Bott Periodicity for Inclusions of Symmetric Spaces

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Abstract. When looking at Bott's original proof of his periodicity theorem for the stable homotopy groups of the orthogonal and unitary groups, one sees in the background a differential geometric periodicity phenomenon. We show that this geometric phenomenon extends to the standard inclusion of the orthogonal group into the unitary group. Standard inclusions between other classical Riemannian symmetric spaces are considered as well. An application to homotopy theory is also discussed.

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Contents

1 Introduction

Bott's original proof of his periodicity theorem [Bo-59] is differential geometric in its nature. It relies on the observation that in a compact Riemannian symmetric space P one can choose two points p and q in "special position" such that the connected components of the space of shortest geodesics in P joining p and q are again compact symmetric spaces. Set $P_0 = P$ and let P_1 be one of the resulting connected components. This construction can be repeated inductively: given points p_j, q_j in "special position" in P_j, then P_{j+1} is one of the connected components of the space of shortest geodesic segments in P_j between p_j and q_j. If we start this iterative process with the classical groups

$$P_0 := \mathrm{SO}_{16n}, \quad \tilde{P}_0 := \mathrm{U}_{16n}, \quad \bar{P}_0 := \mathrm{Sp}_{16n}$$

and make at each step appropriate choices of the two points and of the connected component, one obtains

$$P_8 = \mathrm{SO}_n, \quad \tilde{P}_2 = \mathrm{U}_{8n}, \quad \bar{P}_8 = \mathrm{Sp}_n.$$

Each of the three processes can be continued, provided that n is divisible by a sufficiently high power of 2. We obtain (periodically) copies of a special orthogonal, unitary, and symplectic group after every eighth, second, respectively eighth iteration. These purely geometric periodicity phenomena are the key ingredients of Bott's proof of his periodicity theorems [Bo-59] for the stable homotopy groups $\pi_i(\mathrm{O})$, $\pi_i(\mathrm{U})$, and $\pi_i(\mathrm{Sp})$ (see also the remark at the end of this section).

In his book [Mi-69], Milnor constructed totally geodesic embeddings

$$P_{k+1} \subset P_k, \quad \tilde{P}_{k+1} \subset \tilde{P}_k, \quad \bar{P}_{k+1} \subset \bar{P}_k,$$

for all $k = 0, 1, \ldots, 7$. In each case, the inclusion is given by the map which assigns to a geodesic its midpoint (cf. [Qu-10] and [Ma-Qu-10], see also Section 2 below).

The goal of this paper is to establish connections between the following three chains of symmetric spaces:

$$\begin{aligned} &P_0 \supset P_1 \supset P_2 \supset \ldots \supset P_8, \\ &\tilde{P}_0 \supset \tilde{P}_1 \supset \tilde{P}_2 \supset \ldots \supset \tilde{P}_8, \\ &\bar{P}_0 \supset \bar{P}_1 \supset \bar{P}_2 \supset \ldots \supset \bar{P}_8. \end{aligned}$$

We will refer to them as the SO-, U-, respectively Sp-*Bott chains.* Starting with the natural inclusions

$$P_0 = \mathrm{SO}_{16n} \subset \mathrm{U}_{16n} = \tilde{P}_0 \quad \text{and} \quad \tilde{P}_0 = \mathrm{U}_{16n} \subset \mathrm{Sp}_{16n} = \bar{P}_0$$

we show that the iterative process above provides inclusions

$$P_j \subset \tilde{P}_j \quad \text{and} \quad \tilde{P}_j \subset \bar{P}_j$$

for all $j = 0, 1, \dots, 8$. These are all canonical reflective inclusions of symmetric spaces, i. e. they can be realized as fixed point sets of isometric involutions (see Appendix A, especially Tables 5 and 6 and Subsections A.1 - A.16) and make the following diagrams commutative:

$$\begin{tikzcd}
P_0 \arrow[d, "\cap"] & P_1 \arrow[l, "\supset"'] \arrow[d, "\cap"] & P_2 \arrow[l, "\supset"'] \arrow[d, "\cap"] & \cdots \arrow[l, "\supset"'] & P_8 \arrow[l, "\supset"'] \arrow[d, "\cap"] \\
\tilde{P}_0 & \tilde{P}_1 \arrow[l, "\supset"'] & \tilde{P}_2 \arrow[l, "\supset"'] & \cdots \arrow[l, "\supset"'] & \tilde{P}_8 \arrow[l, "\supset"']
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{P}_0 \arrow[d, "\cap"] & \tilde{P}_1 \arrow[l, "\supset"'] \arrow[d, "\cap"] & \tilde{P}_2 \arrow[l, "\supset"'] \arrow[d, "\cap"] & \cdots \arrow[l, "\supset"'] & \tilde{P}_8 \arrow[l, "\supset"'] \arrow[d, "\cap"] \\
\bar{P}_0 & \bar{P}_1 \arrow[l, "\supset"'] & \bar{P}_2 \arrow[l, "\supset"'] & \cdots \arrow[l, "\supset"'] & \bar{P}_8 \arrow[l, "\supset"']
\end{tikzcd}$$

Moreover, the vertical inclusions are periodic, with period equal to 8. Concretely, we show that up to isometries, the inclusions

$$P_8 \subset \tilde{P}_8 \quad \text{and} \quad \tilde{P}_8 \subset \bar{P}_8$$

are again the natural inclusions

$$\mathrm{SO}_n \subset \mathrm{U}_n \quad \text{and} \quad \mathrm{U}_n \subset \mathrm{Sp}_n$$

(see Theorems 4.1, 4.3 and Remark 4.5 below). We mention that all inclusions in the two diagrams above are actually reflective. For example, notice that $P_4 = \mathrm{Sp}_{2n}$, $\tilde{P}_4 = \mathrm{U}_{4n}$, and $\bar{P}_4 = \mathrm{SO}_{8n}$; the inclusions

$$P_4 \subset \tilde{P}_4 \quad \text{and} \quad \tilde{P}_4 \subset \bar{P}_4$$

are essentially the usual subgroup inclusions

$$\mathrm{Sp}_{2n} \subset \mathrm{U}_{4n} \quad \text{and} \quad \mathrm{U}_{4n} \subset \mathrm{SO}_{8n}$$

(see Remark 4.4).

Remark. We recall that the celebrated periodicity theorem of Bott [Bo-59] concerns the stable homotopy groups $\pi_i(\mathrm{O})$, $\pi_i(\mathrm{U})$, and $\pi_i(\mathrm{Sp})$ of the orthogonal, unitary, respectively symplectic groups. Concretely, one has the following group isomorphisms:

$$\pi_i(\mathrm{O}) \simeq \pi_{i+8}(\mathrm{O}), \quad \pi_i(\mathrm{U}) \simeq \pi_{i+2}(\mathrm{U}), \quad \pi_i(\mathrm{Sp}) \simeq \pi_{i+8}(\mathrm{Sp}),$$

for all $i \geq 0$. If we now consider the standard inclusions

$$\mathrm{O}_n \subset \mathrm{U}_n \quad \text{and} \quad \mathrm{U}_n \subset \mathrm{Sp}_n \tag{1.1}$$

then the maps induced between homotopy groups, that is $\pi_i(\mathrm{O}_n) \to \pi_i(\mathrm{U}_n)$ and $\pi_i(\mathrm{U}_n) \to \pi_i(\mathrm{Sp}_n)$ are stable relative to n within the "stability range". One can see that the resulting maps

$$f_i : \pi_i(\mathrm{O}) \to \pi_i(\mathrm{U}) \quad \text{and} \quad g_i : \pi_i(\mathrm{U}) \to \pi_i(\mathrm{Sp}),$$

are periodic in the following sense:

$$f_{i+8} = f_i \quad \text{and} \quad g_{i+8} = g_i. \tag{1.2}$$

These facts are basic in homotopy theory and can be proved using techniques described e.g. in [May-77, Ch. 1]. We provide an alternative, more elementary proof of Equation (1.2) and determine the maps f_i and g_i explicitly, by using only the long exact homotopy sequence of the principal bundles $\mathrm{O}_n \to \mathrm{U}_n \to \mathrm{U}_n/\mathrm{O}_n$ and $\mathrm{U}_n \to \mathrm{Sp}_n \to \mathrm{Sp}_n/\mathrm{U}_n$, combined with the explicit knowledge of the stable homotopy groups of O, U, Sp, U/O, and Sp/U (the details can be found in Section 5, see especially Theorems 5.3 and 5.6, Remarks 5.4 and 5.7, and Tables 1 and 3). The present paper shows that the results stated by Equation (1.2) are just direct consequences of the abovementioned differential geometric periodicity phenomenon, in the spirit of Bott's original proof of his periodicity theorems. Besides the inclusions given by Equation (1.1) we will also consider the following ones, which are described in detail in Appendix A, Subsections A.1 - A.16:

$$\begin{array}{lll}
\mathrm{O}_{2n}/\mathrm{U}_n \subset \mathrm{G}_n(\mathbb{C}^{2n}), & \mathrm{U}_{2n}/\mathrm{Sp}_n \subset \mathrm{U}_{2n}, & \mathrm{G}_n(\mathbb{H}^{2n}) \subset \mathrm{G}_{2n}(\mathbb{C}^{4n}), \\
\mathrm{Sp}_n \subset \mathrm{U}_{2n}, & \mathrm{Sp}_n/\mathrm{U}_n \subset \mathrm{G}_n(\mathbb{C}^{2n}), & \mathrm{U}_n/\mathrm{O}_n \subset \mathrm{U}_n, \\
\mathrm{G}_n(\mathbb{R}^{2n}) \subset \mathrm{G}_n(\mathbb{C}^{2n}), & \mathrm{G}_n(\mathbb{C}^{2n}) \subset \mathrm{Sp}_{2n}/\mathrm{U}_n, & \mathrm{U}_n \subset \mathrm{U}_{2n}/\mathrm{O}_{2n}, \\
\mathrm{G}_n(\mathbb{C}^{2n}) \subset \mathrm{G}_{2n}(\mathbb{R}^{4n}), & \mathrm{U}_n \subset \mathrm{O}_{2n}, & \mathrm{G}_n(\mathbb{C}^{2n}) \subset \mathrm{O}_{4n}/\mathrm{U}_{2n}, \\
\mathrm{U}_n \subset \mathrm{U}_{2n}/\mathrm{Sp}_n, & \mathrm{G}_n(\mathbb{C}^{2n}) \subset \mathrm{G}_n(\mathbb{H}^{2n}). &
\end{array}$$

For each of them we will prove a periodicity result similar to those described by Equation (1.1). The precise statements are Corollaries 5.5 and 5.8.

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2 Bott periodicity from a geometric viewpoint

In this section we review the original (geometric) proof of Bott's periodicity theorem. We adapt the original treatment in [Bo-59] to our needs and therefore change it slightly. More precisely, we will use ideas of Milnor [Mi-69], as well as the concept of centriole, which was defined by Chen and Nagano [Ch-Na-88] (see also [Na-88], [Na-Ta-91], and [Bu-92]).

2.1 The geometry of centrioles.

Let P be a compact connected symmetric space and o a point in P. We say that (P, o) is a *pointed symmetric space*. As already mentioned in the introduction, a key role is played by the space of all shortest geodesic segments in P from o to a point in P which belongs to a certain "special" class. It turns out that this class consists of the poles of (P, o), (cf. [Qu-10] and [Ma-Qu-10]). The notion of pole is described by the following definition. First, for any $p \in P$ we denote by $s_p : P \to P$ the corresponding geodesic symmetry.

Definition 2.1 *A* pole *of the pointed symmetric space (P, o) is a point $p \in P$ with the property that $s_p = s_o$ and $p \neq o$.*

Let G be the identity component of the isometry group of P. This group acts transitively on P. We denote by K the G-stabilizer of o and by K_e its identity component. The following result is related to [Lo-69, Vol. II, Ch. VI, Proposition 2.1 (b)].

Lemma 2.2 *If p is a pole of (P, o), then $k.p = p$ for all $k \in K_e$.*

Proof. The map $\sigma : G \to G$, $\sigma(g) = s_o g s_o$ is an involutive group automorphism of G whose fixed point set G^σ has the same identity component K_e as K. Since p is a pole, we have $\sigma(g) = s_p g s_p$ and the fixed point set G^σ has the same identity component as the stabilizer G_p of p in G. Consequently, $K_e \subset G_p$. □

Example 2.3 Any compact connected Lie group G can be equipped with a bi-invariant metric and becomes in this way a Riemannian symmetric space (cf. e.g. [Mi-69, Section 21]). The geodesic symmetry at $g \in G$ is the map $s_g : G \to G$, $s_g(x) = gx^{-1}g$, $x \in G$. An immediate consequence is a description of the poles of G: they are exactly those g which lie in the center of G and whose square is equal to the identity of G. We also note that the identity component of the isometry group of G is $G \times G/\Delta(Z(G))$, where $\Delta(Z(G)) := \{(z, z) \ : \ z \in Z(G)\}$. Here $G \times G$ acts on G via

$$(g_1, g_2).h := g_1 h g_2^{-1} \quad g_1, g_2, h \in G \tag{2.1}$$

and the kernel of this action is equal to $\Delta(Z(G))$. Finally, the stabilizer of the identity element e of G is $\Delta(G)/\Delta(Z(G))$.

Remark 2.4 Not any pointed compact symmetric space admits a pole. For example, consider the Grassmannian $\mathrm{G}_k(\mathbb{K}^{2m})$, where $0 \leq k \leq 2m$ and $\mathbb{K} =$

$\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. It has a canonical structure of a Riemannian symmetric space. One can show that $\mathrm{G}_k(\mathbb{K}^{2m})$ has a pole if and only if $k = m$. Indeed, let us first consider an element V of $\mathrm{G}_m(\mathbb{K}^{2m})$. Then a pole of the pointed symmetric space $(\mathrm{G}_m(\mathbb{K}^{2m}), V)$ is $V^\perp$, the orthogonal complement of V in $\mathbb{K}^{2m}$.
From now on, we will assume that $k \neq m$. We take into account the general fact that if a compact symmetric space P has a pole, then there is a non-trivial Riemannian double covering $P \to P'$ (see e.g. [Ch-Na-88, Proposition 2.9] or [Qu-10, Lemma 2.15]). Now, none of the spaces $\mathrm{G}_k(\mathbb{K}^{2m})$ is a covering of another space, in other words, all $\mathrm{G}_k(\mathbb{K}^{2m})$ are adjoint symmetric spaces. To prove this, we need to consider the following two situations. If $\mathbb{K} = \mathbb{R}$, we note that the symmetric space $\mathrm{G}_k(\mathbb{R}^{2m})$ has the Dynkin diagram of type $\mathfrak{b}$, hence it has exactly one simple root with coefficient equal to 1 in the expansion of the highest root (see [He-01], Table V, p. 518, Table IV, p. 532 and the table on p. 477). On the other hand, $\mathrm{G}_k(\mathbb{R}^{2m})$ is covered by the Grassmannian of all oriented k-subspaces in $\mathbb{R}^{2m}$. By using the theorem of Takeuchi [Ta-64], the latter space is simply connected, and $\mathrm{G}_k(\mathbb{R}^{2m})$ is its adjoint symmetric space. If $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{H}$, we note that the symmetric space $\mathrm{G}_k(\mathbb{K}^{2m})$ has Dynkin diagram of type $\mathfrak{bc}$; by using again [Ta-64], we deduce that $\mathrm{G}_k(\mathbb{K}^{2m})$ is at the same time simply connected and an adjoint symmetric space.

Recall that spaces of shortest geodesic segments with prescribed endpoints in a symmetric space are an important tool in Bott's proof of his periodicity theorem [Bo-59]. We can identify such spaces with submanifolds by mapping a shortest geodesic segment to its midpoint. We therefore have a closer look at these spaces. The objects described in the following definition are slightly more general, in the sense that the geodesic segments are not required to be shortest (we will return to this assumption at the end of this subsection).

Definition 2.5 *Let p be a pole of (P, o). The set $C_p(P, o)$ of all midpoints of geodesics in P from o to p is called a* centrosome. *The connected components of a centrosome are called* centrioles.

For more on these notions we refer to [Ch-Na-88] and [Na-88]. The following result is a consequence of [Na-88, Proposition 2.12 (ii)] (see also [Qu-10, Proposition 2.16] or [Qu-11, Proposition 2]).

Lemma 2.6 *Any centriole in a compact symmetric space is a reflective, hence totally geodesic submanifold.*

We recall that a submanifold of a Riemannian manifold is called *reflective* if it is a connected component of the fixed point set of an isometric involution. Reflective submanifolds of irreducible simply connected Riemannian symmetric spaces have been classified by Leung in [Le-74] and [Le-79] . This classification in the special case when the symmetric space is a compact simple Lie group will be an important tool for us (see Appendix C).
Although the following result appears to be known (see [Ch-Na-88] and [Na-88]), we decided to include a proof of it, for the sake of completeness.

Lemma 2.7 *Let p be a pole of (P, o). The centrioles of (P, o) relative to p are orbits of the canonical K_e-action on P.*

Proof. Let C be a connected component of $C_p(P, o)$ and take $x \in C$. There exists a geodesic $\gamma : \mathbb{R} \to P$ such that $\gamma(0) = o, \gamma(1) = x$, and $\gamma(2) = p$. For any $k \in K_e$, the restriction of the map $k.\gamma : \mathbb{R} \to P$ to the interval $[0, 2]$ is a geodesic segment between o and p (see Lemma 2.2). Thus the point $k.\gamma(1)$ is in $C_p(p, o)$. Since K_e is connected, we deduce that $K_e.x \subset C$.
Let us now prove the converse inclusion. Take $y \in C$ and consider a geodesic $\mu : \mathbb{R} \to C$ such that $\mu(0) = x$ and $\mu(1) = y$. By Lemma 2.6, μ is a geodesic in P as well. We consider the one-parameter subgroup of transvections along μ which is given by $\tau_\mu : \mathbb{R} \to G$, $\tau_\mu(t) := s_{\mu(t/2)} \circ s_{\mu(0)}$ (see e.g. [Sa-96, Lemma 6.2]).
Claim. $\tau_\mu(t) \in K_e$, for all $t \in \mathbb{R}$.
Indeed, since $\mu(0)$ and $\mu(t/2)$ are both midpoints of geodesic segments between o and p, we have $s_{\mu(0)}.o = p$ and $s_{\mu(t/2)}.p = o$. Hence, $\tau_\mu(t).o = o$. We deduce that $\tau_\mu(t) \in K$. Since $\tau_\mu(0)$ is the identity transformation of P, we actually have $\tau_\mu(t) \in K_e$.
The claim along with the fact that $\mu(0) = x$ implies that $\tau_\mu(1).x = s_{\mu(1/2)} \circ s_{\mu(0)}.x = s_{\mu(1/2)}.x = y$. Thus $y \in K_e.x$. □

From Lemmata 2.2 and 2.7, we see that whenever a centriole in $C_p(P, o)$ contains a midpoint of a shortest geodesic segment between o and p, then this centriole consists of midpoints of such shortest geodesic segments only. Such centrioles are called *s-centrioles*. (For further properties of s-centrioles we refer to [Qu-11].)

2.2 The SO-Bott chain

We outline Milnor's description [Mi-69, Section 24] of this chain. The chain starts with $P_0 = \mathrm{SO}_{16n}$. We then consider the space of all orthogonal complex structures in SO_{16n}, that is,

$$\Omega_1 := \{J \in \mathrm{SO}_{16n} \ : \ J^2 = -I\}.$$

This space has two connected components, which are both diffeomorphic to $\mathrm{SO}_{16n}/\mathrm{U}_{8n}$. We pick any of these two components and denote it by P_1. For $2 \le k \le 7$ we construct the spaces $P_k \subset \mathrm{SO}_{16n}$ inductively, as follows: Assume that P_k has been constructed and pick a base-point $J_k \in P_k$. We define P_{k+1} as the top-dimensional connected component of the space

$$\Omega_{k+1} := \{J \in P_k \ : \ JJ_k = -J_k J\}.$$

In this way we construct $P_2, \ldots, P_7$. Finally, we pick $J_7 \in P_7$ and define P_8 as any of the two connected components of the space

$$\Omega_8 := \{J \in P_7 \ : \ JJ_7 = -J_7 J\}.$$

(Note the latter space is diffeomorphic to the orthogonal group O_n, thus it has two components that are diffeomorphic). It turns out that $P_1, \ldots, P_8$ are submanifolds of SO_{16n}, whose diffeomorphism types can be described as follows: $P_0 = \mathrm{SO}_{16n}$, $P_1 = \mathrm{SO}_{16n}/\mathrm{U}_{8n}$, $P_2 = \mathrm{U}_{8n}/\mathrm{Sp}_{4n}$, $P_3 = \mathrm{G}_{2n}(\mathbb{H}^{4n}) = \mathrm{Sp}_{4n}/(\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n})$, $P_4 = \mathrm{Sp}_{2n}$, $P_5 = \mathrm{Sp}_{2n}/\mathrm{U}_{2n}$, $P_6 = \mathrm{U}_{2n}/\mathrm{O}_{2n}$, $P_7 = \mathrm{G}_n(\mathbb{R}^{2n}) = \mathrm{SO}_{2n}/\mathrm{S}(\mathrm{O}_n \times \mathrm{O}_n)$, and $P_8 = \mathrm{SO}_n$. The details can be found in [Mi-69, Section 24].

For our future goals it is useful to have an alternative description of the SO-Bott chain. This is presented by the following two lemmata.

Lemma 2.8 *For any $0 \le k \le 7$, the subspace P_k of SO_{16n} is invariant under the automorphism of SO_{16n} given by $X \mapsto -X$.*

Proof. First, Ω_k is obviously invariant under $X \mapsto -X$, $X \in \mathrm{SO}_{16n}$. The decisive argument is the information provided by the last paragraph on p. 137 in [Mi-69]: for any $X \in \Omega_k$, there exists a path in Ω_k from X to $-X$. □

Let us now equip SO_{16n} with the bi-invariant metric induced by

$$\langle X, Y \rangle = -\mathrm{tr}(XY), \tag{2.2}$$

for all X, Y in the Lie algebra $\mathfrak{o}_{16n}$ of SO_{16n}. Then $P_1, \ldots, P_8$ are totally geodesic submanifolds of SO_{16n} (see [Mi-69, Lemma 24.4]). Fix $k \in \{0, 1, \ldots, 7\}$ and set $J_0 := I$. From Example 2.3 we deduce that $-J_k$ is a pole of (SO_{16n}, J_k). By the previous lemma, $-J_k$ lies in P_k and, since the latter space is totally geodesic in SO_{16n}, $-J_k$ is a pole of (P_k, J_k). The following lemma follows from the Remark on p. 138 in [Mi-69].

Lemma 2.9 *For any $k \in \{0, 1, \ldots, 7\}$, the space P_{k+1} is an s-centriole of (P_k, J_k) relative to the pole $-J_k$.*

Remark 2.10 As we will show in Proposition B.1 (b), P_8 is isometric to SO_n, the latter being equipped with the standard bi-invariant metric multiplied by a certain scalar. Assume that n is an even integer and pick $J_8 \in P_8$. With the method used in the proof of Lemma 2.8 one can show that $-J_8$ is in P_8 as well (indeed by the footnote on p. 142 in [Mi-69], there exists an orthogonal complex structure $J \in \mathrm{SO}_{16n}$ which anti-commutes with $J_1, \ldots, J_7$). As in Lemma 2.9, $-J_8$ is a pole of (P_8, J_8) and, by using Example 2.3 for $G = \mathrm{SO}_n$, it is the only one. We conclude that the SO-Bott chain can be extended and is periodic, in the sense that if n is divisible by a "large" power of 16, then every eighth element of the chain is isometric to a certain special orthogonal group equipped with a bi-invariant metric.

2.3 The Sp-Bott chain

This is obtained from the SO-chain by taking P_4 as the initial element. More precisely, we replace n by $8n$ and, in this way, P_4 is diffeomorphic to

Sp_{16n}. This is the first term of the Sp-chain, call it $\bar{P}_0$. Here is the list of all terms of the chain, described up to diffeomorphism: $\bar{P}_0 = \mathrm{Sp}_{16n}, \bar{P}_1 = \mathrm{Sp}_{16n}/\mathrm{U}_{16n}, \bar{P}_2 = \mathrm{U}_{16n}/\mathrm{O}_{16n}, \bar{P}_3 = \mathrm{G}_{8n}(\mathbb{R}^{16n}) = \mathrm{SO}_{16n}/\mathrm{S}(\mathrm{O}_{8n} \times \mathrm{O}_{8n}), \bar{P}_4 = \mathrm{SO}_{8n}, \bar{P}_5 = \mathrm{SO}_{8n}/\mathrm{U}_{4n}, \bar{P}_6 = \mathrm{U}_{4n}/\mathrm{Sp}_{2n}, \bar{P}_7 = \mathrm{G}_n(\mathbb{H}^{2n}) = \mathrm{Sp}_{2n}/\mathrm{Sp}_n \times \mathrm{Sp}_n$, and $\bar{P}_8 = \mathrm{Sp}_n$. As explained in the previous subsection, these are Riemannian manifolds obtained by successive applications of the centriole construction. The starting point is $P_0 = \mathrm{Sp}_{16n}$ with the Riemannian metric which is described at the beginning of Section 3: by Proposition B.1 (a), this metric is the same as the submanifold metric on P_4, up to a scalar multiple.

2.4 Poles and centrioles in U_{2q}

Let q be an integer, $q \geq 1$. We equip the unitary group U_{2q} with the bi-invariant metric induced by the inner product

$$\langle X, Y \rangle = -\mathrm{tr}(XY), \tag{2.3}$$

for all X, Y in the Lie algebra $\mathfrak{u}_{2q}$ of U_{2q}. The center of U_{2q} is

$$Z(\mathrm{U}_{2q}) = \{zI \;:\; z \in \mathbb{C}, |z| = 1\}.$$

From Example 2.3, the pointed symmetric space (U_{2q}, I) has exactly one pole, namely the matrix $-I$. By Lemma 2.7, the centrioles of (U_{2q}, I) are certain orbits of the conjugation action of U_{2q} on itself, since they coincide with the orbits of the action of $\mathrm{U}_{2q}/Z(\mathrm{U}_{2q})$.

Let us describe explicitly the s-centrioles. We first describe the shortest geodesic segments in U_{2q} between I and $-I$, that is, $\gamma : [0,1] \to \mathrm{U}_{2q}$ such that $\gamma(0) = I$ and $\gamma(1) = -I$. Any such γ is U_{2q}-conjugate to the 1-parameter subgroup

$$\gamma_k : \; t \mapsto \exp\left[t \begin{pmatrix} \pi i I_k & 0 \\ 0 & -\pi i I_{2q-k} \end{pmatrix}\right], t \in \mathbb{R} \tag{2.4}$$

restricted to the interval $[0,1]$, for some $0 \leq k \leq 2q$ (see [Mi-69, Section 23]). Consequently, any s-centriole is of the form $\mathrm{U}_{2q}.\gamma_k\left(\frac{1}{2}\right)$, that is, the U_{2q}-conjugacy class of

$$\exp\left[\frac{1}{2}\begin{pmatrix} \pi i I_k & 0 \\ 0 & -\pi i I_{2q-k} \end{pmatrix}\right] = \begin{pmatrix} iI_k & 0 \\ 0 & -iI_{2q-k} \end{pmatrix}.$$

The U_{2q}-stabilizer of this matrix is $\mathrm{U}_k \times \mathrm{U}_{2q-k}$, hence one can identify the orbit with $\mathrm{U}_{2q}/\mathrm{U}_k \times \mathrm{U}_{2q-k}$, which is just the Grassmannian $\mathrm{G}_k(\mathbb{C}^{2q})$. If we equip the orbit with the submanifold Riemannian metric, then the (transitive) conjugation action of U_{2q} on it is isometric, in other words, the metric is U_{2q}-invariant. Note that up to a scalar there is a unique such metric on $\mathrm{G}_k(\mathbb{C}^{2q})$ and it makes this space into a symmetric space.

We will be especially interested in the centriole corresponding to $k = q$, which we call the *top-dimensional s-centriole*. Concretely, this is the U_{2q}-conjugacy class of the matrix

$$A_q := \begin{pmatrix} iI_q & 0 \\ 0 & -iI_q \end{pmatrix} \tag{2.5}$$

and it is isometric to the Grassmannian $\mathrm{G}_q(\mathbb{C}^{2q})$ equipped with a canonical symmetric space metric.

Finally, note that if instead of I the base point is an arbitrary element A of U_{2q}, then the only pole of (U_{2q}, A) is the matrix $-A$. The corresponding centrioles are $A(\mathrm{U}_{2q}.\gamma_k\left(\frac{1}{2}\right))$, that is, A-left translates in U_{2q} of the conjugacy classes described above. As before, they are all s-centrioles.

Remark 2.11 The top-dimensional s-centriole of (U_{2q}, A) relative to $-A$ is invariant under the automorphism of U_{2q} given by $X \mapsto -X$. The reason is that the matrix $-A_q$ is U_{2q}-conjugate to A_q.

2.5 Poles and centrioles in $\mathrm{G}_q(\mathbb{C}^{2q})$

We regard the Grassmannian $\mathrm{G}_q(\mathbb{C}^{2q})$ as the top-dimensional s-centriole of (U_{2q}, I) relative to $-I$, that is, the conjugacy class in U_{2q} of the matrix A_q described by Equation (2.5). Note that, by Remark 2.11, if A is in $\mathrm{G}_q(\mathbb{C}^{2q})$, then $-A$ is in $\mathrm{G}_q(\mathbb{C}^{2q})$, too.

Lemma 2.12 *If $A \in \mathrm{G}_q(\mathbb{C}^{2q})$, then the pointed symmetric space $(\mathrm{G}_q(\mathbb{C}^{2q}), A)$ has only one pole, which is $-A$.*

Proof. First, observe that the geodesic symmetries s_A and s_{-A} of U_{2q} are identically equal (see Example 2.3). By Lemma 2.6, $\mathrm{G}_q(\mathbb{C}^{2q})$ is a totally geodesic submanifold of U_{2q}. Hence, $-A$ is a pole of $(\mathrm{G}_q(\mathbb{C}^{2q}), A)$. We claim that the pointed symmetric space $(\mathrm{G}_q(\mathbb{C}^{2q}), A)$ has at most one pole. Indeed, let π be the Cartan map of $\mathrm{G}_q(\mathbb{C}^{2q})$, i.e. the map that assigns to each point its geodesic symmetry. It is known that this is a Riemannian covering onto its image, the latter being a compact symmetric space. Observe that the fundamental group of the adjoint space of $\mathrm{G}_q(\mathbb{C}^{2q})$ is $\mathbb{Z}_2$. We prove this by using the same kind of argument as in the second half of Remark 2.4: the Dynkin diagram of the symmetric space $\mathrm{G}_q(\mathbb{C}^{2q})$ is of type $\mathfrak{c}$, hence there is exactly one simple root with coefficient equal to 1 in the expansion of the highest root (see [He-01], Table V, p. 518, Table IV, p. 532 and the table on p. 477); we use again the theorem of Takeuchi [Ta-64]. Since $\mathrm{G}_q(\mathbb{C}^{2q})$ is simply connected and we have $\pi(A) = \pi(-A)$ we deduce that π is a double covering. Finally, we take into account that any pole of $(\mathrm{G}_q(\mathbb{C}^{2q}), A)$ is in the pre-image $\pi^{-1}(\pi(A))$. □

We note that this lemma is related to [Na-92, Proposition 2.23 (i)].

Remark 2.13 Recall that, by definition, $\mathrm{G}_q(\mathbb{C}^{2q})$ is the space of all q-dimensional complex vector subspaces of $\mathbb{C}^{2q}$. The lemma above implies readily that if V is such a vector space, then the pointed symmetric space $(\mathrm{G}_q(\mathbb{C}^{2q}), V)$

has only one pole, which is $V^\perp$, the orthogonal complement of V in $\mathbb{C}^{2q}$ relative to the usual Hermitian inner product.

As a next step, we look at s-centrioles in $\mathrm{G}_q(\mathbb{C}^{2q})$. Since $\mathrm{G}_q(\mathbb{C}^{2q})$ is an irreducible and simply connected symmetric space, there is a *unique* s-centriole of $(\mathrm{G}_q(\mathbb{C}^{2q}), A_q)$ relative to the pole $-A_q$ (see Theorem 1.2 and the subsequent remark in [Ma-Qu-10]). To describe it, we first find a shortest geodesic segment from A_q to $-A_q$ in $\mathrm{G}_q(\mathbb{C}^{2q})$. Let us consider the curve $\gamma : [0,1] \to \mathrm{G}_q(\mathbb{C}^{2q}) \subset \mathrm{U}_{2q}$,

$$\gamma(t) = \exp\left[t \begin{pmatrix} 0 & \frac{\pi i}{2} I_q \\ \frac{\pi i}{2} I_q & 0 \end{pmatrix}\right].A_q = \begin{pmatrix} \cos(\frac{\pi t}{2}) I_q & i\sin(\frac{\pi t}{2}) I_q \\ i\sin(\frac{\pi t}{2}) I_q & \cos(\frac{\pi t}{2}) I_q \end{pmatrix}.A_q,$$

where the dot indicates the conjugation action. Observe that $\gamma(0) = A_q$ and $\gamma(1) = -A_q$. We claim that γ is a shortest geodesic segment between A_q and $-A_q$ in $\mathrm{G}_q(\mathbb{C}^{2q})$. Indeed, for any $t \in [0,1]$ the matrix $\gamma'(t)$ is U_{2q}-conjugate with the Lie bracket of the matrices

$$\begin{pmatrix} 0 & \frac{\pi i}{2} I_q \\ \frac{\pi i}{2} I_q & 0 \end{pmatrix}$$

and A_q, which is equal to

$$\begin{pmatrix} 0 & \pi i I_q \\ \pi i I_q & 0 \end{pmatrix}.$$

Thus the length of γ relative to the bi-invariant metric on U_{2q} given by Equation (2.3) is equal to $\pi\sqrt{2q}$, which means that γ is a shortest path in U_{2q} between A_q and $-A_q$ (see [Mi-69, p. 127] or Lemma 3.1 below). Since the length of the vector $\gamma'(t)$ is independent of t, γ is a geodesic segment. Its midpoint is

$$\gamma\left(\frac{1}{2}\right) = \begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix}. \tag{2.6}$$

In view of Lemma 2.7, the centriole we are interested in is the orbit of $\gamma(\frac{1}{2})$ under the K_e-action. Since $K_e = (\mathrm{U}_q \times \mathrm{U}_q)/Z(\mathrm{U}_{2q})$, this is the same as the orbit of $\gamma(\frac{1}{2})$ under conjugation by $\mathrm{U}_q \times \mathrm{U}_q \subset \mathrm{U}_{2q}$. One can easily see that this orbit consists of all matrices of the form

$$\begin{pmatrix} 0 & -C^{-1} \\ C & 0 \end{pmatrix}$$

where C is in U_q. Multiplication from the left by the matrix given by Equation (2.6) induces an isometry between the latter orbit and the subspace of U_{2q} formed by all matrices

$$\begin{pmatrix} C & 0 \\ 0 & C^{-1} \end{pmatrix},$$

with $C \in \mathrm{U}_q$.

We deduce that if we equip the s-centriole of $(\mathrm{G}_q(\mathbb{C}^{2q}), A_q)$ relative to $-A_q$ with the submanifold metric, then it becomes isometric to U_q, where the latter is endowed with the bi-invariant metric induced by

$$\langle X, Y\rangle = -2\mathrm{tr}(XY), \tag{2.7}$$

$X, Y \in \mathfrak{u}_q$. Moreover, if instead of A_q the base point is an arbitrary element A of $\mathrm{G}_q(\mathbb{C}^{2q})$, then the only pole of $(\mathrm{G}_q(\mathbb{C}^{2q}), A)$ is the matrix $-A$. The corresponding centriole is obtained from the previous one by conjugation with B, where $B \in \mathrm{U}_{2q}$ satisfies $A = BA_qB^{-1}$. Thus this centriole has the same isometry type as the previous one.

Remark 2.14 We saw that there is a natural isometric identification between the centriole of $(\mathrm{G}_q(\mathbb{C}^{2q}), A)$ relative to $-A$ and U_q. One can also see from the previous considerations that this centriole is invariant under the isometry $X \mapsto -X$, $X \in \mathrm{U}_{2q}$, and the isometry induced on U_q is $X' \mapsto -X'$, $X' \in \mathrm{U}_q$.

2.6 The U-Bott chain

The following chain of inclusions results from the previous two subsections. We start with $\tilde{P}_0 := \mathrm{U}_{2q}$, equipped with the bi-invariant Riemannian metric defined by Equation (2.3). The top-dimensional s-centriole of $(\tilde{P}_0, I)$ relative to $-I$ is denoted by $\tilde{P}_1$. Pick $J_1 \in \tilde{P}_1$. (The reason why the elements of $\tilde{P}_1$ are denoted by J is explained in Appendix A, particularly Definition A.1 and Equation (A.1).) By Remark 2.11, $-J_1$ is in $\tilde{P}_1$, too. We denote by $\tilde{P}_2$ the s-centriole of $(\tilde{P}_1, J_1)$ relative to the pole $-J_1$. We have

$$\tilde{P}_0 \supset \tilde{P}_1 \supset \tilde{P}_2.$$

The elements of the chain are described by the following isometries:

$$\tilde{P}_0 \simeq \mathrm{U}_{2q},\ \tilde{P}_1 \simeq \mathrm{G}_q(\mathbb{C}^{2q}),\ \tilde{P}_2 \simeq \mathrm{U}_q,$$

where $\mathrm{G}_q(\mathbb{C}^{2q})$ carries the (symmetric space) metric induced via its embedding in U_{2q} and U_q is endowed with the metric described by Equation (2.7).
We now take $q = 8n$ and repeat the construction above three more times. By always choosing the top-dimensional centriole, we ensure that all our spaces are invariant under the map $\mathrm{U}_{16n} \to \mathrm{U}_{16n}$, $X \mapsto -X$ (see Remarks 2.11 and 2.14 above). We proceed as follows:
First we pick $J_2 \in \tilde{P}_2$ as a base point. Then $-J_2$ is a pole of $(\tilde{P}_2, J_2)$. Indeed, we know that the geodesic symmetries s_{J_2} and s_{-J_2} of U_{16n} are equal (see Example 2.3) and $\tilde{P}_2$ is a totally geodesic submanifold of U_{16n}.
After that, we consider the top-dimensional s-centriole of $(\tilde{P}_2, J_2)$ relative to $-J_2$ and denote it by $\tilde{P}_3$. As before, we have the identification

$$\tilde{P}_3 \simeq \mathrm{G}_{4n}(\mathbb{C}^{8n}).$$

In the same way, we construct $\tilde{P}_4, \ldots, \tilde{P}_8$, by picking J_{k-1} in $\tilde{P}_{k-1}$ and defining $\tilde{P}_k$ as the top-dimensional centriole of $(\tilde{P}_{k-1}, J_{k-1})$ relative to $-J_{k-1}$, for all $k = 4, \ldots, 8$. We have the identifications:

$$\tilde{P}_5 \simeq \mathrm{G}_{2n}(\mathbb{C}^{4n}),\ \tilde{P}_6 \simeq \mathrm{U}_{2n},\ \tilde{P}_7 \simeq \mathrm{G}_n(\mathbb{C}^{2n}),\ \tilde{P}_8 \simeq \mathrm{U}_n,$$

where each $\tilde{P}_k$ carries the submanifold metric. Similarly to Equation (2.7), one can see that the Riemannian metric induced on U_n via the diffeomorphism $\tilde{P}_8 \simeq \mathrm{U}_n$ coincides with the bi-invariant metric on U_n induced by

$$\langle X, Y\rangle = -16\mathrm{tr}(XY), \tag{2.8}$$

$X, Y \in \mathfrak{u}_n$.
In this way we have constructed the U-Bott chain, which is $\tilde{P}_0 \supset \tilde{P}_1 \supset \ldots \supset \tilde{P}_8$.

2.7 Bott's periodicity theorems

Bott's original proof (see [Bo-59]) uses the space of paths between two points in a Riemannian manifold.

Definition 2.15 *If M is a Riemannian manifold and p, q are two points in M, we denote by $\Omega(M; p, q)$ the space of piecewise smooth paths $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$.*

The space $\Omega(M; p, q)$ has a topology which is induced by a certain canonical metric (the details can be found for instance in [Mi-69, Section 17]).
Let (P, o) be again a pointed compact symmetric space, p a pole of it, and $Q \subset P$ one of the corresponding s-centrioles. Recall that Q consists of midpoints of geodesics in P from o to p. We have a continuous injection

$$j : Q \to \Omega(P; o, p) \tag{2.9}$$

that assigns to $q \in Q$ the unique shortest geodesic segment $[0, 1] \to M$ from o to p whose midpoint is q. This induces a map

$$j_* : \pi_i(Q) \to \pi_i(\Omega(P; o, p)) = \pi_{i+1}(P)$$

between homotopy groups. Bott's proof [Bo-59] relies on the fact that this map is an isomorphism for all $i > 0$ that are smaller than a certain number which can be calculated explicitly in concrete situations, including all the situations we have described in Subsections 2.2, 2.3, and 2.6. The main tool is Morse theory, see also Milnor's book [Mi-69] (for a different approach we address to [Mit-88]).
We now apply the result above for the elements of the SO-chain, see Subsection 2.2. For all $i = 1, 2, \ldots$ sufficiently smaller than n, we obtain

$$\pi_i(\mathrm{SO}_n) = \pi_i(P_8) \simeq \pi_{i+1}(P_7) \simeq \ldots \simeq \pi_{i+7}(P_1) \simeq \pi_{i+8}(P_0) = \pi_{i+8}(\mathrm{SO}_{16n}).$$

This yields the following isomorphism between stable homotopy groups:

$$\pi_k(\mathrm{O}) \simeq \pi_{k+8}(\mathrm{O}),$$

for all $k = 0, 1, 2, \ldots$. This is Bott's periodicity theorem for the orthogonal group. Similarly, for the unitary and symplectic groups, one has

$$\pi_k(\mathrm{U}) \simeq \pi_{k+2}(\mathrm{U}) \quad \text{and} \quad \pi_k(\mathrm{Sp}) \simeq \pi_{k+8}(\mathrm{Sp})$$

for all $k = 0, 1, 2, \ldots$.

3 Inclusions between Bott chains

In this section we link the three Bott chains constructed above. The following lemmata are key ingredients that make this process possible. We recall that for any $q \geq 1$ the Lie group U_{2q} carries the bi-invariant Riemann metric described by Equation (2.3). We regard SO_{2q} as a Lie subgroup of U_{2q} and endow it with the submanifold metric (note that for $q = 8n$ this is the same as the metric described by Equation (2.2)). For $r \geq 1$ we also consider the symplectic group Sp_r, which is defined as the space of all $\mathbb{H}$-linear automorphisms of $\mathbb{H}^n$ that preserve the norm of a vector. As explained in Subsection A.5, this group has a canonical embedding into U_{2r}. More precisely, Sp_r can be identified with the subgroup of U_{2r} that consists of all matrices of the form

$$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}$$

which are in U_{2r}, where A and B are $r \times r$ matrices with complex entries (see [Br-tD-85, Ch. I, Section 1.11]). Yet another canonical embedding, which we also need here, is the one of U_r into Sp_r, see Subsection A.9. Concretely, U_r can be considered as the subgroup of Sp_r consisting of all matrices which are of the above form with $B = 0$ and $A \in \mathrm{U}_r$.

For future use we also mention that Sp_r lies in U_{2r} and U_r lies in Sp_r as fixed point sets of certain involutive group automorphisms. More precisely, let us consider the element

$$K_r := \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$$

of U_{2r} and the group automorphism of U_{2r} given by $X \mapsto K_r \overline{X} K_r^{-1}$, where $\overline{X}$ is the complex conjugate of X: the automorphism is involutive and its fixed point set is just Sp_r. In the same vein, let us consider the element

$$A_r := \begin{pmatrix} iI_r & 0 \\ 0 & -iI_r \end{pmatrix}$$

of Sp_r and the corresponding (inner) automorphism of Sp_r, $\bar{\tau}(X) := A_r X A_r^{-1}$: this automorphism is involutive as well and its fixed point set is equal to U_r

(note that A_r has also been used in Subsections 2.4 and 2.5 and is also relevant in Subsection A.15).
Let us now consider the inner product on $\mathfrak{u}_{2r}$ given by

$$\langle X, Y\rangle = -\frac{1}{2}\mathrm{tr}(XY), \quad X, Y \in \mathfrak{u}_{2r}.$$

Note that the bi-invariant Riemannian metric induced on U_{2r} is different from the one defined by Equation (2.3). However, we are exclusively interested in the subspace metrics on Sp_r and U_r. On the last space, the induced metric is bi-invariant and satisfies

$$\langle X, Y\rangle = -\mathrm{tr}(XY),$$

for all $X, Y \in \mathfrak{u}_r$, i.e. this metric is the one given by Equation (2.3).

Lemma 3.1 *Relative to the metrics defined above, we have:*

$$\mathrm{dist}_{\mathrm{SO}_{2q}}(I, -I) = \mathrm{dist}_{\mathrm{U}_{2q}}(I, -I) = \pi\sqrt{2q},$$
$$\mathrm{dist}_{\mathrm{U}_r}(I, -I) = \mathrm{dist}_{\mathrm{Sp}_r}(I, -I) = \pi\sqrt{r}.$$

Proof. The length of a shortest geodesic segment in U_{2q} between I and $-I$ has been calculated in [Mi-69, Section 23]. It is equal to $\pi\sqrt{2q}$. By [Mi-69, Section 24], a shortest geodesic segment in SO_{2q} from I to $-I$ is

$$[0,1] \to \mathrm{SO}_{2q},\ t \mapsto \exp\left[t\pi\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}\right].$$

Its length is also equal to $\pi\sqrt{2q}$.
To justify the second equation in the lemma, we just note that

$$[0,1] \to \mathrm{U}_{2r},\ t \mapsto \exp\left[t\begin{pmatrix} \pi i I_r & 0 \\ 0 & -\pi i I_r \end{pmatrix}\right]$$

is a shortest geodesic segment in U_{2r} from I to $-I$. The image of this geodesic lies entirely in $\mathrm{U}_r \subset \mathrm{Sp}_r$ and is consequently shortest in both U_r and Sp_r. Its length can be calculated as before, by using [Mi-69, Section 23]. $\square$

The next lemma concerns the SO-Bott chain, which has been constructed in Subsection 2.2. The result can be found in [Mi-69, p. 137]. Since it plays an important role in our development, we state it separately.

Lemma 3.2 *If we equip each P_k, $k = 1, 2, \dots, 7$ with the submanifold metric, then we have*

$$\mathrm{dist}_{\mathrm{SO}_{16n}}(I, -I) = \mathrm{dist}_{P_1}(J_1, -J_1) = \ldots = \mathrm{dist}_{P_7}(J_7, -J_7).$$

This result can also be deduced from [Qu-Ta-11]. Relevant to this context is [Na-Ta-91, Remark 3.2 b)], too.
We are now ready to construct the inclusions between the three Bott chains.

3.1 Including P_k into $\tilde{P}_k$

We start by recalling that P_1 is one of the two s-centrioles of (SO_{16n}, I) relative to the pole $-I$ (see Subsection 2.2). Also recall that $\tilde{P}_1$ is the top-dimensional s-centriole of (U_{16n}, I) relative to the pole $-I$ (see Subsection 2.6). By Lemma 3.1, P_1 is contained in one of the s-centrioles of (U_{16n}, I) relative to $-I$, call it $\tilde{P}_1'$.

Claim. $\tilde{P}_1' = \tilde{P}_1$, i.e. $P_1 \subset \tilde{P}_1$.

Both J_1 and $-J_1$ are in P_1, thus also in $\tilde{P}_1'$. The geodesic symmetries s_{J_1} and s_{-J_1} of U_{16n} are equal. Since $\tilde{P}_1'$ is a totally geodesic submanifold of U_{16n}, the restrictions of the two geodesic symmetries to $\tilde{P}_1'$ are equal, too. Therefore, $-J_1$ is a pole of the pointed symmetric space $(\tilde{P}_1', J_1)$. On the other hand, $\tilde{P}_1'$ is isometric to one of the symmetric spaces $\mathrm{G}_k(\mathbb{C}^{16n})$, where $0 \leq k \leq 16n$ (see Subsection 2.4). It is known that amongst these Grassmannians there is just one which admits a pole relative to a given base point, namely the one corresponding to $k = 8n$ (see Remark 2.4). This finishes the proof of the claim.

Note that the following diagram is commutative:

$$\begin{tikzcd}
P_1 \arrow[r, "\jmath_1"] \arrow[d, "\cap"] & \Omega(P_0; I, -I) \arrow[d, "\cap"] \\
\tilde{P}_1 \arrow[r, "\tilde{\jmath}_1"] & \Omega(\tilde{P}_0; I, -I)
\end{tikzcd}$$

where the horizontal arrows are inclusion maps and the vertical arrows are given by Equation (2.9).

Recall that P_2 is an s-centriole of (P_1, J_1) relative to $-J_1$. By Lemmata 3.1 and 3.2, any shortest geodesic segment in P_1 which joins J_1 and $-J_1$ is also shortest in $\tilde{P}_1$. Since $\tilde{P}_2$ is the unique s-centriole of $(\tilde{P}_1, J_1)$ relative to $-J_1$ (see Subsection 2.5), we have

$$P_2 \subset \tilde{P}_2. \tag{3.1}$$

Again, we have a commutative diagram, which is:

$$\begin{tikzcd}
P_2 \arrow[r, "\jmath_2"] \arrow[d, "\cap"] & \Omega(P_1; J_1, -J_1) \arrow[d, "\cap"] \\
\tilde{P}_2 \arrow[r, "\tilde{\jmath}_2"] & \Omega(\tilde{P}_1; J_1, -J_1)
\end{tikzcd}$$

In the same way we prove that we have the inclusions

$$P_k \subset \tilde{P}_k \tag{3.2}$$

for all $k = 3, \ldots, 8$.

3.2 The inclusion $P_k \subset \tilde{P}_k$ as fixed points of the complex conjugation

We will use the following notations.
Notations. Let A be a topological space. If a is an element of A, then A_a denotes the connected component of A which contains a. If σ is a map from A to A then $A^\sigma := \{x \in A \ : \ \sigma(x) = x\}$.
The main tool we will use in this subsection is the following lemma.

Lemma 3.3 *Let $(\tilde{P}, o)$ be a compact connected pointed symmetric space, p a pole of $(\tilde{P}, o)$ and $\gamma_0 : [0,1] \to \tilde{P}$ a geodesic segment which is shortest between $\gamma_0(0) = o$ and $\gamma_0(1) = p$. Set $j_0 := \gamma_0\left(\frac{1}{2}\right)$ and denote by $\tilde{Q}$ the centriole of $(\tilde{P}, o)$ relative to p which contains j_0 (see Definition 2.5). Let also σ be an isometry of $\tilde{P}$. Assume that $\sigma(o) = o$, $\sigma(p) = p$, and set $P := (\tilde{P}^\sigma)_o$. Also assume that the trace of γ_0 is contained in P. Then:*
(a) p is a pole of (P, o),
(b) $\tilde{Q}$ is σ-invariant,
(c) $(\tilde{Q}^\sigma)_{j_0} = (C_p(P, o))_{j_0}$.

Proof. (a) Since p is a pole of $(\tilde{P}, o)$, the geodesic reflections $s_o^{\tilde{P}}$ and $s_p^{\tilde{P}}$ are equal. But P is a totally geodesic submanifold of $\tilde{P}$, hence the geodesic reflections $s_o^P = s_o^{\tilde{P}}|_P$ and $s_p^P = s_p^{\tilde{P}}|_P$ are equal as well.
(b) Take $x \in C_p(\tilde{P}, o)$. Then there exists a geodesic segment $\gamma : [0,1] \to \tilde{P}$ with $\gamma(0) = o$, $\gamma(1) = p$, and $\gamma\left(\frac{1}{2}\right) = x$. The path $\sigma \circ \gamma : [0,1] \to \tilde{P}$ is also a geodesic segment. It joins $\sigma \circ \gamma(0) = o$ with $\sigma \circ \gamma(1) = p$. Thus, its midpoint $\sigma(x)$ lies in $C_p(\tilde{P}, o)$ as well. We have shown that σ leaves $C_p(\tilde{P}, o)$ invariant and induces a homeomorphism of it. Consequently, σ maps $P = C_p(\tilde{P}, o)_{j_0}$ onto a connected component of $C_p(\tilde{P}, o)$. This must be $C_p(\tilde{P}, o)_{j_0}$, because $\sigma(j_0) = j_0$.
(c) Since P is a totally geodesic submanifold of $\tilde{P}$, we deduce that $C_p(P, o) \subset C_p(\tilde{P}, o) \cap \tilde{P}^\sigma$, hence $C_p(P, o)_{j_0} \subset C_p(\tilde{P}, o)_{j_0} \cap \tilde{P}^\sigma = \tilde{Q}^\sigma$. We have shown that $C_p(P, o)_{j_0} \subset (\tilde{Q}^\sigma)_{j_0}$.
Let us now prove the opposite inclusion. Take j an arbitrary element of $\tilde{Q}^\sigma$. There exists $\gamma : [0,1] \to \tilde{P}$ a geodesic segment with $\gamma(0) = o$, $\gamma(1) = p$, and $\gamma\left(\frac{1}{2}\right) = j$. Since $\tilde{Q}$ is an s-centriole, we can assume that γ is shortest between o and p. This implies that the restriction of γ to the interval $\left[0, \frac{1}{2}\right]$ is a shortest geodesic segment between o and j; moreover, it is the *unique* shortest geodesic segment $\left[0, \frac{1}{2}\right] \to \tilde{P}$ between o and j (cf. e.g. [Ga-Hu-La-04, Corollary 2.111]). On the other hand, the curve $\sigma \circ \gamma : \left[0, \frac{1}{2}\right] \to \tilde{P}$ is a shortest geodesic segment with the properties

$$\sigma \circ \gamma(0) = \sigma(o) = o \quad \text{and} \quad \sigma \circ \gamma\left(\frac{1}{2}\right) = \sigma(j) = j.$$

Consequently, we have $\sigma \circ \gamma = \gamma$, and therefore the trace of γ is contained in P. This implies that $j \in C_p(P, o)$. We have shown that $\tilde{Q}^\sigma \subset C_p(P, o)$. This implies readily the desired conclusion. □

Let us denote by τ the (isometric) group automorphism of U_{16n} given by complex conjugation. That is, $\tau : \mathrm{U}_{16n} \to \mathrm{U}_{16n}$,

$$\tau(X) := \overline{X}, \quad X \in \mathrm{U}_{16n}.$$

Note that the fixed point set of τ is O_{16n}.
By collecting results we have proved in this subsection and the previous one, we can now state the following theorem.

Theorem 3.4 *For any $k \in \{0,1,\dots,8\}$, the space $\tilde{P}_k$ is τ-invariant and we have*

$$P_k = (\tilde{P}_k^\tau)_{J_k}. \tag{3.3}$$

The following diagram is commutative:

$$\begin{tikzcd} P_0 \arrow[d, "\cap"] & P_1 \arrow[l, "\supset"'] \arrow[d, "\cap"] & P_2 \arrow[l, "\supset"'] \arrow[d, "\cap"] & \cdots \arrow[l, "\supset"'] & P_8 \arrow[l, "\supset"'] \arrow[d, "\cap"] \\ \tilde{P}_0 & \tilde{P}_1 \arrow[l, "\supset"'] & \tilde{P}_2 \arrow[l, "\supset"'] & \cdots \arrow[l, "\supset"'] & \tilde{P}_8 \arrow[l, "\supset"'] \end{tikzcd} \tag{3.4}$$

where the two horizontal components are the SO- *and the* U-*Bott chains, and the vertical arrows are the inclusions $P_k \subset \tilde{P}_k$, $k \in \{0,1,\dots,8\}$, induced by Equation (3.3). The following diagram is also commutative*

$$\begin{tikzcd} P_{\ell+1} \arrow[r, "\jmath_{\ell+1}"] \arrow[d, "\cap"] & \Omega(P_\ell; J_\ell, -J_\ell) \arrow[d, "\cap"] \\ \tilde{P}_{\ell+1} \arrow[r, "\tilde{\jmath}_{\ell+1}"] & \Omega(\tilde{P}_\ell; J_\ell, -J_\ell) \end{tikzcd} \tag{3.5}$$

where the maps $\jmath_{\ell+1}$ and $\tilde{\jmath}_{\ell+1}$ are the canonical inclusions given by Equation (2.9), for all $\ell \in \{0,1,\dots,7\}$.

3.3 The inclusions $\tilde{P}_k \subset \bar{P}_k$

We start with the standard inclusion $\tilde{P}_0 = \mathrm{U}_{16n} \subset \mathrm{Sp}_{16n} = \bar{P}_0$. The Sp-Bott chain defined in Subsection 2.2 can be described in terms of the complex structures $J_1, \dots, J_8 \in \mathrm{SO}_{16n}$ above as follows: $\bar{P}_{k+1}$ is an s-centriole of $(\bar{P}_k, J_k)$ relative to $-J_k$, for all $k = 0,1,\dots,7$ (as already mentioned in Subsection 2.2, the main reference for this construction is [Mi-69, Section 24]; see also [Es-08], Section 19, especially pp. 43–44). With the methods of Subsection 3.1 one can show that we have the totally geodesic embeddings $\tilde{P}_k \subset \bar{P}_k$, for all $k = 0,1,\dots,8$.
As mentioned at the beginning of this section, U_{16n} lies in Sp_{16n} as the fixed point set of the (involutive, inner) group automorphism $\bar{\tau} : \mathrm{Sp}_{16n} \to \mathrm{Sp}_{16n}$, $\bar{\tau}(X) := A_{8n} X A_{8n}^{-1}$. In the same way as in Subsection 3.2, we can prove the following analogue of Theorem 3.4:

Theorem 3.5 *For any* $k \in \{0,1,\dots,8\}$*, the space* $\bar{P}_k$ *is* $\bar{\tau}$*-invariant and we have*

$$\tilde{P}_k = (\bar{P}_k^{\bar{\tau}})_{J_k}. \tag{3.6}$$

The following diagram is commutative:

$$\begin{tikzcd}
\tilde{P}_0 \arrow[d, "\cap"] & \tilde{P}_1 \arrow[l, "\supset"'] \arrow[d, "\cap"] & \tilde{P}_2 \arrow[l, "\supset"'] \arrow[d, "\cap"] & \cdots \arrow[l, "\supset"'] & \tilde{P}_8 \arrow[l, "\supset"'] \arrow[d, "\cap"] \\
\bar{P}_0 & \bar{P}_1 \arrow[l, "\supset"'] & \bar{P}_2 \arrow[l, "\supset"'] & \cdots \arrow[l, "\supset"'] & \bar{P}_8 \arrow[l, "\supset"']
\end{tikzcd} \tag{3.7}$$

where the two horizontal components are the U- *and the* Sp-*Bott chains, and the vertical arrows are the inclusions* $\tilde{P}_k \subset \bar{P}_k$*,* $k \in \{0,1,\dots,8\}$*, induced by Equation (3.6). The following diagram is also commutative*

$$\begin{tikzcd}
\tilde{P}_{\ell+1} \arrow[r, "\tilde{\jmath}_{\ell+1}"] \arrow[d, "\cap"] & \Omega(\tilde{P}_\ell; J_\ell, -J_\ell) \arrow[d, "\cap"] \\
\bar{P}_{\ell+1} \arrow[r, "\bar{\jmath}_{\ell+1}"] & \Omega(\bar{P}_\ell; J_\ell, -J_\ell)
\end{tikzcd}$$

where the maps $\tilde{\jmath}_{\ell+1}$ *and* $\bar{\jmath}_{\ell+1}$ *are the canonical inclusions given by Equation (2.9), for all* $\ell \in \{0,1,\dots,7\}$*.*

Remark 3.6 We note in passing that all maps in the commutative diagrams described by Equations (3.4) and (3.7) are inclusions of reflective submanifolds.

4 Periodicity of inclusions between Bott chains

4.1 The inclusion $P_8 \subset \tilde{P}_8$

We have the isometries:

$$P_8 \simeq \mathrm{SO}_n \text{ and } \tilde{P}_8 \simeq \mathrm{U}_n.$$

The first is discussed in Proposition B.1 (b) and the second in Subsection 2.6. Note that $\tilde{P}_1$ is actually contained in SU_{16n} (see Subsection 2.5). Thus, from Theorem 3.4 we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{SO}_{16n} \arrow[d, "\cap"] & P_8 \arrow[l, "\supset"'] \arrow[d, "\cap"] \\
\mathrm{SU}_{16n} & \tilde{P}_8 \arrow[l, "\supset"']
\end{tikzcd}$$

where all arrows are inclusion maps, as follows: $P_8 \subset P_0 = \mathrm{SO}_{16n}$; $\tilde{P}_8 \subset \tilde{P}_1 \subset \mathrm{SU}_{16n}$; SO_{16n} is contained in SU_{16n} as the identity component of the fixed point set of τ, the latter being the complex conjugation; finally, by Theorem 3.4, the space $\tilde{P}_8$ is τ-invariant and P_8 is a connected component of the fixed point set $\tilde{P}_8^\tau$. We will prove the following result.

Theorem 4.1 *There exists an isometry $\psi : \tilde{P}_8 \to \mathrm{U}_n$ which maps P_8 to SO_n and makes the following diagram commutative:*

$$\begin{tikzcd} P_8 \arrow[r, "\psi|_{P_8}"] \arrow[d, "\cap"] & \mathrm{SO}_n \arrow[d, "\cap"] \\ \tilde{P}_8 \arrow[r, "\psi"] & \mathrm{U}_n \end{tikzcd}$$

Here the inclusions $P_8 \subset \tilde{P}_8$ and $\mathrm{SO}_n \subset \mathrm{U}_n$ are the one mentioned in the diagram (3.4), respectively the standard one (see e.g. Subsection A.1).

The rest of this subsection is devoted to the proof of this theorem. First pick $J \in P_8$ and denote

$$\mathfrak{p} = T_J\tilde{P}_8.$$

Let $R : \mathfrak{p} \times \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ be the curvature tensor of $\tilde{P}_8$ at the point J. It is a Lie triple in the sense of Loos [Lo-69, Vol. I]. Let $\mathfrak{c}$ be the center of this Lie triple, that is,

$$\mathfrak{c} = \{\eta \in \mathfrak{p} \ : \ R(\eta, x)y = 0 \text{ for all } x, y \in \mathfrak{p}\}.$$

We also denote by $\check{\mathfrak{p}}$ the orthogonal complement of $\mathfrak{c}$ in $\mathfrak{p}$ relative to the Riemann metric $\langle \ , \ \rangle_J$ of $\tilde{P}_8$ at the point J. Both elements of the splitting

$$\mathfrak{p} = \mathfrak{c} \oplus \check{\mathfrak{p}}$$

are Lie subtriples of $\mathfrak{p}$. Recall from Subsection 2.6 that there exists an isometry

$$\varphi : \tilde{P}_8 \to \mathrm{U}_n,$$

where U_n is equipped with the bi-invariant Riemannian metric described by Equation (2.8). Thus, the center $\mathfrak{c}$ is a 1-dimensional vector subspace of $\mathfrak{p}$. Let $\tau_* : \mathfrak{p} \to \mathfrak{p}$ be the differential of $\tau|_{\tilde{P}_8}$ at J. It is a Lie triple automorphism of $\mathfrak{p}$ that preserves the inner product $\langle \ , \ \rangle_J$. Thus it leaves both the center $\mathfrak{c}$ and its orthogonal complement $\check{\mathfrak{p}}$ invariant. The fixed point set of τ_*, call it $\mathrm{Fix}(\tau_*)$, is a Lie sub-triple which splits as:

$$\mathrm{Fix}(\tau_*) = \mathrm{Fix}(\tau_*|_{\mathfrak{c}}) \oplus \mathrm{Fix}(\tau_*|_{\check{\mathfrak{p}}}).$$

The first term of the splitting above is contained in the center of $\mathrm{Fix}(\tau_*)$. On the other hand, P_8 is the connected component of J in the fixed point set of $\tau|_{\tilde{P}_8} : \tilde{P}_8 \to \tilde{P}_8$. Therefore we have $\mathrm{Fix}(\tau_*) = T_JP_8$; as P_8 is isometric to SO_n (see the beginning of this section), T_JP_8 is isomorphic to the Lie triple of SO_n.

The latter Lie triple has no center, since SO_n is a semi-simple symmetric space. Consequently, we have $\mathrm{Fix}(\tau_*|_{\mathfrak{c}}) = \{0\}$. Both τ and τ_* are involutive, thus

$$\tau_*(x) = -x, \text{ for all } x \in \mathfrak{c}. \tag{4.1}$$

Consequently,

$$\mathrm{Fix}(\tau_*) = \mathrm{Fix}(\tau_*|_{\check{\mathfrak{p}}}).$$

We denote by $\check{P}_8$ the complete connected totally geodesic subspace of $\tilde{P}_8$ corresponding to the Lie sub-triple $\check{\mathfrak{p}}$. It is mapped by φ isometrically onto SU_n, the latter being equipped with the restriction of the bi-invariant metric given by Equation (2.8). The space $\check{P}_8$ is τ-invariant and we have

$$(\check{P}_8^\tau)_J = (\tilde{P}_8^\tau)_J = P_8. \tag{4.2}$$

We need the following lemma.

Lemma 4.2 *There exists an isometry $\varphi : \tilde{P}_8 \to \mathrm{U}_n$ such that $\varphi(J) = I_n$ and $\varphi(P_8) = \mathrm{SO}_n$. Moreover, there exists $A \in \mathrm{SU}_n$ which satisfies $A = A^T$ such that*

$$\varphi(\tau(p)) = A\overline{\varphi(p)}A^{-1}, \tag{4.3}$$

for all $p \in \check{P}_8$.

Proof. Let $\varphi : \tilde{P}_8 \to \mathrm{U}_n$ be the isometry above. The condition $\varphi(J) = I_n$ is achieved after modifying φ suitably, that is, multiplying it pointwise by $\varphi(J)^{-1}$. This proves the first claim in the lemma.
We now prove the second claim. To this end, we first recall that $\varphi|_{\check{P}_8} : \check{P}_8 \to \mathrm{SU}_n$ is an isometry, where SU_n is equipped with the restriction of the bi-invariant metric given by Equation (2.8). Thus, the map $\tau' := \varphi \circ \tau \circ \varphi^{-1}|_{\mathrm{SU}_n}$ is an involutive isometry of SU_n. Moreover, the identity element I_n is in the fixed point set $\mathrm{SU}_n^{\tau'}$. From Proposition C.1 we deduce that there exists an involutive group automorphism μ of SU_n such that either

$$\tau'(X) = \mu(X), \text{ for all } X \in \mathrm{SU}_n \tag{4.4}$$

or

$$\tau'(X) = \mu(X)^{-1}, \text{ for all } X \in \mathrm{SU}_n. \tag{4.5}$$

Moreover, in the second case the space $(\mathrm{SU}_n^{\tau'})_{I_n}$ is isometric to $\mathrm{SU}_n/\mathrm{SU}_n^\mu$, where the last space has the canonical symmetric space metric. Assume that we are in the second case. From Equation (4.2), SO_n would be isometric to $\mathrm{SU}_n/\mathrm{SU}_n^\mu$. The involutive group automorphisms of SU_n are classified, see e.g. [Wo-84, p. 281 and p. 290]. It turns out that the group SU_n^μ is isomorphic to $\mathrm{S}(\mathrm{U}_k \times \mathrm{U}_{n-k})$, for some $0 \leq k \leq n$, or to SO_n, or to $\mathrm{Sp}_{n/2}$, if n is divisible by 2. None of the corresponding quotients is a symmetric space isometric to SO_n.

We deduce that Equation (4.4) holds. Once again from the classification of the involutive group automorphisms of SU_n mentioned above ([Wo-84, p. 290]), we deduce readily the presentation of τ described by Equation (4.3). $\square$

We are now ready to prove the main result of this subsection.
Proof of Theorem 4.1. Let $\varphi : \tilde{P}_8 \to \mathrm{U}_n$ be the isometry mentioned in Lemma 4.2.
Claim. Equation (4.3) holds actually for all $p \in \tilde{P}_8$.
Indeed, both $\varphi \circ \tau$ and $A\overline{\varphi}A^{-1}$ are isometries $\tilde{P}_8 \to \mathrm{U}_n$, which map J to I_n. It remains to show that their differentials at J are identically equal. By Equation (4.3) they are equal on the last component of the splitting $T_J\tilde{P}_8 = \mathfrak{c} \oplus \check{\mathfrak{p}}$. In fact, they are also equal on $\mathfrak{c}$, in the sense that for any $x \in \mathfrak{c}$ we have

$$(d\varphi)_J \circ \tau_*(x) = A\overline{(d\varphi)_J(x)}A^{-1}.$$

This can be justified as follows. First, by Equation (4.1), the left-hand side is equal to $-(d\varphi)_J(x)$. Second, since $\varphi : \tilde{P}_8 \to \mathrm{U}_n$ is an isometry, $(d\varphi)_J$ is a Lie triple isomorphism between $T_J\check{P}_8$ and $T_{I_n}\mathrm{U}_n$, thus it maps x to the center of $T_{I_n}\mathrm{U}_n$, which is the space of all purely imaginary multiples of the identity; hence we have $\overline{(d\varphi)_J(x)} = -(d\varphi)_J(x)$ and this matrix commutes with A.
Let us now consider the map $c : \mathrm{U}_n \to \mathrm{U}_n$, $c(X) = A\overline{X}A^{-1}$, and observe that the following diagram is commutative:

$$\begin{tikzcd}
\tilde{P}_8 \arrow[r, "\varphi"] \arrow[d, "\tau"] & \mathrm{U}_n \arrow[d, "c"] \\
\tilde{P}_8 \arrow[r, "\varphi"] & \mathrm{U}_n
\end{tikzcd}$$

Since $\varphi(J) = I_n$, we deduce that φ maps $(\tilde{P}_8^\tau)_J$ to $(\mathrm{U}_n)^c_{I_n}$. The latter set, that is, the fixed point set of c, has been determined explicitly in [Wo-84, p. 290]: it is of the form $B\mathrm{O}_nB^{-1}$, for some $B \in \mathrm{U}_n$. The connected component of I_n in this space is $B\mathrm{SO}_nB^{-1}$. On the other hand, by Equation (4.2), we have $(\tilde{P}_8^\tau)_J = P_8$. Thus φ maps P_8 isometrically onto $B\mathrm{SO}_nB^{-1}$. In conclusion, the map $\psi : \tilde{P}_8 \to \mathrm{U}_n$, $\psi(X) = B^{-1}\varphi(X)B$, has all the desired properties. $\square$

4.2 The inclusion $\tilde{P}_8 \subset \bar{P}_8$

The following result is analogous to Theorem 4.1:

Theorem 4.3 *There exists an isometry $\chi : \bar{P}_8 \to \mathrm{Sp}_n$ which maps $\tilde{P}_8$ to U_n and makes the following diagram commutative:*

$$\begin{tikzcd}
\tilde{P}_8 \arrow[r, "\chi|_{\tilde{P}_8}"] \arrow[d, "\cap"] & \mathrm{U}_n \arrow[d, "\cap"] \\
\bar{P}_8 \arrow[r, "\chi"] & \mathrm{Sp}_n
\end{tikzcd}$$

Here the inclusions $\tilde{P}_8 \subset \bar{P}_8$ and $\mathrm{U}_n \subset \mathrm{Sp}_n$ are the one mentioned in the diagram (3.7), respectively the standard one (see e.g. Section A.9 and the beginning of Section 3).

This can be proved by using the same method as in Subsection 4.1. In fact, the proof is even simpler in this case, since, unlike U_n, the symmetric space Sp_n is semisimple, i.e. the corresponding Lie triple has no center.

Remark 4.4 In the same spirit and with the same methods as in Theorems 4.1 and 4.3, one can show that for the embeddings $P_4 \subset \tilde{P}_4$ and $\tilde{P}_4 \subset \bar{P}_4$ one obtains commutative diagrams

$$\begin{tikzcd} P_4 \arrow[r, "\simeq"] \arrow[d, "\cap"] & \mathrm{Sp}_{2n} \arrow[d, "\cap"] \\ \tilde{P}_4 \arrow[r, "\simeq"] & \mathrm{U}_{4n} \end{tikzcd} \qquad \begin{tikzcd} \tilde{P}_4 \arrow[r, "\simeq"] \arrow[d, "\cap"] & \mathrm{U}_{4n} \arrow[d, "\cap"] \\ \bar{P}_4 \arrow[r, "\simeq"] & \mathrm{SO}_{8n} \end{tikzcd}$$

where the horizontal arrows indicate isometries. More precisely, the spaces $P_4, \tilde{P}_4$, and $\bar{P}_4$ have the submanifold metrics arising from the three Bott chains and the spaces Sp_{4n}, U_{8n}, and SO_{8n} have the metrics described earlier in this paper (see the beginning of Section 3) up to appropriate rescalings. The inclusions $P_4 \subset \tilde{P}_4$, $\tilde{P}_4 \subset \bar{P}_4$ are those mentioned in the diagrams (3.4) respectively (3.7) and the inclusions $\mathrm{Sp}_{2n} \subset \mathrm{U}_{4n}$ and $\mathrm{U}_{4n} \subset \mathrm{SO}_{8n}$ are standard, i.e. those described in Subsections A.5, respectively A.13.

Remark 4.5 Assume that in the above context n is divisible by 16. As we have already pointed out (see Remark 2.10 and Sections 2.3 and 2.6), each of the three Bott chains can be extended using the centriole construction. One obtains:

$$\begin{aligned} P_0 \supset P_1 \supset P_2 \supset \ldots \supset P_{16}, \\ \tilde{P}_0 \supset \tilde{P}_1 \supset \tilde{P}_2 \supset \ldots \supset \tilde{P}_{16}, \\ \bar{P}_0 \supset \bar{P}_1 \supset \bar{P}_2 \supset \ldots \supset \bar{P}_{16}, \end{aligned}$$

where we have isometries

$$P_{16} \simeq \mathrm{SO}_{n/16}, \quad \tilde{P}_{16} \simeq \mathrm{U}_{n/16}, \quad \bar{P}_{16} \simeq \mathrm{Sp}_{n/16}.$$

Theorems 4.1 and 4.3 imply that the centriole constructions can be performed in such a way that we have

$$P_k \subset \tilde{P}_k, \quad \tilde{P}_k \subset \bar{P}_k, \quad 8 \le k \le 16,$$

and these inclusions are again those described by Tables 5 and 6, up to some obvious changes of the subscripts. This observation is one of the main achievements of our paper. We can express it in a more informal manner, by saying that the inclusions $P_{k+8} \subset \tilde{P}_{k+8}, \tilde{P}_{k+8} \subset \bar{P}_{k+8}$ are the same as $P_k \subset \tilde{P}_k$, respectively $\tilde{P}_k \subset \bar{P}_k$.

5 Application: periodicity of maps between homotopy groups

In this section we apply the main results of this paper, which are differential geometric, to the topology of classical Riemannian symmetric spaces. The results we prove here, i.e. Theorems 5.3 and 5.6, followed by Corollaries 5.5 and 5.8, are in fact just common knowledge in homotopy theory (one can prove them using techniques described e.g. in [May-77, Ch. 1]). The goal of our approach is to provide more insight concerning these results, by indicating that there is a differential geometric periodicity phenomenon that stays behind them, similar to the periodicity phenomenon that stays behind Bott's classical periodicity theorems [Bo-59].

We start by recalling that a simple application of the long exact homotopy sequence of the principal bundle $\mathrm{U}_m \to \mathrm{U}_{m+1} \to \mathbb{S}^{2m+1}$ shows that the homotopy groups $\pi_i(\mathrm{U}_m)$ are m-stable. More precisely, they remain unchanged up to an isomorphism for any m which is larger than $\frac{i}{2}$. We denote by $\pi_i(\mathrm{U})$ the resulting group, or rather, isomorphism class of groups. The Bott periodicity theorem [Bo-59] for the unitary group says that $\pi_i(\mathrm{U}) = \pi_{i+2}(\mathrm{U})$, for $i = 0, 1, 2, \ldots$. There is also a version of this result for the orthogonal and symplectic group. First of all, we have $\pi_i(\mathrm{O}_m) \simeq \pi_i(\mathrm{O}_{m+1}) =: \pi_i(\mathrm{O})$ for all m and i such that $m \geq i+1$. The periodicity theorem in this case says that $\pi_i(\mathrm{O}) = \pi_{i+8}(\mathrm{O})$, for $i = 0, 1, 2, \ldots$. Similarly, $\pi_i(\mathrm{Sp}) = \pi_{i+8}(\mathrm{Sp})$, for $i = 0, 1, 2, \ldots$ (see [Mi-69, Section 24]).

5.1 The maps induced by $\mathrm{O}_m \hookrightarrow \mathrm{U}_m$

Let us consider the canonical embedding map $\imath_m : \mathrm{O}_m \hookrightarrow \mathrm{U}_m$. Let $f_i^m := (\imath_m)_* : \pi_i(\mathrm{O}_m) \to \pi_i(\mathrm{U}_m)$ be the map between homotopy groups induced by $\imath_m$. The following notion will be used in this section.

Definition 5.1 *Let $\mathcal{A}, \mathcal{A}', \mathcal{B}$, and $\mathcal{B}'$ be groups and $f : \mathcal{A} \to \mathcal{B}$, $f' : \mathcal{A}' \to \mathcal{B}'$ group homomorphisms. We say that f is equivalent to f' and denote $f \sim f'$ if there exist group isomorphisms $g : \mathcal{A} \to \mathcal{A}'$ and $h : \mathcal{B} \to \mathcal{B}'$ that make the following diagram commutative:*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \downarrow g & & \downarrow h \\ \mathcal{A}' & \xrightarrow{f'} & \mathcal{B}' \end{array}$$

We will need the following result.

Lemma 5.2 *The equivalence class modulo $\sim$ of the map $f_i^m : \pi_i(\mathrm{O}_m) \to \pi_i(\mathrm{U}_m)$ is stable. That is, modulo the equivalence relation $\sim$, the map f_i^m is independent of m for all $m \geq i+1$.*

Proof. Let us consider the commutative diagram

$$\begin{array}{ccc} \mathrm{O}_m & \xrightarrow{\imath_m} & \mathrm{U}_m \\ \downarrow & & \downarrow \\ \mathrm{O}_{m+1} & \xrightarrow{\imath_{m+1}} & \mathrm{U}_{m+1} \end{array}$$

The vertical arrows indicate the canonical inclusion maps, given by

$$A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

for any $m \times m$ orthogonal matrix A. By functoriality we obtain the following commutative diagram.

$$\begin{array}{ccc} \pi_i(\mathrm{O}_m) & \xrightarrow{f_i^m} & \pi_i(\mathrm{U}_m) \\ \downarrow & & \downarrow \\ \pi_i(\mathrm{O}_{m+1}) & \xrightarrow{f_i^{m+1}} & \pi_i(\mathrm{U}_{m+1}) \end{array}$$

We only need to recall that for any $m \geq i+1$ both vertical arrows are isomorphisms (to show that the map $\pi_i(\mathrm{O}_m) \to \pi_i(\mathrm{O}_{m+1})$ is an isomorphism for $m \geq i+1$, one uses the long exact sequence of the principal bundle $\mathrm{O}_m \to \mathrm{O}_{m+1} \to \mathbb{S}^m$). □

Let us denote by f_i the *equivalence class* of the map f_i^m, for $m \geq i+1$. Before stating the main result of this subsection, let us note that both the domain and the codomain of the map $f_i^m : \pi_i(\mathrm{O}_m) \to \pi_i(\mathrm{U}_m)$ are periodic relative to i, with period equal to 8. The following theorem says that the map f_i^m itself is periodic (modulo $\sim$).

Theorem 5.3 *We have $f_i = f_{i+8}$, for all $i \geq 0$.*

Proof. Let us first assume that $i > 0$. We use the notations which have been established in the previous sections. The commutative diagram (3.5) induces by functoriality

$$\begin{array}{ccccc} \pi_i(P_{k+1}) & \xrightarrow{(\jmath_{k+1})_*} & \pi_i(\Omega(P_k)) & \xrightarrow{\simeq} & \pi_{i+1}(P_k) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_i(\tilde{P}_{k+1}) & \xrightarrow{(\tilde{\jmath}_{k+1})_*} & \pi_i(\Omega(\tilde{P}_k)) & \xrightarrow{\simeq} & \pi_{i+1}(\tilde{P}_k) \end{array} \tag{5.1}$$

for all $k \in \{0, 1, \ldots, 7\}$. Recall that $P_0 = \mathrm{SO}_{16n}$, $\tilde{P}_0 = \mathrm{U}_{16n}$, and both $(\jmath_{k+1})_*$ and $(\tilde{\jmath}_{k+1})_*$ are isomorphisms for any i which is sufficiently small compared to n (see Subsection 2.7 and the references therein). Since $\pi_{i+8}(\mathrm{SO}_{16n}) = \pi_{i+8}(\mathrm{O}_{16n})$, we obtain the diagram:

$$\begin{tikzcd} \pi_{i+8}(\mathrm{O}_{16n}) \arrow[r, "\simeq"] \arrow[d, "f^{i+8}_{16n}"] & \pi_i(P_8) \arrow[d] \\ \pi_{i+8}(\mathrm{U}_{16n}) \arrow[r, "\simeq"] & \pi_i(\tilde{P}_8) \end{tikzcd}$$

Finally, from Theorem 4.1 we deduce that we have a commutative diagram of the form

$$\begin{tikzcd} \pi_i(P_8) \arrow[r, "\simeq"] \arrow[d] & \pi_i(\mathrm{SO}_n) \arrow[d, "f^n_i"] \\ \pi_i(\tilde{P}_8) \arrow[r, "\simeq"] & \pi_i(\mathrm{U}_n) \end{tikzcd}$$

We only need to use the fact that $\pi_i(\mathrm{SO}_n) = \pi_i(\mathrm{O}_n)$.
We now analyze the case $i = 0$. We have $\pi_0(\mathrm{U}) = \pi_8(\mathrm{U}) = \{0\}$, thus the maps f_0 and f_8 are clearly equal. This finishes the proof of the theorem. $\square$

i mod 8	0	1	2	3	4	5	6	7
$\pi_i(\mathrm{O})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_i(\mathrm{U})$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
f_i	0	0	0	$k \mapsto 2k$	0	0	0	id

Table 1: The stable maps between homotopy groups induced by $\mathrm{O}_m \hookrightarrow \mathrm{U}_m$.

To calculate the maps f_i explicitly, we can use the long exact homotopy sequence of the principal bundle $\mathrm{O}_m \to \mathrm{U}_m \to \mathrm{U}_m/\mathrm{O}_m$. This information is described in Table 1 (where we have used the table from [Mi-69, p. 142]).
Justifications are needed only for the maps f_3 and f_7. Let us calculate the map $f_3 : \pi_3(\mathrm{O}) \to \pi_3(\mathrm{U})$. Since $\pi_4(\mathrm{U}/\mathrm{O}) = 0$ and $\pi_3(\mathrm{U}/\mathrm{O}) = \mathbb{Z}_2$ (cf. e.g. [Bo-59, Section 1]), we obtain the following exact sequence:

$$0 \to \mathbb{Z} \stackrel{f_3}{\to} \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

This implies the desired description of f_3. As about f_7, the relevant exact sequence is

$$0 \to \mathbb{Z} \stackrel{f_7}{\to} \mathbb{Z} \to 0.$$

Remark 5.4 Note that the two exact sequences above can be used to show that for any $j = 0, 1, 2, \dots$, the map $f_{8j+3} : \mathbb{Z} \to \mathbb{Z}$ is given by $k \mapsto 2k$, $k \in \mathbb{Z}$, and $f_{8j+7} : \mathbb{Z} \to \mathbb{Z}$ is the identity map. Therefore this simple argument gives an alternative proof to Theorem 5.3.

We can combine Theorem 5.3 above with the commutative diagram given by (5.1) and the results concerning the exact expressions of the embeddings $P_k \subset \tilde{P}_k$, $k = 1, 2, \ldots, 8$ obtained in Appendix A (see Table 5 and Subsections A.2 - A.8). We deduce:

Corollary 5.5 *Let $A_m \hookrightarrow B_m$ be given by any of the inclusions*

$$\begin{array}{lll} \mathrm{O}_{2m}/\mathrm{U}_m \subset \mathrm{G}_m(\mathbb{C}^{2m}), & \mathrm{U}_{2m}/\mathrm{Sp}_m \subset \mathrm{U}_{2m}, & \mathrm{G}_m(\mathbb{H}^{2m}) \subset \mathrm{G}_{2m}(\mathbb{C}^{4m}), \\ \mathrm{Sp}_m/\mathrm{U}_m \subset \mathrm{G}_m(\mathbb{C}^{2m}), & \mathrm{U}_m/\mathrm{O}_m \subset \mathrm{U}_m, & \mathrm{G}_m(\mathbb{R}^{2m}) \subset \mathrm{G}_m(\mathbb{C}^{2m}), \\ \mathrm{Sp}_m \subset \mathrm{U}_{2m}. & & \end{array}$$

Then the maps $\pi_i(A_m) \to \pi_i(B_m)$ induced between the stable homotopy groups are stable relative to m and periodic relative to i, with period equal to 8.

The exact expression of the stable maps $\pi_i(A_m) \to \pi_i(B_m)$ can be deduced from the table above by finding n and k such that P_k and $\tilde{P}_k$ are equal to A_m respectively B_m for a certain m which depends on n (see Table 5 for $1 \leq k \leq 7$). The only embedding for which this is not possible is $\mathrm{O}_{2m}/\mathrm{U}_m \subset \mathrm{G}_m(\mathbb{C}^{2m})$. In this case, we note that $P_1 = \mathrm{SO}_{2m}/\mathrm{U}_m$, where $m = 8n$ (see Subsection 2.2 or Table 5). Consequently, $\pi_i(\mathrm{O}_{2m}/\mathrm{U}_m) = \pi_i(P_1)$ for any $i \neq 0$ and therefore in this case the map $\pi_i(\mathrm{O}_{2m}/\mathrm{U}_m) \to \pi_i(\mathrm{G}_m(\mathbb{C}^{2m}))$ is equivalent to $\pi_i(P_1) \to \pi_i(\tilde{P}_1)$ in the sense of Definition 5.1. For $i \equiv 0 \mod 8$, we note that $\pi_i(\mathrm{G}_m(\mathbb{C}^{2m})) = \{0\}$, hence the map $\pi_i(\mathrm{O}_{2m}/\mathrm{U}_m) \to \pi_i(\mathrm{G}_m(\mathbb{C}^{2m}))$ is identically zero. To deal with any of the remaining six inclusions we just take $k \in \{2, 3, \ldots, 7\}$ and use inductively the commutative diagram (5.1) to deduce that the map $\pi_i(P_k) \to \pi_i(\tilde{P}_k)$ is equivalent to $\pi_{i+k}(\mathrm{O}_m) \to \pi_{i+k}(\mathrm{U}_m)$ (here $m = 16n$ is in the stability range). For instance the stable maps between homotopy groups induced by the inclusion $\mathrm{Sp}_m \subset \mathrm{U}_{2m}$ are described in Table 2 (see also Remark 4.4).

i mod 8	0	1	2	3	4	5	6	7
$\pi_i(\mathrm{Sp})$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$\pi_i(\mathrm{U})$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\pi_i(\mathrm{Sp}) \to \pi_i(\mathrm{U})$	0	0	0	id	0	0	0	$k \mapsto 2k$

Table 2: The stable maps between homotopy groups induced by $\mathrm{Sp}_m \hookrightarrow \mathrm{U}_{2m}$.

5.2 The maps induced by $\mathrm{U}_m \hookrightarrow \mathrm{Sp}_m$

In the same way as in the previous subsection, we consider the inclusion map $\mathrm{U}_m \to \mathrm{Sp}_m$ and the maps $g_i^m : \pi_i(\mathrm{U}_m) \to \pi_i(\mathrm{Sp}_m)$ induced between homotopy groups. As in Lemma 5.2, if we fix i and take any m which is sufficiently larger than i, all of these group homomorphisms are equivalent in the sense of Definition 5.1. Denote by g_i the equivalence class of these maps. The following result can be proved with the same methods as Theorem 5.3.

Theorem 5.6 *We have* $g_{i+8} = g_i$.

Table 3 describes the maps g_i explicitly.

i mod 8	0	1	2	3	4	5	6	7
$\pi_i(\mathrm{U})$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\pi_i(\mathrm{Sp})$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
g_i	0	0	0	$k \mapsto 2k$	0	$k \mapsto k$ mod 2	0	id

Table 3: The stable maps between homotopy groups induced by $\mathrm{U}_m \hookrightarrow \mathrm{Sp}_m$.

Remark 5.7 The results in Table 3 have been obtained as direct consequences of the long exact homotopy sequence of the principal bundle $\mathrm{U}_m \to \mathrm{Sp}_m \to \mathrm{Sp}_m/\mathrm{U}_m$ and the knowledge of $\pi_i(\mathrm{Sp}/\mathrm{U})$, $i = 0, 1, 2, \ldots$. In fact, this long exact sequence can also be used to give an alternative proof of the periodicity of the maps $\pi_i(\mathrm{U}) \to \pi_i(\mathrm{Sp})$.

In the same way as Corollary 5.5, we can prove the following result (this time using Table 6 in Appendix A and Subsections A.10 - A.16).

Corollary 5.8 *Let* $A_m \hookrightarrow B_m$ *be given by any of the inclusions*

$$\begin{array}{lll} \mathrm{G}_m(\mathbb{C}^{2m}) \subset \mathrm{Sp}_{2m}/\mathrm{U}_{2m}, & \mathrm{U}_m \subset \mathrm{U}_{2m}/\mathrm{O}_m, & \mathrm{G}_m(\mathbb{C}^{2m}) \subset \mathrm{G}_{2m}(\mathbb{R}^{4m}), \\ \mathrm{G}_m(\mathbb{C}^{2m}) \subset \mathrm{O}_{4m}/\mathrm{U}_{2m}, & \mathrm{U}_m \subset \mathrm{U}_{2m}/\mathrm{Sp}_m, & \mathrm{G}_m(\mathbb{C}^{2m}) \subset \mathrm{G}_m(\mathbb{H}^{2m}), \\ \mathrm{U}_m \subset \mathrm{O}_{2m}. & & \end{array}$$

Then the maps $\pi_i(A_m) \to \pi_i(B_m)$ *induced between the stable homotopy groups are stable relative to* m *and periodic relative to* i*, with period equal to 8.*

These maps $\pi_i(A_m) \to \pi_i(B_m)$ mentioned above can be described explicitly, by using Table 3 and the fact that $\pi_i(\tilde{P}_k) \to \pi_i(\bar{P}_k)$ is equivalent to $\pi_{i+k}(\mathrm{U}_m) \to \pi_{i+k}(\mathrm{Sp}_m)$. For example, the stable maps $\pi_i(\mathrm{U}_m) \to \pi_i(\mathrm{O}_{2m})$ are described in Table 4.

i mod 8	0	1	2	3	4	5	6	7
$\pi_i(\mathrm{U})$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\pi_i(\mathrm{O})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_i(\mathrm{U}) \to \pi_i(\mathrm{O})$	0	$k \mapsto k$ mod 2	0	id	0	0	0	$k \mapsto 2k$

Table 4: The stable maps between homotopy groups induced by $\mathrm{U}_m \hookrightarrow \mathrm{O}_{2m}$.

A Standard inclusions of symmetric spaces

Explicit descriptions of the spaces in the SO-Bott chain have been obtained by Milnor in [Mi-69, Section 24] using orthogonal complex structures of $\mathbb{R}^{16n}$, i.e. elements J of O_{16n} with the property that $J^2 = -I$. A similar construction works for any matrix Lie group, as it has been pointed out in [Qu-10]. For our needs, we describe the U-Bott chain in these terms. We start with the following definition.

Definition A.1 *An element $J \in \mathrm{U}_{16n}$ is a* complex structure *if $J^2 = -I$.*

Like in the case of the orthogonal group (see [Mi-69, Lemma 24.1]) we can identify complex structures in U_{16n} with midpoints of shortest geodesic segments in U_{16n} from I to $-I$. More specifically, recall from Subsection 2.4 that $-I$ is a pole of (U_{16n}, I) and the space of shortest geodesic segments in U_{16n} from I to $-I$ is the union of all conjugacy orbits $\mathrm{U}_{16n}.\gamma_k|_{[0,1]}$, $0 \le k \le 16n$ (see Equation (2.4)).

Lemma A.2 *The set of all midpoints of the geodesic segments in the union $\bigcup_{0\le k\le 16n} \mathrm{U}_{16n}.\gamma_k|_{[0,1]}$ coincides with the set of all complex structures in U_{2q}.*

Proof. Let $\gamma|_{[0,1]} : [0,1] \to \mathrm{U}_{16n}$ be a geodesic segment in the union above: it satisfies $\gamma(0) = I$ and $\gamma(1) = -I$. Then $\gamma : \mathbb{R} \to \mathrm{U}_{16n}$ is a one-parameter subgroup. Thus we have

$$\gamma\left(\frac{1}{2}\right)^2 = \gamma(1) = -I.$$

This means that $\gamma\left(\frac{1}{2}\right)$ is a complex structure. To prove the converse inclusion, take $J \in \mathrm{U}_{16n}$ such that $J^2 = -I$. Then the eigenvalues of J are $\pm i$, hence J is U_{16n}-conjugate to a matrix of the form

$$\begin{pmatrix} iI_k & 0 \\ 0 & -iI_{16n-k} \end{pmatrix}$$

for some $k \in \{0, 1, \dots, 16n\}$. The converse inclusion is proved. $\square$

Recall from Subsection 2.6 that $\tilde{P}_1$ is the top-dimensional s-centriole of (U_{16n}, I) and J_1 is an element of $\tilde{P}_1$. The previous lemma says that the union of all s-centrioles in U_{16n} from I to $-I$ is the same as the set of all complex structures in U_{16n}. We deduce that

$$\tilde{P}_1 = \{J \in \mathrm{U}_{16n} \ : \ J^2 = -I\}_{J_1}, \tag{A.1}$$

where we have used the notation established at the beginning of Subsection 3.2.

We saw afterwards that we have the isometry $\tilde{P}_1 \simeq \mathrm{G}_{8n}(\mathbb{C}^{16n})$, where $\tilde{P}_1$ is equipped with the metric induced by its embedding in U_{16n} and $\mathrm{G}_{8n}(\mathbb{C}^{16n})$ with the usual symmetric space metric. We defined $\tilde{P}_2$ as the (unique) s-centriole of $(\tilde{P}_1, J_1)$ relative to $-J_1$ and then we fixed an element J_2 of $\tilde{P}_2$.

Lemma A.3 *The centrosome $C_{-J_1}(\tilde{P}_1, J_1)$ can be expressed as:*

$$C_{-J_1}(\tilde{P}_1, J_1) = \{J \in \tilde{P}_1 \ : \ JJ_1 = -J_1 J\}. \tag{A.2}$$

Consequently, $\tilde{P}_2 = \{J \in \tilde{P}_1 \ : \ JJ_1 = -J_1 J\}_{J_2}$.

Proof. We first take $J = \gamma(\frac{1}{2})$, where $\gamma : \mathbb{R} \to \tilde{P}_1$ is a geodesic such that

$$\gamma(0) = J_1 \quad \text{and} \quad \gamma(1) = -J_1.$$

But $\tilde{P}_1$ is a totally geodesic submanifold of U_{16n}, thus γ is a geodesic in U_{16n}. We deduce that there exists x in $\mathfrak{u}_{16n}$ such that

$$\gamma(t) = J_1 \exp(2tx), \ t \in \mathbb{R}.$$

The condition $\gamma\left(\frac{1}{2}\right) = J$ implies that $J_1 \exp(x) = J$. Multiplying from the left by J_1 and taking into account that $J_1^2 = -I$ gives $\exp(x) = -J_1 J$. Consequently, we have

$$-J_1 = \gamma(1) = J_1 \exp(x) \exp(x) = J(-J_1 J) = -JJ_1 J.$$

This implies that $JJ_1 = -J_1 J$.
We now prove the converse inclusion. Take $J \in \tilde{P}_1$ such that $JJ_1 = -J_1 J$. Let $\gamma : \mathbb{R} \to \tilde{P}_1$ be a geodesic with the property that

$$\gamma(0) = J_1 \quad \text{and} \quad \gamma\left(\frac{1}{2}\right) = J.$$

Claim. $\gamma(1) = -J_1$.
Indeed, the curve $J_1^{-1}\gamma$ is a geodesic in U_{16n}, thus a one-parameter group. In other words, we have

$$J_1^{-1}\gamma(t) = \exp(2tx), \ t \in \mathbb{R},$$

where $x \in \mathfrak{u}_{16n}$. This implies that $J = \gamma(\frac{1}{2}) = J_1 \exp(x)$, hence $\exp(x) = J_1^{-1} J = -J_1 J = JJ_1$, and consequently

$$\gamma(1) = J_1 \exp(2x) = J_1 \exp(x) \exp(x) = J(JJ_1) = -J_1.$$

□

Note that the previous two results are special cases of [Qu-10, Lemmata 3.1 and 3.2].
We have the isometry $\tilde{P}_2 \simeq \mathrm{U}_{8n}$, where $\tilde{P}_2$ has the submanifold metric and U_{8n} the bi-invariant metric described by Equation (2.7) with $q = 4n$ (see Subsection 2.5). The pair $(\tilde{P}_2, J_2)$ has exactly one pole, which is $-J_2$. We defined $\tilde{P}_3$ as the top-dimensional s-centriole of $(\tilde{P}_2, J_2)$ relative to $-J_2$ and we

fixed $J_3 \in \tilde{P}_3$. Since $\tilde{P}_2$ is a totally geodesic submanifold of U_{16n}, we can use the same reasoning as in the proof of Lemma A.3 to show that

$$\tilde{P}_3 = \{J \in \tilde{P}_2 \ : \ JJ_2 = -J_2J\}_{J_3}.$$

In the same way, for any $k \in \{2, \ldots, 8\}$ we have

$$\tilde{P}_k = \{J \in \tilde{P}_{k-1} \ : \ JJ_{k-1} = -J_{k-1}J\}_{J_k}.$$

These new presentations of the spaces $\tilde{P}_1, \ldots, \tilde{P}_8$, along with those obtained by Milnor in [Mi-69, Section 24] for the spaces $P_1, \ldots, P_8$ lead us to descriptions of the embeddings between Bott chains which are discussed in Section 3. Concretely, they are given by Tables 5 and 6 together with the list A.1 - A.16.

k	P_k	$\tilde{P}_k$	$P_k \subset \tilde{P}_k$
0	SO_{16n}	U_{16n}	A.1 composed with $\mathrm{SO}_{16n} \subset \mathrm{O}_{16n}$
1	$\mathrm{SO}_{16n}/\mathrm{U}_{8n}$	$\mathrm{G}_{8n}(\mathbb{C}^{16n})$	A.2 composed with $\mathrm{SO}_{16n}/\mathrm{U}_{8n} \subset \mathrm{O}_{16n}/\mathrm{U}_{8n}$
2	$\mathrm{U}_{8n}/\mathrm{Sp}_{4n}$	U_{8n}	A.3
3	$\mathrm{G}_{2n}(\mathbb{H}^{4n})$	$\mathrm{G}_{4n}(\mathbb{C}^{8n})$	A.4
4	Sp_{2n}	U_{4n}	A.5
5	$\mathrm{Sp}_{2n}/\mathrm{U}_{2n}$	$\mathrm{G}_{2n}(\mathbb{C}^{4n})$	A.6
6	$\mathrm{U}_{2n}/\mathrm{O}_{2n}$	U_{2n}	A.7
7	$\mathrm{G}_n(\mathbb{R}^{2n})$	$\mathrm{G}_n(\mathbb{C}^{2n})$	A.8
8	SO_n	U_n	A.1 composed with $\mathrm{SO}_n \subset \mathrm{O}_n$

Table 5: Inclusions between the SO-Bott chain and the U-Bott chain.

k	$\tilde{P}_k$	$\bar{P}_k$	$\tilde{P}_k \subset \bar{P}_k$
0	U_{16n}	Sp_{16n}	A.9
1	$\mathrm{G}_{8n}(\mathbb{C}^{16n})$	$\mathrm{Sp}_{16n}/\mathrm{U}_{16n}$	A.10
2	U_{8n}	$\mathrm{U}_{16n}/\mathrm{O}_{16n}$	A.11
3	$\mathrm{G}_{4n}(\mathbb{C}^{8n})$	$\mathrm{G}_{8n}(\mathbb{R}^{16n})$	A.12
4	U_{4n}	SO_{8n}	A.13
5	$\mathrm{G}_{2n}(\mathbb{C}^{4n})$	$\mathrm{SO}_{8n}/\mathrm{U}_{4n}$	A.14
6	U_{2n}	$\mathrm{U}_{4n}/\mathrm{Sp}_{2n}$	A.15
7	$\mathrm{G}_n(\mathbb{C}^{2n})$	$\mathrm{G}_n(\mathbb{H}^{2n})$	A.16
8	U_n	Sp_n	A.9

Table 6: Inclusions between the U-Bott chain and the Sp-Bott chain.

In what follows we will be frequently using, without pointing it out each time, presentations of certain classical symmetric spaces as given in [Mi-69, Section 24] (see also [Es-08, Section 19]).

A.1 The inclusion $\mathrm{O}_r \subset \mathrm{U}_r$

To any orthogonal isomorphism $A : \mathbb{R}^r \to \mathbb{R}^r$ one attaches its complex-linear extension $A^c : \mathbb{C}^r \to \mathbb{C}^r$, which is defined by $A^c(u + iv) := A(u) + iA(v)$, for all $u, v \in \mathbb{R}^r$. One can see that A^c preserves the norm of a vector in $\mathbb{C}^r$ with respect to the canonical Hermitian product. Thus, A^c lies in U_r.

A.2 The inclusion $\mathrm{O}_{2r}/\mathrm{U}_r \subset \mathrm{G}_r(\mathbb{C}^{2r})$

The quotient $\mathrm{O}_{2r}/\mathrm{U}_r$ is identified with the space of all orthogonal complex structures of $\mathbb{R}^{2r}$, that is, of all $J \in \mathrm{O}_{2r}$ with the property that $J^2 = -I$. The inclusion $\mathrm{O}_{2r}/\mathrm{U}_r \subset \mathrm{G}_r(\mathbb{C}^{2r})$ assigns to any such J the eigenspace $E_i(J^c) = \{v \in \mathbb{C}^{2r} \ : \ J^c(v) = iv\}$.

A.3 The inclusion $\mathrm{U}_{2r}/\mathrm{Sp}_r \subset \mathrm{U}_{2r}$

Fix $J_0 \in \mathrm{O}_{4r}$ an orthogonal complex structure of $\mathbb{R}^{4r}$. Let $J_0^c : \mathbb{C}^{4r} \to \mathbb{C}^{4r}$ be its complex linear extension. The eigenspaces $V^+ := E_i(J_0^c)$ and $V^- := E_{-i}(J_0^c)$ are complex vector subspaces of $\mathbb{C}^{4r}$ of dimension equal to $2r$, since the complex conjugation is an ($\mathbb{R}$-linear) isomorphism between V^+ and V^-. The quotient $\mathrm{U}_{2r}/\mathrm{Sp}_r$ can be identified with the space of all orthogonal complex structures J of $\mathbb{R}^{4r}$ that anticommute with J_0. The complex linear extension J^c of such a J maps V^+ to V^-, being obviously a unitary isomorphism. The inclusion $\mathrm{U}_{2r}/\mathrm{Sp}_r \hookrightarrow \mathrm{U}_{2r}$ assigns to J the map $J^c|_{V^+} : V^+ \to V^-$, where both V^+ and V^- are identified with $\mathbb{C}^{2r}$.

A.4 The inclusion $\mathrm{G}_r(\mathbb{H}^{2r}) \subset \mathrm{G}_{2r}(\mathbb{C}^{4r})$

We first identify $\mathbb{C}$ with the subspace of $\mathbb{H}$ consisting of all quaternions $a + bi + cj + dk$ with $c = d = 0$. This allows us to equip $\mathbb{H}^{2r}$ with the structure of complex vector space induced by multiplication with complex numbers from the right. It also allows us to embed $\mathbb{C}^{2r}$ into $\mathbb{H}^{2r}$. In this way we obtain the following identification of complex vector spaces: $\mathbb{H}^{2r} = \mathbb{C}^{2r} \oplus j\mathbb{C}^{2r} = \mathbb{C}^{4r}$. The Grassmannian $\mathrm{G}_r(\mathbb{H}^{2r})$ consists of all right $\mathbb{H}$-submodules of $\mathbb{H}^{2r}$ of dimension equal to r. The map $\mathrm{G}_r(\mathbb{H}^{2r}) \hookrightarrow \mathrm{G}_{2r}(\mathbb{C}^{4r})$ attaches to any such submodule $V \subset \mathbb{H}^{2r}$ the space V itself, regarded as a $2r$-dimensional complex subspace of $\mathbb{C}^{4r}$.

A.5 The inclusion $\mathrm{Sp}_r \subset \mathrm{U}_{2r}$

As explained before, we can regard $\mathbb{H}^r = \mathbb{C}^r \oplus j\mathbb{C}^r = \mathbb{C}^{2r}$ as a complex vector spaces. The map $\mathrm{Sp}_r \hookrightarrow \mathrm{U}_{2r}$ assigns to any symplectic ($\mathbb{H}$-linear on the right) isomorphism $A : \mathbb{H}^r \to \mathbb{H}^r$ the map A itself, regarded as a unitary ($\mathbb{C}$-linear) isomorphism $\mathbb{C}^{2r} \to \mathbb{C}^{2r}$. A description of this embedding in matrix form can be found for instance [Br-tD-85, Ch. I, Section 1.11] (see also the beginning of Section 3).

A.6 The inclusion $\mathrm{Sp}_r/\mathrm{U}_r \subset \mathrm{G}_r(\mathbb{C}^{2r})$

The quotient $\mathrm{Sp}_r/\mathrm{U}_r$ can be identified with the space of all complex forms of the quaternionic space $\mathbb{H}^r$, that is, all $V \subset \mathbb{H}^r$ which is a complex vector subspace relative to the identification $\mathbb{H}^r = \mathbb{C}^r + j\mathbb{C}^r = \mathbb{C}^{2r}$ mentioned above and satisfies $\mathbb{H}^r = V \oplus jV$. The inclusion map $\mathrm{Sp}_r/\mathrm{U}_r \hookrightarrow \mathrm{G}_r(\mathbb{C}^{2r})$ assigns to any such V the space V itself.

A.7 The inclusion $\mathrm{U}_r/\mathrm{O}_r \subset \mathrm{U}_r$

The quotient $\mathrm{U}_r/\mathrm{O}_r$ can be identified with the space of all real forms of $\mathbb{C}^r$, that is, all real vector subspaces $V \subset \mathbb{C}^r$ such that $\mathbb{C}^r = V \oplus iV$. This can be further identified with the space of all orthogonal ($\mathbb{R}$-linear) automorphisms of $\mathbb{C}^r = \mathbb{R}^{2r}$ that are anti-complex linear and square to I: the identification is given by attaching to such an automorphism its 1-eigenspace. If we fix an anti-complex linear orthogonal automorphism B_0 of $\mathbb{R}^{2n}$, then

$$\mathrm{U}_r/\mathrm{O}_r = \{B_0 A \ : \ A \in \mathrm{U}_r, (B_0A)^2 = I\}.$$

The inclusion map $\mathrm{U}_r/\mathrm{O}_r \to \mathrm{U}_r$ maps B_0A to A.

A.8 The inclusion $\mathrm{G}_r(\mathbb{R}^{2r}) \subset \mathrm{G}_r(\mathbb{C}^{2r})$

This map assigns to any r-dimensional real vector subspace of $\mathbb{R}^{2r}$ the space $V \otimes \mathbb{C}$, which is an r-dimensional complex vector subspace of $\mathbb{C}^{2r}$.

A.9 The inclusion $\mathrm{U}_r \subset \mathrm{Sp}_r$

Let R_i and R_j be the maps $\mathbb{H}^r \to \mathbb{H}^r$ given by multiplication from the right by the quaternionic units i and j. Then Sp_r can be characterized as the space of all $\mathbb{R}$-linear endomorphisms of $\mathbb{H}^r$ which commute with R_i and R_j and preserve the norm of any vector in $\mathbb{H}^r$ relative to the canonical inner product of $\mathbb{H}^r$. Let us consider the splitting $\mathbb{H}^r = \mathbb{C}^r \oplus j\mathbb{C}^r$. The group U_r consists of all $\mathbb{R}$-linear endomorphisms of $\mathbb{C}^r$ which commute with R_i and preserve the norm of any vector in $\mathbb{C}^r$ relative to the canonical Hermitian product of $\mathbb{C}^r$. The desired embedding $\mathrm{U}_r \hookrightarrow \mathrm{Sp}_r$ is given by

$$\mathrm{U}_r \ni A \mapsto A^h \in \mathrm{Sp}_r,$$

where $A^h : \mathbb{H}^r \to \mathbb{H}^r$ is determined by:

$$A^h(v + jw) := Av + j(\bar{A}w), \quad v, w \in \mathbb{C}^r.$$

(One can easily verify that A^h lies in Sp_r.)

A.10 The inclusion $\mathrm{G}_r(\mathbb{C}^{2r}) \subset \mathrm{Sp}_{2r}/\mathrm{U}_{2r}$

The quotient $\mathrm{Sp}_{2r}/\mathrm{U}_{2r}$ can be identified with the space of all complex forms of $\mathbb{H}^{2r}$, that is, of all real vector subspace $X \subset \mathbb{H}^{2r}$ with the property that $R_iX = X$, i.e. X is a complex vector subspace of $\mathbb{H}^{2r}$, and $\mathbb{H}^{2r} = X \oplus R_jX$ (orthogonal direct sum). The inclusion $\mathrm{G}_r(\mathbb{C}^{2r}) \hookrightarrow \mathrm{Sp}_{2r}/\mathrm{U}_{2r}$ assigns to the r-dimensional complex vector subspace $V \subset \mathbb{C}^{2r}$ the space $V \oplus R_jV^\perp$, where $V^\perp$ is the orthogonal complement of V in $\mathbb{C}^{2r}$.

A.11 The inclusion $\mathrm{U}_r \subset \mathrm{U}_{2r}/\mathrm{O}_{2r}$

Recall that $\mathrm{U}_{2r}/\mathrm{O}_{2r}$ is the space of all real forms of $\mathbb{C}^{2r}$ (see Subsection A.7). Also recall that $\mathbb{H}^r$ is a complex vector space relative to multiplication by complex numbers from the right, the dimension being equal to $2r$. Let us now consider the splitting $\mathbb{H}^r = \mathbb{C}^r \oplus R_j\mathbb{C}^r$. The inclusion $\mathrm{U}_r \hookrightarrow \mathrm{U}_{2r}/\mathrm{O}_{2r}$ can be described as follows:

$$\mathrm{U}_r \ni A \mapsto V := \{v + R_jAv \ : \ v \in \mathbb{C}^r\}.$$

Note that V described by this equation is a real form of $\mathbb{H}^r$, where the latter is a complex vector space in the way mentioned above. Indeed, this follows readily from the fact that $Vi = \{v - R_jAv \ : \ v \in \mathbb{C}^r\}$.

A.12 The inclusion $\mathrm{G}_r(\mathbb{C}^{2r}) \subset \mathrm{G}_{2r}(\mathbb{R}^{4r})$

This map assigns to a complex n-dimensional vector subspace $V \subset \mathbb{C}^{2r}$ the space V itself, viewed as a real vector subspace of $\mathbb{C}^{2r} = \mathbb{R}^{4r}$.

A.13 The inclusion $\mathrm{U}_r \subset \mathrm{SO}_{2r}$.

We identify $\mathbb{C}^r = \mathbb{R}^r \oplus i\mathbb{R}^r$ with $\mathbb{R}^{2r}$ and make the following elementary observations: a $\mathbb{C}$-linear transformation of $\mathbb{C}^r$ is also $\mathbb{R}$-linear; the norm of a vector in $\mathbb{C}^r$ relative to the standard Hermitian inner product is equal to its norm in $\mathbb{R}^{2r}$ relative to the standard Euclidean inner product. We are lead to the subgroup embedding $\mathrm{U}_r \hookrightarrow \mathrm{O}_{2r}$. Since U_r is connected, we actually get $\mathrm{U}_r \hookrightarrow \mathrm{SO}_{2r}$.

A.14 The inclusion $\mathrm{G}_r(\mathbb{C}^{2r}) \subset \mathrm{SO}_{4r}/\mathrm{U}_{2r}$

We start with the embedding $\mathrm{U}_{2r} \subset \mathrm{SO}_{4r}$ described in Subsection A.13. It induces the inclusion $\{J \in \mathrm{U}_{2r} \ : \ J^2 = -I\} \subset \{J \in \mathrm{SO}_{4r} \ : \ J^2 = -I\}$. The first space can be identified with the Grassmannian of all complex vector subspaces in $\mathbb{C}^{2r}$ (see Lemma A.2 and Subsection 2.4). Among its connected components we can find $\mathrm{G}_r(\mathbb{C}^{2r})$. This is contained in one of the two connected components of $\{J \in \mathrm{SO}_{4r} \ : \ J^2 = -I\}$. They are both diffeomorphic to $\mathrm{SO}_{4r}/\mathrm{U}_{2r}$. The desired embedding is now clear.

A.15 The inclusion $\mathrm{U}_r \subset \mathrm{U}_{2r}/\mathrm{Sp}_r$

We first consider

$$A_r := \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix} \in \mathrm{U}_{2r},$$

which is an orthogonal complex structure of $\mathbb{R}^{4r}$ via the embedding described at A.13. Denote by $\mathrm{U}(\mathbb{R}^{4r}, A_r)$ the set of all elements of O_{4r} which commute with A_r. This is a subgroup of O_{4r} which is isomorphic to U_{2r}. It acts transitively, via group conjugation, on the set of all $J \in \mathrm{O}_{4r}$ with $J^2 = -I$ and $A_r J = -JA_r$. Moreover, the stabilizer of any J is isomorphic to Sp_r. In this way we obtain the identification

$$\{J \in \mathrm{O}_{4r} \ : \ J^2 = -I, JA_r = -A_r J\} = \mathrm{U}_{2r}/\mathrm{Sp}_r.$$

The embedding $\mathrm{U}_r \hookrightarrow \mathrm{U}_{2r}/\mathrm{Sp}_r$ assigns to an arbitrary $X \in \mathrm{U}_r$ the matrix

$$A := \begin{pmatrix} 0 & -X^{-1} \\ X & 0 \end{pmatrix} \in \mathrm{U}_{2r},$$

which is regarded as an element of O_{4r} in the same way as before, i.e. by using the embedding A.13. (One can easily verify that $A^2 = -I$ and $A_r A = -AA_r$.)

A.16 The inclusion $\mathrm{G}_r(\mathbb{C}^{2r}) \subset \mathrm{G}_r(\mathbb{H}^{2r})$

We consider again the embedding $\mathbb{C}^{2r} \subset \mathbb{H}^{2r}$ defined in Subsection A.4. The embedding $\mathrm{G}_r(\mathbb{C}^{2r}) \hookrightarrow \mathrm{G}_r(\mathbb{H}^{2r})$ assigns to a complex r-dimensional vector subspace $V \subset \mathbb{C}^{2r}$ the space $V \otimes_{\mathbb{C}} \mathbb{H} = \{v + wj \ : \ v, w \in V\}$, which is an $\mathbb{H}$-linear subspace of $\mathbb{H}^{2r}$ of dimension r.

B The isometry types of P_4 and P_8

For any $r \geq 1$, we consider the *standard* bi-invariant Riemannian metrics on each of the groups SO_r, U_r, and Sp_r. By definition, they are given by $\langle X, Y\rangle = -\mathrm{tr}(XY)$, for any X, Y in the Lie algebra of SO_r, respectively U_r; as for Sp_r, the metric is induced by its canonical embedding in U_{2r}, where the latter group is equipped with the standard metric divided by two (see also the beginning of Section 3).
The SO-Bott chain $P_0, P_1, \ldots, P_8$ has been defined in Subsection 2.2. Recall that $P_1, \ldots, P_8$ are totally geodesic submanifolds of $P_0 = \mathrm{SO}_{16n}$, the latter space being equipped with the standard metric. The main goal of this section is to prove the following result.

Proposition B.1 *(a) If we equip P_4 with the submanifold metric, then P_4 is isometric to Sp_{2n}, where the metric on the latter space is eight times the standard one.*
(b) If we equip P_8 with the submanifold metric, then P_8 is isometric to SO_n, where the metric on the latter space is sixteen times the standard one.

Proof. The proof will be divided into two steps. In the first step, we show that the results stated by the proposition hold true for a particular choice of complex structures $J_1, \ldots, J_7$. (Afterwards we will address the general situation.) Concretely, we write

$$\mathbb{H}^{4n} = \mathbb{R}^{4n} \oplus \mathbb{R}^{4n} i \oplus \mathbb{R}^{4n} j \oplus \mathbb{R}^{4n} k,$$

and identify in this way $\mathbb{H}^{4n} = \mathbb{R}^{16n}$. Take $J_1 := R_i$ and $J_2 := R_j$, that is, multiplication on $\mathbb{H}^{4n}$ from the right by the quaternionic units i, respectively j. We take J_3 in such a way, that $J_1J_2J_3$ is given by

$$J_1J_2J_3(q_1, \ldots, q_{4n}) := (q_1, \ldots, q_{2n}, -q_{2n+1}, \ldots, -q_{4n}),$$

for all $(q_1, \ldots, q_{4n}) \in \mathbb{H}^{4n}$.
We now perform Milnor's construction of the space P_4 (cf. [Mi-69, p. 139], see also Subsection 2.2 above). First, note that the eigenspace decomposition of $J_1J_2J_3 : \mathbb{H}^{4n} \to \mathbb{H}^{4n}$ is $\mathbb{H}^{4n} = \mathbb{H}^{2n} \oplus (\mathbb{H}^{2n})^\perp$, where $\mathbb{H}^{2n}$ stands here for the space of all vectors in $\mathbb{H}^{4n}$ with the last $2n$ entries equal to 0 and $(\mathbb{H}^{2n})^\perp$ is the space of all vectors in $\mathbb{H}^{4n}$ with the first $2n$ entries equal to 0. The space P_4 consists of all $J \in P_3$ which anticommute with J_3. If J is such a transformation, then J_3J maps $\mathbb{H}^{2n}$ to $(\mathbb{H}^{2n})^\perp$ as a $\mathbb{H}$-linear map (relative to scalar multiplication from the right) that preserves the norm of any vector.
Let us now consider the subgroup of SO_{16n} which consists of all $\mathbb{R}$-linear endomorphisms of $\mathbb{R}^{16n}$ that are $\mathbb{H}$-linear, i.e. commute with J_1 and J_2, and preserve the norm of a vector. This group is just Sp_{4n}. We prefer to see its elements as $4n \times 4n$ matrices, say A, with entries in $\mathbb{H}$, such that $AA^* = I_{4n}$. From the above observation, $J_3P_4 := \{J_3J \ : \ J \in P_4\}$ is the same as the space of all elements of Sp_{4n} of the form

$$\begin{pmatrix} 0 & -C^{-1} \\ C & 0 \end{pmatrix}.$$

Consider

$$B_{2n} := \begin{pmatrix} 0 & I_{2n} \\ -I_{2n} & 0 \end{pmatrix},$$

which is an element of Sp_{4n}. By translating our set J_3P_4 from the left by B_{2n}, we obtain

$$B_{2n}(J_3P_4) = \left\{ \begin{pmatrix} C & 0 \\ 0 & C^{-1} \end{pmatrix} \ : \ C \in \mathrm{Sp}_{2n} \right\}.$$

It is clear that P_4, as a submanifold of SO_{16n}, is isometric to $B_{2n}(J_3P_4)$, and the latter is a subspace of Sp_{4n}. More precisely, it is the image of the embedding $\mathrm{Sp}_{2n} \to \mathrm{Sp}_{4n}$,

$$C \mapsto \begin{pmatrix} C & 0 \\ 0 & C^{-1} \end{pmatrix}. \tag{B.1}$$

The metric on Sp_{4n} induced by its embedding in SO_{16n} is equal to the standard metric multiplied by 4. (Indeed, Sp_{4n} is contained in the subspace of all elements of SO_{16n} which commute with J_1, which is U_{8n}, and the resulting embedding $\mathrm{Sp}_{4n} \subset \mathrm{U}_{8n}$ is just the one described at the beginning of Section 3; moreover, the Riemannian metric on U_{8n} induced by its embedding in SO_{16n} is twice its standard metric.) Thus, the submanifold metric induced on Sp_{2n} via the embedding (B.1) is the standard one multiplied by 8. Finally note that, from the previous considerations, Sp_{2n} equipped with this metric is isometric to the subspace P_4 of SO_{16n}.

We will now prove point (b) of Proposition B.1 for a particular choice of J_5, J_6, and J_7. As it has been pointed out by Eschenburg in [Es-08, Section 19], we may assume that $P_4 = \mathrm{Sp}_{2n}$ and P_5, P_6, P_7, P_8 are subspaces of Sp_{2n} defined as follows: first, $P_5 := \{J' \in \mathrm{Sp}_{2n} \ : \ (J')^2 = -I\}$; then, for $\ell = 5, 6$, or 7, pick $J'_\ell \in P_\ell$ and define $P_{\ell+1}$ as one of the top-dimensional components of the space $\{J' \in P_\ell \ : \ J'J'_\ell + J'_\ell J' = 0\}$. In what follows Sp_{2n} is regarded as the space of all $\mathbb{R}$-linear endomorphisms of $\mathbb{H}^{2n}$ which preserve the norm of a vector and commute with R'_i and R'_j, the operators given by multiplication from the right by i, respectively j.

We first consider $J'_5 : \mathbb{H}^{2n} \to \mathbb{H}^{2n}$ given by multiplication from the left by the negative of the quaternionic unit i, that is

$$J'_5(q_1, \dots, q_{2n}) := -i(q_1, \dots, q_{2n}),$$

for all $(q_1, \dots, q_{2n}) \in \mathbb{H}^{2n}$. One can see that J'_5 is an element of Sp_{2n} and satisfies $(J'_5)^2 = -I$. Note that the 1-eigenspace of $R'_i J'_5$ is $\mathbb{C}^{2n}$, which is canonically embedded in $\mathbb{H}^{2n}$. Next, we take $J'_6 : \mathbb{H}^{2n} \to \mathbb{H}^{2n}$ given by multiplication from the left by $-j$:

$$J'_6(q_1, \dots, q_{2n}) := -j(q_1, \dots, q_{2n}),$$

for all $(q_1, \dots, q_{2n}) \in \mathbb{H}^{2n}$. As before, J'_6 is in Sp_{2n} and $(J'_6)^2 = I$. We also have $J'_5 J'_6 = -J'_6 J'_5$. The composed map $R'_j J'_6$ leaves $\mathbb{C}^{2n}$ invariant and the 1-eigenspace of $R'_j J'_6|_{\mathbb{C}^{2n}}$ is $\mathbb{R}^{2n}$, which is canonically embedded in $\mathbb{C}^{2n}$. Finally, we choose J'_7 to be the map $\mathbb{H}^{2n} \to \mathbb{H}^{2n}$,

$$J'_7(q_1, \dots, q_{2n}) := -k(q_1, \dots, q_n, -q_{n+1}, \dots, -q_{2n}),$$

for all $(q_1, \dots, q_{2n}) \in \mathbb{H}^{2n}$. This new map is in Sp_{2n}, it squares to $-I$, and it anticommutes with both J'_5 and J'_6. The composed map $R'_k J'_7$ leaves $\mathbb{R}^{2n}$ invariant and we have

$$R'_k J'_7(x_1, \dots, x_{2n}) = (x_1, \dots, x_n, -x_{n+1}, \dots, -x_{2n})$$

for all $(x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$. Consequently, the 1-eigenspace of $R'_k J'_7|_{\mathbb{R}^{2n}}$ is $\mathbb{R}^n$, that is, the subspace of $\mathbb{R}^{2n}$ consisting of all vectors with the last n components equal to 0. The (-1)-eigenspace of $R'_k J'_7|_{\mathbb{R}^{2n}}$ is $(\mathbb{R}^n)^\perp$, the orthogonal complement of $\mathbb{R}^n$ in $\mathbb{R}^{2n}$.

We are especially interested in the embedding of P_8 in Sp_{2n}. By [Mi-69, p. 141] (see also [Es-08, Section 19, item 8']), one can identify P_8 with one of the two connected components of the space of all orthogonal transformations from $\mathbb{R}^n$

to $(\mathbb{R}^n)^\perp$; the identification is given by $J' \mapsto J'_7 J'|_{\mathbb{R}^n}$. We deduce that $J'_7 P_8$ is one of the two connected components of the subspace of Sp_{2n} consisting of all matrices of the form

$$\begin{pmatrix} 0 & -D^{-1} \\ D & 0 \end{pmatrix}$$

where $D \in \mathrm{O}_n$. We may assume that $J'_7 P_8$ is the space of all matrices of the form above with $D \in \mathrm{SO}_n$. Let us now consider the matrix

$$B_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and observe that

$$B_n(J'_7 P_8) = \left\{ \begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix} \ : \ D \in \mathrm{SO}_n \right\}.$$

The subspaces P_8 and $B_n(J'_7 P_8)$ of Sp_{2n} are isometric. We only need to characterize the submanifold metric on SO_n induced by the embedding $\mathrm{SO}_n \to \mathrm{Sp}_{2n}$,

$$D \mapsto \begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix},$$

where Sp_{2n} is equipped with the standard metric multiplied by eight (by point (a)). To this end we first look at the subspaces U_{2n} and SO_{2n} of Sp_{2n}: the metric induced on U_{2n} is eight times its canonical metric (see the beginning of Section 3), thus also the metric on SO_{2n} is eight times its canonical metric. Consequently, the metric on SO_n we are interested in is equal to the standard one multiplied by 16.

If $J_1, \ldots, J_7$ are now arbitrary, then the results stated by Proposition B.1 remain true. Indeed, one can easily see that in this general set-up, the spaces P_0 and P_1 are the same as above, whereas each of $P_2, \ldots, P_8$ differ from the ones described above by group conjugation inside SO_{16n}. □

C Simple Lie groups as symmetric spaces and their involutions

Let G be a compact, connected, and simply connected simple Lie group. Equipped with a bi-invariant metric, G becomes a Riemannian symmetric space, as explained in Example 2.3. The following result has been proved in [Le-74] (see the proof of Theorem 3.3 in that paper). Since it plays an essential role in our Subsection 4, we decided to state it separately and give the details of the proof.

Proposition C.1 *Let $\tau : G \to G$ be an isometric involution with the property that $\tau(e) = e$, where e is the identity element of G. Then there exists an involutive group automorphism $\mu : G \to G$ such that either*

$$\tau(g) = \mu(g) \text{ for all } g \in G$$

or

$$\tau(g) = \mu(g)^{-1} \textit{ for all } g \in G.$$

Moreover, in the second case, the space $(G^\tau)_e$ *(the connected component through* e *of the fixed point set* G^τ*) is a totally geodesic submanifold of* G *which is isometric to* G/G^μ*. Here* G/G^μ *is equipped with the symmetric space structure induced by some bi-invariant metric on* G*.*

PROOF. Let $\hat{G}$ be the identity component of the isometry group of G. Then τ induces the involutive group automorphism $\hat{\tau} : \hat{G} \to \hat{G}$, $f \mapsto \tau \circ f \circ \tau$. Let $\hat{\mathfrak{g}}$ be the Lie algebra of $\hat{G}$ and denote by $\hat{\tau}_* : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ the differential map of $\hat{\tau}$ at the point $\hat{e}$, which is the identity element of $\hat{G}$. We know that $\hat{G} = (G \times G)/\Delta(Z(G))$, where $Z(G)$ is the center of G. Thus, if we denote the Lie algebra of G by $\mathfrak{g}$, then we have $\hat{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{g}$. Consider the map $\sigma : G \times G \to G \times G$, $\sigma(g_1, g_2) = (g_2, g_1)$, for all $g_1, g_2 \in G$ along with its differential map at the identity element, that is $\sigma_* := (d\sigma)_e : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$, $\sigma_*(x_1, x_2) = (x_2, x_1)$, for all $x_1, x_2 \in \mathfrak{g}$.
Claim 1. $\hat{\tau}_* \circ \sigma_* = \sigma_* \circ \hat{\tau}_*$.
Indeed, σ_* can also be described as the differential at e of the map $\hat{G} \to \hat{G}$, $f \mapsto s_e \circ f \circ s_e$, where s_e is the geodesic symmetry of G at e (see [He-01, Ch. IV, Theorem 3.3]). We only need to notice that the automorphism $\hat{G} \to \hat{G}$ described above commutes with $\hat{\tau}$. In turn, this follows from the fact that $\tau \circ s_e = s_e \circ \tau$ (both sides of the equation are isometries of G whose value at e is e and whose differential map at e is equal to $-(d\tau)_e$).
Let us now observe that $\hat{\tau}_*(\mathfrak{g} \times \{0\})$ is an ideal of $\mathfrak{g} \times \mathfrak{g}$. It can only be equal to $\mathfrak{g} \times \{0\}$ or to $\{0\} \times \mathfrak{g}$, since $\mathfrak{g}$ is a simple Lie algebra.
Case 1. $\hat{\tau}_*(\mathfrak{g} \times \{0\}) = \mathfrak{g} \times \{0\}$. There exists $\mu : \mathfrak{g} \to \mathfrak{g}$ an involutive Lie algebra automorphism such that $\hat{\tau}_*(x, 0) = (\mu(x), 0)$, for all $x \in \mathfrak{g}$. From Claim 1 we deduce that $\hat{\tau}_*(0, x) = (0, \mu(x))$, for all $x \in \mathfrak{g}$, thus

$$\hat{\tau}_*(x_1, x_2) = (\mu(x_1), \mu(x_2)),$$

for all $x_1, x_2 \in \mathfrak{g}$. We consider the group automorphism of G whose differential at e is μ and denote it also by μ. We have

$$\hat{\tau}([g_1, g_2]) = [\mu(g_1), \mu(g_2)],$$

for all $g_1, g_2 \in G$, where the brackets [,] indicate the coset modulo $\Delta(Z(G))$. Using the identification $\hat{G} = (G \times G)/\Delta(Z(G))$ and the explicit form of its action on G given by Equation (2.1), this implies

$$\tau(g_1 \tau(h) g_2^{-1}) = \mu(g_1) h \mu(g_2)^{-1},$$

for all $g_1, g_2, h \in G$. Thus, $\tau(g) = \mu(g)$ for all $g \in G$.
Case 2. $\hat{\tau}_*(\mathfrak{g} \times \{0\}) = \{0\} \times \mathfrak{g}$. This time, there exists $\mu : \mathfrak{g} \to \mathfrak{g}$ an involutive Lie algebra automorphism such that $\hat{\tau}_*(x, 0) = (0, \mu(x))$, for all $x \in \mathfrak{g}$. From Claim 1 we deduce that $\hat{\tau}_*(0, x) = (\mu(x), 0)$, for all $x \in \mathfrak{g}$, thus

$$\hat{\tau}_*(x_1, x_2) = (\mu(x_2), \mu(x_1)),$$

for all $x_1, x_2 \in \mathfrak{g}$. Again, we consider the group automorphism $\mu : G \to G$ whose differential at e is μ. This time we have

$$\hat{\tau}([g_1, g_2]) = [\mu(g_2), \mu(g_1)],$$

which implies

$$\tau(g_1\tau(h)g_2^{-1}) = \mu(g_2)h\mu(g_1)^{-1},$$

for all $g_1, g_2, h \in G$. This implies $\tau(g) = \mu(g)^{-1}$, for all $g \in G$.

We now prove the last assertion in the proposition. We are in Case 2. Consider the action of G on G^τ given by $G \times G^\tau \to G^\tau$, $g.x := gx\tau(g)$, for all $g \in G$ and $x \in G^\tau$. Since G is connected, it leaves $(G^\tau)_e$ invariant. The corresponding action is isometric, where $(G^\tau)_e$ is equipped with the submanifold metric induced by its embedding in G.

Claim 2. The action $G \times (G^\tau)_e \to (G^\tau)_e$ is transitive.

To justify this, we take $x \in (G^\tau)_e$ and show that there exists $g \in G$ with $x = g.e = g\tau(g)$. Indeed, let $\gamma : \mathbb{R} \to (G^\tau)_e$ be a geodesic in $(G^\tau)_e$ with $\gamma(0) = e$ and $\gamma(1) = x$. Since $(G^\tau)_e$ is a totally geodesic subspace of G, γ is a geodesic in G, hence we have $\gamma(t) = \exp(tX)$, for all $t \in \mathbb{R}$, where X is an element of $\mathfrak{g}$. From $\tau(\gamma(t)) = \gamma(t)$ for all $t \in \mathbb{R}$ we deduce that $(d\tau)_e(X) = X$. Then $g := \exp(\frac{1}{2}X)$ is in $(G^\tau)_e$ and we have $g\tau(g) = g^2 = \exp(X) = x$.

It only remains to observe that the stabilizer of e under the action mentioned in the claim is G^μ. □

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