On a Theorem of Lehrer and Zhang

Jun Hu, Zhankui Xiao

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Abstract. Let K be an arbitrary field of characteristic not equal to 2. Let $m, n \in \mathbb{N}$ and V be an m dimensional orthogonal space over K. There is a right action of the Brauer algebra $\mathfrak{B}_n(m)$ on the n-tensor space $V^{\otimes n}$ which centralizes the left action of the orthogonal group $O(V)$. Recently G.I. Lehrer and R.B. Zhang defined certain quasi-idempotents E_i in $\mathfrak{B}_n(m)$ (see (1.1)) and proved that the annihilator of $V^{\otimes n}$ in $\mathfrak{B}_n(m)$ is always equal to the two-sided ideal generated by $E_{[(m+1)/2]}$ if $\operatorname{char} K = 0$ or $\operatorname{char} K > 2(m+1)$. In this paper we extend this theorem to arbitrary field K with $\operatorname{char} K \neq 2$ as conjectured by Lehrer and Zhang. As a byproduct, we discover a combinatorial identity which relates to the dimensions of Specht modules over the symmetric groups of different sizes and a new integral basis for the annihilator of $V^{\otimes m+1}$ in $\mathfrak{B}_{m+1}(m)$.

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1 Introduction

Let $\mathbb{N}$ be the set of non-negative integers. Let x be an indeterminate over $\mathbb{Z}$ and $0 < n \in \mathbb{N}$. The Brauer algebra $\mathfrak{B}_n(x)$ over $\mathbb{Z}[x]$ was introduced by Richard Brauer (see [1]) when he studied how the n-tensor space $V^{\otimes n}$ decomposes into irreducible modules over the orthogonal group $O(V)$ or the symplectic group $Sp(V)$, where V is an orthogonal vector space or a symplectic vector space. It was defined as the free $\mathbb{Z}[x]$-module on the basis of the set Bd_n of all Brauer n-diagrams, graphs on $2n$ vertices, and n edges with the property that every vertex is incident to precisely one edge. The multiplication of two Brauer n-diagrams is defined using natural concatenation of diagrams. Precisely, we compose two diagrams D_1, D_2 by identifying the bottom row of vertices in

D_1 with the top row of vertices in D_2. The result is a graph, with a certain number, $n(D_1, D_2)$, of interior loops. After removing the interior loops and the identified vertices, retaining the edges and remaining vertices, we obtain a new Brauer n-diagram $D_1 \circ D_2$, the composite diagram. Then we define $D_1 \cdot D_2 = x^{n(D_1,D_2)} D_1 \circ D_2$. For example, let d be the following Brauer 5-diagram.

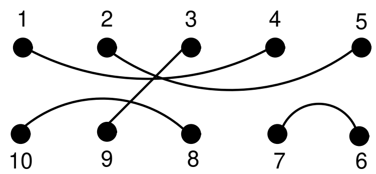

Figure 1.1

Let d' be the following Brauer 5-diagram.

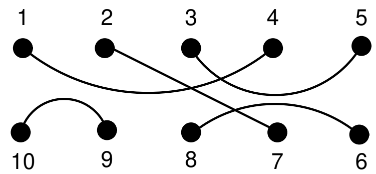

Figure 1.2

Then dd' is equal to

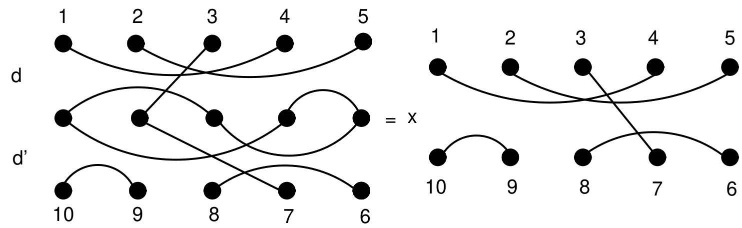

Figure 1.3

In general, the multiplication of two elements in $\mathfrak{B}_n(x)$ is given by the linear extension of a product defined on diagrams. For each integer i with $1 \leq i \leq 2n$, we define $i^- := 2n + 1 - i$. The Brauer algebra $\mathfrak{B}_n(x)$ is a free $\mathbb{Z}[x]$-module with rank $(2n-1) \cdot (2n-3) \cdots 3 \cdot 1$. For any $\mathbb{Z}[x]$-algebra R with x specialized to $\delta \in R$, we define $\mathfrak{B}_n(\delta)_R := R \otimes_{\mathbb{Z}[x]} \mathfrak{B}_n(x)$.

Now let K be an arbitrary field of characteristic not equal to 2. Let m, n be two positive integers and V an m dimensional orthogonal space over K. Let

$\mathfrak{B}_n(m)$ be the specialized Brauer algebra with parameter $m \cdot 1_K$. There is a right action of $\mathfrak{B}_n(m)$ on the n-tensor space $V^{\otimes n}$ which commutes with the left action of the orthogonal group $O(V)$. If $K = \mathbb{C}$, then by a well-known result of Brauer [1], the canonical homomorphism $\varphi : \mathfrak{B}_n(m) \to \mathrm{End}_{O(V)}(V^{\otimes n})$ is surjective. In general, as long as K is an infinite field of characteristic not equal to 2, the surjection still holds and we actually have a characteristic-free version of the Schur–Weyl duality between $\mathfrak{B}_n(m)$ and $KO(V)$ on $V^{\otimes n}$. For the proof as well as the symplectic version of these results, we refer the readers to [7], [10] and [11].

The above Schur–Weyl duality is closely related to the second fundamental theorem in invariant theory for $O(V)$. By [12] and [27], $\mathfrak{B}_n(m)$ is semisimple if and only if $m \geq n-1$. From the representation theoretic point of view, it is desirable to describe the radical of $\mathfrak{B}_n(m)$ in the non-semisimple case. By [16], the kernel of φ is closely related to the radical of $\mathfrak{B}_n(m)$. Therefore, it is important to understand the kernel of φ. Note that φ is not injective if and only if $n \geq m+1$. In [16, Theorem 4.8], using the invariant theory for $O(V)$, Gavarini showed that the kernel of φ is spanned by some diagrammatic minors of order $m+1$ (which are certain alternating sum of some Brauer n-diagrams). Note that, however, those diagrammatic minors are not necessarily K-linearly independent. In [11, Theorem 1.4, Theorem 6.9], an integral basis for the kernel of φ was obtained. The Brauer algebra $\mathfrak{B}_n(m)$ can be endowed with a right $\mathfrak{S}_{2n}$-module structure in a way such that $\mathrm{Ker}\,\varphi$ is an $\mathfrak{S}_{2n}$-submodule (cf. [11] and [15]). So far, to the best of our knowledge, it remains an open question on whether or not there exists a characteristic-free basis for $\mathrm{Ker}\,\varphi$ which consists of some diagrammatic minors of order $m+1$.

In [20, Corollary 5.9], we proved in the symplectic case that $\mathrm{Ker}\,\varphi$ is always generated by one specific diagrammatic Pfaffian of order $2m+2$. In the quantized type C case, we proved (in [20, Proposition 5.6]) a similar statement under the assumption that the quantum parameter q is generic. Recently, G.I. Lehrer and R.B. Zhang have studied extensively the orthogonal case in [22] by connecting it with the second fundamental theorem of invariant theory for the orthogonal group. For each Brauer n-diagram $D \in \mathrm{Bd}_n$, the vertices of D are arranged in two rows: the top and bottom rows. The vertices in top row are labeled by the indices $1, 2, \cdots, n$ from left to right; while the vertices in bottom row are labeled by the indices $1^-, \cdots, n^-$ from left to right. The following key definitions are due to them.

Definition 1.1. ([22, Definition 4.2]) *Let $a, b \in \mathbb{N}$ such that $1 \leq a+b \leq n$. Let $\mathrm{Bd}(a,b)$ be the set of all Brauer n-diagrams D such that:*

(1) for each integer s with $a+b+1 \leq s \leq n$, D connects the vertex labeled by s with the vertex labeled by s^-; and

(2) for each integer s with $s \in \{1, 2, \cdots, a, (a+1)^-, (a+2)^-, \cdots, (a+b)^-\}$, D connects the vertex labeled by s with the vertex labeled by t for $t \in \{1^-, 2^-, \cdots, a^-, a+1, a+2, \cdots, a+b\}$.

We define

$$E_{a,b} := \sum_{D\in \mathrm{Bd}(a,b)} \mathrm{sign}(D)D, \quad E_i := E_{i,m+1-i},\ \forall\ 0\le i\le m+1.$$

We refer the reader to Definition 2.2 for the definition of sign(D).[1] Lehrer and Zhang have proved a number of important properties about those E_i. In particular, they have proved the following theorem in [22].

Theorem 1.2. ([22, Proposition 4.4, Theorems 4.3, 9.4]) *Assume that $m < n$. Then for each integer i with $0 \le i \le [(m+1)/2]$, $E_i \in \operatorname{Ker}\varphi$. Furthermore, if* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > 2(m+1)$ *then* $\operatorname{Ker}\varphi$ *is the two-sided ideal of* $\mathfrak{B}_n(m)$ *generated by* $E_{[(m+1)/2]}$.

Furthermore, Lehrer and Zhang have conjectured in [22, Remark 9.5] that the second statement of the above theorem is true for arbitrary field K with $\operatorname{char} K \ne 2$. The main result in this paper is a proof of this conjecture. In other words, we extend Lehrer and Zhang's theorem to arbitrary field K with $\operatorname{char} K \ne 2$. That is,

Theorem 1.3. *Let K be an arbitrary field of characteristic other than two. Then* $\operatorname{Ker}\varphi$ *is always equal to the two-sided ideal generated by* $E_{[(m+1)/2]}$.

As a byproduct, we discover (in Theorem 4.15 and Corollary 4.17) a combinatorial identity which connects to the dimensions of some Specht modules over the symmetric group $\mathfrak{S}_{2m+2}$ and the symmetric group $\mathfrak{S}_{m+1}$. We get (in Corollary 4.19) a new integral basis for the annihilator of $V^{\otimes m+1}$ in $\mathfrak{B}_{m+1}(m)$. The content is organized as follows. In Section 2 we recall some basic knowledge about the structure and representation theory of the Brauer algebras as well as some related combinatorics which are needed later. In Section 3 we prove that the annihilator of $V^{\otimes n}$ in $\mathfrak{B}_n(m)$ is equal to the two-sided ideal generated by $E_0, E_1, \ldots, E_{[(m+1)/2]}$. The proof makes essential use of the integral basis of $\operatorname{Ker}\varphi$ obtained in [11]. In Section 4 we prove our main result Theorem 1.3. The proof will proceed in three steps. The main strategy to prove Theorem 1.3 is to transform it into a statement about identification between certain two-sided ideals in the symmetric group algebra $K\mathfrak{S}_n$. For the latter, we make use of the Young seminormal basis and the Murphy basis theory of the symmetric group algebras as well as the first main result obtained in Section 3.

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[1] At a first look, the definition of E_i which we give here seems to be different with [22, Definition 4.2]. But they are indeed the same. The equivalence between the two definitions follows from a simple counting by the proof given in the paragraph directly below [22, (4.5)].

2 The Brauer algebra

The Brauer algebra $\mathfrak{B}_n(x)$ can be alternatively defined as the unital associative $\mathbb{Z}[x]$-algebra with generators $s_1, \cdots, s_{n-1}, e_1, \cdots, e_{n-1}$ and relations (see [13]):

$$\begin{gathered} s_i^2 = 1,\ e_i^2 = xe_i,\ e_is_i = e_i = s_ie_i, \quad \forall\, 1 \le i \le n-1, \\ s_is_j = s_js_i,\ s_ie_j = e_js_i,\ e_ie_j = e_je_i, \quad \forall\, 1 \le i < j-1 \le n-2, \\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1},\ e_ie_{i+1}e_i = e_i,\ e_{i+1}e_ie_{i+1} = e_{i+1},\ \forall\, 1 \le i \le n-2, \\ s_ie_{i+1}e_i = s_{i+1}e_i,\ e_{i+1}e_is_{i+1} = e_{i+1}s_i, \quad \forall\, 1 \le i \le n-2. \end{gathered}$$

Note that the subalgebra of $\mathfrak{B}_n(x)$ generated by $s_1, \cdots, s_{n-1}$ is isomorphic to the symmetric group algebra $\mathbb{Z}[x]\mathfrak{S}_n$ of $\mathfrak{S}_n$ over $\mathbb{Z}[x]$ and $s_1, \cdots, s_{n-1}$ are the standard Coxeter generators. Let $\ell : \mathfrak{S}_n \to \mathbb{N}$ be the length function on $\mathfrak{S}_n$ so that $\ell(w) = k$ if k is minimal such that $w = s_{i_1} \dots s_{i_k}$, for some s_{i_j} with $1 \le i_j < n$.

For each integer $1 \le j < n$, the generator s_j corresponds to the Brauer n-diagram with edges connecting the vertices j (respectively, $j+1$) on the top row with $(j+1)^-$ (respectively, j^-) on the bottom row, and all other edges are vertical, connecting the vertices k and k^- on the top and bottom rows for all $k \neq i, i+1$; the generator e_j corresponds to the Brauer n-diagram with horizontal edges connecting the vertices $j, j+1$ (resp., $j^-, (j+1)^-$) on the top rows (resp., bottom rows), and all other edges are vertical, connecting the vertices k and k^- on the top and bottom rows for all $k \neq j, j+1$.

Let R be a commutative integral domain which is an $\mathbb{Z}[x]$-algebra such that x is specialized to $\delta \in R$. Then both the symmetric group algebra $R\mathfrak{S}_n$ and the Brauer algebra $\mathfrak{B}_n(\delta)_R$ are cellular algebras over R (see [26] and [17]). To recall their cellular structures we need some combinatorics. A composition of n is a sequence of nonnegative integer $\lambda = (\lambda_1, \lambda_2, \cdots)$ with $\sum_{i\ge1}\lambda_i = n$. A composition $\lambda = (\lambda_1, \lambda_2, \cdots)$ of n is said to be a partition if $\lambda_1 \ge \lambda_2 \ge \cdots$. In this case, we write $\lambda \vdash n$ and $|\lambda| = n$. We use $\mathcal{P}_n$ to denote the set of all the partitions of n. For any composition λ of n, the conjugate of λ is defined to be a partition $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$, where $\lambda'_j := \#\{i | \lambda_i \ge j\}$ for each $j \ge 1$. We use $\mathfrak{S}_\lambda$ to denote the standard Young subgroup of $\mathfrak{S}_n$ corresponding to λ. That is

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\{1,2,\cdots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\lambda_1+2,\cdots,\lambda_1+\lambda_2\}} \times \cdots .$$

Let λ be a composition of n. The Young diagram of λ is defined to be the set

$$[\lambda] := \big\{(i,j) \bigm| 1 \le j \le \lambda_i\big\}.$$

The elements of $[\lambda]$ are called nodes of λ. A λ-tableau is a bijection $\mathfrak{t} : [\lambda] \to \{1, 2, \cdots, n\}$. The symmetric group $\mathfrak{S}_n$ acts on the set of λ-tableaux from the right hand side by letter permutations. If λ is a partition, then the conjugate of $\mathfrak{t}$ is define to be the λ'-tableau $\mathfrak{t}'$ such that $\mathfrak{t}'(i,j) := \mathfrak{t}(j,i)$ for any $(i,j) \in [\lambda']$. The λ-tableau $\mathfrak{t}$ is row standard if $\mathfrak{t}(i,j) \le \mathfrak{t}(i,k)$ whenever $j \le k$. $\mathfrak{t}$ is standard if both $\mathfrak{t}$ and $\mathfrak{t}'$ are row-standard. Let $\mathrm{Std}(\lambda)$ be the set of standard λ-tableaux.

We denote by $\mathfrak{t}^\lambda$ (respectively, $\mathfrak{t}_\lambda$) the standard λ-tableau in which the numbers $1, 2, \cdots, n$ appear in order along successive rows (respectively, columns). If $\mathfrak{t}$ is a λ-tableau then let $d(\mathfrak{t}) \in \mathfrak{S}_n$ such that $\mathfrak{t}^\lambda d(\mathfrak{t}) = \mathfrak{t}$ and we shall write $\mathrm{Shape}(\mathfrak{t}) = \lambda$. Note that $\mathfrak{S}_\lambda$ is the row stabilizer of $\mathfrak{t}^\lambda$. We use $\mathcal{D}_\lambda$ to denote the set of distinguished right coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$. Then for any $d \in \mathcal{D}_\lambda$ and $w \in \mathfrak{S}_\lambda$ we have that $\ell(wd) = \ell(w) + \ell(d)$. Let $w_\lambda \in \mathfrak{S}_n$ such that $\mathfrak{t}^\lambda w_\lambda = \mathfrak{t}_\lambda$. Then $w_\lambda \in \mathcal{D}_\lambda$.

We define

$$X_\lambda := \sum_{w \in \mathfrak{S}_\lambda} w, \quad Y_\lambda := \sum_{w \in \mathfrak{S}_\lambda} (-1)^{\ell(w)} w.$$

Let τ be the R-algebra automorphism of $R\mathfrak{S}_n$ which is defined on generators by $\tau(s_i) = -s_i$ for any $1 \leq i < n$. It is clear that $\tau^2 = \mathrm{id}$ and $\tau(Y_\lambda) = X_\lambda$.

Let $\lambda \vdash n$. For any $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$, we define $X_{\mathfrak{s}\mathfrak{t}} := d(\mathfrak{s})^{-1} X_\lambda d(\mathfrak{t})$. By a well-known result of Murphy [26], the set $\{X_{\mathfrak{s}\mathfrak{t}} | \lambda \vdash n, \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ is a basis of $R\mathfrak{S}_n$. We call it the *Murphy basis* of $R\mathfrak{S}_n$. It is a cellular basis of $R\mathfrak{S}_n$ in the sense of [17]. Note also that the set $\{Y_{\mathfrak{s}\mathfrak{t}} := d(\mathfrak{s})^{-1} Y_\lambda d(\mathfrak{t}) | \lambda \vdash n, \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ is a cellular basis of $R\mathfrak{S}_n$ too. We call it the *Y Murphy basis* of $R\mathfrak{S}_n$. For both cellular bases the cell modules (i.e., Specht modules) of $R\mathfrak{S}_n$ are labeled by the partitions in $\mathcal{P}_n$.

For any $\lambda, \mu \in \mathcal{P}_n$, we write $\lambda \trianglerighteq \mu$ if $\sum_{j=1}^i \lambda_j \geq \sum_{j=1}^i \mu_j$ for any $i \geq 1$. If $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$, then we write $\lambda \triangleright \mu$. We use $(R\mathfrak{S}_n)^{\trianglerighteq \lambda}$ (respectively, $(R\mathfrak{S}_n)^{\triangleright \lambda}$) to denote the free R-submodule of $R\mathfrak{S}_n$ spanned by the Murphy basis elements of the form $X_{\mathfrak{u}\mathfrak{v}}$ with $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\mu)$ and $\mu \trianglerighteq \lambda$ (respectively, $\mu \triangleright \lambda$). Then both $(R\mathfrak{S}_n)^{\trianglerighteq \lambda}$ and $(R\mathfrak{S}_n)^{\triangleright \lambda}$ are two-sided ideals of $R\mathfrak{S}_n$.

We now recall the cellular structure of the Brauer algebra $\mathfrak{B}_n(m)$. Let f be an integer with $0 \leq f \leq [n/2]$, where $[n/2]$ is the largest non-negative integer not bigger than $n/2$. We define

$$\mathcal{D}_f := \left\{ d \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} (2j-1)d < (2j)d \text{ for } 1 \leq j \leq f \\ (1)d < (3)d < \cdots < (2f-1)d \\ (2f+1)d < (2f+2)d < \cdots < (n)d \end{array} \right\}.$$

For each $\lambda \in \mathcal{P}_{n-2f}$, we denote by $\mathrm{Std}_{2f}(\lambda)$ the set of all the standard λ-tableaux with entries in $\{2f+1, \cdots, n\}$. The initial tableau $\mathfrak{t}_f^\lambda$ in this case has the numbers $2f+1, \cdots, n$ in order along successive rows. Again, for each $\mathfrak{t} \in \mathrm{Std}_{2f}(\lambda)$, let $d(\mathfrak{t})$ be the unique element in $\mathfrak{S}_{\{2f+1,\cdots,n\}} \subseteq \mathfrak{S}_n$ with $\mathfrak{t}_f^\lambda d(\mathfrak{t}) = \mathfrak{t}$. Let $\sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}}$ and $d_1, d_2 \in \mathcal{D}_f$. Then $d_1^{-1} e_1 e_3 \cdots e_{2f-1} \sigma d_2$ corresponds to the Brauer n-diagram where the top horizontal edges connect $(2i-1)d_1$ and $(2i)d_1$, the bottom horizontal edges connect $\big((2i-1)d_2\big)^-$ and $\big((2i)d_2\big)^-$, for $i = 1, 2, \cdots, f$, and the vertical edges connects $(j)d_1$ with $\big((j)d_2\big)^-$ for $j = 2f+1, 2f+2, \cdots, n$.

Lemma 2.1. ([10, Corollary 3.3]) *With the above notations, the set*

$$\left\{ d_1^{-1} e_1 e_3 \cdots e_{2f-1} \sigma d_2 \;\middle|\; 0 \le f \le [n/2],\ \sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}},\ d_1, d_2 \in \mathcal{D}_f \right\}.$$

is a basis of the Brauer algebra $\mathfrak{B}_n(x)_R$, which coincides with the natural basis given by Brauer n-diagrams.

Definition 2.2. *Let $D = d_1^{-1} e_1 e_3 \cdots e_{2f-1} \sigma d_2 \in \mathrm{Bd}_n$, where $0 \le f \le [n/2]$, $\sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}}$, $d_1, d_2 \in \mathcal{D}_f$. Then we define $\ell(D) := \ell(d_1) + \ell(d_2) + \ell(\sigma)$ and* $\mathrm{sign}(D) := (-1)^f (-1)^{\ell(D)}$.

Remark 2.3. *1) We can always draw the Brauer diagram as a "nice diagram" (i.e., in a way such that two edges intersect at most once and there are no self-intersections and no three edges intersect at one point, etc, see [14, 1.1]). If D is represented by a "nice diagram" with f horizontal edges in each row and $n(D)$ is the number of crossings of edges, then* $\mathrm{sign}(D) = (-1)^{f+n(D)}$. *Moreover, if $D \in \mathfrak{S}_n$ then $n(D)$ coincides with the length function on $\mathfrak{S}_n$ which we introduced before.*
2) For any $D_1, D_2 \in \mathrm{Bd}_n$, note that in general

$$\mathrm{sign}(D_1 D_2) \neq \mathrm{sign}(D_1)\,\mathrm{sign}(D_2).$$

3) Our definition of $\mathrm{sign}(D)$ *coincides with that of $\varepsilon(D)$ in [16, 1.4].*

Note that, however, the above basis is not a cellular basis for $\mathfrak{B}_n(x)$. But if we replace the σ in the above basis by a Murphy basis element of $R\mathfrak{S}_{\{2f+1,\cdots,n\}}$ then we will get a cellular basis of $\mathfrak{B}_n(x)$. Precisely, the set

$$\left\{ d_1^{-1} e_1 e_3 \cdots e_{2f-1} \big(d(\mathfrak{s})^{-1} X_\lambda^{(f)} d(\mathfrak{t}) \big) d_2 \;\middle|\; \begin{array}{l} 0 \le f \le [n/2],\ \lambda \in \mathcal{P}_{n-2f} \\ \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}_{2f}(\lambda), d_1, d_2 \in \mathcal{D}_f \end{array} \right\},$$

where $X_\lambda^{(f)} := \sum_{w \in \mathfrak{S}_\lambda^{(f)}} w$ and

$$\mathfrak{S}_\lambda^{(f)} := \mathfrak{S}_{\{2f+1,\cdots,2f+\lambda_1\}} \times \mathfrak{S}_{\{2f+\lambda_1+1,\cdots,2f+\lambda_1+\lambda_2\}} \times \cdots ,$$

is a cellular basis of the Brauer algebra $\mathfrak{B}_n(x)_R$. The cell modules of $\mathfrak{B}_n(x)_R$ are labeled by the set of pairs (f, λ), where $0 \le f \le [n/2]$ and $\lambda \vdash n - 2f$. For any two pairs $(f, \lambda), (g, \mu)$ with $0 \le f, g \le [n/2]$ and $\lambda \in \mathcal{P}_{n-2f}, \mu \in \mathcal{P}_{n-2g}$, we define $(f, \lambda) \trianglerighteq (g, \mu)$ if either $f > g$ or $f = g$ and $\lambda \trianglerighteq \mu$. If $(f, \lambda) \trianglerighteq (g, \mu)$ and $(f, \lambda) \neq (g, \mu)$, then we write $(f, \lambda) \rhd (g, \mu)$. We use $(\mathfrak{B}_n(x))^{\trianglerighteq (f,\lambda)}$ (respectively, $(\mathfrak{B}_n(x))^{\rhd (f,\lambda)}$) to denote the free R-submodule of $\mathfrak{B}_n(x)$ spanned by the cellular basis elements corresponding to those $(g, \mu, d_1, d_2, \mathfrak{s}, \mathfrak{t})$ with $\mu \in \mathcal{P}_{n-2g}$, $d_1, d_2 \in \mathcal{D}_g$, $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\mu)$ and $(g, \mu) \trianglerighteq (f, \lambda)$ (respectively, $(g, \mu) \rhd (f, \lambda)$). Then both $(\mathfrak{B}_n(x))^{\trianglerighteq (f,\lambda)}$ and $(\mathfrak{B}_n(x))^{\rhd (f,\lambda)}$ are two-sided ideals of $\mathfrak{B}_n(x)$. In particular, if we denote by $\mathfrak{B}_n(x)^{(f)}$ the two-sided ideal of $\mathfrak{B}_n(x)$ generated by $e_1e_3\cdots e_{2f-1}$, then

$$\mathfrak{B}_n(x)^{(f)} = \sum_{\lambda\vdash n-2f} (\mathfrak{B}_n(x))^{\trianglerighteq(f,\lambda)}$$

is spanned by the cellular basis elements which it contains. Henceforth, we shall write $\mathfrak{B}_n^{(f)}$ instead of $\mathfrak{B}_n(x)^{(f)}$ for simplicity.

The Brauer algebra $\mathfrak{B}_n(x)$ and its specialized version have been studied in a number of references, e.g., [1], [2], [3], [4], [5], [6], [7], [10], [12], [14], [15], [16], [19], [20], [25], [27] and [28]. In the set up of Schur–Weyl duality for orthogonal groups, we only need certain specialized Brauer algebras. Let K be a field of characteristic not equal to 2. Let $m \in \mathbb{N}$ and V an m-dimensional orthogonal space over K. Let $\mathfrak{B}_n(m)_{\mathbb{Z}} := \mathbb{Z}\otimes_{\mathbb{Z}[x]}\mathfrak{B}_n(x)$, where $\mathbb{Z}$ is regarded as $\mathbb{Z}[x]$-algebra by specifying x to m. Let $\mathfrak{B}_n(m) := K\otimes_{\mathbb{Z}}\mathfrak{B}_n(m)_{\mathbb{Z}}$, where K is regarded as $\mathbb{Z}$-algebra in the natural way. Then there is a right action of $\mathfrak{B}_n(m)$ on the n-tensor space $V^{\otimes n}$ which commutes with the natural left action of $O(V)$. We recall the definition of this action. Let $\delta_{i,j}$ denote the value of the usual Kronecker delta. We fix an ordered basis $\{v_1, v_2, \cdots, v_m\}$ of V such that

$$(v_i, v_j) = \delta_{i,m+1-j}, \quad \forall\ 1 \le i, j \le m.$$

The right action of $\mathfrak{B}_n(m)$ on $V^{\otimes n}$ is defined on generators by

$$\begin{aligned}(v_{i_1}\otimes\cdots\otimes v_{i_n})s_j &:= v_{i_1}\otimes\cdots\otimes v_{i_{j-1}}\otimes v_{i_{j+1}}\otimes v_{i_j}\otimes v_{i_{j+2}}\otimes\cdots\otimes v_{i_n},\\ (v_{i_1}\otimes\cdots\otimes v_{i_n})e_j &:= \delta_{i_j,m+1-i_{j+1}} v_{i_1}\otimes\cdots\otimes v_{i_{j-1}}\otimes\left(\sum_{k=1}^{m} v_k\otimes v_{m+1-k}\right)\\ &\qquad \otimes v_{i_{j+2}}\otimes\cdots\otimes v_{i_n}.\end{aligned}$$

Let $\overline{K}$ be the algebraic closure of K. Set $V_{\overline{K}} := \overline{K}\otimes_K V$. Then by the main results in [1], [2], [3], [7], [11] and [29], we know that there is a Schur–Weyl duality between $\mathfrak{B}_n(m)_{\overline{K}}$ and $O(V_{\overline{K}})$ on $V_{\overline{K}}^{\otimes n}$. In particular, we have two surjective homomorphisms as follows:

$$\varphi_{\overline{K}} : \mathfrak{B}_n(m)_{\overline{K}} \to \operatorname{End}_{O(V_{\overline{K}})}(V_{\overline{K}}^{\otimes n}), \quad \psi_{\overline{K}} : \overline{K}O(V_{\overline{K}}) \to \operatorname{End}_{\mathfrak{B}_n(m)_{\overline{K}}}(V_{\overline{K}}^{\otimes n}).$$

Furthermore, $\varphi_{\overline{K}}$ is injective if and only if $m \ge n$. If $m < n$ then $\dim\operatorname{Ker}\varphi_{\overline{K}}$ is independent of the characteristic of the field K (as long as $\operatorname{char} K \neq 2$).

From now on until the end of this section, we assume that $m < n$. The main results in [11] actually implies that $\dim_{\overline{K}}\operatorname{Ker}\varphi_{\overline{K}} = \dim_K \operatorname{Ker}\varphi$ and $\operatorname{Ker}\varphi_{\overline{K}} = \overline{K}\otimes_K \operatorname{Ker}\varphi$ because [11, Theorem 1.4, Theorem 6.9] gave an integral basis for $\operatorname{Ker}\varphi_{\overline{K}}$. In particular, $\dim\operatorname{Ker}\varphi$ is independent of the characteristic of the field K (as long as $\operatorname{char} K \neq 2$). In the following sections we shall sometimes use $\operatorname{Ann}_{\mathfrak{B}_n(m)}(V^{\otimes n})$ to denote the annihilator of $V^{\otimes n}$ in $\mathfrak{B}_n(m)$. By definition, $\operatorname{Ann}_{\mathfrak{B}_n(m)}(V^{\otimes n}) = \operatorname{Ker}\varphi$.

3 The annihilator of n-tensor space

In this section, we shall prove that the annihilator of $V^{\otimes n}$ in $\mathfrak{B}_n(m)$ is equal to the two-sided ideal generated by $E_0, E_1, \dots, E_{[(m+1)/2]}$. This generalizes the early result (for the case $\operatorname{char} K = 0$) of Lehrer–Zhang [22, Theorem 6.1].

We first recall a definition and a result given in [22].

Definition 3.1. ([22, Lemma 4.1]) *Let $S = (i_1, \cdots, i_N), S' = (j_1, \cdots, j_N)$ be two N-tuples of integers such that $\{i_1, \cdots, i_N\}, \{j_1, \cdots, j_N\}$ are two disjoint subsets of $\{1, 2, \cdots, 2n\}$. Let β be any pairing of the vertices $\bigl\{1, 2, \cdots, 2n\bigr\} \setminus \{i_1, \cdots, i_N, j_1, \cdots, j_N\}$. For $w \in \mathfrak{S}_N$, let $D_w(S, S', \beta)$ be the Brauer diagram with edges $\bigl\{(i_k, j_{\pi(k)}) \bigm| k = 1, 2, \cdots, N\bigr\} \sqcup \beta$. We define*

$$b(S, S', \beta) := \sum_{w \in \mathfrak{S}_N} \operatorname{sign}(w) D_w(S, S', \beta) \in \mathfrak{B}_n(m).$$

Lemma 3.2. ([22, (4.5)]) *Let $S_{a,b} := (1, 2, \cdots, a, (a+1)^-, (a+2)^-, \cdots, (a+b)^-)$, $S'_{a,b} := (a+1, a+2, \cdots, a+b, 1^-, 2^-, \cdots, a^-)$, β_{a+b} be the pairing $(a+b+1, (a+b+1)^-)$, $(a+b+2, (a+b+2)^-), \cdots, (n, n^-)$. Then we have that*

$$E_{a,b} = b(S_{a,b}, S'_{a,b}, \beta_{a+b}).$$

The advantage of the above alternative description of $E_{a,b}$ lies in that the sign $\operatorname{sign}(w)$ before $D_w(S, S', \beta)$ depends only on w which is more easier to be handled than the sign $\operatorname{sign}(D_w(S, S', \beta))$. More precisely, up to a sign, $b(S, S', \beta)$ depends only on β and the two subsets $\{i_1, \cdots, i_N\}$, $\{j_1, \cdots, j_N\}$ but not on the orderings on these two subsets.

For any $h \in \mathfrak{B}_n(m)$, we use $\langle h \rangle$ to denote the two-sided ideal of $\mathfrak{B}_n(m)$ generated by h. For any finite set S, we use $|S|$ to denote the cardinality of S.

Lemma 3.3. *Let $a, b \in \mathbb{N}$ such that $1 \le a+b \le n$. Then there exist two elements $w_1, w_2 \in \mathfrak{S}_{a+b}$ such that $E_{b,a} = \pm w_1 E_{a,b} w_2$. In particular, $\langle E_{a,b} \rangle = \langle E_{b,a} \rangle$.*

Proof. This is clear by Lemma 3.2. In fact, we can take w_1 to be the Brauer n-diagram which has

$$\begin{aligned} &\{1, (a+1)^-\}, \ \ \{2, (a+2)^-\}, \cdots, \{b, (a+b)^-\},\\ &\{b+1, 1^-\}, \ \ \{b+2, 2^-\}, \cdots, \{b+a, a^-\},\\ &\{r, r^-\}, \quad \text{for all } r \in \{a+b+1, a+b+2, \cdots, n\}, \end{aligned}$$

as its (vertical) edges; and w_2 to be the Brauer n-diagram which has

$$\begin{aligned} &\{1, (b+1)^-\}, \ \ \{2, (b+2)^-\}, \cdots, \{a, (b+a)^-\},\\ &\{a+1, 1^-\}, \ \ \{a+2, 2^-\}, \cdots, \{a+b, b^-\},\\ &\{r, r^-\}, \quad \text{for all } r \in \{a+b+1, a+b+2, \cdots, n\}, \end{aligned}$$

as its vertical edges. $\square$

The proof of the next lemma uses the original definition of $E_{a,b}$.

Lemma 3.4. *For any positive integers a, b with $1 \le a+b \le n$, we have that $E_{a,b} \in \langle E_{a,b-1}\rangle \cap \langle E_{a-1,b}\rangle$.*

Proof. For each $k \in \{1, 2, \cdots, a, (a+1)^-, \cdots, (a+b)^-\}$, we use $\mathrm{Bd}(k; a, b)$ to denote the subset of the Brauer diagrams in $\mathrm{Bd}(a, b)$ which have the edge $\{k, a+b\}$.
If $k = i \in \{1, 2, \cdots, a\}$ then we use d to denote the Brauer n-diagram which has $\{k, a+b\}$ and $\{k^-, (a+b)^-\}$ as its only horizontal edges and

$$\{r, r^-\}, \quad \text{for all } r \in \{1, 2, \cdots, n\} \backslash \{k, a+b\},$$

as its vertical edges. It is clear that $\mathrm{sign}(d) = -1$. By the concatenation rule of Brauer diagrams, it is easy to see that

$$\big\{D \bigm| D \in \mathrm{Bd}(k; a, b)\big\} = \big\{dD'' \bigm| D'' \in \mathrm{Bd}(a, b-1)\big\}.$$

We claim that

$$\sum_{D \in \mathrm{Bd}(k;a,b)} \mathrm{sign}(D) D = -dE_{a,b-1} \in \langle E_{a,b-1}\rangle.$$

To prove this claim, it suffices to show that for each $D'' \in \mathrm{Bd}(a, b-1)$,

$$\mathrm{sign}(dD'') = \mathrm{sign}(d)\,\mathrm{sign}(D'') = -\,\mathrm{sign}(D''). \tag{3.5}$$

Note that when concatenating a "nice diagram" for d with a "nice diagram" for D'' and transforming it into a "nice diagram" for dD'', the only transformation which can possibly change the parity of $\ell(d) + 1 + \ell(D'') + f$ (where $2f$ is the number of horizontal edges of D'') is the following type of edge which was drawn in red color (where $D'' \in \mathrm{Bd}(a, b-1)$):

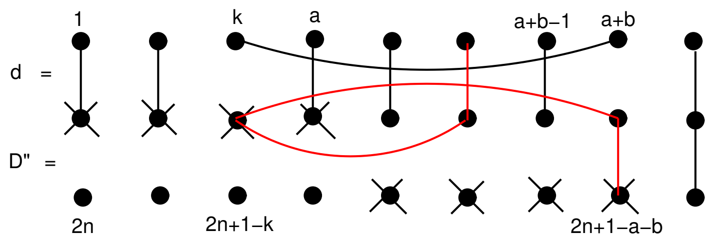

Figure 1.4

That is, we need to eliminate the self-intersection in the following picture.

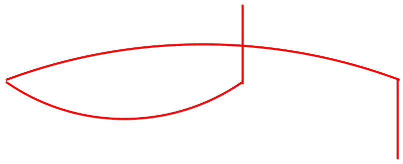

Figure 1.5

However, by eliminating the above self-intersection and making it into an edge in a "nice diagram" for dD'' has the effect of removing one horizontal edge on the top rows of D'' together with eliminating $2k-1$ crossing on this concatenation diagram for some $k \in \mathbb{N}$. To be more precise, when we eliminate the self-intersection in Figure 1.5, the immediate effect is that we will remove one top horizontal edge of D'' as well as one crossing from the concatenation diagram. However, there are possibly some more crossings which will be removed. These are the crossings arising from the vertices inside the area circled by the edge in Figure 1.5. If a vertex γ inside the area connects with another vertex which is also inside the circled area then these two interior vertices will contribute two crossings with the red line which will finally be eliminated; otherwise γ itself will connect with two different vertices outside the circled area and hence will produce two crossings with the red line which will finally be eliminated. To sum all, the sign finally remains unchanged. This proves (3.5).

If $k = i^- \in \{(a+1)^-, (a+2)^-, \cdots, (a+b)^-\}$ then we use d' to denote the Brauer n-diagram which has

$$\begin{aligned}&\{i,(i+1)^-\},\quad \{i+1,(i+2)^-\},\cdots,\{a+b-1,(a+b)^-\},\quad \{a+b,i^-\},\\ &\{r,r^-\},\quad \text{for all } r\in\{1,2,\cdots,i-1\}\sqcup\{a+b+1,a+b+2,\cdots,n\},\end{aligned}$$

as its (vertical) edges. It is clear that $\operatorname{sign}(D_2) = (-1)^{a+b-i}$. Then by a similar argument as in the case $k = i$, we can deduce that

$$\sum_{D\in \mathrm{Bd}(k;a,b)} \operatorname{sign}(D)D = (-1)^{a+b-i}E_{a,b-1}d' \in \langle E_{a,b-1}\rangle.$$

Therefore, we have that

$$E_{a,b} = \sum_{k\in\{1,2,\cdots,a,(a+1)^-,\cdots,(a+b)^-\}}\ \sum_{D\in \mathrm{Bd}(k;a,b)} \operatorname{sign}(D)D \in \langle E_{a,b-1}\rangle.$$

This proves $E_{a,b} \in \langle E_{a,b-1}\rangle$. It remains to prove that $E_{a,b} \in \langle E_{a-1,b}\rangle$. Exchanging the roles of a and b and using Lemma 3.3, we see that

$$\langle E_{a,b}\rangle = \langle E_{b,a}\rangle \subseteq \langle E_{b,a-1}\rangle = \langle E_{a-1,b}\rangle.$$

as required. This completes the proof of the lemma. □

Note that if $a \geq 1$ and $b = 0$ (respectively, if $a = 0$ and $b \geq 1$) then, by the theory of symmetric group, we have that $E_{a,0} \in \langle E_{a-1,0}\rangle$ (respectively, $E_{0,b} \in \langle E_{0,b-1}\rangle$).

Let $A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}$ be four subsets of indices such that

(a) $A^{(i)} \cap A^{(j)} = \emptyset$ for any $1 \leq i \neq j \leq 4$; and

(b) $A^{(1)} \sqcup A^{(3)} \subseteq \{1, 2, \cdots, n\}$, $A^{(2)} \sqcup A^{(4)} \subseteq \{1^-, 2^-, \cdots, n^-\}$; and

(c) $|A^{(1)}| + |A^{(2)}| = |A^{(3)}| + |A^{(4)}|$.

Recall that $i^- = 2n + 1 - i$ for each $1 \leq i \leq 2n$. We set $n_0 := |A^{(1)}| + |A^{(2)}|$, and

$$\{a_1, a_2, \cdots, a_{2n-2n_0}\} := \{1, 2, \cdots, 2n\} \setminus \bigsqcup_{k=1}^{4} A^{(k)}.$$

Let $(i_1, j_1, i_2, j_2, \cdots, i_{n-n_0}, j_{n-n_0})$ be a fixed permutation of $\{a_1, \cdots, a_{2n-2n_0}\}$. Set

$$\begin{aligned} &\mathbf{i} := (i_1, i_2, \cdots, i_{n-n_0}), \quad \mathbf{j} := (j_1, j_2, \cdots, j_{n-n_0}). \\ &a_{11} := |A^{(1)}|,\ a_{12} := |A^{(2)}|,\ a_{21} := |A^{(3)}|,\ a_{22} := |A^{(4)}|. \end{aligned}$$

Let $\beta_{\mathbf{i},\mathbf{j}}$ be the pairing $(i_1, j_1), (i_2, j_2), \cdots, (i_{n-n_0}, j_{n-n_0})$. We fix an ordering on $A^{(1)} \sqcup A^{(2)}$ and an ordering on $A^{(3)} \sqcup A^{(4)}$ respectively. We define S_A, S'_A to be the corresponding n_0-tuples with respect to the two orderings. As we said before, for different choices of orderings, $b(S_A, S'_A, \beta_{\mathbf{i},\mathbf{j}})$ differs only by a sign.

Lemma 3.6. *With notations as above, we have that*

$$b(S_A, S'_A, \beta_{\mathbf{i},\mathbf{j}}) \in \langle E_{a_{11}+k, a_{12}-k}\rangle,$$

for some integers $0 \leq k \leq \min\{a_{12}, n - a_{11}\}$.

Proof. Assume that

$$\begin{aligned} &S_A = (q_1, \cdots, q_{a_{11}}, p^-_{a_{22}+1}, \cdots, p^-_{a_{22}+a_{12}}), \\ &S'_A = (q_{a_{11}+1}, \cdots, q_{a_{11}+a_{21}}, p^-_1, \cdots, p^-_{a_{22}}), \\ &\{q_{a_{11}+a_{21}+1}, \cdots, q_n\} = \{i_1, \cdots, i_{n-n_0}, j_1, \cdots, j_{n-n_0}\} \cap \{1, 2, \cdots, n\}, \\ &\{p^-_{a_{12}+a_{22}+1}, \cdots, p^-_n\} = \{i_1, \cdots, i_{n-n_0}, j_1, \cdots, j_{n-n_0}\} \cap \{1^-, 2^-, \cdots, n^-\}, \end{aligned}$$

where $a_{11} + a_{12} = a_{21} + a_{22} = n_0$. In other words,

$$\begin{aligned} &A^{(1)} = \{q_1, \cdots, q_{a_{11}}\}, \quad A^{(2)} = \{p^-_{a_{22}+1}, \cdots, p^-_{a_{22}+a_{12}}\}, \\ &A^{(3)} = \{q_{a_{11}+1}, \cdots, q_{a_{11}+a_{21}}\}, \quad A^{(4)} = \{p^-_1, \cdots, p^-_{a_{22}}\}. \end{aligned}$$

We divide the proof into three cases:

Case 1. $a_{11} = a_{22}$ and $a_{12} = a_{21}$. In this case, We use σ_1 to denote the Brauer diagram which has the following edges

$$\begin{aligned}&\{k, q_k^-\}, \quad \text{for } 1 \le k \le a_{11} + a_{21},\\ &\{a_{11} + a_{21} + l, q_{a_{11}+a_{21}+l}^-\}, \quad \text{for } 1 \le l \le n - a_{11} - a_{21},\end{aligned}$$

and use σ_2 to denote the Brauer diagram which has the following edges

$$\begin{aligned}&\{p_k, k^-\}, \quad \text{for } 1 \le k \le a_{12} + a_{22},\\ &\{p_{a_{12}+a_{22}+l}, (a_{12} + a_{22} + l)^-\}, \quad \text{for } 1 \le l \le n - a_{12} - a_{22}.\end{aligned}$$

Then σ_1, σ_2 are both elements in the symmetric group $\mathfrak{S}_n$.
The pairing $\beta_{\mathbf{i},\mathbf{j}}$ and the elements σ_1, σ_2 determine a pairing β on the set of vertices $\{n_0+1, n_0+2, \cdots, n, (n_0+1)^-, (n_0+2)^-, \cdots, n^-\}$, and hence a Brauer $(n - n_0)$-diagram D. Since the number of top horizontal edges of D is the same as the number of the bottom horizontal edges of D, we can clearly write $D = D_1 D_0 D_2$ such that $D_1, D_0, D_2 \in \mathrm{Bd}_{n-n_0}$ and D_0 contains only the vertical edges of the form (k, k^-) with $n_0 + 1 \le k \le n$. We extend the Brauer diagrams $D_1, D_2 \in \mathrm{Bd}_{n-n_0}$ to be Brauer diagrams $D_1', D_2' \in \mathrm{Bd}_n$ by adding the vertical edges (k, k^-) with $1 \le k \le n_0$ to their left-hand sides.
Then it follows directly from the alternative description of $E_{a,b}$ given in Lemma 3.2 that

$$\sigma_1 b(S_A, S_A', \beta_{\mathbf{i},\mathbf{j}})\sigma_2 = D_1' E_{a_{11},a_{12}} D_2'.$$

It follows (by Lemma 3.3) that

$$b(S_A, S_A', \beta_{\mathbf{i},\mathbf{j}}) = \sigma_1^{-1} D_1' E_{a_{11},a_{12}} D_2' \sigma_2^{-1} \in \langle E_{a_{11},a_{12}} \rangle = \langle E_{a_{12},a_{11}} \rangle,$$

as required.
Case 2. $a_{11} > a_{22}$ and $a_{12} < a_{21}$. We set $d := a_{11} - a_{22}$. Then $d \in \mathbb{N}$ and $d \ge 1$. We use induction on d. It is clear that $a_{11} \ge 1$. In this case, for each Brauer diagram D'' involved in $b(S_A, S_A', \beta_{\mathbf{i},\mathbf{j}})$, since $a_{11} + a_{21} > a_{12} + a_{22}$, there must be some bottom horizontal edges in D'' of the form $\{p_a^-, p_b^-\}$ with $a, b > a_{12} + a_{22}$. In other words, there must exist a pairing $(i_{s_0}, j_{s_0}) \in \beta_{\mathbf{i},\mathbf{j}}$ such that $i_{s_0}, j_{s_0} \in \{1^-, 2^-, \cdots, n^-\}$. Let D_3 be the Brauer diagram which has the horizontal edges $\{i_{s_0}^-, j_{s_0}^-\}, \{i_{s_0}, j_{s_0}\}$ and vertical edges $\{j, j^-\}$ for any $j \in \{1, 2, \cdots, n\} \setminus \{i_{s_0}^-, j_{s_0}^-\}$. Define

$$\begin{aligned}&S_B := S_A, \quad S_B' := (S_A' \setminus \{q_{a_{11}+a_{21}}\}) \cup \{i_{s_0}\},\\ &\beta' := \big\{(i_s, j_s) | 1 \le s \le n - n_0, (i_s, j_s) \ne (i_{s_0}, j_{s_0})\big\} \sqcup \{(q_{a_{11}+a_{21}}, j_{s_0})\}.\end{aligned}$$

Then it is clear that

$$b(S_A, S_A', \beta_{\mathbf{i},\mathbf{j}}) = b(S_B, S_B', \beta') D_3 \in \langle b(S_B, S_B', \beta') \rangle.$$

Now we are in a position to apply induction hypothesis to $b(S_B, S_B', \beta')$. This proves the lemma in Case 2.

Case 3. $a_{11} < a_{22}$ and $a_{12} > a_{21}$. We set $d := a_{22} - a_{11}$. Then $d \in \mathbb{N}$ and $d \geq 1$. We use induction on d. It is clear that $a_{12} \geq 1$. In this case, for each Brauer diagram $\widehat{D}$ involved in $b(S_A, S'_A, \beta_{\mathbf{i},\mathbf{j}})$, since $a_{11} + a_{21} < a_{12} + a_{22}$, there must be some top horizontal edges in $\widehat{D}$ of the form $\{q_a, q_b\}$ with $a, b > a_{11} + a_{21}$. In other words, there must exist a pairing $(i_{s_0}, j_{s_0}) \in \beta_{\mathbf{i},\mathbf{j}}$ such that $i_{s_0}, j_{s_0} \in \{1, 2, \cdots, n\}$. Let D'_3 be the Brauer diagram which has the horizontal edges $\{i^-_{s_0}, j^-_{s_0}\}, \{i_{s_0}, j_{s_0}\}$ and vertical edges $\{j, j^-\}$ for any $j \in \{1, 2, \cdots, n\} \setminus \{i_{s_0}, j_{s_0}\}$. Define

$$
\begin{aligned}
&S_B := (S_A \setminus \{p^-_{a_{12}+a_{22}}\}) \cup \{i_{s_0}\}, \quad S'_B := S'_A,\\
&\beta' := \big\{(i_s, j_s) | 1 \leq s \leq n - n_0, (i_s, j_s) \neq (i_{s_0}, j_{s_0})\big\} \sqcup \{(j_{s_0}, p^-_{a_{12}+a_{22}})\}.
\end{aligned}
$$

Then it is clear that

$$b(S_A, S'_A, \beta_{\mathbf{i},\mathbf{j}}) = D'_3 b(S_B, S'_B, \beta') \in \langle b(S_B, S'_B, \beta') \rangle.$$

Now we are in a position to apply induction hypothesis to $b(S_B, S'_B, \beta')$. This proves the lemma in Case 3. Hence we complete the proof of the lemma. $\square$

For the sake of simplicity, we shall abbreviate the partition $(\underbrace{a, \cdots, a}_{k \text{ copies}})$ as (a^k).

Definition 3.7. ([11, Theorem 1.4]) *We set*

$$
\begin{aligned}
(2\mathcal{P}_n)' &:= \big\{\tilde{\lambda} := (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \cdots) \vdash 2n \bigm| \lambda = (\lambda_1, \lambda_2, \cdots) \in \mathcal{P}_n\big\},\\
T_m &:= \big\{(\nu, \mathfrak{t}) \bigm| \mathfrak{t} \in \mathrm{Std}(\tilde{\nu}), (m+1, 1^{n-m-1}) \trianglelefteq \nu \in \mathcal{P}_n\big\}.
\end{aligned}
$$

Now we are in a position to state the main result of this section.

Theorem 3.8. *Let K be an arbitrary field of characteristic other than two. If $n > m$, then*

$$\mathrm{Ann}_{\mathfrak{B}_n(m)}\left(V^{\otimes n}\right) = \left\langle E_0, E_1, \cdots, E_{[\frac{m+1}{2}]} \right\rangle.$$

Proof. By [11, Theorem 1.4 and Theorem 6.9], we know that $\mathrm{Ann}_{\mathfrak{B}_n(m)}\left(V^{\otimes n}\right)$ has a basis consisting of elements of the form $Y_{\nu,\mathfrak{t}}$, where $(\nu, \mathfrak{t}) \in T_m$. It remains to show (by the first statement of Theorem 1.2) that each $Y_{\nu,\mathfrak{t}}$ belongs the two-sided ideal generated by $E_0, E_1, \cdots, E_{[\frac{m+1}{2}]}$.

Let $(\nu, \mathfrak{t}) \in T_m$. By the definition (see [11, §6])

$$Y_{\nu,\mathfrak{t}} = (-1)^{\ell(d(\mathfrak{t}))} Y_\nu * d(\mathfrak{t}) = (-1)^{\ell(d(\mathfrak{t}))} \sum_{\mathbf{i},\mathbf{j}} (\pm b(S_A, S'_A, \beta_{\mathbf{i},\mathbf{j}})),$$

where "$*$" denotes the permutation action of $\mathfrak{S}_{2n}$ on $\mathfrak{B}_n(m)$ (see [11, Section 6]), and

$$
\begin{aligned}
A^{(1)} &:= \{(i)d(\mathfrak{t}) \mid i = 1, 2, 3, \cdots, \nu_1\} \cap \{1, 2, \cdots, n\},\\
A^{(2)} &:= \{(i)d(\mathfrak{t}) \mid i = 1, 2, 3, \cdots, \nu_1\} \cap \{1^-, 2^-, \cdots, n^-\},\\
A^{(3)} &:= \{(i^-)d(\mathfrak{t}) \mid i = 1, 2, 3, \cdots, \nu_1\} \cap \{1, 2, \cdots, n\},\\
A^{(4)} &:= \{(i^-)d(\mathfrak{t}) \mid i = 1, 2, 3, \cdots, \nu_1\} \cap \{1^-, 2^-, \cdots, n^-\},
\end{aligned}
$$

with $|A^{(1)}|+|A^{(2)}|=|A^{(3)}|+|A^{(4)}|=\nu_1$, S_A, S'_A are defined by using certain prefixed ordering on the sets $A^{(1)}\sqcup A^{(2)}$, $A^{(3)}\sqcup A^{(4)}$respectively, and

$$\mathbf{i}:=(i_1,i_2,\cdots,i_{n-\nu_1}),\quad \mathbf{j}:=(j_1,j_2,\cdots,j_{n-\nu_1})$$

such that $(i_1,j_1,i_2,j_2,\cdots,i_{n-\nu_1},j_{n-\nu_1})$ runs over a subset of permutations of the integers in $\{1,2,\cdots,2n\}\setminus\bigsqcup_{k=1}^4 A^{(k)}$.

By Lemma 3.6, we obtain

$$b(S_A,S'_A,\beta_{\mathbf{i},\mathbf{j}})\in\left\langle E_{|A^{(1)}|+k,|A^{(2)}|-k}\right\rangle,$$

for some integers k with $0\leq k\leq\min\{|A^{(2)}|,n-|A^{(1)}|\}$. Note that the condition $(\nu,\mathfrak{t})\in T_m$ implies that $|A^{(1)}|+|A^{(2)}|=\nu_1\geq m+1$. It follows from Lemma 3.4 that $b(S_A,S'_A,\beta_{\mathbf{i},\mathbf{j}})$ belongs to the two-sided ideal generated by $E_0,E_1,\cdots,E_{m+1}$.

On the other hand, it is clear that for any integer $[\frac{m+1}{2}]+1\leq i\leq m+1$, there exists $\sigma'_1,\sigma'_2\in\mathfrak{S}_n$, such that $E_i=\pm\sigma'_1E_{m+1-i}\sigma'_2\in\langle E_{m+1-i}\rangle$. As a consequence, we get that

$$b(S_A,S'_A,\beta_{\mathbf{i},\mathbf{j}})\in\langle E_0,E_1,\cdots,E_{[(m+1)/2]}\rangle,$$

as required. This completes the proof of the theorem. $\square$

4 Proof of Theorem 1.3

In this section we shall give the main result of this paper. That is, the proof of Theorem 1.3.

We shall proceed the proof in three steps. The first step is to prove a statement (Theorem 4.10) about identification between certain two-sided ideals in the symmetric group algebra $K\mathfrak{S}_n$. To this end, we need to recall the seminormal basis ([18], [9], [24]) of the symmetric group algebra. We shall follow the approach in [24]. Note that [24] only consider the seminormal basis of the (cyclotomic) Hecke algebra with $q\neq 1$. The symmetric group case (i.e., $q=1$) is similar and may be proved using the same arguments. The only real difference between the cases $q\neq 1$ and $q=1$ is the choice of content function: if $q\neq 1$ then $\mathrm{cont}_{\mathfrak{t}}(k)=\xi^{c-r}$, when $\mathfrak{t}(r,c)=k$, and if $q=1$ then, instead, $\mathrm{cont}_{\mathfrak{t}}(k)=c-r$. Analogous minor 'logarithmic' adjustments are required in the argument below when $q=1$.

Set $L_1:=0$ and define $L_{i+1}:=s_iL_is_i+s_i$ for $i=1,\cdots,n-1$. The elements $L_1,\cdots,L_n$ are called the Jucys–Murphy operators of the symmetric group $\mathfrak{S}_n$. Let $\lambda\vdash n$ and $\mathfrak{t}\in\mathrm{Std}(\lambda)$. For any integer $1\leq k\leq n$, we define $\mathrm{cont}_{\mathfrak{t}}(k)=j-i$ if k appears in row i and column j in $\mathfrak{t}$. Let

$$\mathcal{R}(k):=\left\{d\in\mathbb{Z}\;\middle|\;|d|<k\text{ and }d\neq 0\text{ if }k=2,3\right\},$$

which is the complete set of possible contents $\mathrm{cont}_{\mathfrak{t}}(k)$ as $\mathfrak{t}$ runs over the set of standard tableaux.

Definition 4.1. ([24, Definition 2.4]) *Let $\lambda \vdash n$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$.*

(i) Let $F_{\mathfrak{t}} := \prod_{k=1}^{n} \prod_{\substack{c\in\mathcal{R}(k)\\ \mathrm{cont}_{\mathfrak{t}}(k)\neq c}} \dfrac{L_k - c}{\mathrm{cont}_{\mathfrak{t}}(k) - c}$.

(ii) Let $f_{\mathfrak{s}\mathfrak{t}} := F_{\mathfrak{s}} X_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{t}}$.

Let $\lambda \vdash n$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$. For each integer $1 \le k \le n$ we use $\mathfrak{t}_k$ to denote the subtableau of $\mathfrak{t}$ which contains $\{1, 2, \cdots, k\}$. If $\gamma = (i, j) \in [\lambda]$ such that $[\lambda]\setminus\{\gamma\}$ is again the Young diagram of a partition μ. Then we call γ a removable node of λ and an addable node of μ. For any two nodes $\gamma = (i, j), \gamma' = (i', j')$ we say that γ is below γ', or γ' is above γ if $i > i'$.

Definition 4.2. ([24, (2.8)][2], [21, 3.15]) *Let $\lambda \vdash n$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$. For $k = 1, \cdots, n$, let $\mathscr{A}_{\mathfrak{t}}(k)$ be the set of addable nodes of the partition* $\mathrm{Shape}(\mathfrak{t}_k)$ *which are below $\mathfrak{t}^{-1}(k)$. Similarly, let $\mathscr{R}_{\mathfrak{t}}(k)$ be the set of removable nodes of* $\mathrm{Shape}(\mathfrak{t}_k)$ *which are below $\mathfrak{t}^{-1}(k)$. Now define*

$$\gamma_{\mathfrak{t}} = \prod_{k=1}^{n} \frac{\prod_{\alpha\in\mathscr{A}_{\mathfrak{t}}(k)} \big(\mathrm{cont}_{\mathfrak{t}}(k) - \mathrm{cont}(\alpha)\big)}{\prod_{\rho\in\mathscr{R}_{\mathfrak{t}}(k)} \big(\mathrm{cont}_{\mathfrak{t}}(k) - \mathrm{cont}(\rho)\big)} \quad \in \mathbb{Q},$$

and $\tilde{f}_{\mathfrak{s}\mathfrak{t}} := \gamma_{\mathfrak{t}}^{-1} f_{\mathfrak{s}\mathfrak{t}}$ for any $\mathfrak{s} \in \mathrm{Std}(\lambda)$.

Lemma 4.3. ([24, (2.9)]) *Let $\lambda \vdash n$. Then*

$$\gamma_{\mathfrak{t}^\lambda} = [\lambda]! := \prod_{i\ge 1} \lambda_i!\,.$$

Theorem 4.4. ([24, (2.14), (2.15)])

(1) $\big\{\tilde{f}_{\mathfrak{s}\mathfrak{t}} \bigm| \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda), \lambda \vdash n\big\}$ is a basis of matrix units in $\mathbb{Q}\mathfrak{S}_n$.

(2) Let $\lambda \vdash n$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$, then $F_{\mathfrak{t}} = f_{\mathfrak{t}\mathfrak{t}}/\gamma_{\mathfrak{t}}$ and $F_{\mathfrak{t}}$ is a primitive idempotent in $\mathbb{Q}\mathfrak{S}_n$ with $S^\lambda \cong F_{\mathfrak{t}}\mathbb{Q}\mathfrak{S}_n$.

(3) For any $\lambda \vdash n$ let $F_\lambda := \sum_{\mathfrak{t}\in\mathrm{Std}(\lambda)} F_{\mathfrak{t}}$. Then F_λ is a primitive central idempotent in $\mathbb{Q}\mathfrak{S}_n$.

(4) $\big\{F_\lambda \bigm| \lambda \vdash n\big\}$ is a complete set of primitive central idempotent in $\mathbb{Q}\mathfrak{S}_n$ and

$$1 = \sum_{\lambda\vdash n} F_\lambda = \sum_{\mathfrak{t}\ \textit{standard}} F_{\mathfrak{t}}.$$

[2]We remark that there is a typos in [24, Page 704, Line 9], $y \prec x$ there should be replaced by $y \succ x$, cf. [21, 3.15].

Lemma 4.5. ([24, Proposition 2.6]) *Let $\lambda \vdash n$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$. Then*

$$f_{\mathfrak{s}\mathfrak{t}} \equiv X_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{u},\mathfrak{v} \in \mathrm{Std}(\lambda) \\ \mathfrak{u} \rhd \mathfrak{s}, \mathfrak{v} \rhd \mathfrak{t}}} a_{\mathfrak{u}\mathfrak{v}} X_{\mathfrak{u}\mathfrak{v}} \pmod{(\mathbb{Q}\mathfrak{S}_n)^{\rhd\lambda}},$$

where $a_{\mathfrak{u},\mathfrak{v}} \in \mathbb{Q}$ for each $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\lambda)$.

Definition 4.6. ([8, Section 4]) *Let $\lambda \vdash n$. We define*

$$z_\lambda := X_\lambda w_\lambda Y_{\lambda'}.$$

Lemma 4.7. ([8], [26]) *Let $\lambda \vdash n$ and $w \in \mathfrak{S}_n$. Then*

(1) $(\mathbb{Z}\mathfrak{S}_n)^{\rhd\lambda} Y_{\lambda'} = 0 = Y_{\lambda'} (\mathbb{Z}\mathfrak{S}_n)^{\rhd\lambda}$;

(2) If $w \neq w_\lambda$ and $\ell(w) \leq \ell(w_\lambda)$, then $X_\lambda w Y_{\lambda'} = 0$ in $\mathbb{Z}\mathfrak{S}_n$.

Proof. (1) follows from [26, Lemma 4.12]. It remains to prove (2). Assume that $w \neq w_\lambda$ and $\ell(w) \leq \ell(w_\lambda)$. If $\ell(w) < \ell(w_\lambda)$, then by [26, Corollary 4.13] we see that $X_\lambda w Y_{\lambda'} = 0$. Now assume $\ell(w) = \ell(w_\lambda)$. Then by [8, Lemma 1.5] $\mathfrak{t}^\lambda w \notin \mathrm{Std}(\lambda)$ because $w \neq w_\lambda$.
For any $\mathfrak{t} \in \mathrm{Std}(\lambda)$ and integer $1 \leq k < n$, it is well-known that

$$X_\lambda d(\mathfrak{t}) s_k = \begin{cases} X_\lambda d(\mathfrak{t}), & \text{if } k, k+1 \text{ are in the same row of } \mathfrak{t}; \\ X_\lambda d(\mathfrak{t} s_k), & \text{if } \mathfrak{t} s_k \in \mathrm{Std}(\lambda), \end{cases}$$

and if $k, k+1$ are in the same column of $\mathfrak{t}$, then (by [23, Corollary 3.21])

$$X_\lambda d(\mathfrak{t}) s_k \equiv -X_\lambda d(\mathfrak{t}) + \sum_{\mathfrak{t} \lhd \mathfrak{v} \in \mathrm{Std}(\lambda)} r_{\mathfrak{v}} X_\lambda d(\mathfrak{v}) \pmod{(\mathbb{Z}\mathfrak{S}_n)^{\rhd\lambda}},$$

where $r_{\mathfrak{v}} \in \mathbb{Z}$ for each $\mathfrak{v}$. By [23, Theorem 3.8], $\mathfrak{t} \lhd \mathfrak{v} \in \mathrm{Std}(\lambda)$ only if $\ell(d(\mathfrak{t})) > \ell(d(\mathfrak{v}))$. As a result (of the fact $\mathfrak{t}^\lambda w \notin \mathrm{Std}(\lambda)$), we see that

$$X_\lambda w \equiv \sum_{\substack{\mathfrak{t} \in \mathrm{Std}(\lambda) \\ \ell(d(\mathfrak{t})) < \ell(w) = \ell(w_\lambda)}} b_{\mathfrak{t}} X_\lambda d(\mathfrak{t}) \pmod{(\mathbb{Z}\mathfrak{S}_n)^{\rhd\lambda}},$$

where $b_{\mathfrak{t}} \in \mathbb{Z}$ for each $\mathfrak{t}$. Now, applying the result (1) (which we have just proved) and [26, Corollary 4.13] again,

$$X_\lambda w Y_{\lambda'} = \sum_{\substack{\mathfrak{t} \in \mathrm{Std}(\lambda) \\ \ell(d(\mathfrak{t})) < \ell(w) = \ell(w_\lambda)}} b_{\mathfrak{t}} X_\lambda d(\mathfrak{t}) Y_{\lambda'} = 0,$$

as required. This completes the proof of (2). □

Lemma 4.8. ([24, (3.13)]) *Let $\lambda \vdash n$. Then*

$$z_\lambda = \gamma_{\mathfrak{t}^{\lambda'}} f_{\mathfrak{t}^\lambda \mathfrak{t}_\lambda}.$$

Note that in the right hand side of the above lemma the coefficient is $\gamma_{\mathfrak{t}^{\lambda'}}$ instead of $\gamma'_{\mathfrak{t}^{\lambda'}}$ because we have specialized q to 1.

The next lemma is a key observation to the proof of Theorem 4.10.

Lemma 4.9. *Let $\lambda = (n-k,k) \vdash n$, where $k \in \mathbb{Z}$ such that $0 \leq k \leq n/2$. Then*

(1) $z_\lambda = 2^k f_{\mathfrak{t}^\lambda \mathfrak{t}_\lambda} \equiv 2^k X_{\mathfrak{t}^\lambda \mathfrak{t}_\lambda} (\mathrm{mod}\ (\mathbb{Z}\mathfrak{S}_n)^{\rhd\lambda})$.

(2) $Y_{\lambda'} w_{\lambda'} X_\lambda w_\lambda Y_{\lambda'} = 2^k Y_{\lambda'} w_{\lambda'} X_\lambda w_\lambda$.

Proof. By Lemma 4.8 and Lemma 4.3, we get that $z_\lambda = 2^k f_{\mathfrak{t}^\lambda \mathfrak{t}_\lambda}$. Applying Lemma 4.5, we get that

$$z_\lambda = 2^k f_{\mathfrak{t}^\lambda \mathfrak{t}_\lambda} \equiv 2^k X_{\mathfrak{t}^\lambda \mathfrak{t}_\lambda} \pmod{(\mathbb{Q}\mathfrak{S}_n)^{\rhd\lambda}}.$$

Since $z_\lambda, 2^k X_{\mathfrak{t}^\lambda \mathfrak{t}_\lambda} \in \mathbb{Z}\mathfrak{S}_n$, we deduce that

$$z_\lambda = 2^k f_{\mathfrak{t}^\lambda \mathfrak{t}_\lambda} \equiv 2^k X_{\mathfrak{t}^\lambda \mathfrak{t}_\lambda} \pmod{(\mathbb{Z}\mathfrak{S}_n)^{\rhd\lambda}}.$$

This proves (1). Note that $w_\lambda^{-1} = w_{\lambda'}$. By [8, (4.1)] and applying the anti-automorphism $*$ we know that $Y_{\lambda'} w_{\lambda'} (\mathbb{Z}\mathfrak{S}_n)^{\rhd\lambda} = 0$. Now (2) follows from this equality and (1). □

Let λ, μ be two compositions of n. A λ-tableau of type μ is a map $\mathtt{S} : [\lambda] \to \{1, 2, \cdots, d\}$ such that $\mu_i = \#\{\gamma \in [\lambda] | \mathtt{S}(\gamma) = i\}$ for $i \geq 1$. A λ-tableau $\mathtt{S}$ of type μ is row semistandard if the entries in each row of $\mathtt{S}$ are non-decreasing from left to right; $\mathtt{S}$ is semistandard if (i) λ is a partition; and (ii) $\mathtt{S}$ is row semistandard and the entries in each column of $\mathtt{S}$ are strictly increasing from top to bottom. If $\mathfrak{t} \in \mathrm{Std}(\lambda)$ then we define $\mu(\mathfrak{t})$ to be the λ-tableau obtained from $\mathfrak{t}$ by replacing each entry i in $\mathfrak{t}$ by r if i appears in row r of $\mathfrak{t}^\mu$. It is clear that $\mu(\mathfrak{t})$ is a row semistandard λ-tableau of type μ. Recall our definition (see Section 2) of X_λ, Y_λ for each composition λ. For each integer i with $0 \leq i \leq n$, we set

$$X_i := X_{(i,n-i)}, \quad Y_i := Y_{(i,n-i)}.$$

For any $h \in K\mathfrak{S}_n$, we use $\langle h \rangle_0$ to denote the two-sided ideal of $K\mathfrak{S}_n$ generated by h. The next theorem is the first step in the direction towards the proof of Theorem 1.3.

Theorem 4.10. *Let K be a field of characteristic other than two. Let a be an integer with $0 \leq a \leq n/2$. Then we have that*

$$\langle X_{n-a} \rangle_0 = \big(K\mathfrak{S}_n\big)^{\unrhd(n-a,a)}.$$

Proof. By the cellular structure of $K\mathfrak{S}_n$, we see that $\langle X_{n-a} \rangle_0 \subseteq \big(K\mathfrak{S}_n\big)^{\unrhd(n-a,a)}$ and

$$\dim\big(K\mathfrak{S}_n\big)^{\unrhd(n-a,a)} = n_a := \sum_{(n-a,a) \unlhd \lambda \vdash n} \Big(\#\,\mathrm{Std}(\lambda)\Big)^2. \tag{4.11}$$

To prove the theorem, it suffices to find at least n_a K-linear independent elements in the two-sided ideal $\langle X_{n-a}\rangle_0$.

Let $\lambda \vdash n$ be a partition such that $\lambda \trianglerighteq \mu := (n-a, a)$. Then $\lambda := (n-k, k)$, where $k \in \mathbb{Z}$ with $0 \leq k \leq a \leq n/2$. Let $\mathtt{S}_k$ be the following (unique) semistandard λ-tableau of type μ:

$$\mathtt{S}_k := \begin{array}{ll} \overbrace{1, 1, \ldots, 1,}^{n-a \text{ copies}} & \overbrace{2, \ldots, 2,}^{a-k \text{ copies}} \\ \underbrace{2, \ldots, 2}_{k \text{ copies}} & \end{array}$$

We define $S_0(\lambda) := \{\mathfrak{t} \in \mathrm{Std}(\lambda) \mid \mu(\mathfrak{t}) = \mathtt{S}_k\}$. By [26, Section 7],

$$\sum_{\mathfrak{t} \in S_0(\lambda)} X_\lambda d(\mathfrak{t}) \in (K\mathfrak{S}_n) X_{n-a} \subseteq \langle X_{n-a}\rangle_0. \tag{4.12}$$

It is clear that $S_0(\lambda) \neq \emptyset$. We divide the remaining proof into three steps:

Step 1. We claim that $Y_{\lambda'} w_{\lambda'} X_\lambda \Big(\sum_{\mathfrak{t} \in S_0(\lambda)} d(\mathfrak{t})\Big) K\mathfrak{S}_n = Y_{\lambda'} w_{\lambda'} X_\lambda K\mathfrak{S}_n$.
Let $\mathfrak{t}_0 \in S_0(\lambda)$ such that $\ell(d(\mathfrak{t}_0))$ is maximal. Then $\ell(d(\mathfrak{s})) \leq \ell(d(\mathfrak{t}_0))$ for any $\mathfrak{s} \in S_0(\lambda)$. Furthermore, by [8, Lemma 1.5],

$$\ell(w_\lambda) = \ell(d(\mathfrak{t}_0)) + \ell(d(\mathfrak{t}_0)^{-1} w_\lambda).$$

We set $w := d(\mathfrak{t}_0)^{-1} w_\lambda$. By Lemma 4.7, we deduce that $X_\lambda d(\mathfrak{s}) w Y_{\lambda'} = 0$ for any $\mathfrak{t}_0 \neq \mathfrak{s} \in S_0(\lambda)$. Now multiplying $wY_{\lambda'}$ and applying Lemma 4.7 and Lemma 4.9, we get that

$$\begin{aligned} & Y_{\lambda'} w_{\lambda'} X_\lambda \Big(\sum_{\mathfrak{t} \in S_0(\lambda)} d(\mathfrak{t})\Big) w Y_{\lambda'} \\ &= Y_{\lambda'} w_{\lambda'} X_\lambda w_\lambda Y_{\lambda'} \qquad = 2^k Y_{\lambda'} w_{\lambda'} X_\lambda w_\lambda. \end{aligned}$$

Since $\operatorname{char} K \neq 2$, 2^k is invertible in K. The above equality implies that

$$Y_{\lambda'} w_{\lambda'} X_\lambda \Big(\sum_{\mathfrak{t} \in S_0(\lambda)} d(\mathfrak{t})\Big) K\mathfrak{S}_n = Y_{\lambda'} w_{\lambda'} X_\lambda w_\lambda K\mathfrak{S}_n = Y_{\lambda'} w_{\lambda'} X_\lambda K\mathfrak{S}_n,$$

as required. This proves our claim. Furthermore, it is well-known that the elements in $\{Y_{\lambda'} w_{\lambda'} X_\lambda d(\mathfrak{t}) | \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ form a K-basis of $Y_{\lambda'} w_{\lambda'} X_\lambda K\mathfrak{S}_n$. In particular, we have that $\dim Y_{\lambda'} w_{\lambda'} X_\lambda K\mathfrak{S}_n = \#\,\mathrm{Std}(\lambda)$.

Step 2. We define

$$N_\lambda := \frac{X_\lambda \Big(\sum_{\mathfrak{t} \in S_0(\lambda)} d(\mathfrak{t})\Big) K\mathfrak{S}_n}{X_\lambda \Big(\sum_{\mathfrak{t} \in S_0(\lambda)} d(\mathfrak{t})\Big) K\mathfrak{S}_n \bigcap (K\mathfrak{S}_n)^{\rhd\lambda}}.$$

We claim that $n_\lambda := \dim N_\lambda \geq \#\,\mathrm{Std}(\lambda)$.

In fact, by Lemma 4.7, the left multiplication by $Y_{\lambda'}w_{\lambda'}$ induces a surjective homomorphism from N_λ onto $Y_{\lambda'}w_{\lambda'}X_\lambda\Big(\sum_{\mathfrak{t}\in S_0(\lambda)} d(\mathfrak{t})\Big)K\mathfrak{S}_n$. By the main result of Step 1, we know that $Y_{\lambda'}w_{\lambda'}X_\lambda\Big(\sum_{\mathfrak{t}\in S_0(\lambda)} d(\mathfrak{t})\Big)K\mathfrak{S}_n = Y_{\lambda'}w_{\lambda'}X_\lambda K\mathfrak{S}_n$ and has dimension $\#\,\mathrm{Std}(\lambda)$. It follows that $n_\lambda := \dim N_\lambda \geq \#\,\mathrm{Std}(\lambda)$, as required. This proves our claim.

As a consequence, we can find $u_1, \cdots, u_{n_\lambda} \in \mathfrak{S}_n$ such that the natural image of the following elements

$$X_\lambda\Big(\sum_{\mathfrak{t}\in S_0(\lambda)} d(\mathfrak{t})\Big)u_1, \cdots, X_\lambda\Big(\sum_{\mathfrak{t}\in S_0(\lambda)} d(\mathfrak{t})\Big)u_{n_\lambda}$$

in N_λ form a K-basis of N_λ. For each $\mathfrak{s}\in\mathrm{Std}(\lambda)$ and each integer $1\leq j\leq n_\lambda$, we define

$$v_{\mathfrak{s},j} := d(\mathfrak{s})^{-1}X_\lambda\Big(\sum_{\mathfrak{t}\in S_0(\lambda)} d(\mathfrak{t})\Big)u_j.$$

By construction, it is clear that $v_{\mathfrak{s},j}\in\langle X_{n-a}\rangle_0$.

Step 3. We claim that the elements in the following set

$$\big\{v_{\mathfrak{s},j} \bigm| \mathfrak{s}\in\mathrm{Std}(\lambda), \lambda\vdash n, 1\leq j\leq n_\lambda\big\} \tag{4.13}$$

are K-linearly independent.

In fact, assume that

$$\sum_{\substack{\lambda\vdash n,\mathfrak{s}\in\mathrm{Std}(\lambda)\\ 1\leq j\leq n_\lambda}} c_{\mathfrak{s},j}v_{\mathfrak{s},j} = \sum_{\substack{\lambda\vdash n,\mathfrak{s}\in\mathrm{Std}(\lambda)\\ 1\leq j\leq n_\lambda}} c_{\mathfrak{s},j}d(\mathfrak{s})^{-1}X_\lambda\Big(\sum_{\mathfrak{t}\in S_0(\lambda)} d(\mathfrak{t})\Big)u_j = 0, \tag{4.14}$$

where $c_{\mathfrak{s},j}\in K$ for each $\mathfrak{s}, j$. Set

$$\Sigma_0 := \big\{(\mathfrak{s}, j) \bigm| \mathfrak{s}\in\mathrm{Std}(\lambda), \lambda\vdash n, 1\leq j\leq n_\lambda, c_{\mathfrak{s},j}\neq 0\big\}.$$

Suppose that $\Sigma_0\neq\emptyset$. We choose an $\mathfrak{s}$ such that $(\mathfrak{s}, j)\in\Sigma_0$ for some j and $\mathrm{Shape}(\mathfrak{s})=\lambda$ is minimal under the dominance order "$\lhd$". By the property of the cellular basis $\{X_{\mathfrak{s},\mathfrak{t}}\}$ we know that

$$X_\lambda\Big(\sum_{\mathfrak{t}\in S_0(\lambda)} d(\mathfrak{t})\Big)u_j \equiv \sum_{\mathfrak{t}\in\mathrm{Std}(\lambda)} r_{j,\mathfrak{t}}X_{\mathfrak{t}^\lambda,\mathfrak{t}} \pmod{(K\mathfrak{S}_n)^{\rhd\lambda}},$$

and hence

$$v_{\mathfrak{s},j} \equiv \sum_{\mathfrak{t}\in\mathrm{Std}(\lambda)} r_{j,\mathfrak{t}}X_{\mathfrak{s},\mathfrak{t}} \pmod{(K\mathfrak{S}_n)^{\rhd\lambda}}.$$

By the main result in Step 2, we know that $X_\lambda\Big(\sum_{\mathfrak{t}\in S_0(\lambda)} d(\mathfrak{t})\Big)u_j \notin (K\mathfrak{S}_n)^{\rhd\lambda}$. It follows that at least one of those coefficients $r_{j,\mathfrak{t}}$ is nonzero. Combing this fact and the minimality of λ and (4.14) we can deduce that for each $\mathfrak{s} \in \mathrm{Std}(\lambda)$ and each $1 \le j \le n_\lambda$,

$$\sum_{1\le j\le n_\lambda} c_{\mathfrak{s},j} d(\mathfrak{s})^{-1} X_\lambda \Big(\sum_{\mathfrak{t}\in S_0(\lambda)} d(\mathfrak{t}) \Big) u_j \in (K\mathfrak{S}_n)^{\rhd\lambda}.$$

Now applying the main result of Step 2 again, we deduce that $c_{\mathfrak{s},j} = 0$ for each $1 \le j \le n_\lambda$, a contradiction to the definition of Σ_0. This completes the proof of our claim.

As a consequence, we see that the two-sided ideal $\langle X_{n-a}\rangle_0$ contains at least

$$\sum_{(n-a,a)\unlhd\lambda\vdash n} n_\lambda(\#\,\mathrm{Std}(\lambda)) \ge \sum_{(n-a,a)\unlhd\lambda\vdash n} (\#\,\mathrm{Std}(\lambda))^2 = n_a$$

K-linearly independent elements. This implies that we must have that

$$\dim\langle X_{n-a}\rangle_0 = n_a = \dim(K\mathfrak{S}_n)^{\unrhd(n-a,a)}$$

and hence $\langle X_{n-a}\rangle_0 = (K\mathfrak{S}_n)^{\unrhd(n-a,a)}$. This completes the proof of the theorem. $\square$

Our second step in the direction towards Theorem 1.3 is the proof of the following purely combinatorial identity.

Theorem 4.15. *Let K be a field of arbitrary characteristic. Let $S^{(m+1,m+1)}$ (respectively, $S^{(m+1-k,k)}$) be the Specht module of the symmetric group algebra $K\mathfrak{S}_{2m+2}$ (respectively, $K\mathfrak{S}_{m+1}$) corresponding to $(m+1,m+1)$ (respectively, $(m+1-k,k)$, where $0\le k\le [(m+1)/2]$. Then we have that*

$$\sum_{k=0}^{[(m+1)/2]} \big(\dim_K S^{(m+1-k,k)}\big)^2 = \dim_K S^{(m+1,m+1)}.$$

Proof. We shall give a representation theoretic argument to prove the above combinatorial identity. Note that the dimension of each Specht module is independent of the field. We can assume without loss of generality that $K = \mathbb{C}$.

We use $\mathfrak{B}_{m+1}(m)_{\mathbb{C}}$ to denote the specialized Brauer algebra over $\mathbb{C}$ with parameter m. In other words, we assume that $n = m+1$ for the moment. Then by [12], $\mathfrak{B}_{m+1}(m)_{\mathbb{C}}$ is semisimple. We have a natural surjective homomorphism $\varphi_{\mathbb{C}} : \mathfrak{B}_{m+1}(m)_{\mathbb{C}} \twoheadrightarrow \mathrm{End}_{\mathbb{C}O(V_{\mathbb{C}})}\big(V_{\mathbb{C}}^{\otimes m+1}\big)$, where $V_{\mathbb{C}}$ is an orthogonal $\mathbb{C}$-vector space with dimension m. Note that $\mathbb{C}\mathfrak{S}_{2m+2}$ is semisimple and each Specht module $\mathbb{C}\mathfrak{S}_{2m+2}$ over is irreducible. By [11] we see that

$$\dim \mathrm{Ker}\,\varphi_{\mathbb{C}} = \dim_{\mathbb{C}} S_{\mathbb{C}}^{(m+1,m+1)}, \tag{4.16}$$

where we use $S_{\mathbb{C}}^{(m+1,m+1)}$ to denote the simple (Specht) module of $\mathbb{C}\mathfrak{S}_{2m+2}$ corresponding to the partition $(m+1,m+1)$. As before, we use $\mathfrak{B}_{m+1}^{(1)}$ to denote the two-sided ideal of $\mathfrak{B}_{m+1}(m)_{\mathbb{C}}$ generated by e_1. Equivalently, $\mathfrak{B}_{m+1}^{(1)}$ is the subspace of $\mathfrak{B}_{m+1}(m)_{\mathbb{C}}$ spanned by those Brauer $(m+1)$-diagrams which contain horizontal edges. We claim that $\operatorname{Ker}\varphi_{\mathbb{C}} \cap \mathfrak{B}_{m+1}^{(1)} = \{0\}$.

In fact, by [22, Theorem 4.3], we know that $\operatorname{Ker}\varphi_{\mathbb{C}}$ is equal to the two-sided ideal $\langle E_{[(m+1)/2]}\rangle$. It suffices to show that $\langle E_{[(m+1)/2]}\rangle \cap \mathfrak{B}_{m+1}^{(1)} = \{0\}$. Since $\mathfrak{B}_{m+1}^{(1)}$ is a two-sided ideal of the semisimple $\mathbb{C}$-algebra $\mathfrak{B}_{m+1}(m)_{\mathbb{C}}$, we have that $\mathfrak{B}_{m+1}(m)_{\mathbb{C}} = \mathfrak{B}_{m+1}^{(1)} \oplus I_0$ for some two-sided ideal I_0 of $\mathfrak{B}_{m+1}(m)_{\mathbb{C}}$. In particular, $xI_0 = 0 = I_0 x$ for any $x \in \mathfrak{B}_{m+1}^{(1)}$. We define

$$J_0 := \left\{ y \in \mathfrak{B}_{m+1}(m)_{\mathbb{C}} \;\middle|\; yx = 0 = xy \text{ for any } x \in \mathfrak{B}_{m+1}^{(1)} \right\}.$$

Then J_0 is a two-sided ideal of $\mathfrak{B}_{m+1}(m)_{\mathbb{C}}$ and $I_0 \subseteq J_0$. Thus $\mathfrak{B}_{m+1}(m)_{\mathbb{C}} = J_0 + \mathfrak{B}_{m+1}^{(1)}$. Since $J_0 x = 0$ for any $x \in \mathfrak{B}_{m+1}^{(1)}$, it follows that every right simple submodule of J_0 is a simple module over $\mathfrak{B}_{m+1}(m)_{\mathbb{C}}/\mathfrak{B}_{m+1}^{(1)} \cong \mathbb{C}\mathfrak{S}_{m+1}$. Using the Wedderburn theorem for semisimple algebras we get that

$$J_0 = \bigoplus_{\lambda \vdash m+1} \left(S_{\mathbb{C}}^{\lambda}\right)^{\oplus a_\lambda},$$

where $a_\lambda \in \mathbb{N}$ such that $0 \le a_\lambda \le \dim S_{\mathbb{C}}^{\lambda}$ for each $\lambda \vdash m+1$. As a result, we deduce that

$$\dim J_0 \le \sum_{\lambda \vdash m+1} (\dim S_{\mathbb{C}}^{\lambda})^2 = \dim_{\mathbb{C}} \mathbb{C}\mathfrak{S}_{m+1} = \dim_{\mathbb{C}} \mathfrak{B}_{m+1}(m)_{\mathbb{C}} - \dim_{\mathbb{C}} \mathfrak{B}_{m+1}^{(1)},$$

because $\mathfrak{B}_{m+1}(m)_{\mathbb{C}}/\mathfrak{B}_{m+1}^{(1)} \cong \mathbb{C}\mathfrak{S}_{m+1}$. This in turn forces

$$J_0 \bigoplus \mathfrak{B}_{m+1}^{(1)} = J_0 + \mathfrak{B}_{m+1}^{(1)} = \mathfrak{B}_{m+1}(m)_{\mathbb{C}}.$$

In particular, $J_0 \cap \mathfrak{B}_{m+1}^{(1)} = \{0\}$. On the other hand, by [22, Corollary 5.13] we know that $\langle E_{[(m+1)/2]}\rangle \subseteq J_0$, from which our claim follows at once.

We write $a_0 := m + 1 - [(m+1)/2]$. Then $a_0 \ge m + 1 - a_0$ and hence $(a_0, m+1-a_0)$ is a partition of $m+1$. Note that (by Lemma 3.3)

$$\begin{aligned} \langle E_{[(m+1)/2]}\rangle &= \langle E_{[(m+1)/2],a_0}\rangle = \langle E_{a_0,[(m+1)/2]}\rangle = \langle E_{a_0}\rangle, \\ E_{a_0} &= E_{a_0,m+1-a_0} \equiv Y_{a_0,m+1-a_0} \equiv Y_{a_0} \pmod{\mathfrak{B}_{m+1}^{(1)}}. \end{aligned}$$

By Theorem 4.10, we know that $\langle Y_{a_0}\rangle_0 = (\mathbb{C}\mathfrak{S}_{m+1})^{\trianglerighteq (a_0,m+1-a_0)}$. Combining these fact together with the equality

$$\langle E_{[(m+1)/2]}\rangle \cap \mathfrak{B}_{m+1}^{(1)} = \operatorname{Ker}\varphi_{\mathbb{C}} \cap \mathfrak{B}_{m+1}^{(1)} = \{0\}$$

which we have just proved, we can deduce that

$$\begin{aligned}\dim \operatorname{Ker}\varphi_{\mathbb{C}} &= \dim\langle E_{[(m+1)/2]}\rangle = \dim(\mathbb{C}\mathfrak{S}_{m+1})^{\trianglerighteq(a_0,m+1-a_0)}\\ &= \sum_{k=0}^{[(m+1)/2]} \left(\dim_{\mathbb{C}} S_{\mathbb{C}}^{(m+1-k,k)}\right)^2.\end{aligned}$$

Finally, comparing the above equality with (4.16), we complete the proof of the theorem. □

By the well-known hook formula (cf. [23]) for the dimension of Specht modules, we get that

$$\dim S^{(m+1)} = 1, \quad \dim S^{(m,1)} = m,$$

and for each $2 \le k \le [(m+1)/2]$,

$$\begin{aligned}\dim S^{(m+1-k,k)} &= \frac{(m+1)!}{\prod_{(i,j)\in[(m+1-k,k)]} h_{i,j}^{(m+1-k,k)}}\\ &= \frac{(m+1)m\cdots(m-k+4)(m-k+3)(m-2k+2)}{k!},\end{aligned}$$

where $h_{i,j}^{\lambda} := \lambda_i + \lambda'_j - i - j + 1$ is the (i,j)-hook length of the partition λ. Similarly,

$$\begin{aligned}\dim S^{(m+1,m+1)} &= \frac{(2m+2)!}{\prod_{(i,j)\in[(m+1,m+1)]} h_{i,j}^{(m+1,m+1)}}\\ &= \frac{(2m+2)(2m+1)\cdots(m+3)}{(m+1)!}.\end{aligned}$$

As a result, we have the following corollary.

Corollary 4.17. *We have the following identity:*

$$\begin{aligned}1+m^2+\sum_{k=2}^{[(m+1)/2]}\left(\frac{(m+1)m\cdots(m-k+3)(m+2-2k)}{k!}\right)^2\\ =\frac{(2m+2)(2m+1)\cdots(m+3)}{(m+1)!}.\end{aligned}$$

Proof. This follows directly from Theorem 4.15 and the hook length formulae for the dimensions of Specht modules over symmetric groups. □

Now we are in the final step to prove Theorem 1.3.

Theorem 4.18. *Let K be an arbitrary field of characteristic other than two. Then the annihilator of $V^{\otimes n}$ in $\mathfrak{B}_n(m)$ is the two-sided ideal generated by $E_{[(m+1)/2]}$.*

Proof. By Theorem 3.8, it suffices to show that for each integer i with $0 \leq i < [(m+1)/2]$, E_i lies in the two-sided ideal of $\mathfrak{B}_n(m)$ generated by $E_{[(m+1)/2]}$. For simplicity, we write $a := [(m+1)/2]$. By Lemma 3.3, we know that $\langle E_j \rangle = \langle E_{m+1-j} \rangle$ for each $0 \leq j \leq m+1$. Therefore, it suffices to show that for each integer i with $0 \leq i < a$, $E_{(m+1-i,i)}$ lies in the two-sided ideal of $\mathfrak{B}_n(m)$ generated by $E_{m+1-a} := E_{(m+1-a,a)}$. Note that for each integer $0 \leq i \leq m+1$, $E_i \in \mathfrak{B}_{m+1}(m) \subseteq \mathfrak{B}_n(m)$. Without loss of generality, we can assume that $n = m+1$ henceforth.

We consider the natural homomorphism $\varphi_K : \mathfrak{B}_{m+1}(m) \to \mathrm{End}_K(V^{\otimes m+1})$. By Corollary 4.17, we have that

$$\begin{aligned}(m+1)_a &:= 1 + m^2 + \sum_{k=2}^{[(m+1)/2]} \left(\frac{(m+1)m \cdots (m-k+3)(m+2-2k)}{k!} \right)^2 \\ &= \frac{(2m+2)(2m+1)\cdots(m+3)}{(m+1)!}.\end{aligned}$$

By [11, Lemma 7.1], Theorem 4.15 and Corollary 4.17, we have that

$$\dim_K \mathrm{Ann}_{\mathfrak{B}_{m+1}(m)}(V^{\otimes m+1}) = \dim_K \mathrm{Ker}\, \varphi_K = (m+1)_a.$$

On the other hand, since $E_{m+1-a} \equiv Y_{m+1-a} \pmod{\mathfrak{B}_n^{(1)}}$, we deduce (from Theorem 4.15, Corollary 4.17 and the Y-Murphy basis for $K\mathfrak{S}_{m+1}$) that $\langle E_{m+1-a} \rangle$ contains at least $(m+1)_a$ K-linearly independent elements. Since $\langle E_{m+1-a} \rangle \subseteq \mathrm{Ker}\, \varphi_K$, it follows that $\langle E_{m+1-a} \rangle = \mathrm{Ker}\, \varphi_K$ and hence for each integer $0 \leq i \leq [(m+1)/2]$, $E_i \in \langle E_{m+1-a} \rangle = \langle E_a \rangle = \langle E_{[(m+1)/2]} \rangle$, as required. This completes the proof of the Theorem. □

The following corollary give a new integral basis for the annihilator of $V^{\otimes m+1}$ in $\mathfrak{B}_{m+1}(m)$.

Corollary 4.19. *Let K be a field of characteristic other than two. Then the elements in the following set*

$$\left\{ d(\mathfrak{s})^{-1} E_{[(m+1)/2]} d(\mathfrak{t}) \;\middle|\; \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(m+1-k, k), 0 \leq k \leq [(m+1)/2] \right\}$$

form a K-basis of $\mathrm{Ann}_{\mathfrak{B}_{m+1}(m)}(V^{\otimes m+1})$

Proof. This follows directly from the proof of Theorem 4.18. □

References

[1] R. Brauer, *On algebras which are connected with semisimple continuous groups*, *Ann. Math.* 38 (1937), 857–872.

[2] W.P. Brown, *An algebra related to the orthogonal group*, Michigan Math. J., 3 (1955–1956), 1–22.

[3] ——, *The semisimplicity of* ω_f^n, Ann. Math., 63 (1956), 324–335.

[4] A. Cox and M.De. Visscher, *Diagrammatic Kazhdan-Lusztig theory for the (walled) Brauer algebra*, preprint, arXiv:1009.4064.

[5] A. Cox, M.De. Visscher and P. Martin, *A geometric characterisation of the blocks of the Brauer algebra*, J. London Math. Soc., (2) 80 (2009), 471–494.

[6] ——, *The blocks of the Brauer algebra in characteristic zero*, Representation Theory, 13 (2009), 272–308.

[7] C. De Concini and C. Procesi, *A characteristic free approach to invariant theory*, Adv. Math., 21 (1976), 330–354.

[8] R. Dipper and G.D. James, *Representations of Hecke algebras of general linear groups*, Proc. London. Math. Soc., (3) 52 (1986), 20–52.

[9] ——, *Blocks and idempotents of Hecke algebras of general linear groups*, Proc. London. Math. Soc., (3) 54 (1987), 57–82.

[10] R. Dipper, S. Doty and J. Hu, *Brauer algebras, symplectic Schur algebras and Schur-Weyl duality*, Trans. Amer. Math. Soc., (1) 360 (2008), 189–213.

[11] S. Doty and J. Hu, *Schur-Weyl duality for orthogonal groups*, Proc. London Math. Soc., 98 (2009), 679–713.

[12] W. Doran, D. Wales and P. Hanlon, *On the semisimplicity of the Brauer centralizer algebras*, J. Algebra, 211 (1999), 647–685.

[13] J. Enyang, *Cellular bases for the Brauer and Birman–Murakami–Wenzl algebras*, J. Algebra, 281 (2004), 413–449.

[14] S. Fishel, I. Grojnowski, *Canonical bases for the Brauer centralizer algebra*, Math. Res. Lett., (1) 2 (1995), 15–26.

[15] F. Gavarini, *A Brauer algebra theoretic proof of Littlewood's restriction rules*, J. Algebra, 212 (1999), 240–271.

[16] ——, *On the radical of Brauer algebras*, Math. Zeit., 260 (2008), 673–697.

[17] J.J. Graham and G.I. Lehrer, *Cellular algebras*, Invent. Math., 123 (1996), 1–34.

[18] P.N. Hoefsmit, *Representations of Hecke algebras of finite groups with BN-pair of classical type*, Thesis. University of British of Columbia. 1974.

[19] J. Hu, *Specht filtrations and tensor spaces for the Brauer algebra*, J. Algebr. Comb., (2) 28 (2008), 281–312.

[20] J. Hu and Z.K. Xiao, *On tensor spaces for Birman-Murakami-Wenzl algebras*, J. Algebra, 324 (2010), 2893–2922.

[21] G.D. James and A. Mathas, *The Jantzen sum formula for cyclotomic q–Schur algebras*, Trans. Amer. Math. Soc., 352 (2000), 5381–5404.

[22] G.I. Lehrer and R.B. Zhang, *The second fundamental theorem of Invariant theory for the orthogonal group*, preprint, arXiv:math.GR/1102.3221.

[23] A. Mathas, *Iwahori–Hecke algebras and Schur algebras of the symmetric group*, Univ. Lecture Notes, 15, Amer. Math. Soc., 1999.

[24] ——, *Matrix units and generic degrees for the Ariki-Koike algebras*, J. Algebra, 281 (2004), 695–730.

[25] P. Martin, *The decomposition matrices of the Brauer algebra over the complex field*, preprint, arXiv:0908.1500.

[26] G.E. Murphy, *The representations of Hecke algebras of type A_n*, J. Algebra, 173 (1995), 97–121.

[27] H.B. Rui and M. Si, *A criterion on the semisimple Brauer algebras II*, J. Comb. Theory, Ser. A, 113 (2006), 1199–1203.

[28] H. Wenzl, *On the structure of Brauer's centralizer algebras*, Ann. Math., 128 (1988), 173–193.

[29] H. Weyl, *The classical groups, their invariants and representations*, Princeton University Press, 1946.

Jun Hu
School of Mathematics
Beijing Institute of Technology
Beijing, 100081
P.R. China
&
School of Mathematics and Statistics
University of Sydney
NSW, 2006, Australia
junhu303@yahoo.com.cn

Zhankui Xiao
School of Mathematical Sciences
Huaqiao University
Quanzhou, Fujian, 362021
P. R. China
zhkxiao@gmail.com