On the Algebraic K-Theory of Some Homogeneous Varieties

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Abstract.

The K-theory of inner twisted forms of homogeneous varieties G/H with connected reductive algebraic groups $H \subset G$ of the same rank is computed. We provide an explicit isomorphism with the K-theory of certain central simple algebras associated to the considered variety, as a consequence one has that $K_0(G/H)$ is a free abelian group of rank $[W(G) : W(H)]$. The result is used for the computation of the K-theory of some affine homogeneous varieties including an octonionic projective plane and quaternionic projective spaces.

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1 Introduction.

It is known that the K-theory of homogeneous projective varieties could be expressed by means of the K-theory of central simple algebras. The most fundamental result concerns the case of a projective space and states that there is an isomorphism

$$K_*(k)[t]/t^{n+1} \xrightarrow{\sim} K_*(\mathbb{P}^n).$$

The case of Severi-Brauer varieties was treated by Quillen [Q, §8], and one has

$$\bigoplus_{i=0}^{n} K_*(A^{\otimes i}) \xrightarrow{\sim} K_*(\mathbb{P}^n_\gamma),$$

where A is a central simple algebra defined by the cocycle γ. Swan [Sw] computed the K-theory of a smooth projective quadric and showed that there is an analogous isomorphism involving some Clifford algebras. It was shown by Panin [Pan1] that one has an analogous isomorphism for every homogeneous projective variety and one can express its K-theory in the terms of the K-theory of certain separable algebras. In the present paper we provide a unified approach to the K-theory of homogeneous varieties and compute it for inner forms of G/H with connected reductive $H \subset G$ of the same rank. The main result is theorem 4 which claims that there is an isomorphism

$$K_*((G/H)_\gamma) \xrightarrow{\sim} \bigoplus_{i=1}^{r} K_*(A(\lambda_i)_\gamma),$$

with $r = [W(G) : W(H)]$ and central simple algebras $A(\lambda_i)_\gamma$ associated to $(G/H)_\gamma$ in some canonical way. In the last section of the present paper it is shown that the known results concerning the homogeneous projective varieties could be derived from this theorem, although it deals with the affine varieties. An essential role in our computations plays a well-known fact that the choice of a rational point on a homogeneous variety induces an equivalence between the category of equivariant vector bundles over the homogeneous variety and the category of finite dimensional representations of the stabilizer of the chosen point. Another important ingredient is the spectral sequence constructed by Merkurjev [Mer] that allows to pass from the equivariant K-theory to the ordinary one. It turns out that when the groups have the same rank the spectral sequence degenerates and provides a very explicit answer. In order to show that the sequence degenerates we use a theorem proved by Steinberg [St] which states that in our case the representation ring $R(H)$ is a free module over $R(G)$. We give a new proof of the last theorem which provides us a good basis consisting of the irreducible representations such that we can handle it in the twisted case.

Note that there is a decent classification of the connected reductive subgroups containing the maximal torus [BT, § 3]. They correspond to quasi-closed (for $char\, k = 0$ one can say closed) symmetric subsets in the root system of the group G, so one can explicitly write down the varieties covered by theorem 4. For example we can compute by hand the K-theory for the variety $G(E_6)/G(A_2 + A_2 + A_2)$ with the inclusion provided by $3A_2 \subset E_6$. The K_0 in this case is a free abelian group of rank 240.

In the article everything is settled over a field k of an arbitrary characteristic. Algebraic groups are supposed to be smooth algebraic varieties over the field k. The text is organized as follows. In the second section we recall some well-known facts on the representation theory of reductive groups, including the combinatorics concerning roots, weights and the Weyl group.

In the next section we introduce some useful combinatorics arising from a reductive subgroup of maximal rank. We define a linear order on the dominant weights and prove key lemmas providing the technical tool for the new approach

to the Steinberg theorem.
In section 4 we show that with the given order one can choose some set resembling the Gröbner basis and could carry out the division relative to the chosen elements. Using the above idea we construct a basis for the representation ring in theorem 2 and show that there is a natural freedom in the choice of the basis. The introduced division algorithm provides an explicit method for the calculation of the multiplicative structure on the obtained free module.
The fifth section contains some examples, from a vivid two-dimensional case involving G_2 to the non-obvious series of C_n root systems.
In section 6 we recall basic notions from the equivariant K-theory and present the spectral sequence constructed by Merkurjev. The following section deals with a split case of the homogeneous varieties, a degeneration of the spectral sequence is demonstrated and the isomorphism for the K-theory is constructed. Section 8 deals with the twisted forms, separable algebras are introduced and the main result is proved by means of the splitting principle [Pan2].
In the last section we use the developed technique towards concrete examples. First of all the relations with the known results are presented and the K-theory for twisted flags is computed. Then we turn to the case of the characteristic zero and show that the K-theory for any homogeneous variety with the stabilizer connected and having the maximal rank could be computed without the assumption on reductiveness. Also some affine homogeneous examples are considered, including an octonionic projective plane and quaternionic projective spaces.
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2 Representations of reductive algebraic groups.

In this section we fix the notations and recall some well-known facts concerning the representation theory of split reductive algebraic groups. A comprehensive survey of this theme could be found in [Jan], and a classic reference for the semisimple case is [Hum2].
Let G be a connected split reductive algebraic group and let $T \subset G$ be a split maximal torus of G. Let $W(G,T) = N_G(T)/Z_G(T)$ be the Weyl group of G. Since all split maximal tori are conjugate, $W(G,T)$ does not depend on the choice of torus T so we will as usual denote it by $W(G)$. Let

$$X^*(T) = \mathrm{Hom}(T, G_m) \cong \mathbb{Z}^{rk(T)}, \quad \mathrm{Ch} = \mathrm{Hom}(Z(G), G_m)$$

be the character groups of torus T and center $Z(G)$ respectively. Recall that the Weyl group $W(G)$ acts faithfully on $X^*(T)$ and that there is a natural Weyl-equivariant Ch grading on $X^*(T)$.
Let $Rep_k(G)$ be the category of finite dimensional k-rational representations of G and let $R(G) = K_0(Rep_k(G))$ be the representation ring of G. Recall that as

an additive group $R(G)$ is a free abelian group generated by the isomorphism classes of irreducible representations. The following result is well-known (for example, see [Se, Théorème 4]).

Theorem 1. *Let G be a connected split reductive algebraic group and let $T \subset G$ be a split maximal torus of G. Then there is a ring isomorphism $R(T) \cong \mathbb{Z}[X^*(T)]$ where the last one denotes a group ring. Moreover, the restriction of representations induces $R(G) \cong \mathbb{Z}[X^*(T)]^{W(G)}$.*

We need some more combinatorial data on the connection between representations and characters group ring.
There is a root system Φ in $X^*(T)$ and one could construct a positive definite bilinear form $(-\,,-)$ on

$$V = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{rk(G)}$$

such that the Weyl group is generated by the reflections $\{w_\alpha, \alpha \in \Phi\}$. If G is semisimple then there is a canonical choice of this bilinear form. In general case we proceed as follows. For the root datum $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ and the pairing $\langle -\,,-\rangle \colon X^*(T) \times X_*(T) \to \mathbb{Z}$ the Weyl group $W(G)$ is generated by the reflections

$$s_\alpha\lambda = \lambda - \langle \lambda, \alpha^\vee\rangle\alpha,\ \alpha \in \Phi.$$

Set

$$X_0(T) = \{\, \lambda \in X^*(T) |\, \langle \lambda, \alpha^\vee\rangle = 0\ \ \forall \alpha \in \Phi\}\,.$$

Then $V = \mathbb{R}\Phi \oplus X_0(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The space $\mathbb{R}\Phi$ possesses a canonical bilinear form. Choose an arbitrary positive definite bilinear form on $X_0(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The required form $(-\,,-)$ is an orthogonal sum of these two forms.
The hyperplanes H_α orthogonal to the roots $\alpha \in \Phi$ divide V into chambers which are the fundamental domains for the Weyl group action. The hyperplanes adjacent to the chamber are called walls of this chamber. Fix a set of simple roots $\Pi \subset \Phi$ and positive roots Φ^+. Let

$$\mathcal{C}(G) = \{v \in V | (v, \alpha) \geq 0, \alpha \in \Pi\}$$

be the fundamental Weyl chamber. Walls of the fundamental Weyl chamber $\mathcal{C}(G)$ coincide with the hyperplanes orthogonal to the simple roots. Let

$$\Lambda_G^+ = \mathcal{C}(G) \cap X^*(T)$$

be a cone of the dominant weights. Note that in the semisimple case G is simply connected iff there is an isomorphism of semigroups $\Lambda_G^+ \cong (\mathbb{N}_0^+)^{rk(G)}$, where $\mathbb{N}_0^+$ stands for the additive semigroup of non-negative integers.
Let $\lambda \in \Lambda_G^+$ be a dominant weight. Theorem 1 states that there is a bijection between such weights and irreducible G-modules, so we will denote by $V_G(\lambda)$ the corresponding G-module.
At last, recall that there is a partial order on $X^*(T)$ which is defined by the choice of simple roots Π: $\mu \preceq_\Pi \lambda$ if and only if $\lambda - \mu$ is a sum of positive

roots. The interaction between this order and Weyl action is stated in the next lemma.

Lemma 1. *Let $\lambda \in \Lambda_G^+$, $w \in W(G)$ then $w(\lambda) \preceq_\Pi \lambda$.*

Proof. If G is semisimple we can use [Hum1, Lemma 13.2A]. In general case consider the inclusion of the derived group $\mathcal{D}G \subset G$ with maximal torus $T_1 \subset T$ [Jan, 1.18]. It induces an orthogonal projection

$$\pi\colon X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}\Phi \cong X^*(T_1) \otimes_{\mathbb{Z}} \mathbb{R}$$

mapping simple roots to simple roots and Λ_G^+ to $\Lambda_{\mathcal{D}G}^+$. We have $\lambda = \lambda_0 + \lambda_1$ for some $\lambda_0 \in X_0(T) \otimes \mathbb{R}$ and $\lambda_1 \in \mathbb{R}\Phi$ such that $\lambda_1 = \pi(\lambda_1) \in \Lambda_{\mathcal{D}G}^+$. Then

$$\lambda - w(\lambda) = \lambda_0 + \lambda_1 - w(\lambda_0) - w(\lambda_1) = \lambda_1 - w(\lambda_1) \in \mathbb{R}\Phi$$

and

$$\lambda_1 - w(\lambda_1) = \pi(\lambda_1 - w(\lambda_1)) = \pi(\lambda_1) - w(\pi(\lambda_1)).$$

Using the semisimple case we conclude that $\pi(\lambda_1) - w(\pi(\lambda_1))$ is a sum of positive roots of $\mathcal{D}G$ which coincide with the positive roots of G.

$\square$

3 Subgroup combinatorics.

In this section we introduce the necessary combinatorics that we need in order to prove theorem 2. The main goal is to order dominant weights of the subgroup and show that there are several weights with good properties relative to the order.

Let G be a connected split semisimple simply connected group of rank r, let $T \subset G$ be a split maximal torus of G and let $T \subset H \subset G$ be a connected split reductive subgroup of maximal rank. Evidently, in this setting there is an inclusion of the Weyl groups $W(H) \subset W(G)$. Hence we have the corresponding combinatorial data introduced in the previous section: the lattice $X^*(T) \subset V$ in the euclidean space, the root system $\Phi \subset X^*(T)$ and the actions of the Weyl groups $W(H) \subset W(G)$ on the V. The following lemma shows that we can choose compatible fundamental Weyl chambers and the corresponding cones of dominant weights

$$\begin{tikzcd} \mathcal{C}(G) \arrow[r, hook] & \mathcal{C}(H) \\ \Lambda_G^+ \arrow[r, hook] \arrow[u, hook] & \Lambda_H^+ \arrow[u, hook] \end{tikzcd}$$

Let $r = [W(G) : W(H)]$ be the Weyl group index.

Lemma 2. *Any Weyl chamber of H is a union of r Weyl chambers of G.*

Proof. It is clear that any wall for the $W(H)$ action is a wall for the $W(G)$ action, so in order to get the chambers of G we need to subdivide the H chambers. The number of G subchambers is independent on a H chamber since $W(H)$ rearranges the G chambers and acts transitively on the H chambers. Since the number of the chambers coincides with the order of Weyl group, the number of subchambers equals to the index r. $\square$

So we choose some chambers $\mathcal{C}(G) \subset \mathcal{C}(H)$ and elements

$$e = w_1, w_2, ..., w_r \in W(G),$$

such that

$$\mathcal{C}(H) = \bigcup_{1 \le i \le r} w_i \mathcal{C}(G), \quad \Lambda_H^+ = \bigcup_{1 \le i \le r} w_i \Lambda_G^+.$$

Let $\omega_1, ..., \omega_s$ be the fundamental weights corresponding to $\mathcal{C}(G)$ and let Π, Π' be the sets of simple roots for G, H respectively.

Definition 1. Let $\mu \in X^*(T)$ be some weight. Define

$$H(\mu) = \{H_\alpha, \alpha \in \Phi^+ | \exists i : (\mu, \alpha) \cdot (\omega_i, \alpha) < 0\} = \{H_\alpha, \alpha \in \Phi^+ | (\mu, \alpha) < 0\}$$

to be the set of walls which separate μ from $\mathcal{C}(G)$. Let $H(w\mathcal{C}(G)) = H(\mu)$ for some interior weight $\mu \in w\mathcal{C}(G)^o$.

Remark 1. The set $H(\mu)$ somehow measures a combinatorial spherical distance from μ to $\mathcal{C}(G)$, the furthest weights are separated by the most hyperplanes. Also note that $\#H(w\mathcal{C}(G)) = l(w)$, the usual length of an element of a Weyl group, which is defined to be the number of simple reflections in the shortest word representing w.

Lemma 3. *Let $\mu \in \Lambda_H^+$ be a dominant weight. Then there exists such i that $\mu \in w_i \Lambda_G^+$ and $H(w_i \mathcal{C}(G)) = H(\mu)$.*

Proof. If μ belongs to the interior of some chamber we should take that chamber. Otherwise we can choose an arbitrary $\nu \in \mathcal{C}(G)^o$ and draw a segment connecting the points corresponding to μ and ν. Since ν is interior for $\mathcal{C}(G)$ this segment does not belong to hyperplanes H_α and we should take the chamber $w_i\mathcal{C}(G)$ which interior it crosses first, starting from μ. There are no hyperplanes separating the chosen chamber from μ so $H(w_i\mathcal{C}(G)) = H(\mu)$. $\square$

Lemma 4. *Let $\mu, \lambda \in w_i \Lambda_G^+$ for some i and $w \in W(G)$. Suppose that there exists a hyperplane H_α separating λ and $w\mu$, i.e. such that $(\lambda, \alpha) \cdot (w\mu, \alpha) < 0$. Then $(w\mu, \lambda) < (\mu, \lambda)$.*

Proof. First of all we multiply the weights by w_i^{-1} and consider

$$\mu' = w_i^{-1}\mu,\ \lambda' = w_i^{-1}\lambda,\ w' = w_i^{-1} w w_i,\ \alpha' = w_i^{-1}\alpha.$$

It follows that $\mu', \lambda' \in \Lambda_G^+$ and

$$(\lambda', \alpha') \cdot (w'\mu', \alpha') = (w_i^{-1}\lambda, w_i^{-1}\alpha) \cdot (w_i^{-1}w\mu, w_i^{-1}\alpha) = (\lambda, \alpha) \cdot (w\mu, \alpha),$$

and by the same vein

$$(w'\mu', \lambda') = (w\mu, \lambda),\ (\mu', \lambda') = (\mu, \lambda).$$

So from now on we suppose $w_i = e$.
Note that by lemma 1 $\mu - w\mu$ equals to a sum of positive roots and λ is a sum of fundamental weights with nonnegative coefficients, so in general $(w\mu - \mu, \lambda) \leq 0$, and we need to show that the difference $w\mu - \mu$ is not orthogonal to λ. First of all $w\mu \notin \Lambda_G^+$ (i.e. not equals μ) since there are no hyperplanes crossing $\mathcal{C}(G)$.
We can find a sequence $\alpha_1, \alpha_2, ..., \alpha_n \in \Phi^+$ such that the following conditions hold, where $s_i = w_{\alpha_i} w_{\alpha_{i-1}} \dots w_{\alpha_1}$ and $s_0 = e$.

a. $w = s_n = w_{\alpha_n} w_{\alpha_{n-1}} \dots w_{\alpha_1}$.

b. For every $1 \leq i < n$ the hyperplane H_{α_i} is a wall of $s_{i-1}\mathcal{C}(G)$ and separates it and $\mathcal{C}(G)$ from $s_n\mathcal{C}(G)$

This presentations divides w into the sequence of flips, and every flip drives the chamber further from $\mathcal{C}(G)$.
We claim that the roots α_i should be the roots corresponding to hyperplanes in $H(w\mathcal{C}(G))$ written in an appropriate order. Indeed, there exists a hyperplane $H_{\alpha_1} \in H(w\mathcal{C}(G))$ which is the wall of $\mathcal{C}(G)$, otherwise $w\mathcal{C}(G) = \mathcal{C}(G)$ and $w = e$, contradicting $\mu \neq w\mu$. Note that

$$H(s_1\mathcal{C}(G)) = \{H_{\alpha_1}\} \subset H(w\mathcal{C}(G)),$$

and whenever $s_1 \neq w$ we can find $H_{\alpha_2} \in H(w\mathcal{C}(G)) \setminus H(s_1\mathcal{C}(G))$ satisfying the condition (b), i.e. it should be a wall of $s_1\mathcal{C}(G)$, the separating part is valid since we look at the separating hyperplanes. Now one has

$$H(s_2\mathcal{C}(G)) = \{H_{\alpha_1}, H_{\alpha_2}\} \subset H(w\mathcal{C}(G)).$$

If $s_2 \neq w$ we can find $\alpha_3 \in H(w\mathcal{C}(G)) \setminus H(s_2\mathcal{C}(G))$ and so on.
For the above roots α_i one has

$$(w\mu - \mu, \lambda) = (\sum_{i=1}^{n} s_i\mu - s_{i-1}\mu, \lambda) = \sum_{i=1}^{n} c_i(\alpha_i, \lambda),$$

where $c_i = -2\frac{(s_{i-1}\mu, \alpha_i)}{(\alpha_i, \alpha_i)}$. From the condition (b) it follows that H_{α_i} does not separate $s_{i-i}\mathcal{C}(G)$ from $\mathcal{C}(G)$, so $(s_{i-1}\mu, \alpha_i) \geq 0$, hence c_i is nonpositive.
In general $(\alpha_i, \lambda) \geq 0$ so it is sufficient to show that there exists some α_i such that $(\alpha_i, \lambda) \neq 0$ and $(s_{i-1}\mu, \alpha_i) \neq 0$ simultaneously. The first condition is equivalent to $\lambda \notin H_{\alpha_i}$ and the second means that $s_i\mu \neq s_{i-1}\mu$. Now suppose that there is no such α_i, then we can get from μ to $w\mu$ by reflections w_{α_i}

such that $\lambda \in H_{\alpha_i}$. Then $w\mu$ and λ lie in the same chamber and there are no hyperplanes separating them. So by contradiction we can find such i that $c_i(\alpha_i, \lambda) < 0$ and this finishes the proof.

$\square$

Now we are ready to introduce a good order on Λ_H^+ which uses $W(G)$ action and hence somehow connects $W(G)$ orbits with H weights.

DEFINITION 2. Let $\mu_1, \mu_2 \in \Lambda_H^+$, we say that $\mu_1 \preceq' \mu_2$ if and only if one of the following conditions holds:

1. $\mu_1 = \mu_2$
2. $(\mu_1, \mu_1) < (\mu_2, \mu_2)$
3. $(\mu_1, \mu_1) = (\mu_2, \mu_2)$ and $H(\mu_1) \supsetneq H(\mu_2)$

Remark 2. The meaning of the above definition is that a dominant weight is smaller if the vector is shorter or the combinatorial spherical distance to $\mathcal{C}(G)$ is greater.

LEMMA 5.

1. *$\preceq'$ defines a partial order on Λ_H^+.*
2. *For any $\mu \in \Lambda_H^+$ there are only finitely many μ' such that $\mu \npreceq' \mu'$.*
3. *Let $\mu_1, \mu_2 \in \Lambda_H^+$ and $\mu_1 \prec_{\Pi'} \mu_2$. Then $\mu_1 \prec' \mu_2$.*

Proof.

1. is checked by hand.
2. Follows from the fact that there are finitely many weights μ' such that $(\mu', \mu') \leq (\mu, \mu)$.
3. There exists $\beta \in X^*(T)$ such that β equals to a sum of positive roots and $\mu_2 = \mu_1 + \beta$. Then $(\mu_2, \mu_2) = (\mu_1, \mu_1) + (\beta, \beta) + 2(\mu_1, \beta)$. The last term is nonnegative since the scalar product of a simple root and a dominant weight is nonnegative and so is the scalar product of a positive root and a dominant weight. So $\mu_1 \preceq' \mu_2$ follows from the examining of their lengths.

$\square$

DEFINITION 3. Let $\preceq$ be an arbitrary linear extension of the order $\preceq'$ on Λ_H^+, i.e. such a linear order that from $\mu_1 \preceq' \mu_2$ it follows that $\mu_1 \preceq \mu_2$.

Remark 3. Part (3) of the previous lemma is valid for $\preceq$ too and the part (2) transforms into the property that there are only finitely many μ' such that $\mu' \preceq \mu$.

The next lemma introduces basic and in some sense minimal and indecomposable elements $\lambda_i \in \Lambda_H^+$, one for the each chamber $w_i\mathcal{C}(G)$.

Lemma 6. *For every i there exists an element $\lambda_i \in w_i\Lambda_G^+$ such that*

1. *$H(\lambda_i) = H(w_i\mathcal{C}(G))$.*
2. *For every $\mu \in w_i\Lambda_G^+$, $H(\mu) = H(w_i\mathcal{C}(G))$ one has $\mu - \lambda_i \in w_i\Lambda_G^+$.*
3. *For every $\mu \in w_i\Lambda_G^+$ one has $\lambda_i + w_jw_i^{-1}\mu \preceq \lambda_i + \mu$.*

Proof. The set of weights $\{\mu \in w_i\Lambda_G^+ \,|\, H(\mu) \neq H(w_i\mathcal{C}(G))\}$ is just an intersection of $w_i\Lambda_G^+$ with the union of chamber $w_i\mathcal{C}(G)$ walls which separate it from the $\mathcal{C}(G)$. Indeed, the only chance for the weight to have the lesser number of walls separating it from $\mathcal{C}(G)$ is to belong to a such wall, and every weight lying on this wall has the lesser number of separating hyperplanes.
Now we use the fact that G is simply connected so $w_i\Lambda_G^+ \cong (\mathbb{N}_0^+)^s$. The walls of the chamber correspond to the hyperplanes where some coordinate equals 0, so the weights, which have the same $H(\mu)$ as the chamber, correspond to the points with certain coordinates, say $1, ..., l$, strictly greater then 0. Let λ_i be the element corresponding to the point with first l coordinates equal 1 and others equal 0. From the above it follows that we get (1) and (2), so we turn to (3).
First of all note that all weights really lie in the Λ_H^+, so we can try to compare them. Examine their lengths:

$$(\lambda_i + w_jw_i^{-1}\mu, \lambda_i + w_jw_i^{-1}\mu) = (\lambda_i, \lambda_i) + 2(\lambda_i, w_jw_i^{-1}\mu) + (w_jw_i^{-1}\mu, w_jw_i^{-1}\mu) =$$

$$= (\lambda_i, \lambda_i) + 2(\lambda_i, w_jw_i^{-1}\mu) + (\mu, \mu),$$

$$(\lambda_i + \mu, \lambda_i + \mu) = (\lambda_i, \lambda_i) + 2(\lambda_i, \mu) + (\mu, \mu).$$

Applying lemma 1 to $w_i^{-1}\lambda_i, w_i^{-1}\mu \in \Lambda_G^+$ and $w_i^{-1}w_j \in W(G)$ we have

$$(\lambda_i, w_jw_i^{-1}\mu) = (w_i^{-1}\lambda_i, w_i^{-1}w_jw_i^{-1}\mu) \leq (w_i^{-1}\lambda_i, w_i^{-1}\mu) = (\lambda, \mu)$$

and, consequently,

$$(\lambda_i + w_jw_i^{-1}\mu, \lambda_i + w_jw_i^{-1}\mu) \leq (\lambda_i + \mu, \lambda_i + \mu).$$

Now look at $H(\lambda_i)$. Observe that $H(\lambda_i) = H(\lambda_i + \mu)$. Indeed, $\lambda_i + \mu \in w_i\Lambda_G^+$ and from the first part of the lemma it follows that $H(\lambda_i) \supset H(\lambda_i + \mu)$. The opposite inclusion follows from the fact that since μ and λ_i lie in the same chamber there are no hyperplanes H_α separating them, i.e. one has $(\lambda_i, \alpha) \cdot (\mu, \alpha) \geq 0$. For every $H_\alpha \in H(\lambda_i)$ one has $(\lambda_i, \alpha) < 0$, so $(\mu, \alpha) \leq 0$ and $(\lambda_i + \mu, \alpha) < 0$, then $H_\alpha \in H(\lambda_i + \mu)$.
First suppose that there exists some $H_\alpha \in H(\lambda_i)$ such that $(w_jw_i^{-1}\mu, \alpha) > 0$. Since $H_\alpha \in H(\lambda_i)$ one has $(\lambda_i, \alpha) < 0$ and

$$(\lambda_i, \alpha) \cdot (w_jw_i^{-1}\mu, \alpha) < 0.$$

Then we are in the setting of lemma 4 with a slight change of notation, so $(\lambda_i, w_j w_i^{-1}\mu) < (\lambda_i, \mu)$ hence $\lambda_i + w_j w_i^{-1}\mu \preceq \lambda_i + \mu$.
Otherwise for all $H_\alpha \in H(\lambda_i)$ one has $(w_j w_i^{-1}\mu, \alpha) \leq 0$ and since $(\lambda_i, \alpha) < 0$ one has $(\lambda_i + w_j w_i^{-1}\mu, \alpha) < 0$, so $H(\lambda_i) \subset H(w_j w_i^{-1}\mu + \lambda_i)$ and

$$H(\lambda_i + \mu) \subset H(w_j w_i^{-1}\mu + \lambda_i).$$

The last two sets coincide only if $w_j w_i^{-1}\mu + \lambda_i \in w_i \Lambda_G^+$ and the second part of the lemma in this case yields $w_j w_i^{-1}\mu \in w_i \Lambda_G^+$ that means $\mu = w_j w_i^{-1}\mu$. In any case one has $\lambda_i + w_j w_i^{-1}\mu \preceq \lambda_i + \mu$.

□

Remark 4. Note that from the above construction one gets $\lambda_1 = 0 \in X^*(T)$.

4 Restriction of representations.

In this section we study the representation restriction homomorphism on the representation rings and prove theorem 2.
Let $T \subset H \subset G$ be the same groups as in the previous section. From theorem 1 we get the following commutative diagram.

$$\begin{tikzcd}
R(G) \arrow[r, hook] \arrow[d, "\simeq"] & R(H) \arrow[r, hook] \arrow[d, "\simeq"] & R(T) \arrow[d, "\simeq"] \\
\mathbb{Z}[X^*(T)]^{W(G)} \arrow[r, hook] & \mathbb{Z}[X^*(T)]^{W(H)} \arrow[r, hook] & \mathbb{Z}[X^*(T)]
\end{tikzcd}$$

Recall that $Z(G) \subset T$, hence all the rings above are $Ch = X^*(Z(G))$-graded. We are interested in the $R(G)$-module structure on $R(H)$ and its connection with the grading.
We need the following easy lemma from commutative algebra.

Lemma 7. *Let $S \subset R$ be integral domains and let $\lambda_1, ..., \lambda_r \in R$ generate R as S-module. Set $Q(R)$ and $Q(S)$ to be the fraction fields of R and S respectively and let $[Q(R) : Q(S)] = r$. Then R is a free S-module with basis $\lambda_1, ..., \lambda_r$.*

Proof. R is finitely generated as an S-module hence it is integral over S. Then $R \otimes_S Q(S)$ is an integral domain integral over the field $S \otimes_S Q(S) = Q(S)$, hence itself is a field [AM, Proposition 5.7] so $R \otimes_S Q(S) \cong Q(R)$.
We have the following short exact sequence induced by $\lambda_1, ..., \lambda_r$.

$$N \rightarrowtail S^k \twoheadrightarrow R\,,$$

hence

$$N \otimes_S Q(S) \rightarrowtail Q(S)^k \twoheadrightarrow R \otimes_S Q(S)\,.$$

The last term is isomorphic to $Q(R)$ and comparing dimensions one can see that $N \otimes_S Q(S) = 0$ hence $N = 0$ and we get the claim of the lemma. □

We consider $X^*(T)$ as an additive group, so we will write the element of $\mathbb{Z}[X^*(T)]$ corresponding to weight μ in a such way: x^μ.

Definition 4. For $\mu \in X^*(T)$ we will denote by $(x^\mu)^{W(H)}$ the sum in the $\mathbb{Z}[X^*(T)]$ of all elements corresponding to the weights in a $W(H)$-orbit of μ and for any monomial $ax^\mu \in \mathbb{Z}[X^*(T)]$ by $(ax^\mu)^{W(H)} = a(x^\mu)^{W(H)}$ we denote the similar orbit but with a coefficient.

With the above notation one has the unique decomposition of

$$f = \sum (a_{\mu_j} x^{\mu_j})^{W(H)} \in \mathbb{Z}[X^*(T)]^{W(H)}$$

into the sum of monomial orbits with distinct $\mu_j \in \Lambda_H^+$. Recall that we have a linear order $\preceq$ on μ_j introduced in the previous section.

Definition 5. Let $f = \sum (a_{\mu_j} x^{\mu_j})^{W(H)}$, then define the degree $\deg(f) = \max\limits_j \mu_j$ to be the maximal μ_j in the decomposition and the leading orbit $\mathrm{lo}(f) = (a_{\deg(f)} x^{\deg(f)})^{W(H)}$ to be the orbit of the maximal monomial.

We will use an analogous notation for the group G.

Lemma 8. *For $\mu_1, \mu_2 \in \Lambda_H^+$ we have*

$$\mathrm{lo}\left((x^{\mu_1})^{W(H)} (x^{\mu_2})^{W(H)} \right) = \left(x^{\mu_1+\mu_2} \right)^{W(H)}.$$

Proof. Expanding the orbits we obtain

$$(x^{\mu_1})^{W(H)} (x^{\mu_2})^{W(H)} = \sum_{s_1 \in S_1, s_2 \in S_2} x^{s_1\mu_1 + s_2\mu_2}$$

for some sets $S_i \subset W(H)$ of representatives of cosets $W(H)/\mathrm{Stab}_{W(H)}(\mu_i)$. We can choose S_i such that $e \in S_1, S_2$. By lemma 1 we have

$$s_1\mu_1 \preceq_\Pi \mu_1, \quad s_2\mu_2 \preceq_\Pi \mu_2,$$

and for nontrivial s_i the relation is strict. Hence $s_1\mu_1 + s_2\mu_2 \prec_\Pi \mu_1 + \mu_2$ and applying lemma 5 we get $s_1\mu_1 + s_2\mu_2 \prec \mu_1 + \mu_2$. ∎

Theorem 2. *Let G be a split semisimple simply connected group and let H be a connected split reductive subgroup of the maximal rank (i.e. H contains a split maximal torus T of G). Then $R(H)$ is a free $R(G)$-module of rank $[W(G) : W(H)]$ and there is a* Ch*-homogeneous basis.*

Proof. First of all we will deal with the weight realization of the rings of representations, i.e. with the following sequence.

$$\mathbb{Z}[X^*(T)]^{W(G)} \hookrightarrow \mathbb{Z}[X^*(T)]^{W(H)} \hookrightarrow \mathbb{Z}[X^*(T)]$$

In the previous section in lemma 6 we have constructed some λ_i and we claim that the orbits $(x^{\lambda_i})^{W(H)}$ form a homogeneous basis of $\mathbb{Z}[X^*(T)]^{W(H)}$ over $\mathbb{Z}[X^*(T)]^{W(G)}$.

a. Homogeneity. It is the easiest part since it follows at once from the equivariance of the $W(H)$ action.

b. $(x^{\lambda_i})^{W(H)}$ generate $\mathbb{Z}[X^*(T)]^{W(H)}$ as a $\mathbb{Z}[X^*(T)]^{W(G)}$-module. We will show by induction on $\deg(f)$ that $f \in \mathbb{Z}[X^*(T)]^{W(H)}$ could be expressed as a linear combination of $(x^{\lambda_i})^{W(H)}$ with $\mathbb{Z}[X^*(T)]^{W(G)}$ coefficients. Note that $\lambda_1 = 0 \in X^*(T)$ and

$$(x^{\lambda_1})^{W(H)} = 1 \in \mathbb{Z}[X^*(T)]^{W(H)},$$

so we have the constants. Now suppose that we can express as linear combinations all $f \in \mathbb{Z}[X^*(T)]^{W(H)}$ such that $\deg(f) \prec \mu_0$ and we need to write down such an expression for $(x^{\mu_0})^{W(H)}$.

By lemma 3 we have some chamber $w_l\mathcal{C}(G)$ such that $\mu_0 \in w_l\Lambda_G^+$ and $H(\mu_0) = H(w_l\mathcal{C}(G))$, hence, by lemma 6 $\nu = \mu_0 - \lambda_l \in w_l\Lambda_G^+$. Choose a subset $\{w_j\}$ of $\{w_i\}$ such that we have all the distinct $w_j w_l^{-1}\nu$. Then by subdividing the $W(G)$-orbit into the $W(H)$-orbits we have the following equality.

$$\operatorname{lo}\left((x^{\nu})^{W(G)}\left(x^{\lambda_l}\right)^{W(H)}\right) = \operatorname{lo}\left(\sum_j \left(\left(x^{w_j w_l^{-1}\nu}\right)^{W(H)}\left(x^{\lambda_l}\right)^{W(H)}\right)\right).$$

Looking only at the leading orbits of the summands in the last expression and applying lemma 8 we get

$$\operatorname{lo}\left((x^{\nu})^{W(G)}\left(x^{\lambda_l}\right)^{W(H)}\right) = \operatorname{lo}\left(\left(\sum_j x^{w_j w_l^{-1}\nu+\lambda_l}\right)^{W(H)}\right),$$

and finally, by lemma 6, we have

$$\operatorname{lo}\left(\left(\sum_j x^{w_j w_l^{-1}\nu+\lambda_l}\right)^{W(H)}\right) = \left(x^{\nu+\lambda_l}\right)^{W(H)} = (x^{\mu_0})^{W(H)}.$$

Hence

$$\deg\left((x^{\mu_0})^{W(H)} - (x^{\nu})^{W(G)}\left(x^{\lambda_l}\right)^{W(H)}\right) \prec \mu_0$$

and we can use the induction.

c. $(x^{\lambda_i})^{W(H)}$ are linearly independent. >From the sequence

$$\mathbb{Z}[X^*(T)]^{W(G)} \hookrightarrow \mathbb{Z}[X^*(T)]^{W(H)} \hookrightarrow \mathbb{Z}[X^*(T)],$$

one gets a sequence of the fraction fields

$$\begin{tikzcd}
Q(\mathbb{Z}[X^*(T)]^{W(G)}) \arrow[r, hook] \arrow[d, equal] & Q(\mathbb{Z}[X^*(T)]^{W(H)}) \arrow[r, hook] \arrow[d, equal] & Q(\mathbb{Z}[X^*(T)]) \arrow[d, equal] \\
Q(\mathbb{Z}[X^*(T)])^{W(G)} \arrow[r, hook] & Q(\mathbb{Z}[X^*(T)])^{W(H)} \arrow[r, hook] & Q(\mathbb{Z}[X^*(T)])
\end{tikzcd}$$

with a degree of field extension

$$\left[Q\left(\mathbb{Z}[X^*(T)]\right)^{W(H)} : Q\left(\mathbb{Z}[X^*(T)]\right)^{W(G)}\right] = \left[W(G) : W(H)\right].$$

Hence, by lemma 7 one gets the claim of the theorem.

□

Corollary 1. *In the notation of theorem 2 one has a basis of $R(H)$ over $R(G)$ consisting of the classes of irreducible representations of H.*

Proof. One can take $V_H(\lambda_i)$ and since their leading orbits coincide with the basis constructed in the theorem one gets the claim. □

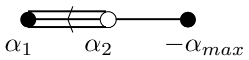
Remark 5. We can choose various chambers $\mathcal{C}(G)$ and the different choices produce different bases. Also the proof of the theorem gives an explicit algorithm for the calculation of the coefficients of decomposition with respect to the chosen basis, so, for example, in every particular case one can write down the multiplication table for the basis, yet it seems that there is no elegant general formula.

5 Examples: $A_1 + A_1 \subset G_2$, $B_4 \subset F_4$ and $C_1 + C_{n-1} \subset C_n$.

In this section we compute several examples of bases. Every reductive subgroup containing the maximal torus is defined by some quasi-closed root subset [BT, § 3], so we use the root system notation. Every maximal root subsystem of full rank corresponds to a node in the Dynkin diagram and the subsystem diagram is just an extended Dynkin diagram with the chosen node removed. We label simple roots in a way of [Bou].

5.1 $A_1 + A_1 \subset G_2$

In this example we take the subsystem in G_2 defined by the short simple root and the maximal one. The corresponding Dynkin diagram is the next one, with the white node removed.



We label the roots in a way shown above and $\alpha_{max} = 3\alpha_1 + 2\alpha_2$. The fundamental chamber $\mathcal{C}(G_2)$ is a chamber spanned by the fundamental weights

$$\omega_1 = \alpha_{max} = 3\alpha_1 + 2\alpha_2, \quad \omega_2 = 2\alpha_1 + \alpha_2.$$

The fundamental chamber $\mathcal{C}(A_1+A_1)$ should contain $\mathcal{C}(G_2)$ so it is the quarter of the plane bounded by α_1 and α_{max}. Note, by the way, that the considered group $G(A_1+A_1)$ is not simply connected since there are no weights $\frac{1}{2}\alpha_1$ and $\frac{1}{2}\alpha_{max}$ in our lattice and one can see that

$$G(A_1+A_1) = (SL_2 \times SL_2)/\mu_2$$

with μ_2 embedded diagonally.
The chamber $\mathcal{C}(A_1+A_1)$ subdivides into the G_2 chambers in the following way:

$$\mathcal{C}(A_1+A_1) = \mathcal{C}(G_2) \cup w_{\alpha_2}\mathcal{C}(G_2) \cup w_{\alpha_1+\alpha_2}w_{\alpha_2}\mathcal{C}(G_2)$$

So theorem 2 tells us that we should take in each subchamber the shortest of the furthest by spherical distance weights, i.e. the generator for the furthest wall, hence one has

$$0 \in \mathcal{C}(G), \quad 3\alpha_1 + \alpha_2 \in w_{\alpha_2}\mathcal{C}(G), \quad \alpha_1 \in w_{\alpha_1+\alpha_2}w_{\alpha_2}\mathcal{C}(G_2)$$

and the corresponding sums over $W(A_1+A_1)$ would form a basis. The basis from the theorem is the following one:

$$1, \quad x^{3\alpha_1+\alpha_2} + x^{\alpha_2} + x^{-\alpha_2} + x^{-3\alpha_1-\alpha_2}, \quad x^{\alpha_1} + x^{-\alpha_1}.$$

One could compute the corresponding basis consisting of irreducible modules from corollary 1 having the following weight subspaces:

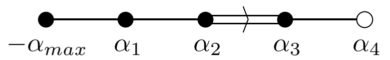
$$\begin{array}{lcl} V(0) & = & 1 \\ V(3\alpha_1+\alpha_2) & = & x^{2\alpha_1+\alpha_2} + x^{\alpha_1+\alpha_2} + x^{-\alpha_1-\alpha_2} + x^{-2\alpha_1-\alpha_2} + \\ & & +x^{3\alpha_1+\alpha_2} + x^{\alpha_2} + x^{-\alpha_2} + x^{-3\alpha_1-\alpha_2} \\ V(\alpha_1) & = & 1 + x^{\alpha_1} + x^{-\alpha_1} \end{array}$$

In fact after identifying $G(A_1+A_1) = (SL_2 \times SL_2)/\mu_2$ one can write down the above representations in more natural way, denoting by W_1, W_2 the regular representations of the factors one has

$$V(0) = S^0W_1 \otimes S^0W_2, \quad V(3\alpha_1+\alpha_2) = S^3W_1 \otimes W_2, \quad V(\alpha_1) = S^2W_1 \otimes S^0W_2.$$

5.2 $B_4 \subset F_4$

In this case we remove the α_4 node from the extended Dynkin diagram of type F_4. One can show that it corresponds to $Spin_9 \subset G(F_4)$.



With the above labelling one has

$$\alpha_{\max} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4.$$

We will shorten the above notation to $(2,3,4,2)$. The fundamental weights defining $\mathcal{C}(F_4)$ are

$$\omega_1 = (2,3,4,2), \quad \omega_2 = (3,6,8,4), \quad \omega_3 = (2,4,6,3), \quad \omega_4 = (1,2,3,2).$$

Choose the simple roots for B_4 in the following way:

$$\alpha'_1 = w_{\alpha_4}(-\alpha_{max}) = (0,1,2,2), \quad \alpha'_2 = \alpha_1, \quad \alpha'_3 = \alpha_2, \quad \alpha'_4 = \alpha_3,$$

hence we have the fundamental weights defining $\mathcal{C}(B_4) \sim \{\omega'_1, \omega'_2, \omega'_3, \omega'_4\}$:

$$\omega'_1 = (1,2,3,2) = \omega_4, \quad \omega'_2 = (2,3,4,2) = \omega_1,$$

$$\omega'_3 = (2,4,5,2), \quad \omega'_4 = (1,2,3,1).$$

Since these weights belong to the considered lattice the chosen $G(B_4)$ is simply connected, so it really is $Spin_9$. Now we compute the subdividing of $\mathcal{C}(B_4)$:

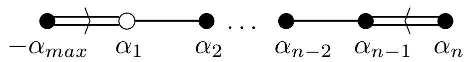
$$\mathcal{C}(B_4) = \mathcal{C}(F_4) \cup w_{\alpha_4}\mathcal{C}(F_4) \cup w_{\alpha_3+\alpha_4} w_{\alpha_4}\mathcal{C}(F_4),$$

$$w_{\alpha_4}\mathcal{C}(B_4) \sim \{\omega_1, \omega_2, \omega_3, \omega'_4\}, \quad w_{\alpha_3+\alpha_4} w_{\alpha_4}\mathcal{C}(B_4) \sim \{\omega_1, \omega_2, \omega'_3, \omega'_4\}.$$

Theorem 2 suggests to look at the elements appeared after flips, since they are the spherically furthest, so the basis would consist of $W(B_4)$ orbits of $0, \omega'_4, \omega'_3$. Another basis comes from the corollary 1 that claims $V(0), V(\omega'_3), V(\omega'_4)$ to be a basis. These representations are just the trivial one, $\Lambda^3 W$ for the regular W and the spin one.

5.3 $C_1 + C_{n-1} \subset C_n$

In this case we remove α_1 node from the extended Dynkin diagram of type C_n and it corresponds to $(Sp_2 \times Sp_{2n-2}) \subset Sp_{2n}$ with the quotient variety $HP^{n-1} = Sp_{2n}/(Sp_2 \times Sp_{2n-2})$ being a quaternionic projective space in notation of section 9.3.



One has

$$\alpha_{\max} = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n = (2,2,\ldots,2,1).$$

The fundamental weights for $\mathcal{C}(C_n)$ are

$$\omega_i = (1,2,\ldots,i-1,i,i,\ldots i,\frac{i}{2}).$$

Choosing $\alpha_2, \alpha_3, \ldots, \alpha_n, \alpha_{max}$ to be the simple roots of $C_1 + C_{n-1}$ one gets the following $\mathcal{C}(C_1 + C_{n-1})$:

$$\omega_i' = (0, 1, \ldots, i-1, i, i, \ldots i, \frac{i}{2}), \quad \omega_n' = (1, 1, \ldots, 1, \frac{1}{2}).$$

The subdivision of $\mathcal{C}(C_1 + C_{n-1})$ is straightforward:

$$\mathcal{C}(C_1 + C_{n-1}) = \mathcal{C}(C_n) \cup w_{\alpha_1}\mathcal{C}(C_n) \cup w_{\alpha_1+\alpha_2}w_{\alpha_1}\mathcal{C}(C_n) \cup \ldots \\ \cdots \cup (w_{\alpha_1+\cdots+\alpha_{n-1}} \ldots w_{\alpha_1+\alpha_2}w_{\alpha_1})\mathcal{C}(C_n),$$

$$(w_{\alpha_1+\cdots+\alpha_i} \ldots w_{\alpha_1+\alpha_2}w_{\alpha_1})\mathcal{C}(C_n) \sim \{\omega_1', \omega_2', \ldots, \omega_i', \omega_{i+1}, \ldots \omega_n\}.$$

Corollary 1 claims $V(0), V(\omega_1'), \ldots V(\omega_{n-1}')$ to be a basis and this representations are just $\Lambda^i W$ for a regular representation W of Sp_{2n-2}.

6 Representations, vector bundles and equivariant K-theory.

In this section we recall some results on the equivariant K-theory. An extensive exposition and further references could be found in [Mer].
Let G be an algebraic group, let $H \subset G$ be a closed subgroup and let $X = G/H$ be the corresponding smooth homogeneous G-variety. There is a well-known tensor equivalence [Mer, Example 2]

$$Rep_k(H) \xrightarrow{\sim} Vect^G(X)$$

between the categories $Rep_k(H)$ of finite dimensional k-rational representations of H and $Vect^G(X)$ of G-equivariant vector bundles over X. The inverse for the above equivalence is given by the fiber over the distinguished rational point eH of X. Further we will use the following notation.

Definition 6. Let $V_H(\lambda)$ be an irreducible representation of H with the highest weight $\lambda \in \Lambda_H^+$, then denote by $\mathcal{V}_H(\lambda)$ the corresponding vector bundle over G/H. For an irreducible representation $V_G(\mu)$ of G with the highest weight $\mu \in \Lambda_G^+$ one can use the restriction of representations, get a representation of H (not necessary irreducible) and then take the corresponding vector bundle $\mathcal{V}_G(\mu)$. Occasionally we will write $V_G(\lambda)$ and $\mathcal{V}_G(\lambda)$ for $\lambda \in \Lambda_H^+$ and it means that one should find $\mu \in \Lambda_G^+$ from a $W(G)$-orbit of λ and then take the corresponding $V_G(\mu)$ and $\mathcal{V}_G(\mu)$.

Remark 6. Note that after forgetting about the G-action the last bundle becomes trivial, i.e. the composition

$$Rep_k(G) \xrightarrow{Res} Rep_k(H) \xrightarrow{\sim} Vect^G(X) \longrightarrow Vect(X)$$

takes G representations to trivial bundles.

Set

$$K_n(G;X) = K_n(Vect^G(X)).$$

The above equivalence yields

$$K_n(G;X) \cong K_n(Rep(H)),$$

in particular

$$K_0(G;X) \cong R(H).$$

Note that $R(H)$ is a $R(G)$-module, hence every $K_n(G,X)$ also is. The following proposition, being a straightforward consequence of [Mer, Theorem 10], compares $K_n(G;X)$ with $K_n(X)$.

Proposition 1. *Let G be a split simply connected semisimple group. Then there is a spectral sequence*

$$E^2_{p,q} = \mathrm{Tor}_p^{R(G)}(\mathbb{Z}, K_q(G;X)) \Longrightarrow K_{p+q}(X).$$

7 K-theory of a homogeneous variety.

In this section we calculate the K-theory of a homogeneous variety $X = G/H$ with connected split reductive algebraic groups $H \subset G$ of the same rank.

Lemma 9. *$K_n(Rep_k(H)) \cong R(H) \otimes_{\mathbb{Z}} K_n(k)$ as $R(H)$-modules.*

Proof. Note that $\operatorname{char} k$ not necessary equals 0, so the reductive group H is not necessary geometrically reductive, i.e. the category $Rep_k(H)$ may be not semisimple. But, nevertheless, all objects of $Rep_k(H)$ have finite length and, thanks to Devissage property of the K-theory, one has

$$K_n(Rep_k(H)) \cong K_n(Rep_k(H)_{ss}),$$

where $Rep_k(H)_{ss}$ stands for the subcategory of semisimple representations. By Shur's Lemma we can pass to a sum of the abelian categories of modules over the endomorphism rings of irreducible representations

$$K_n(Rep_k(H)_{ss}) \cong K_n\left(\bigoplus_{[V_i]} M\left(\mathrm{End}_{Rep}(V_i)\right)[V_i]\right)$$

with $[V_i]$ being the isomorphism classes of irreducible representations. Since $\mathrm{End}_{Rep}(V_i) = k$, the last one equals to

$$K_n\left(\bigoplus_{[V_i]} M\left(k\right)[V_i]\right) \cong \bigoplus K_n(k)\,[V_i] \cong R(H) \otimes_{\mathbb{Z}} K_n(k).$$

$\square$

Proposition 2. *Let G be a connected split simply connected semisimple algebraic group and let $H \subset G$ be a connected split reductive subgroup of the same rank. Then the spectral sequence in proposition 1 degenerates, i.e.*

$$\operatorname{Tor}_p^{R(G)}(\mathbb{Z}, K_n(G;X)) = \begin{cases} K_n(X), & \text{if } p = 0; \\ 0, & \text{if } p > 0. \end{cases}$$

Proof. Due to lemma 9 it is sufficient to show that for $p \geq 1$ one has

$$\operatorname{Tor}_p^{R(G)}(\mathbb{Z}, R(H) \otimes_{\mathbb{Z}} K_n(k)) = 0.$$

Replace $K_n(k)$ with an arbitrary abelian group M. Since Tor commutes with the direct limits we can reduce the problem to the finitely generated abelian groups, and, moreover, to $M = \mathbb{Z}$ or $M = \mathbb{Z}/m\mathbb{Z}$.
In the first case we at once get the claim from theorem 2,

$$\operatorname{Tor}_p^{R(G)}(\mathbb{Z}, R(H)) = 0,$$

since $R(H)$ is a free $R(G)$-module.
In the second case we can write a resolution

$$0 \longrightarrow R(H) \xrightarrow{m} R(H) \longrightarrow R(H) \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0\ ,$$

which is exact since $R(H)$ is a domain of a zero characteristic. Denoting the rank of $R(H)$ over $R(G)$ by r, after tensoring with $\mathbb{Z}$ one still gets an exact sequence

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{m} \mathbb{Z}^r \longrightarrow (\mathbb{Z}/m\mathbb{Z})^r \longrightarrow 0\ .$$

So, we conclude

$$\operatorname{Tor}_p^{R(G)}(\mathbb{Z}, R(H) \otimes \mathbb{Z}/m\mathbb{Z}) = 0,$$

finishing the proof. □

In order to remove the annoying restriction that G should be simply connected we need the following lemma.

Lemma 10. *Let $H \subset G$ be a pair of connected split reductive groups of the same rank. Then there exists a connected split simply connected semisimple group $\widetilde{G}$ and a connected split reductive subgroup $\widetilde{H} \subset \widetilde{G}$ of the same rank such that $\widetilde{G}/\widetilde{H} \cong G/H$.*

Proof. Let $\widetilde{G}$ be a simply connected covering of the derived group $\mathcal{D}G$. There exists a covering

$$Z \rightarrowtail (G_m)^l \times \widetilde{G} \twoheadrightarrow G$$

with a finite kernel Z. Since H contains the maximal torus the preimage of H under this projection contains the factor $(G_m)^l$, so we have the following diagram.

$$\begin{tikzcd}
(G_m)^l \times \widetilde{G} \arrow[r, two heads] & G \\
(G_m)^l \times \widetilde{H} \arrow[r, two heads] \arrow[u, hook] & H \arrow[u, hook]
\end{tikzcd}$$

The above consideration yields

$$\widetilde{G}/\widetilde{H} \cong (G_m)^l \times \widetilde{G}/(G_m)^l \times \widetilde{H} \cong G/H,$$

so we need to show that $\widetilde{H}$ is connected. One has

$$Z \rightarrowtail (G_m)^l \times \widetilde{H} \twoheadrightarrow H$$

with finite central Z and connected H. The identity component of $(G_m)^l \times \widetilde{H}$ contains the maximal torus, hence it contains Z and the connectedness of H yields that the identity component coincides with the whole group. The group $\widetilde{H}$ is connected as the quotient of the connected group. □

Theorem 3. *Let $H \subset G$ be a pair of connected split reductive groups of the same rank. Denote $r = [W(G) : W(H)]$. Then there exist $\mathcal{V}_1, ..., \mathcal{V}_r \in Vect(G/H)$ such that*

$$K_*(G/H) = \bigoplus_{i=1}^{r} K_*(k)[\mathcal{V}_i].$$

Proof. By lemma 10 we can pass to a simply connected semisimple group G and from the proof one has that $[W(G) : W(H)]$ remains the same. Due to proposition 2, the spectral sequence in proposition 1 degenerates, so using lemma 9 one has

$$\begin{aligned} K_n(X) \cong Tor_0^{R(G)}(\mathbb{Z}, R(H) \otimes_{\mathbb{Z}} K_n(k)) = \\ = \mathbb{Z} \otimes_{R(G)} R(H) \otimes_{\mathbb{Z}} K_n(k) = \mathbb{Z}^r \otimes_{\mathbb{Z}} K_n(k). \end{aligned}$$

The above isomorphism is induced by the elements of the basis $R(H)$ over $R(G)$ constructed in theorem 2, so we can take as $\mathcal{V}_i$ the corresponding elements of $Vect(G/H)$. □

Remark 7. In remark 5 we noted that there is an explicit algorithm to write down the multiplication for the basis elements and now it describes the ring structure on the $K_0(G/H)$.

8 The K-theory of twisted forms.

In this section we deal with certain twisted forms of homogeneous varieties $X = G/H$ with connected split reductive groups $H \subset G$ of the same rank. From now on we suppose G to be simply connected semisimple, lemma 10 shows that in fact it is not a restriction. Denote as before $r = [W(G) : W(H)]$. One has an obvious left action of G on G/H and since H contains the maximal torus hence the center of G, this action extends to the action of

$$\overline{G} = G/Z(G).$$

Now fix a 1-cocycle $\gamma : \mathrm{Gal}(k^{sep}/k) \to \overline{G}(k^{sep})$. Twisting the variety with this cocycle we obtain

$$X_\gamma = (G/H)_\gamma.$$

The following lemma provides a splitting variety for this cocycle.

Lemma 11. *For the above cocycle γ there exists a variety Y such that the following conditions hold:*

1. *Y is a smooth projective variety.*
2. *The Euler characteristic $\chi(Y)$ equals to 1.*
3. *For every point (not necessary closed) $y \in Y$ the cocycle $\gamma_{k(y)}$ is a coboundary.*

Proof. Note that $\overline{G}$ is a split semisimple group and we can twist it with γ as well. The last condition is equivalent to the condition that for every point $y \in Y$ the group $(\overline{G}_\gamma)_{k(y)}$ is split. Set $\overline{B}$ to be a Borel subgroup of $\overline{G}$ and consider

$$Y = (\overline{G}/\overline{B})_\gamma.$$

We claim that the variety $Y_F = ((\overline{G}/\overline{B})_\gamma)_F$ has a rational point if and only if $\overline{G}_\gamma$ splits over F. The existence of a rational point on this variety is equivalent to the existence of a Borel subgroup defined over F, which is the stabilizer of this point in $(\overline{G}_\gamma)_F$. The existence of a Borel subgroup means that the group is quasi-split and in our case it is equivalent to be split, since we work with an inner form.
$Y = (\overline{G}/\overline{B})_\gamma$ is clearly a smooth projective variety. In order to compute the Euler characteristic $\chi(\overline{G}/\overline{B})$ it can be shown [Jan, Proposition 4.5] that

$$h^i(O_{\overline{G}/\overline{B}}) = \begin{cases} 0, & \text{if } i > 0; \\ 1, & \text{if } i = 0. \end{cases}$$

so we get the claim. □

The idea lying behind the calculation of the K-theory of a twisted form is quite simple: one needs to construct some candidate for the K-theory and a morphism such that they will produce a correct answer and an isomorphism in the split case. The isomorphism in the split case is written by means of some vector bundles $\mathcal{V}_i$, so in general we want to twist them. And here is the problem – there is no action of $\overline{G}$ on these bundles since the center could act non-trivially. In order to get over that we should tensor $\mathcal{V}_i$ with some bundles to trivialize the center action, and then for the cancellation of this tensoring we should look at the modules over endomorphism algebras of the excessive factors. Twisting these endomorphism algebras we get separable algebras which produce the answer.

Definition 7. Let $V_G(\lambda)$ be a representation of G then we denote

$$A(\lambda) = \mathrm{End}_{\mathrm{k}}(V_G(\lambda)) = V_G(\lambda) \otimes V_G(\lambda)^*$$

the endomorphism algebra of the underlying vector space. There is an obvious diagonal G action on $A(\lambda)$ which extends to the $\overline{G}$ action, so we can twist this algebra and get a separable algebra $A(\lambda)_\gamma$. Also one can pass to the corresponding trivial sheaf of algebras $\mathcal{A}(\lambda)_\gamma$ over X_γ.

Now we fix λ_i from theorem 2 and the corresponding $A(\lambda_i)$. Denote

$$W(\lambda_i) = V_H(\lambda_i) \otimes V_G(\lambda_i)^*,$$

and the corresponding vector bundle $\mathcal{W}(\lambda_i)$. Note that $W(\lambda_i)$ is a right module over $A(\lambda_i)$ through the second factor and so $\mathcal{W}(\lambda_i)$ is. Recall that

$$Z(G) \subset T \subset H$$

and the weights λ_i are Ch-homogeneous. The center $Z(G)$ acts on the $V_H(\lambda_i)$ and $V_G(\lambda_i)^*$ through the opposite characters hence it acts trivially on $W(\lambda_i)$ and $\mathcal{W}(\lambda_i)$ so we can obtain a twisted form $\mathcal{W}(\lambda_i)_\gamma$.
All the considered above structures are agreed, so now we have trivial sheaves of separable algebras $\mathcal{A}(\lambda_i)_\gamma$ and vector bundles $\mathcal{W}(\lambda_i)_\gamma$ that are right $\mathcal{A}(\lambda_i)_\gamma$-modules.

Definition 8. For a variety Z and a separable algebra A let $\mathcal{P}(Z, A)$ be the category of coherent $O_Z \otimes A$-modules which are locally free O_Z-modules. Then we denote

$$K_*(Z, A) = K_*(\mathcal{P}(Z, A)).$$

There is a corresponding notion of $K'_*(Z, A)$ and it satisfies all the usual properties of the K-theory [Mer].

Proposition 3. *Let Z be a variety such that every point $z \in Z$ (not necessary closed) splits γ, i.e. $\gamma_{k(z)}$ is a coboundary. Then in the above notation one has an isomorphism*

$$\sum_{i=1}^{r} \phi_i : \bigoplus_{i=1}^{r} K'_*(Z, A(\lambda_i)_\gamma) \longrightarrow K'_*(X_\gamma \times Z),$$

where
$$\phi_i(U) = p_X^*(\mathcal{W}(\lambda_i)_\gamma) \otimes_{A(\lambda_i)_\gamma} p_Z^*(U).$$

Proof. This is proved by induction on the variety dimension.
Suppose first that $\dim Z = 0$, i.e. Z is a point, then we are in fact in the split case. Let $F = k(Z)$, so we have

$$X_\gamma \times Z = X_F, \quad (A(\lambda_i)_\gamma)_F = A(\lambda_i)_F = \mathrm{End}_F(V_G(\lambda_i) \otimes F),$$
$$(\mathcal{W}(\lambda_i)_\gamma)_F = \mathcal{W}(\lambda_i)_F = \mathcal{V}_H(\lambda_i) \otimes \mathcal{V}_G^*(\lambda_i) \otimes F.$$

Since every module over $\mathrm{End}_F(V_G(\lambda_i) \otimes F)$ is isomorphic to $V_G(\lambda_i) \otimes F^n$ and

$$V_G^*(\lambda_i) \otimes F \otimes_{\mathrm{End}_F(V_G(\lambda_i)\otimes F)} V_G(\lambda_i) \otimes F \cong F,$$

one has

$$\begin{aligned}\phi_i(V_G(\lambda_i) \otimes F^n) = \mathcal{V}_H(\lambda_i) \otimes \mathcal{V}_G^*(\lambda_i) \otimes F \otimes_{\mathrm{End}_F(V_G(\lambda_i)\otimes F)} \mathcal{V}_G(\lambda_i) \otimes F^n = \\ = \mathcal{V}_H(\lambda_i) \otimes F^n.\end{aligned}$$

The above considerations show that we are in the setting of theorem 3 claiming $\sum \phi_i$ to be an isomorphism.
For the dimension greater then 0 we can write the localization sequence for all subvarieties $Z' \subset Z$ of the codimension one, so for the generic point $\eta = \operatorname{Spec} k(Z)$ one has

$$\begin{tikzcd}
\varinjlim_{Z'\subset Z} \bigoplus_{i=1}^{r} K'_*(Z', A(\lambda_i)_\gamma) \arrow[r] \arrow[d] & \bigoplus_{i=1}^{r} K'_*(Z, A(\lambda_i)_\gamma) \arrow[r] \arrow[d, "\sum \phi_i"] & \bigoplus_{i=1}^{r} K'_*(\eta, A(\lambda_i)_\gamma) \arrow[d] \\
\varinjlim_{Z'\subset Z} K'_*(X_\gamma \times Z') \arrow[r] & K'_*(X_\gamma \times Z) \arrow[r] & K'_*(X_\gamma \times \eta)
\end{tikzcd}$$

This sequence extends to the right and to the left with the shifts in the K-theory, and both the side vertical morphisms in each triple are isomorphisms by induction, so using the five lemma one concludes that the middle one is an isomorphism. □

Corollary 2. *In the notation of proposition 3 for a smooth Z one has*

$$\sum_{i=1}^{r} \phi_i : \bigoplus_{i=1}^{r} K_*(Z, A(\lambda_i)_\gamma) \longrightarrow K_*(X_\gamma \times Z).$$

Proof. One has $K_*(Z, A) = K'_*(Z, A)$ and $K_*(X_\gamma \times Z) = K'_*(X_\gamma \times Z)$. □

Theorem 4. *In the above notation there is an isomorphism*

$$\sum_{i=1}^{r} \psi_i : \bigoplus_{i=1}^{r} K_*(A(\lambda_i)_\gamma) \longrightarrow K_*(X_\gamma),$$

where

$$\psi_i(U) = \mathcal{W}(\lambda_i)_\gamma \otimes_{A(\lambda_i)_\gamma} U.$$

Proof. We can insert two copies of our morphism into the following diagram with the middle arrow from corollary 2 and Y being the splitting variety constructed in lemma 11.

$$\begin{tikzcd}
\bigoplus_{i=1}^{r} K_*(A(\lambda_i)_\gamma) \arrow[rrr, "\sum \psi_i"] \arrow[dd, "\chi(Y)\cdot"] \arrow[dr, "p^*"] & & & K_*(X_\gamma) \arrow[dd, "\chi(Y)\cdot"] \arrow[dl, "p^*"'] \\
& \bigoplus_{i=1}^{r} K_*(Y, A(\lambda_i)_\gamma) \arrow[r, "\sum \phi_i"] \arrow[dl, "p_*"'] & K_*(X_\gamma \times Y) \arrow[dr, "p_*"] & \\
\bigoplus_{i=1}^{r} K_*(A(\lambda_i)_\gamma) \arrow[rrr, "\sum \psi_i"] & & & K_*(X_\gamma)
\end{tikzcd}$$

A direct verification shows that the diagram is commutative. The vertical morphisms are just multiplications by $\chi(Y) = 1$ since they are equal to the composition p_*p^* with p being a projection from Y to a point. The above yields that our morphism $\sum \psi_i$ is a retraction of an isomorphism $\sum \phi_i$ hence is an isomorphism itself. □

Remark 8. It can be shown [Pan1] that $K_*(A(\lambda_i))$ depends only on the $Z(G)$ action on $V_G(\lambda_i)$. The explicit description of the arising algebras could be found in [Ti].

9 Examples.

9.1 Twisted flag variety.

The K-theory of twisted flag varieties was computed in [Pan1] and our computation gives the same description for the inner forms. Flag variety is a homogeneous variety G/P with a split semisimple G and a parabolic $P \subset G$, and this definition includes projective spaces, flag varieties in usual sense (for $G = SL_n$), split projective quadrics, etc.

There is a decomposition $P = LU$ into a semidirect product of a Levi subgroup and the unipotent radical of P hence there is a morphism

$$G/L \longrightarrow G/P$$

with the fiber U. Over the distinguished point eP the group P acts on U by

$$l_p u_p \cdot u = l_p u_p u l_p^{-1},$$

so one has a representation of P on the Lie algebra of U, and G/L is the corresponding vector bundle.
Levi subgroup is a reductive subgroup of maximal rank hence theorem 4 gives an explicit answer for the $K_*(G/L)$ and by the homotopy invariance of the K-theory this is an answer for $K_*(G/P)$. Note that the center $Z(G)$ acts trivially on the above bundle so one can twist it and get a new vector bundle

$$(G/L)_\gamma \longrightarrow (G/P)_\gamma,$$

hence the K-theory for an inner form of a flag variety could be computed by our method as well.

9.2 Even dimensional affine quadric.

This case corresponds to the inclusion $SO_{2n} \subset SO_{2n+1}$ and for the root systems it is $D_n \subset B_n$.
First of all we pass to a simply connected group,

$$Spin_{2n+1}/Spin_{2n} = SO_{2n+1}/SO_{2n}.$$

One has $[W(B_n) : W(D_n)] = 2$ so there are two elements in a basis. First element $V(\lambda_1)$ as usual corresponds to the trivial representation of $Spin_{2n}$ and as the second we can take one of the half-spin representations $V(\omega_{n-1})$, $V(\omega_n)$, since the algorithm from theorem 2 suggests one of the fundamental weights having an orbit consisting of two points.
After twisting with γ we get a quadric $X(q)$ defined by a quadratic form q, then the algebra $A(\lambda_2)_\gamma = C_0(q)$ is an even Clifford algebra for the form q [Ti]. Hence one has

$$K_*(X(q)) = K_*(k) \oplus K_*(C_0(q)).$$

This answer coincides with the one obtained in [Sw].

9.3 Quaternionic projective space.

We consider

$$HP^n = Sp_{2n+2}/(Sp_2 \times Sp_{2n})$$

as an algebraic model for the quaternionic projective space. The motivation comes from the fact that $HP^n(\mathbb{C})$ is homotopy equivalent to the usual quaternionic projective space $\mathbb{HP}^n$. An extensive treatment of the quaternionic flag varieties including the simplest case of projective spaces one can find in [PW]. The root systems in this case are $C_1 + C_n \subset C_{n+1}$, so the basis consists of

$$[W(C_{n+1}) : (W(C_1) \times W(C_n))] = n+1$$

elements. We have dealt with this case in section 5.3 and the basis consists of $\Lambda^i W$ for a regular representation W of Sp_{2n}. The center acts trivially on the even degrees and nontrivially on the odd ones, so one has

$$K_*(HP^n_\gamma) = K_*(k)^{\lceil \frac{n+1}{2} \rceil} \oplus K_*(A(\lambda_1)_\gamma)^{\lfloor \frac{n+1}{2} \rfloor}.$$

In the split case it reduces to $K_*(HP^n) = K_*(k)^{n+1}$, and it agrees with the result obtained in [PW].

9.4 Zero characteristic.

In this case we can treat non-reductive groups. When $char\, k = 0$ one has the Levi decomposition $G = L_G U_G$ of group G into a semidirect product of some reductive subgroup and the unipotent radical [McN], which in general fails in the positive characteristic. Also in this case the unipotent radical U_G splits, i.e. it has a filtration with vector factors [KMT] so the underlying variety is $\mathbb{A}^n$, which also can fail over nonperfect fields. Hence for the connected split groups of the same rank $H \subset G$ one has the following triangle.

$$\begin{tikzcd} G/L_H \arrow[r, "p_1"] \arrow[d, "p_2"] & G/H \\ L_G/L_H & \end{tikzcd}$$

The fibres of p_1 and p_2 are isomorphic to U_G and U_H respectively and both are affine spaces. One can show that both the projections define some vector bundles with the trivial action of the center $Z(L_G)$, so one can twist with a $L_G/Z(L_G)$-cocycle γ and from the homotopy invariance obtain that

$$K_*((G/H)_\gamma) = K_*((L_G/L_H)_\gamma).$$

The last one could be computed using the methods introduced in this paper.

9.5 Octonionic projective plain.

It could be shown [Ba] that

$$\mathbb{OP}^2 \cong G(F_4)/Spin(9),$$

where $G(F_4)$ stands for the compact form of a simple algebraic group with the root system F_4. We consider as an algebraic model

$$OP^2 = G(F_4)/Spin_9$$

with split $G(F_4)$. It corresponds to the root systems $B_4 \subset F_4$ treated in section 5.2. One has

$$[W(F_4) : W(B_4)] = 3,$$

and the corresponding representations for $Spin_9$ are $k, V(\omega_3), V(\omega_4)$, i.e. the trivial one, 84-dimensional $\Lambda^3 W$ and 16-dimensional spinor representation. Since the center of $G(F_4)$ is trivial the twisting does not produce interesting algebras, though it changes the variety. Hence one has

$$K_*(OP^2_\gamma) = K_*(k) \oplus K_*(k) \oplus K_*(k).$$

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