The Equivariant Cohomology of Isotropy Actions on Symmetric Spaces

Oliver Goertsches

Received: September 9, 2011

Communicated by Eckhard Meinrenken

Abstract. We show that for every symmetric space G/K of compact type with K connected, the K-action on G/K by left translations is equivariantly formal.

2010 Mathematics Subject Classification: 53C35, 55N25, 57S15

1 Introduction

Given compact connected Lie groups $K \subset G$ of equal rank, it is well-known that the K-action on the homogeneous space G/K is equivariantly formal because the odd de Rham cohomology groups of G/K vanish. (See for example [7] for an investigation of the equivariant cohomology of such spaces.) If however the rank of K is strictly smaller than the rank of G, then the isotropy action is not necessarily equivariantly formal, and in general it is unclear when this is the case.[1] Restricting our attention to symmetric spaces of compact type, we will prove the following theorem.

Theorem. *Let (G, K) be a symmetric pair of compact type, where G and K are compact connected Lie groups. Then the K-action on the symmetric space $M = G/K$ by left translations is equivariantly formal.*

For symmetric spaces of type II, i.e., compact Lie groups, this result is already known, see Section 4.3. More generally, in the case of symmetric spaces of split rank ($\operatorname{rank} G = \operatorname{rank} K + \operatorname{rank} G/K$), the fact that all K-isotropy groups have maximal rank implies equivariant formality, see Section 4.5. However, for the general case we have to rely on an explicit calculation of the dimension of the

[1] A sufficient condition for equivariant formality of the isotropy action was introduced in [17], see Remark 4.2 below. If K belongs to a certain class of subtori of G this condition is in fact an equivalence, see [18].

cohomology of the T-fixed point set M^T, where $T \subset K$ is a maximal torus, in order to use the characterization of equivariant formality via the condition $\dim H^*(M^T) = \dim H^*(M)$. With the help of the notion of compartments introduced in [1] and several results proven therein we will find in Section 4.1 a calculable expression for this dimension, and after reducing to the case of an irreducible simply-connected symmetic space in Section 4.2 we can invoke the classification of such spaces to show equivariant formality in each of the remaining cases by hand. On the way we obtain a formula for the number of compartments in a fixed K-Weyl chamber, see Proposition 4.14.

Acknowledgements. The author wishes to express his gratitude to Augustin-Liviu Mare for interesting discussions on a previous version of the paper.

2 Symmetric spaces

Let G be a connected Lie group and $K \subset G$ a closed subgroup. Then K is said to be a symmetric subgroup of G if there is an involutive automorphism $\sigma : G \to G$ such that K is an open subgroup of the fixed point subgroup G^σ. We will refer to the pair (G, K) as a symmetric pair, and G/K is a symmetric space.

Given a symmetric pair (G, K) with corresponding involution $\sigma : G \to G$, then the Lie algebra $\mathfrak{g}$ decomposes into the (± 1)-eigenspaces of σ: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and the usual commutation relations hold: $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The rank of G/K is by definition the maximal dimension of an abelian subalgebra of $\mathfrak{p}$. Then clearly $\operatorname{rank} G - \operatorname{rank} K \leq \operatorname{rank} G/K$, and if equality holds, then we say that G/K is of split rank.

A symmetric pair (G, K) is called (almost) effective if G acts (almost) effectively on G/K. Given a symmetric pair (G, K), then the kernel $N \subset G$ of the G-action on G/K is contained in K, and $(G/N, K/N)$ is an effective symmetric pair with $(G/N)/(K/N) = G/K$. An almost effective symmetric pair (G, K) (and the corresponding symmetric space G/K) will be called of compact type if G is a compact semisimple Lie group. In this paper only symmetric spaces of compact type will occur. If (G, K) is effective, then G can be regarded as a subgroup of the isometry group of G/K with respect to any G-invariant Riemannian metric on G/K. If (G, K) is additionally of compact type, then this inclusion is in fact an isomorphism between G and the identity component of the isometry group.

3 Equivariant formality

The equivariant cohomology of an action of a compact connected Lie group K on a compact manifold M is by definition the cohomology of the Borel construction

$$H_K^*(M) = H^*(EK \times_K M);$$

we use real coefficients throughout the paper. The projection $EK \times_K M \to EK/K = BK$ to the classifying space BK of K induces on $H^*_K(M)$ the structure of an $H^*(BK)$-algebra.
An action of a compact connected Lie group K on a compact manifold M is called equivariantly formal in the sense of [3] if $H^*_K(M)$ is a free $H^*(BK)$-module. If the K-action on M is equivariantly formal then automatically

$$H^*_K(M) = H^*(M) \otimes H^*(BK) \tag{1}$$

as graded $H^*(BK)$-modules, see [2, Proposition 2.3]. In the following proposition we collect some known equivalent characterizations of equivariant formality.

Proposition 3.1. *Consider an action of a compact connected Lie group K on a compact manifold M, and let $T \subset K$ be a maximal torus. Then the following conditions are equivalent:*

1. *The K-action on M is equivariantly formal.*
2. *The T-action on M is equivariantly formal.*
3. *The cohomology spectral sequence associated to the fibration $ET \times_T M \to BT$ collapses at the E_2-term.*
4. *We have $\dim H^*(M) = \dim H^*(M^T)$.*
5. *The natural map $H^*_T(M) \to H^*(M)$ is surjective.*

Proof. For the equivalence of (1) and (2) see [5, Proposition C.26]. The Borel localization theorem implies that the rank of $H^*_T(M)$ as an $H^*(BT)$-module always equals $\dim H^*(M^T)$. Then [5, Lemma C.24] implies the equivalence of (2), (3), and (4); see also [10, p. 46]. For the equivalence to (5), see [13, p. 148]. □

Note that by [10, p. 46] the inequality $\dim H^*(M^T) \leq \dim H^*(M)$ holds for any T-action on M. Condition (5) in the proposition shows that

Corollary 3.2. *If a compact connected Lie group K acts equivariantly formally on a compact manifold M, then so does every connected closed subgroup of K.*

Applying the gap method to the spectral sequence in Item (3) of Proposition 3.1 we obtain the following well-known sufficient condition for equivariant formality.

Proposition 3.3. *Any action of a compact Lie group K on a compact manifold M with $H^{odd}(M) = 0$ is equivariantly formal.*

4 Isotropy actions on symmetric spaces of compact type

Let G be a compact connected Lie group and $K \subset G$ a compact connected subgroup. Because an equivariantly formal torus action always has fixed points, the only tori $T \subset G$ that can act equivariantly formally on G/K by left translations are those that are conjugate to a subtorus of K. On the other hand, if a maximal torus T of K acts equivariantly formally on G/K, then we know by Corollary 3.2 that all these tori do in fact act equivariantly formally. In the following, we will prove that this indeed happens for symmetric spaces of compact type. More precisely:

Theorem 4.1. *Let (G, K) be a symmetric pair of compact type, where G and K are compact connected Lie groups. Then the K-action on the symmetric space G/K by left translations is equivariantly formal.*

Remark 4.2. The pair (G, K) is a Cartan pair in the sense of [4], see [4, p. 448]. Therefore, [17, Theorem A] shows that a sufficient condition for the K-action on G/K to be equivariantly formal is that the map $H^*(G/K)^{N_G(K)} \to H^*(G)$ induced by the projection $G \to G/K$, where $N_G(K)$ acts on G/K from the right, is injective. It would be interesting to know whether a symmetric pair always satisfies this condition.

4.1 The fixed point set of a maximal torus in K

Let (G, K) be a symmetric pair of compact type, where G and K are compact connected Lie groups. Denote by $\sigma : G \to G$ the corresponding involutive automorphism. Then $M = G/K$ is a symmetric space of compact type. We fix maximal tori $T_K \subset K$ and $T_G \subset G$ such that $T_K \subset T_G$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of the Lie algebra $\mathfrak{g}$ into eigenspaces of σ.
In order to prove Theorem 4.1 we can without loss of generality assume that the symmetric pair (G, K) is effective: if $N \subset K$ is the kernel of the G-action on G/K, then clearly the K-action on $G/K = (G/N)/(K/N)$ is equivariantly formal if and only if the K/N-action is equivariantly formal. (This follows for example from Proposition 3.1 because the fixed point sets of appropriately chosen maximal tori in K and K/N coincide.)

Lemma 4.3. *The T_K-fixed point set in M is $N_G(T_K)/N_K(T_K)$.*

Proof. An element $gK \in M$ is fixed by T_K if and only if $g^{-1}T_K g \subset K$ (i.e., $g^{-1}T_K g$ is a maximal torus in the compact Lie group K), which is the case if and only if there is some $k \in K$ with $k^{-1}g^{-1}T_K gk = T_K$. Thus, $(G/K)^{T_K} = N_G(T_K)/N_G(T_K) \cap K = N_G(T_K)/N_K(T_K)$. ☐

Lemma 4.4 ([15, Proposition VII.3.2]). *T_G is the unique maximal torus in G containing T_K.*

Lemma 4.4 implies that the Lie algebra $\mathfrak{t}_\mathfrak{g}$ of T_G decomposes according to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as $\mathfrak{t}_\mathfrak{g} = \mathfrak{t}_\mathfrak{k} \oplus \mathfrak{t}_\mathfrak{p}$. (In fact, this statement is the first part of the proof of [15, Proposition VII.3.2].)

Proposition 4.5. *Each connected component of M^{T_K} is a torus of dimension* $\operatorname{rank} G - \operatorname{rank} K$.

Proof. Because of Lemma 4.4, the abelian subalgebra $\mathfrak{t}_{\mathfrak{p}} \subset \mathfrak{p}$ is the space of elements in $\mathfrak{p}$ that commute with $\mathfrak{t}_{\mathfrak{k}}$. Thus, Lemma 4.3 implies that the component of M^{T_K} containing eK is $T_G/(T_G \cap K) = T_G/T_K$ (note that the centralizer of T_K in K is exactly T_K), i.e., a $\operatorname{rank} G - \operatorname{rank} K$-dimensional torus. Because the fixed set M^{T_K} is a homogeneous space, all components are diffeomorphic. □

We therefore understand the structure of the T_K-fixed point set M^{T_K} if we know its number of connected components, which we denote by r. In view of condition (4) in Proposition 3.1, we are mostly interested in the dimension of its cohomology. Proposition 4.5 implies immediately:

Proposition 4.6. *We have* $\dim H^*(M^{T_K}) = 2^{\operatorname{rank} G - \operatorname{rank} K} \cdot r$.

In order to get a calculable expression for r we will use several results from [1, Sections 5 and 6] which we now collect. Denote by $\Delta_G = \Delta_{\mathfrak{g}}$ the root system of G with respect to the maximal torus T_G, i.e., the set of nonzero elements $\alpha \in \mathfrak{t}_{\mathfrak{g}}^*$ such that the corresponding eigenspace $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [W, X] = i\alpha(W)X \text{ for all } W \in \mathfrak{t}_{\mathfrak{g}}\}$ is nonzero. Then we have the root space decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{g}}} \mathfrak{g}_\alpha. \tag{2}$$

The $\mathfrak{g}$-Weyl chambers are the connected components of the set $\mathfrak{t}_{\mathfrak{g}} \setminus \bigcup_{\alpha \in \Delta_{\mathfrak{g}}} \ker \alpha$. Because of Lemma 4.4, $\mathfrak{t}_{\mathfrak{k}}$ contains $\mathfrak{g}$-regular elements, hence no root in $\Delta_{\mathfrak{g}}$ vanishes on $\mathfrak{t}_{\mathfrak{k}}$. Therefore some of the $\mathfrak{g}$-Weyl chambers intersect $\mathfrak{t}_{\mathfrak{k}}$ nontrivially, and following [1] we will refer to these intersections as *compartments.* Considering as in [1] the decomposition of $\Delta_{\mathfrak{g}}$ into complementary subsets $\Delta_{\mathfrak{g}} = \Delta' \cup \Delta''$, where

$$\Delta' = \{\alpha \in \Delta_{\mathfrak{g}} \mid \mathfrak{g}_\alpha \not\subset \mathfrak{p}^{\mathbb{C}}\}, \quad \Delta'' = \{\alpha \in \Delta_{\mathfrak{g}} \mid \mathfrak{g}_\alpha \subset \mathfrak{p}^{\mathbb{C}}\}, \tag{3}$$

we have by [1, Lemma 9] that the root system $\Delta_K = \Delta_{\mathfrak{k}}$ of K with respect to T_K is given by

$$\Delta_{\mathfrak{k}} = \{\alpha|_{\mathfrak{t}_{\mathfrak{k}}} \mid \alpha \in \Delta'\}. \tag{4}$$

In particular, $\mathfrak{g}$-regular elements in $\mathfrak{t}_{\mathfrak{k}}$ are also $\mathfrak{k}$-regular, and hence each compartment is contained in a $\mathfrak{k}$-Weyl chamber.

Because of Lemma 4.4, the group $N_G(T_K)$ is a subgroup of $N_G(T_G)$. Both groups have the same identity component T_G, so we may regard the quotient group $N_G(T_K)/T_G$ as a subgroup of the Weyl group $W(G)$ of G. The free action of $W(G)$ on the $\mathfrak{g}$-Weyl chambers induces an action of $N_G(T_K)/T_G$ on the set of compartments. Because any two compartments are G-conjugate [1, Theorem 10], this action is simply transitive on the set of compartments, and it follows that the number of connected components of $N_G(T_K)$ equals the total number of compartments in $\mathfrak{t}_{\mathfrak{k}}$. On the other hand no connected component of $N_G(T_K)$ contains more than one connected component of $N_K(T_K)$.

(It is sufficient to check this for the identity component T_G of $N_G(T_K)$. An element in $N_K(T_K) \cap T_G$ is an element in K centralizing T_K, hence already contained in T_K.) Because the number of connected components of $N_K(T_K)$ equals the number of $\mathfrak{k}$-Weyl chambers, and each $\mathfrak{k}$-Weyl chamber contains the same number of compartments [1, Theorem 10], we have shown the following lemma.

LEMMA 4.7. *The number r of connected components of $M^{T_K} = N_G(T_K)/N_K(T_K)$ is the number of compartments in a fixed $\mathfrak{k}$-Weyl chamber. In particular it only depends on the Lie algebra pair $(\mathfrak{g}, \mathfrak{k})$.*

Let C be a $\mathfrak{g}$-Weyl chamber that intersects $\mathfrak{t}_\mathfrak{k}$ nontrivially. By [1, Lemma 8] the compartment $C \cap \mathfrak{t}_\mathfrak{k}$ can be described explicitly: The involution $\sigma : G \to G$ permutes the $\mathfrak{g}$-Weyl chambers and fixes $\mathfrak{t}_\mathfrak{k}$, hence it fixes C. Let $B = \{\alpha_1, \ldots, \alpha_{\operatorname{rank} G}\}$ be the corresponding simple roots such that C is exactly the set of points where the elements of B take positive values. The involution σ acts as a permutation group on B because for any i the linear form $\alpha_i \circ \sigma$ is again positive on C. Note that for every root $\alpha \in \Delta_\mathfrak{g}$ the linear form $\frac{1}{2}(\alpha + \alpha \circ \sigma)$ vanishes on $\mathfrak{t}_\mathfrak{p}$ and coincides with $\alpha|_{\mathfrak{t}_\mathfrak{k}}$ on $\mathfrak{t}_\mathfrak{k}$. The set $B|_{\mathfrak{t}_\mathfrak{k}} = \{\alpha_i|_{\mathfrak{t}_\mathfrak{k}} \mid i = 1, \ldots, \operatorname{rank} G\}$ is a basis of $\mathfrak{t}_\mathfrak{k}^*$ (in particular it consists of $\dim \mathfrak{t}_\mathfrak{k}$ elements) and the compartment $C \cap \mathfrak{t}_\mathfrak{k}$ is exactly the set of points in $\mathfrak{t}_\mathfrak{k}$ where all $\alpha_i|_{\mathfrak{t}_\mathfrak{k}}$ take positive values. It is a simplicial cone bounded by the hyperplanes $\ker \alpha_i|_{\mathfrak{t}_\mathfrak{k}}$. Any such hyperplane is either a wall of a $\mathfrak{k}$-Weyl chamber or the kernel of a $\mathfrak{g}$-root α_i with $\alpha_i \circ \sigma = \alpha_i$, see (4). In any case, reflection along the hyperplane defines an element of $N_G(T_K)/T_G$ and takes $C \cap \mathfrak{t}_\mathfrak{k}$ to an adjacent compartment. (This argument is taken from the proof of [1, Theorem 10].)
It follows that the action of $N_G(T_K)/T_G$ on the set of compartments described above is generated by the reflections along all hyperplanes $\ker \alpha|_{\mathfrak{t}_\mathfrak{k}}$, where $\alpha \in \Delta_\mathfrak{g}$. Let $\langle \cdot, \cdot \rangle$ be the Killing form on $\mathfrak{g}$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is orthogonal with respect to $\langle \cdot, \cdot \rangle$. We identify $\mathfrak{t}_\mathfrak{g}^*$ with $\mathfrak{t}_\mathfrak{g}$ and $\mathfrak{t}_\mathfrak{k}^*$ with $\mathfrak{t}_\mathfrak{k}$ via $\langle \cdot, \cdot \rangle$. For $\alpha \in \Delta_\mathfrak{g}$, let $H_\alpha \in \mathfrak{t}_\mathfrak{g}$ be the element such that $\alpha(H) = \langle H, H_\alpha \rangle$ for all $H \in \mathfrak{t}_\mathfrak{g}$. Given $X \in \mathfrak{t}_\mathfrak{g}$, we write $X^\mathfrak{k}$ and $X^\mathfrak{p}$ for the $\mathfrak{k}$- and $\mathfrak{p}$-parts of X respectively. Then $H_\alpha^\mathfrak{k}$ corresponds to $\alpha|_{\mathfrak{t}_\mathfrak{k}}$ under the isomorphism $\mathfrak{t}_\mathfrak{k} \cong \mathfrak{t}_\mathfrak{k}^*$.

LEMMA 4.8. *Let $\alpha \in \Delta_\mathfrak{g}$ be a root with $\alpha \circ \sigma \neq \alpha$. Then either*

1. *$\langle H_\alpha, H_{\alpha\circ\sigma} \rangle = 0$ and $|H_\alpha^\mathfrak{p}|^2 = |H_\alpha^\mathfrak{k}|^2$ or*
2. *$2 \cdot \frac{\langle H_\alpha, H_{\alpha\circ\sigma} \rangle}{|H_\alpha|^2} = -1$, $|H_\alpha^\mathfrak{p}|^2 = 3|H_\alpha^\mathfrak{k}|^2$ and $\alpha + \alpha \circ \sigma \in \Delta_\mathfrak{g}$.*

Proof. We have $H_{\alpha\circ\sigma} = H_\alpha^\mathfrak{k} - H_\alpha^\mathfrak{p}$, and because $\Delta_\mathfrak{g}$ is a root system it follows that

$$2 \cdot \frac{\langle H_\alpha, H_{\alpha\circ\sigma} \rangle}{|H_\alpha|^2} = 2 \cdot \frac{|H_\alpha^\mathfrak{k}|^2 - |H_\alpha^\mathfrak{p}|^2}{|H_\alpha^\mathfrak{k}|^2 + |H_\alpha^\mathfrak{p}|^2} \in \mathbb{Z}.$$

Because α and $\alpha \circ \sigma$ are roots of equal length, this integer can only equal 0 or ± 1 [12, Proposition 2.48.(d)]. Further, because $\alpha - \alpha \circ \sigma$ is not a root (by Lemma 4.4 no root vanishes on $\mathfrak{t}_\mathfrak{k}$) and not 0, only the possibilities 0 and -1

remain, and in the latter case we also have that $\alpha+\alpha\circ\sigma\in\Delta_{\mathfrak{g}}$ [12, Proposition 2.48.(e)]. □

Proposition 4.9. *The set* $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}} = \{\,\alpha|_{\mathfrak{t}_{\mathfrak{k}}} \mid \alpha\in\Delta_{\mathfrak{g}}\}$ *is a root system in* $\mathfrak{t}_{\mathfrak{k}}^*$.

Proof. It is clear that $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}}$ spans $\mathfrak{t}_{\mathfrak{k}}^*$. We have to check that for all $\alpha,\beta\in\Delta_{\mathfrak{g}}$, the quantity

$$2\cdot\frac{\langle H_\alpha^{\mathfrak{k}},H_\beta^{\mathfrak{k}}\rangle}{|H_\alpha^{\mathfrak{k}}|^2} \tag{5}$$

is an integer. With respect to the decomposition $\Delta_{\mathfrak{g}}=\Delta'\cup\Delta''$ (see (3)) there are four cases:
If both α and β are elements of Δ', then (5) is an integer because $\alpha|_{\mathfrak{t}_{\mathfrak{k}}}$ and $\beta|_{\mathfrak{t}_{\mathfrak{k}}}$ are $\mathfrak{k}$-roots, see (4). In case α and β are elements of Δ'', then the corresponding vectors H_α and H_β are already elements of $\mathfrak{t}_{\mathfrak{k}}$, so $H_\alpha^{\mathfrak{k}}=H_\alpha$ and $H_\beta^{\mathfrak{k}}=H_\beta$, hence (5) is an integer.
Consider the case that $\alpha\in\Delta''$ and $\beta\in\Delta'$. Then $H_\alpha=H_\alpha^{\mathfrak{k}}\in\mathfrak{t}_{\mathfrak{k}}$, hence

$$2\cdot\frac{\langle H_\alpha^{\mathfrak{k}},H_\beta^{\mathfrak{k}}\rangle}{|H_\alpha^{\mathfrak{k}}|^2}=2\cdot\frac{\langle H_\alpha,H_\beta\rangle}{|H_\alpha|^2}\in\mathbb{Z}.$$

The last case to be considered is that $\alpha\in\Delta'$ and $\beta\in\Delta''$. In this case $H_\beta=H_\beta^{\mathfrak{k}}\in\mathfrak{t}_{\mathfrak{k}}$. It may happen that $H_\alpha\in\mathfrak{t}_{\mathfrak{k}}$, but then the claim would follow as before, so we may assume that $H_\alpha\notin\mathfrak{t}_{\mathfrak{k}}$. It follows that $\alpha\circ\sigma$ is a root different from α. By Lemma 4.8 we have $|H_\alpha^{\mathfrak{p}}|^2=c|H_\alpha^{\mathfrak{k}}|^2$ with $c=1$ or $c=3$. We know that

$$2\cdot\frac{\langle H_\alpha,H_\beta\rangle}{|H_\alpha|^2}=2\cdot\frac{\langle H_\alpha^{\mathfrak{k}},H_\beta^{\mathfrak{k}}\rangle}{|H_\alpha^{\mathfrak{k}}|^2+|H_\alpha^{\mathfrak{p}}|^2}=\frac{2}{1+c}\cdot\frac{\langle H_\alpha^{\mathfrak{k}},H_\beta^{\mathfrak{k}}\rangle}{|H_\alpha^{\mathfrak{k}}|^2}$$

is an integer, hence multiplying with the integer $1+c$ shows that (5) is an integer in this case as well.
Next we have to check that for each $\alpha\in\Delta_{\mathfrak{g}}$ the reflection $s_{\alpha|_{\mathfrak{t}_{\mathfrak{k}}}}:\mathfrak{t}_{\mathfrak{k}}\to\mathfrak{t}_{\mathfrak{k}}$ along $\ker\alpha|_{\mathfrak{t}_{\mathfrak{k}}}$ defined by

$$X\mapsto X-2\cdot\frac{\langle H_\alpha^{\mathfrak{k}},X\rangle}{|H_\alpha^{\mathfrak{k}}|^2}H_\alpha^{\mathfrak{k}} \tag{6}$$

sends $\{H_\beta^{\mathfrak{k}}\mid\beta\in\Delta_{\mathfrak{g}}\}$ to itself. If $H_\alpha\in\mathfrak{t}_{\mathfrak{k}}$ (this includes the case $\alpha\in\Delta''$), then the reflection $s_\alpha:\mathfrak{t}_{\mathfrak{g}}\to\mathfrak{t}_{\mathfrak{g}}$ along $\ker\alpha$ leaves invariant $\mathfrak{t}_{\mathfrak{k}}$, and (6) is nothing but the restriction of this reflection to $\mathfrak{t}_{\mathfrak{k}}$. Thus, $\{H_\beta^{\mathfrak{k}}\mid\beta\in\Delta_{\mathfrak{g}}\}$ is sent to itself.
Let $\alpha\in\Delta'$ with $H_\alpha\notin\mathfrak{t}_{\mathfrak{k}}$. We treat the two cases that can arise by Lemma 4.8 separately: assume first that $\langle H_\alpha,H_{\alpha\circ\sigma}\rangle=0$. In this case the two reflections

s_α and $s_{\alpha\circ\sigma}$ commute and we have, recalling that $H_{\alpha\circ\sigma} = H_\alpha^{\mathfrak{k}} - H_\alpha^{\mathfrak{p}}$,

$$\begin{aligned} s_{\alpha\circ\sigma}\circ s_\alpha(X) &= X - 2\cdot\frac{\langle H_\alpha, X\rangle}{|H_\alpha|^2}H_\alpha - 2\cdot\frac{\langle H_{\alpha\circ\sigma}, X\rangle}{|H_{\alpha\circ\sigma}|^2}H_{\alpha\circ\sigma}\\ &= X - 2\cdot\frac{\langle H_\alpha, X\rangle + \langle H_{\alpha\circ\sigma}, X\rangle}{2|H_\alpha^{\mathfrak{k}}|^2}H_\alpha^{\mathfrak{k}} - 2\cdot\frac{\langle H_\alpha, X\rangle - \langle H_{\alpha\circ\sigma}, X\rangle}{2|H_\alpha^{\mathfrak{p}}|^2}H_\alpha^{\mathfrak{p}}\\ &= X - 2\cdot\frac{\langle H_\alpha^{\mathfrak{k}}, X\rangle}{|H_\alpha^{\mathfrak{k}}|^2}H_\alpha^{\mathfrak{k}} + 2\cdot\frac{\langle H_\alpha^{\mathfrak{p}}, X\rangle}{|H_\alpha^{\mathfrak{p}}|^2}H_\alpha^{\mathfrak{p}}. \end{aligned}$$

In particular for each $\beta\in\Delta_{\mathfrak{g}}$ the vector $H_\beta^{\mathfrak{k}} - 2\cdot\frac{\langle H_\alpha^{\mathfrak{k}}, H_\beta^{\mathfrak{k}}\rangle}{|H_\alpha^{\mathfrak{k}}|^2}H_\alpha^{\mathfrak{k}}$ is the $\mathfrak{k}$-part of some vector H_γ, which shows that (6) sends $\{H_\beta^{\mathfrak{k}}\mid \beta\in\Delta_{\mathfrak{g}}\}$ to itself.
In the second case of Lemma 4.8 we have that $\alpha+\alpha\circ\sigma\in\Delta_{\mathfrak{g}}$, with $\ker(\alpha+\alpha\circ\sigma) = \ker\alpha|_{\mathfrak{t}_{\mathfrak{k}}}\oplus\mathfrak{t}_{\mathfrak{p}}$. Thus, the reflection $s_{\alpha|_{\mathfrak{t}_{\mathfrak{k}}}}$ is nothing but the restriction of $s_{\alpha+\alpha\circ\sigma}$ to $\mathfrak{t}_{\mathfrak{k}}$; in particular it sends $\{H_\beta^{\mathfrak{k}}\mid\beta\in\Delta_{\mathfrak{g}}\}$ to itself. □

Remark 4.10. The root system $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}}$ is not necessarily reduced: if there exists a root $\alpha\in\Delta_{\mathfrak{g}}$ with $\alpha\circ\sigma\neq\alpha$ for which the second case of Lemma 4.8 holds, then it contains $\alpha|_{\mathfrak{t}_{\mathfrak{k}}}$ as well as $2\cdot\alpha|_{\mathfrak{t}_{\mathfrak{k}}}$. This happens for instance for $\mathrm{SU}(2m+1)/\mathrm{SO}(2m+1)$.

Because B is the set of simple roots of $\Delta_{\mathfrak{g}}$ every root $\alpha\in\Delta_{\mathfrak{g}}$ can be written as a linear combination of elements in B with integer coefficients of the same sign. It follows that every restriction $\alpha|_{\mathfrak{t}_{\mathfrak{k}}}\in\Delta|_{\mathfrak{t}_{\mathfrak{k}}}$ is a linear combination of elements in $B|_{\mathfrak{t}_{\mathfrak{k}}}$ of the same kind. We thus have proven the following lemma.

Lemma 4.11. *The $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}}$-Weyl chambers are exactly the compartments. If C is a $\mathfrak{g}$-Weyl chamber that intersects $\mathfrak{t}_{\mathfrak{k}}$ nontrivially, with corresponding set of simple roots $B\subset\Delta_{\mathfrak{g}}$, then $B|_{\mathfrak{t}_{\mathfrak{k}}}$ is the set of simple roots of the root system $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}}$ corresponding to $C\cap\mathfrak{t}_{\mathfrak{k}}$.*

Recall that the $N_G(T_K)/T_G$-action on the set of compartments was shown to be generated by the reflections along all hyperplanes $\ker\alpha|_{\mathfrak{t}_{\mathfrak{k}}}$, where $\alpha\in\Delta_{\mathfrak{g}}$. Thus, we obtain

Corollary 4.12. *The $N_G(T_K)/T_G$-action on the set of compartments is the same as the action of the Weyl group $W(\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}})$. In particular, it is generated by the reflections along the hyperplanes $\ker\alpha_i|_{\mathfrak{t}_{\mathfrak{k}}}$. Furthermore, $r = \frac{|W(\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}})|}{|W(\mathfrak{k})|}$.*

Recall that whereas a reduced root system is determined by its simple roots [12, Proposition 2.66], this is no longer true for nonreduced root systems such as $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}}$, see [12, II.8]. However, the reduced elements in a nonreduced root system always form a reduced root system [12, Lemma 2.91] with the same simple roots and the same Weyl group. Using the following proposition taken from [15] we will identify this reduced root system contained in $\Delta_{\mathfrak{g}}|_{\mathfrak{t}_{\mathfrak{k}}}$ with the root system of a second symmetric subalgebra $\mathfrak{k}'\subset\mathfrak{g}$.

Proposition 4.13 ([15, Proposition VII.3.4]). *There is an extension of $\sigma : \mathfrak{t}_\mathfrak{g} \to \mathfrak{t}_\mathfrak{g}$ to an involutive automorphism $\sigma' : \mathfrak{g} \to \mathfrak{g}$ such that its $\mathbb{C}$-linear extension $\sigma' : \mathfrak{g}^\mathbb{C} \to \mathfrak{g}^\mathbb{C}$ satisfies $\sigma'|_{\mathfrak{g}_\alpha} = \mathrm{id}$ for every root $\alpha \in B$ with $\alpha = \alpha \circ \sigma$. The root system of the fixed point algebra $\mathfrak{k}' = \mathfrak{g}^{\sigma'}$ relative to the maximal abelian subalgebra $\mathfrak{t}_\mathfrak{k}$ has $B|_{\mathfrak{t}_\mathfrak{k}}$ as simple roots.*

The roots of $\mathfrak{k}'$ relative to $\mathfrak{t}_\mathfrak{k}$ are restrictions of certain (not necessarily all) elements in $\Delta_\mathfrak{g}$ to $\mathfrak{t}_\mathfrak{k}$; the restrictions of all elements in B occur. See [15, p. 129] for the root space decomposition of $\mathfrak{k}'$ with respect to $\mathfrak{t}_\mathfrak{k}$. Because the sub-root system of reduced elements in $\Delta_\mathfrak{g}|_{\mathfrak{t}_\mathfrak{k}}$ and the root system of $\mathfrak{k}'$ have the same simple roots, these reduced root systems coincide. In particular we obtain the following formula for r:

Proposition 4.14. *We have $r = \frac{|W(\mathfrak{k}')|}{|W(\mathfrak{k})|}$.*

Example 4.15. If $\operatorname{rank} G = \operatorname{rank} K$, i.e., if T_K is also a maximal torus of G, then the identity on $\mathfrak{g}$ satisfies the conditions of Proposition 4.13. Hence $\mathfrak{k}' = \mathfrak{g}$ and the proposition says $r = \frac{|W(G)|}{|W(K)|}$. This however follows already from Lemma 4.3.

Example 4.16. If G/K is a symmetric space of split rank, i.e., $\operatorname{rank} G = \operatorname{rank} K + \operatorname{rank} G/K$, then σ itself satisfies the conditions of Proposition 4.13. In fact, let $\alpha \in B$ with $\alpha = \alpha \circ \sigma$. In this case α vanishes on $\mathfrak{t}_\mathfrak{p}$, which implies that $\mathfrak{g}_\alpha$ is contained either in $\mathfrak{k}^\mathbb{C}$ or in $\mathfrak{p}^\mathbb{C}$. But if it was contained in $\mathfrak{p}^\mathbb{C}$, then $[\mathfrak{t}_\mathfrak{p}, \mathfrak{g}_\alpha] = 0$ and $[\mathfrak{t}_\mathfrak{p}, \mathfrak{g}_{-\alpha}] = 0$, which would contradict the fact that $\mathfrak{t}_\mathfrak{p}$ is maximal abelian in $\mathfrak{p}$. Thus, we have $r = 1$ in the split rank case. Note that $r = 1$ also follows from [1, Lemma 13], combined with Lemma 4.7.

Example 4.17. The symmetric space G/K', where K' is the connected subgroup of G with Lie algebra $\mathfrak{k}'$, is not always of split rank. Assume as in Remark 4.10 that there exists a root $\alpha \in \Delta_\mathfrak{g}$ with $\alpha \circ \sigma \neq \alpha$ such that $\alpha + \alpha \circ \sigma \in \Delta_\mathfrak{g}$. Let $X \in \mathfrak{g}_\alpha$ be nonzero. Then $[X, \sigma'(X)]$ is a nonzero element in $\mathfrak{g}_{\alpha+\alpha\circ\sigma}$. We have $\sigma'([X, \sigma'(X)]) = -[X, \sigma'(X)]$, thus $[X, \sigma'(X)] \in \mathfrak{p}'$, where $\mathfrak{p}'$ is the -1-eigenspace of σ'. By definition of σ' we have $\mathfrak{t}_\mathfrak{p} \subset \mathfrak{p}'$, but $\mathfrak{t}_\mathfrak{p}$ is not a maximal abelian subspace of $\mathfrak{p}'$ because it commutes with $[X, \sigma'(X)]$. For example, in the case $\mathrm{SU}(2m+1)/\mathrm{SO}(2m+1)$ we have $K' = K$ although the space is not of split rank, see Subsection 4.6.2 below.

We will use below that the symmetric subalgebra $\mathfrak{k}'$ can be determined via the Dynkin diagram of G: σ defines an automorphism of the Dynkin diagram of G (because it is a permutation group of B), which is nontrivial if and only if $\operatorname{rank} \mathfrak{g} > \operatorname{rank} \mathfrak{k}$. One can calculate the root system of $\mathfrak{k}'$ via the fact that by Proposition 4.13 the simple roots of $\mathfrak{k}'$ are given by $B|_{\mathfrak{t}_\mathfrak{k}} = \{\frac{1}{2}(\alpha_i + \alpha_i \circ \sigma) \mid i = 1, \ldots, \operatorname{rank} G\}$.

4.2 Reduction to the irreducible case

Lemma 4.18. *If (G,K) and (G',K') are two effective symmetric pairs of connected compact semisimple Lie groups associated to the same pair of Lie algebras $(\mathfrak{g},\mathfrak{k})$, then the K-action on G/K is equivariantly formal if and only if the K'-action on G'/K' is equivariantly formal.*

Proof. Because K and K' are connected, both $H^*(G/K)$ and $H^*(G'/K')$ are given as the $\mathbb{R}$-algebra of $\mathfrak{k}$-invariant elements in $\Lambda^*\mathfrak{p}$, see [20, Theorem 8.5.8]. In particular $\dim H^*(G/K) = \dim H^*(G'/K')$. Choosing maximal tori $T\subset K$ and $T'\subset K'$, we furthermore know from Propositions 4.6 and 4.7 that $\dim H^*((G/K)^T) = \dim H^*((G'/K')^{T'})$ because (G,K) and (G',K') correspond to the same Lie algebra pair. The statement then follows from Proposition 3.1. □

Lemma 4.19. *Given actions of compact connected Lie groups K_i on compact manifolds M_i ($i = 1 \ldots n$), then the $K_1\times\ldots\times K_n$-action on $M_1\times\ldots\times M_n$ is equivariantly formal if and only if all the K_i-actions on M_i are equivariantly formal.*

Proof. Choose maximal tori $T_i\subset K_i$. Then $T_1\times\ldots\times T_n$ is a maximal torus in $K_1\times\ldots\times K_n$. The claim follows from Proposition 3.1 because the $T_1\times\ldots\times T_n$-fixed point set is exactly the product of the T_i-fixed point sets. □

Lemmas 4.18 and 4.19 imply that for proving Theorem 4.1 it suffices to check it for effective symmetric pairs (G,K) of compact connected Lie groups such that G/K is an irreducible simply-connected symmetric space of compact type. Below we will make use of the classification of such spaces, see [8].

4.3 Lie groups

Given a compact connected Lie group G, the product $G\times G$ acts on G via $(g_1,g_2)\cdot g = g_1 g g_2^{-1}$. The isotropy group of the identity element is the diagonal $D(G)\subset G\times G$. In the language of Helgason [8], we obtain an irreducible symmetric pair $(G\times G, D(G))$ of type II. The $D(G)$-action on $(G\times G)/D(G)$ is nothing but the action of G on itself by conjugation. But for any compact connected Lie group, the action on itself by conjugation is equivariantly formal. In fact, if $T\subset G$ is a maximal torus, then the fixed point set of the T-action, G^T, is T itself, and thus $\dim H^*(G^T) = \dim H^*(T) = 2^{\operatorname{rank} G} = \dim H^*(G)$. For other ways to prove that this action is equivariantly formal see [2, Example 4.6]. For instance, equivariant formality would also follow from Proposition 4.23 below as $(G\times G, D(G))$ is of split rank.

4.4 Inner symmetric spaces

Consider the case that the symmetric space G/K of compact type is inner, i.e., that the involution σ is inner. By [8, Theorem IX.5.6] this is the case if and only if $\operatorname{rank} G = \operatorname{rank} K$. Hence, a maximal torus $T_K \subset K$ is also a maximal torus in G, and the T_K-fixed point set is by Lemma 4.3 a finite set of cardinality $\frac{|W(G)|}{|W(K)|}$. Because of the following classical result (see for example [4, Chapter XI, Theorem VII]), the case of inner symmetric spaces is easy to deal with.

PROPOSITION 4.20. *Given any compact connected Lie groups $K \subset G$, the following conditions are equivalent:*

1. $\operatorname{rank} G = \operatorname{rank} K$.
2. $\chi(G/K) > 0$.
3. $H^{odd}(G/K) = 0$.

It follows from Proposition 3.3 that the K-action on a homogeneous space G/K with $\operatorname{rank} G = \operatorname{rank} K$ is always equivariantly formal. Alternatively, [2, Corollary 4.5] implies that the G-action on G/K is equivariantly formal because all its isotropy groups have rank equal to the rank of G. Then by Corollary 3.2 any closed subgroup of G acts equivariantly formally on G/K.

PROPOSITION 4.21. *If $\operatorname{rank} G = \operatorname{rank} K$, then the K-action on G/K is equivariantly formal. If $T_K \subset K$ is a maximal torus, then the fixed point set of the induced T_K-action consists of exactly $\dim H^*(G/K) = \frac{|W(G)|}{|W(K)|}$ points.*

Remark 4.22. This is not a new result. For an investigation of the (algebra structure of the) equivariant cohomology of homogeneous spaces G/K with $\operatorname{rank} G = \operatorname{rank} K$ see [7], or [9, Section 5] for an emphasis on other coefficient rings.

4.5 SPACES OF SPLIT RANK

Also when G/K is of split rank, i.e., $\operatorname{rank} G = \operatorname{rank} K + \operatorname{rank} G/K$, there is a general argument that implies equivariant formality of the K-action on G/K.

PROPOSITION 4.23. *If G/K is of split rank, then the natural K-action on G/K is equivariantly formal.*

Proof. We will show that every K-isotropy algebra has maximal rank, i.e., rank equal to $\operatorname{rank} \mathfrak{k}$. Then equivariant formality follows from [2, Corollary 3.5].
Consider the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and choose any Ad_K-invariant scalar product on $\mathfrak{p}$ that turns G/K into a Riemannian symmetric space. Then we have an exponential map $\exp : \mathfrak{p} \to G/K$, and it is known that every orbit of the K-action on G/K meets $\exp(\mathfrak{a})$, where $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$. Because G/K is of split rank, there is a maximal torus $T_K \subset K$ such that $\mathfrak{t}_\mathfrak{k} \oplus \mathfrak{a}$ is abelian. The torus T_K acts trivially on $\exp(\mathfrak{a})$. Thus, the K-isotropy algebra of any point in $\exp(\mathfrak{a})$ (and hence of any point in M) has maximal rank. $\square$

In the split-rank case we have $r = 1$ by Example 4.16. We thus have

Proposition 4.24. *If G/K is of split rank then* $\dim H^*(G/K) = 2^{\operatorname{rank} G/K}$. *If $T_K \subset K$ is a maximal torus, then the fixed point set of the induced T_K-action on G/K is a* $\operatorname{rank} G/K$*-dimensional torus (in particular connected).*

4.6 Outer symmetric spaces which are not of split rank

For the remaining cases that are not covered by any of the arguments above, i.e., irreducible simply-connected symmetric spaces of type I that are neither of equal nor of split rank, we do not have a general argument for equivariant formality of the isotropy action. Using the classification of symmetric spaces [8, p. 518], we calculate for each of these spaces the dimension of the cohomology of the T_K-fixed point set and show that it coincides with the dimension of the cohomology of G/K (which we take from the literature), upon which we conclude equivariant formality via Proposition 3.1. Fortunately, there are only three (series of) such symmetric spaces, namely

$$\mathrm{SU}(n)/\mathrm{SO}(n), \quad \mathrm{SO}(2p+2q+2)/\mathrm{SO}(2p+1)\times\mathrm{SO}(2q+1), \text{ and } E_6/\mathrm{PSp}(4),$$

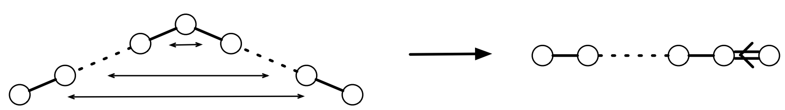
where $n \geq 4$ and $p, q \geq 1$. We have shown with Propositions 4.6 and 4.14 that

$$\dim H^*((G/K)^{T_K}) = 2^{\operatorname{rank}\mathfrak{g}-\operatorname{rank}\mathfrak{k}} \cdot \frac{|W(\mathfrak{k}')|}{|W(\mathfrak{k})|},$$

where the symmetric subalgebra $\mathfrak{k}' \subset \mathfrak{g}$ was introduced in Proposition 4.13. Because in this section we are dealing with outer symmetric spaces, we have $\operatorname{rank}\mathfrak{g} > \operatorname{rank}\mathfrak{k}$, so $\mathfrak{k}' \neq \mathfrak{g}$ is a symmetric subgroup of $\mathfrak{g}$. The orders of the appearing Weyl groups are listed in [11, p. 66].

4.6.1 $\mathrm{SU}(2m)/\mathrm{SO}(2m)$

Let $M = \mathrm{SU}(2m)/\mathrm{SO}(2m)$, where $m \geq 2$, and $T \subset \mathrm{SO}(2m)$ be a maximal torus. The only connected symmetric subgroup of $\mathrm{SU}(2m)$ of rank m different from $\mathrm{SO}(2m)$ is $\mathrm{Sp}(m)$. The fact that $\mathfrak{k}' = \mathfrak{sp}(m)$ can be visualized via the Dynkin diagrams: the involution σ fixes only the middle root of the Dynkin diagram A_{2m-1} of $\mathrm{SU}(2m)$. Hence, after restricting, the middle root becomes a root which is longer than the other roots, and only in C_m there exists a root longer than the others, not in D_m.



We thus may calculate

$$r = \frac{|W(C_m)|}{|W(D_m)|} = \frac{2^m \cdot m!}{2^{m-1}\cdot m!} = 2;$$

note that for this example the number of compartments was also calculated in [1, p. 11]. It is known that $\dim H^*(M) = 2^m$ (see for example [4, p. 493] or [14, Theorem III.6.7.(2)]), hence

$$\dim H^*(M^T) = 2^{2m-1-m} \cdot r = 2^m = \dim H^*(M).$$

Thus, the action is equivariantly formal.

4.6.2 $\mathrm{SU}(2m+1)/\mathrm{SO}(2m+1)$

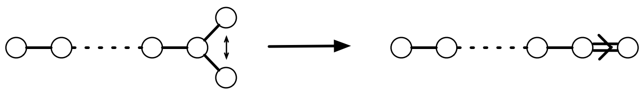
Let $M = \mathrm{SU}(2m+1)/\mathrm{SO}(2m+1)$, where $m \geq 2$, and $T \subset \mathrm{SO}(2m+1)$ be a maximal torus. It is known that $\dim H^*(M) = 2^m$ (see for example [4, p. 493] or [14, Theorem III.6.7.(2)]), hence

$$2^m \cdot r = \dim H^*(M^T) \leq \dim H^*(M) = 2^m$$

for some natural number r. Thus necessarily $r = 1$ (in fact $\mathfrak{k}' = \mathfrak{so}(2m+1)$) and the action is equivariantly formal. Note that this space is also listed as an exception in [1] as it is the only outer symmetric space which is not of split rank such that the corresponding involution fixes no root in the Dynkin diagram (and hence every compartment is a K-Weyl chamber).

4.6.3 $\mathrm{SO}(2p+2q+2)/\mathrm{SO}(2p+1) \times \mathrm{SO}(2q+1)$

Let $M = \mathrm{SO}(2p+2q+2)/\mathrm{SO}(2p+1) \times \mathrm{SO}(2q+1)$, where $p, q \geq 1$, and $T \subset \mathrm{SO}(2p+1) \times \mathrm{SO}(2q+1)$ be a maximal torus. The only connected symmetric subgroups of $\mathrm{SO}(2p+2q+2)$ of rank $p+q$ are $\mathrm{SO}(2p'+1) \times \mathrm{SO}(2q'+1)$, where $p' + q' = p + q$. The involution σ fixes all roots of the Dynkin diagram D_{p+q+1} of $\mathrm{SO}(2p+2q+2)$ but two; after restricting, these two become a single root which is shorter than the others. Because $A_{p+q-1} \oplus A_1$ and D_{p+q} do not appear as the Dynkin diagram of any of the possible symmetric subgroups, the Dynkin diagram of $\mathfrak{k}'$ is forced to be B_{p+q}, which means that $\mathfrak{k}' = \mathfrak{so}(2p+2q+1)$.



We thus have

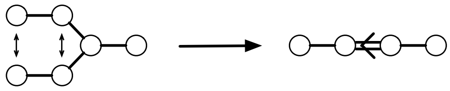
$$r = \frac{|W(B_{p+q})|}{|W(B_p)| \cdot |W(B_q)|} = \frac{2^{p+q} \cdot (p+q)!}{2^p \cdot p! \cdot 2^q \cdot q!} = \binom{p+q}{p}.$$

By [4, p. 496] we have $\dim H^*(M) = 2 \cdot \binom{p+q}{p}$, and it follows that the action is equivariantly formal because of

$$\dim H^*(M^T) = 2^{p+q+1-p-q} \cdot r = 2 \cdot \binom{p+q}{p} = \dim H^*(M).$$

4.6.4 $E_6/\operatorname{PSp}(4)$

Let $M = E_6/\operatorname{PSp}(4)$ and $T \subset \operatorname{PSp}(4)$ be a maximal torus. The only symmetric subalgebra of $\mathfrak{e}_6$ of rank 4 different from $\mathfrak{sp}(4)$ is $\mathfrak{f}_4$.



We obtain

$$r = \frac{|W(F_4)|}{|W(C_4)|} = \frac{2^7 \cdot 3^2}{2^4 \cdot 4!} = 3.$$

It is shown in [19] that $\dim H^*(M) = 12$. Thus,

$$\dim H^*(M^T) = 2^{6-4} \cdot r = 2^2 \cdot 3 = 12 = \dim H^*(M)$$

shows that the action is equivariantly formal.

References

[1] J.-H. Eschenburg, A.-L. Mare and P. Quast, *Pluriharmonic maps into outer symmetric spaces and a subdivision of Weyl chambers*, Bull. Lond. Math. Soc. (2010), doi: 10.1112/blms/bdq070.

[2] O. Goertsches and S. Rollenske, *Torsion in equivariant cohomology and Cohen-Macaulay G-actions*, Transform. Groups. 16 (2011), no. 4, 1063-1080.

[3] M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. 131 (1998), no. 1, 25-83.

[4] W. Greub, S. Halperin, and R. Vanstone, *Connections, curvature, and cohomology. Volume III*, Pure and Applied Mathematics, Vol. 47-III. Academic Press, New York-London, 1976.

[5] V. W. Guillemin, V. L. Ginzburg, and Y. Karshon, *Moment maps, cobordisms, and Hamiltonian group actions*, Mathematical Surveys and Monographs, 96. AMS, Providence, RI, 2002.

[6] V. W. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory*, Springer-Verlag, Berlin 1999.

[7] V. W. Guillemin, T. Holm, and C. Zara, *A GKM description of the equivariant cohomology ring of a homogeneous space*, J. Algebraic Combin. 23 (2006), no. 1, 21–41.

[8] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, 80. Academic Press, Inc., New York-London, 1978.

[9] T. Holm and R. Sjamaar, *Torsion and abelianization in equivariant cohomology*, Transform. Groups 13 (2008), no. 3-4, 585615.

[10] W. Y. Hsiang, *Cohomology theory of topological transformation groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 85. Springer-Verlag, New York-Heidelberg, 1975.

[11] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, New York-Berlin, 1972.

[12] A. W. Knapp, *Lie groups beyond an introduction*, Second edition. Progress in Mathematics, 140. Birkhuser Boston, Inc., Boston, MA, 2002.

[13] J. McCleary, *A user's guide to spectral sequences*, Second edition. Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001.

[14] M. Mimura and H. Toda, *Topology of Lie groups, I and II*, Translations of Mathematical Monographs, 91. American Mathematical Society, Providence, RI, 1991.

[15] O. Loos, *Symmetric spaces. II: Compact spaces and classification*, W. A. Benjamin, Inc., New York-Amsterdam 1969.

[16] T. Sakai, *Riemannian geometry*, Translations of Mathematical Monographs, 149. American Mathematical Society, Providence, RI, 1996.

[17] H. Shiga, *Equivariant de Rham cohomology of homogeneous spaces*, J. Pure Appl. Algebra 106 (1996), no. 2, 173–183.

[18] H. Shiga and H. Takahashi, *Remarks on equivariant cohomology of homogeneous spaces*, Technical report of the Technological University of Nagaoka, 17 (1995), 7-9.

[19] M. Takeuchi, *On Pontrjagin classes of compact symmetric spaces*, J. Fac. Sci. Univ. Tokyo Sect. I 9 (1962), 313–328.

[20] J. A. Wolf, *Spaces of constant curvature*, McGraw-Hill Book Co., New York-London-Sydney, 1967.

Oliver Goertsches
Universität Hamburg
Fachbereich Mathematik
Bundesstraße 55
20146 Hamburg
Germany