Decompositions of Motives of Generalized Severi-Brauer Varieties

Maksim Zhykhovich

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Abstract. Let p be a positive prime number and X be a Severi-Brauer variety of a central division algebra D of degree p^n, with $n \geq 1$. We describe all shifts of the motive of X in the complete motivic decomposition of a variety Y, which splits over the function field of X and satisfies the nilpotence principle. In particular, we prove the motivic decomposability of generalized Severi-Brauer varieties $X(p^m, D)$ of right ideals in D of reduced dimension p^m, $m = 0, 1, \ldots, n-1$, except the cases $p = 2$, $m = 1$ and $m = 0$ (for any prime p), where motivic indecomposability was proven by Nikita Karpenko.

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Contents

1. Introduction

Let F be an arbitrary field and p be a prime numbre. For any integer l, we write $v_p(l)$ for the exponent of the highest power of p dividing l.

Let D be a central division F-algebra of degree p^n, with $n \geq 1$. We write $X(p^m, D)$ for the generalized Severi-Brauer variety of right ideals in D of reduced dimension p^m for $m = 0, 1, \ldots, n$. In particular, $X(p^n, D) = \operatorname{Spec} F$ and $X(1, D)$ is the usual Severi-Brauer variety of D. The generalized Severi-Brauer varieties are twisted forms of grassmannians (see [11, §I.1.C]).

For each integer $m = 0, ..., n$ we define an *upper motive* $M_{m,D}$ in the category of Chow motives with coefficients in $\mathbb{F}_p$. This is the summand of the complete motivic decomposition of the variety $X(p^m, D)$ such that the 0-codimensional Chow group of $M_{m,D}$ is non-zero.

Let A be a central simple F-algebra, such that the p-primary component of A is Brauer equivalent to D. Let $\mathfrak{X}_A$ be the class of finite direct products of projective $(\operatorname{Aut} A)$-homogeneous F-varieties (the class $\mathfrak{X}_A$ includes the generalized Severi-Brauer varieties of the algebra A). Nikita Karpenko proved the following theorem [9, Theorem 3.8]. Any variety X from $\mathfrak{X}_A$ decomposes into a sum of shifts of the motives $M_{m,D}$ with $m \leq v_p(\operatorname{ind} A_{F(X)})$. This theorem shows that the motivic indecomposable summands $M_{m,D}$ of the generalized Severi-Brauer varieties $X(p^m, D)$ are some kind of "basic material" to constuct the motives of more general class of varieties. This gives us a motivation to understand the structure of the upper motives $M_{m,D}$ themselves. It was known that in the cases $m = 0$ (Severi-Brauer case, see Corollary 3.2) and $m = 1$, $p = 2$ ([9, Theorem 4.2]) the motive $M_{m,D}$ coincides with the whole motive of the variety $X(p^m, D)$ (that is, the motive of this variety is indecomposable). Taking into account these cases and the fact that any generalized Severi-Brauer variety $X(p^m, D)$ is p-incompressible [9, Theorem 4.3] (this condition is weaker than motivic indecomposability), one probably expected that the Chow motive with coefficients in $\mathbb{F}_p$ of any variety $X(p^m, D)$ is indecomposable. But, except the two already mentioned cases, the motivic decomposability of generalized Severi-Brauer variety $X(p^m, D)$ was proven in [14].

This article is an extended version of [14]. To show that the motive of the variety $X(p^m, D)$ is decomposable, we prove in [14] that some shifts of $M_{0,D}$ are the motivic summands of $X(p^m, D)$. Let Y be a F-variety satisfying the nilpotence principle and such that it splits over the function field of $X(1, D)$. For example, one can take for Y any generalized Severi-Brauer variety $X(p^m, D)$ and, more generally, any variety from $\mathfrak{X}_A$. The main result of the present article (Theorem 3.4) find all shifts of $M_{0,D}$ in the complete motivic decomposition of the variety Y in terms of some subgroups of rational cycles. These subgroups can be described in the case of generalized Severi-Brauer variety $X(p^m, D)$ (see Proposition 3.7). As consequence, we prove the motivic decomposability of these varieties in Corollary 3.8. With Theorem 3.4 in hand, we find in §4 more examples (comparing to [14]) of complete motivic decompositions of generalized Severi-Brauer varieties $X(p^m, D)$ and therefore we describe the upper motives $M_{m,D}$ in that cases. Theorem 3.4 also permits to prove differently (see Corollary 3.12, [3, Corollary 5]) a particular case of the following conjecture.

Conjecture 1.1. *Let D be a central division F-algebra. Let K/F be a field extension such that D_K is still division. Then $(M_{m,D})_K$ is still indecomposable.*

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2. Chow motives with finite coefficients

A variety is a separated scheme of finite type over a field. Our basic reference for Chow groups and Chow motives (including notations) is [4]. We fix an associative unital commutative ring Λ. Given a variety X over a field F, we write $\mathrm{Ch}(X)$ and $\mathrm{CH}(X)$ respectively for its Chow group with coefficients in Λ and for its integral Chow group. For a field extension L/F we denote by X_L the respective extension of scalars. An element of $\mathrm{Ch}(X_L)$ is called F-rational, if it lies in the image of the homomorphism $\mathrm{Ch}(X) \to \mathrm{Ch}(X_L)$.
Our category of motives is the category $\mathrm{CM}(F,\Lambda)$ of graded Chow motives with coefficients in Λ, [4, definition of § 64]. By a sum of motives we always mean the direct sum. We also write Λ for the motive $M(\mathrm{Spec} F) \in \mathrm{CM}(F,\Lambda)$. A Tate motive is the motive of the form $\Lambda(i)$ with i an integer.
Let X be a smooth complete variety over F and let M be a motive. We call M split if it is a finite sum of Tate motives. We call X split, if its integral motive $M(X) \in \mathrm{CM}(F,\mathbb{Z})$ (and therefore the motive of X with an arbitrary coefficient ring Λ) is split. We call M or X geometrically split, if it splits over a field extension of F. For a geometrically split variety X over F, we denote by $\bar{X}$ the scalar extension of X to a splitting field of its motive and we write $\bar{\mathrm{Ch}}(X)$ for the subring of F-rational cycles in $\mathrm{Ch}(\bar{X})$. Note that the rings $\mathrm{Ch}(\bar{X})$ and $\bar{\mathrm{Ch}}(X)$ are independent on the choice of a splitting field.
Over an extension of F the geometrically split motive M becomes isomorphic to a finite sum of Tate motives. We write $\mathrm{rk}\, M$ and $\mathrm{rk}_i\, M$ for respectively the number of all summands and the number of summands $\Lambda(i)$ in this decomposition, where i is an integer. Note that these two numbers do not depend on the choice of a splitting field extension.
We say that X satisfies the nilpotence principle, if for any field extension E/F and any coefficient ring Λ, the kernel of the change of field homomorphism $\mathrm{End}(M(X)) \to \mathrm{End}(M(X)_E)$ consists of nilpotents. Any projective homogeneous (under an action of a semisimple affine algebraic group) variety is geometrically split and satisfies the nilpotence principle, [4, Theorem 92.4 with Remark 92.3].

A complete decomposition of an object in an additive category is a finite direct sum decomposition with indecomposable summands. We say that the Krull-Schmidt principle holds for a given object of a given additive category, if every direct sum decomposition of the object can be refined to a complete one (in particular, a complete decomposition exists) and there is only one (up to a permutation of the summands) complete decomposition of the object. We have the following theorem:

THEOREM 2.1. ([2, Theorem 3.6 of Chapter I]). *Assume that the coefficient ring Λ is finite. The Krull-Schmidt principle holds for any shift of any summand of the motive of any geometrically split F-variety satisfying the nilpotence principle.*

We will use the following two statements in the next section.

LEMMA 2.2. *Assume that the coefficient ring Λ is a field. Let X be a split variety. Then the bilinear form $\mathfrak{b} : \mathrm{Ch}(X) \times \mathrm{Ch}(X) \to \Lambda$, $\mathfrak{b}(x, y) = \deg(x \cdot y)$ is non-degenerate.*

Proof. Since the motive of X decomposes into a finite sum of Tate motives, we have the following decomposition for the diagonal class $\Delta \in \mathrm{Ch}_{\dim X}(X \times X)$:

$$\Delta = a_1 \times b_1 + ... + a_n \times b_n\,,$$

where $a_1, ..., a_n$ and $b_1, ..., b_n$ are the homogeneous elements in $\mathrm{Ch}(X)$, such that for any $i, j = 1, ..., n$ the degree $\deg(a_i \cdot b_j) \in \Lambda$ is 0 for $i \neq j$ and 1 for $i = j$.
Note that $\dim_\Lambda \mathrm{Ch}(X) = \mathrm{rk}\, M(X) = n < \infty$. Therefore, to prove the lemma it suffices to show that $\mathrm{rad}\,\mathfrak{b} = \{0\}$. Suppose that $x \in \mathrm{rad}\,\mathfrak{b}$ (this means $\mathfrak{b}(x, y) = 0$ for any $y \in \mathrm{Ch}(X)$). Then we have

$$x = \Delta_*(x) = \sum_{i=1}^{n} \deg(x \cdot a_i) b_i = \sum_{i=1}^{n} \mathfrak{b}(x, a_i) b_i = 0\,.$$

□

LEMMA 2.3. *Assume that the coefficient ring Λ is finite. Let X be a variety satisfying the nilpotence principle. Let $f \in \mathrm{End}(M(X))$ and $1_E = f_E \in \mathrm{End}(M(X)_E)$ for some field extension E/F. Then $f^n = 1$ for some positive integer n.*

Proof. Since X satisfies the nilpotence principle, we have $f = 1 + \varepsilon$, where ε is nilpotent. Let n be a positive integer such that $\varepsilon^n = 0 = n\varepsilon$. Then $f^{n^n} = (1 + \varepsilon)^{n^n} = 1$ because the binomial coefficients $\binom{n^n}{i}$ for $i < n$ are divisible by n. □

3. MAIN RESULTS

Let p be a positive prime integer. The coefficient ring Λ is $\mathbb{F}_p$ in this section. Let F be a field. Let D be a central division F-algebra of degree p^n. We write $X(p^m, D)$ for the generalized Severi-Brauer variety of right ideals in D of reduced dimension p^m for $m = 0, 1, \ldots, n$.

Lemma 3.1. *Let E/F be a splitting field extension for $X = X(1, D)$. Then the subgroup of F-rational cycles in $\mathrm{Ch}_{\dim X}(X_E \times X_E)$ is generated by the diagonal class.*

Proof. By [7, Proposition 2.1.1], we have $\bar{\mathrm{Ch}}^i(X) = 0$ for $i > 0$. Since the (say, first) projection $X^2 \to X$ is a projective bundle, we have a (natural with respect to the base field change) isomorphism $\mathrm{Ch}_{\dim X}(X^2) \simeq \mathrm{Ch}(X)$. Passing to $\bar{\mathrm{Ch}}$, we get an isomorphism $\bar{\mathrm{Ch}}_{\dim X}(X^2) \simeq \bar{\mathrm{Ch}}(X) = \bar{\mathrm{Ch}}^0(X)$ showing that $\dim_{\mathbb{F}_p} \bar{\mathrm{Ch}}_{\dim X}(X^2) = 1$. Since the diagonal class in $\bar{\mathrm{Ch}}_{\dim X}(X^2)$ is non-zero, it generates all the group. $\square$

Corollary 3.2. (cf. [7, Theorem 2.2.1]). *The motive with coefficients in $\mathbb{F}_p$ of the Severi-Brauer variety $X = X(1, D)$ is indecomposable.*

Proof. To prove that our motive is indecomposable it is enough to show that $\mathrm{End}(M(X))= \mathrm{Ch}_{\dim X}(X \times X)$ does not contain nontrivial projectors. Let $\pi \in \mathrm{Ch}_{\dim X}(X \times X)$ be a projector. By Lemma 3.1, π_E is zero or equal to 1_E. Since X satisfies the nilpotence principle, π is nilpotent in the first case, but also idempotent, therefore π is zero. Lemma 2.3 gives us $\pi = 1$ in the second case. $\square$

Nikita Karpenko proved the motivic indecomposability of generalized Severi-Brauer varieties also in the case $p = 2$, $m = 1$.

Theorem 3.3. (cf. [9, Theorem 4.2]). *Let D be a central division F-algebra of degree 2^n with $n \geq 1$. Then the motive with coefficients in $\mathbb{F}_2$ of the variety $X(2, D)$ is indecomposable.*

Corollary 3.8 of the following main theorem will show that Corollary 3.2 and Theorem 3.3 give us the only cases when the motive of generalized Severi-Brauer variety is indecomposable.

Theorem 3.4. *Let D be a central division F-algebra of degree p^n with $n \geq 1$. Let X be the Severi-Brauer variety $X(1, D)$ and Y be a variety satisfying nilpotence principle, such that Y is split over the function field of X. Then for any integer k the number of copies $M(X)(k)$ in the complete motivic decomposition of Y is equal to $\dim_{\mathbb{F}_p} f_* \bar{\mathrm{Ch}}_{\dim Y - k}(X \times Y)$, where f is a projection onto the second factor.*

Proof. We fix an integer k and we note the motive $M(X)(k)$ simply by M. Let r be the number of copies of M in the complete motivic decomposition of Y. We note $V := f_* \bar{\mathrm{Ch}}_{\dim Y - k}(X \times Y)$ and $r' := \dim_{\mathbb{F}_p} V$. We want to show that $r = r'$.

Let $A_1, ..., A_m$ and $B_1, ..., B_n$ be the motives. We recall that a morphism between the motives $\bigoplus_{i=1}^m A_i$ and $\bigoplus_{j=1}^n B_j$ is given by an $n \times m$-matrix of morphisms $A_i \to B_j$. The composition of morphisms is the matrix multiplication.

The motive $M^{\oplus r}$ is a summand of the motive $M(Y)$. Therefore there exist two morphisms $\alpha = (\alpha_1, ..., \alpha_r)^t \in \mathrm{Hom}(M^{\oplus r}, M(Y))$ and $\beta = (\beta_1, ..., \beta_r) \in \mathrm{Hom}(M(Y), M^{\oplus r})$, such that

$$\beta \circ \alpha = (\beta_j \circ \alpha_i)_{1 \leq i,j \leq r} = (\delta_{i,j})_{1 \leq i,j \leq r} ,$$

where $(\delta_{i,j})_{1 \leq i,j \leq r}$ is the identity morphism in $\mathrm{Hom}(M^{\oplus r}, M^{\oplus r})$ (that is $\delta_{i,j}$ is zero if $i \neq j$ and $\delta_{i,j}$ is the diagonal class Δ in $\mathrm{Corr}_0(X, X)$ if $i = j$).
Let $E = F(X)$, then E/F is a splitting field extension for the varieties X and Y (here we use the condition of the theorem) and $X_E \simeq \mathbb{P}^d$, where $d = p^n - 1$. We know that $\Delta_E = \sum_{i=0}^{d} h^i \times h^{d-i}$, where h is the hyperplane class in $\mathrm{Ch}^1(X_E)$. For any $1 \leq i \leq r$ we have

$$(\beta_i)_E \circ (\alpha_i)_E = (\delta_{i,i})_E = \Delta_E =$$
$$= h^0 \times h^d + \sum_{i=1}^{d} h^i \times h^{d-i} = [X_E] \times [pt] + \sum_{i=1}^{d} h^i \times h^{d-i} ,$$

where $[pt]$ is the class of a rational point in $\mathrm{Ch}(X_E)$. Therefore the correspondences $\beta_i \in \mathrm{Ch}_{\dim Y - k}(Y_E \times X_E)$ and $\alpha_i \in \mathrm{Ch}_{d+k}(X_E \times Y_E)$ have to be of the following form:

$$(\beta_i)_E = b_i \times [pt] + \dots , \tag{3.5}$$

where $b_i \in \mathrm{Ch}^k(Y_E)$ is non-zero and where "…" stands for a linear combination of only those terms whose first factor has codimension $> k$,

$$(\alpha_i)_E = [X_E] \times b_i^* + \dots , \tag{3.6}$$

where $b_i^* \in \mathrm{Ch}_k(Y_E)$ is such that $\deg(b_i \cdot b_i^*) = 1$ and where "…" stands for a linear combination of only those terms whose second factor has dimension $> k$. For any $i \neq j$ we have $(\beta_j)_E \circ (\alpha_i)_E = 0$, this implies that $\deg(b_j \cdot b_i^*) = 0$. Therefore the system of vectors $\{b_1^*, ..., b_r^*\}$ from the vector space $\mathrm{Ch}(Y_E)$ is dual to the system of vectors $\{b_1, ..., b_r\}$ with respect to the bilinear form $\mathfrak{b} : \mathrm{Ch}(Y_E) \times \mathrm{Ch}(Y_E) \to \mathbb{F}_p$, $\mathfrak{b}(x_1, x_2) = \deg(x_1 \cdot x_2)$. It follows that the vectors $b_1, ..., b_r$ are linearly independent. Since $b_i = f_*((\beta_i^t)_E)$, then $b_i \in V$ for any $1 \leq i \leq r$. Therefore $r \leq r'$.
Let now $b_1, ..., b_{r'}$ be a basis of V. We want to show that $M^{\oplus r'}$ is a motivic summand of Y. By the definition of V, there exist correspondences $\beta_1, ..., \beta_{r'} \in \mathrm{Ch}_{\dim Y - k}(Y \times X)$ of the form (3.5), such that $b_i = f_*((\beta_i^t)_E)$. Since the variety Y_E is split, then by Lemma 2.2 the bilinear form $\mathfrak{b}$ is non-degenerate. It follows that there exists a system of vectors $\{b_1^*, ..., b_{r'}^*\}$ from the vector space $\mathrm{Ch}(Y_E)$, which is dual to the system of vectors $\{b_1, ..., b_{r'}\}$. For any $1 \leq i \leq r'$ we construct the correspondence $\alpha_i \in \mathrm{Ch}_{d+k}(X \times Y)$, such that $(\alpha_i)_E$ is of the form (3.6), by the following way. The pull-back homomorphism

$$g : \mathrm{Ch}(X \times Y) \to \mathrm{Ch}(Y_{F(X)}) = \mathrm{Ch}(Y_E)$$

with respect to the morphism $Y_{F(X)} = (\mathrm{Spec}\, F(X)) \times Y \to X \times Y$ given by the generic point of X is surjective by [4, Corollary 57.11]. We define

$\alpha_i \in \mathrm{Ch}(X \times Y)$ as a cycle whose image in $\mathrm{Ch}(Y_E)$ under the surjection g is b_i^*. We have $(\alpha_i)_E = [X_E] \times b_i^* + \dots$, so $(\alpha_i)_E$ is of the form (3.6).
The r'-tuples $(\alpha_1, ..., \alpha_{r'})^t$ and $(\beta_1, ..., \beta_{r'})$ give us respectively two morphisms $\alpha \in \mathrm{Hom}(M^{\oplus r'}, M(Y))$ and $\beta \in \mathrm{Hom}(M(Y), M^{\oplus r'})$. By the construction of α and β, the matrix $(\mathrm{mult}((\beta_j)_E \circ (\alpha_i)_E))_{1\leq i,j\leq r}$ is an identity matrix. Then, by Lemma 3.1, $\beta_E \circ \alpha_E = ((\beta_j)_E \circ (\alpha_i)_E)_{1\leq i,j\leq r} = 1_E$, where we note simply by 1 the identity morphism $((\delta_{i,j}))_{1\leq i,j\leq r}$ in $\mathrm{Hom}(M^{\oplus r}, M^{\oplus r})$. Let $\mathfrak{X}$ be a disjoint union of r' copies of X, then $\mathrm{Hom}(M(\mathfrak{X}), M(\mathfrak{X})) = \mathrm{Hom}(M^{\oplus r'}, M^{\oplus r'})$. According to [4, Theorem 92.4] the variety $\mathfrak{X}$ satisfies the nilpotence principle. By Lemma 2.3, there exist a positive integer n, such that $(\beta \circ \alpha)^n = 1$ (we apply Lemma 2.3 to the variety $\mathfrak{X}$ and to the morphism $\beta \circ \alpha \in \mathrm{Hom}(M(\mathfrak{X}), M(\mathfrak{X})))$. The morphisms α and $(\beta \circ \alpha)^{n-1} \circ \beta$ give the isomorphism between the motive $M^{\oplus r'}$ and a direct summand of $M(Y)$. Therefore $r' \geq r$ and then finally $r' = r$. □

Proposition 3.7. *Let D be a central division F-algebra of degree p^n with $n \geq 1$. Let X and Y be respectively the varieties $X(1, D)$ and $X(p^m, D)$, $0 \leq m < n$. Let E/F be a splitting field extension for the variety X, let T_1 and T_{p^m} be the tautological bundles of rank 1 and p^m on X_E and Y_E respectively. Then the subring of F-rational cycles in $\mathrm{Ch}(X_E \times Y_E)$ is generated by the Chern classes of the vector bundle $T_1 \boxtimes (-T_{p^m})^\vee$ (we lift the bundles T_1 and T_{p^m} on $X_E \times Y_E$ and then take a product).*

Proof. Let Tav be the tautological vector bundle on X. The product $X \times Y$ considered over X (via the first projection) is isomorphic (as a scheme over X) to the Grassmann bundle $G_r(Tav)$ of r-dimensional subspaces in Tav (cf. [6, Proposition 4.3]), where $r = p^n - p^m$. Let T be the tautological r-dimensional vector bundle on $G_r(Tav)$. By [5, Example 14.6.6], the Chow ring $\mathrm{Ch}(G_r(Tav))$ as an algebra over $\mathrm{Ch}(X)$ is generated by Chern classes $c_0(T), c_1(T), ..., c_r(T)$. By [7, Proposition 2.1.1], we have $\bar{\mathrm{Ch}}(X) = \bar{\mathrm{Ch}}^0(X) = \mathbb{Z} \cdot [X_E]$. Therefore the Chow ring $\bar{\mathrm{Ch}}(X \times Y) \simeq \bar{\mathrm{Ch}}(G_r(Tav))$ is generated (as a ring) by Chern classes $c_0(T_E), ..., c_r(T_E)$. Since there exists an isomorphism (cf. [6, Proposition 4.3]): $T_E \simeq T_1 \boxtimes (-T_{p^m})^\vee$, we are done. □

Corollary 3.8. *The motive with coefficients in $\mathbb{F}_p$ of the variety $X(p^m, D)$ is decomposable for $p = 2$, $1 < m < n$ and for $p > 2$, $0 < m < n$. In these cases $M(X(1, D))(k)$ is a summand of $M(X(p^m, D))$ for $2 \leq k \leq p^n - p^m$.*

Proof. We use the notations: $X = X(1, D)$, $Y = X(p^m, D)$, $d = \dim(X(1, D)) = p^n - 1$, $r = p^n - p^m$. Let $E = F(X)$, then E/F is a splitting field extension for the variety X (and also for Y). Over the field E the algebra D becomes isomorphic to $\mathrm{End}_E(V)$ for some E-vector space V of dimension $d + 1 = p^n$. We have $X_E \simeq \mathbb{P}^d(V)$ and $Y_E \simeq G_{p^m}(V)$. Let T_1 and T_{p^m} be the tautological bundles of rank 1 and p^m on X_E and Y_E respectively. We note by T the r-dimensional vector bundle $T_1 \boxtimes (-T_{p^m})^\vee$ on $X_E \times Y_E$. By Proposition

3.7, the ring $\bar{\mathrm{Ch}}(X \times Y)$ is generated by Chern classes of the vector bundle T. Let $h = c_1(T_1) \in \mathrm{Ch}^1(X_E)$ (then $-h$ is the hyperplane class in $\mathrm{Ch}^1(X_E)$) and $c_i = c_i((-T_{p^m})^\vee) \in \mathrm{Ch}^i(Y_E)$, $0 \le i \le r$. Then by [5, Remark 3.2.3(b)]

$$c_t(T) = c_t(T_1 \boxtimes (-T_{p^m})^\vee) = \sum_{i=0}^{r}(1 + (h \times 1)t)^{r-i}(1 \times c_i)t^i\,. \tag{3.9}$$

It follows from the conditions of the corollary that the binomial coefficients $\binom{p^n-p^m}{2}$, $\binom{p^n-p^m}{p^m-1}$ are divisible by p and $\binom{p^n-p^m-1}{p^m-2} \equiv (-1)^{p^m-2} \bmod p$. Therefore

$$c_1(T) = (p^n - p^m)h \times 1 + 1 \times c_1 = 1 \times c_1\,,$$

$$c_2(T) = \binom{p^n - p^m}{2} h^2 \times 1 + (p^n - p^m - 1)h \times c_1 + 1 \times c_2 = -h \times c_1 + 1 \times c_2\,,$$

$$\begin{aligned} c_{p^m-1}(T) &= \binom{p^n - p^m}{p^m - 1} h^{p^m-1} \times 1 + \binom{p^n - p^m - 1}{p^m - 2} h^{p^m-2} \times c_1 + \ldots = \\ &= (-1)^{p^m-2} h^{p^m-2} \times c_1 + \ldots\,, \end{aligned}$$

where "…" stands for a linear combination of only those terms whose second factor has codimension > 1. For the top Chern class we have:

$$c_r(T) = \sum_{i=0}^{r} h^{r-i} \times c_i\,.$$

For any integer $k \ge 2$ we define $\beta_k = c_r(T)c_{p^m-1}(T)c_2(T)c_1(T)^{k-2} = (-h)^d \times c_1^k + \ldots = [pt] \times c_1^k + \ldots$, where "…" stands for a linear combination of only those terms whose second factor has codimension $> k$ and where $[pt]$ is the class of a rational point in $\mathrm{Ch}(X_E)$. Let $f : X \times Y \to X$ be a projection onto the first factor. The cycle β_k is F-rational and $f_*(\beta_k) = c_1^k$. By [5, Example 14.6.6], the cycle c_1^k is non-zero for $2 \le k \le p^n - p^m$. Therefore $\dim_{\mathbb{F}_p} f_* \bar{\mathrm{Ch}}_{\dim Y - k}(X \times Y) \ge 1$ for $2 \le k \le p^n - p^m$. The statement follows from Theorem 3.4. □

REMARK 3.10. The Corollary 3.8 also gives us some information about the integral motive of the variety $X(p^m, D)$. Indeed, according to [12, Corollary 2.7] the decomposition of $M(X(p^m, D))$ with coefficients in $\mathbb{F}_p$ lifts (and in a unique way) to the coefficients $\mathbb{Z}/p^N\mathbb{Z}$ for any $N \ge 2$. Then by [12, Theorem 2.16] it lifts to $\mathbb{Z}$ (uniquely for $p = 2$ and $p = 3$ and non-uniquely for $p > 3$). See also Remark 4.14.

REMARK 3.11. Let l be an integer such that $0 < l < p^n$ and $\gcd(l, p) = 1$. The complete decomposition of the motive $M(X(l, D))$ with coefficients in $\mathbb{F}_p$ is described in [1, Proposition 2.4].

COROLLARY 3.12. *Let D be a central division F-algebra of p-primary index. Let K/F be a field extension, such that D_K is still division. Then the motive $(M_{1,D})_K$ is still indecomposable.*

Proof. We note by X and Y respectively the varieties $X(1,D)$ and $X(p,D)$. We note by M the motive $M(X)$. By [9, Theorem 3.8] the complete motivic decomposition of the variety Y consists of the motive $M_{1,D}$ and of the sum of motives M (we neglect the shifts in this proof). Suppose that the motive $(M_{1,D})_K$ is decomposable, then by the same theorem, M_K is a summand of $(M_{1,D})_K$. Therefore, the number of motives M_K in the complete motivic decomposition of Y_K is greater then the number of motives M in the complete motivic decomposition of Y. Let E/K be a splitting field extension for the algebra D. By Proposition 3.7, the subspace of K-rational cycles in $\mathrm{Ch}(X_E \times Y_E)$ coincides with the subspace of F-rational cycles in $\mathrm{Ch}(X_E \times Y_E)$. Therefore the Theorem 3.4 gives a contradiction. □

4. Complete motivic decompositions

In the Corollary 3.8 we proved that the motive of the variety $X(p^m, D)$ is decomposable for $p = 2$, $1 < m < n$ and for $p > 2$, $0 < m < n$. Moreover, in these cases the Corollary 3.8 gives us a list of some motivic summands of the variety $X(p^m, D)$. By duality, we can extend this list. It happens, that in two small-dimensional cases $p = 3$, $m = 1$, $n = 2$ and $p = 2$, $m = 2$, $n = 3$ this is already a complete list of indecomposable motivic summands of the variety $X(p^m, D)$. Note that in general it is not true (see Example 4.8).

Example 4.1. In this example we describe the complete motivic decomposition of $Y := X(3, D)$ for a division F-algebra D of degree 9. We note by X the variety $X(1, D)$ and by M the motive $M(X)$. Note that $\dim X = 8$ and $\dim Y = 18$.
By [9, Theorem 3.8], any indecomposable motivic summand of Y, besides the upper motive $M_{1,D}$, is some shift of M. By Corollary 3.8, the motives $M(2)$, $M(3)$, $M(4)$, $M(5)$, $M(6)$ and by duality $M(8)$, $M(7)$ are direct summands of $M(Y)$. Suppose that there is at least one more motive $M(t)$ (for some integer $t \geq 0$) in the complete motivic decomposition of Y. Since by [9, Theorem 4.3] the variety Y is 3-incompressible, we have

$$\mathrm{rk}_0 M(Y) = \mathrm{rk}_0 M_{1,D} = \mathrm{rk}_{\dim Y} M_{1,D} = \mathrm{rk}_{\dim Y} M(Y) = 1\,.$$

It follows that $\mathrm{rk}_0 M(t) = \mathrm{rk}_{\dim Y} M(t) = 0$. We have

$$1 \leq t \leq \dim Y - \dim X - 1 = 9\,.$$

Since the decomposition of any of eight motives $M(2)$, $M(3)$, ..., $M(8), M(t)$ into the sum of Tate motives over the splitting field contains a Tate motive $\mathbb{F}_3(9)$, then $\mathrm{rk}_9 M_{1,D} \leq \mathrm{rk}_9 M(Y) - 8$. According to [13, §2.5], we have $\mathrm{rk}_9 M(Y) = 8$, therefore $\mathrm{rk}_9 M_{1,D} = 0$.
By [8, Corollary 10.19], we have the following motivic decomposition of Y over the function field $L = F(Y)$:

$$M(Y)_L = \bigoplus_{i+j+k=3} M(X(i,C) \times X(j,C) \times X(k,C))\,, \tag{4.2}$$

where C is a central division L-algebra (of degree 3) Brauer-equivalent to D_L. Note that the triples $(3,0,0)$, $(0,3,0)$, $(0,0,3)$ correspond to three Tate motives $\mathbb{F}_3$, $\mathbb{F}_3(9)$ and $\mathbb{F}_3(18)$. Let $\widetilde{M} = M(X(1,C))$, then by [8, Example 10.20], $M_L = \widetilde{M} \oplus \widetilde{M}(3) \oplus \widetilde{M}(6)$. It follows that the complete decomposition of M_L does not contain $\mathbb{F}_3(9)$. Therefore $\mathbb{F}_3(9)$ is a direct motivic summand of $(M_{1,D})_L$ and we have a contradiction with $\mathrm{rk}_9\, M_{1,D} = 0$.
The complete motivic decomposition of the variety $X(3,D)$ with coefficients in $\mathbb{F}_3$ is the following one:
(4.3)
$$M(X(3,D)) = M_{1,D} \oplus M(2) \oplus M(3) \oplus M(4) \oplus M(5) \oplus M(6) \oplus M(7) \oplus M(8)\,.$$

Example 4.4. Similarly, as in the previous example, we can find the complete motivic decomposition of $Y := X(4,D)$ for a division F-algebra D of degree 8. We note by M the motive $M(X(1,D))$.
By Corollary 3.8, the motives $M(2)$, $M(3)$, $M(4)$ and by duality $M(7)$, $M(6)$, $M(5)$ are direct summands of $M(Y)$. We have

$$M(X(4,D)) = M(2) \oplus ... \oplus M(7) \oplus N$$

for some motive N. Assume that N is decomposable. Then by [9, Theorem 3.8], and Theorems 3.2, 3.3, the motive N has an indecomposable summand which is some shift of either $M_{0,D} = M$ or $M_{1,D} = M(X(2,D))$. But the second case is impossible because

$$70 = \binom{8}{4} = \mathrm{rk}\, M(Y) < 6\,\mathrm{rk} M + \mathrm{rk}\, M(X(2,D)) = 6 \cdot 8 + \binom{8}{2} = 76$$

(see [9, Example 2.18] for the computations of ranks). Therefore $M(t)$ is a summand of N for some integer t.
According to [8, Corollary 10.19], we can write the complete decomposition of N over the function field $L = F(Y)$:

$$N_L = \mathbb{F}_2 \oplus \widetilde{M}(1) \oplus M(X(2,C))(4) \oplus M(X(2,C))(8) \oplus \widetilde{M}(12) \oplus \mathbb{F}_2(16)\,,$$

where C is a central division L-algebra (of degree 4) Brauer-equivalent to D_L and where $\widetilde{M} = M(X(1,C))$. It follows from this decomposition that the motive $M(t)_L = \widetilde{M}(t) \oplus \widetilde{M}(t+4)$ can not be a direct summand of N_L. We have a contradiction. Therefore the motive N is indecomposable and $N \simeq M_{2,D}$.
Now we can write the complete motivic decomposition of $X(4,D)$ with coefficients in $\mathbb{F}_2$:

(4.5) $$M(X(4,D)) = M_{2,D} \oplus M(2) \oplus M(3) \oplus M(4) \oplus M(5) \oplus M(6) \oplus M(7)\,.$$

Let us consider the following class of generalized Severi-Brauer varieties.

Definition 4.6. We say that the generalized Severi-Brauer variety $X(p^m, D)$ is *of type* 0, if the complete decomposition of $M(X(p^m,D))$ consists only of the upper motive $M_{m,D}$ and some (possibly zero) shifts of the motive $M_{0,D} = M(X(1,D))$.

For example, by [9, Theorem 3.8], the variety $X(p,D)$ is of this type. Let Y be a generalized Severi-Brauer $X(p^m,D)$ variety of type 0. By Theorem 3.4, the subspace of F-rational cycles in $\mathrm{Ch}(X_E \times Y_E)$ describes the complete motivic decomposition of Y, where $X = X(1,D)$, $E = F(X)$. Note that the structure of the ring $\mathrm{Ch}(X_E \times Y_E) = \mathrm{Ch}(X_E) \times \mathrm{Ch}(Y_E)$ is well-known (cf. [5, § 14]) and by Proposition 3.7 we can compute the subring $\bar{\mathrm{Ch}}(X \times Y) \subset \mathrm{Ch}(X_E \times Y_E)$. Therefore we can say that the complete motivic decomposition of any generalized Severi-Brauer variety $X(p^m,D)$ of type 0 can be "theoretically" found in a finite time using computer.

Remark 4.7. We do not possess a single example of a variety $X(p^m,D)$, which is not of type 0. Therefore, it may happen that the generalized Severi-Brauer variety $X(p^m,D)$ is always of type 0 (for any division F-algebra D of degree p^n and for any integer m, $0 \le m \le n$). Note that if this is true, then Conjecture 1.1 holds (one can follow the lines of the proof of Corollary 3.12).

Example 4.8. Let D be a central division F-algebra of degree 27. In this example we find complete motivic decomposition of the variety $Y = X(3,D)$, which is of type 0. We take the same notations as in the proof of Corollary 3.8: $X = X(1,D)$, $E = F(X)$, $T = T_1 \boxtimes (-T_3)^\vee$, where T_1 and T_3 are the tautological bundles of rank 1 and 3 on X_E and Y_E respectively (the vector bundle T is of the rank 24). We note also by V_* the graded $\mathbb{F}_3$-vector space $f_* \bar{\mathrm{Ch}}_{\dim Y - *}(X \times Y)$, where f is a projection onto the second factor.
By Theorem 3.4, for any integer k the number of motives $M(k)$ in the complete motivic decomposition of Y is equal to $\dim_{\mathbb{F}_3} V_k$, where $M = M(X)$. By duality, this number is also equal to the number of motives $M(\dim Y - \dim X - k) = M(46-k)$ in the same decomposition. Therefore the vector space $V_{\le 23}$ describes the complete motivic decomposition of Y.
Let $h = c_1(T_1) \in \mathrm{Ch}^1(X_E)$ and $c_i = c_i((-T_3)^\vee) \in \mathrm{Ch}^i(Y_E)$, $0 \le i \le 24$. Using the formula 3.9 we can compute the following Chern classes of the vector bundle T:

$$c_1(T) = 1 \times c_1\,, \quad c_2(T) = -h \times c_1 + 1 \times c_2\,,$$
$$c_7(T) = 1 \times c_7\,, \quad c_8(T) = -h \times c_7 + 1 \times c_8\,,$$
$$c_{24}(T) = h^{24} \times 1 + \textstyle\sum_{i=1}^{24} h^{24-i} \times c_i\,.$$

We have:

$$c_1^2 = f_*\big(c_{24}(T)(c_2(T))^2\big) \in V_2\,, \quad c_1c_7 = f_*\big(c_{24}(T)c_2(T)c_8(T)\big) \in V_8\,,$$
$$c_7^2 = f_*\big(c_{24}(T)(c_8(T))^2\big) \in V_{14}\,.$$

Also we have the following property:

$$x \in V_* \Rightarrow (xc_1 \in V_{*+1} \quad \text{and} \quad xc_7 \in V_{*+7})\,. \tag{4.9}$$

Indeed, if $x \in V_*$, then $x = f_*(y)$ for some $y \in \bar{\mathrm{Ch}}_{\dim Y - *}(X_E \times Y_E)$. Therefore $xc_1 = f_*(y \cdot c_1(T)) \in V_{*+1}$ and $xc_7 = f_*(y \cdot c_7(T)) \in V_{*+7}$.
This property gives us the following elements in V_i:

(4.10)
$$\begin{array}{llll} c_1^i & \text{if } 2 \le i \le 7\,, & c_1^i, c_1^{i-7}c_7, c_1^{i-14}c_7^2 & \text{if } 14 \le i \le 20\,, \\ c_1^i, c_1^{i-7}c_7 & \text{if } 8 \le i \le 13\,, & c_1^i, c_1^{i-7}c_7, c_1^{i-14}c_7^2, c_1^{i-21}c_7^3 & \text{if } 21 \le i \le 23\,. \end{array}$$

We also define a sequence b_i, $i \in \mathbb{Z}$:

$$b_i = \begin{cases} 0 & \text{if } i < 2 \\ 1 & \text{if } 2 \le i \le 7 \\ 2 & \text{if } 8 \le i \le 13 \\ 3 & \text{if } 14 \le i \le 20 \\ 4 & \text{if } 21 \le i \le 23 \\ b_{46-i} & \text{if } i > 23. \end{cases}$$

Note that for any $i \le 23$ the number of elements lying in V_i from the list 4.10 is equal to b_i.

We are going to show that all elements from the list 4.10 are linearly independent (to apply then Theorem 3.4). The $\mathbb{F}_3$-vector space V_* is a subspace of $\mathrm{Ch}^*(Y_E)$. We note $\tilde{c_i} = c_i(T_3^\vee)$, $i = 1, 2, 3$, where T_3 is the tautological bundle of rank 3 on Y_E. According to [5, Example 14.6.6] the graded ring $\mathrm{Ch}^*(Y_E)$ is generated by Chern classes $\tilde{c_1}, \tilde{c_2}, \tilde{c_3}, c_1, ..., c_{24}$ modulo the homogeneous relations

$$c_r + c_{r-1}\tilde{c_1} + c_{r-2}\tilde{c_2} + c_{r-3}\tilde{c_3} = 0 \quad \text{for} \quad r = 1, ..., 27\,, \tag{4.11}$$

where $c_i = 0$ for $i < 0$ or $i > 24$. It follows that the graded ring $\mathrm{Ch}^*(Y_E)$ is generated by only three elements $\tilde{c_1}, \tilde{c_2}, \tilde{c_3}$ modulo some homogeneous relations of degree greater than 23. Therefore we have an isomorphism:

$$\mathrm{Ch}^{*\,\le 23}(Y_E) \simeq \mathbb{F}_3[\tilde{c_1}, \tilde{c_2}, \tilde{c_3}]_{\le 23}\,.$$

Using relations 4.11 we can compute that $c_1 = -\tilde{c_1}$ and $c_7 = -\tilde{c_1}^7 + \tilde{c_1}^4\tilde{c_3} - \tilde{c_1}^3\tilde{c_2}^2 + \tilde{c_1}\tilde{c_2}^3$.

Now we consider the elements from the list 4.10 as polynomials in $\mathbb{F}_3[\tilde{c_1}, \tilde{c_2}, \tilde{c_3}]$. To show that all of them are linearly independent, it suffices to check this for four elements $c_1^{21}, c_1^{14}c_7, c_1^7c_7^2, c_7^3$ (they are in our list) from V_{21}. Since the polynomial ring $\mathbb{F}_3[\tilde{c_1}, \tilde{c_2}, \tilde{c_3}]$ is factorial and c_7 is not divisible by $\tilde{c}_1^2$, then for any $\alpha, \beta, \gamma, \delta \in \mathbb{F}_3$ we have

$$\alpha c_1^{21} + \beta c_1^{14}c_7 + \gamma c_1^7 c_7^2 + \delta c_7^3 = 0 \quad \Longrightarrow \quad \alpha = \beta = \gamma = \delta = 0\,.$$

Since all elements from the list 4.10 are linearly independent, then $\dim_{\mathbb{F}_3} V_i \ge b_i$ for $i \le 23$. Therefore for any integer i the motive $M^{\oplus b_i}(i)$ is a direct summand of $M(Y)$. Indeed, the statement follows from Theorem 3.4 for $i \le 23$ and by duality it is also true for $i > 23$. We have

$$M(Y) = \oplus_{i\in\mathbb{Z}} M^{\oplus b_i}(i) \oplus N \tag{4.12}$$

for some motive N over F.

Now we want to understand the complete decomposition of N. Let L be a function field of the variety Y and C be a central division L-algebra (of degree

9) Brauer-equivalent to D_L. Using the motivic decomposition similar to the decomposition 4.2 from Example 4.1, we can show that the complete decomposition of $M(Y)_L$ contains three indecomposable motives: $M_{1,C}$, $M_{1,C}(27)$, $M_{1,C}(54)$. Moreover any other summand in the complete motivic decomposition of $M(Y)_L$ is a shift of the motive $\widetilde{M} := M(X(1,C))$. We know that $M_L = \widetilde{M} \oplus \widetilde{M}(9) \oplus \widetilde{M}(18)$. It follows that $N_L = M_{1,C} \oplus M_{1,C}(27) \oplus M_{1,C}(54) \oplus N'$ for some motive N' over L and N' is a sum of shifts of the motive $\widetilde{M}$. Note that if $M(k)$ is direct summand of N for some integer k, then $M_L(k)$ is a direct summand of N'.

Let S be a direct summand of the motive of a geometrically split variety. We write $P(S,t)$ for the Poincaré polynomial of S:

$$P(S,t) = \sum_{i\geq 0} (\mathrm{rk}_i\, S) \cdot t^i .$$

Let us find the Poincaré polynomial of the motive N'. We have

$$P(N',t) = P(M(Y),t) - (1+t^{27}+t^{54})P(M_{1,C},t) - \sum_{i\geq 0} b_i t^i P(M,t) .$$

Using the following formulas

$$P(M(Y),t) = \frac{(1-t^{27})(1-t^{26})(1-t^{25})}{(1-t)(1-t^2)(1-t^3)} \text{ , (according to [13, §2.5]),}$$

$$P(M,t) = \frac{1-t^{27}}{1-t} = \sum_{i=0}^{26} t^i ,$$

$$P(M_{1,C},t) = t^6 + t^{12} + \sum_{i=0}^{26} t^i \text{ , (by Example 4.3) ,}$$

we can compute $P(N',t)$. Since N' is a sum of shifts of the motive $\widetilde{M}$ then $P(N',t)$ is divisible by $P(\widetilde{M},t) = (1-t^9)/(1-t) = 1+t+...+t^8$. Let $Q(t)$ be a quotient of these two polynomials. After computations, we have

$$Q(t) = t^7+t^{13}+t^{16}+t^{18}+t^{19}+t^{20}+t^{22}+t^{24}+t^{26}+t^{28}+t^{29}+t^{30}+t^{34}+t^{35}+ t^{36}+t^{38}+t^{40}+t^{42}+t^{44}+t^{45}+t^{46}+t^{48}+t^{51}+t^{57} .$$

Now if $M(k)$ is direct summand of N for some integer k, then $M_L(k) = \widetilde{M}(k) \oplus \widetilde{M}(k+9) \oplus \widetilde{M}(k+18)$ is a direct summand of N'. Therefore in this case the decomposition of $Q(t)$ contains $t^k + t^{k+9} + t^{k+18} = P(M_L(k),t)/P(\widetilde{M},t)$. Only two values $k = 20$ and $k = 26$ satisfy this condition. Note that if complete decomposition of the motive N contains $M(20)$ then by duality it contains also $M(26)$. It follows that the question of the complete motivic decomposition of Y reduces to the question either $\dim_{\mathbb{F}_3} V_{20} = 3$ or $\dim_{\mathbb{F}_3} V_{20} = 4$? Let us show that we are in the second case.

Consider the following cycle e from V_{20}

$$e = f_*(c_2^{11}(-T)c_3^8(-T)) = -\tilde{c}_1^{17}\tilde{c}_3 + \tilde{c}_1^{16}\tilde{c}_2^2 - \tilde{c}_1^{14}\tilde{c}_2^3 - \tilde{c}_1^{14}\tilde{c}_3^2 - \tilde{c}_1^{13}\tilde{c}_2^2\tilde{c}_3 - \tilde{c}_1^{12}\tilde{c}_2^4 + \tilde{c}_1^{11}\tilde{c}_2^3\tilde{c}_3 - \tilde{c}_1^{11}\tilde{c}_3^3 - \tilde{c}_1^{10}\tilde{c}_2^5 - \tilde{c}_1^2\tilde{c}_2^9\,,$$

where $c_2(-T) = -h\tilde{c}_1 + \tilde{c}_2$, $c_3(-T) = h^3 + h^2\tilde{c}_1 + h\tilde{c}_2 + \tilde{c}_3$. The cycle e as a polynomial in $\mathbb{F}_3[\tilde{c}_1, \tilde{c}_2, \tilde{c}_3]$ is not divisible by $\tilde{c}_1^3$. It follows that the cycle e could not be a linear combination of three cycles $c_1^{20}, c_1^{13}c_7, c_1^6c_7^2$ from the list 4.10. Therefore $\dim_{\mathbb{F}_3} V_{20} = 4$.

Consider a sequence $(a_i)_{i\in\mathbb{Z}}$ defined by

$$a_i = \begin{cases} b_i + 1 & \text{if } i = 20 \text{ or } i = 26 \\ b_i & \text{else.} \end{cases}$$

The complete motivic decomposition of the variety Y is the following

$$M(Y) = \oplus_{i\in\mathbb{Z}} M^{\oplus a_i} \oplus M_{1,D}\,. \tag{4.13}$$

REMARK 4.14. We have the same decompositions as (4.3), (4.5) and (4.13) for the motives with the integral coefficients. To show this one can apply [12, Corollary 2.7] and then [12, Theorem 2.16].

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Maksim Zhykhovich
UPMC Univ Paris 06
UMR 7586
Institut de Mathématiques
de Jussieu
(boite courrier 247)
4 place Jussieu
F-75252 Paris CEDEX 05
France