(\beta), \text{ where } \beta \in I_m^W/I_m^{i+1} \cap I_m^W.$$

Applying ϕ_i^{-1} to the both sides we obtain

$$\tau_i \cdot (i-1)! \cdot \tilde{\alpha} = \beta \in I_m^W/I_m^{i+1} \cap I_m^W$$

Applying $\mathfrak{c}_m$ to the both sides and observing that $I_m^W = \ker \mathfrak{c}_m$ we obtain that $\tau_i \cdot (i-1)! \cdot \alpha = 0$. □

Let ${}_\xi X$ be a twisted form of the variety X by means of a cocycle $\xi \in Z^1(k, G)$. By [Pa94, Thm. 2.2.(2)] the restriction map $K_0({}_\xi X) \to K_0(X)$ (here we identify $K_0(X)$ with the $K_0(X \times_k \bar{k})$ over the algebraic closure $\bar{k}$) is an isomorphism. Since the characteristic classes commute with restrictions, this induces an isomorphism between the γ-filtrations, i.e. $\gamma^i({}_\xi X) \simeq \gamma^i(X)$ for every $i \geq 0$, and between the respective quotients

$$\gamma^i({}_\xi X)/\gamma^{i+1}({}_\xi X) \simeq \gamma^i(X)/\gamma^{i+1}(X) \quad \text{for every } i \geq 0.$$

In view of this fact Theorem 6.5 implies that

6.7. Corollary. *Let G be a split simple simply connected group of type B_n $(n \geq 3)$ or D_n $(n \geq 4)$. Then for every $\xi \in Z^1(k, G)$ the torsion in $\gamma^4({}_\xi X)/\gamma^5({}_\xi X)$ is annihilated by 24.*

Consider the topological filtration on $K_0(Y)$, where Y is a smooth projective variety, given by the ideals

$$\tau^i(Y) := \langle [\mathcal{O}_V] \mid V \hookrightarrow Y,\, codim_V Y \geq i \rangle.$$

It is known (see [FuLa, Ch.V, Thm. 3.9]) that $\gamma^i(Y) \subseteq \tau^i(Y)$ for every $i \geq 0$.

Given an Abelian group M let $e(M)$ denote the exponent of its torsion subgroup. The following exact sequences of Abelian groups

(7) (i) $\gamma^i/\gamma^{i+1} \hookrightarrow \tau^i/\gamma^{i+1} \twoheadrightarrow \tau^i/\gamma^i$ and (ii) $\tau^{i+1}/\gamma^{i+1} \hookrightarrow \tau^i/\gamma^{i+1} \twoheadrightarrow \tau^i/\tau^{i+1}$,

where $\tau^i = \tau^i(Y)$, $\gamma^i = \gamma^i(Y)$, lead to the recursive divisibility for each $i \geq 1$

$$e(\tau^i/\gamma^{i+1}) \mid e(\gamma^i/\gamma^{i+1}) \cdot e(\tau^i/\gamma^i) \mid e(\gamma^i/\gamma^{i+1}) \cdot e(\tau^{i-1}/\gamma^i)$$

which gives

$$e(\tau^i/\gamma^{i+1}) \mid e(\gamma^i/\gamma^{i+1}) \cdot e(\gamma^{i-1}/\gamma^i) \cdot \ldots \cdot e(\gamma^1/\gamma^2). \tag{8}$$

By the Riemann-Roch theorem [Fu, Ex.15.3.6], the composition

$$\mathrm{CH}^i(Y) \to \tau^i/\tau^{i+1} \xrightarrow{c_i} \mathrm{CH}^i(Y)$$

is the multiplication by $(-1)^{i-1}(i-1)!$, therefore, by (7).(ii) the torsion subgroup of $\mathrm{CH}^i(Y)$ is annihilated by $(i-1)! \cdot e(\tau^i/\tau^{i+1}) \mid (i-1)! \cdot e(\tau^i/\gamma^{i+1})$. Combining this with the formula (8) and Theorem 6.5 we obtain

6.8. Corollary. *Let G be a split simple simply connected group. Then for every $\xi \in Z^1(k,G)$ the torsion in $\mathrm{CH}^i({}_\xi X)$ for $i = 2,3,4$ is annihilated by the integer*

$$(i-1)! \cdot \prod_{j=2}^{i} \tau_j (j-1)!$$

except for $i = 4$ and G is of type B_n $(n \geq 3)$ or D_n $(n \geq 5)$ where it is annihilated by 2^7.

References

[B4-6] Bourbaki, N. *Lie groups and Lie algebras: Chapters* 4-6, Springer-Verlag, Berlin, (2002).

[B7-9] ———, *Lie groups and Lie algebras: Chapters* 7-9, Spinger-Verlag, Berlin 2005.

[Br91] Braden, H. W. *Integral pairings and Dynkin indices*, J. London Math. Soc. (2) 43 (1991), 313–323.

[CPZ] Calmès, B., Petrov, V., Zainoulline, K. *Invariants, torsion indices and oriented cohomology of complete flags*, Preprint arXiv:0905.1341 (2010), 36pp.

[De74] Demazure, M. *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. (4) 7 (1974), 53–88.

[De73] Demazure, M. *Invariants symétriques entiers des groupes de Weyl et torsion*, Invent. Math. 21 (1973), 287–301.

[Dy57] Dynkin, E. B. *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl., Ser. II, 6 (1957), 111–244.

[Fu] Fulton, W. *Intersection Theory*, Springer-Verlag, New-York (1984).

[FuLa] Fulton, W., Lang, S. *Riemann-Roch Algebra*, Springer-Verlag, New-York (1985).

[GZ10] Garibaldi, S., Zainoulline, K. *The gamma-filtration and the Rost invariant*, Preprint arXiv:1007.3482 (2010), 19pp.

[GZ11] Gille, S., Zainoulline, K. *Equivariant pretheories and invariants of torsors.* to appear in Transf. groups.

[GN04] Gross, B. H., Nebe, G. *Globally maximal arithmetic groups*, J. Algebra 272 (2004), no. 2, 625–642.

[Hu] Humphreys, J. *Reflection groups and Coxeter groups.* Cambridge studies in Advanced Math. 29, Cambridge Univ. Press (1990).

[KNR] Kumar, S., Narasimhan, M. S., Ramanathan, A. *Infinite Grassmannians and moduli spaces of G-bundles*, Math. Ann. 300 (1994), 41–75.

[LS97] Laszlo, Y., Sorger, C. *The line bundles on the moduli of parabolic G-bundles over curves and their sections*, Ann. Ec. Norm. Sup. (4) 30 (1997), 499–525.

[Pa94] Panin, I. A. *On the algebraic K-theory of twisted flag varieties*, K-Theory 8 (1994), no. 6, 541–585.

[SGA6] SGA 6, Théorie des intersections et Théorème de Riemann-Roch, Lecture Notes in Math. 225, Springer-Verlag. 1971.

Sanghoon Baek
Department of Mathematics
and Statistics
University of Ottawa
sbaek@uottawa.ca

Erhard Neher
Department of Mathematics
and Statistics
University of Ottawa
erhard.neher@uottawa.ca

Kirill Zainoulline
Department of Mathematics
and Statistics
University of Ottawa
585 King Edward
Ottawa ON K1N 6N5
Canada
kirill@uottawa.ca