Moduli Spaces of Flat Connections and Morita Equivalence of Quantum Tori

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Abstract. We study moduli spaces of flat connections on surfaces with boundary, with boundary conditions given by Lagrangian Lie subalgebras. The resulting symplectic manifolds are closely related with Poisson-Lie groups and their algebraic structure (such as symplectic groupoid structure) gets a geometrical explanation via 3-dimensional cobordisms. We give a formula for the symplectic form in terms of holonomies, based on a central extension of the gauge group by closed 2-forms. This construction is finally used for a certain extension of the Morita equivalence of quantum tori to the world of Poisson-Lie groups.

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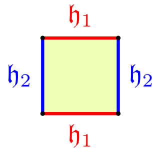
1 Introduction

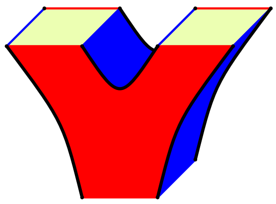
Let $\mathfrak{g}$ be a Lie algebra with an invariant inner product $\langle\cdot,\cdot\rangle$ (of any signature). It gives rise to two interesting types of symplectic manifolds. The first type are moduli spaces of flat $\mathfrak{g}$-connections on oriented surfaces. The second type are symplectic manifolds connected with Poisson-Lie groups such as the Lu-Weinstein double symplectic groupoid [9] (the symplectic groupoid integrating a Poisson-Lie group) corresponding to a Manin triple

$$\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}.$$

We shall notice that these "Poisson-Lie type" symplectic manifolds are, in fact, themselves moduli spaces of flat connections, if we allow surfaces with boundary and impose boundary conditions on the flat connections. To get the Lu-Weinstein double groupoid, the surface is a square, with boundary conditions as on the picture:



We shall provide formulas for symplectic forms using holonomies of the flat connections, in the spirit of Alekseev-Malkin-Meinrenken [1]. The basic idea is that the symplectic form can be interpreted as the integral over the surface of the curvature of a certain connection. The integral is then readily computed in terms of parallel transport. Moreover we shall describe how 3dim bodies give rise to Lagrangian submanifolds; for example, this picture gives one of the two products in Lu-Weinstein double groupoid:



This is a symplectic version of Chern-Simons TQFT in the sense of D. Freed [5], with appropriate boundary conditions.
The motivation for this work was to give a symplectic description of Morita equivalence of quantum tori, and moreover, to extend this Morita equivalence from Abelian T-duality to Poisson-Lie T-duality [7] (though just on the symplectic level, without performing geometrical quantization). This is done in the final section.

Aknowledgements

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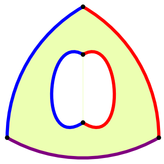
2 Colored surfaces and moduli spaces

Let G be a connected Lie group and $\langle,\rangle$ an Ad-invariant inner product (of any signature) on its Lie algebra $\mathfrak{g}$.
We shall consider compact oriented surfaces Σ with corners (i.e. locally looking like $(\mathbb{R}_{\geq 0})^2$). We shall assume that none of the components of Σ is closed and

that on each component of $\partial\Sigma$ there is at least one corner.[1] The boundary of Σ is thus split by the corners into a finite number of arcs.
For each arc a we choose a Lie subalgebra $\mathfrak{h}_a \subset \mathfrak{g}$ which is Lagrangian w.r.t. $\langle , \rangle$ (i.e. $\mathfrak{h}_a^\perp = \mathfrak{h}_a$). Let $H_a \subset G$ be the corresponding connected Lie subgroup. We demand for every corner x of Σ that if a and b are the arcs meeting at x then $\mathfrak{h}_a \cap \mathfrak{h}_b = 0$. We shall call such a Σ (together with the choice of subalgebras) a *colored surface*.



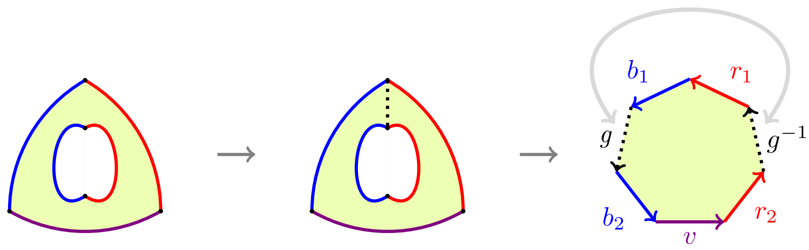
For every colored surface Σ we define a symplectic manifold $\mathcal{M}_\Sigma$. Let us give three equivalent definitions of the manifold $\mathcal{M}_\Sigma$; its symplectic form is defined in Section 3.

2.1 $\mathcal{M}_\Sigma$ via cuts of Σ

Let us first describe $\mathcal{M}_\Sigma$ in a way which depends on a choice of certain cuts of Σ.
We keep cutting Σ along paths connecting corners until we get a polygon. For every side s of the polygon we choose an element $g_s \in G$ such that:

1. if s is an arc of the boundary of Σ then $g_s \in H_s$
2. if s and s' are the two sides which are the result of a cut then $g_{s'} = g_s^{-1}$
3. the product of all g_s's along the boundary of the polygon (in their natural cyclic order) is equal to 1.

An assignment $s \mapsto g_s$ satisfying these properties is, by definition, a point in $\mathcal{M}_\Sigma$.



$b_1\, g\, b_2\, v\, r_2\, g^{-1} r_1 = 1$

[1]We impose these assumptions only for simplicity reasons, as they imply that the moduli spaces defined below are non-singular.

$\mathcal{M}_\Sigma$ can thus be described as the preimage of 1 under a map

$$\prod_a H_a \times G^{\#\text{cuts}} \to G. \tag{1}$$

The map is a submersion and thus $\mathcal{M}_\Sigma$ is a manifold.

2.2 $\mathcal{M}_\Sigma$ as a space of groupoid morphisms

Let us now describe $\mathcal{M}_\Sigma$ without using cuts. Let $X \subset \Sigma$ be the set of corners of Σ and let $\Pi_1(\Sigma, X)$ be the fundamental groupoid of Σ (the set of objects of $\Pi_1(\Sigma, X)$ is X and morphisms are homotopy classes of paths between corners). Every arc of the boundary can be seen as a morphism in $\Pi_1(\Sigma, X)$. Then

$$\mathcal{M}_\Sigma = \{F : \Pi_1(\Sigma, X) \to G;\ F(a) \in H_a \text{ for every arc } a\}.$$

2.3 $\mathcal{M}_\Sigma$ as a moduli space of flat connections

Finally, let us describe $\mathcal{M}_\Sigma$ as a moduli space of flat connections. Let $\pi : P \to \Sigma$ be a principal G-bundle. For every arc a we choose a reduction of $P|_a$ to $H_a \subset G$, i.e. a submanifold $P_a \subset \pi^{-1}(a)$ which is a principal H_a-bundle over a. For every corner $x \in \Sigma$ we choose a point $p_x \in P_a \cap P_b$ where a and b are the arcs meeting at x. Let us call $\pi : P \to \Sigma$ with its additional structure a *colored G-bundle* over Σ.

Let us consider connections on P which restrict to connections (i.e. to $\mathfrak{h}_a$-valued 1-forms) on every P_a; we shall call such a connection a *colored connection.* $\mathcal{M}_\Sigma$ can then be described the moduli space of colored flat connections on colored G-bundles over Σ.

The groupoid morphism $\Pi_1(\Sigma, X) \to G$ corresponding to a colored flat connection is given by parallel transport (the fiber of P over any corner x is trivialized by the choice of the point p_x). In the opposite direction, if $F : \Pi_1(\Sigma, X) \to G$ satisfies $F(a) \in H_a$ for every arc a then the corresponding flat colored G-bundle is construced as follows. Let $p : \hat{\Sigma} \to \Sigma$ be a universal cover of Σ with a chosen corner $y_0 \in \hat{\Sigma}$ and let $x_0 = p(y_0)$. Let $\tilde{P} = \hat{\Sigma} \times G \to \hat{\Sigma}$ be the trivial G-bundle, with the trivial flat connection. By restriction of F we have a homomorphism $\pi_1(\Sigma, x_0) \to G$ and we define the flat G-bundle $P \to \Sigma$ as $P = \tilde{P}/G$. The reduction of $\tilde{P}$ over a corner $y \in \hat{\Sigma}$ is $(y, F([p \circ \gamma_{y_0 y}]))$, where $\gamma_{y_0 y}$ is a path (unique up to homotopy) from y_0 to y. These reductions then extend uniquely to a coloring of $\tilde{P}$, and the coloring descends to a coloring of P. It is clear that every flat colored G-bundle $P \to \Sigma$ arises in this way, as the flat connection on p^*P can be used to trivialize it.

Notice that $\mathcal{M}_\Sigma$ is the disjoint union over the isomorphism classes of colored G-bundles of the moduli spaces with fixed colored G-bundle class.

If P is the trivial G-bundle $P = \Sigma \times G$ and its coloring is also trivial (i.e. $P_a = a \times H_a$, $p_x = (x, 1)$) then a colored connection can be described as a 1-form $A \in \Omega^1(\Sigma) \otimes \mathfrak{g}$ such that the restriction of A to any arc a is in $\Omega^1(a) \otimes \mathfrak{h}_a$.

The space of these flat connections modulo the gauge transformations (by maps $g : \Sigma \to G$ such that $g(x) = 1$ for every corner x and $g(a) \subset H_a$ for every arc a) is a connected component of $\mathcal{M}_\Sigma$.

3 Symplectic form in terms of holonomies

3.1 Symplectic form on moduli spaces of flat connections

Let $P \to \Sigma$ be a colored G-bundle. Colored connections on P form an affine space $\mathcal{A}_{\mathrm{col}}(P)$ modeled on $\Omega^1_{\mathrm{col}}(\Sigma, Ad_P)$, where $\Omega_{\mathrm{col}}(\Sigma, Ad_P) \subset \Omega(\Sigma, Ad_P)$ is the space of forms that restrict to $\Omega(a, Ad_{P_a})$ on every arc $a \subset \partial\Sigma$.
If A is a flat colored connection on P then the covariant differential d_A makes $\Omega_{\mathrm{col}}(\Sigma, Ad_P)$ to a complex and we have a natural isomorphism

$$T_{[P,A]}\mathcal{M}_\Sigma \cong H^1(\Omega_{\mathrm{col}}(\Sigma, Ad_P), d_A),$$

where $[P, A] \in \mathcal{M}_\Sigma$ denotes the isomorphism class of (P, A). The antisymmetric pairing

$$\omega([\alpha], [\beta]) = \int_\Sigma \langle \alpha \wedge \beta \rangle$$

on $T_{[P,A]}\mathcal{M}_\Sigma$ ($\alpha, \beta \in \Omega^1_{\mathrm{col}}(\Sigma, Ad_P)$) is non-degenerate by Poincaré–Verdier duality.
The moduli space $\mathcal{M}_\Sigma$ becomes in this way a symplectic manifold. To see that ω is smooth and closed (we already checked that it is non-degenerate), let us choose an open subset $U \subset \mathcal{M}_\Sigma$ which admits a smooth family of colored flat connections $\phi : U \to \mathcal{A}_{\mathrm{col}}(P)$, $\phi : x \mapsto A_x$, such that $[P, A_x] = x$. Then

$$\omega = \phi^* \omega_{\mathcal{A}} \tag{2}$$

where $\omega_{\mathcal{A}} \in \Omega^2(\mathcal{A}_{\mathrm{col}}(P))$ is the constant (hence closed) 2-form

$$\omega_{\mathcal{A}}(\alpha, \beta) = \int_\Sigma \langle \alpha \wedge \beta \rangle$$

on the affine space $\mathcal{A}_{\mathrm{col}}(P)$; ω is therefore closed. The symplectic form ω is a straightforward generalization of the symplectic form of Atiyah-Bott [2] and Goldman [6] who considered closed surfaces.

Remark 3.1. The symplectic manifold $(\mathcal{M}_\Sigma, \omega)$ is best described as the symplectic reduction of $(\mathcal{A}_{\mathrm{col}}(P), \omega_{\mathcal{A}})$ by the group of the automorphisms of P preserving the coloring. Making this statement precise is, however, rather technical. Here I present the formal part of the story, ignoring the problems with infinite-dimensional manifolds:
To simplify notations, let us discuss the case when the colored G-bundle $P \to \Sigma$ is trivial. Let us recall how symplectic forms appear on moduli spaces of flat connections.

The group of smooth maps $\Sigma \to G$ acts affinely on $\Omega^1(\Sigma)\otimes\mathfrak{g}$ by gauge transformations ($g:\Sigma\to G$, $\alpha\in\Omega^1(\Sigma)\otimes\mathfrak{g}$)

$$g\cdot\alpha = g^{-1}dg + g^{-1}\alpha g$$

and this action preserves $\omega_{\mathcal{A}}$.
The infinitesimal action of a $t:\Sigma\to\mathfrak{g}$ is generated by the Hamiltonian

$$H_t(\alpha)=\int_\Sigma \langle t,F\rangle + \int_{\partial\Sigma}\langle t,\alpha\rangle \tag{3}$$

where $F=d\alpha+\alpha^2$ is the curvature of the $\mathfrak{g}$-connection α; the Poisson bracket of two such Hamiltonians is

$$\{H_{t_1},H_{t_2}\}=H_{[t_1,t_2]}+c(t_1,t_2)$$

where

$$c(t_1,t_2)=\int_{\partial\Sigma}\langle t_1,dt_2\rangle. \tag{4}$$

Notice that the cocycle (4) vanishes on the Lie algebra

$$\{t:\Sigma\to\mathfrak{g};t(a)\subset\mathfrak{h}_a \text{ for every arc } a\}$$

of infinitesimal gauge transformations preserving the coloring. The moment map (3) is 0 at $\alpha\in\Omega^1(\Sigma)\otimes\mathfrak{g}$ iff

$$\alpha \text{ is flat and } \alpha|_a\in\Omega^1(a)\otimes\mathfrak{h}_a \text{ for every arc } a.$$

The symplectic reduction is thus the part of the moduli space $\mathcal{M}_\Sigma$ coming from the trivial colored G-bundle.

3.2 Central extension by closed 2-forms

Let M be a manifold and let $\Omega^2_{cl}(M)$ denote the space of closed 2-forms on M. Let us recall that the Lie algebra $\mathfrak{g}(M)$ of smooth maps $M\to\mathfrak{g}$ has a central extension $\tilde{\mathfrak{g}}(M)$ by $\Omega^2_{cl}(M)$: as a vector space,

$$\tilde{\mathfrak{g}}(M)=\mathfrak{g}(M)\oplus\Omega^2_{cl}(M),$$

and the bracket is

$$[(t_1,\omega_1),(t_2,\omega_2)]=([t_1,t_2],\langle dt_1\wedge dt_2\rangle). \tag{5}$$

The corresponding group $\tilde{G}(M)$, a central extension of $G(M)$ (the group of smooth maps $M\to G$) by $\Omega^2_{cl}(M)$, can be described as follows: its elements are pairs

$$(g,\omega),\quad g:M\to G,\ \omega\in\Omega^2(M),\ d\omega=\frac{1}{2}g^*\eta \tag{6}$$

where the invariant 3-form η on G is given by

$$\eta(u,v,w) = \langle [u,v], w\rangle.$$

The product in the group is

$$(g_1,\omega_1)(g_2,\omega_2) = (g_1 g_2,\, \omega_1+\omega_2+\frac{1}{2}\langle g_1^{-1}dg_1 \wedge dg_2\, g_2^{-1}\rangle) \tag{7}$$

and the inverse

$$(g,\omega)^{-1} = (g^{-1}, -\omega). \tag{8}$$

Finally, let us also introduce an auxiliary group $\tilde{G}_{big}(M) \supset \tilde{G}(M)$:

$$\tilde{G}_{big}(M) = G(M) \times \Omega^2(M), \tag{9}$$

with the product and inverse given by the same formulas (7), (8). The map

$$\tilde{G}_{big}(M) \to \Omega^3_{cl}(M), \quad (g,\omega) \mapsto d\omega - \frac{1}{2} g^*\eta$$

is a group morphism and $\tilde{G}(M)$ is its kernel.

3.3 Symplectic form in terms of holonomies

Let us cut Σ until we get a polygon (as in Section 2). For each side s of the polygon we have a map $\gamma_s : \mathcal{M}_\Sigma \to G$ (the holonomy along the side).

Theorem 3.1. *The symplectic form ω on $\mathcal{M}_\Sigma$ is given by*

$$(1,\omega) = \prod_s (\gamma_s, 0)$$

where the product is taken in the group $\tilde{G}_{big}(\mathcal{M}_\Sigma)$ (see Equation (7)) and the sides of the polygon are taken in their natural (cyclic) order.

The idea of the proof is that ω is the integral of the curvature of a $\tilde{\mathfrak{g}}(\mathcal{M}_\Sigma)$-valued connection on Σ, and hence can be expressed in terms of the holonomies g_s's. The proof is in Section 3.6. The formula for ω is a generalization of a similar formula of Alekseev-Malkin-Meinrenken [1] for the case of closed surfaces.

3.4 Integral of curvature

Let

$$C \to \tilde{K} \to K$$

be a central extension of Lie groups. Let $\tilde{P} \to D$ be a principal $\tilde{K}$-bundle over a disk D and let $P \to D$ be the corresponding K-bundle, $P = \tilde{P}/C$. Suppose that A is a flat connection on P and $\tilde{A}$ is a (non-flat) connection on $\tilde{P}$ lifting A. The curvature $\tilde{F}$ of $\tilde{A}$ is a $\mathfrak{c}$-valued 2-form on D and its integral is

$$C \ni \exp \int_D \tilde{F} = \mathrm{hol}_{\partial D} \tilde{A}. \tag{10}$$

The proof of this simple claim is obvious: trivialize $\tilde{P} \to D$ (and hence $P \to D$) in such a way that the connection A on $P = D \times K$ becomes trivial. Such a trivialization can be achieved e.g. by by the parallel transport of $\tilde{A}$ along straight lines starting at the center of the disc D. Formula (10) then becomes Stokes theorem.

We shall use (10) for the central extension

$$\Omega^2_{cl}(M) \to \tilde{G}(M) \to G(M).$$

The disk will be the result of cutting Σ and M will run over certain open subsets of $\mathcal{M}_\Sigma$.

3.5 Symplectic form as integral of curvature

Let Σ be a colored surface. Let $U \subset \mathcal{M}_\Sigma$ be an open subset, $P \to \Sigma$ a colored G-bundle and A_x a smooth family of colored flat connections on P parametrized by $x \in U$, such that the class of (P, A_x) is x. $\mathcal{M}_\Sigma$ can be covered by such open subsets U.

Using the inclusion $G \to \tilde{G}(U)$, $g \mapsto (g, 0)$, we lift P to a principal $\tilde{G}(U)$-bundle $\tilde{P}_U \to \Sigma$. Similarly, the inclusion $G \to G(U)$ lifts P to a principal $G(U)$-bundle $P_U \to \Sigma$ and $P_U = \tilde{P}_U / \Omega^2_{closed}(U)$.

The family A_x can be seen as a flat connection A on P_U, and $\tilde{A} = (A, 0)$ as a (non-flat) connection on $\tilde{P}_U$. The curvature $\tilde{F}$ of $\tilde{A}$ is a $\Omega^2_{cl}(U)$-valued 2-form on Σ, and the integral of $\tilde{F}$ is (using (2) and (5)) the symplectic form on $U \subset \mathcal{M}_\Sigma$:

$$\omega = \int_\Sigma \tilde{F}. \tag{11}$$

Remark 3.2. To speak properly about principal $G(U)$- and $\tilde{G}(U)$ bundles, we should understand in what sense $G(U)$ and $\tilde{G}(U)$ are Lie groups. However, we don't need to do it. A principal $G(U)$-bundle ober Σ is given by an open cover $\{V_\alpha\}$ of Σ and by a cocycle $V_\alpha \cap V_\beta \to G(U)$, i.e. by a cocycle of smooth maps $(V_\alpha \cap V_\beta) \times U \to G$. In our case the maps are constant on U. A connection on such a bundle is given by 1-forms $A_\alpha \in \Omega^1(V_\alpha, \mathfrak{g}(U))$, compatible on the overlaps $V_\alpha \cap V_\beta$. A 1-form $A_\alpha \in \Omega^1(V_\alpha, \mathfrak{g}(U))$ is by definition a family of 1-forms in $\Omega^1(V_\alpha, \mathfrak{g})$, smoothly parametrized by U. That is how the family A_x is seen as a flat connection A on P_U. Notice also that since P can be trivialized when we cut Σ to a dics, $P(U)$ and $\tilde{P}(U)$ are also trivialized.

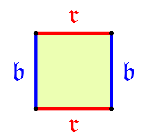
Let us also remark that we are not constructing $G(\mathcal{M}_\Sigma)$- and $\tilde{G}(\mathcal{M}_\Sigma)$-bundles over Σ. Formula (11) gives the symplectic form only on our open subsets $U \subset \mathcal{M}_\Sigma$, but these open subsets cover $\mathcal{M}_\Sigma$.

3.6 Symplectic form in terms of holonomies (proof)

Proof of Theorem 3.1. It follows immediately from (11) and (10). We cut Σ to a polygon. If $s \subset \partial\Sigma$ then $(\gamma_s, 0)$ is the holonomy of $\tilde{A}$ along s (since $\mathfrak{h}_s$ is isotropic and thus the cocycle in (5) vanishes). If s comes from a cut then the holonomy of $\tilde{A}$ along s is (γ_s, β) for some 2-form β. However, the holonomy along the other side coming from the same cut is its inverse; we can thus replace β with 0 and the product of holonomies will not change. This proves Theorem 3.1 for open $U \subset \mathcal{M}_\Sigma$ satisfying the condition of Section 3.5, and since they cover $\mathcal{M}_\Sigma$, it proves it for entire $\mathcal{M}_\Sigma$ □

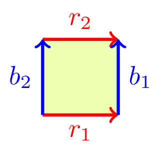
3.7 Examples

Example 3.1. Let Σ be a square colored by a Manin triple $\mathfrak{r}, \mathfrak{b} \subset \mathfrak{g}$:



Let us denote the holonomies as on the picture, i.e.

$$\mathcal{M}_\Sigma = \{(r_1, r_2, b_1, b_2) \in R \times R \times B \times B;\ r_1 b_1 = b_2 r_2\} :$$



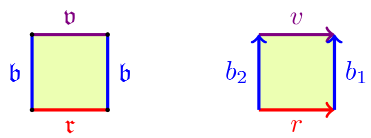
In $\tilde{G}(\mathcal{M}_\Sigma)$ we have $(r_1, 0)(b_1, 0) = (r_1 b_1, \langle r_1^{-1} dr_1 \wedge db_1 b_1^{-1}\rangle/2)$ and $(b_2, 0)(r_2, 0) = (b_2 r_2, \langle b_2^{-1} db_2 \wedge dr_2 r_2^{-1}\rangle/2)$. The symplectic form on $\mathcal{M}_\Sigma$ is thus

$$\omega = \frac{1}{2}\langle r_1^{-1} dr_1 \wedge db_1 b_1^{-1}\rangle - \frac{1}{2}\langle b_2^{-1} db_2 \wedge dr_2 r_2^{-1}\rangle.$$

This symplectic manifold $(\mathcal{M}_\Sigma, \omega)$ is the Lu-Weinstein double symplectic groupoid [9] corresponding to the triple $R, B \subset G$. This fact was already noticed by the author in [12]. A similar interpretation of the Lu-Weinstein double groupoid was found by P. Boalch in [3] using irregular connections.

In one of the groupoid structures, the space of objects is B and a point in $\mathcal{M}_\Sigma$ is an arrow from b_2 to b_1; in the other groupoid structure, the roles of R and B are exchanged. The groupoid products are given by concatenation of squares (either horizontal or vertical); they will be explained more properly in the following section.

EXAMPLE 3.2. Let now Σ be a square colored as follows:



The symplectic form is

$$\omega = \frac{1}{2}\langle r^{-1}dr \wedge db_1 b_1^{-1}\rangle - \frac{1}{2}\langle b_2^{-1}db_2 \wedge dv\, v^{-1}\rangle.$$

The symplectic manifold $(\mathcal{M}_\Sigma, \omega)$ is again a well-known object: it is the symplectic groupoid integrating the homogeneous Poisson space given by $R, B, V \subset G$ via Drinfeld's classification [4]. This symplectic groupoid was discovered by Jiang-Hua Lu [8].

EXAMPLE 3.3. Now let Σ be a triangle.

In this case

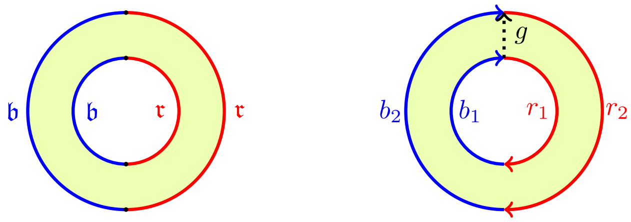
$$\mathcal{M}_\Sigma = \{(r, b, v) \in R \times B \times V;\ rbv = 1\}.$$

The symplectic form is

$$\omega = \frac{1}{2}\langle v^{-1}dv \wedge db\, b^{-1}\rangle.$$

This symplectic manifold is, up to covering, the big symplectic leaf in the homogeneous Poisson space given by $R, B, V \subset G$. It will play a role when we discuss Morita equivalence.

EXAMPLE 3.4. Finally, let us discuss the simplest Σ that requires a cut.



$$\mathcal{M}_\Sigma = \{(r_1, r_2, b_1, b_2, g) \in R^2 \times B^2 \times G;\ r_1 b_1 g = g\, r_2 b_2\}$$
$$(1, \omega) = (r_1, 0)(b_1, 0)(g, 0)\left((g, 0)(r_2, 0)(b_2, 0)\right)^{-1}$$

The symplectic manifold $(\mathcal{M}_\Sigma, \omega)$ is the double symplectic groupoid integrating the Drinfeld double given by the triple $R, B \subset G$, i.e. the Lu-Weinstein groupoid of the triple $R \times B, G_{\text{diag}} \subset G \times G$.

4 Painted bodies and Lagrangian submanifolds

In this section we shall discuss how cobordisms of painted surfaces give rise to Lagrangian submanifolds in the moduli spaces. These Lagrangian submanifolds will turn the moduli spaces into interesting algebraic objects, such as (double) groupoids, modules, etc. The Lagrangian submanifold will consist of those flat connection on the surface that can be extended to flat connections on the 3dim manifold (cobordism). This construction is a straightforward generalization of the symplectic Chern-Simons theory of D. Freed [5], who considered closed surfaces.

4.1 Painted bodies

Let us consider a compact oriented 3dim manifold with corners (i.e. locally looking as $(\mathbb{R}_{\geq 0})^3$). Its boundary is divided to vertices (corners), edges and faces. For some of the faces we choose a Lagrangian Lie subalgebra of $\mathfrak{g}$ (we shall call such a face painted). We shall require the following. Whenever two faces meet along an edge then at least one of them is painted, and if both are painted, then the two subalgebras are transverse. At each vertex should meet two painted and one unpainted face. Finally, the unpainted part of the boundary should be a colored surface, i.e. each of its components should have boundary and on each of the boundary circles there should be a corner. We shall call such a manifold a *painted body.* The unpainted part of the boundary of a body X will be denoted Σ_X.

4.2 Flat connections on a painted body

Let X be a painted body. We shall consider principal G-bundles $P \to X$ with a reduction to H_a over every painted face a and with a section over every edge between painted faces. We then consider flat connections compatible with the reductions. Let $\mathcal{L}_X \subset \mathcal{M}_{\Sigma_X}$ denote the set of equivalence classes of flat colored connections on Σ_X that are extensible to X. We shall call $\mathcal{L}_X$ *smooth* if it is a submanifold and moreover it can be locally lifted to a smooth family of flat connections on X.

Theorem 4.1. *If $\mathcal{L}_X \subset \mathcal{M}_{\Sigma_X}$ is smooth, it is a Lagrangian submanifold.*

Proof. We shall prove that the formal tangent spaces to $\mathcal{L}_X$ are Lagrangian in the tangent spaces of $\mathcal{M}_{\Sigma_X}$. If $\mathcal{L}_X$ is smooth then these formal tangent spaces are the actual tangent spaces.
Let P be a painted G-bundle over X, $Ad_P \to X$ the vector bundle associated to the adjoint representation of G on $\mathfrak{g}$, and let

$$\Omega_{\mathrm{col}}(X, Ad_P) \subset \Omega(X, Ad_P)$$

be the space of Ad_P-valued differential forms that take values in the corresponding subalgebra of $\mathfrak{g}$ when restricted to a painted face of X. Let A be a

flat colored connection; then d_A makes $\Omega_{\mathrm{col}}(X, Ad_P)$ into a complex. Let us denote this complex $\Omega_A(X)$ and its cohomology $H_A(X)$.
Let $P' = P|_{\Sigma_X}$ and $A' = A|_{\Sigma_X}$. Let us consider the short exact sequence

$$0 \to \Omega_{A,0}(X) \to \Omega_A(X) \to \Omega_{A'}(\Sigma_X) \to 0$$

(where $\Omega_{A,0}(X)$ are the forms vanishing at Σ_X) and the following piece of the resulting long exact sequence:

$$H^1_A(X) \to H^1_{A'}(\Sigma_X) \to H^2_{A,0}(X). \tag{12}$$

We have

$$H^1_{A'}(\Sigma_X) = T_{[P',A']}\mathcal{M}_{\Sigma_X},$$

and the image of the first arrow is the formal $T_{[P',A']}\mathcal{L}_X$.
By Poincaré duality the dual of (12) is obtained just by reversing the arrows:

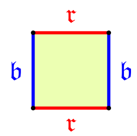
$$H^2_{A,0}(X) \leftarrow H^1_{A'}(\Sigma_X) \leftarrow H^1_A(X)$$

(in particular, the identification of $H^1_{A'}(\Sigma_X)$ with its dual is via the symplectic form). As a consequence, the image of the first arrow in (12) is a Lagrangian subspace. □

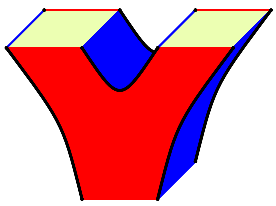
In all the examples that we consider below, $\mathcal{L}_X$ is easily seen to be smooth.

4.3 Examples

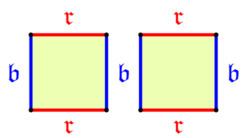
As we noticed above, Lu-Weinstein's double symplectic groupoid corresponding to a Manin triple $B, R \subset G$ is the moduli space for the surface



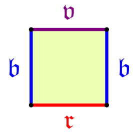
The graph of one of the products in this double groupoid is $\mathcal{L}_X$ where X is



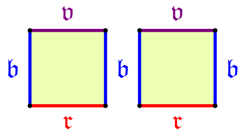
In other words, the product is given by gluing squares along the adjacent sides on the picture



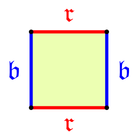
The other product is obtained when we exchange the colors.
The moduli space of



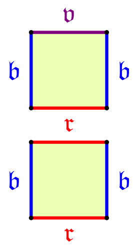
is a symplectic groupoid via the gluing



It is a symplectic groupoid integrating the Poisson B-homogeneous space corresponding to the quadruple $R, B, V \subset G$. It is also a module of



via the gluing



Similar pictures can be drawn for the double symplectic groupoid integrating the Drinfeld double and also for its R-matrix in the sense of Weinstein and Xu; see [12] for details.

5 Morita equivalence of quantum tori and beyond

This last section is a bit speculative. On the other hand, it describes the motivation for the constructions described above, so it is included anyway.

5.1 Morita equivalence of quantum tori

Recall that two algebras A and B are said to be Morita equivalent if their categories of modules are linearly equivalent. Equivalently, there exist a $A \otimes$

$B^{\rm op}$-module M and a $B\otimes A^{\rm op}$ module N such that $M\otimes_B N\cong A$ and $N\otimes_A M\cong B$.

Let $\theta_{ij}=-\theta_{ji}$, $1\le i,j\le n$, be a skew-symmetric matrix with real elements. We suppose that the graph of the corresponding linear map $\mathbb{R}^n\to\mathbb{R}^n$ intersects $\mathbb{Z}^{2n}\subset\mathbb{R}^{2n}$ only in $0\in\mathbb{Z}^{2n}$. To the matrix θ we associate the algebra $\mathbb{T}^n_\theta$ (a quantum torus) generated by elements u_i ($1\le i\le n$) and their inverses, modulo relations $u_iu_j=\exp(2\pi\sqrt{-1}\theta_{ij})u_ju_i$. A famous result of Rieffel and Schwarz [10] says that the algebra $\mathbb{T}^n_\theta$ is Morita equivalent to $\mathbb{T}^n_{\theta^{-1}}$.[2]

The quantum torus $\mathbb{T}^n_\theta$ can be seen as a quantization of the n-dimensional torus $\mathbb{T}^n$ with the constant Poisson structure given by θ. The following natural questions are due to A. Schwarz and A. Weinstein (motivated by an extension of T-duality [11] to Poisson-Lie T-duality [7]).

1. Is there a generalization of Morita equivalence when $\mathbb{T}^n$ is replaced by a quantum group H and $\mathbb{T}^n_\theta$ by a torsor of H?

2. Is there a symplectic/Poisson version of Morita equivalence for tori with constant Poisson structure? Can it be extended to Poisson-Lie groups, giving a symplectic/Poisson analog of Question 1?

We shall give an answer to Question 2. It will provide a conjectural answer to Question 1.

5.2 H-Morita equivalence

Let H be a Hopf algebra. Let A be an associative algebra in the (monoidal) category H-mod of left H-modules. In other words, A is an H-module and the product $A\otimes A\to A$ is a morphism of H-modules.

Let $A\text{-mod}_H$ be the category of A-modules in H-mod, i.e. the category of vector spaces V which are modules of both A and H, such that $A\otimes V\to V$ is a morphism of H-modules. Let A-mod be the category of A-modules in the category of vector spaces. We have the forgetful functor $\mathrm{res}: A\text{-mod}_H\to A\text{-mod}$ and its left adjoint $\mathrm{ind}: A\text{-mod}\to A\text{-mod}_H$.

Let now B be an algebra in the (monoidal) category H-comod of right H-comodules. We have the category $B\text{-mod}^H$ of B-modules in H-comod and the category B-mod. Now we have the forgetful functor $\mathrm{cores}: B\text{-mod}^H\to B\text{-mod}$ and its right adjoint $\mathrm{coind}: B\text{-mod}\to B\text{-mod}^H$.

DEFINITION 5.1. *We shall say that A and B are H-*Morita equivalent *if there are equivalences of linear categories A-mod $\to B$-mod^H and A-mod$_H\to B$-mod*

[2] it implies a Morita equivalence of $\mathbb{T}^n_\theta$ with $\mathbb{T}^n_{A\cdot\theta}$ for any $A\in SO(n,n;\mathbb{Z})$, where $SO(n,n;\mathbb{Z})$ acts on the graph of θ in $\mathbb{R}^{2n}$, provided the transformed graph is again a graph

such that the diagram

$$\begin{tikzcd} A\text{-}mod \arrow[r] \arrow[d, "\mathrm{ind}"'] & B\text{-}mod^H \arrow[d, "\mathrm{cores}"] \\ A\text{-}mod_H \arrow[r] & B\text{-}mod \end{tikzcd}$$

commutes up to a natural isomorphism, or equivalently, such that

$$\begin{tikzcd} A\text{-}mod \arrow[r] & B\text{-}mod^H \\ A\text{-}mod_H \arrow[r] \arrow[u, "\mathrm{res}"] & B\text{-}mod \arrow[u, "\mathrm{coind}"'] \end{tikzcd}$$

commutes up to a natural isomorphism.

The simplest example is when $A = k$ is trivial (k is the base field) and $B = H$. The category H-mod^H is called the category of Hopf modules of H. A linear equivalence $F : k\text{-mod} \to H\text{-mod}^H$ making the diagram

$$\begin{tikzcd} k\text{-mod} \arrow[r, "F"] \arrow[d, "\mathrm{ind}"'] & H\text{-mod}^H \arrow[d, "\mathrm{cores}"] \\ H\text{-mod} \arrow[r, "="] & H\text{-mod} \end{tikzcd}$$

commutative (up to a natural isomorphism) is due to Sweedler [14]; F is given simply by $F(V) = H \otimes V$.

Proposition 5.1. *An H-Morita equivalence is equivalent to a vector space M which is a right A-module and left B-module, satisfying the compatibility relation $b \cdot (m \cdot a) = (b_{(1)} \cdot m) \cdot (b_{(2)} \cdot a)$ for all $a \otimes m \otimes b \in A \otimes M \otimes B$, where $b \mapsto b_{(1)} \otimes b_{(2)} \in B \otimes H$ is the H-comodule structure of B.*

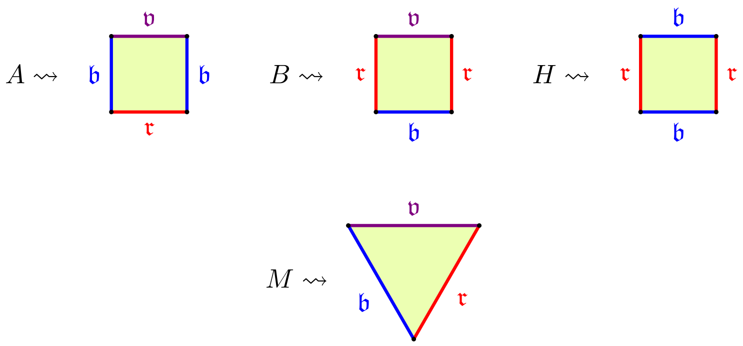
Proof. If M is given then $M \otimes H$ is a B-A-bimodule via $b \cdot (m \otimes h) = (b_{(1)} \cdot m) \otimes (b_{(2)} \cdot h)$, $(m \otimes h) \cdot a = (m \cdot (h_{(2)} \cdot a)) \otimes h_{(1)}$. It induces an H-Morita equivalence as follows. The functor A-mod $\to B$-mod^H is given by $V \mapsto V \otimes_A (M \otimes H)$, and A-mod$_H \to B$-mod by $W \mapsto \big(W \otimes_A (M \otimes H)\big)^H$, where X^H (for a H-module X) is $\{x \in X; (\forall h \in H)\, h \cdot x = \epsilon(h)x\}$.
If an H-Morita equivalence is given then $M = F(A)^H$, where F is the functor A-mod $\to B$-mod^H. ∎

5.3 Conjectural answer to Question 1

Suppose again that G is a connected Lie group and its Lie algebra $\mathfrak{g}$ has an invariant inner product $\langle,\rangle$. Let $\mathfrak{r}, \mathfrak{b} \subset \mathfrak{g}$ be a Manin triple and let R and B be the corresponding Poisson-Lie groups. Suppose also that $\mathfrak{v}$ is another Lagrangian subalgebra with the property that $\mathfrak{v} \cap \mathfrak{r}$ is the Lie algebra of a closed connected subgroup $R_{\mathfrak{v}} \subset R$ and similarly, $\mathfrak{v} \cap \mathfrak{b}$ is the Lie algebra of a closed connected subgroup $B_{\mathfrak{v}} \subset B$. By Drinfeld's classification of Poisson homogeneous spaces [4] the homogeneous space $R/R_{\mathfrak{v}}$ has a Poisson structure such that the map $R \times (R/R_{\mathfrak{v}}) \to R/R_{\mathfrak{v}}$ is Poisson, and similarly for $B/B_{\mathfrak{v}}$.
Below we shall prove a symplectic version of the following loosely stated conjecture: If Hopf algebra H is a (suitable) quantization of the Lie bialgebra corresponding to the Manin triple $\mathfrak{r}, \mathfrak{b} \subset \mathfrak{g}$ and algebras A and B are (suitable) quantizations of the Poisson manifolds $R_{\mathfrak{v}}$ and $B_{\mathfrak{v}}$ respectively, then A and B are H-Morita equivalent.
In particular, if $\mathfrak{v} = \mathfrak{r}$ we get $A = k$ and $B = H$, i.e. Sweedler's example of H-Morita equivalence.
Let us notice that we prove the symplectic version of the conjecture only under the additional assumption that $\mathfrak{v}$ is transverse to both $\mathfrak{r}$ and $\mathfrak{b}$. The general case would require colored surfaces where we allow adjacent subalgebras to have non-trivial intersection. The corresponding colored G-bundles would have a reduction over the corresponding corner to the group exponentiating the intersection. We shall treat this generalization elsewhere.

5.4 Symplectic H-Morita equivalence

In this final section we provide a symplectic analogue of our conjectural answer to Question 1. Vector spaces are replaced by symplectic manifolds of the form $\mathcal{M}_\Sigma$ as follows:

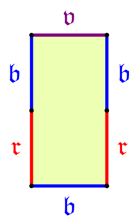


In other words, A is replaced by the symplectic groupoid integrating the Poisson homogeneous space $R_{\mathfrak{v}}$, B by the symplectic groupoid integrating $B_{\mathfrak{v}}$, H by the double symplectic groupoid integrating both R and B, and M by the moduli space of the displayed triangle.

Let us recall the definition of Morita equivalence of symplectic groupoids [15]. Let I denote the groupoid with two objects 0 and 1 and with a unique morphism $0 \to 1$. Let Γ be a Lie groupoid with a groupoid morphism $\Gamma \to \mathsf{I}$. Γ splits naturally to 4 components: Γ_{ij} $(i, j \in \{0,1\})$ is the space of arrows lying over the unique morphism $i \to j$. Let X_i denote the space of objects of Γ lying over $i \in \{0,1\}$.

Definition 5.2. *If the maps $\Gamma_{01} \to X_1$ and $\Gamma_{10} \to X_0$ are surjective then the groupoids $\Gamma_{00} \rightrightarrows X_0$ and $\Gamma_{11} \rightrightarrows X_1$ are said to be* Morita equivalent *via the bimodules Γ_{01} and Γ_{10}. If the groupoid Γ is symplectic then we have a* Morita equivalence of symplectic groupoids*.*

Let $\mathfrak{r}, \mathfrak{b}, \mathfrak{v} \subset \mathfrak{g}$ be as above and let $R, B, V \subset G$ be the corresponding groups. Let $X_0 = R \times B$ and let the arrows $(r_1, b_1) \to (r_2, b_2)$ in Γ_{00} be $(v, r) \in V \times R$ such that $r_1 b_1 v = b\, r_2 b_2$; composition of arrows is by $(v, r)(v', r') = (vv', rr')$. The groupoid Γ_{00} can be seen as $\mathcal{M}_\Sigma$ for the surface

$\Gamma_{00} \sim$ 

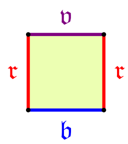
which makes it to a symplectic groupoid. The groupoid composition is by (horizontal) gluing of rectangles.

The symplectic groupoid Γ_{00} integrates the following Poisson structure on $R \times B$. We have the Poisson action $B \times B_\mathfrak{v} \to B_\mathfrak{v}$. The forgetful functor

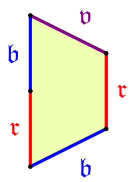
$$\text{Poisson manifolds with a moment map to } R \to \text{Poisson } \mathfrak{b}\text{-manifolds}$$

has a right adjoint F (see [13]), and $F(B_\mathfrak{v}) = R \times B$ is our Poisson manifold. It is a semi-classical analog of the crossed product $A \rtimes H$, where A is a quantization of $B_\mathfrak{v}$ (an associative algebra) and H a quantization of the Lie bialgebra $\mathfrak{b}$ (a Hopf algebra). Notice that $A \rtimes H$-mod is equivalent to A-mod$_H$.

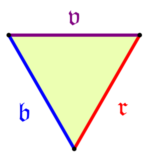
Let us suppose for simplicity that the map $R \times B \to G$, $(r, b) \mapsto rb$, is a diffeomorphism. The symplectic groupoid Γ_{00} is Morita equivalent to the symplectic groupoid Γ_{11} integrating the Poisson manifold $R_\mathfrak{v}$, i.e. $\mathcal{M}_\Sigma$ for the surface

$\Gamma_{11} \sim$ 

The bimodule Γ_{01} is $\mathcal{M}_\Sigma$ for the surface

$\Gamma_{01} \sim$ 

with the bimodule structure given by gluing along the vertical sides.
This Morita equivalence of symplectic groupoids is analogous to a Morita equivalence $A\text{-mod}_H \to B\text{-mod}$. If we exchange $\mathfrak{r}$ and $\mathfrak{b}$ then we get a similar Morita equivalence, analogous to $A\text{-mod} \to B\text{-mod}^H$, and these two Morita equivalences of symplectic groupoids are easily seen to form a commutative square analogous to H-Morita equivalence of A and B. Finally, the moduli space for the triangle



is analogous to the vector space M; the A- and B-module structure of M is given by gluing squares to the triangle along the edge of the corresponding color.
In the case when $R, B = \mathbb{T}^n$, $G = R \times B$, all these symplectic manifolds are of the form $\mathbb{R}^{2m}/\mathbb{Z}^k$, with constant symplectic structure. They can be therefore easily geometrically quantized (provided the symplectic form is integral on $\mathbb{Z}^k$) and these quantizations are compatible with the groupoid/module structures, so we get an H-Morita equivalence. If we choose the Planck constant so that the geometrical quantization H of the double symplectic groupoid $R \times B$ is trivial, we get the standard proof of Morita equivalence of quantum tori [10]: In the simplest case of 2-dimensional tori, A is given by the relation $uv = e^{2\pi i\theta} vu$, B by $u'v' = e^{2\pi i/\theta} v'u'$, and the bimodule is $M = L^2(\mathbb{R})$ given by $(u \cdot f)(x) = f(x+1)$, $(v \cdot f)(x) = e^{2\pi i\theta x} f(x)$, $(u' \cdot f)(x) = f(x + 1/\theta)$, $(v' \cdot f)(x) = e^{2\pi i x} f(x)$.

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