Documenta Mathematica

Journal der
Deutschen Mathematiker-Vereinigung
Gegründet 1996

Band 16 · 2011

Documenta Mathematica, Journal der Deutschen Mathematiker-Vereinigung, veröffentlicht Forschungsarbeiten aus allen mathematischen Gebieten und wird in traditioneller Weise referiert. Es wird indiziert durch Mathematical Reviews, Science Citation Index Expanded, Zentralblatt für Mathematik.

Artikel können als TEX-Dateien per E-Mail bei einem der Herausgeber eingereicht werden. Hinweise für die Vorbereitung der Artikel können unter der unten angegebenen WWW-Adresse gefunden werden.

Documenta Mathematica, Journal der Deutschen Mathematiker-Vereinigung, publishes research manuscripts out of all mathematical fields and is refereed in the traditional manner. It is indexed in Mathematical Reviews, Science Citation Index Expanded, Zentralblatt für Mathematik.

Manuscripts should be submitted as TEX -files by e-mail to one of the editors. Hints for manuscript preparation can be found under the following web address.

`http://www.math.uni-bielefeld.de/documenta`

ISSN 1431-0635 (Print), ISSN 1431-0643 (Internet)

SPARC
Leading Edge

Documenta Mathematica is a Leading Edge Partner of SPARC, the Scholarly Publishing and Academic Resource Coalition of the Association of Research Libraries (ARL), Washington DC, USA.

Address of Technical Managing Editor: Ulf Rehmann, Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Typesetting in TEX.

Documenta Mathematica

Band 16, 2011

Emerton's Jacquet Functors for Non-Borel Parabolic Subgroups

Richard Hill and David Loeffler

Received: May 30, 2010
Revised: October 18, 2010

Communicated by Peter Schneider

Abstract. This paper studies Emerton's Jacquet module functor for locally analytic representations of p-adic reductive groups, introduced in [Eme06a]. When P is a parabolic subgroup whose Levi factor M is not commutative, we show that passing to an isotypical subspace for the derived subgroup of M gives rise to essentially admissible locally analytic representations of the torus $Z(M)$, which have a natural interpretation in terms of rigid geometry. We use this to extend the construction in of eigenvarieties in [Eme06b] by constructing eigenvarieties interpolating automorphic representations whose local components at p are not necessarily principal series.

2010 Mathematics Subject Classification: 11F75, 22E50, 11F70
Keywords and Phrases: Eigenvarieties, p-adic automorphic forms, completed cohomology

1 Introduction

1.1 Background

Let $\mathfrak{G}$ be a reductive group over a number field F. The automorphic representations of the group $\mathfrak{G}(\mathbb{A})$, where $\mathbb{A}$ is the adèle ring of F, are central objects of study in number theory. In many cases, it is expected that the set $\Pi(\mathfrak{G})$ of automorphic representations contains a distinguished subset $\Pi(\mathfrak{G})^{\mathrm{arith}}$ of representations which are (in some sense) "definable over $\overline{\mathbb{Q}}$". The subject of this paper is the p-adic interpolation properties of these representations (and their

[0]The second author is grateful for the support of EPSRC Postdoctoral Fellowship EP/F04304X/2.

associated Hecke eigenvalues). Following the pioneering work of Coleman and Coleman-Mazur [Col96, Col97, CM98] for the automorphic representations attached to modular forms with nonzero Hecke eigenvalue at p, it is expected that these Hecke eigenvalues should be parametrised by p-adic rigid spaces (eigenvarieties).

A very general construction of eigenvarieties is provided by the work of Emerton [Eme06b], using the cohomology of arithmetic quotients of $\mathfrak{G}$. For any fixed open compact subgroup $K_f \subseteq \mathfrak{G}(\mathbb{A}_f)$ (where $\mathbb{A}_f$ is the finite adèles of F), and K_∞° the identity component of a maximal compact subgroup of $\mathfrak{G}(F \otimes \mathbb{R})$, the quotients $Y(K_f) = \mathfrak{G}(F)\backslash\mathfrak{G}(\mathbb{A})/K_f K_\infty^\circ$ are real manifolds, equipped with natural local systems $\mathcal{V}_X$ for each algebraic representation X of $\mathfrak{G}$. The cohomology groups $H^i(Y(K_f), \mathcal{V}_X)$ are finite-dimensional, and passing to the direct limit over K_f gives an admissible smooth representation $H^i(\mathcal{V}_X)$ of $\mathfrak{G}(\mathbb{A}_f)$. Every irreducible subquotient of $H^i(\mathcal{V}_X)$ is the finite part of an automorphic representation; we say that the representations arising in this way are *cohomological* (in degree i).

Emerton's construction proceeds in two major steps. Fix a prime $\mathfrak{p}$ above p and an open compact subgroup $K^{(\mathfrak{p})} \subseteq \mathfrak{G}(\mathbb{A}_f^{(\mathfrak{p})})$ (a "tame level"). Firstly, from the spaces $H^i(Y(K^{(\mathfrak{p})}K_\mathfrak{p}), \mathcal{V}_X)$ for various open compact subgroups $K_\mathfrak{p} \subseteq G = \mathfrak{G}(F_\mathfrak{p})$, Emerton constructs Banach space representations $\tilde{H}^i(K^{(\mathfrak{p})})$ of G. For any complete subfield L of $F_\mathfrak{p}$, the spaces $\tilde{H}^i(K^{(\mathfrak{p})})_{\mathrm{la}}$ of locally L-analytic vectors are locally L-analytic representations of G, and there are natural maps

$$H^i(\mathcal{V}_X)^{K^{(\mathfrak{p})}} \to \mathrm{Hom}_\mathfrak{g}(X', \tilde{H}^i(K^{(\mathfrak{p})})_{\mathrm{la}}) \tag{1.1}$$

where $\mathfrak{g} = \mathrm{Lie}\, G$. In many cases, these maps are known to be isomorphisms; if this holds, the automorphic representations which are cohomological in degree i are exactly those which appear as subquotients of $\mathrm{Hom}_\mathfrak{g}(X', \tilde{H}^i(K^{(\mathfrak{p})})_{\mathrm{la}})$ for some X and tame level $K^{(\mathfrak{p})}$.

The second step in the construction is to extract the desired information from the space $\tilde{H}^i(K^{(\mathfrak{p})})_{\mathrm{la}}$. This is carried out by applying the Jacquet module functor of [Eme06a], for a Borel subgroup $B \subseteq G$. This then produces an essentially admissible locally analytic representation of the Levi factor M of B, which is a torus. There is an anti-equivalence of categories between essentially admissible locally analytic representations of M and coherent sheaves on the rigid-analytic space $\widehat{M}$ parametrising characters of M. The eigenvariety $E(i, K^{(\mathfrak{p})})$ is then constructed from this sheaf by passing to the relative spectrum of the unramified Hecke algebra $\mathcal{H}^{\mathrm{sph}}$ of $K^{(\mathfrak{p})}$; points of this variety correspond to characters $(\kappa, \lambda) \in \widehat{M} \times \mathrm{Spec}\, \mathcal{H}^{\mathrm{sph}}$ such that the $(M = \kappa, \mathcal{H}^{\mathrm{sph}} = \lambda)$-eigenspace of $J_B(\tilde{H}^i(K^{(\mathfrak{p})})_{\mathrm{la}})$ is nonzero. Hence if the map (1.1) above is an isomorphism, there is a point of $E(i, K^{(\mathfrak{p})})$ for each automorphic representation $\pi = \bigotimes_v \pi_v$ which is cohomological in degree i with $(\pi_f^{(\mathfrak{p})})^{K^{(\mathfrak{p})}} \otimes J_B(\pi_\mathfrak{p}) \neq 0$.

1.2 Statement of the main result

In this paper, we consider the situation where B is replaced by a general parabolic subgroup P of G. This extends the scope of the theory in two ways: firstly, it may happen that no Borel subgroup exists (G may not be quasi-split); and even if a Borel subgroup exists, there will usually be automorphic representations for which $J_B(\pi_{\mathfrak{p}}) = 0$, which do not appear in Emerton's eigenvariety.
As above, we choose a number field F, a connected reductive group $\mathfrak{G}$ over F, and a prime $\mathfrak{p}$ of F above the rational prime p. Let $\mathcal{G} = \mathfrak{G} \times_F F_{\mathfrak{p}}$, a reductive group over $F_{\mathfrak{p}}$, and $G = \mathcal{G}(F_{\mathfrak{p}})$. Let us choose a parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ (not necessarily arising from a parabolic subgroup of $\mathfrak{G}$), with unipotent radical $\mathcal{N}$; and let $\mathcal{M}$ be a Levi factor of $\mathcal{P}$, with centre $\mathcal{Z}$ and derived subgroup $\mathcal{D}$. We write $G = \mathcal{G}(F_{\mathfrak{p}})$, and similarly for P, M, D, Z. We choose a complete extension L of $\mathbb{Q}_p$ contained in $F_{\mathfrak{p}}$, so G, P, M, D, Z are locally L-analytic groups.
Let $\Gamma = D \times \mathfrak{G}(\mathbb{A}_f^{\mathfrak{p}}) \times \pi_0$, where π_0 is the component group of $\mathfrak{G}(F \otimes \mathbb{R})$. Let us choose an open compact subgroup $U \subseteq \Gamma$ (this is the most natural notion of a "tame level" in this context), and a finite-dimensional irreducible algebraic representation W of $\mathcal{M}$. As we will explain below, the Hecke algebra $\mathcal{H}(\Gamma//U)$ can be written as a tensor product $\mathcal{H}^{\mathrm{ram}} \otimes \mathcal{H}^{\mathrm{sph}}$, where $\mathcal{H}^{\mathrm{sph}}$ is commutative, and $\mathcal{H}^{\mathrm{ram}}$ is finitely-generated (and supported at a finite set of places S).

Theorem (Theorem 6.3). *There exists a rigid-analytic subvariety $\mathcal{E}(i, P, W, U)$ of $\widehat{Z} \times \operatorname{Spec} \mathcal{H}^{\mathrm{sph}}$, endowed with a coherent sheaf $\overline{\mathcal{F}}(i, P, W, U)$ with a right action of $\mathcal{H}^{\mathrm{ram}}$, such that:*

1. *The natural projection $\mathcal{E}(i, P, W, U) \to \mathfrak{z}'$ has discrete fibres. In particular, the dimension of $\mathcal{E}(i, P, W, U)$ is at most equal to the dimension of Z.*
2. *The point $(\chi, \lambda) \in \widehat{Z} \times \operatorname{Spec} \mathcal{H}^{\mathrm{sph}}$ lies in $\mathcal{E}(i, P, W, U)$ if and only if the $(Z = \chi, \mathcal{H}^{\mathrm{sph}} = \lambda)$-eigenspace of $\operatorname{Hom}_U\left(W, J_P(\tilde{H}^i)_{\mathrm{la}}\right)$ is nonzero. If this is so, the fibre of $\overline{\mathcal{F}}(i, P, W, U)$ at (χ, λ) is isomorphic as a right $\mathcal{H}^{\mathrm{ram}}$-module to the dual of that eigenspace.*
3. *If there is a compact open subgroup $G_0 \subseteq G$ such that $(\tilde{H}^i_{\mathrm{la}})^{U^{(\mathfrak{p})}}$ is isomorphic as a G_0-representation to a finite direct sum of copies of $C^{\mathrm{la}}(G_0)$ (where $U^{(\mathfrak{p})} = U \cap \mathfrak{G}(\mathbb{A}_f^{\mathfrak{p}})$), then $\mathcal{E}(i, P, W, U)$ is equidimensional, of dimension equal to the rank of Z.*

Now let us suppose that W is absolutely irreducible, and write $\Pi(i, P, W, U)$ for the set of irreducible smooth $\mathfrak{G}(\mathbb{A}_f) \times \pi_0$-representations π_f such that $J_P(\pi_f)^U \neq 0$, and π_f appears as a subquotient of the cohomology space $H^i(\mathcal{V}_X)$ for some irreducible algebraic representation X of G such that $(X')^N \cong W \otimes \chi$ for a character χ. To any such π_f, we may associate the point $(\theta\chi, \lambda) \in \widehat{Z} \times \operatorname{Spec} \mathcal{H}^{\mathrm{sph}}$, where θ is the smooth character by which Z acts on $J_P(\pi_p)$, and λ the character by which $\mathcal{H}^{\mathrm{sph}}$ acts on $J_P(\pi_f)^U$. Let $E(i, P, W, U)_{\mathrm{cl}}$ denote the set of points of $\widehat{Z} \times \operatorname{Spec} \mathcal{H}^{\mathrm{sph}}$ obtained in this way from representations $\pi_f \in \Pi(i, P, W, U)$.

Corollary (Corollary 6.4). *If the map* (1.1) *is an isomorphism in degree i for all irreducible algebraic representations X such that $(X')^N$ is a twist of W, then $E(i, P, W, U)_{\mathrm{cl}} \subseteq \mathcal{E}(i, P, W, U)$. In particular, the Zariski closure of $E(i, P, W, U)_{\mathrm{cl}}$ has dimension at most* $\dim Z$.

In the special case when $\mathfrak{G}(F \otimes \mathbb{R})$ is compact modulo centre, a related statement has been proved (by very different methods) by the second author [Loe11]. If P_1 and P_2 are two different choices of parabolic, with $P_1 \supseteq P_2$, we have a relation between the eigenvarieties attached to P_1 and P_2 under a mild additional hypothesis, namely that the tame level be of the form $U^{(\mathfrak{p})} \times U_{\mathfrak{p}}$, with $U^{(\mathfrak{p})}$ an open compact subgroup away from $\mathfrak{p}$ and $U_{\mathfrak{p}}$ an open compact subgroup of $D_1 = [M_1, M_1]$ which admits a certain decomposition with respect to the parabolic $P_2 \cap D_1$ (see §5.2 below). In this situation, we have the following:

Theorem (Theorem 6.5). *If U is of the above type, then the space $\mathcal{E}(i, P_1, W, U)$ is equal to the union of two subvarieties $\mathcal{E}(i, P_1, W, U)_{P_2-\mathrm{fs}}$ and $\mathcal{E}(i, P_1, W, U)_{P_2-\mathrm{null}}$, which are respectively endowed with sheaves of $\mathcal{H}^{\mathrm{ram}}$-modules $\overline{\mathcal{F}}(i, P, W, U)_{P_2-\mathrm{fs}}$ and $\overline{\mathcal{F}}(i, P, W, U)_{P_2-\mathrm{null}}$ whose direct sum is $\overline{\mathcal{F}}(i, P, W, U)$.*
If $\pi_f \in \Pi(i, P, W, U)$ and π_f is not annihilated by the map (1.1)*, then the point of $\mathcal{E}(i, P_1, W, U)$ corresponding to π_f lies in the former subvariety if $J_{P_2}(\pi_{\mathfrak{p}}) \neq 0$, and in the latter if $J_{P_2}(\pi_{\mathfrak{p}}) = 0$. Moreover, there is a closed subvariety of $\mathcal{E}(i, P_2, W^{N_{12}}, U \cap D_2)$ whose image in $\widehat{Z}_1 \times \operatorname{Spec} \mathcal{H}^{\mathrm{sph}}$ is $\mathcal{E}(i, P_1, W, U)_{P_2-\mathrm{fs}}$.*

2 Preliminaries

2.1 Notation and definitions

Let p be a prime. Let $K \supseteq \mathbb{Q}_p$ be a complete discretely valued field, which will be the coefficient field for all the representations we consider, and L a finite extension of $\mathbb{Q}_p$ contained in K. If V is a locally convex K-vector space, we let V' denote the continuous dual of V. We write V'_b for V' endowed with the strong topology (which is the only topology on V' we shall consider).
Let S be an abstract semigroup. A *topological representation* of S is a locally convex Hausdorff topological K-vector space V endowed with a left action of S by continuous operators. If S has a topology, we say that the representation is *separately continuous* if the orbit map of each $v \in V$ is a continuous map $S \to V$, and *continuous* if the map $S \times V \to V$ is continuous. In particular, this applies when S is a topological K-algebra and V is an S-module, in which case we shall refer to V as a separately continuous or continuous topological S-module.
If G is a locally compact topological group and V is a continuous representation of G, then V' is a module over the algebra $D(G)$ of measures on G [Eme04,

5.1.7], defined as $C(G)'$ where $C(G)$ is the space of continuous K-valued functions on G. If G is a locally p-adic analytic group, then for any open compact subgroup $H \subseteq G$, the subalgebra $D(H)$ is Noetherian, and we say V is *admissible continuous* [ST02a, Lemma 3.4] if V is a Banach space and V' is finitely generated over $D(H)$ for one (and hence every) open compact H.

If G is a locally L-analytic group, in the sense of [ST02b], then we say the representation V is *locally analytic* if it is a continuous G-representation on a space of compact type, and the orbit maps are locally L-analytic functions $G \to V$. This implies [Eme04, 5.1.9] that V_b' is a separately continuous topological module over the topological K-algebra $D^{\mathrm{la}}(G)$ of distributions on G, defined as $C^{\mathrm{la}}(G)_b'$ where $C^{\mathrm{la}}(G)$ is the space of locally L-analytic K-valued functions on G. For H an open compact subgroup, the subalgebra $D^{\mathrm{la}}(H)$ is a Fréchet-Stein algebra [ST03, 5.1], so the category of coadmissible $D^{\mathrm{la}}(H)$-modules is defined [ST03, §3]; we say V is *admissible locally analytic* if V_b' is coadmissible as a module over $D^{\mathrm{la}}(H)$ for one (and hence every) open compact H.

Finally, if G is a locally L-analytic group for which $Z = Z(G)$ is topologically finitely generated, we say the representation V is *Z-tempered* if it is locally analytic and can be written as an increasing union of Z-invariant BH-spaces. This implies that for any open compact subgroup $H \subseteq G$, V_b' is a jointly continuous topological module over the algebra $D^{\mathrm{ess}}(H, Z(G)) = D^{\mathrm{la}}(H) \mathbin{\widehat{\otimes}}_{D^{\mathrm{la}}(Z\cap H)} C^{\mathrm{an}}(\widehat{Z})$, where $\widehat{Z}$ is the rigid space[1] parametrising characters of Z. The algebra $D^{\mathrm{ess}}(H, Z(G))$ is also a Fréchet-Stein algebra [Eme04, 5.3.22], and we say V is *essentially admissible locally analytic* if V_b' is coadmissible as a module over $D^{\mathrm{ess}}(H, Z(G))$ for one (and hence every) open compact H.

We write $\mathrm{Rep}_{\mathrm{top}}(G)$ for the category of topological representations of G, with morphisms being G-equivariant continuous linear maps. We consider the following full subcategories:

- $\mathrm{Rep}_{\mathrm{cts}}(G)$: continuous representations
- $\mathrm{Rep}_{\mathrm{cts,ad}}(G)$: admissible continuous representations
- $\mathrm{Rep}_{\mathrm{top,c}}(G)$: topological representations on compact type spaces
- $\mathrm{Rep}_{\mathrm{la,c}}(G)$: locally analytic representations
- $\mathrm{Rep}^{z}_{\mathrm{la,c}}(G)$: Z-tempered representations
- $\mathrm{Rep}_{\mathrm{la,ad}}(G)$: admissible locally analytic representations
- $\mathrm{Rep}_{\mathrm{ess}}(G)$: essentially admissible locally analytic representations
- $\mathrm{Rep}_{\mathrm{cts,fd}}(G)$: finite-dimensional continuous representations
- $\mathrm{Rep}_{\mathrm{la,fd}}(G)$: finite-dimensional locally analytic representations

[1]The space $\widehat{Z}$ is in fact defined over L, but we shall always consider it as a rigid space over K by base extension.

Each of these categories is stable under passing to closed G-invariant submodules. The categories $\mathrm{Rep}_{\mathrm{cts,ad}}(G)$, $\mathrm{Rep}_{\mathrm{la,ad}}(G)$ and $\mathrm{Rep}_{\mathrm{ess}}(G)$ have the additional property that all morphisms are strict, with closed image.
The definition of $\mathrm{Rep}_{\mathrm{top}}$ and $\mathrm{Rep}_{\mathrm{top,c}}$ makes sense if G is only assumed to be a semigroup. We will need one more category of representations of semigroups: if S is a semigroup which contains a locally L-analytic subgroup S_0, we define $\mathrm{Rep}^z_{\mathrm{la,c}}(S)$ to be the full subcategory of $\mathrm{Rep}_{\mathrm{top,c}}(S)$ of representations which are locally analytic as representations of S_0, and can be written as an increasing union of $Z(S)$-invariant BH-subspaces. We will, in fact, only use this when either S is a group (in which case the definition reduces to the definition of $\mathrm{Rep}^z_{\mathrm{la,c}}$ above) or S is commutative.

Remark. If $V \in \mathrm{Rep}_{\mathrm{top}}(G)$, V' naturally carries a right action of G. Hence we follow the conventions of [Eme04, §5.1] by defining the algebra structures on $D(G)$ and its cousins in such a way that the Dirac distributions satisfy $\delta_g \star \delta_h = \delta_{hg}$, so all of our modules are left modules. The alternative is to consider the contragredient action on V', which is the convention followed in [ST02b, ST03]; we do not adopt this approach here as we will occasionally wish to consider semigroups rather than groups.

2.2 Smooth and locally isotypical vectors

We now present a slight generalisation of the theory of [Eme04, §7].
Let G be a locally compact topological group and $H \trianglelefteq G$ closed. We suppose that G admits a countable basis of neighbourhoods of the identity consisting of open compact subgroups; this is automatic if G is locally p-adic analytic, for instance. The action of any $g \in G$ on H by conjugation gives a homeomorphism from H to itself, so the conjugation action of G preserves the set of open compact subgroups of H.

Definition 2.1. *Let V be an (abstract) K-vector space with an action of G. We say a vector $v \in V$ is H-smooth if there is an an open compact subgroup U of H such that $Uv = v$.*

Our assumptions imply that the space $V_{H-\mathrm{sm}}$ of H-smooth vectors is G-invariant.

Definition 2.2 ([Eme04, 7.1.1]). *Suppose $V \in \mathrm{Rep}_{\mathrm{top}}(G)$. We define*

$$V_{H-\mathrm{st.sm}} = \varinjlim_{\substack{U \subseteq H \\ U \text{ open}}} V^U,$$

equipped with the locally convex inductive limit topology.

Clearly $V_{H-\mathrm{st.sm}}$ can be identified with $V_{H-\mathrm{sm}}$ as an abstract K-vector space, but the inductive limit topology on the former is generally finer than the subspace topology on the latter. It is clear that the action of G on V induces a topological action on $V_{H-\mathrm{st.sm}}$, so $(-)_{H-\mathrm{st.sm}}$ is a functor from $\mathrm{Rep}_{\mathrm{top}}(G)$ to itself, and the natural injection $V_{H-\mathrm{st.sm}} \hookrightarrow V$ is G-equivariant. We say V is strictly H-smooth if this map is a topological isomorphism.

Proposition 2.3.

(i) If $V \in \mathrm{Rep}_{\mathrm{cts}}(G)$, then $V_{H-\mathrm{st.sm}} \in \mathrm{Rep}_{\mathrm{cts}}(G)$.

(ii) If $V \in \mathrm{Rep}_{\mathrm{top,c}}(G)$, then $V_{H-\mathrm{st.sm}}$ is of compact type and the natural map $V_{H-\mathrm{st.sm}} \to V$ is a closed embedding.

Proof. To show (i), we argue as in [Eme04, 7.1.10]. We let G_0 be an open compact subgroup of G and $(H_i)_{i\geq 0}$ a decreasing sequence of open compact subgroups of H satisfying $\bigcap_i H_i = \{1\}$ and with each H_i normal in G_0; it is clear that we may do this, by our assumption on G. We set $H_i = G_i \cap H$. Then V^{H_i} is a G_0-invariant closed subspace of V, and letting V_i denote the kernel of the "averaging" map $V^{H_i} \to V^{H_{i-1}}$, we have $V^{H-\mathrm{st.sm}} = \bigoplus_i V_i$. Since each V_i is in $\mathrm{Rep}_{\mathrm{cts}}(G_0)$, $V_{H-\mathrm{st.sm}} \in \mathrm{Rep}_{\mathrm{cts}}(G_0)$, which implies it is in $\mathrm{Rep}_{\mathrm{cts}}(G)$. Statement (ii) depends only on V as an H-representation, so we are reduced to the case of [Eme04, 7.1.3]. $\square$

It follows from (ii) that for $V \in \mathrm{Rep}_{\mathrm{top,c}}(G)$ we do not need to distinguish between $V_{H-\mathrm{st.sm}}$ and $V_{H-\mathrm{sm}}$. Moreover, we see that if $V \in \mathrm{Rep}_{\mathrm{la,c}}(G)$ or any of the subcategories of admissible representations introduced above, $V_{H-\mathrm{st.sm}}$ has the same property.

Definition 2.4. *Let V, W be abstract K-vector spaces with an action of G. We say a vector $v \in V$ is locally (H, W)-isotypic if there is an integer n, an open compact subgroup U of H, and a U-equivariant linear map $W^n \to V$ whose image contains v.*

The locally (H, W)-isotypic vectors clearly form a G-invariant subspace of V, since H is normal in G. By construction, this is the image of the evaluation map $\mathrm{Hom}_{H-\mathrm{sm}}(W, V) \otimes_K W \to V$, where $\mathrm{Hom}_{H-\mathrm{sm}}(W, V)$ denotes the subspace of H-smooth vectors in $\mathrm{Hom}_K(W, V) = W' \otimes_K V$ with its diagonal G-action.
If V and W are in $\mathrm{Rep}_{\mathrm{top}}(G)$, with W finite-dimensional, then $\mathrm{Hom}_K(W, V)$ has a natural topology (as a direct sum of finitely many copies of V) and we write $\mathrm{Hom}_{H-\mathrm{st.sm}}(W, V)$ for $\mathrm{Hom}_K(W, V)_{H-\mathrm{st.sm}}$, with its inductive limit topology as above. Then $\mathrm{Hom}_{H-\mathrm{st.sm}}(W, V) \otimes_K W$ is an object of $\mathrm{Rep}_{\mathrm{top}}(G)$ with a natural morphism to V.
We let $V_{(H,W)-\mathrm{liso}}$ denote the image of $\mathrm{Hom}_{H-\mathrm{st.sm}}(W, V) \otimes_K W$ in V, endowed with the quotient topology from the source (which is generally finer than the subspace topology on the target). We say V is strictly locally (H, W)-isotypical if the map $V_{(H,W)-\mathrm{liso}} \to V$ is a topological isomorphism.

Definition 2.5. *We say W is H-good if W is finite-dimensional, and for any open compact subgroup $U \subseteq H$, $\mathrm{End}_U(W) = \mathrm{End}_H(W) = \mathrm{End}_G(W)$.*

Proposition 2.6. *Suppose W is H-good, with $B = \mathrm{End}_G(W)$. Then for any representation V of G on an abstract K-vector space, the natural map*

$$\mathrm{Hom}_K(W, V)_{H-\mathrm{sm}} \otimes_B W \to V$$

is a G-equivariant injection. Dually, for any abstract right B-module X with a B-linear G-action which is smooth restricted to H, the natural map

$$X \to \mathrm{Hom}_K(W, X \otimes_B W)_{H-\mathrm{sm}}$$

is an isomorphism.

Proof. If $G = H$, the first statement is [Eme04, 4.2.4] (the assumption in *op.cit.* that W be algebraic is only used to show that W is H-good). For the general case, the map exists and is injective at the level of H-representations, so it suffices to note that the assumption on W implies that the left-hand side has a well-defined G-action, for which the map is G-equivariant.
For the second part, it suffices to show that the map restricts to an isomorphism $X^U \to \mathrm{Hom}_U(W, X \otimes_B W)$ for any open $U \subseteq H$. Since W is faithful as a B-module by construction, the natural map is an injection. Since X is smooth as an H-representation, any vector in the left-hand side is in $\mathrm{Hom}_U(W, X^{U'} \otimes_B W)$ for some U', which we may assume to be normal in U. However, we have

$$\mathrm{Hom}_U(W, X^{U'} \otimes_B W) \subseteq \mathrm{Hom}_{U'}(W, X^{U'} \otimes_B W) = X^{U'} \otimes_B \mathrm{Hom}_{U'}(W, W).$$

and since W is H-good, we have $\mathrm{Hom}_{U'}(W, W) = B$, so $\mathrm{Hom}_{U'}(W, X^{U'} \otimes_B W) = X^{U'}$. Passing to U/U'-invariants gives the result. □

Combining the preceding results shows that for W an H-good representation, the two functors

$$\mathrm{Hom}_{H-\mathrm{st.sm}}(W, -) \quad \text{and} \quad - \otimes_B W$$

are mutually inverse equivalences between the categories of strictly locally (H, W)-isotypical representations of G and strictly H-smooth G-representations on right B-modules.

Proposition 2.7. *If H is a locally L-analytic group, and V is in $\mathrm{Rep}_{\mathrm{top}}(G) \cap \mathrm{Rep}_{\mathrm{la,c}}(H)$, then there is a topological isomorphism $V_{H-\mathrm{st.sm}} \cong V^{\mathfrak{h}}$, where $\mathfrak{h}$ is the Lie algebra of H. More generally, if W is an H-good locally analytic representation of G, $V_{(H,W)-\mathrm{liso}} \cong \mathrm{Hom}_{\mathfrak{h}}(W, V) \otimes_B W$.*

Proof. Clear from proposition 2.3(i), since a vector $v \in V$ is in $V_{H-\mathrm{sm}}$ if and only if it is $\mathfrak{h}$-invariant. □

3 Preservation of admissibility

3.1 Spaces of invariants

In this section we consider a group G and a normal subgroup H, and consider the functor of H-invariants $V \mapsto V^H : \mathrm{Rep}_{\mathrm{top}}(G) \to \mathrm{Rep}_{\mathrm{top}}(G/H)$. Our aim is to show that this preserves the various subcategories of admissible representations introduced in the previous section.

PROPOSITION 3.1. *If V is an admissible Banach representation of a locally p-adic analytic group G, and $H \trianglelefteq G$ is a closed normal subgroup, then V^H is an admissible Banach representation of G/H.*

Proof. Suppose first G is compact, so $D(G)$ is Noetherian. Since H is normal and acts continuously on V, V^H is a G-invariant closed subspace; so $(V^H)'$ is a $D(G)$-module quotient of a finitely-generated $D(G)$-module, and hence is a finitely-generated $D(G)$-module. However, the closed embedding $C(G/H) \hookrightarrow C(G)$ dualises to a surjection $D(G) \to D(G/H)$, and it is clear that the $D(G)$-action on $(V^H)'$ factors through this surjection. Hence $(V^H)'$ is finitely-generated over $D(G/H)$. In the general case, let G_0 be a compact open subgroup of G and $H_0 = G_0 \cap H$. Then G_0/H_0 is an open compact subgroup of G/H. By the above, V^{H_0} is an admissible continuous G_0/H_0-representation. Since V^H is a closed G_0/H_0-invariant subspace of V^{H_0} it is also admissible continuous as a representation of G_0/H_0 and hence of G/H. □

We now suppose G is a locally L-analytic group. We write $H \trianglelefteq_L G$ to mean that H is a closed normal subgroup of G and the $\mathbb{Q}_p$-subspace $\mathrm{Lie}(H) \subseteq \mathrm{Lie}(G)$ is in fact an L-subspace, so H and G/H also inherit locally L-analytic structures.

PROPOSITION 3.2. *If V is an admissible locally analytic representation of G, and $H \trianglelefteq_L G$. Then V^H is an admissible locally analytic representation of G/H.*

Proof. As above, we may assume G is compact. As in the Banach case, we note that V^H is a closed G-invariant subspace of V, so it is an admissible locally analytic G-representation [ST03, 6.4(ii)] on which the action of G factors through G/H. Hence the action of $D^{\mathrm{la}}(G)$ on $(V^H)'$ factors through $D^{\mathrm{la}}(G/H)$. Since the natural map $C^{\mathrm{la}}(G/H) \to C^{\mathrm{la}}(G)$ is a closed embedding, $D^{\mathrm{la}}(G/H)$ is a Hausdorff quotient of $D^{\mathrm{la}}(G)$ and hence a coadmissible $D^{\mathrm{la}}(G)$-module, and so by [ST03, 3.8] we see that $(V^H)'_b$ is coadmissible as a $D(G/H)$-module as required. □

We now assume that G is a locally L-analytic group with $Z(G)$ topologically finitely generated, and $H \trianglelefteq_L G$. In this case $Z(G/H)$ may be much larger than $Z(G)/(Z(G) \cap H)$, as in the case of $\mathbb{Q}_p^\times \ltimes \mathbb{Q}_p$; so an element of $\mathrm{Rep}^z_{\mathrm{la,c}}(G)$ on which H acts trivially need not lie in $\mathrm{Rep}^z_{\mathrm{la,c}}(G/H)$. Moreover, it is not obvious that $Z(G/H)$ need be topologically finitely generated if $Z(G)$ is so. We shall therefore assume that G is a direct product $H \times J$, with $H, J \trianglelefteq_L G$, and $Z(H)$ and $Z(J)$ are both topologically finitely generated.

PROPOSITION 3.3. *In the above situation, for any essentially admissible locally analytic G-representation V, the space V^H is an essentially admissible locally analytic representation of J.*

Proof. By [Eme04, 6.4.11], any closed invariant subspace of an essentially admissible representation is essentially admissible; so it suffices to assume that $V = V^H$. Let $J_0 \subseteq J$ and $H_0 \subseteq H$ be open compact subgroups. Then $G_0 = J_0 \times H_0$ is an open compact subgroup of G. We have $Z(G) = Z(H) \times Z(J)$, and hence $\widehat{Z(G)} = \widehat{Z(H)} \times \widehat{Z(J)}$.
We now unravel the tensor products to find that the algebra

$$D^{\mathrm{ess}}(G_0, Z(G)) = D^{\mathrm{la}}(G_0) \mathop{\widehat{\otimes}}_{D^{\mathrm{la}}(G_0 \cap Z(G))} C^{\mathrm{an}}(\widehat{Z(G)})$$

decomposes as

$$\begin{aligned}
&\left(D^{\mathrm{la}}(H_0) \mathop{\widehat{\otimes}}_K D^{\mathrm{la}}(J_0)\right) \mathop{\widehat{\otimes}}_{D^{\mathrm{la}}(H_0 \cap Z(H)) \widehat{\otimes}_K D^{\mathrm{la}}(J_0 \cap Z(J))} \left(C^{\mathrm{an}}(\widehat{Z(H)}) \mathop{\widehat{\otimes}}_K C^{\mathrm{an}}(\widehat{Z(J)})\right) \\
&= \left(D^{\mathrm{la}}(H_0) \mathop{\widehat{\otimes}}_{D^{\mathrm{la}}(H_0 \cap Z(H))} C^{\mathrm{an}}(\widehat{Z(H)})\right) \mathop{\widehat{\otimes}}_K \left(D^{\mathrm{la}}(J_0) \mathop{\widehat{\otimes}}_{D^{\mathrm{la}}(J_0 \cap Z(J))} C^{\mathrm{an}}(\widehat{Z(J)})\right) \\
&= D^{\mathrm{ess}}(H_0, Z(H)) \mathop{\widehat{\otimes}}_K D^{\mathrm{ess}}(J_0, Z(J)).
\end{aligned}$$

By assumption, the action of $D^{\mathrm{ess}}(H_0, Z(H))$ on V_b' factors through the augmentation map to K; so the action of $D^{\mathrm{ess}}(G_0, Z(G))$ factors through $D^{\mathrm{ess}}(J_0, Z(J))$. Since $D^{\mathrm{ess}}(J_0, Z(J))$ is a Hausdorff quotient of $D^{\mathrm{ess}}(G_0, Z(G))$, it is a coadmissible $D^{\mathrm{ess}}(G_0, Z(G))$-algebra, and thus V_b' is a coadmissible $D^{\mathrm{ess}}(J_0, Z(J))$-module as required. □

3.2 ADMISSIBLE REPRESENTATIONS OF PRODUCT GROUPS

In this section, we'll recall the theory presented in [Eme04, §7] of representations of groups of the form $G \times \Gamma$, where G is a locally L-analytic group and Γ an arbitrary locally profinite (locally compact and totally disconnected) topological group. This will allow us to give more "global" formulations of the results of the previous section.
Let $*$ denote one of the set {"admissible Banach", "admissible locally analytic", "essentially admissible locally analytic"}, so we shall speak of "$*$-admissible representations". Whenever we consider essentially admissible representations we will assume that the groups concerned have topologically finitely generated centre, so the concept is well-defined.

DEFINITION 3.4 ([Eme04, 7.2.1]). *A $*$-admissible representation of (G, Γ) is a locally convex K-vector space V with an action of $G \times \Gamma$ such that*

- *For each open compact subgroup $U \subseteq \Gamma$, V^U has property $*$ as a representation of G (in the subspace topology);*
- *V is a strictly smooth Γ-representation in the sense of definition 2.1.*

Remark. Our terminology is slightly different from that of [Eme04], where such representations are described as $*$-admissible representations of $G \times \Gamma$. We adopt the formulation above in order to avoid ambiguity when Γ is also a locally analytic group.

The results of the preceding section can be combined to prove:

Proposition 3.5. *If G and H are locally L-analytic groups, V is a $*$-representation of $G \times H$, and $Z(H)$ is compact if $* =$ "essentially admissible locally analytic", then the space*

$$V_{H-\mathrm{st.sm}} = \varinjlim_{\substack{U \subseteq H \\ \textit{open compact}}} V^U$$

is a $*$-admissible representation of (G, H).

Proof. Since the natural maps $V^U \hookrightarrow V^{U'}$ for $U' \subseteq U$ are closed embeddings, the map $V^U \hookrightarrow V_{H-\mathrm{st.sm}}$ is also a closed embedding [Bou87, page II.32]; and its image is clearly $\left(V_{H-\mathrm{st.sm}}\right)^U$, so it suffices to check that V^U has property $*$ for each U.
In the admissible Banach case, this is clear from proposition 3.1. In the admissible locally analytic case, it likewise follows from proposition 3.2. In the essentially admissible case, it suffices to note that the assumption on $Z(H)$ implies that V is essentially admissible as a representation of $G \times H$ if and only if it is essentially admissible as a representation of $G \times U$ for any open compact $U \subseteq H$; so we are in the situation of proposition 3.3. $\square$

A slightly more general version of this applies to groups of the form $G \times H \times J$, where G and H are locally L-analytic and J is an arbitrary locally compact topological group.

Theorem 3.6. *Let V be a $*$-admissible representation of $(G \times H, J)$, where $Z(H)$ is compact in the essentially admissible case. Then $V_{H-\mathrm{st.sm}}$ is a $*$-admissible representation of $(G, H \times J)$.*

Proof. We have

$$V_{H-\mathrm{st.sm}} = (V_{J-\mathrm{st.sm}})_{H-\mathrm{st.sm}} = \varinjlim_{U \subseteq H, U' \subseteq J} V^{U \times U'},$$

which is clearly a strict inductive limit; and $V^{U \times U'}$ is the U-invariants in the $*$-admissible $G \times H$-representation $V^{U'}$, and hence an admissible G-representation. The open compact subgroups of $H \times J$ of the form $U \times U'$ are cofinal in the family of all open compact subgroups, so $V_{H-\mathrm{st.sm}}$ is a $*$-admissible $(G, H \times J)$-representation as required. $\square$

We write $\mathrm{Rep}_{\mathrm{cts,ad}}(G, \Gamma)$ for the category of admissible continuous (G, Γ)-representations, and similarly for the other admissibility conditions.

3.3 Ordinary parts and Jacquet modules

Let $\mathcal{G}$ be a connected reductive algebraic group over L, and $\mathcal{P}$ a parabolic subgroup of $\mathcal{G}$ with Levi factor $\mathcal{M}$. We write $\mathcal{Z} = Z(\mathcal{M})$, $\mathcal{D} = \mathcal{M}^{ss}$. We use Roman letters G, P, M, Z, D for the L-points of these, which are locally L-analytic groups. Note that the multiplication map $Z \times D \to M$ has finite kernel and cokernel, and hence a representation of M has property $*$ if and only if it has the corresponding property as a representation of $Z \times D$.
Suppose that $V \in \mathrm{Rep}_{\mathrm{cts,adm}}(G)$. We say V is unitary if the topology of V can be defined by a G-invariant norm (or equivalently if V contains a G-invariant separated open lattice); this is automatic if G is compact, but not otherwise. The category $\mathrm{Rep}_{\mathrm{u,adm}}(G)$ of unitary admissible Banach representations of G over K is equivalent to $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(\mathcal{O}_K)_{\mathbb{Q}}$, where $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(\mathcal{O}_K)$ is the category considered in [Eme10, 2.4.5] and the subscript $\mathbb{Q}$ denotes the category with the same objects but all Hom-spaces tensored with $\mathbb{Q}$.
In [Eme10, §3], Emerton constructs the ordinary part functor

$$\mathrm{Ord}_P : \mathrm{Mod}_G^{\varpi-\mathrm{adm}}(\mathcal{O}_K) \to \mathrm{Mod}_M^{\varpi-\mathrm{adm}}(\mathcal{O}_K).$$

This functor is additive, so it extends to a functor

$$\mathrm{Ord}_P : \mathrm{Rep}_{\mathrm{u,adm}}(G) \to \mathrm{Rep}_{\mathrm{u,adm}}(M).$$

It is easy to extend this to representations of product groups of the type considered above. Let Γ be a locally profinite topological group, and V a unitary admissible Banach (G,Γ)-representation (i.e. admitting a $G\times\Gamma$-invariant norm). We define

$$\mathrm{Ord}_P(V) = \varinjlim_{\substack{U\subseteq\Gamma \\ \text{open}}} \mathrm{Ord}_P(V^U).$$

Given any subgroups $U' \subseteq U$, there is an "averaging" map $\pi : V^{U'} \to V^U$; and we may write $V^{U'}$ as a locally convex direct sum $V^{U'} = V^U \oplus V^\pi$, where V^π denotes the kernel of π. Since the ordinary part functor commutes with direct sums, we find that $\mathrm{Ord}_P(V^{U'}) = \mathrm{Ord}_P(V^U) \oplus \mathrm{Ord}_P(V^\pi)$; thus the natural map $\mathrm{Ord}_P(V^U) \to \mathrm{Ord}_P(V^{U'})$ is a closed embedding, and if $U' \trianglelefteq U$, we have $\mathrm{Ord}_P(V^{U'})^U = \mathrm{Ord}_P(V^U)$. Passing to the direct limit, we have $\mathrm{Ord}_P(V)^U = \mathrm{Ord}_P(V^U)$, and $\mathrm{Ord}_P(V)$ is an admissible Banach (M,Γ)-representation.
An identical argument applies to the Jacquet module functor $J_P : \mathrm{Rep}_{\mathrm{ess}}(G) \to \mathrm{Rep}_{\mathrm{ess}}(M)$ of [Eme06a] (and indeed to any functor which preserves direct sums). Combining this with theorem 3.6 above, we have:

Proposition 3.7.

(i) If $V \in \mathrm{Rep}_{\mathrm{u,ad}}(G,\Gamma)$ *and* $W \in \mathrm{Rep}_{\mathrm{cts,fd}}(M)$*, then*

$$\mathrm{Hom}_{D-\mathrm{st.sm}}(W, \mathrm{Ord}_P V) \in \mathrm{Rep}_{\mathrm{cts,ad}}(Z, D\times\Gamma).$$

Moreover, $\mathrm{Hom}_{D-\mathrm{st.sm}}(W, \mathrm{Ord}_P V)$ *is unitary if* W *is.*

(ii) If $V \in \mathrm{Rep}_{\mathrm{ess}}(G, \Gamma)$ *and* $W \in \mathrm{Rep}_{\mathrm{la,fd}}(M)$*, and* $\mathfrak{d} = \mathrm{Lie}\, D$*, then*

$$\mathrm{Hom}_{D-\mathrm{st.sm}}(W, J_P V) = \mathrm{Hom}_{\mathfrak{d}}(W, J_P V) \in \mathrm{Rep}_{\mathrm{ess}}(Z, D \times \Gamma).$$

4 Jacquet modules of admissible representations

As in section 3.3 above, let G be the L-points of a connected reductive algebraic group over L, and P a parabolic subgroup with Levi subgroup M. Proposition 3.7(ii) gives us a copious supply of essentially admissible locally analytic representations of the torus $Z = Z(M)$: for any $V \in \mathrm{Rep}_{\mathrm{ess}}(G)$, any open compact $U \subseteq D = M^{ss}$, and any finite-dimensional M-representation W, $\mathrm{Hom}_U(W, J_P V) = (W' \otimes_K J_P V)^U \in \mathrm{Rep}_{\mathrm{ess}}(Z)$. These correspond, by the equivalence of categories of [Eme06b, 2.3.2], to coherent sheaves on the rigid space $\widehat{Z}$. For $V \in \mathrm{Rep}_{\mathrm{ess}}(Z)$, we will write $\mathrm{Exp}\, V$ for the support of the sheaf corresponding to V, a reduced rigid subspace of $\widehat{Z}$.
In this section, we'll prove two results describing the geometry of the rigid spaces $\mathrm{Exp}\,\mathrm{Hom}_U(W, J_P V)$, for $U \subseteq D$ open compact, under additional assumptions on V. These generalise the corresponding results in [Eme06a] when P is a Borel subgroup.

4.1 Compact maps

We begin by generalising some results from [Eme06a, §2.3] on compact endomorphisms of topological modules. Recall that a topological K-algebra is said to be of compact type if it can be written as an inductive limit of Banach algebras, with injective transition maps that are both algebra homomorphisms and compact as maps of topological K-vector spaces. If A is such an algebra, then a topological A-module is said to be of compact type if it is of compact type as a topological K-vector space.
In this situation, we have the following definition of a compact morphism (*op.cit.*, def. 2.3.3):

Definition 4.1. *A continuous A-linear morphism $\phi : M \to N$ between compact type topological A-modules is said to be* A-compact *if there is a commutative diagram*

$$\begin{tikzcd} M \arrow[rr, "\phi"] \arrow[dr, "\alpha"'] & & N \\ & N_1 \arrow[ur, "\beta"'] & \\ V \arrow[ur, "\gamma"] \arrow[uurr, dashed, bend right] & & \end{tikzcd} \tag{4.1}$$

where N_1 is a compact type topological A-module, α and β are continuous A-linear maps, V is a compact type K-vector space, and γ is a continuous K-

linear map for which $A \mathbin{\widehat{\otimes}}_K V \to N_1$ is surjective, and the composite dashed arrow is compact as a map of compact type K-vector spaces.

Lemma 4.2. *If M is a compact type module over a compact type topological K-algebra A; $\phi : M \to M$ is an A-compact map; N is a finitely-generated module over a finite-dimensional K-algebra B; and $\psi : N \to N$ is K-linear, then the map $\phi \otimes \psi : M \otimes_K N \to M \otimes_K N$ is $(A \otimes_K B)$-compact.*

Proof. We may assume without loss of generality that ψ is the identity, by [Eme06a, 2.3.4(i)]. This case follows immediately by tensoring each of the spaces in the diagram with N. $\square$

Lemma 4.3. *Let $\sigma : A \to A'$ be a finite morphism of compact type topological K-algebras, and $\phi : M \to N$ a morphism of topological A'-modules which is A'-compact. Then ϕ is A-compact.*

Proof. By assumption, we have a diagram as in lemma 4.1, where the map $A' \mathbin{\widehat{\otimes}}_K V \to N_1$ is surjective. Let $a_1, \dots, a_k$ be a set of elements generating A' as an A-module, let $V' = V^k$, and define the map $\gamma' : V' \to N_1$ by $(v_1, \dots, v_k) \mapsto \sum a_i \gamma(v_i)$.
Then it is clear that $1 \mathbin{\widehat{\otimes}} \gamma'$ gives a surjection $A \mathbin{\widehat{\otimes}}_K V^k \to N_1$. Furthermore, the composite $\phi \circ \gamma' : V' \to N$ is the map $(v_1, \dots, v_k) \mapsto \sum \beta(a_i \gamma(v))$. As β is a morphism of A'-modules, this equals $\sum a_i (\beta \circ \gamma)(v)$, which is clearly compact (since $\beta \circ \gamma$ is). So the map $\gamma' : V' \to N_1$ witnesses ϕ as an A-compact map. $\square$

4.2 Twisted distribution algebras

Let L be a finite extension of $\mathbb{Q}_p$, and G a locally L-analytic group. Let $(H_n)_{n \ge 0}$ be a decreasing sequence of good L-analytic open subgroups of G, in the sense of [Eme04, §5.2], such that

- the subgroups H_n form a basis of neighbourhoods of the identity in G;
- H_n is normal in H_0 for all n;
- the inclusion $H_{n+1} \hookrightarrow H_n$ extends to a morphism of rigid spaces between the underlying affinoid rigid analytic groups $\mathbb{H}_{n+1} \hookrightarrow \mathbb{H}_n$, which is relatively compact.

Such a sequence certainly always exists, since the choice of H_0 determines a Lie $\mathcal{O}_L$-lattice $\mathfrak{h}$ in the Lie algebra of G, and we may take H_n to be the subgroup attached to the sublattice $\pi^n \mathfrak{h}$. We may use this sequence to write the topological K-algebra $A := D^{\mathrm{la}}(H_0) = C^{\mathrm{la}}(H_0)'_b$ as an inverse limit of the spaces $A_n := D(\mathbb{H}_n^\circ, H_0) = \left[C(H_0)_{\mathbb{H}_n^\circ-\mathrm{an}}\right]'_b$. For all n, A_n is a compact type topological K-algebra, and the sequence $(A_n)_{n \ge 0}$ is a weak Fréchet-Stein structure on A.
We begin with a construction related to the "untwisting isomorphism" of [Eme04, 3.2.4]. Let (ρ, W) be any finite-dimensional K-representation of H_0,

and let $E = \mathrm{End}_K W$. We consider the following commutative diagram of K-vector spaces:

$$\begin{tikzcd} & K[H_0] \otimes_K E \arrow[dd, "\gamma"', "\substack{g\otimes m\mapsto \\ g\otimes\rho(g)m}"] \\ K[H_0] \arrow[ur, "g\mapsto g\otimes 1", "\alpha"'] \arrow[dr, "\beta", "g\mapsto g\otimes\rho(g)"'] & \\ & K[H_0] \otimes_K E \end{tikzcd} \tag{4.2}$$

Here α and β are ring homomorphisms, and although γ is not a ring homomorphism, it satisfies the relation $\gamma(\alpha(x)y) = \beta(x)\gamma(y)$, so it intertwines the two $K[H_0]$-module structures on $K[H_0] \otimes_K E$ given by α and β. Furthermore γ is clearly invertible.

We now assume that (ρ, W) is locally analytic (when W is equipped with its unique Hausdorff locally convex topology).[2] Hence there is an integer $n(\rho)$ such that $W_{\mathbb{H}_n^\circ-\mathrm{an}} = W$ for all $n \geq n(\rho)$.

Proposition 4.4. *Let $n \geq n(\rho)$. Then there exist unique continuous maps $\alpha_n, \beta_n : A_n \to A_n \otimes_K \mathrm{End}(W)$ and $\gamma_n : A_n \otimes_K \mathrm{End}(W) \xrightarrow{\sim} A_n \otimes_K \mathrm{End}\, W$ extending the maps α, β, γ above.*

Proof. Taking the (algebraic) K-dual of the diagram (4.2), we have a diagram

$$\begin{tikzcd} & \mathcal{F}(H_0, E') \arrow[dl, "\alpha'"] \\ \mathcal{F}(H_0, K) & \\ & \mathcal{F}(H_0, E') \arrow[ul, "\beta'"'] \arrow[uu, "\gamma'"] \end{tikzcd}$$

where for K-vector space V, $\mathcal{F}(H_0, V)$ indicates the K-vector space of arbitrary functions $H_0 \to V$. One finds that for a function $f : H_0 \to E'$, we have $\alpha'(f)(m) = f(m)(1)$ and $\beta'(f)(m) = f(m)(\rho(m))$, while $\gamma'(f)(m) = x \mapsto f(\rho(m)x)$. All of these maps manifestly preserve the subspaces of $\mathbb{H}_n^\circ$-analytic functions for $n \geq n(\rho)$, and are continuous for the natural topologies of these subspaces; so there are corresponding maps between the duals of these subspaces, as required. $\square$

Corollary 4.5. *For each $n \geq n(\rho)$, the map β_n makes $B_n = A_n \otimes_K \mathrm{End}\, W$ a finitely-generated topological A_n-module, and the natural map $B_{n+1} \to B_n$ induces an isomorphism $A_n \widehat{\otimes}_{A_{n+1}} B_{n+1} \xrightarrow{\sim} B_n$.*

[2] If $L = \mathbb{Q}_p$ this is equivalent to the (*a priori* weaker) assumption that (ρ, W) is continuous. This follows from the p-adic analogue of Cartan's theorem, which states that any continuous homomorphism between two $\mathbb{Q}_p$-analytic groups is locally analytic; see [Ser92, Part II, §V.9].

Proof. This is clearly true for the A_n-module structure on B_n given by α_n, so it follows for the β_n-structure (since the untwisting isomorphisms γ_n and γ_{n+1} are compatible with the map $B_{n+1} \to B_n$). □

Proposition 4.6. *Let $n \geq n(\rho)$ and let X be a compact type topological A_n-module. Then the diagonal H_0-action on $X \otimes_K W$ extends to a topological A_n-module structure. Moreover, if $n \geq n(\rho)+1$, we have an isomorphism of topological A_{n-1}-modules*

$$A_{n-1} \mathop{\widehat{\otimes}}_{A_n} (X \otimes_K W) \xrightarrow{\sim} (A_{n-1} \mathop{\widehat{\otimes}}_{A_n} X) \otimes_K W.$$

Proof. We clearly have commuting, K-linear, continuous actions of A_n and $\operatorname{End} W$ on $X \otimes_K W$, so we obtain an action of $A_n \otimes_K \operatorname{End} W$. Pulling back via the map β_n, we obtain an A_n-module structure, which clearly restricts to the diagonal action of H_0. The isomorphism follows from the last statement of the preceding corollary via the associativity of the tensor product, since

$$\begin{aligned} & A_{n-1} \mathop{\widehat{\otimes}}_{A_n} (X \otimes_K W) \\ =& (A_{n-1} \mathop{\widehat{\otimes}}_{A_n} B_n) \mathop{\widehat{\otimes}}_{B_n} (X \otimes_K W) \\ =& B_{n-1} \mathop{\widehat{\otimes}}_{B_n} (X \otimes_K W) \\ =& (A_{n-1} \mathop{\widehat{\otimes}}_{A_n} X) \otimes_K W. \end{aligned}$$

□

4.3 Twisted Jacquet modules

We now return to the situation considered above, so G is the group of L-points of a reductive algebraic group $\mathcal{G}$ over L as above, with P a parabolic subgroup, M a Levi subgroup of P, N the unipotent radical, and $Z = Z(M)$. We choose a sequence $(H_n)_{n \geq 0}$ of good L-analytic open subgroups of G admitting rigid analytic Iwahori decompositions $\mathbb{H}_n = \overline{\mathbb{N}}_n \times \mathbb{M}_n \times \mathbb{N}_n$, as in [Eme06a, 4.1.6]. We also impose the additional condition that $\mathbb{M}_n = \mathbb{Z}_n \times \mathbb{D}_n$ where $\mathbb{Z}_n$ and $\mathbb{D}_n$ are the affinoid subgroups underlying good analytic open subgroups of Z and of $D = M^{ss}$; it is clear that we can always do this (by exactly the same method as in Emerton's case). We let Z^+ be the submonoid $\{z \in Z(M) : zN_0z^{-1} \subseteq N_0\}$ of Z.

Our starting point is the following, which is part of the proof of [Eme06a, 4.2.23]:

Proposition 4.7. *Let V be an admissible locally analytic representation of G. Then for all $n \geq 0$, the action of $M_0 \times Z^+$ on the space*

$$U_n = \left(D(\mathbb{H}_n^\circ, H_0) \mathop{\widehat{\otimes}}_{D^{\mathrm{la}}(H_0)} V_b' \right)_{N_0}$$

extends to an $A_n[Z^+]$*-module structure. Moreover, the transition map* $A_n \mathbin{\widehat{\otimes}}_{A_{n+1}} U_{n+1} \to U_n$ *is* A_n*-compact and* Z^+*-equivariant, and there is some* $z \in Z^+$ *(independent of* n*) such that there exists a map* $\alpha : U_n \to A_n \mathbin{\widehat{\otimes}}_{A_{n+1}} U_{n+1}$ *making the following diagram commute:*

$$\begin{tikzcd} A_n \mathbin{\widehat{\otimes}}_{A_{n+1}} U_{n+1} \arrow[r] \arrow[d, "\mathrm{id} \mathbin{\widehat{\otimes}} z"] & U_n \arrow[d, "z"] \arrow[dl, dashed, "\alpha"] \\ A_n \mathbin{\widehat{\otimes}}_{A_{n+1}} U_{n+1} \arrow[r] & U_n . \end{tikzcd} \tag{4.3}$$

We now let $\tilde{U}_n = U_n \otimes_K W$, where (W, ρ) is a fixed, finite-dimensional, continuous representation of M. By the last proposition of the preceeding section (taking the groups there denoted by G and H_i to be those we are now calling M and M_i), we have a diagonal A_n-module structure on $\tilde{U}_n$, and there is also a diagonal action of Z^+ on $\tilde{U}_n$ commuting with the M_0-action.

Proposition 4.8. *For any* $n \geq n(\rho)$ *the following holds:*

- $\tilde{U}_n$ *is a compact type topological* A_n*-module, and the action of* Z^+ *is* A_n*-linear.*
- *There is an* $A_{n+1}[Z^+]$*-linear map* $U_{n+1} \to U_n$ *such that the induced map* $A_n \mathbin{\widehat{\otimes}}_{A_{n+1}} \tilde{U}_{n+1} \to \tilde{U}_n$ *is* A_n*-compact.*
- *For any good* $z \in Z^+$*, we can find a map* $\tilde{\alpha} : U_n \to A_n \mathbin{\widehat{\otimes}}_{A_{n+1}} \tilde{U}_{n+1}$ *such that the diagram corresponding to* (4.3) *commutes.*

Also, the direct limit $\varprojlim U_n$ *(with respect to the transition maps above) is isomorphic as a topological* $A[Z^+]$*-module to* $(V^{N_0} \otimes W')'_b$.

Proof. Since $\tilde{U}_n$ is isomorphic to $(U_n)^{\oplus \dim W}$ as a topological K-vector space, it is certainly of compact type, and we have already observed that it is a topological A_n-module for all $n \geq n(\rho)$. Furthermore the Z^+-action commutes with the M_0-action, and thus it must be A_n-linear by continuity.
Moreover, we have an A_n-compact map $A_n \mathbin{\widehat{\otimes}}_{A_{n+1}} U_{n+1} \to U_n$. Tensoring with the identity map gives a morphism of $A_n \otimes \operatorname{End} W$-modules $\big(A_n \mathbin{\widehat{\otimes}}_{A_{n+1}} U_{n+1}\big) \otimes_K W \to U_n \otimes_K W$, which is $A_n \otimes_K \operatorname{End} W$-compact by lemma 4.2. But the map $\beta : A_n \to A_n \otimes_K \operatorname{End} W$ is a finite morphism, so by lemma 4.3, this map is A_n-compact.
Finally, we know that there exists a map $\alpha : U_n \to A_n \mathbin{\widehat{\otimes}}_{A_{n+1}} U_{n+1}$ through which z factors, and it is clear that if we define $\tilde{\alpha}$ to be the map $\alpha \otimes \rho(z)$ then the diagram corresponding to (4.3) commutes. $\square$

The preceding proposition asserts precisely that the hypotheses of [Eme06a, 3.2.24] are satisfied, and that proposition (and its proof) give us the following:

Corollary 4.9. *The space $X = \left[(V^{N_0} \otimes_K W')_{\mathrm{fs}}\right]'_b$ is a coadmissible $C^{\mathrm{an}}(\widehat{Z}) \mathbin{\widehat{\otimes}}_K A$-module, where $(-)_{\mathrm{fs}}$ denotes the finite-slope-part functor $\mathrm{Rep}_{\mathrm{top,c}}(Z^+) \to \mathrm{Rep}^z_{\mathrm{la,c}}(Z)$ of [Eme06a, 3.2.1].*

Moreover, if $(Y_n)_{n \geq 0}$ is any increasing sequence of affinoid subdomains of $\widehat{Z}$ whose union is the entire space, then for any $n \geq n(\rho)$ we have

$$\left(C^{\mathrm{an}}(Y_n)^\dagger \mathbin{\widehat{\otimes}_K} A_n\right) \mathbin{\widehat{\otimes}_{(C^{\mathrm{an}}(\widehat{Z}) \widehat{\otimes}_K A)}} X = C^{\mathrm{an}}(Y_n)^\dagger \mathbin{\widehat{\otimes}_{K[Z^+]}} \tilde{U}_n.$$

By [Eme04, 3.2.9] we have $X = \left[(V^{N_0} \otimes_K W')_{\mathrm{fs}}\right]'_b = \left[(V^{N_0})_{\mathrm{fs}} \otimes_K W'\right]'_b = [J_P(V) \otimes_K W']'_b$, so the above corollary gives us a description of the strong dual of the W-twisted Jacquet module.

We can now prove the first of the two main theorems of this section. Proposition 3.7(ii) above shows that for any $V \in \mathrm{Rep}_{\mathrm{ess}}(G)$, $(J_P(V) \otimes_K W') \in \mathrm{Rep}_{\mathrm{ess}}(Z, D)$. Equivalently, for any open compact subgroup $\Gamma \subseteq D$, the space $(J_P(V) \otimes_K W')^\Gamma$ is an essentially admissible locally analytic Z-representation, and hence corresponds to a coherent sheaf on $\widehat{Z}$. The previous corollary allows us to describe the support of this sheaf when V is admissible:

Theorem 4.10. *Suppose V is an admissible locally analytic G-representation, W is a finite-dimensional locally analytic representation of M, and Γ is an open compact subgroup of D. Let $E \subseteq \widehat{Z}$ be the support of the coherent sheaf on $\widehat{Z}$ corresponding to $(J_P(V) \otimes_K W')^\Gamma$. Then the natural map $E \to (\mathrm{Lie}\, Z)'$ (induced by the differentiation map $\widehat{Z} \to (\mathrm{Lie}\, Z)'$) has discrete fibres.*

Proof. Since we are free to replace the sequence (H_n) of subgroups of G with a cofinal subsequence, we may assume that $\Gamma \supseteq D_0$. So it suffices to prove the result for $\Gamma = D_0$. Furthermore, since the differentiation map $\widehat{Z}_0 \to (\mathrm{Lie}\, Z)'$ has discrete fibres, it suffices to show that for any character χ of Z_0, the rigid space

$$\mathrm{Exp}\, (J_P(V) \otimes_K W')^{D_0, Z_0 = \chi} \subseteq \widehat{Z}$$

is discrete. If χ does not extend to a character of M, then this space is clearly empty, so there is nothing to prove; otherwise, let us fix such an extension, which gives us an isomorphism $(J_P(V) \otimes_K W')^{D_0, Z_0 = \chi} = [J_P(V) \otimes_K (W \otimes_K \chi)']^{M_0}$. So we may assume without loss of generality that χ is the trivial character, and it suffices to show that

$$C^{\mathrm{an}}(Y_n)^\dagger \mathbin{\widehat{\otimes}_{C^{\mathrm{an}}(\widehat{Z})}} \left[(J_P(V) \otimes_K W')^{M_0}\right]'_b$$

is finite-dimensional over K for all n, or (equivalently) all sufficiently large n.

If we take the completed tensor product of both sides of the formula in corollary 4.9 with $C^{\mathrm{an}}(Y_n)^\dagger$, regarded as a $C^{\mathrm{an}}(Y_n)^\dagger \mathbin{\widehat{\otimes}}_K A_n$-algebra via the augmentation map $A_n \to K$, we have

$$C^{\mathrm{an}}(Y_n)^\dagger \underset{\left(C^{\mathrm{an}}(Y_n)^\dagger\, \widehat{\otimes}_K\, A_n\right)}{\widehat{\otimes}} \left(C^{\mathrm{an}}(Y_n)^\dagger \underset{K}{\widehat{\otimes}} A_n\right) \underset{\left(C^{\mathrm{an}}(\widehat{Z})\, \widehat{\otimes}_K\, A\right)}{\widehat{\otimes}} [J_P(V) \otimes_K W']'_b$$
$$= C^{\mathrm{an}}(Y_n)^\dagger \underset{\left(C^{\mathrm{an}}(Y_n)^\dagger\, \widehat{\otimes}_K\, A_n\right)}{\widehat{\otimes}} \left(C^{\mathrm{an}}(Y_n)^\dagger \underset{K[Z^+]}{\widehat{\otimes}} \tilde{U}_n\right). \quad (4.4)$$

The left-hand side of (4.4) simplifies as

$$C^{\mathrm{an}}(Y_n)^\dagger \underset{\left(C^{\mathrm{an}}(Y_n)^\dagger\, \widehat{\otimes}_K\, A_n\right)}{\widehat{\otimes}} \left(C^{\mathrm{an}}(Y_n)^\dagger \underset{K}{\widehat{\otimes}} A_n\right) \underset{\left(C^{\mathrm{an}}(\widehat{Z})\, \widehat{\otimes}_K\, A\right)}{\widehat{\otimes}} [J_P(V) \otimes_K W']'_b$$
$$= C^{\mathrm{an}}(Y_n)^\dagger \underset{\left(C^{\mathrm{an}}(\widehat{Z})\, \widehat{\otimes}_K\, A\right)}{\widehat{\otimes}} [J_P(V) \otimes_K W']'_b$$
$$= C^{\mathrm{an}}(Y_n)^\dagger \underset{C^{\mathrm{an}}(\widehat{Z})}{\widehat{\otimes}} \left(K \underset{A}{\widehat{\otimes}} [J_P(V) \otimes_K W']'_b\right)$$
$$= C^{\mathrm{an}}(Y_n)^\dagger \underset{C^{\mathrm{an}}(\widehat{Z})}{\widehat{\otimes}} \left[(J_P(V) \otimes_K W')^{M_0}\right]'_b.$$

Meanwhile, the right-hand side of (4.4) is

$$C^{\mathrm{an}}(Y_n)^\dagger \underset{\left(C^{\mathrm{an}}(Y_n)^\dagger\, \widehat{\otimes}_K\, A_n\right)}{\widehat{\otimes}} \left(C^{\mathrm{an}}(Y_n)^\dagger \underset{K[Z^+]}{\widehat{\otimes}} \tilde{U}_n\right)$$
$$= C^{\mathrm{an}}(Y_n)^\dagger \underset{K[Z^+]}{\widehat{\otimes}} \left(K \underset{A_n}{\widehat{\otimes}} \tilde{U}_n\right).$$

Any $z \in Z^+$ that induces an A_n-compact endomorphism of $\tilde{U}_n$ will induce a K-compact endomorphism of $K \widehat{\otimes}_{A_n} \tilde{U}_n$, by [Eme06a, 2.3.4(ii)]. Such a z does exist, by hypothesis. Hence $C^{\mathrm{an}}(Y_n)^\dagger \widehat{\otimes}_{K[Z^+]} \left(K \widehat{\otimes}_{A_n} \tilde{U}_n\right)$ is finite-dimensional over K, by [Eme06a, 2.3.6]. Comparing the two sides of (4.4), we are done. □

We also have a version of [Eme06a, 4.2.36] in this context.

Theorem 4.11. *If V is an admissible locally analytic representation of G such that there is an isomorphism of H-representations $V \xrightarrow{\sim} C^{\mathrm{la}}(H)^r$, for some open compact $H \subseteq G$ and some $r \in \mathbb{N}$, then for any W and Γ, $E = \operatorname{Exp}(J_P(V) \otimes_K W')^\Gamma$ is equidimensional of dimension d, where d is the dimension of Z.*

Proof. As in [Eme06a], we may assume (by replacing the sequence $(G_n)_{n\geq 0}$ with a cofinal subsequence if necessary) that $H = H_0$ and $\Gamma \supseteq D_0$. But then we can identify $(J_P(V) \otimes_K W')^\Gamma$ with a direct summand of $(J_P(V) \otimes_K W')^{D_0}$; this identifies $\operatorname{Exp}(J_P(V) \otimes_K W')^\Gamma$ with a union of irreducible components of $\operatorname{Exp}(J_P(V)\otimes_K W')^{D_0}$. We may therefore assume that in fact $\Gamma = D_0$. As a final reduction, letting $U_n = \left(D(\mathbb{H}_n^\circ, H_0)\mathbin{\widehat{\otimes}}_{D^{\mathrm{la}}(H_0)} V_b'\right)_{N_0}$ as before, we note that the untwisting isomorphism $U_n \xrightarrow{\sim} D(\overline{\mathbb{N}}_n^\circ, \overline{N}_0)^r \mathbin{\widehat{\otimes}}_K A_n$ (equation 4.2.39 in [Eme04]) can be extended to an isomorphism $U_n \otimes_K W \to D(\overline{\mathbb{N}}_n^\circ, \overline{N}_0)^{r \dim W} \mathbin{\widehat{\otimes}}_K A_n$. We thus assume that W is the trivial representation.

Following Emerton, we choose Banach spaces W_n such that the map $D(\overline{\mathbb{N}}_{n+1}^\circ, \overline{N}_0)^r \to D(\overline{\mathbb{N}}_n^\circ, \overline{N}_0)^r$ factors through W_n, and (exactly as in the Borel case) for a suitable $z \in Z^+$ we have

$$J_P(V)_b' \xrightarrow{\sim} \varprojlim_n K\{\{z, z^{-1}\}\} \mathbin{\widehat{\otimes}}_{K[z]} (W_n \mathbin{\widehat{\otimes}}_K A_n),$$

for some A_n-linear action of z on $W_n \mathbin{\widehat{\otimes}}_K A_n$ which factors through $D(\overline{\mathbb{N}}_{n+1}^\circ, \overline{N}_0)^r \mathbin{\widehat{\otimes}}_K A_n$. Taking the completed tensor product with the map $A_n \to D(\mathbb{Z}_n^\circ, Z_0)$ given by the augmentation map of D_0, we have

$$\left[J_P(V)^{D_0}\right]_b' \xrightarrow{\sim} \varprojlim_n K\{\{z, z^{-1}\}\} \mathbin{\widehat{\otimes}}_{K[z]} W_n \mathbin{\widehat{\otimes}}_K D(\mathbb{Z}_n^\circ, Z_0).$$

Let us write $\widehat{Z}_0$ as an increasing union of affinoid subdomains $(X_n)_{n\geq 0}$, such that the natural map $D^{\mathrm{la}}(Z_0) \xrightarrow{\sim} C^{\mathrm{an}}(\widehat{Z}_0) \to C^{\mathrm{an}}(X_n)$ factors through $D(\mathbb{Z}_n^\circ, Z_0)$. Extending scalars from $D(\mathbb{Z}_n^\circ, Z_0)$ to $C^{\mathrm{an}}(\widehat{Z})$ via this map, the above formula becomes

$$\left[J_P(V)^{D_0}\right]_b' = \varprojlim_n K\{\{z, z^{-1}\}\} \mathbin{\widehat{\otimes}}_{K[z]} W_n \mathbin{\widehat{\otimes}}_K C^{\mathrm{an}}(X_n).$$

The action of z on $W_n \mathbin{\widehat{\otimes}}_K C^{\mathrm{an}}(X_n)$ is a $C^{\mathrm{an}}(X_n)$-compact morphism of an orthonormalizable $C^{\mathrm{an}}(X_n)$-Banach module, so the result follows by the methods of [Buz07]. ☐

5 Change of parabolic

We now consider the problem of relating the geometric objects arising from the above construction for two distinct parabolic subgroups.

5.1 Transitivity of Jacquet functors

Let us recall the definition of the finite-slope-part functor, which we have already seen in the previous section. We let Z be a topologically finitely generated abelian locally L-analytic group, and Z^+ an open submonoid of Z which generates Z as a group. Then we have the following functor $\mathrm{Rep}_{\mathrm{top,c}}(Z^+) \to \mathrm{Rep}^z_{\mathrm{la,c}}(Z)$:

Definition 5.1 ([Eme06a, 3.2.1]). *For any object $V \in \mathrm{Rep}_{\mathrm{top,c}}(Z^+)$, we define*

$$V_{\mathrm{fs}} = \mathcal{L}_{b,Z^+}(C^{\mathrm{an}}(\widehat{Z}), V),$$

endowed with the action of Z on the first factor.

Lemma 5.2. *Let Z be a topologically finitely generated abelian group and Y a closed subgroup, and suppose Y^+ and Z^+ are submonoids of Y and Z satisfying the conditions above, with $Y^+ \subseteq Y \cap Z^+$. Then for all $V \in \mathrm{Rep}_{\mathrm{top,c}}(Z^+)$, the natural map $V_{Y-\mathrm{fs}} \to V$ induces an isomorphism*

$$(V_{Y-\mathrm{fs}})_{Z-\mathrm{fs}} \xrightarrow{\sim} V_{Z-\mathrm{fs}}.$$

Proof. Consider the canonical Z^+-equivariant map $V_{Z-\mathrm{fs}} \to V$. We note that $V_{Z-\mathrm{fs}}$ is in $\mathrm{Rep}^z_{\mathrm{la.c}}(Z)$, and hence *a fortiori* in $\mathrm{Rep}^z_{\mathrm{la.c}}(Y)$. Hence by the universal property of [Eme06a, 3.2.4(ii)], the above map factors through $V_{Y-\mathrm{fs}}$. The factored map is still Z^+-equivariant, so by a second application of the universal property it factors through $(V_{Y-\mathrm{fs}})_{Z-\mathrm{fs}}$. This gives a continuous Z-equivariant map $V_{Z-\mathrm{fs}} \to (V_{Y-\mathrm{fs}})_{Z-\mathrm{fs}}$, which is clearly inverse to the map in the statement of the proposition. ☐

Now suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are parabolic subgroups of the reductive group $\mathcal{G}$ over L, with $\mathcal{P}_1 \supseteq \mathcal{P}_2$. We let $\mathcal{N}_1, \mathcal{N}_2$ be their unipotent radicals, so we have a chain of inclusions $\mathcal{G} \supseteq \mathcal{P}_1 \supseteq \mathcal{P}_2 \supseteq \mathcal{N}_2 \supseteq \mathcal{N}_1$.
Let us choose a Levi subgroup $\mathcal{M}_2$ of $\mathcal{P}_2$, so $\mathcal{P}_2 = \mathcal{N}_2 \rtimes \mathcal{M}_2$. There is a unique Levi subgroup $\mathcal{M}_1$ of $\mathcal{P}_1$ containing $\mathcal{M}_2$; and $\mathcal{P}_{12} = \mathcal{P}_2 \cap \mathcal{M}_1$ is a parabolic subgroup of $\mathcal{M}_1$ of which $\mathcal{M}_2$ is also a Levi factor. We write $\mathcal{Z}_1$, $\mathcal{Z}_2$ for the centres of $\mathcal{M}_1$ and $\mathcal{M}_2$.
All of the above are algebraic groups over L, so their L-points are locally L-analytic groups; we denote these groups of points by the corresponding Roman letters.

Theorem 5.3.

1. *For any unitary admissible continuous G-representation V, there is a unique isomorphism of admissible continuous M_2-representations*

$$\mathrm{Ord}_{P_{12}}(\mathrm{Ord}_{P_1} V) = \mathrm{Ord}_{P_2} V$$

commuting with the canonical lifting maps from both sides into V^{N_2}.

2. *For any essentially admissible locally analytic G-representation V, there is a unique isomorphism of essentially admissible locally analytic M_2-representations*

$$J_{P_{12}}(J_{P_1} V) = J_{P_2} V$$

commuting with the canonical lifting maps.

Proof. We begin by proving the second statement. We have $N_2 = N_1 \rtimes N_{12}$, where $N_{12} = N_2 \cap M_1$ is the unipotent radical of P_{12}. Let $N_{2,0}$ be an open compact subgroup of N_2 which has the form $N_{1,0} \rtimes N_{12,0}$, for open compact subgroups of the two factors; such subgroups certainly exist, since the conjugation action of N_1 on N_{12} is continuous.

For $i = 1, 2$ we write M_i^+ for the submonoid of elements $m \in M_i$ for which $mN_{i,0}m^{-1} \subseteq N_{i,0}$ and $m^{-1}\overline{N}_{i,0}m \subseteq \overline{N}_{i,0}$, and $Z_i = M_i^+ \cap Z_i$. Then we have $M_2^+ \subseteq M_1^+$, and in particular $Z_1^+ \subseteq Z_2^+$.
We have

$$J_{P_1}V = \mathcal{L}_{b,Z_1^+}\left(C^{\mathrm{an}}(\widehat{Z}_1), V^{N_{1,0}}\right)$$

endowed with the action of $M_1 = Z_1 \times_{Z_1^+} M_1^+$ determined by the actions of Z_1 on $C^{\mathrm{an}}(\widehat{Z}_1)$ and M_1^+ on $V^{N_{1,0}}$. The restriction of this action to $N_{12,0}$ is simply the action on the right factor (since $N_{12,0} \subseteq M_{1,0} \subseteq M_1^+$) and hence

$$(J_{P_1}V)^{N_{12,0}} = \mathcal{L}_{b,Z_1^+}\left(C^{\mathrm{an}}(\widehat{Z}_1), (V^{N_{1,0}})^{N_{12,0}}\right) = \mathcal{L}_{b,Z_1^+}\left(C^{\mathrm{an}}(\widehat{Z}_1), V^{N_{2,0}}\right).$$

The Hecke operator construction of [Eme06a, §3.4] gives two actions of M_2^+ on $V^{N_{2,0}}$, given respectively by $m \circ v = \pi_{N_{2,0}}mv$ and $m \circ v = \pi_{N_{12,0}}\pi_{N_{1,0}}mv$, where the operators $\pi_{N_{i,0}}$ are the averaging operators with respect to Haar measure on the subgroups $N_{i,0}$. Since $N_{2,0} = N_{12,0} \ltimes N_{1,0}$, and the Haar measure on the product is the product of the Haar measures on the factors, these two actions coincide. Applying the preceding lemma with $Z = Z_2$ and $Y = Z_1$ gives the result.
The statement for the ordinary part functor can be proved along similar lines, but it is easier to note that the functor Ord_P is right adjoint to the parabolic induction functor $\mathrm{Ind}_{\overline{P}}^G$ [Eme10, 4.4.6], for $\overline{P}$ an opposite parabolic to P. Since a composition of adjunctions is an adjunction, it suffices to check instead that $\mathrm{Ind}_{\overline{P}_1}^G \mathrm{Ind}_{\overline{P}_{12}}^{M_1} U = \mathrm{Ind}_{\overline{P}_2}^G U$ for any $U \in \mathrm{Rep}_{\mathrm{u,adm}}(M_2)$. We may identify $C(G, C(M_1, U))$ with $C(G \times M_1, U)$. Evaluating at $1 \in M_1$ gives a map to $C(G, U)$, and it is easy to check that this restricts to an isomorphism between the subspaces realising the two induced representations. $\square$

5.2 Hecke algebras and the canonical lifting

We now turn to studying the Jacquet functor in a special case; later we will combine this with the transitivity result above to deduce a general statement. As before, let $\mathcal{G}$ be a reductive algebraic group over L, and let $\mathcal{H} = [\mathcal{G}, \mathcal{G}]$, a semisimple group. There is a bijection between parabolics of $\mathcal{G}$ and $\mathcal{H}$, given by $\mathcal{P} \mapsto \mathcal{P}' = \mathcal{P} \cap \mathcal{H}$ and $\mathcal{P}' \mapsto \mathcal{P} = N_{\mathcal{G}}(\mathcal{P}')$.
We also choose an opposite parabolic $\overline{\mathcal{P}}$, determining a Levi subgroup $\mathcal{M}$ of $\mathcal{P}$, and also a Levi $\mathcal{M}'$ of $\mathcal{P}'$ in the obvious way. Write $\mathcal{Z}_{\mathcal{M}}$, $\mathcal{Z}_{\mathcal{M}'}$ and $\mathcal{Z}_{\mathcal{G}}$ for the centres of these subgroups, so $\mathcal{Z}_{\mathcal{M}}$ is isogenous to $\mathcal{Z}_{\mathcal{M}'} \times \mathcal{Z}_{\mathcal{G}}$. As before, we use Roman letters for the L-points of these algebraic groups.
Let H_0 be an open compact subgroup of H. We say H_0 is *decomposed* with respect to P' and $\overline{P}'$ if the product of the subgroups $M_0' = H_0 \cap M'$, $N_0 = H_0 \cap N$ and $\overline{N}_0 = H_0 \cap \overline{N}$ is H_0, for any ordering of the factors.
We say an element $m \in M$ is *positive* (for H_0) if $mN_0m^{-1} \subseteq N_0$ and $m^{-1}\overline{N}_0m \subseteq \overline{N}_0$ (see [Bus01, §3.1]). Let $M^{\oplus} \subseteq M$ be the monoid of positive elements, and $Z_M^{\oplus}$ its intersection with Z_M; and let $\mathcal{H}^{\oplus}(M_0')$ denote the

subalgebra of the Hecke algebra $\mathcal{H}(M_0')$ supported on $M'^{+} = M^{+} \cap M'$. Note that $M^{\oplus}$ is contained in the monoid M^{+} of the previous section, and clearly has finite index therein.
We say an element $z \in Z_M$ is *strongly positive* if the sequences $z^n N_0 z^{-n}$ and $z^{-n}\overline{N}_0 z^n$ tend monotonically to $\{1\}$; if this holds, then z^{-1} and $M^{\oplus}$ together generate M. Such elements exist in abundance; any element whose pairing with the roots corresponding to P has sufficiently large valuation will suffice. In particular, there exist strongly positive elements in $Z_{M'}$.

Proposition 5.4. *For any essentially admissible G-representation V, we have $J_P(V) = \left(V^{N_0}\right)_{Y-\mathrm{fs}}$, where Y is any closed subgroup of M that contains a strongly positive element. In particular, $J_P(V) = J_{P'}(V)$.*

Proof. For any open compact $N_0 \subseteq N$, [Eme06a, lemma 3.2.29] and the discussion following it shows that V^{N_0} is in the category denoted therein by $\mathrm{Rep}^{z}_{\mathrm{la,c}}(Z_M^{+})$; thus the hypotheses of [Eme06a, prop 3.2.28] are satisfied for the subgroup $Y = Z_{M'}$. The conclusion of that proposition then states that $J_P(V) = (V^{N_0})_{Z_M-\mathrm{fs}} = (V^{N_0})_{Y-\mathrm{fs}}$. ◻

We now lighten the notation somewhat by writing superscript $+$ for $\oplus$, since the proposition shows that the distinction between M^{+} and $M^{\oplus}$ is unimportant from the perspective of Jacquet modules.

Proposition 5.5. *Let j be the morphism $\mathcal{H}^{+}(M_0') \to \mathcal{H}(H_0)$ constructed in [Bus01, §3.3]. Then the natural inclusion $V^{H_0} \hookrightarrow V^{M_0' N_0}$ is $\mathcal{H}^{+}(M_0')$-equivariant, where $\mathcal{H}^{+}(M_0')$ acts via j on the first space and via its inclusion into $\mathcal{H}(M_0')$ on the second.*

Proof. Easy check. ◻

Proposition 5.6. *For any essentially admissible locally analytic G-representation V which is smooth as an H-representation, the above inclusion induces an isomorphism*

$$(V^{H_0})_{Z_{M'}-\mathrm{fs}} \xrightarrow{\sim} (V^{M_0' N_0})_{Z_{M'}-\mathrm{fs}} = J_P(V)^{M_0'}.$$

Moreover, there exists a direct sum decomposition

$$V^{H_0} = (V^{H_0})_{Z_{M'}-\mathrm{fs}} \oplus (V^{H_0})_{Z_{M'}-\mathrm{null}}$$

where the summands are closed subspaces of V^{H_0}, stable under the action of Z_G and $\mathcal{H}(M_0')$.

Proof. Let $Q = V^{M_0' N_0}/V^{H_0}$. By the left-exactness of the finite slope part functor [Eme06a, 3.2.6(ii)], there is a closed embedding

$$(V^{M_0' N_0})_{Z_{M'}-\mathrm{fs}}/(V^{H_0})_{Z_{M'}-\mathrm{fs}} \hookrightarrow Q_{Z_{M'}-\mathrm{fs}}.$$

But since V is smooth as an H-representation, every element $v \in V^{M_0' N_0}$ is in fact in $V^{U M_0' N_0}$ for some open $U \subseteq \overline{N}$; any such U contains a $Z_{M'}^+$-conjugate of $\overline{N}_0$, so there is some $z \in Z^+$ such that $zv \in V^{\overline{N}_0 M_0'}$. Our hypothesis that H_0 is decomposed implies that the averaging operator $\pi_{N_0} : V^{\mathfrak{n}} \to V^{N_0}$ preserves $V^{\overline{N}_0 M_0'}$, so $z \circ v = \pi_{N_0}(zv) \in V^{H_0}$. Therefore Q is $Z_{M'}^+$-torsion, and thus clearly $Q_{Z-\mathrm{fs}} = 0$.

For the second statement, let z be any strongly positive element of $Z_{M'}$. By [Bus01, Theorem 1], there exists an integer n (depending only on P, H_0 and z) such that for any smooth H-representation V, the action of z on V^{H_0} via j satisfies

$$V^{H_0} = z^n V^{H_0} \oplus \operatorname{Ker}(z^n \mid V^{H_0}),$$

with z invertible on the subspace $z^n V^{H_0}$. For representations V as in the statement, the subspace $\operatorname{Ker}(z^n \mid V^{H_0})$ is clearly closed, and moreover z^n gives a continuous map from the essentially admissible Z_G-representation V^{H_0} to itself, so its image is also closed. □

In particular, since V^{H_0} is an essentially admissible Z_G-representation, $J_P(V)^{M_0'}$ is essentially admissible as a Z_G-representation, not just as a representation of the larger group $Z_G \times Z_{M'}/(Z_{M'} \cap H_0)$.

Remark. If H_0 satisfies the stronger conditions of [Bus01, §1.2], we obtain a finer decomposition of V^{H_0} into a direct sum of closed Z_G-subrepresentations corresponding to Bernstein components of H.

5.3 Jacquet modules of locally isotypical representations

We now extend the results on H-smooth representations above to certain locally H-isotypical representations.

Proposition 5.7. *If W is a twist of an absolutely irreducible algebraic representation of $\mathcal{G}$, and $P = MN$ is a parabolic subgroup of G with $[M, M] = D$, then $\operatorname{End}_{\mathfrak{d}}(W^N) = K$, so in particular the M-representation W^N is D-good.*

Proof. The twist is of no consequence, so suppose that W is algebraic. Let us choose a maximal torus T in M, and a field $K' \supset K$ over which M is split; then there is a Borel subgroup $B \subseteq P$ defined over K' with Levi factor T. The theory of highest weights then shows that W is absolutely irreducible if and only if the highest weight space of W is 1-dimensional; applying this condition to W and to the M-representation W^N, we deduce that W^N is absolutely irreducible as an M-representation. Since M is isogenous to $D \times Z(M)$ and all absolutely irreducible representations of $Z(M)$ are one-dimensional, it follows that W^N is in fact absolutely irreducible as a D-representation. □

Proposition 5.8. *If $W \in \operatorname{Rep}_{\mathrm{la,fd}}(G)$ is H-good, with $B = \operatorname{End}_{\mathfrak{h}}(W) = \operatorname{End}_G W$, and furthermore $W^{\mathfrak{n}} = W^N$, then for any $X \in \operatorname{Rep}^z_{\mathrm{la,c}}(G)$ which is smooth as an H-representation and has a right action of B, we have*

$$J_P(X \otimes_B W) = J_P(X) \otimes_B W^N.$$

Proof. Compare [Eme06a, 4.3.4]. Since X is smooth as an H-representation it is certainly smooth as an N-representation. Arguing as in the proof of proposition 2.6, we have $(X \otimes_B W)^{N_0} = X^{N_0} \otimes_B W^{N_0}$, which by assumption equals $X^{N_0} \otimes_B W^N$. Passing to finite-slope parts now yields the result. $\square$

The condition $W^{\mathfrak{n}} = W^N$ is certainly satisfied for any W that is algebraic as a representation of N (since any open subgroup of N is Zariski-dense).

Proposition 5.9. *Let W be a twist of an absolutely irreducible algebraic representation of G, and let $V \in \mathrm{Rep}^{z}_{\mathrm{la,c}}(P)$ be locally (H, W)-isotypical. Then $J_P(V)$ is locally (D, W^N)-isotypical, and*

$$\mathrm{Hom}_{\mathfrak{d}}(W^N, J_P(V)) = J_P(\mathrm{Hom}_{\mathfrak{h}}(W, V)).$$

Proof. Let $X = \mathrm{Hom}_{\mathfrak{h}}(W, V)$. By proposition 2.6, we have $V = X \otimes_K W$; so by proposition 5.8 and the remark following, $J_P(V) = J_P(X) \otimes_K W^N$. Since W^N is D-good, we can apply the converse implication of proposition 2.6 to deduce that $J_P(X) = \mathrm{Hom}_{\mathfrak{d}}(W^N, J_P(V))$ as required. $\square$

5.4 Combining the constructions

We now summarize the results of the above analysis.

Theorem 5.10. *For any $V \in \mathrm{Rep}_{\mathrm{ess}}(G)$, we have:*

1. *For any parabolic subgroup $P \subseteq G$ with Levi subgroup M, any finite-dimensional $W \in \mathrm{Rep}_{\mathrm{la,c}}(M)$, and any open compact subgroup $U \subseteq D = [M, M]$, there is a coherent sheaf $\mathcal{F}(V, P, W, U)$ on $\widehat{Z(M)}$ with a right action of $\mathcal{H}(U)$, whose fibre at a character $\chi \in \widehat{Z(M)}$ is isomorphic (as a right $\mathcal{H}(U)$-module) to the dual of the space $\mathrm{Hom}_U(W, J_P V)^{Z(M)=\chi}$. In particular, a character χ lies in the subvariety $\mathcal{S}(V, P, W, U) = \mathrm{support}\, \mathcal{F}(V, P, W, U)$ if and only if this eigenspace is nonzero.*

2. *If $V \in \mathrm{Rep}_{\mathrm{la,ad}}(G)$, then the projection $\mathcal{S}(V, P, W, U) \to (\mathrm{Lie}\, Z)'$ has discrete fibres.*

3. *If V is isomorphic as an H-representation to $C^{\mathrm{la}}(H)^m$ for some m and some open compact $H \subseteq G$, then $\mathcal{S}(V, P, W, U)$ is equidimensional of dimension $\dim Z$.*

4. *If P_1, P_2 are parabolics with $P_1 \supseteq P_2$ as above, W is an absolutely irreducible algebraic representation of M_1, and U is an open compact subgroup of D_1 which is decomposed with respect to the parabolic $P_2 \cap D_1$, then there is a decomposition*

$$\mathcal{F}(V, P_1, U, W) = \mathcal{F}(V, P_1, U, W)_{Z_2-\mathrm{null}} \oplus \mathcal{F}(V, P_1, U, W)_{Z_2-\mathrm{fs}},$$

where the latter factor is isomorphic to a quotient of the pushforward to Z_1 of the sheaf

$$\mathcal{F}(V, P_2, W^N, U \cap D_2)$$

on Z_2.

Proof. The only statement still requiring proof is the last one. Let $Y = (J_{P_1} V)_{D_1, W-\mathrm{liso}}$. The closed embedding $Y \hookrightarrow J_{P_1}(V)$ induces by functoriality a closed embedding $J_{P_{12}} Y \hookrightarrow J_{P_{12}}(J_{P_1} V)$. The right-hand side is simply $J_{P_2} V$, by theorem 5.3. Thus we have a closed embedding

$$\mathrm{Hom}_{\mathfrak{d}_2}(W^N, J_{P_{12}} Y) \hookrightarrow \mathrm{Hom}_{\mathfrak{d}_2}(W^N, J_{P_2} V).$$

The left-hand side is isomorphic, by proposition 5.9, to $J_{P_{12}}[\mathrm{Hom}_{\mathfrak{d}_1}(W, Y)]$. We may now apply proposition 5.6 to the M_1-representation $\mathrm{Hom}_{\mathfrak{d}_1}(W, Y) = \mathrm{Hom}_{\mathfrak{d}_1}(W, J_{P_1} V)$, to deduce that there is a direct sum decomposition

$$\mathrm{Hom}_U(W, J_{P_1} V) = \mathrm{Hom}_U(W, J_{P_1} V)_{Z_2-\mathrm{fs}} \oplus \mathrm{Hom}_U(W, J_{P_1} V)_{Z_2-\mathrm{null}}$$

and the first direct summand is isomorphic as a Z_2-representation to a closed subspace of

$$\mathrm{Hom}_{U \cap M_2'}(W^N, J_{P_{12}} Y) \subseteq \mathrm{Hom}_{U \cap D_2}(W^N, J_{P_2} V).$$

Dualising, we obtain the stated relation between the sheaves $\mathcal{F}(\dots)$. □

6 Application to completed cohomology

6.1 Construction of eigenvarieties

Let us now fix a number field F, a connected reductive group $\mathfrak{G}$ over F, and a prime $\mathfrak{p}$ of F above p. Let $\mathcal{G} = \mathfrak{G} \times_F F_{\mathfrak{p}}$, a reductive group over $F_{\mathfrak{p}}$, and $G = \mathcal{G}(F_{\mathfrak{p}})$. Let us choose a parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ (not necessarily arising from a parabolic subgroup of $\mathfrak{G}$), and set $P = \mathcal{P}(F_{\mathfrak{p}})$, and similarly for M, N, D, Z as above. We suppose our base field L is a subfield of $F_{\mathfrak{p}}$, so G, P, M, N, D, Z are locally L-analytic groups.

We recall from [Eme04, 2.2.16] the construction of the completed cohomology spaces $\tilde{H}^i$ for each cohomological degree $i \geq 0$, which are unitary admissible Banach representations of $(G, \mathfrak{G}(\mathbb{A}_f^{\mathfrak{p}}) \times \pi_0)$, where π_0 is the group of components of $\mathfrak{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$. The following is immediate from the above:

Proposition 6.1. *Let $\Gamma = D \times \mathfrak{G}(\mathbb{A}_f^{\mathfrak{p}}) \times \pi_0$. For any $i \geq 0$, we have:*

1. *For any $W \in \mathrm{Rep}_{\mathrm{cts,fd}}(M)$, the space*

$$\mathrm{Hom}_{D-\mathrm{st.sm}}(W, \mathrm{Ord}_P \tilde{H}^i)$$

is an admissible continuous (Z, Γ)-representation.

2. For any $W \in \mathrm{Rep}_{\mathrm{la,fd}}(M)$, the space

$$\mathrm{Hom}_{\mathfrak{d}}(W, J_P \tilde{H}^i_{\mathrm{la}})$$

is an essentially admissible locally L-analytic (Z, Γ)-representation.

Let us fix an open compact subgroup $U \subseteq \Gamma$ (this is the most natural notion of a "tame level" in this context). Then we can use the above result to define an eigenvariety of tame level U, closely following [Eme06b, §2.3].
Let v be a (finite or infinite) prime of S. We set

$$\Gamma_v = \begin{cases} \mathfrak{G}(F_v) & \text{if } v \nmid \infty \text{ and } v \neq \mathfrak{p} \\ D & \text{if } v = \mathfrak{p} \\ \pi_0(\mathfrak{G}(F_v)) & \text{if } v \mid \infty. \end{cases}$$

Then $\Gamma = \prod'_v \Gamma_v$. Let us set $U_v = U \cap \Gamma_v$. We say v is *unramified* (for U) if v is finite, $v \neq \mathfrak{p}$, and U_v is a hyperspecial maximal compact subgroup of Γ_v. Let S be the (clearly finite) set of ramified primes, and $\Gamma^S = \prod_{v \notin S} \Gamma_v$, $\Gamma_S = \prod_{v \in S} \Gamma_v$.
It is easy to see that $U = U_S \times U^S$, where $U^S = U \cap \Gamma^S$ and similarly $U_S = U \cap \Gamma_S$, and hence we have a tensor product decomposition of Hecke algebras

$$\mathcal{H}(\Gamma//U) = \mathcal{H}(\Gamma_S//U_S) \otimes \mathcal{H}(\Gamma^S//U^S) =: \mathcal{H}^{\mathrm{ram}} \otimes \mathcal{H}^{\mathrm{sph}}.$$

As is well known, the algebra $\mathcal{H}^{\mathrm{sph}}$ is commutative (but not finitely generated over K), while $\mathcal{H}^{\mathrm{ram}}$ is finitely generated (but not commutative in general).
By construction, $\mathcal{H}(\Gamma//U)$ acts on the essentially admissible Z-representation $\mathrm{Hom}_U(W, J_P \tilde{H}^i_{\mathrm{la}})$, and hence it also acts on the corresponding sheaf $\mathcal{F}(i, P, W, U)$ on $\widehat{Z}$.

DEFINITION 6.2. *Let $\mathcal{E}(i, P, W, U)$ be the relative spectrum $\underline{\mathrm{Spec}}\,\mathcal{A}$, where $\mathcal{A}$ is the $\mathcal{O}_{\widehat{Z}}$-subsheaf of $\underline{\mathrm{End}}\,\mathcal{F}(i, P, W, U)$ generated by the image of $\mathcal{H}^{\mathrm{sph}}$.*

For the definition of the relative spectrum, see [Con06, Thm 2.2.5]. By definition $\mathcal{E}(i, P, W, U)$ is a rigid space over K, endowed with a finite morphism $\pi : \mathcal{E}(i, P, W, U) \to \widehat{Z}$ and an isomorphism of sheaves of $\mathcal{O}_{\widehat{Z}}$-algebras $\mathcal{A} \cong \pi_* \mathcal{O}_{\mathcal{E}(i,P,W,U)}$. Consequently, $\mathcal{F}(i, P, W, U)$ lifts to a sheaf $\overline{\mathcal{F}}(i, P, W, U)$ on $\mathcal{E}(i, P, W, U)$.
We can regard $\mathcal{E}(i, P, W, U)$ as a subvariety of $\widehat{Z}_K \times \mathrm{Spec}\,\mathcal{H}^{\mathrm{sph}}$ (although the latter will not be a rigid space if $\mathfrak{G}$ is not the trivial group); in particular, a K-point of $\mathcal{E}(i, P, W, U)$ gives rise to a homomorphism $\lambda : \mathcal{H}^{\mathrm{sph}} \to K$.
We record the following properties of this construction, which are precisely analogous to [Eme06b, 2.3.3]:

THEOREM 6.3.

1. *The natural projection $\mathcal{E}(i,P,W,U) \to \mathfrak{z}'$ has discrete fibres. In particular, the dimension of $\mathcal{E}(i,P,W,U)$ is at most equal to the dimension of Z.*

2. *The action of $\mathcal{H}^{\mathrm{ram}}$ on $\mathcal{F}(i,P,W,U)$ lifts to an action on $\overline{\mathcal{F}}(i,P,W,U)$, and the fibre of $\overline{\mathcal{F}}(i,P,W,U)$ at a point $(\chi,\lambda) \in \widehat{Z} \times \operatorname{Spec}\mathcal{H}^{\mathrm{sph}}$ is isomorphic as a right $\mathcal{H}^{\mathrm{ram}}$-module to the dual of the $(Z=\chi, \mathcal{H}^{\mathrm{sph}}=\lambda)$-eigenspace of $\operatorname{Hom}_U(W, J_P\widetilde{H}^i_{\mathrm{la}})$. In particular, the point (χ,λ) lies in $\mathcal{E}(i,P,W,U)$ if and only if this eigenspace is non-zero.*

3. *If there is a compact open subgroup $G_0 \subseteq G$ such that $(\tilde{H}^i_{\mathrm{la}})^{U^{(\mathfrak{p})}}$ is isomorphic as a G_0-representation to a finite direct sum of copies of $C^{\mathrm{la}}(G_0)$ (where $U^{(\mathfrak{p})} = U \cap \mathfrak{G}(\mathbb{A}_f^{\mathfrak{p}})$), then $\mathcal{E}(i,P,W,U)$ is equidimensional, of dimension equal to the rank of Z.*

Remark. The hypothesis in the last point above is always satisfied when $i=0$ and $\mathfrak{G}(F\otimes\mathbb{R})$ is compact, since for any open compact subgroup $U^{(\mathfrak{p})} \subseteq \mathfrak{G}(\mathbb{A}_f^{\mathfrak{p}})$, the image of $G(F)\cap U^{(\mathfrak{p})}$ in G is a discrete cocompact subgroup Λ, and the $U^{(\mathfrak{p})}$-invariants $\tilde{H}^0(U^{(\mathfrak{p})})$ are isomorphic as a representation of G and as a $\mathcal{H}(U^{(\mathfrak{p})})$-module to $C(\Lambda\backslash G)$. This case is considered extensively in an earlier publication of the second author [Loe11].

Now let us suppose G is split over K, and fix an irreducible (and therefore absolutely irreducible) algebraic representation W of M. We let $\Pi(P,W,U)$ denote the set of irreducible smooth $\mathfrak{G}(\mathbb{A}_f)\times\pi_0$-representations π_f such that $J_P(\pi_f)^U \ne 0$, and π_f appears as a subquotient of the cohomology space $H^i(\mathcal{V}_X)$ of [Eme06b, §2.2] for some irreducible algebraic representation X of G with $(X')^N \cong W\otimes\chi$ for some character χ. To any such π_f, we may associate the point $(\theta\chi,\lambda)\in\widehat{Z}\times\operatorname{Spec}\mathcal{H}^{\mathrm{sph}}$, where θ is the smooth character by which Z acts on $J_P(\pi_p)$, and λ the character by which $\mathcal{H}^{\mathrm{sph}}$ acts on $J_P(\pi_f)^U$. Let $E(i,P,W,U)_{\mathrm{cl}}$ denote the set of points of $\widehat{Z}\times\operatorname{Spec}\mathcal{H}^{\mathrm{sph}}$ obtained in this way from representations $\pi_f \in \Pi(i,P,W,U)$.

Corollary 6.4. *If the map* (1.1) *is an isomorphism for all irreducible algebraic representations X such that $(X')^N$ is a twist of W, then $E(i,P,W,U)_{\mathrm{cl}} \subseteq \mathcal{E}(i,P,W,U)$. In particular, the Zariski closure of $E(i,P,W,U)_{\mathrm{cl}}$ has dimension at most* $\dim Z$.

Proof. Let $\pi_f\in\Pi(i,P,W,U)$. Then the locally algebraic $(G,\mathfrak{G}(\mathbb{A}_f^{(\mathfrak{p})})\times\pi_0)$-representation $\pi_f\otimes X'$ appears in $H^i(\mathcal{V}_X)\otimes_K X'$. By assumption, the latter embeds as a closed subrepresentation of $\widetilde{H}^i_{\mathrm{la}}$. The Jacquet functor is exact restricted to locally X'-algebraic representations (since this is so for smooth representations). Moreover, the functor $\operatorname{Hom}_{\mathfrak{d}}(W,-)$ is exact restricted to locally W-algebraic representations, so $\operatorname{Hom}_{\mathfrak{d}}\left[W, J_P(\pi_f)\otimes_K (X')^N\right]$ appears as a subquotient of $\operatorname{Hom}_{\mathfrak{d}}\left[W, J_P(\tilde{H}^i_{\mathrm{la}})\right]$. Since $(X')^N = W\otimes\chi$, the former space is simply $J_P(\pi_f)\otimes_K\chi$, so the point $(\theta\chi,\lambda)$ appears in $\mathcal{E}(i,P,W,U)$ as required. ∎

Remarks. 1. The entire construction can also be carried out with the spaces $\tilde{H}^i$ replaced by the compactly supported versions $\tilde{H}^i_{\mathrm{c}}$ or the parabolic versions $\tilde{H}^i_{\mathrm{par}}$; we then obtain analogues of the above proposition for the compactly supported or parabolic cohomology of the arithmetic quotients.

2. It suffices to check that the map (1.1) is an isomorphism for $L = \mathbb{Q}_p$. This is known to hold in many cases, e.g. in degree $i = 0$ for any $\mathfrak{G}$, and in degree 1 for $\mathrm{GL}_2(\mathbb{Q})$ (as shown in [Eme06b]) or for a semisimple and simply connected group (as shown by the first author in [Hil07]). The "weak Emerton criterion" of [Hil07, defn. 2] suffices to prove corollary 6.4 when W is not a character; this is known in many more cases, e.g. when $i = 2$ and the congruence kernel of $\mathfrak{G}$ is finite. When W is a character $\chi : M \to \mathbb{G}_m$, the weak Emerton criterion implies that the points $E(i, P, W, U)_{\mathrm{cl}}$ are contained in the union of $\mathcal{E}(i, P, W, U)$ and the single point $(\chi^{-1}, 1)$.

Theorem 6.5. *Suppose $P_1 \supseteq P_2$ are two parabolics, and $U = U^{(\mathfrak{p})} \times U_{\mathfrak{p}}$, where $U^{(\mathfrak{p})} \subseteq \mathfrak{G}(\mathbb{A}_f^{(\mathfrak{p})}) \times \pi_0$ and $U_{\mathfrak{p}} \subseteq D_1$ is decomposed with respect to $P_2 \cap D_1$. Then $\mathcal{E}(i, P_1, W, U)$ is equal to a union of two closed subvarieties*

$$\mathcal{E}(i, P_1, W, U)_{P_2-\mathrm{fs}} \cap \mathcal{E}(i, P_1, W, U)_{P_2-\mathrm{null}},$$

which are respectively equipped with sheaves of $\mathcal{H}^{\mathrm{ram}}$-modules $\overline{\mathcal{F}}(i, P, W, U)_{P_2-\mathrm{fs}}$ and $\overline{\mathcal{F}}(i, P, W, U)_{P_2-\mathrm{null}}$ whose direct sum is $\overline{\mathcal{F}}(i, P, W, U)$.
The element of $\mathcal{H}^{\mathrm{ram}}$ corresponding to any strictly positive element of Z_2 acts invertibly on $\overline{\mathcal{F}}(i, P, W, U)_{P_2-\mathrm{fs}}$ and nilpotently on $\overline{\mathcal{F}}(i, P, W, U)_{P_2-\mathrm{null}}$; and there is a subvariety of $\mathcal{E}(i, P_2, W^{N_{12}}, U \cap D_2)$ whose image in $\widehat{Z}_1 \times \operatorname{Spec} \mathcal{H}^{\mathrm{sph}}$ coincides with $\mathcal{E}(i, P_1, W, U)_{P_2-\mathrm{fs}}$.

Proof. By theorem 5.10, we may decompose $\mathcal{F}(i, P_1, W, U)$ as a direct sum of a null part and a finite slope part; this decomposition is clearly functorial, and hence it is preserved by the action of the Hecke algebra $\mathcal{H}^{\mathrm{sph}}$, so we may define the spaces $\mathcal{E}(i, P_1, W, U)_{P_2-\mathrm{fs}}$ and $\mathcal{E}(i, P_1, W, U)_{P_2-\mathrm{null}}$ to be the relative spectra of the Hecke algebra acting on the two summands.
For the final statement, we note that there is a quotient $\mathcal{Q}$ of $\mathcal{F}(i, P_2, W^{N_{12}}, U \cap D_2)$, corresponding to the Z_2-subrepresentation

$$J_{P_{12}} \left(\mathrm{Hom}_{\mathfrak{d}_1}(W, J_{P_1} \widetilde{H}^i_{\mathrm{la}}) \right)^{U \cap M_2} \subseteq \mathrm{Hom}_{\mathfrak{d}_2}(W^{N_{12}}, J_{P_2} \widetilde{H}^i_{\mathrm{la}})^{U \cap D_2}$$

such that the pushforward of $\mathcal{Q}$ to $\widehat{Z}_1$ is isomorphic to $\mathcal{F}(i, P_1, W, U)_{P_2-\mathrm{fs}}$. This isomorphism clearly commutes with the action of $\mathcal{H}^{\mathrm{sph}}$ on both sides, from which the result follows. ☐

7 Acknowledgements

The second author would like to thank Kevin Buzzard and Matthew Emerton for illuminating conversations; the participants in the autum 2009 "P-adic

Analysis" study group at DPMMS, University of Cambridge, for their assistance in learning the necessary theory; and Sarah for her love and support.

REFERENCES

[Bou87] Nicolas Bourbaki, *Topological vector spaces. Chapters 1–5*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1987, Translated from the French by H. G. Eggleston and S. Madan. MR 910295.

[Bus01] Colin J. Bushnell, *Representations of reductive p-adic groups: localization of Hecke algebras and applications*, J. London Math. Soc. (2) 63 (2001), no. 2, 364–386. MR 1810135.

[Buz07] Kevin Buzzard, *Eigenvarieties*, L-functions and Galois representations (Durham, 2004), London Math. Soc. Lecture Notes, vol. 320, Cambridge Univ. Press, 2007, pp. 59–120. MR 2392353.

[Col96] Robert F. Coleman, *Classical and overconvergent modular forms*, Invent. Math. 124 (1996), no. 1-3, 215–241. MR 1369416.

[Col97] ———, *p-adic Banach spaces and families of modular forms*, Invent. Math. 127 (1997), no. 3, 417–479. MR 1431135.

[CM98] Robert F. Coleman and Barry Mazur, *The eigencurve*, Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Notes, vol. 254, Cambridge Univ. Press, 1998, pp. 1–113. MR 1696469.

[Con06] Brian Conrad, *Relative ampleness in rigid geometry*, Ann. Inst. Fourier (Grenoble) 56 (2006), no. 4, 1049–1126. MR 2266885.

[Eme04] Matthew Emerton, *Locally analytic vectors in representations of locally p-adic analytic groups*, Mem. Am. Math. Soc. (to appear), 2004.

[Eme06a] ———, *Jacquet modules of locally analytic representations of p-adic reductive groups. I. Construction and first properties*, Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 5, 775–839. MR 2292633.

[Eme06b] ———, *On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms*, Invent. Math. 164 (2006), no. 1, 1–84. MR 2207783.

[Eme10] ———, *Ordinary parts of admissible representations of p-adic reductive groups. I. Construction and first properties*, Astérisque 331 (2010), 355–402. MR 2667882.

[Hil07] Richard Hill, *Construction of eigenvarieties in small cohomological dimensions for semi-simple, simply connected groups*, Doc. Math. 12 (2007), 363–397. MR 2365907.

[Loe11] David Loeffler, *Overconvergent algebraic automorphic forms*, Proc. London Math. Soc. 102 (2011), no. 2, 193–228.

[ST02a] Peter Schneider and Jeremy Teitelbaum, *Banach space representations and Iwasawa theory*, Israel J. Math. 127 (2002), 359–380. MR 1900706.

[ST02b] ———, *Locally analytic distributions and p-adic representation theory, with applications to* GL_2, J. Am. Math. Soc. 15 (2002), no. 2, 443–468. MR 1887640.

[ST03] ———, *Algebras of p-adic distributions and admissible representations*, Invent. Math. 153 (2003), no. 1, 145–196. MR 1990669.

[Ser92] Jean-Pierre Serre, *Lie algebras and Lie groups (1964 lectures given at Harvard University)*, second ed., Lecture Notes in Mathematics, vol. 1500, Springer-Verlag, Berlin, 1992. MR 1176100.

Richard Hill
Department of Mathematics
University College London
Gower Street
London WC1E 6BT, UK
rih@math.ucl.ac.uk

David Loeffler
Warwick Mathematics Institute
Zeeman Building
University of Warwick
Coventry CV4 7AL, UK
d.loeffler.01@cantab.net

Log-Growth Filtration and Frobenius Slope Filtration of F-Isocrystals at the Generic and Special Points

Bruno Chiarellotto[1] and Nobuo Tsuzuki[2]

Received: October 9, 2009
Revised: February 25, 2010

Communicated by Peter Schneider

Abstract. We study, locally on a curve of characteristic $p > 0$, the relation between the log-growth filtration and the Frobenius slope filtration for F-isocrystals, which we will indicate as φ-∇-modules, both at the generic point and at the special point. We prove that a bounded φ-∇-module at the generic point is a direct sum of pure φ-∇-modules. By this splitting of Frobenius slope filtration for bounded modules we will introduce a filtration for φ-∇-modules (PBQ filtration). We solve our conjectures of comparison of the log-growth filtration and the Frobenius slope filtration at the special point for particular φ-∇-modules (HPBQ modules). Moreover we prove the analogous comparison conjecture for PBQ modules at the generic point. These comparison conjectures were stated in our previous work [CT09]. Using PBQ filtrations for φ-∇-modules, we conclude that our conjecture of comparison of the log-growth filtration and the Frobenius slope filtration at the special point implies Dwork's conjecture, that is, the special log-growth polygon is above the generic log-growth polygon including the coincidence of both end points.

2010 Mathematics Subject Classification: Primary 12H25; Secondary 11S80.
Keywords and Phrases: p-adic differential equations, logarithmic growth, Frobenius slopes, Newton polygon.

[1]The author is supported by Progetto di Ateneo Padova "Rapresentations and Motives" and Progetto Eccellenza "Differential methods in arithmetic geometry and algebra" (Cariparo Found.).

[2]The author is supported by Grant-in-Aid for Exploratory Research and Core-to-Core Program 18005 of JSPS.

Contents

1 Introduction

The local behavior of p-adic linear differential equations is, in one sense, very easy. If the equation has a geometric origin (i.e., if it is furnished with a Frobenius structure), then the radius of convergence of solutions at any nonsingular point is at least 1. In general, the p-adic norm of the coefficients a_n in the Taylor series of a solution is an increasing function on n. However, one knows that some solutions are p-adically integral power series. B.Dwork discovered these mysterious phenomena and introduced a measure, called logarithmic growth (or log-growth, for simplicity), for power series in order to investigate this delicate difference systematically (see [Dw73] and [Ka73, Section 7]). He studied the log-growth of solutions of p-adic linear differential equations both at the generic point and at special points (see [Ro75], [Ch83]), and asked whether the behaviors are similar to those of Frobenius slopes or not. He conjectured that the Newton polygon of log-growth of solutions at a special point is above the Newton polygon of log-growth of solutions at the generic point [Dw73, Conjecture 2]. We refer to it as Conjecture $\mathbf{LG}_{\mathrm{Dw}}$ when there are not Frobenius structures, and as Conjecture $\mathbf{LGF}_{\mathrm{Dw}}$ where there are Frobenius structures (see Conjecture 2.7). He also proved that the Newton polygon of log-growth of solutions at the generic (resp. special) point coincides with the Newton polygon of Frobenius slopes in the case of hypergeometric Frobenius-differential systems if the systems are nonconstant, thus establishing the conjecture in these cases [Dw82, 9.6, 9.7, 16.9].

On the other hand P.Robba studied the generic log-growth of differential modules defined over the completion of $\mathbb{Q}(x)$ under the p-adic Gauss norm by introducing a filtration on them via p-adic functional analysis [Ro75] (see Theorem 2.2). His theory works on more general p-adically complete fields, for example our field $\mathcal{E}$.

Let k be a field of characteristic $p > 0$, let $\mathcal{V}$ be a discrete valuation ring with residue field k, and let K be the field of fractions of $\mathcal{V}$ such that the characteristic of K is 0. In [CT09] we studied Dwork's problem on the log-growth for φ-∇-modules over $\mathcal{E}$ or $K[\![x]\!]_0$ which should be seen as localizations of F-isocrystals on a curve over k with coefficients in K. Here $K[\![x]\!]_0$ is the ring of bounded functions on the unit disk around $x = 0$, $\mathcal{E}$ is the p-adically complete field which is the field of fractions of $K[\![x]\!]_0$, and φ (resp. ∇) indicates the Frobenius structure (resp. the connection) (See the notation and terminology introduced in Section 2). We gave careful attention to Dwork's result on the comparison between the log-growth and the Frobenius slopes of hypergeometric Frobenius-differential equations. We compared the log-growth and the Frobenius slopes at the level of filtrations.

Let M be a φ-∇-module over $K[\![x]\!]_0$. Let $M_\eta = M \otimes_{K[\![x]\!]_0} \mathcal{E}$ be a φ-∇-module over $\mathcal{E}$ which is the generic fiber of M and let $V(M)$ be the φ-module over K consisting of horizontal sections on the open unit disk. Denote by M_η^λ the log-growth filtration on M_η at the generic point indexed by $\lambda \in \mathbb{R}$, and by $V(M)^\lambda$ be the log-growth filtration with real indices on the φ-module $V(M)$. Furthermore, let $S_\lambda(\,\cdot\,)$ be the Frobenius slope filtration such that $S_\lambda(\,\cdot\,)/S_{\lambda-}(\,\cdot\,)$ is pure of slope λ.

We proved that the log-growth filtration is included in the orthogonal part of the Frobenius slope filtration of the dual module under the natural perfect pairing $M_\eta \otimes_{\mathcal{E}} M_\eta^\vee \to \mathcal{E}$ (resp. $V(M) \otimes_K V(M^\vee) \to K$) at the generic point (resp. the special point) [CT09, Theorem 6.17] (see the precise form in Theorem 2.3):

$$M_\eta^\lambda \subset (S_{\lambda-\lambda_{\max}}(M_\eta^\vee))^\perp \qquad (\text{resp.}\, V(M)^\lambda \subset (S_{\lambda-\lambda_{\max}}(V(M^\vee)))^\perp)$$

for any $\lambda \in \mathbb{R}$ if $\lambda_{\max}$ is the highest Frobenius slope of M_η. We then conjectured: (a) the rationality of log-breaks λ (both at the generic and special fibers) and (b) if the bounded quotient M_η/M_η^0 is pure as a φ-module then the inclusion relation becomes equality both at the generic and special points [CT09, Conjectures 6.8, 6.9]. The hypothesis of (b) will be called the condition of being "pure of bounded quotient" (PBQ) in Definition 5.1. Note that there are examples with irrational breaks, and that both $M^{\lambda-} \supsetneq M^\lambda$ and $M^\lambda \supsetneq M^{\lambda+}$ can indeed occur for log-growth filtrations in absence of Frobenius structures [CT09, Examples 5.3, 5.4]. We state the precise forms of our conjectures in Conjecture 2.4 on $\mathcal{E}$ and Conjecture 2.5 on $K[\![x]\!]_0$, and denote the conjectures by $\mathbf{LGF}_{\mathcal{E}}$ and $\mathbf{LGF}_{K[\![x]\!]_0}$, respectively. We have proved our conjectures $\mathbf{LGF}_{\mathcal{E}}$ and $\mathbf{LGF}_{K[\![x]\!]_0}$ if M is of rank ≤ 2 [CT09, Theorem 7.1, Corollary 7.2], and then we established Dwork's conjecture $\mathbf{LGF}_{\mathrm{Dw}}$ if M is of rank ≤ 2 [CT09, Corollary 7.3].

Let us now explain the results in the present paper. First we characterize bounded φ-∇-modules over $\mathcal{E}$ by using Frobenius structures (Theorem 4.1):

(1) A bounded φ-∇-module M over $\mathcal{E}$ (i.e., $M^0 = 0$, which means that all the solutions on the generic disk are bounded) is isomorphic to a direct sum of several pure φ-∇-modules if the residue field k of $\mathcal{V}$ is perfect.

Note that the assertion corresponding to (1) is trivial for a φ-∇-module M over $K[\![x]\!]_0$ such that M_η is bounded by Christol's transfer theorem (see [CT09, Proposition 4.3]). This characterization implies the existence of a unique increasing filtration $\{P_i(M)\}$ of φ-∇-modules M over $\mathcal{E}$ such that $P_i(M)/P_{i-1}(M)$ is the maximally PBQ submodule of $M/P_{i-1}(M)$ (Corollary 5.5). This filtration is called the PBQ filtration. When we start with a φ-∇-module M over $K[\![x]\!]_0$, we can introduce a similar PBQ filtration for M, i.e., a filtration consisting of φ-∇-submodules over $K[\![x]\!]_0$ whose generic fibers will induce the PBQ filtration of the generic fiber M_η (Corollary 5.10). To this end we use an argument of A.J. de Jong in [dJ98] establishing the full faithfulness of the forgetful functor from the category of φ-∇-modules over $K[\![x]\!]_0$ to the category of φ-∇-modules over $\mathcal{E}$.

The need to study horizontality behavior for the PBQ condition with respect to the special and generic points leads us to introduce a new condition for φ-∇-modules over $K[\![x]\!]_0$, namely, the property of being "horizontally pure of bounded quotient " (which, for simplicity, we abbreviate as HPBQ, cf. Definition 6.1). Then in Theorem 6.5 we prove that

(2) our conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (see 2.5) on the comparison between the log-growth filtration and the Frobenius slope filtration at the special point holds for a HPBQ module.

A HPBQ module should be understood as a φ-∇-module for which the bounded quotient is horizontal and pure with respect to the Frobenius. Our method of proof will lead us to introduce the related definition of equislope φ-∇-modules over $K[\![x]\!]_0$ (Definition 6.7): they admit a filtration as φ-∇-modules over $K[\![x]\!]_0$ which induces the Frobenius slope filtration at the generic point. Note that a PBQ equislope object is HPBQ. Using this result, we prove in Theorem 7.1 that

(3) our conjecture $\mathbf{LGF}_{\mathcal{E}}$ (see 2.4) on comparison between the log-growth filtration and the Frobenius slope filtration at the generic point holds for PBQ modules over $\mathcal{E}$.

Indeed, for a φ-∇-module M over $\mathcal{E}$, the induced φ-∇-module $M_\tau = M \otimes_{\mathcal{E}} \mathcal{E}_t[\![X-t]\!]_0$ (where $\mathcal{E}_t[\![X-t]\!]_0$ is the ring of bounded functions on the open unit disk at generic point t) is equislope. For the proof of comparison for HPBQ modules, we use an explicit calculation of log-growth for solutions of certain Frobenius equations (Lemma 4.8) and a technical induction argument.
For a submodule L of a φ-∇-module M over $\mathcal{E}$ with $N = M/L$, the induced right exact sequence

$$L/L^\lambda \to M/M^\lambda \to N/N^\lambda \to 0$$

is also left exact for any λ if L is a maximally PBQ submodule of M by Proposition 2.6. Since there do exist PBQ filtrations, the comparison between the log-growth filtrations and the Frobenius slope filtrations for PBQ modules both at the generic point and at the special point implies the rationality of breaks (Theorem 7.2 and Proposition 7.3) as well as Dwork's conjecture (Theorem 8.1) that the special log-growth polygon lies above the generic log-growth polygon (including the coincidence of both end points):

(4) Our conjecture of comparison between the log-growth filtration and the Frobenius slope filtration at the special point (Conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$, 2.5) implies Dwork's conjecture (Conjecture $\mathbf{LGF}_{\mathrm{Dw}}$, 2.7).

As an application, we have the following theorem (Theorem 8.8) without any assumptions.

(5) The coincidence of both log-growth polygons at the generic and special points is equivalent to the coincidence of both Frobenius slope polygons at the generic and special points.

Let us also mention some recent work on log-growth. Y.André ([An08]) proved the conjecture $\mathbf{LG}_{\mathrm{Dw}}$ of Dwork without Frobenius structures, that is, the log-growth polygon at the special point is above the log-growth filtration at the generic point for ∇-modules, but without coincidence of both end points. (Note that his convention on the Newton polygon is different from ours, see Remark 2.8). He used semi-continuity of log-growth on Berkovich spaces. K.Kedlaya

defined the log-growth at the special point for regular singular connections and studied the properties of log-growth [Ke09, Chapter 18].
This paper is organized in the following manner. In Section 2 we recall our notation and results from [CT09]. In Section 3 we establish the independence of the category of φ-∇-modules over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) of the choices of Frobenius on $\mathcal{E}$ (resp. $K[\![x]\!]_0$). In Section 4 we study when the Frobenius slope filtration of φ-∇-modules over $\mathcal{E}$ is split and prove (1) above. In Section 5 we introduce the notion of PBQ and prove the existence of PBQ filtrations. In Section 6 we study the log-growth filtration for HPBQ φ-∇-modules over $K[\![x]\!]_0$ and prove the comparison (2) between the log-growth filtration and the Frobenius slope filtration. This comparison implies the comparison (3) for PBQ φ-∇-modules over $\mathcal{E}$ in Section 7. In Section 8 we show that (4) our conjecture of comparison at the special point implies Dwork's conjecture.

2 PRELIMINARIES

We fix notation and recall the terminology in [CT09]. We also review Dwork's conjecture and our conjectures.

2.1 NOTATION

Let us fix the basic notation which follows from [CT09].

p : a prime number.

K : a complete discrete valuation field of mixed characteristic $(0, p)$.

$\mathcal{V}$: the ring of integers of K.

k : the residue field of $\mathcal{V}$.

$\mathfrak{m}$: the maximal ideal of $\mathcal{V}$.

$|\ |$: a p-adically absolute value on K and its extension as a valuation field, which is normalized by $|p| = p^{-1}$.

q : a positive power of p.

σ : (q-)Frobenius on K, i.e., a continuous lift of q-Frobenius endomorphism ($a \mapsto a^q$ on k). We suppose the existence of Frobenius on K. We also denote by σ a K-algebra endomorphism on $\mathcal{A}_K(0, 1^-)$, which is an extension of Frobenius on K, such that $\sigma(x)$ is bounded and $|\sigma(x) - x^q|_0 < 1$. Then $K[\![x]\!]_\lambda$ is stable under σ. We also denote by σ the unique extension of σ on $\mathcal{E}$, which is a Frobenius on $\mathcal{E}$. In the case we only discuss φ-∇-modules over $\mathcal{E}$, one can take a Frobenius σ on K such that $\sigma(x) \in \mathcal{E}$ with $|\sigma(x) - x^q|_0 < 1$.

$\widehat{K^{\mathrm{perf}}}$: the p-adic completion of the inductive limit K^{perf} of $K \xrightarrow{\sigma} K \xrightarrow{\sigma} \cdots$. Then $\widehat{K^{\mathrm{perf}}}$ is a complete discrete valuation field such that the residue field of the ring of integers of $\widehat{K^{\mathrm{perf}}}$ is the perfection of k and that the value group of $\widehat{K^{\mathrm{perf}}}$ coincides with the value group of K. The Frobenius σ uniquely extend to $\widehat{K^{\mathrm{perf}}}$. Moreover, taking the p-adic completion $\widehat{K^{\mathrm{al}}}$ of the maximally unramified extension K^{al} of K^{perf}, we have a canonical extension of K as a discrete valuation field with the same value group such that the residue field of the ring of integers is algebraically closed and the Frobenius extends on it. We use the same symbol σ for Frobenius on the extension.

q^{λ} : an element of K with $\log_q |q^{\lambda}| = -\lambda$ for a rational number λ such that $\sigma(q^{\lambda}) = q^{\lambda}$. Such a q^{λ} always exists if the residue field k is algebraically closed and $\lambda \in \log_q |K^{\times}|$. In particular, if k is algebraically closed, then there exists an extension L of K as a discrete valuation field with an extension of Frobenius such that q^{λ} is contained in L for a fix λ. In this paper we freely extend K as above.

$\mathcal{A}_K(c, r^-)$: the K-algebra of analytic functions on the open disk of radius r at the center c, i.e.,

$$\mathcal{A}_K(c, r^-) = \left\{ \sum_{n=0}^{\infty} a_n (x-c)^n \in K[\![x-c]\!] \;\middle|\; \begin{array}{l} |a_n|\gamma^n \to 0 \text{ as } n \to \infty \\ \text{for any } 0 < \gamma < r \end{array} \right\}.$$

$K[\![x]\!]_0$: the ring of bounded power series over K, i.e.,

$$K[\![x]\!]_0 = \left\{ \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}_K(0, 1^-) \;\middle|\; \sup_n |a_n| < \infty \right\}.$$

An element of $K[\![x]\!]_0$ is said to be a bounded function.

$K[\![x]\!]_{\lambda}$: the Banach K-module of power series of log-growth λ in $\mathcal{A}_K(0, 1^-)$ for a nonnegative real number $\lambda \in \mathbb{R}_{\geq 0}$, i.e.,

$$K[\![x]\!]_{\lambda} = \left\{ \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}_K(0, 1^-) \;\middle|\; \sup_n |a_n|/(n+1)^{\lambda} < \infty \right\},$$

with a norm $|\sum_{n=0}^{\infty} a_n x^n|_{\lambda} = \sup_n |a_n|/(n+1)^{\lambda}$. $K[\![x]\!]_{\lambda}$ is a $K[\![x]\!]_0$-modules. $K[\![x]\!]_{\lambda} = 0$ for $\lambda < 0$ for the convenient. An element $f \in K[\![x]\!]_{\lambda}$ which is not contained in $K[\![x]\!]_{\gamma}$ for $\gamma < \lambda$ is said to be exactly of log-growth λ.

$\mathcal{E}$: the p-adic completion of the field of fractions of $K[\![x]\!]_0$ under the Gauss norm $|\;|_0$, i.e.,

$$\mathcal{E} = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \;\middle|\; a_n \in K,\, \sup_n |a_n| < \infty,\, |a_n| \to 0 \,(\text{as } n \to -\infty) \right\}.$$

$\mathcal{E}$ is a complete discrete valuation field under the Gauss norm $|\ |_0$ in fact K is discrete valuated. The residue field of the ring $\mathcal{O}_{\mathcal{E}}$ of integers of $\mathcal{E}$ is $k(\!(x)\!)$.

t : a generic point of radius 1.

$\mathcal{E}_t$: the valuation field corresponding to the generic point t, i.e., the same field as $\mathcal{E}$ in which x is replaced by t: we emphasize t in the notation with the respect to [CT09]. We regard the Frobenius σ as a Frobenius on $\mathcal{E}_t$.

$\mathcal{E}_t[\![X-t]\!]_0$: the ring of bounded functions in $\mathcal{A}_{\mathcal{E}_t}(t, 1^-)$. Then

$$\tau : \mathcal{E} \to \mathcal{E}_t[\![X-t]\!]_0 \qquad \tau(f) = \sum_{n=0}^{\infty} \frac{1}{n!}(\frac{d^n}{dx^n} f)|_{x=t}(X-t)^n$$

is a K-algebra homomorphism which is equivariant under the derivations $\frac{d}{dx}$ and $\frac{d}{dX}$. The Frobenius σ on $\mathcal{E}_t[\![X-t]\!]_0$ is defined by σ on $\mathcal{E}_t$ and $\sigma(X-t) = \tau(\sigma(x)) - \sigma(x)|_{x=t}$. τ is again σ-equivariant.

For a function f on R and for a matrix $A = (a_{ij})$ with entries in R, we define $f(A) = (f(a_{ij}))$. In case where f is a norm $|\ |$, then $|A| = \sup_{i,j} |a_{ij}|$. We use 1 (resp. 1_r) to denote the unit matrix of suitable degree (resp. of degree r). For a decreasing filtration $\{V^\lambda\}$ indexed by the set $\mathbb{R}$ of real numbers, we put

$$V^{\lambda-} = \cap_{\mu<\lambda} V^\mu, \qquad V^{\lambda+} = \cup_{\mu>\lambda} V^\mu.$$

We denote by $W_{\lambda-} = \cup_{\mu<\lambda} W_\mu$ and $W_{\lambda+} = \cap_{\mu>\lambda} W_\mu$ the analogous objects for an increasing filtration $\{W_\lambda\}_\lambda$, respectively.

2.2 Terminology

We recall some terminology and results from [CT09].
Let R be either K (K might be $\mathcal{E}$) or $K[\![x]\!]_0$. A φ-module over R consists of a free R-module M of finite rank and an R-linear isomorphism $\varphi : \sigma^* M \to M$. For a φ-module over K, there is an increasing filtration $\{S_\lambda(M)\}_{\lambda\in\mathbb{R}}$ which is called the Frobenius slope filtration. Then there is a sequence $\lambda_1 < \cdots < \lambda_r$ of real numbers, called the Frobenius slopes of M, such that $S_{\lambda_i}(M)/S_{\lambda_i-}(M)$ is pure of slope λ_i and $M \otimes \widehat{K^{\mathrm{al}}} \cong \oplus_i S_{\lambda_i}(M) \otimes_K \widehat{K^{\mathrm{al}}}/S_{\lambda_i-}(M) \otimes_K \widehat{K^{\mathrm{al}}}$ is the Dieudonné-Manin decomposition as φ-modules over $\widehat{K^{\mathrm{al}}}$. We call λ_1 the first Frobenius slope and λ_r the highest Frobenius slope, respectively.
Let R be either $\mathcal{E}$ or $K[\![x]\!]_0$. A φ-∇-module over R consists of a φ-module (M, φ) over R and a K-connection $\nabla : M \to M \otimes_R \Omega_R$, where $\Omega_R = R dx$, such that $\varphi \circ \sigma^*(\nabla) = \nabla \circ \varphi$. For a basis $(e_1, \cdots, e_r)$, the matrices A and G with entries R,

$$\varphi(1 \otimes e_1, \cdots, 1 \otimes e_r) = (e_1, \cdots, e_r)A, \qquad \nabla(e_1, \cdots, e_r) = (e_1, \cdots, e_r)G dx$$

are called the Frobenius matrix and the connection matrix of R, respectively. Then one has

$$\frac{d}{dx}A + GA = (\frac{d}{dx}\sigma(x))A\sigma(G) \tag{FC}$$

by the horizontality of φ. We denote the dual of M by $M^\vee$.
Let M be a φ-∇-module over $K[\![x]\!]_0$. We define the K-space

$$V(M) = \{s \in M \otimes_{K[\![x]\!]_0} \mathcal{A}_K(0, 1^-) \mid \nabla(s) = 0\}$$

of horizontal sections and the K-space of solutions,

$$\mathrm{Sol}(M) = \mathrm{Hom}_{K[\![x]\!]_0[\nabla]}(M, \mathcal{A}_K(0, 1^-)),$$

on the unit disk. Both $\dim_K V(M)$ and $\dim_K \mathrm{Sol}(M)$ equal to $\mathrm{rank}_{K[\![x]\!]_0} M$ by the solvability. If one fixes a basis of M, the solution Y of the equations

$$\begin{cases} A(0)\sigma(Y) = YA \\ \frac{d}{dx}Y = YG \\ Y(0) = 1 \end{cases}$$

in $\mathcal{A}_K(0, 1^-)$ is a solution matrix of M, where $A(0)$ and $Y(0)$ are the constant terms of A and Y, respectively. The log-growth filtration $\{V(M)^\lambda\}_{\lambda\in\mathbb{R}}$ is defined by the orthogonal space of the K-space $\mathrm{Sol}_\lambda(M) = \mathrm{Hom}_{K[\![x]\!]_0[\nabla]}(M, K[\![x]\!]_\lambda)$ under the natural bilinear perfect pairing

$$V(M) \times \mathrm{Sol}(M) \to K.$$

Then $V(M)^\lambda = 0$ for $\lambda >> 0$ by the solvability of M and the log-growth filtration is a decreasing filtration of $V(M)$ as φ-modules over K. The following proposition allows one to change the coefficient field K to a suitable extension K'.

Proposition 2.1 ([CT09, Proposition 1.10]) *Let M be a φ-module over $K[\![x]\!]_0$. For any extension K' over K as a complete discrete valuation field with an extension of Frobenius, there is a canonical isomorphism $V(M \otimes_{K[\![x]\!]_0} K'[\![x]\!]_0) \cong V(M) \otimes_K K'$ as log-growth filtered φ-modules.*

The induced φ-∇-module $M_\eta = M \otimes_{K[\![x]\!]_0} \mathcal{E}$ over $\mathcal{E}$ is said to be the generic fiber of M, and the K-module $V(M)$ is called the special fiber of M.

Let M be a φ-∇-module over $\mathcal{E}$. We denote by M_τ the induced φ-∇-module $M \otimes_{\mathcal{E}} \mathcal{E}_t[\![X - t]\!]_0$ over $\mathcal{E}_t[\![X - t]\!]_0$. Applying the theory of Robba [Ro75], we have a decreasing filtration $\{M^\lambda\}_{\lambda\in\mathbb{R}}$ of M as φ-∇-modules over $\mathcal{E}$ which is characterized by the following universal property.

Theorem 2.2 [Ro75, 2.6, 3.5] (See [CT09, Theorem 3.2].) *For any real number λ, M/M^λ is the maximum quotient of M such that all solutions of log-growth λ of M_τ on the generic unit disk come from the solutions of $(M/M^\lambda)_\tau$.*

The filtration $\{M^\lambda\}$ is called the log-growth filtration of M. Note that $M^\lambda = M$ for $\lambda < 0$ by definition and $M^\lambda = 0$ for $\lambda >> 0$ by the solvability. The quotient module M/M^0 is called the bounded quotient, and, in particular, if $M^0 = 0$, then M is called bounded.

Our main theorem in [CT09] is the following:

Theorem 2.3 ([CT09, Theorem 6.17])

(1) *Let M be a φ-∇-module over $\mathcal{E}$. If $\lambda_{\max}$ is the highest Frobenius slope of M, then $M^\lambda \subset (S_{\lambda-\lambda_{\max}}(M^\vee))^\perp$.*

(2) *Let M be a φ-∇-module over $K[\![x]\!]_0$. If $\lambda_{\max}$ is the highest Frobenius slope of M_η, then $V(M)^\lambda \subset (S_{\lambda-\lambda_{\max}}(V(M^\vee)))^\perp$.*

Here $S^\perp$ denotes the orthogonal space of S under the natural bilinear perfect pairing

$$M \otimes_{\mathcal{E}} M^\vee \to \mathcal{E} \text{ or } V(M) \otimes_K V(M^\vee) \to K.$$

We conjectured that equalities hold in Theorem 2.3 if M is PBQ (Definition 5.1) in [CT09], and proved them if M is of rank ≤ 2 [CT09, Theorem 7.1, Corollary 7.2].

Conjecture 2.4 ([CT09, Conjectures 6.8]) *Let M be a φ-∇-module over $\mathcal{E}$.*

(1) *All breaks of log-growth filtration of M are rational and $M^\lambda = M^{\lambda+}$ for any λ.*

(2) *Let $\lambda_{\max}$ be the highest Frobenius slope of M. If M/M^0 is pure as φ-module (PBQ in Definition 5.1 (1)), then $M^\lambda = (S_{\lambda-\lambda_{\max}}(M^\vee))^\perp$.*

We denote Conjecture 2.4 above by $\mathbf{LGF}_{\mathcal{E}}$.

Conjecture 2.5 ([CT09, Conjectures 6.9]) *Let M be a φ-∇-module over $K[\![x]\!]_0$.*

(1) *All breaks of log-growth filtration of $V(M)$ are rational and $V(M)^\lambda = V(M)^{\lambda+}$ for any λ.*

(2) *Let $\lambda_{\max}$ be the highest Frobenius slope of M_η. If M_η/M_η^0 is pure as φ-module (PBQ in Definition 5.1 (2)), then $V(M)^\lambda = (S_{\lambda-\lambda_{\max}}(V(M)^\vee))^\perp$.*

We denote Conjecture 2.5 above by $\mathbf{LGF}_{K[\![x]\!]_0}$.

Note that we formulate the theorem and the conjecture in the case where $\lambda_{\max} = 0$ in [CT09]. However, the theorem holds for an arbitrary $\lambda_{\max}$ by Proposition 2.1 (and the conjecture should also hold). Moreover, it suffices to establish the conjecture when the residue field k of $\mathcal{V}$ is algebraically closed.

In section 7 we will reduce the conjecture $\mathbf{LGF}_{\mathcal{E}}(1)$ (resp. $\mathbf{LGF}_{K[\![x]\!]_0}(1)$) to the conjecture $\mathbf{LGF}_{\mathcal{E}}(2)$ (resp. $\mathbf{LGF}_{K[\![x]\!]_0}(2)$) by applying the proposition below to the PBQ filtration which is introduced in section 5. The following proposition is useful for attacking log-growth questions by induction.

Proposition 2.6 *Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-∇-modules over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) and let $\lambda_{\max}$ be the highest Frobenius slope of M and L (resp. M_η and L_η).*

(1) *Suppose that $L^\lambda = (S_{\lambda-\lambda_{\max}}(L^\vee))^\perp$ for λ. Then the induced sequence*

$$0 \to L/L^\lambda \to M/M^\lambda \to N/N^\lambda \to 0$$

is exact.

(2) *Suppose that $V(L)^\lambda = (S_{\lambda-\lambda_{\max}}(V(L)^\vee))^\perp$ for λ. Then the induced sequence*

$$0 \to V(L)/V(L)^\lambda \to V(M)/V(M)^\lambda \to V(N)/V(N)^\lambda \to 0$$

is exact.

Proof. (1) Since

$$L/L^\lambda \to M/M^\lambda \to N/N^\lambda \to 0$$

is right exact by [CT09, Proposition 3.6], we have only to prove the injectivity of the first morphism. There is an inclusion relation

$$M^\lambda \subset (S_{\lambda-\lambda_{\max}}(M^\vee))^\perp = S_{(\lambda_{\max}-\lambda)-}(M)$$

by Theorem 2.3 and the equality

$$L^\lambda = (S_{\lambda-\lambda_{\max}}(L^\vee))^\perp = S_{(\lambda_{\max}-\lambda)-}(L).$$

holds by our hypothesis on L. Since the Frobenius slope filtrations are strict for any morphism, the bottom horizontal morphism in the natural commutative diagram

$$\begin{array}{ccc} L/L^\lambda & \longrightarrow & M/M^\lambda \\ =\downarrow & & \downarrow \\ L/S_{(\lambda_{\max}-\lambda)-}(L) & \longrightarrow & M/S_{(\lambda_{\max}-\lambda)-}(M) \end{array}$$

is injective. Hence we have the desired injectivity.
(2) The proof here is similar to that of (1) on replacing [CT09, Proposition 3.6] by [CT09, Proposition 1.8]. $\square$

2.3 Dwork's conjecture

We recall Dwork's conjecture. We have proved it in the case where M is of rank ≤ 2 [CT09, Corollary 7.3].

Conjecture 2.7 ([Dw73, Conjecture 2], [CT09, Conjecture 4.9]) *Let M be a φ-∇-module over $K[\![x]\!]_0$. Then the special log-growth is above the generic log-growth polygon (with coincidence at both endpoints).*

We denote Conjecture 2.7 above by $\mathbf{LGF}_{\mathrm{Dw}}$. We will prove that the conjecture $\mathbf{LGF}_{\mathrm{Dw}}$ follows from the conjectures $\mathbf{LGF}_{\mathcal{E}}$ and $\mathbf{LGF}_{K[\![x]\!]_0}$ in section 8. There is also a version of Dwork's conjecture without Frobenius structures, we denote it by $\mathbf{LG}_{\mathrm{Dw}}$.

Let us recall the definition of the log-growth polygon: the generic log-growth polygon is the piecewise linear curve defined by the vertices

$$(0,0), (\dim_{\mathcal{E}} \tfrac{M_\eta}{M_\eta^{\lambda_1+}}, \lambda_1 \dim_{\mathcal{E}} \tfrac{M_\eta^{\lambda_1-}}{M_\eta^{\lambda_1+}}), \cdots, (\dim_{\mathcal{E}} \tfrac{M_\eta}{M_\eta^{\lambda_i+}}, \textstyle\sum_{j=1}^{i} \lambda_j \dim_{\mathcal{E}} \tfrac{M_\eta^{\lambda_j-}}{M_\eta^{\lambda_j+}}),$$
$$\cdots, (\dim_{\mathcal{E}} M_\eta, \textstyle\sum_{j=1}^{r} \lambda_j \dim_{\mathcal{E}} \tfrac{M_\eta^{\lambda_j-}}{M_\eta^{\lambda_j+}}),$$

where $0 = \lambda_1 < \cdots < \lambda_r$ are breaks (i.e., $M^{\lambda-} \neq M^{\lambda+}$) of the log-growth filtration of M_η. The special log-growth polygon is defined in the same way using the log-growth filtration of $V(M)$.

Remark 2.8 (1) *The convention of André's polygon of log-growth [An08] is different from ours. His polygon at the generic fiber is $\sum_{j=1}^{r} \lambda_j \dim_{\mathcal{E}} \frac{M^{\lambda_j-}}{M^{\lambda_j+}}$ below our polygon in the direction of the vertical axis and the starting point of the polygon is $(\dim_{\mathcal{E}} M, 0)$, and the same at the special fiber. André proved the conjecture $\mathbf{LG}_{\mathrm{Dw}}$ except the coincidence of both endpoints in [An08].*

(2) *If the special log-growth polygon lies above the generic log-growth polygon in both conventions of André's and ours, then both endpoints coincide with each other. However even if this is the case, we cannot prove $M_\eta^{\lambda} = M_\eta^{\lambda+}$ (resp. $V(M)^{\lambda} = V(M)^{\lambda+}$) for a break λ.*

3 Choices of Frobenius

Let us recall the precise form of equivalence between categories of φ-∇-modules with respect to different choices of Frobenius on $\mathcal{E}$ (resp. $K[\![x]\!]_0$) (see [Ts98a, Section 3.4] for example). We will use it in the next section.

3.1 Comparison morphism $\vartheta_{\sigma_1,\sigma_2}$

Let σ_1 and σ_2 be Frobenius maps on $\mathcal{E}$ (resp. $K[\![x]\!]_0$) such that the restriction of each σ_i to K is the given Frobenius on K. Let M be a φ-∇-module. We define an $\mathcal{E}$-linear (resp. $K[\![x]\!]_0$-linear) morphism

$$\vartheta_{\sigma_1,\sigma_2} : \sigma_1^* M \to \sigma_2^* M$$

by

$$\vartheta_{\sigma_1,\sigma_2}(a \otimes m) = a \sum_{n=0}^{\infty} (\sigma_2(x) - \sigma_1(x))^n \otimes \frac{1}{n!} \nabla(\frac{d^n}{dx^n})(m).$$

Since M is solvable and $|\sigma_2(x) - \sigma_1(x)| < 1$, the right hand side converges in $\sigma_2^* M$. As a matrix representation, the transformation matrix is

$$H = \sum_{n=0}^{\infty} \sigma_2(G_n) \frac{(\sigma_2(x) - \sigma_1(x))^n}{n!}$$

for the induced basis $1 \otimes e_1, \cdots, 1 \otimes e_r$, where G is the matrix of connection, $G_0 = 1$ and $G_{n+1} = GG_n + \frac{d}{dx} G_n$ for $n \geq 0$.

Proposition 3.1 *Let $\sigma_1, \sigma_2, \sigma_3, \sigma$ be Frobenius maps of $\mathcal{E}$ (resp. $K[\![x]\!]_0$) as above. Then we have the cocycle conditions:*

(1) $\vartheta_{\sigma_2,\sigma_3} \circ \vartheta_{\sigma_1,\sigma_2} = \vartheta_{\sigma_1,\sigma_3}$.

(2) $\vartheta_{\sigma,\sigma} = \mathrm{id}_{\sigma^* M}$.

Proposition 3.2 *Let M be a φ-∇-module pure of slope λ over $\mathcal{E}$ and let A be the Frobenius matrix of M with respect to a basis. Suppose that $|A - q^{\lambda} 1|_0 \leq q^{-\mu}$ for $\mu \geq \lambda$. Then the representation matrix H of the comparison morphism $\vartheta_{\sigma_1,\sigma_2}$ with respect to the bases which are the pull-backs by σ_1 and σ_2 respectively, satisfies $|H - 1|_0 < q^{\lambda-\mu}$.*

Proof. By replacing the Frobenius φ by $q^{-\lambda}\varphi$, we may assume that $\lambda = 0$. The assertion then follows from the fact that under these assumptions the solution matrix Y at the generic point satisfies $Y \equiv 1 \,(\mathrm{mod}\, (X-t)\mathfrak{m}^n \mathcal{O}_{\mathcal{E}_t}[\![X - t]\!])$. Here n is the least integer such that $|\mathfrak{m}^n| \leq q^{-\mu}$. □

3.2 Equivalence of categories

Let R be either $\mathcal{E}$ or $K[\![x]\!]_0$ and let σ_1 and σ_2 be Frobenius maps on R as in the previous subsection. We define a functor

$$\vartheta^*_{\sigma_1,\sigma_2} : (\varphi\text{-}\nabla\text{-modules over } (R, \sigma_2)) \to (\varphi\text{-}\nabla\text{-modules over } (R, \sigma_1))$$

by $(M, \nabla, \varphi) \mapsto (M, \nabla, \varphi \circ \vartheta_{\sigma_1,\sigma_2})$. Here $\vartheta_{\sigma_1,\sigma_2}$ is defined as in the previous section. The propositions of the previous subsection then give

Theorem 3.3 *$\vartheta^*_{\sigma_1,\sigma_2}$ is an equivalence of categories which preserves tensor products and duals. Moreover, $\vartheta^*_{\sigma_1,\sigma_2}$ preserves the Frobenius slope filtration and the log-growth filtration of M (resp. $V(M)$) for a φ-∇-module M over $\mathcal{E}$ (resp. $K[\![x]\!]_0$).*

4 Boundedness and splitting of the Frobenius slope filtration

4.1 Splitting theorem

Theorem 4.1 *Suppose that the residue field k of $\mathcal{V}$ is perfect. A φ-∇-module M over $\mathcal{E}$ is bounded if and only if M is a direct sum of pure φ-∇-modules, that is,*

$$M \cong \oplus_{i=1}^{r} S_{\lambda_i}(M)/S_{\lambda_i -}(M)$$

as φ-∇-modules, where $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ are Frobenius slopes of M.

Since any pure φ-∇-module over $\mathcal{E}$ is bounded by [CT09, Corollary 6.5]. Hence, Theorem 4.1 above follows from the next proposition.

Proposition 4.2 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-∇-modules over $\mathcal{E}$ such that both L and N are pure of Frobenius slope λ and ν, respectively. If one of the conditions*

(1) $\nu - \lambda < 0$;

(2) $\nu - \lambda > 1$;

(3) *M is bounded and* $0 < \nu - \lambda \leq 1$,

holds, then the exact sequence is split, that is, $M \cong L \oplus N$ as φ-∇-modules.

In the case (1) the assertion easily follows from the fact that, for $a \in \mathcal{E}$ with $|a|_0 < 1$, $a\sigma$ is a contractive operator on the p-adic complete field $\mathcal{E}$. The rest of this section will be dedicated to proving the assertion in cases (2) and (3).

4.2 Descent of splittings

Proposition 4.3 *Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-modules over $\mathcal{E}$ such that L and N are pure and the two slopes are different. Let $\mathcal{E}'$ be one of the following:*

(i) *$\mathcal{E}'$ is a p-adic completion of an unramified extension of $\mathcal{E}$;*

(ii) *$\mathcal{E}'$ is the p-adic completion of $\mathcal{E} \otimes_K K'$ for some extension K' of K as a complete discrete valuation field with an extension σ' of σ such that, if G is the group of continuous automorphisms of K' over K, then the invariant subfield of K' by the action of G is K.*

If the exact sequence is split over $\mathcal{E}'$, then it is split over $\mathcal{E}$. The same holds for φ-∇-modules over $\mathcal{E}$.

Proof. In each case we may assume that $\mathcal{E}$ is the invariant subfield of $\mathcal{E}'$ by the action of continuous automorphism group G. Let $e_1, \cdots, e_r, e_{r+1}, \cdots, e_{r+s}$ be a basis of M over $\mathcal{E}$ such that $e_1, \cdots, e_r$ is a basis of L. Put

$$\varphi(e_1, \cdots, e_r, e_{r+1}, \cdots, e_{r+s}) = (e_1, \cdots, e_r, e_{r+1}, \cdots, e_{r+s}) \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where A_{11} is of degree r and A_{22} is of degree s, respectively, and all entries of A_{11}, A_{12} and A_{22} are contained in $\mathcal{E}$. By the hypothesis of splitting over $\mathcal{E}'$ there exists a matrix Y with entries in $\mathcal{E}'$ such that

$$A_{11}\sigma(Y) - YA_{22} + A_{12} = 0.$$

For any $\rho \in G$, $\rho(Y)$ also gives a splitting. Hence $A_{11}\sigma(Y - \rho(Y)) = (Y - \rho(Y))A_{22}$. By the assumption on slopes, $\rho(Y) = Y$. Therefore, all entries of Y are contained in $\mathcal{E}$ and the exact sequence is split over $\mathcal{E}$. □

Definition 4.4 *An extension $\mathcal{E}'$ (resp. K') of $\mathcal{E}$ (resp. K) is allowable if $\mathcal{E}'$ is a finitely successive extension of $\mathcal{E}$ (resp. K) of type in (i) or (ii) (resp. (ii)) of Proposition 4.3.*

4.3 Preparations

In this subsection we assume that the residue field k of $\mathcal{V}$ is algebraically closed. Moreover we assume that the Frobenius on $\mathcal{E}$ (resp. $K[\![x]\!]_0$) is defined by $\sigma(x) = x^q$. For an element $a = \sum_n a_n x^n$ in $\mathcal{E}$ (resp. $K[\![x]\!]$) we define the subseries $a^{(q)}$ by $\sum_n a_{qn} x^{qn}$.

Lemma 4.5 *Let $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ be an invertible matrix of degree $r + s$ over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) with A_{11} of degree r and A_{22} of degree s such that the matrix satisfies the conditions:*

(i) *$A_{11} = A_{11}^{(q)}$ and $A_{11} = P^{-1}$ for a matrix P over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) with $|P|_0 < 1$,*

(ii) *$A_{22} = A_{22}^{(q)}$ and $|A_{22} - 1_s|_0 < 1$.*

Suppose that $A_{12}^{(q)} \neq 0$. Then there exists an $r \times s$ matrix Y over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) with $|Y|_0 < |A_{12}^{(q)}|_0$ such that, if one puts $B = A_{11}\sigma(Y) - YA_{22} + A_{12}$, then $|B^{(q)}|_0 < |A_{12}^{(q)}|_0$. Moreover, there exists an $r \times s$ matrix Y over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) such that if one defines B_{12} by

$$\begin{pmatrix} A_{11} & B_{12} \\ 0 & A_{22} \end{pmatrix} = \begin{pmatrix} 1_r & -Y \\ 0 & 1_s \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} 1_r & \sigma(Y) \\ 0 & 1_s \end{pmatrix},$$

then $B_{12}^{(q)} = 0$.

PROOF. Take a matrix Y such that $\sigma(Y) = -PA_{12}^{(q)}$. Such a Y exists since the residue field k of $\mathcal{V}$ is perfect. Then $|Y|_0 < |A_{12}^{(q)}|_0$ and $B = A_{11}\sigma(Y) - YA_{22} + A_{12} = A_{11}PA_{12}^{(q)} - YA_{22} + A_{12} = A_{12} - A_{12}^{(q)} - YA_{22}$. Hence $|B^{(q)}|_0 = |YA_{22}^{(q)}|_0 < |A_{12}^{(q)}|_0$ and we have the first assertion. Applying the first assertion inductively on the value $|A_{12}^{(q)}|_0$, we have a desired matrix Y of the second assertion since $\mathcal{E}$ (resp. $K[\![x]\!]_0$) is complete under the norm $|\ |_0$. $\square$

We give a corollary of the preceding lemma for φ-∇-modules over $\mathcal{E}$.

PROPOSITION 4.6 *Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-∇-modules over $\mathcal{E}$. Suppose that N is pure of Frobenius slope ν and all Frobenius slopes of L are less than ν. Then there exist an allowable extension $\mathcal{E}'$ of $\mathcal{E}$ and a basis $e_1, \cdots, e_r, e_{r+1}, \cdots, e_{r+s}$ of $M \otimes_{\mathcal{E}} \mathcal{E}'$ with respect to the exact sequence such that, if one fixes an element x' in the ring $\mathcal{O}_{\mathcal{E}'}$ of integers of $\mathcal{E}'$ whose image gives a uniformizer of the residue field of $\mathcal{O}_{\mathcal{E}'}$ and a Frobenius σ' on $\mathcal{E}'$ with $\sigma'(x') = x'^q$, then the Frobenius matrix $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ of $M \otimes_{\mathcal{E}} \mathcal{E}'$ with respect to σ' (here we use Theorem 3.3) has the following form:*

(i) $A_{11} = A_{11}^{(q)}$ *and* $A_{11} = P^{-1}$ *for a matrix* $|P|_0 < q^{\nu}$,

(ii) $A_{22} = A_{22}^{(q)}$ *and* $|A_{22} - q^{\nu}1_s|_0 < q^{-\nu}$,

(iii) $A_{12}^{(q)} = 0$,

where $a^{(q)}$ is defined by using the parameter x'. Moreover, one can replaces the inequality $|A_{22} - q^{\nu}1_s|_0 < q^{-\nu}$ in (ii) by the inequality $|A_{22} - q^{\nu}1_s|_0 < q^{-\nu}\eta$ for a given $0 < \eta \leq 1$ (the extension $\mathcal{E}'$ depends on η).

PROOF. Since k is algebraically closed, there is a uniformizer π of K such that $\sigma(\pi) = \pi$. Let K_m be a Galois extension $K(\pi^{1/m}, \zeta_m)$ of K for a positive integer m, where ζ_m denotes a primitive m-th root of unity. Then σ on K extends on K_m. If we choose a positive integer m such that $m/\log_q|\pi|$ is a common multiple of denominators of ν and the highest Frobenius slope of L, then ν and the highest Frobenius slope of L are contained in $\log_q|K_m^{\times}|$. Hence we may assume that $\nu = 0$ and all Frobenius slopes of the twist $\pi\varphi_L$ of the Frobenius φ_L of L are less than or equal to 0.
Let $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ be a Frobenius matrix of M with respect to the given exact sequence. Since any φ-module over $\mathcal{E}$ has a cyclic vector [Ts96, Proposition 3.2.1], we may assume that $A_{22} \in \mathrm{GL}_s(\mathcal{O}_{\mathcal{E}})$ by $\nu = 0$. Then there is a matrix $X \in \mathrm{GL}_s(\mathcal{O}_{\mathcal{E}'})$ such that $X^{-1}A_{22}\sigma(X) \equiv 1_s \pmod{\mathfrak{m}\mathcal{O}_{\mathcal{E}'}}$ for some finite unramified extension $\mathcal{E}'$ over $\mathcal{E}$ by [Ts98b, Lemma 5.2.2]. By applying the existence of a cyclic vector again, we may assume that the all entries of Frobenius matrix of $L^{\vee}$ are contained in $\mathfrak{m}\mathcal{O}_{\mathcal{E}}$ by the hypothesis on Frobenius slopes of L.

Now we fix a parameter x' of $\mathcal{E}'$ and change a Frobenius σ' on $\mathcal{E}'$ such that $\sigma'(x') = x'^q$. The the hypothesis of the matrices A_{11} and A_{12} are stable by Theorem 3.3. If one replaces the basis $(e_1, \cdots, e_{r+s})$ by $(e_1, \cdots, e_{r+s})A$, then the Frobenius matrix becomes $\sigma'(A)$. Since the hypothesis in Lemma 4.5 hold in our Frobenius matrix A, we have the assertion. □

Now a variant of Proposition 4.6 for φ-∇-modules over $K[\![x]\!]_0$, which we use it in section 6, is given.

Proposition 4.7 *Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-∇-modules over $K[\![x]\!]_0$. Suppose that N_η is pure of Frobenius slope ν and all Frobenius slopes of L_η are less than ν. Then there exist an allowable extension K' of K with an extension of Frobenius σ' and a basis $e_1, \cdots, e_r, e_{r+1}, \cdots, e_{r+s}$ of $M \otimes_{K[\![x]\!]_0} K'[\![x]\!]_0$ with respect to the exact sequence such that the Frobenius matrix $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ of $M \otimes_{K[\![x]\!]_0} K'[\![x]\!]_0$ with respect to σ' has the following form:*

(i) *$A_{11} = A_{11}^{(q)}$ and $A_{11} = P^{-1}$ for a matrix $|P|_0 < q^\nu$,*

(ii) *$A_{22} = q^\nu 1_s$,*

(iii) *$A_{12}^{(q)} = 0$*

Proof. We may assume $\mu = 0$ and the highest Frobenius slope of L_η is contained in $\log_q |K_m^\times|$ as in the proof of Proposition 4.6. Then N is a direct sum of copies of the unit object $(K[\![x]\!]_0, d, \sigma)$'s since k is algebraically closed. In order to find the matrix P, we apply the isogeny theorem [Ka79, Theorem 2.6.1] and the existence of a free lattice over $\mathcal{V}[\![x]\!]$ in [dJ98, Lemma 6.1] for $L^\vee$. The rest is again same as the proof of Proposition 4.6. □

Lemma 4.8 *Let ν be a nonnegative rational number. Suppose that $y \in xK[\![x]\!]$ satisfies a Frobenius equation*

$$y - q^{-\nu} a\sigma(y) = f.$$

for $a \in K$ with $|a| = 1$ and for $f = \sum_n f_n x^n \in xK[\![x]\!]$.

(1) *Suppose that $f^{(q)} = 0$. If $f \in K[\![x]\!]_\nu \setminus \{0\}$, then $y \in K[\![x]\!]_\nu \setminus K[\![x]\!]_{\nu-}$, and if $f \in K[\![x]\!]_\lambda \setminus K[\![x]\!]_{\lambda-}$ for $\lambda > \nu$, then $y \in K[\![x]\!]_\lambda \setminus K[\![x]\!]_{\lambda-}$.*

(2) *Let l be a nonnegative integer with $q \not| \, l$. If $f \in K[\![x]\!]_0$ and $|f_l| > |q^\nu f|_0 = q^{-\nu}|f|_0 \neq 0$, then $y \in K[\![x]\!]_\nu \setminus K[\![x]\!]_{\nu-}$.*

Proof. Since the residue field k of $\mathcal{V}$ is algebraically closed, we may assume that $a = 1$. Formally in $K[\![x]\!]$,

$$y = \sum_n \sum_{m=0}^{\infty} (q^{-\nu})^m \sigma^m(f_n) x^{q^m n}$$

is a solution of the equation.
(1) If $q^m n = q^{m'} n'$, then $m = m'$ and $n = n'$ because $q \not| n, n'$. Hence, $y \neq 0$. By considering a subseries $\sum_{m=0}^{\infty} (q^{-\nu})^m \sigma^m(f_n) x^{q^m n}$ for $f_n \neq 0$, y is of log-growth equal to or greater than ν. Moreover, we have

$$|(q^{-\nu})^m \sigma^m(f_n)|/(q^m n + 1)^\nu = |f_n|/(n + 1/q^m)^\nu$$

Hence, if $f \in K[\![x]\!]_\nu$, then y is exactly of log-growth ν. Suppose $f \in K[\![x]\!]_\lambda \setminus K[\![x]\!]_{\lambda-}$. Since for each m, n

$$|(q^{-\nu})^m \sigma^m(f_n)|/(q^m n + 1)^\lambda = |f_n|/(q^{m(1-\nu/\lambda)} n + 1/q^{m\nu/\lambda})^\lambda,$$

the log-growth of y is exactly λ.
(2) There exists $z \in xK[\![x]\!]_0$ with $|z|_0 \leq |q^\nu f|_0 = q^{-\nu}|f|_0$ such that, if $g = f - z + q^{-\nu}\sigma(z) = \sum_n g_n x^n$, then $g^{(q)} = 0$ and $g_l \neq 0$ by the same construction of the proof of Lemma 4.5. The assertion now follows from (1). □

4.4 Proof of Proposition 4.2

Replacing K by an extension, we may assume that k is algebraically closed and that $\lambda = 0$, $\nu > 0$ and $\nu \in \log_q |K^\times|$ by Proposition 4.3 (see the beginning of proof of Proposition 4.6). We may also assume $\sigma(x) = x^q$ by Theorem 3.3. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ be a Frobenius matrix of M with respect to the basis which is compatible with the given extension (i.e., the $(1,1)$-part (resp. $(2,2)$-part) corresponds to L (resp. N)) and let $G = \begin{pmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{pmatrix}$ be the matrix of the connection, respectively. The commutativity of Frobenius and connection (the relation (FC) in section 2.2) gives the relation

$$1^\circ \qquad \frac{d}{dx}A_{12} + G_{11}A_{12} + G_{12}A_{22} = qx^{q-1}(A_{11}\sigma(G_{12}) + A_{12}\sigma(G_{22}))$$

of the $(1,2)$-part of the matrix. We may assume that

2° $A_{11} = A_{11}^{(q)}$, $|A_{11} - 1_r|_0 \leq q^{-1}$ and hence $|G_{11}|_0 < q^{-1}$ (r is rank of L);

3° $A_{22} = A_{22}^{(q)}$, $|A_{22} - q^\nu 1_s|_0 \leq q^{-\nu-1}$ and $|G_{22}|_0 < q^{-1}$ (s is rank of N);

4° $A_{12}^{(q)} = 0$.

by Proposition 4.6 Note that both inequalities $|G_{11}|_0 < q^{-1}$ and $|G_{11}|_0 < q^{-1}$ above follow from the relation (FC) in section 2.2 for L and N, respectively. When $\nu \neq 1$, we will first prove $A_{12} = 0$ and then prove $G_{12} = 0$. When $\nu = 1$, we will first prove $G_{12} = 0$ and then prove $A_{12} = 0$. Hence, we will have a splitting in all cases.

4.4.1 The case where $\nu > 1$

Suppose $\nu > 1$ (and $\lambda = 0$). Assume that $A_{12} \neq 0$. By 4° we have $|\frac{d}{dx}A_{12}|_0 > |qA_{12}|_0 = q^{-1}|A_{12}|_0$. Then $|G_{11}A_{12}|_0 < q^{-1}|A_{12}|_0 < |\frac{d}{dx}A_{12}|_0$ and $|qx^{q-1}A_{12}\sigma(G_{22})|_0 < q^{-1}|A_{12}|_0 < |\frac{d}{dx}A_{12}|_0$. On the other hand, $|G_{12}A_{22}|_0 < |qx^{q-1}A_{11}\sigma(G_{12})|_0$ by $\nu > 1$ since A_{11} (resp. A_{22}) is a unit matrix (resp. a unit matrix times q^ν) modulo $\mathfrak{m}\mathcal{O}_\mathcal{E}$ (resp. $q^\nu\mathfrak{m}\mathcal{O}_\mathcal{E}$) by 2° (resp. 3°). So we have

$$\frac{d}{dx}A_{12} \equiv qx^{q-1}A_{11}\sigma(G_{12}) \ (\mathrm{mod}\, q^{-\log_q|\frac{d}{dx}A_{12}|_0}\mathfrak{m}\mathcal{O}_\mathcal{E})$$

But, on comparing the x-adic order of both sides, this is seen to be impossible by 2°, 3° and 4°. Hence $A_{12} = 0$. Now the commutativity of Frobenius and connection (the relation 1°) is just

$$G_{12}A_{22} = qx^{q-1}A_{11}\sigma(G_{12}).$$

Since any morphism between pure φ-modules with different Frobenius slopes are 0, we have $G_{12} = 0$ by $\nu > 1$.

4.4.2 The case where $0 < \nu < 1$

Suppose $0 < \nu < 1$ (and $\lambda = 0$). Assuming that $A_{12} \neq 0$, we will show the existence of unbounded solutions on the generic disk by applying Lemma 4.8 (2). This is a contradiction to our hypothesis of boundedness of M, and thus we must have $A_{12} = 0$. Since $\nu \neq 1$, we again have $G_{12} = 0$ by the slope reason. Therefore, the extension is split.

Assume that $A_{12} = \sum_n A_{12,n}x^n \neq 0$. Since $|G_{12}A_{22}|_0 = q^{-\nu}|G_{12}|_0$, $|qx^{q-1}A_{11}\sigma(G_{12})|_0 = q^{-1}|G_{12}|_0$, and $|\frac{d}{dx}A_{12}|_0 > q^{-1}|A_{12}|_0$ by 3°, 2° and our hypothesis, respectively, the formula 4° gives estimates

5° $\quad q^{-1}|A_{12}|_0 < q^{-\nu}|G_{12}|_0 = |G_{12}A_{22}|_0 = |\frac{d}{dx}A_{12}|_0 \leq |A_{12}|_0.$

We also claim that

6° $\quad$ there is a positive integer m with $q \not| m$ such that $|\frac{1}{m!}\frac{d^m}{dx^m}A_{12}|_0 = |A_{12}|_0$

by 1°. Indeed, let l be an integer such that $|A_{12,l}| = |A_{12}|_0$. When $l > 0$, we put $m = l$. Then the coefficient of $\frac{1}{m!}\frac{d^l}{dx^l}A_{12}$ in the 0-th term x^0 is $A_{12,l}$ and we have $|\frac{1}{l!}\frac{d^l}{dx^l}A_{12}|_0 = |A_{12,l}| = |A_{12}|_0$. When $l < 0$, we put $m = q^{-l} + l$ (remark that any sufficient large power of q can be replaced by q^{-l}). Then the coefficient of $\frac{1}{m!}\frac{d^m}{dx^m}A_{12}$ in the $l - m(= -q^{-l})$-th term x^{l-m} is $(-1)^m\begin{pmatrix} m-l-1 \\ m\end{pmatrix}A_{12,l}$ and we have $|\frac{1}{m!}\frac{d^m}{dx^m}A_{12}|_0 = |A_{12,l}| = |A_{12}|_0$ since $(-1)^m\begin{pmatrix} m-l-1 \\ m\end{pmatrix}$ is a p-adic unit.

In proving the assertion, we will consider the following two cases for A_{12}:

(i) $|\frac{d}{dx}A_{12}|_0 > q^{-\nu}|A_{12}|_0$.

(ii) $|\frac{d}{dx}A_{12}|_0 \leq q^{-\nu}|A_{12}|_0$. (Hence we have $|G_{12}|_0 \leq |A_{12}|_0$ by 5°)

In order to prove the existence of unbounded solutions above, let us reorganize the matrix representation by using changes of basis of M, a change of Frobenius and an extension of scalar field. Let us consider the induced φ-∇-module $M_\tau = M \otimes_{\mathcal{E}} \mathcal{E}_t[\![X-t]\!]_0$ over the bounded functions $\mathcal{E}_t[\![X-t]\!]_0$ at the generic disk. Since L_τ and N_τ are pure, we have bounded solution matrices Y_{11} of L and Y_{22} of N, that is,

$$7^\circ \quad \begin{array}{l} L : \begin{cases} A_{11}(t)\sigma(Y_{11}) = Y_{11}\tau(A_{11}) \\ \dfrac{d}{dX}Y_{11} = Y_{11}\tau(G_{11}) \\ Y_{11} \in 1_r + q(X-t)\mathrm{Mat}_r(\mathcal{O}_{\mathcal{E}_t}[\![X-t]\!]) \end{cases} \\ N : \begin{cases} A_{22}(t)\sigma(Y_{22}) = Y_{22}\tau(A_{22}) \\ \dfrac{d}{dX}Y_{22} = Y_{22}\tau(G_{22}) \\ Y_{22} \in 1_s + q(X-t)\mathrm{Mat}_s(\mathcal{O}_{\mathcal{E}_t}[\![X-t]\!]) \end{cases} \end{array}$$

by 2° and 3°. Note that $\tau(f) = \sum_n \frac{1}{n!}(\frac{d^n}{dx^n}f)(t)(X-t)^n$ for $f \in \mathcal{E}$ and it is an isometry. Consider a change of basis of M_τ by the matrix $Y^{-1} = \begin{pmatrix} Y_{11}^{-1} & 0 \\ 0 & Y_{22}^{-1} \end{pmatrix}$. Then the new Frobenius matrix and the new connection matrix are as follows:

$$\begin{aligned} A^\tau &= YA\sigma(Y)^{-1} = \begin{pmatrix} A_{11}(t) & Y_{11}\tau(A_{12})\sigma(Y_{22})^{-1} \\ 0 & A_{22}(t) \end{pmatrix} \\ G^\tau &= Y\tfrac{d}{dX}Y^{-1} + YGY^{-1} = \begin{pmatrix} 0 & Y_{11}\tau(G_{12})Y_{22}^{-1} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Let us put $A_{12}^\tau = \sum_n A_{12,n}^\tau (X-t)^n$ (resp. G_{12}^τ) to be the $(1,2)$-part of the Frobenius matrix A^τ (resp. G^τ), and define $B_{12}^\tau = \sum_{n>0} A_{12,n}^\tau (X-t)^n$ by the subseries of positive powers. Then we have

$$8^\circ \qquad |B_{12}^\tau|_0 = |A_{12}|_0$$

$$9^\circ \qquad |G_{12}^\tau|_0 = |\tau(G_{12})|_0 = |G_{12}|_0.$$

by 6° and $Y \equiv 1_{r+s} \,(\mathrm{mod}\, q(X-t)\mathcal{O}_{\mathcal{E}_t}[\![X-t]\!])$.

Now we consider a change of Frobenius. At first our Frobenius on $\mathcal{E}$ is given by $\sigma(x) = x^q$. Hence the induced Frobenius on the generic disk is given by $\sigma(X-t) = ((X-t)+t)^q - t^q$. Let us replace σ by the Frobenius $\widetilde{\sigma}$ defined by $\widetilde{\sigma}(X-t) = (X-t)^q$. Note that

$$10^\circ \qquad \sigma(X-t) - \widetilde{\sigma}(X-t) \equiv qt^{q-1}(X-t) \,(\mathrm{mod}\, p(X-t)^2\mathcal{O}_{\mathcal{E}_t}[\![X-t]\!]).$$

Since $|\frac{1}{n!}\frac{d^{n-1}}{dX^{n-1}}G_{12}^\tau|_0 \leq |n|^{-1}|G_{12}|_0$ and $|p^n/n| \leq |p|$ for all $n \geq 1$, the matrix H of comparison transform $\vartheta_{\widetilde{\sigma},\sigma}^*(M_\tau)$ in section 3.1 satisfies the congruence relation

11°
$$\begin{aligned} H &= 1_{r+s} + \sum_{n=1}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & \sigma(\frac{d^{n-1}}{dX^{n-1}} G_{12}^{\tau}) \\ 0 & 0 \end{pmatrix} (\sigma(X-t) - \widetilde{\sigma}(X-t))^n \\ &\equiv 1_{r+s} + qt^{q-1} \begin{pmatrix} 0 & \sigma(G_{12}(t)) \\ 0 & 0 \end{pmatrix} (X-t) \\ &\qquad\qquad (\mathrm{mod}\, pq^{-\log_q |G_{12}|_0}(X-t)^2 \mathcal{O}_{\mathcal{E}_t}[\![X-t]\!]) \end{aligned}$$

by 9° and 10°. Our Frobenius matrix of M_τ with respect to the Frobenius $\widetilde{\sigma}$ is

$$\widetilde{A} = A^\tau H = \begin{pmatrix} A_{11}(t) & A_{12}^\tau + A_{11}(t)H_{12} \\ 0 & A_{22}(t) \end{pmatrix}$$

by the definition of the equivalence (Theorem 3.3), where $H_{12} = \sum_n H_{12,n}(X-t)^n$ is the $(1,2)$-part of H. If we put $\widetilde{A}_{12} = \sum_n \widetilde{A}_{12,n}(X-t)^n$ to be the $(1,2)$-part of $\widetilde{A}$ and put $\widetilde{B}_{12} = \sum_{n>0} \widetilde{A}_{12,n}(X-t)^n$, then

12° there is a positive integer m with $q \not| m$ such that $|\widetilde{A}_{12,m}|_0 > q^{-\nu}|\widetilde{B}_{12}|_0$.

Indeed, in the case (i) for A_{12}, since $\widetilde{A}_{12,1} = A_{12,1}^\tau + A_{11}(t)H_{12,1}$ and $|H_{12,1}|_0 \leq q^{-1}|G_{12}|_0$, we have $|\widetilde{A}_{12,1}|_0 = |A_{12,1}^\tau|_0 = |\frac{d}{dx}A_{12}|_0$ by 5° and 11°. On the other hand, $|\widetilde{B}_{12}|_0 \leq \max\{|B_{12}^\tau|_0, |H_{12}|_0\} \leq \max\{|A_{12}|_0, |p||G_{12}|_0\} < q^\nu|\frac{d}{dx}A_{12}|_0$ by 5°, 8° and 11° because of our hypothesis (i), $|\frac{d}{dx}A_{12}|_0 > q^{-\nu}|A_{12}|_0$. Hence we can take $m = 1$. In the case (ii), we take a positive integer m such as 6°. Since $|G_{12}|_0 \leq |A_{12}|_0$ by the hypothesis (ii), we have $|\widetilde{B}_{12}|_0 \leq \max\{|B_{12}^\tau|_0, |H_{12}|_0\} = |A_{12}|_0$ by 8° and 11°.

By Proposition 2.1 we may replace $\mathcal{E}_t$ by the p-adic completion $\widehat{\mathcal{E}_t^{\mathrm{ur}}}$ of the maximally unramified extension of $\mathcal{E}_t$. Then we may assume that $\widetilde{A}_{11} = 1_r$ and $\widetilde{A}_{22} = q^\nu 1_s$ since the solutions of both $(1,1)$-part and $(2,2)$-part is 1 modulo q by 2° and 3°. The solution matrix of $M_\tau \otimes_{\mathcal{E}_t} \widehat{\mathcal{E}_t^{\mathrm{ur}}}$ has a form $Z = \begin{pmatrix} 1_r & Z_{12} \\ 0 & 1_s \end{pmatrix}$ satisfying $\widetilde{A}|_{X=t}\widetilde{\sigma}(Z) = Z\widetilde{A}$ and $Z_{12}|_{X=t} = 0$. In particular, Z_{12} satisfies the relation

$$\widetilde{\sigma}(Z_{12}) = q^\nu Z_{12} + \widetilde{B}_{12}.$$

On applying Lemma 4.8 (2) to Z_{12}, one sees that one of entries of Z_{12} must be exactly of log-growth ν by 12°. Hence the non-vanishing of A_{12} implies that M is unbounded.

This completes the proof for the case $0 < \nu < 1$.

4.4.3 The case where $\nu = 1$

Suppose that $\nu = 1$. Suppose that $G_{12} \neq 0$. Let us develop $G_{12} = \sum_n G_{12,n}x^n$ and let m be the least integer such that $|G_{12,m}| = |G_{12}|_0$. If $A_{12} \neq 0$, we have $|\frac{d}{dx}A_{12}|_0 > q^{-1}|A_{12}|_0$ by 4°. So the relation 1° induces a congruence

$$13^\circ \qquad \frac{d}{dx}A_{12} + qG_{12} \equiv qx^{q-1}\sigma(G_{12}) \ (\mathrm{mod}\, q^{1+\log_q|G_{12}|_0}\mathfrak{m}\mathcal{O}_{\mathcal{E}})$$

by 2° and 3°. This congruence 13° also holds when $A_{12} = 0$.
Suppose that $m < -1$. The least power of x which should appear in the right hand side of the congruence 13° above is $qm + q - 1$. Since $qm + q - 1 < m$, this is precluded by 4°.
Suppose that $m = -1$. Then

$$14^\circ \qquad \begin{aligned} \tau(G_{12}) &= \sum_{n=0}^{\infty} \frac{1}{n!}(\frac{d^n}{dx^n}G_{12})(t)(X-t)^n \\ &= \sum_{n=0}^{\infty}(G_{12,-1}\frac{(-1)^n}{t^n} + q^{-\log_q|G_{12}|_0}\mathrm{Mat}(t^{1-n}\mathcal{V}[\![t]\!]_0 + \mathfrak{m}\mathcal{O}_{\mathcal{E}}))(X-t)^n. \end{aligned}$$

Let us calculate the solution matrix of M_τ by using 7° as in the previous case. By changing a basis of M_τ by the invertible matrix $Y = \begin{pmatrix} Y_{11}^{-1} & 0 \\ 0 & Y_{22}^{-1} \end{pmatrix}$ as before, our differential equation becomes

$$\frac{d}{dX}\begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & Y_{11}\tau(G_{12})Y_{22}^{-1} \\ 0 & 0 \end{pmatrix}.$$

in Z, that is, $\frac{d}{dX}Z = Y_{11}\tau(G_{12})Y_{22}^{-1}$. Since all the coefficients of all the power series which appear on the entries of $Y_{11}\tau(G_{12})Y_{22}^{-1}$ do not vanish modulo $q^{-\log_q|G_{12}|_0}\mathfrak{m}\mathcal{O}_{\mathcal{E}}$ by 7° and 14°, at least one of entries of Z is exactly of log-growth 1. This contradicts to our hypothesis of boundness of M. Hence, $m \neq -1$.
Suppose that $m > 0$. Then we have

$$G_{12} \equiv -q^{-1}x^{-1}\left(x\tfrac{d}{dx}A_{12} + \sigma(x\tfrac{d}{dx}A_{12}) + \sigma^2(x\tfrac{d}{dx}A_{12}) + \cdots\right) \quad (\mathrm{mod}\, q^{-\log_q|G_{12}|_0}\mathfrak{m}\mathcal{O}_{\mathcal{E}})$$

by 4° and 13°. The case where $A_{12} = 0$ is impossible since $G_{12} \neq 0$. If $A_{12} \neq 0$, then we have a solution exactly of log-growth 1 on the generic disk by the similar construction in the case $m = -1$. This contradicts our hypothesis. Therefore, we have $G_{12} = 0$ in any case.
Now we prove $A_{12} = 0$. Suppose that $A_{12} \neq 0$. Then the relation 1° is

$$\frac{d}{dx}A_{12} + G_{11}A_{12} = qx^{q-1}A_{12}\sigma(G_{22}).$$

This is impossible by 2°, 3° and 4°. Hence, $A_{12} = 0$.
This completes the proof of Proposition 4.2. □

Remark 4.9 *There is another proof of Proposition 4.2: one can reduce Proposition 4.2 to the case where $q = p$, that is, the Frobenius σ is a p-Frobenius. Then, in the proof of the case $0 < \nu < 1$, it is enough to discuss only in the case $|\frac{d}{dx}A_{12}|_0 = |A_{12}|_0$.*

5 PBQ φ-∇-modules

5.1 Definition of PBQ φ-∇-modules

Definition 5.1 *(Definition of "PBQ" φ-∇-modules)*

(1) *A φ-∇-module M over $\mathcal{E}$ is said to be pure of bounded quotient (called PBQ for simplicity) if M/M^0 is pure as a φ-module.*

(2) *A φ-∇-module M over $K[\![x]\!]_0$ is said to be pure of bounded quotient (called PBQ for simplicity) if the generic fiber M_η of M is PBQ as a φ-∇-module over $\mathcal{E}$.*

The notion "PBQ" depends only on the Frobenius slopes of the bounded quotient of the generic fiber of φ-∇-modules. As we saw in Theorem 4.1, the bounded quotient of the generic fiber always admits a splitting by Frobenius slopes when it has different slopes.

Example 5.2 (1) *A bounded φ-∇-module M over $\mathcal{E}$ is PBQ if and only if M is pure as a φ-module. In particular, any φ-∇-module M over $\mathcal{E}$ of rank 1 is PBQ.*

(2) *Any φ-∇-module M over $\mathcal{E}$ of rank 2 which is not bounded is PBQ [CT09, Theorem 7.1].*

(3) *Let us fix a Frobenius on σ with $\sigma(x) = x^q$. Let M be a φ-∇-module over $K[\![x]\!]_0$ with a basis (e_1, e_2, e_3) such that the Frobenius matrix A and the connection matrix G are defined by*

$$A = \begin{pmatrix} 1 & -q^{1/2}x & -qx \\ 0 & q^{1/2} & 0 \\ 0 & 0 & q \end{pmatrix}, \ G = \begin{pmatrix} 0 & \sum_{n=0}^{\infty} q^{n/2}x^{q^n-1} & \sum_{n=0}^{\infty} x^{q^n-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then M_η is not bounded and M is not PBQ. Indeed, the $K[\![x]\!]_0$-submodule L generated by e_1 is a φ-∇-submodule of M such that the quotient $(M/L)_\eta$ is bounded and $(M/L)_\eta$ is not pure. On the other hand the dual $M^\vee$ of M is PBQ.

Proposition 5.3 *Any quotient of PBQ φ-∇-modules over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) is PBQ.*

Proof. Let M be a PBQ φ-∇-module over $\mathcal{E}$ and let M' be a quotient of M. The assertion follows from that the natural morphism $M/M^0 \to M'/(M')^0$ is surjective by [CT09, Corollary 3.5]. □

5.2 Existence of the maximally PBQ φ-∇-submodules over $\mathcal{E}$

Proposition 5.4 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let M be a φ-∇-module over $\mathcal{E}$ with highest Frobenius slope $\lambda_{\max}$ and let N' be a φ-∇-submodule of $M/S_{\lambda_{\max}-}(M)$. Then there is a unique φ-∇-submodule N of M such that N is PBQ and the natural morphism $N/N^0 \to M/S_{\lambda_{\max}-}(M)$ gives an isomorphism between N/N^0 and N'. When $N' = M/S_{\lambda_{\max}-}(M)$, we call the corresponding N the maximally PBQ submodule of M.*

Proof. First we prove the uniqueness of N. Let N_1 and N_2 be a PBQ submodule of M such that both natural morphisms $N_1/N_1^0 \to M/S_{\lambda_{\max}-}(M) \leftarrow N_2/N_2^0$ give isomorphisms with N'. Let N be the image of $N_1 \oplus N_2 \to M$ $(a, b) \mapsto a+b$. Then N is PBQ by Proposition 5.3. Since $N_1/N_1^0 \oplus N_2/N_2^0 \to N/N^0$ is surjective by [CT09, Proposition 3.6], the natural morphism $N/N^0 \to M/S_{\lambda_{\max}-}(M)$ gives an isomorphism with N'. If N_1 (resp. N_2) is not N, then the quotient N/N_1 (resp. N/N_2) has a bounded solution at the generic disk whose Frobenius slope is different from $\lambda_{\max}$. But this is impossible because N is PBQ. Hence $N = N_1 = N_2$.
Now we prove the existence of N. We use the induction on the dimension of M. Let $f : M \to M/M^0$ be a natural surjection. Since M/M^0 is bounded, $M/S_{\lambda_{\max}-}(M)$ is a direct summand of M/M^0 by the maximality of slopes by Theorem 4.1. Put $L = f^{-1}(N')$. If L is PBQ, then one can put $N = L$. If L is not PBQ, then L is a proper submodule of M and there is a PBQ submodule L' of L such that $L'/(L')^0 \cong L/S_{\lambda_{\max}-}(L) = N'$ by the induction hypothesis. □

Corollary 5.5 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let M be a φ-∇-module over $\mathcal{E}$. Then there is a unique filtration $0 = P_0(M) \subsetneq P_1(M) \subsetneq \cdots \subsetneq P_r(M) = M$ of φ-∇-modules over $\mathcal{E}$ such that $P_i(M)/P_{i-1}(M)$ is the maximally PBQ submodule of $M/P_{i-1}(M)$ for any $i = 1, \cdots, r$. We call $\{P_i(M)\}$ the PBQ filtration of M.*

5.3 Existence of the maximally PBQ φ-∇-submodules over $K[\![x]\!]_0$

Theorem 5.6 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let M be a φ-∇-module over $K[\![x]\!]_0$. Then there is a unique φ-∇-submodule N of M over $K[\![x]\!]_0$ such that the generic fiber N_η of N is the maximally PBQ submodule of the generic fiber M_η of M. We call N the maximally PBQ submodule of M.*

Proof. The proof of uniqueness of the maximally PBQ submodules is same to the proof of Proposition 5.4.
We prove the existence of the maximally PBQ submodules by induction on the rank of M. If M is of rank 1, then the assertion is trivial. For general M, if M is PBQ, then there is nothing to prove. Suppose that M is not PBQ. Then there is a direct summand L_η of M_η/M_η^0 such that L_η is pure with the Frobenius slope which is less than the highest slope $\lambda_{\max}$ of M by Theorem

4.1. Consider the composite of natural morphisms $M \to M_\eta/M_\eta^0 \to L_\eta$. It is not injective by Lemma 5.7 below. Put M' to be the kernel. Then M' is a φ-∇-submodule of M such that $M'_\eta/S_{\lambda_{\max}}(M'_\eta) \cong M_\eta/S_{\lambda_{\max}}(M_\eta)$. By the induction hypothesis there is a maximally PBQ submodule N of M' which becomes the maximally PBQ submodule N of M. □

Lemma 5.7 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let M be a φ-module over $K[\![x]\!]_0$ such that the highest Frobenius slope of the generic fiber M_η of M is $\lambda_{\max}$. Suppose that there exists an injective $K[\![x]\!]_0$-homomorphism $f : M \to L_\eta$ which is φ-equivariant, i.e., $\varphi \circ f = f \circ \varphi$ for a pure φ-module L_η over $\mathcal{E}$. Then the Frobenius slope of L_η is $\lambda_{\max}$.*

Proof. In [dJ98, Corollary 8.2] A.J. de Jong proved this assertion when L_η is a generic fiber of a rank 1 pure φ-∇ module L over $K[\![x]\!]_0$. (Indeed, he proved a stronger assertion.) We give a sketch of the proof which is due to [dJ98, Propositions 5.5, 6.4 and 8.1]. Our $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$, resp. $\widetilde{\mathcal{E}}$, resp. $\widetilde{\mathcal{E}}^\dagger$ introduced below) corresponds to Γ (resp. Γ_c, resp. Γ_2, resp. $\Gamma_{2,c}$) in [dJ98]. We also remark that $\widetilde{\mathcal{E}}^\dagger$ is the extended bounded Robba ring $\widetilde{\mathcal{R}}^{\mathrm{bd}}$ in [Ke08, 2.2].
We may assume that the residue field k of $\mathcal{V}$ is algebraically closed and all slopes of M are contained in the value group of $\log_q|K^\times|$. We may also assume that $\sigma(x) = x^q$ by Theorem 3.3. Let us define K-algebras

$$\begin{aligned}
\widetilde{\mathcal{E}} &= \left\{ \sum_{n\in\mathbb{Q}} a_n x^n \;\middle|\; \begin{array}{l} a_n \in K,\ \sup_n|a_n| < \infty,\ |a_n| \to -\infty\ (n \to -\infty), \\ \{n \mid |a_n| \geq \alpha\} \text{ is a well-ordered set with respect to} \\ \text{the order } \leq \text{ for any } \alpha \in \mathbb{R}. \end{array} \right\} \\
\widetilde{\mathcal{E}}^\dagger &= \left\{ \sum_{n\in\mathbb{Q}} a_n x^n \in \widetilde{\mathcal{E}} \;\middle|\; |a_n|\eta^n \to 0\ (n \to -\infty) \text{ for some } 0 < \eta < 1. \right\}.
\end{aligned}$$

Both $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{E}}^\dagger$ are discrete valuation fields such that both ring of integers have a same residue field

$$k(\!(x^{\mathbb{Q}})\!) = \left\{ \sum_{n\in\mathbb{Q}} a_n x^n \;\middle|\; \begin{array}{l} a_n \in k,\ \{n \mid a_n \neq 0\} \text{ is a well-ordered set} \\ \text{with respect to the order } \leq . \end{array} \right\},$$

which includes an algebraic closure of $k(\!(x)\!)$ [Ke01], and that the p-adic completion of $\widetilde{\mathcal{E}}^\dagger$ is $\widetilde{\mathcal{E}}$. $\widetilde{\mathcal{E}}$ is naturally an $\mathcal{E}$-algebra and σ naturally extends to $\widetilde{\mathcal{E}}$ by $\sigma(\sum_n a_n x^n) = \sum_n \sigma(a_n) x^{qn}$. Put

$$\mathcal{E}^\dagger = \widetilde{\mathcal{E}}^\dagger \cap \mathcal{E}.$$

Then $\mathcal{E}^\dagger$ is stable under σ and the K-derivation d/dx. We also denote by $\mathcal{O}_{\widetilde{\mathcal{E}}^\dagger}$ the ring of integer of $\widetilde{\mathcal{E}}^\dagger$.
By explicit calculations we have the following sublemmas.

Sublemma 5.8 *For* $0 < \eta < 1$ *and for* $\sum_{n \in \mathbb{Q}} a_n x^n \in \widetilde{\mathcal{E}}^\dagger$*, let us consider a condition:*

$$(*)_\eta : \quad \mathrm{sup}_n |a_n| \max\{\eta^n, 1\} \leq 1.$$

If f *and* g *in* $\widetilde{\mathcal{E}}^\dagger$ *satisfy the condition* $(*)_\eta$*, then so are* $f+g$ *and* fg*. Moreover, if* $f = \sum_n a_n x^n$ *satisfies the condition* $(*)_\eta$ *and* $|a_0| = 1$*, then so is* f^{-1}*.*

Note that, if $\eta < \mu$, then the condition $(*)_\eta$ implies the condition $(*)_\mu$.

Sublemma 5.9 (1) *Let* $A = 1 + B$ *be a square matrix such that* 1 *is the unit matrix and all entries of* B *contained in* $\mathfrak{m}^n \mathcal{O}_{\widetilde{\mathcal{E}}^\dagger}$ *for a positive integer* n*. Suppose that all entries of* A *satisfy the condition* $(*)_\eta$ *in Sublemma 5.8. Then there is a matrix* $Y = 1 + Z$ *with* $A\sigma(Y) = Y$ *such that all entries of* Z *are contained in* $\mathfrak{m}^n \mathcal{O}_{\widetilde{\mathcal{E}}^\dagger}$ *and satisfy the condition* $(*)_{\eta^q}$*.*

(2) *Let* C *be a matrix such that all entries are contained in* $\mathfrak{m}^n \mathcal{O}_{\widetilde{\mathcal{E}}^\dagger}$ *for a nonnegative integer* n *and satisfy the condition* $(*)_\eta$*. Then there is a matrix* Z *satisfying* $\sigma(Z) - Z = C$ *such that all entries of* Z *are contained in* $\mathfrak{m}^n \mathcal{O}_{\widetilde{\mathcal{E}}^\dagger}$ *and satisfy the condition* $(*)_{\eta^q}$*.*

Proof. (1) follows from (2) by considering a congruence equation $A\sigma(Y) \equiv Y \pmod{\mathfrak{m}^l \mathcal{O}_{\widetilde{\mathcal{E}}^\dagger}}$ inductively on l.
(2) Since the residue field k of $\mathcal{V}$ is perfect, σ is bijective. Put $C = \sum_n C_n x^n = C_- + C_0 + C_+$, where they are subseries of negative powers, a constant term, and subseries of positive powers, respectively. The series $Z_- = \sum_{n<0} \sum_{i=1}^\infty \sigma^{-i}(C_n) x^{n/q^i}$ converges and all entries of Z_- satisfies the condition $(*)_{\eta^q}$, and the equation $\sigma(Z_-) - Z_- = C_-$ holds. Since k is algebraically closed, there is a matrix Z_0 over $\mathcal{V}$ with $|Z_0| \leq |C_0|$ such that $\sigma(Z_0) - Z_0 = C_0$. The series $Z_+ = -\sum_{i=0}^\infty \sigma(C_+)$ converges and satisfies $\sigma(Z_+) - Z_+ = C_+$. Hence, $Z = Z_- + Z_0 + Z_+$ is the desired solution. □

If $N^\dagger$ is a φ-∇-submodule of $M \otimes_{K[\![x]\!]_0} \mathcal{E}^\dagger$ over $\mathcal{E}^\dagger$, then there is a φ-∇-submodule N of M over $K[\![x]\!]_0$ with $N \otimes_{K[\![x]\!]_0} \mathcal{E}^\dagger \cong N$ by [dJ98, Proposition 6.4]. Hence, the induced morphism $M \otimes_{K[\![x]\!]_0} \mathcal{E}^\dagger \to L_\eta$ is also injective. Moreover, since $\widetilde{\mathcal{E}}^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E} \to \widetilde{\mathcal{E}}$ is injective (the similar proof of [dJ98, Proposition 8.1] works), the induced morphism $M \otimes_{K[\![x]\!]_0} \widetilde{\mathcal{E}}^\dagger \to L_\eta \otimes_{\mathcal{E}} \widetilde{\mathcal{E}}$ is again injective. Let $\lambda_1 < \cdots < \lambda_r (= \lambda_{\max})$ be Frobenius slopes of M_η. One can prove that there exists an increasing filtration $0 = \widetilde{M}_0 \subsetneq \widetilde{M}_1 \subsetneq \cdots \subsetneq \widetilde{M}_r = M \otimes_{K[\![x]\!]_0} \widetilde{\mathcal{E}}^\dagger$ of φ-modules over $\widetilde{\mathcal{E}}^\dagger$ such that $(\widetilde{M}_i / \widetilde{M}_{i-1}) \otimes_{\widetilde{\mathcal{E}}^\dagger} \widetilde{\mathcal{E}}$ is pure of slope λ_{r-i+1}. This existence of filtration of opposite direction corresponds to Proposition 5.5 in [dJ98]. Indeed, since the residue field $k(\!(x^{\mathbb{Q}})\!)$ includes an algebraic closure of $k(\!(x)\!)$, there is a basis of $M \otimes_{K[\![x]\!]_0} \widetilde{\mathcal{E}}^\dagger$ such that the Frobenius matrix of

$M \otimes_{K[\![x]\!]_0} \widetilde{\mathcal{E}}^\dagger$ has a form

$$\begin{pmatrix} q^{\lambda_1}1 & & \\ & \ddots & \\ & & q^{\lambda_r}1 \end{pmatrix} + (\text{a square matrix with entries in } \mathfrak{m}^n \mathcal{O}_{\widetilde{\mathcal{E}}})$$

by Dieudonné-Manin classification of φ-modules and the density of $\widetilde{\mathcal{E}}^\dagger$ in $\widetilde{\mathcal{E}}$. Here q^λ is a element of K with $\log_q |q^\lambda| = -\lambda$, 1 is the unit matrix with a certain size (the first matrix is a diagonal matrix), and n is sufficiently large. One can find a basis of $M \otimes_{K[\![x]\!]_0} \widetilde{\mathcal{E}}^\dagger$ such that the Frobenius matrix of $M \otimes_{K[\![x]\!]_0} \widetilde{\mathcal{E}}^\dagger$ is a lower triangle matrix

$$\begin{pmatrix} q^{\lambda_1}1 & & 0 \\ & \ddots & \\ * & & q^{\lambda_r}1 \end{pmatrix}$$

by Sublemmas 5.8 and 5.9. Hence, one has a filtration of opposite direction. Since $\widetilde{M_1}$ is pure of slope $\lambda_r = \lambda_{\max}$ and the inclusion $\widetilde{M_1} \subset L_\eta \otimes_{\mathcal{E}} \widetilde{\mathcal{E}}$ is φ-equivariant, the slope of L_η must be $\lambda_{\max}$. □

Corollary 5.10 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let M be a φ-∇-module over $K[\![x]\!]_0$. Then there is a unique filtration $0 = P_0(M) \subsetneq P_1(M) \subsetneq \cdots \subsetneq P_r(M) = M$ as φ-∇-modules over $K[\![x]\!]_0$ such that $P_i(M)/P_{i-1}(M)$ is the maximally PBQ submodule of $M/P_{i-1}(M)$ for any $i = 1, \cdots, r$. We call $\{P_i(M)\}$ the PBQ filtration of M.*

Example 5.11 *Let M be a φ-∇-module over $K[\![x]\!]_0$ which is introduced in Example 5.2 (3). If $P_1(M)$ is a φ-∇-submodule of M over $K[\![x]\!]_0$ generated by e_1 and e_3, the sequence $0 = P_0(M) \subsetneq P_1(M) \subsetneq P_2(M) = M$ is the PBQ filtration of M.*

6 Log-growth and Frobenius slope for HPBQ φ-∇-modules over $K[\![x]\!]_0$

6.1 Log-growth for HPBQ φ-∇-modules

Definition 6.1 (1) *A φ-∇-module M over $K[\![x]\!]_0$ is horizontal of bounded quotient (HBQ for simplicity) if there is a quotient N of M as a φ-∇-module over $K[\![x]\!]_0$ such that the canonical surjection induces an isomorphism $M_\eta/M_\eta^0 \cong N_\eta$ at the generic fiber.*

(2) *A φ-∇-module M over $K[\![x]\!]_0$ is horizontally pure of bounded quotient (HPBQ for simplicity) if M is PBQ and HBQ.*

Example 6.2 (1) *A bounded φ-∇-module M over $K[\![x]\!]_0$ is HBQ. A bounded φ-∇-module M over $K[\![x]\!]_0$ is HPBQ if and only if M_η is pure as a φ-module.*

(2) *Let M be a φ-∇-module M over $K[\![x]\!]_0$ of rank 2 which arises from the first crystalline cohomology of a projective smooth family E of elliptic curves over* $\operatorname{Spec} k[\![x]\!]$. *Then M is HBQ if and only if either (i) E is a non-isotrivial family over* $\operatorname{Spec} k[\![x]\!]$ *and the special fiber E_s of E is ordinary or (ii) E is an isotrivial family over* $\operatorname{Spec} k[\![x]\!]$. *In the case (i) M is HPBQ, but in the case (ii) M is HPBQ if and only if E is an isotrivial family of supersingular elliptic curves.*

(3) *Let M be a φ-∇-module over $K[\![x]\!]_0$ which is introduced in Example 5.2 (3). Then M is HBQ but is not HPBQ. The dual $M^\vee$ of M is HPBQ.*

Proposition 6.3 *Let M be a φ-∇-module over $K[\![x]\!]_0$. Then M is HBQ if and only if*

$$\dim_K V(M)/V(M)^0 = \dim_{\mathcal{E}} M_\eta/M_\eta^0.$$

Moreover, when M is HBQ, the natural pairing $M \otimes_K \mathrm{Sol}_0(M) \to K[\![x]\!]_0$ induces an isomorphism

$$M_\eta/M_\eta^0 \cong V(M)/V(M)^0 \otimes_K \mathcal{E}$$

as φ-∇-modules.

Proof. Suppose that M is HBQ. Let N be the quotient as in Definition 6.1 (1). Since N_η is bounded, we have $V(N)^0 = 0$ by Christol's transfer theorem (see [CT09, Proposition 4.3]) and $\dim_K V(M)/V(M)^0 \geq \dim_K V(N)/V(N)^0 = \mathrm{rank}_{K[\![x]\!]_0} N = \dim_{\mathcal{E}} M_\eta/M_\eta^0$. On the other hand, one knows an inequality $\dim_K V(M)/V(M)^0 \leq \dim_{\mathcal{E}} M_\eta/M_\eta^0$ by [CT09, Proposition 4.10]. Hence, we have an equality $\dim_K V(M)/V(M)^0 = \dim_{\mathcal{E}} M_\eta/M_\eta^0$. Now we prove the inverse. The natural pairing $M \otimes_K \mathrm{Sol}_0(M) \to K[\![x]\!]_0$ induces the surjection $M \to V(M)/V(M)^0 \otimes_K K[\![x]\!]_0$. If $\dim_K V(M)/V(M)^0 = \dim_{\mathcal{E}} M_\eta/M_\eta^0$, we have an isomorphism $M_\eta/M_\eta^0 \cong V(M)/V(M)^0 \otimes_K \mathcal{E}$ since $V(M)/V(M)^0 \otimes_K \mathcal{E}$ is bounded. □

Since any quotient of bounded φ-∇-modules over $\mathcal{E}$ is again bounded, the proposition below follows from the chase of commutative diagrams.

Proposition 6.4 *Any quotient of HBQ φ-∇-modules over $K[\![x]\!]_0$ is HBQ. In particular, any quotient of HPBQ modules is HPBQ.*

Proof. We may assume that the residue field of $\mathcal{V}$ is algebraically closed and $q^{\lambda_{\max}} \in K$. Since M is HBQ, there is a surjection $M \to V(M)/V(M)^0 \otimes_K K[\![x]\!]_0$ by Propoition 6.3 whose kernel is denoted by L. Then $M_\eta^0 = L_\eta$. If $f: M \to N$ be the given surjection, $N/f(L)$ is a quotient of $V(M)/V(M)^0 \otimes_K K[\![x]\!]_0$ and hence a direct sum of copies of $(K[\![x]\!]_0, q^\lambda\sigma, d)$ for some λ. Since f gives a surjection from M_η^0 to N_η^0 by [CT09, Proposition 3.6], we have

$$\dim_K V(N)/V(N)^0 \geq \mathrm{rank}_{K[\![x]\!]_0} N/f(L) = N_\eta/N_\eta^0.$$

On the other hand, $\dim_K V(N)/V(N)^0 \leq \dim_{\mathcal{E}} N_\eta/N_\eta^0$ by [CT09, Proposition 4.10]. Hence $\dim_K V(N)/V(N)^0 = \dim_{\mathcal{E}} N_\eta/N_\eta^0$. The rest follows from Proposition 5.3. □

Note that the notion PBQ is determined only by the generic fiber. On the other hand, for "HPBQ", the bounded quotient is horizontal.

Theorem 6.5 *Let M be a φ-∇-module M over $K[\![x]\!]_0$ which is HPBQ. Then the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (see 2.5) holds for M.*

Proof. We have only to prove the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}(2)$ for M. Then the property of Frobenius slopes implies the conjecture the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}(1)$ for M. We may assume that the residue field of $\mathcal{V}$ is algebraically closed and all Frobenius slopes of $V(M)$ are contained in the valued group $\log_q|K^\times|$ by Proposition 2.1. We may also assume that our Frobenius σ is defined by $\sigma(x) = x^q$ by Theorem 3.3. Let us denote by $\lambda_{\max}$ the highest Frobenius slope of M_η (= the highest Frobenius slope of $V(M)$). Let $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ be a filtration of M as φ-∇-modules over $K[\![x]\!]_0$ such that $M_i/M_{i-1}\,(i = 1, \cdots, r)$ is irreducible (i.e., it has no nontrivial φ-∇-submodule over $K[\![x]\!]_0$). We will prove the induction on r. If $r = 1$, then $M \cong (K[\![x]\!]_0, q^{\lambda_{\max}}\sigma, d)$ and the assertion is trivial.

Now suppose $r > 1$. We may also assume $\dim_K V(M)/V(M)^0 = 1$, hence $M_r/M_{r-1} \cong (K[\![x]\!]_0, q^{\lambda_{\max}}\sigma, d)$. Indeed, suppose that $s = \dim_K V(M)/V(M)^0 > 1$. By our assumption, there is a φ-∇-submodule L' over $K[\![x]\!]_0$ such that the highest Frobenius slope of L' is $\lambda_{\max}$ with multiplicity 1 (note that L' is M_{r-s+1}). Take the maximally PBQ submodule L of L'. Then L is HPBQ such that the highest Frobenius slope is $\lambda_{\max}$ with multiplicity 1. Since both highest Frobenius slopes of L and M/L are $\lambda_{\max}$, the assertion follows from the induction hypothesis by Propositions 2.6 and 6.4.

Since all Frobenius slopes of $(M_{r-1})_\eta$ are less than $\lambda_{\max}$, one can take a basis $e_1, \cdots, e_s$ of M such that the Frobenius matrix $A = \begin{pmatrix} A_1 & B \\ 0 & q^{\lambda_{\max}} \end{pmatrix}$ (A_1 is the Frobenius matrix of M_{r-1}) satisfies (i) all entries of A_1 are contained in $K[\![x]\!]_0 \cap x^q K[\![x^q]\!]$ and (ii) all entries of B are contained in $xK[\![x]\!]_0 \setminus x^q K[\![x^q]\!] \cup \{0\}$ by Proposition 4.7. Moreover $B \neq 0$ by Lemma 6.6 below since M is PBQ. Let G be the matrix of connection of M. Then the identification

$$\mathrm{Sol}(M) = \left\{ y \in \mathcal{A}_K(0, 1^-) \;\middle|\; \frac{d}{dx} y = yG \right\}$$

is given by $f \mapsto (f(e_1), \cdots, f(e_s))$. The inclusion relation in Theorem 2.3 for the solution space is

$$\mathrm{Sol}_\lambda(M) \supset S_{\lambda - \lambda_{\max}}(\mathrm{Sol}(M)).$$

Then it is sufficient to prove the inclusion is equal for all λ. The φ-module is a direct sum of 1-dimensional φ-spaces, on which φ acts by $q^\delta\sigma$ for some rational number δ such that $\lambda_{\max} - \delta$ is a Frobenius slope of M, by our assumption of K. Let $f \in \mathrm{Sol}_\lambda(M)$ with $\varphi(f) = q^\delta f$. Then the restriction of f on M_{r-1} gives a $(\varphi, \frac{d}{dx})$-equivariant morphism

$$M_{r-1} \to (\mathcal{A}_K(0,1^-), q^{-\delta}\sigma, d).$$

The kernel L of f is a φ-∇-module over $K[\![x]\!]_0$ and f is a solution of M/L of log-growth λ.

Suppose that $L \neq 0$. Then the length of M/L is smaller than M and M/L is HPBQ by Proposition 6.4. Considering f as a solution of M/L, we have $\delta \leq \lambda - \lambda_{\max}$ by the hypothesis of induction.

Suppose that $L = 0$. The Frobenius relation $\varphi(f) = q^\delta f$ is equivalent to

$$q^{-\delta}\sigma(f(e_1), \cdots, f(e_s)) = (f(e_1), \cdots, f(e_s))A.$$

By the assumption of A_1 we have $f(e_i) \in \mathcal{A}_K(0,1^-) \cap x^q K[\![x^q]\!]$. Let us focus on the s-th entry, then it is

$$q^{-\delta}\sigma(f(e_s)) = q^{\lambda_{\max}} f(e_s) + (f(e_1), \cdots, f(e_{s-1}))B.$$

Since the highest Frobenius slope of M_{r-1} is less than $\lambda_{\max}$, the log-growth of the restriction of f on M_{r-1} is of log-growth less than $\lambda_{\max} + \delta$, and so is $(f(e_1), \cdots, f(e_{s-1}))B$. Since f is injective, $(f(e_1), \cdots, f(e_{s-1}))B \in \mathcal{A}_K(0,1^-) \setminus x^q K[\![x^q]\!]$ is not 0. Hence, $f(e_s)$ is exactly of log-growth $\lambda_{\max} + \delta$ by Lemma 4.8 (1). This provides an inequality $\lambda_{\max} + \delta \leq \lambda$, and we have $\delta \leq \lambda - \lambda_{\max}$.

Therefore, $f \in S_{\lambda-\lambda_{\max}}(\mathrm{Sol}(M))$. This completes the proof of Theorem 6.5. □

Lemma 6.6 *Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-∇-modules over $K[\![x]\!]_0$. If the exact sequence is split as φ-modules, then it is split as φ-∇-modules.*

Proof. Let $A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$ and $G = \begin{pmatrix} G_1 & H \\ 0 & G_2 \end{pmatrix}$ be the matrices of Frobenius and connection, respectively. We should prove that $B = 0$ implies $H = 0$. It is sufficient to prove the assertion above as $\mathcal{A}_K(0,1^-)$-modules with Frobenius and connection. Solving the differential modules L and N, we may assume that A_1 and A_2 are constant regular matrices and $G_1 = G_2 = 0$. Then the horizontality of Frobenius structure means he relation

$$HA_2 = qx^{q-1}A_1\sigma(H).$$

Then we have $H = 0$ by comparing the x-adic order of both sides. □

6.2 Equislope φ-∇-modules over $K[\![x]\!]_0$

Definition 6.7 *A φ-∇-module M over $K[\![x]\!]_0$ is equislope if there is an increasing filtration $\{S_\lambda(M)\}_{\lambda\in\mathbb{R}}$ of φ-∇-module over $K[\![x]\!]_0$ such that $S_\lambda(M)\otimes\mathcal{E}$ gives the Frobenius slope filtration of the generic fiber M_η of M. We also call $\{S_\lambda(M)\}_{\lambda\in\mathbb{R}}$ the Frobenius slope filtration of M.*

By [Ka79, 2.6.2] (see [CT09, Theorem 6.21]) we have

Proposition 6.8 *A φ-∇-module M over $K[\![x]\!]_0$ is equislope if and only if both the special polygon and generic polygon of Frobenius slopes of M coincides with each other.*

Corollary 6.9 *Any subquotients, direct sums, extensions, tensor products, duals of equislope φ-∇-modules over $K[\![x]\!]_0$ are equislope.*

Proposition 6.10 *Let M be an equislope φ-∇-module over $K[\![x]\!]_0$.*

(1) *M is HBQ. In particular, if M is PBQ, then M is HPBQ.*

(2) *If $V(M)/V(M)^0$ is pure as a φ-module, then M is HPBQ.*

Proof. (1) We may assume that the residue field of $\mathcal{V}$ is algebraically closed and all slopes of M_η is contained in the value group $\log_q|K^\times|$ of $K^\times$ by Proposition 2.1. Let us take a φ-∇-submodule L such that its generic fiber L_η is M_η^0. Such an L exists by Lemma 6.11 below. Since $(M/L)_\eta\cong M_\eta/L_\eta$ is bounded, M is HBQ by definition.
(2) The assertion follows from (1) and Proposition 6.3. $\square$

Lemma 6.11 *Let M be an equislope φ-∇-module over $K[\![x]\!]_0$. Suppose that the residue field of $\mathcal{V}$ is algebraically closed and all slopes of M_η are contained in the valued group $\log_q|K^\times|$. The map taking generic fibers gives a bijection from the set of φ-∇-submodules of M to the set of φ-∇-submodules of M_η.*

Proof. Since the functor from the category φ-∇-module over $K[\![x]\!]_0$ to the category φ-∇-module over $\mathcal{E}$ is fully faithful, it is sufficient to prove the surjectivity [dJ98, Theorem 1.1].
We may assume that $\sigma(x)=x^q$ by Theorem 3.3. We use the induction on the number of Frobenius slopes of M in order to prove the existence of a submodule N over $K[\![x]\!]_0$ for a given submodule N_η over $\mathcal{E}$. Suppose that M is pure of slope λ. There are a basis $e_1,\cdots,e_r$ of M such that the Frobenius matrix is $q^\lambda 1_r$ since M is bounded. Let N_η be a φ-∇-submodule of M_η over $\mathcal{E}$ which is generated by $(e_1,\cdots,e_r)P$ for $P\in\mathrm{Mat}_{rs}(\mathcal{E})$ with $s=\dim_{\mathcal{C}}N_\eta$. Since N_η is a φ-submodule, there is a $B\in\mathrm{GL}_s(\mathcal{E})$ such that $q^\lambda\sigma(P)=PB$. Since $\mathrm{rank}(P)=s$, there is a regular minor Q of P of degree s such that $q^\lambda\sigma(Q)=QB$. If one puts $R=PQ^{-1}\in\mathrm{Mat}_{rs}(\mathcal{E})$, then $\sigma(R)=R$. Hence, $R\in\mathrm{Mat}_{rs}(K)$. Since $(e_1,\cdots,e_r)R$ is a basis of N_η such that $(e_1,\cdots,e_r)R$

are included in M, the submodule N is given by the $K[\![x]\!]_0$-submodule of M generated by $(e_1, \cdots, e_r)R$.
Let λ_1 be the first slopes of M_η. By the induction hypothesis there are a φ-∇-submodule N_1 of $S_{\lambda_1}(M)$ such that the generic fiber $(N_1)_\eta$ of N_1 is $N_\eta \cap S_{\lambda_1}(M_\eta)$ and a φ-∇-submodule N_2 of $M/S_{\lambda_1}(M)$ such that the generic fiber of N_2 is $N_\eta/(S_{\lambda_1}(M_\eta) \cap N_\eta) = N_\eta/(N_1)_\eta$. Let N_3 be the inverse image of N_2 by the surjection $M/N_1 \to M/S_{\lambda_1}(M)$. Since the intersection of $N_\eta/(N_1)_\eta$ and $S_{\lambda_1}(M_\eta)/(N_1)_\eta$ is 0 in $M_\eta/(N_1)_\eta$, $(N_3)_\eta$ is a direct sum of $N_\eta/(N_1)_\eta$ and $S_\lambda(M_\eta)/(N_1)_\eta$. By applying the fully faithfulness of the functor from the category of φ-∇-modules over $K[\![x]\!]_0$ to the category of φ-∇-modules over $\mathcal{E}$ [dJ98, Theorem 1.1], there is a direct summand N_4 of N_3 as φ-∇-module over $K[\![x]\!]_0$ such that the generic fiber of N_4 is $N_\eta/(N_1)_\eta$. Then the inverse image N of N_4 by the surjection $M \to M/N_1$ is our desired one. $\square$

Theorem 6.12 *The conjecture* $\mathbf{LGF}_{K[\![x]\!]_0}$ *(see 2.5) holds for any equislope and PBQ* φ-∇*-module over* $K[\![x]\!]_0$.

Proof. The assertion follows from Theorem 6.5 and Proposition 6.10 (1). $\square$

7 Log-growth filtration and Frobenius filtration at the generic point

7.1 The log-growth of PBQ φ-∇-modules over $\mathcal{E}$

Theorem 7.1 *The conjecture* $\mathbf{LGF}_{\mathcal{E}}$ *(see 2.4) holds for any PBQ* φ-∇*-module over* $\mathcal{E}$.

Proof. Let M be a PBQ φ-∇-module over $\mathcal{E}$ such that $\lambda_{\max}$ is the highest Frobenius slope of M, and let us consider a φ-∇-module $M_\tau = M \otimes_{\mathcal{E}} \mathcal{E}_t[\![X-t]\!]_0$ over the $\mathcal{E}_t$-algebra $\mathcal{E}_t[\![X-t]\!]_0$ of bounded functions on the generic disk. Then M_τ is equislope since $\{(S_\lambda(M))_\tau\}$ gives a Frobenius slope filtration of M_τ. Moreover, since M is PBQ, $\mathrm{Sol}_0(M, \mathcal{A}_{\mathcal{E}_t}(t, 1^-))$ is a pure φ-module. Hence $V(M_\tau)/V(M_\tau)^0$ is pure, and M_τ is HPBQ by Proposition 6.10 (2). Applying Theorem 6.5 to M_τ, we have

$$\begin{aligned}
\dim_{\mathcal{E}} M/M^\lambda &= \dim_{\mathcal{E}_t} \mathrm{Sol}_\lambda(M, \mathcal{A}_{\mathcal{E}_t}(t, 1^-)) = \dim_{\mathcal{E}_t} V(M_\tau)/V(M_\tau)^\lambda \\
&= \dim_{\mathcal{E}_t} V(M_\tau^\vee) - \dim_{\mathcal{E}_t} (S_{\lambda-\lambda_{\max}}(V(M_\tau^\vee)))^\perp \\
&= \dim_{\mathcal{E}} M^\vee - \dim_{\mathcal{E}} (S_{\lambda-\lambda_{\max}}(M^\vee))^\perp \\
&= \dim_{\mathcal{E}} M^\vee/(S_{\lambda-\lambda_{\max}}(M^\vee))^\perp
\end{aligned}$$

for any λ. Hence, $M^\lambda = (S_{\lambda-\lambda_{\max}}(M^\vee))^\perp$ by Theorem 2.3. Therefore, the conjecture $\mathbf{LGF}_{\mathcal{E}}$ holds for M. $\square$

7.2 Rationality of breaks of log-growth filtrations

Theorem 7.2 *Let M be a φ-∇-module over $\mathcal{E}$ and let λ be a break of log-growth filtration of M, i.e., $M^{\lambda-} \supsetneq M^{\lambda+}$. Then λ is rational and $M^{\lambda} = M^{\lambda+}$. In other words, the conjecture $\mathbf{LGF}_{\mathcal{E}}$ (1) (see 2.4) holds for any φ-∇-modules over $\mathcal{E}$.*

Proof. We may assume that the residue field k of $\mathcal{V}$ is perfect by Proposition 2.1. Suppose that $\lambda_{\max}$ be the maximal Frobenius slope of M. If M is PBQ, then $M^{\lambda} = (S_{\lambda-\lambda_{\max}}(M^{\vee}))^{\perp} = S_{(\lambda_{\max}-\lambda)-}(M)$ for any λ by Theorem 7.1. Then we have

$$M^{\lambda+} = \cup_{\mu>\lambda} S_{(\lambda_{\max}-\mu)-}(M) = \cup_{\mu>\lambda} S_{(\lambda_{\max}-\mu)}(M) = S_{(\lambda_{\max}-\lambda)-}(M) = M^{\lambda}.$$

If λ is a break of log-growth filtration, then

$$S_{\lambda_{\max}-\lambda}(M) = S_{(\lambda_{\max}-\lambda)+}(M) = M^{\lambda-} \supsetneq M^{\lambda} = S_{(\lambda_{\max}-\lambda)-}(M)$$

and λ is also a Frobenius slope filtration. Hence λ is rational.
For a general M, we use the induction on the length of the PBQ filtration of M. Let L be the maximally PBQ submodule of M and suppose $N = M/L$. Then we have the assertion by Proposition 2.6 (1), the PBQ case and the induction hypothesis on L and N. □

Proposition 7.3 *Suppose that the residue field of $\mathcal{V}$ is perfect. Let M be a φ-∇-module over $K[\![x]\!]_0$ and let λ be a break of log-growth filtration of $V(M)$, i.e., $V(M)^{\lambda-} \supsetneq V(M)^{\lambda+}$, and let $\{P_i(M)\}$ be the PBQ filtration of M. Suppose that the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (2) (see 2.5) holds for all $P_i(M)/P_{i-1}(M)$. Then λ is rational and $V(M)^{\lambda} = V(M)^{\lambda+}$. In particular, the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (2) implies the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (1) for any φ-∇-modules over $K[\![x]\!]_0$.*

Proof. The proof is similar to that of Theorem 7.2 by replacing Proposition 2.6 (1) by Proposition 2.6 (2). □

8 Toward Dwork's conjecture $\mathbf{LGF}_{\mathrm{Dw}}$

8.1 The comparison at the special point and Dwork's conjecture $\mathbf{LGF}_{\mathrm{Dw}}$

Theorem 8.1 *The conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (2) (see 2.5) implies the conjecture $\mathbf{LGF}_{\mathrm{Dw}}$ (see 2.7), that is, the special log-growth polygon lies above the generic log-growth polygon (and they have the same endpoints).*

The theorem above follows from the proposition below by Proposition 2.1.

Proposition 8.2 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let M be a φ-∇-module over $K[\![x]\!]_0$ and let $\{P_i(M)\}$ be the PBQ filtration of M. Suppose that the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (2) (see 2.5) holds for all $P_i(M)/P_{i-1}(M)$. Then the special log-growth polygon of M lies above the generic log-growth polygon of M (and they have the same endpoints).*

Proof. For the PBQ φ-∇-modules arising from the PBQ filtration of M, the log-growth polygons at the generic (resp. special) fiber coincides with the Newton polygon of Frobenius slopes of the dual at the generic (resp. special) fiber under the suitable shifts of Frobenius actions by Theorem 7.1 (resp. our hypothesis). The assertion follows from Proposition 2.6, Lemma 8.3 below and the fact that the special Newton polygon of Frobenius slopes is above the generic Newton polygon of Frobenius slopes and they have the same endpoints. $\square$

Lemma 8.3 *Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-∇-modules over $K[\![x]\!]_0$ such that the induced sequences*

$$\begin{array}{ccccccccc} 0 & \to & L_\eta/L_\eta^\lambda & \to & M_\eta/M_\eta^\lambda & \to & N_\eta/N_\eta^\lambda & \to & 0 \\ 0 & \to & V(L)/V(L)^\lambda & \to & V(M)/V(M)^\lambda & \to & V(N)/V(N)^\lambda & \to & 0 \end{array}$$

on both the generic fiber and the special fiber are exact for any λ.

(1) *If the special log-growth polygon lies above the generic log-growth polygon (the endpoints might be different) for both L and N, then the same holds for M.*

(2) *If the special log-growth polygon and the generic log-growth polygon have the same endpoints for both L and N, then the same holds for M.*

(3) *Suppose that the special log-growth polygon lies above the generic log-growth polygon for both L and N. Then both the special and the generic log-growth polygons coincide with each other for M if and only if the same hold for L and N.*

Proof. Let r be the rank of M. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$ be breaks of log-growth filtration of M_η with multiplicities, and put $b_0(M_\eta) = 0$ and

$$b_j(M_\eta) = \lambda_1 + \cdots + \lambda_j$$

for $1 \leq j \leq r$. Then the generic log-growth polygon of M is a polygon which connects points $(0, b_0(M_\eta)), (1, b_1(M_\eta)), \cdots, (r, b_r(M_\eta))$ by lines. We also define $b_j(V(M))$ for the special log-growth of M. Then the exactness for any λ implies the equality

$$b_j(M_\eta) = \min \left\{ b_i(L_\eta) + b_k(N_\eta) \;\middle|\; \begin{array}{c} 0 \leq i \leq \operatorname{rank} L,\ 0 \leq k \leq \operatorname{rank} N, \\ i + k = j \end{array} \right\}$$

for all $0 \leq j \leq r$, and the same holds for the special log-growth. The special log-growth polygon lies above the generic log-growth polygon for M if and only if $b_j(M_\eta) \leq b_j(V(M))$ for all j, the special log-growth polygon and the generic log-growth polygon have the same endpoints for M if and only if $b_r(M_\eta) = b_r(V(M))$, and both the special and the generic log-growth polygons coincide with each other for M if and only if $b_j(M_\eta) = b_j(V(M))$ for all j. Hence we have the assertions. □

Remark 8.4 *If L is supposed to be HPBQ in the short exact sequence of the previous lemma, then the induced sequences are automatically exact for all λ: in fact one has Theorems 7.1 and 6.5 and can apply Proposition 2.6.*

Remark 8.5 *If one assumes that the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (2) (see 2.5) for any PBQ φ-∇-module over $K[\![x]\!]_0$ of rank $\leq r$, then the proofs of Proposition 7.3 and Theorem 8.1 works for any φ-∇-module over $K[\![x]\!]_0$ of rank $\leq r$.*

8.2 Dwork's conjecture in the HBQ cases

Lemma 8.6 *Let M be a HBQ φ-∇-module over $K[\![x]\!]_0$ and let N be a φ-∇-submodule of M over $K[\![x]\!]_0$ which is PBQ. Then N is HPBQ. In particular, suppose that the residue field of $\mathcal{V}$ is perfect and let $\{P_i(M)\}$ be the PBQ filtration of M, then $P_i(M)/P_{i-1}(M)$ is HPBQ for all i.*

Proof. We have $\dim_K V(M)/V(M)^0 = \dim_{\mathcal{E}} M_\eta/M_\eta^0$ and $\dim_K V(M/N)/V(M/N)^0 = \dim_{\mathcal{E}} (M/N)_\eta/(M/N)_\eta^0$ by Proposition 6.3 since the quotient M/N is HBQ by Proposition 6.4. Comparing the induced exact sequence $0 \to N_\eta/N_\eta^0 \to M_\eta/M_\eta^0 \to (M/N)_\eta/(M/N)_\eta^0 \to 0$ at the generic point by Theorem 7.1 and Proposition 2.6 (1) to the corresponding right exact sequence at the special point, we have an inequality $\dim_K V(N)/V(N)^0 \geq \dim_{\mathcal{E}} N_\eta/N_\eta^0$. On the contrary, we know the inequality $\dim_K V(N)/V(N)^0 \leq \dim_{\mathcal{E}} N_\eta/N_\eta^0$ by [CT09, Proposition 4.10]. Hence, $\dim_K V(N)/V(N)^0 = \dim_{\mathcal{E}} N_\eta/N_\eta^0$ and N is HPBQ.
The rest follows from the first part and Proposition 6.4. □

Theorem 8.7 *Let M be a HBQ φ-∇-module over $K[\![x]\!]_0$. Then the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (1) (see 2.5) and the conjecture $\mathbf{LGF}_{\mathrm{Dw}}$ (see 2.7) hold for M.*

Proof. The assertions follows from the similar arguments of Theorems 7.2 and 8.1, respectively, by using Theorem 6.5 and Lemma 8.6. □

8.3 When do the generic and special log-growth polygons coincide?

Theorem 8.8 *Let M be a φ-∇-module over $K[\![x]\!]_0$. The special log-growth polygon and the generic log-growth polygon coincide with each other if and only if M is equislope.*

Proof. We may assume that the residue field of $\mathcal{V}$ is algebraically closed by Proposition 2.1. Let $\{P_i(M)\}$ be the PBQ filtration of M (Theorem 5.6). Each condition (i) the coincidence of special and generic log-growth polygons or (ii) equislope implies that $P_i(M)/P_{i-1}(M)$ is HPBQ and $M/P_i(M)$ is HBQ for all i by Propositions 6.3, 6.4, and Lemma 8.6 for (i) and by Corollary 6.9 and Proposition 6.10 (1) for (ii). Then we can apply Lemma 8.3 (3) inductively on i by Remark 8.4 and Theorem 8.7. Hence it is sufficient to prove the assertion when M is HPBQ by Corollary 6.9. Then the coincidence of the log-growth filtration and the Frobenius slope filtration both at the special point (Theorem 8.7) and at the generic point (Theorem 7.1) implies our desired equivalence. □

Example 8.9 (1) *Let M be a φ-∇-module over $K[\![x]\!]_0$ such that M_η is bounded. Then there is a φ-module L over K such that $M \cong L \otimes_K K[\![x]\!]_0$ by Christol's transfer theorem (see [CT09, Proposition 4.3]). Hence, M is equislope.*

(2) *Let M be a φ-∇-module over $K[\![x]\!]_0$ of rank 2 such that M_η is not bounded. Then we have identities $M^\lambda = (S_{\lambda-\lambda_{\max}}(M^\vee))^\perp$ and $V(M)^\lambda = (S_{\lambda-\lambda_{\max}}(V(M^\vee)))^\perp$ for any λ [CT09, Theorem 7.1], where $\lambda_{\max}$ is the highest Frobenius slope of M_η. Hence the special log-growth polygon and the generic log-growth polygon coincide with each other if and only if M is equislope.*

References

[An08] André, Y. Dwork's conjecture on the logarithmic growth of solutions of p-adic differential equations, Compos. Math. 144 (2008), no. 2, 484–494.

[CT09] Chiarellotto, B., Tsuzuki, N., Logarithmic growth and Frobenius filtrations for solutions of p-adic differential equations, J. Math. Inst. Jussieu 8 (2009), 465–505.

[Ch83] Christol, G. *Modules différentiels et équations différentielles p-adiques*, Queen's papers in Pure and Applied Math. , 66, Queen's Univ., Kingston ON, 1983.

[Dw73] Dwork, B.M. On p-adic differential equations III, Invent. Math. 20 (1973), 35–45.

[Dw82] Dwork, B.M. *Lectures on p-adic differential equations*, Grundlehren der mathematischen Wissenschaften, Volume 253 (Springer, 1982).

[dJ98] de Jong, A.J., Homomorphisms of Barsotti-Tate groups and crystals in positive characteristic, Invent. Math. 134 (1998), no. 2, 301–333.

[Ka73] Katz, N.M., Travaux de Dwork, Séminaire Bourbaki (1971/1972), Exp. No. 409, 167–200, Lecture Notes in Math., 317 (1973), Springer, Berlin.

[Ka79] Katz, N.M., Slope filtrations of F-isocrystals, Astérisque 63 (1979), 113–164.

[Ke01] Kedlaya, K.S., The algebraic closure of the power series field in positive characteristic, Proc. Amer. Math. Soc. 129 No. 12 (2002), 3461–3470.

[Ke08] Kedlaya, K.S., Slope filtration for relative Frobenius, Astérisque 319 (2008), 259–301.

[Ke09] Kedlaya, K.S., *p-adic Differential Equations*, book draft (version of 19 Jan 09).

[Ro75] Robba, Ph. On the Index of p-adic differential operators I, Ann. of Math. 101 (1975), 280–316.

[Ts96] Tsuzuki, N., The overconvergence of morphisms of étale ϕ-∇-spaces on a local field, Compositio Math. 103 no. 2 (1996), 227–239.

[Ts98a] Tsuzuki, N. Slope filtration of quasi-unipotent overconvergent F-isocrystals, Annales de l'institut Fourier, Grenoble 48 (1998), 379–412.

[Ts98b] Tsuzuki, N. Finite local monodromy of overconvergent unit-root F-isocrystals on a curve, Amer. J. Math. 120 no. 6 (1998), 1165–1190.

Bruno Chiarellotto
Dipartimento Matematica
Università di Padova
Via Trieste 63
35121 Padova
Italy
chiarbru@math.unipd.it

Nobuo Tsuzuki
Mathematical Institute
Tohoku University
Aramaki Aza-Aoba 6-3
Aobaku Sendai 980-8578
Japan
tsuzuki@math.tohoku.ac.jp

Multigraded Factorial Rings and Fano Varieties with Torus Action

Jürgen Hausen, Elaine Herppich, Hendrik Süss

Received: November 15, 2009
Revised: December 12, 2010

Communicated by Thomas Peternell

Abstract. In a first result, we describe all finitely generated factorial algebras over an algebraically closed field of characteristic zero that come with an effective multigrading of complexity one by means of generators and relations. This enables us to construct systematically varieties with free divisor class group and a complexity one torus action via their Cox rings. For the Fano varieties of this type that have a free divisor class group of rank one, we provide explicit bounds for the number of possible deformation types depending on the dimension and the index of the Picard group in the divisor class group. As a consequence, one can produce classification lists for fixed dimension and Picard index. We carry this out expemplarily in the following cases. There are 15 non-toric surfaces with Picard index at most six. Moreover, there are 116 non-toric threefolds with Picard index at most two; nine of them are locally factorial, i.e. of Picard index one, and among these one is smooth, six have canonical singularities and two have non-canonical singularities. Finally, there are 67 non-toric locally factorial fourfolds and two one-dimensional families of non-toric locally factorial fourfolds. In all cases, we list the Cox rings explicitly.

2010 Mathematics Subject Classification: 13A02, 13F15, 14J45

Keywords and Phrases: Multigraded factorial ring, Fano variety, torus action

Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. A first aim of this paper is to determine all finitely generated factorial $\mathbb{K}$-algebras R with an effective complexity one multigrading $R = \oplus_{u \in M} R_u$ satisfying $R_0 = \mathbb{K}$; here effective complexity one multigrading means that with $d := \dim R$ we have $M \cong \mathbb{Z}^{d-1}$ and the $u \in M$ with $R_u \neq 0$ generate M as a $\mathbb{Z}$-module. Our result extends work by Mori [23] and Ishida [17], who settled the cases $d = 2$ and $d = 3$.

An obvious class of multigraded factorial algebras as above is given by polynomial rings. A much larger class is obtained as follows. Take a sequence $A = (a_0, \ldots, a_r)$ of vectors $a_i \in \mathbb{K}^2$ such that (a_i, a_k) is linearly independent whenever $k \neq i$, a sequence $\mathfrak{n} = (n_0, \ldots, n_r)$ of positive integers and a family $L = (l_{ij})$ of positive integers, where $0 \le i \le r$ and $1 \le j \le n_i$. For every $0 \le i \le r$, we define a monomial

$$f_i \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ \in \ \mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i],$$

for any two indices $0 \le i, j \le r$, we set $\alpha_{ij} := \det(a_i, a_j)$, and for any three indices $0 \le i < j < k \le r$, we define a trinomial

$$g_{i,j,k} \ := \ \alpha_{jk} f_i \ + \ \alpha_{ki} f_j \ + \ \alpha_{ij} f_k \ \in \ \mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i].$$

Note that the coefficients of $g_{i,j,k}$ are all nonzero. The triple $(A, \mathfrak{n}, L)$ then defines a $\mathbb{K}$-algebra

$$R(A, \mathfrak{n}, L) \ := \ \mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i] \ / \ \langle g_{i,i+1,i+2};\ 0 \le i \le r-2 \rangle.$$

It turns out that $R(A, \mathfrak{n}, L)$ is a normal complete intersection, see Proposition 1.2. In particular, it is of dimension

$$\dim R(A, \mathfrak{n}, L) \quad = \quad n_0 + \ldots + n_r \ - \ r \ + \ 1.$$

If the triple $(A, \mathfrak{n}, L)$ is *admissible*, i.e., the numbers $\gcd(l_{i1}, \ldots, l_{in_i})$, where $0 \le i \le r$, are pairwise coprime, then $R(A, \mathfrak{n}, L)$ admits a canonical effective complexity one grading by a lattice K, see Construction 1.7. Our first result is the following.

Theorem 1.9. *Up to isomorphy, the finitely generated factorial $\mathbb{K}$-algebras with an effective complexity one grading $R = \oplus_M R_u$ and $R_0 = \mathbb{K}$ are*

(i) *the polynomial algebras $\mathbb{K}[T_1, \ldots, T_d]$ with a grading $\deg(T_i) = u_i \in \mathbb{Z}^{d-1}$ such that $u_1, \ldots, u_d$ generate $\mathbb{Z}^{d-1}$ as a lattice and the convex cone on $\mathbb{Q}^{d-1}$ generated by $u_1, \ldots, u_d$ is pointed,*

(ii) *the $(K \times \mathbb{Z}^m)$-graded algebras $R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$, where $R(A, \mathfrak{n}, L)$ is the K-graded algebra defined by an admissible triple $(A, \mathfrak{n}, L)$ and $\deg S_j \in \mathbb{Z}^m$ is the j-th canonical base vector.*

The further paper is devoted to normal (possibly singular) d-dimensional Fano varieties X with an effective action of an algebraic torus T. In the case $\dim T = d$, we have the meanwhile extensively studied class of toric Fano varieties, see [3], [27] and [4] for the initiating work. Our aim is to show that the above Theorem provides an approach to classification results for the case $\dim T = d-1$, that means Fano varieties with a complexity one torus action. Here, we treat the case of divisor class group $\mathrm{Cl}(X) \cong \mathbb{Z}$; note that in the toric setting this gives precisely the weighted projective spaces. The idea is to consider the Cox ring

$$\mathcal{R}(X) \quad = \bigoplus_{D \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

The ring $\mathcal{R}(X)$ is factorial, finitely generated as a $\mathbb{K}$-algebra and the T-action on X gives rise to an effective complexity one multigrading of $\mathcal{R}(X)$ refining the $\mathrm{Cl}(X)$-grading, see [5] and [15]. Consequently, $\mathcal{R}(X)$ is one of the rings listed in the first Theorem. Moreover, X can be easily reconstructed from $\mathcal{R}(X)$; it is the homogeneous spectrum with respect to the $\mathrm{Cl}(X)$-grading of $\mathcal{R}(X)$. Thus, in order to construct Fano varieties, we firstly have to figure out the Cox rings among the rings occuring in the first Theorem and then find those, which belong to a Fano variety; this is done in Propositions 1.11 and 2.5.

In order to produce classification results via this approach, we need explicit bounds on the number of deformation types of Fano varieties with prescribed discrete invariants. Besides the dimension, in our setting, a suitable invariant is the *Picard index* $[\mathrm{Cl}(X) : \mathrm{Pic}(X)]$. Denoting by $\xi(\mu)$ the number of primes less or equal to μ, we obtain the following bound, see Corollary 2.2: for any pair $(d, \mu) \in \mathbb{Z}^2_{>0}$, the number $\delta(d, \mu)$ of different deformation types of d-dimensional Fano varieties with a complexity one torus action such that $\mathrm{Cl}(X) \cong \mathbb{Z}$ and $\mu = [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ hold is bounded by

$$\delta(d, \mu) \quad \leq \quad (6d\mu)^{2\xi(3d\mu)+d-2} \mu^{\xi(\mu)^2+2\xi((d+2)\mu)+2d+2}.$$

In particular, we conclude that for fixed $\mu \in \mathbb{Z}_{>0}$, the number $\delta(d)$ of different deformation types of d-dimensional Fano varieties with a complexity one torus action $\mathrm{Cl}(X) \cong \mathbb{Z}$ and Picard index μ is asymptotically bounded by d^{Ad} with a constant A depending only on μ, see Corollary 2.4.

In fact, in Theorem 2.1 we even obtain explicit bounds for the discrete input data of the rings $R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$. This allows us to construct all Fano varieties X with prescribed dimension and Picard index that come with an effective complexity one torus action and have divisor class group $\mathbb{Z}$. Note that, by the approach, we get the Cox rings of the resulting Fano varieties X for free. In Section 3, we give some explicit classifications. We list all non-toric surfaces X with Picard index at most six and the non-toric threefolds X with Picard index up at most two. They all have a Cox ring defined by a single relation; in fact, for surfaces the first Cox ring with more than one relation occurs for Picard index 29, and for the threefolds this happens with Picard index 3, see Proposition 3.5 as well as Examples 3.4 and 3.7. Moreover, we determine all locally factorial fourfolds X, i.e. those of Picard index one: 67 of them occur sporadic and there are two one-dimensional families. Here comes the result on the locally factorial threefolds; in the table, we denote by w_i the $\mathrm{Cl}(X)$-degree of the variable T_i.

Theorem 3.2. *The following table lists the Cox rings $\mathcal{R}(X)$ of the three-dimensional locally factorial non-toric Fano varieties X with an effective two torus action and* $\mathrm{Cl}(X) = \mathbb{Z}$.

No.	$\mathcal{R}(X)$	$(w_1, \dots, w_5)$	$(-K_X)^3$
1	$\mathbb{K}[T_1, \dots, T_5] \ / \ \langle T_1 T_2^5 + T_3^3 + T_4^2 \rangle$	$(1,1,2,3,1)$	8
2	$\mathbb{K}[T_1, \dots, T_5] \ / \ \langle T_1 T_2 T_3^4 + T_4^3 + T_5^2 \rangle$	$(1,1,1,2,3)$	8
3	$\mathbb{K}[T_1, \dots, T_5] \ / \ \langle T_1 T_2^2 T_3^3 + T_4^3 + T_5^2 \rangle$	$(1,1,1,2,3)$	8
4	$\mathbb{K}[T_1, \dots, T_5] \ / \ \langle T_1 T_2 + T_3 T_4 + T_5^2 \rangle$	$(1,1,1,1,1)$	54
5	$\mathbb{K}[T_1, \dots, T_5] \ / \ \langle T_1 T_2^2 + T_3 T_4^2 + T_5^3 \rangle$	$(1,1,1,1,1)$	24
6	$\mathbb{K}[T_1, \dots, T_5] \ / \ \langle T_1 T_2^3 + T_3 T_4^3 + T_5^4 \rangle$	$(1,1,1,1,1)$	4
7	$\mathbb{K}[T_1, \dots, T_5] \ / \ \langle T_1 T_2^3 + T_3 T_4^3 + T_5^2 \rangle$	$(1,1,1,1,2)$	16
8	$\mathbb{K}[T_1, \dots, T_5] \ / \ \langle T_1 T_2^5 + T_3 T_4^5 + T_5^2 \rangle$	$(1,1,1,1,3)$	2
9	$\mathbb{K}[T_1, \dots, T_5] \ / \ \langle T_1 T_2^5 + T_3^3 T_4^3 + T_5^2 \rangle$	$(1,1,1,1,3)$	2

Note that each of these varieties X is a hypersurface in the respective weighted projective space $\mathbb{P}(w_1, \dots, w_5)$. Except number 4, none of them is quasismooth in the sense that $\operatorname{Spec} \mathcal{R}(X)$ is singular at most in the origin; quasismooth hypersurfaces of weighted projective spaces were studied in [21] and [7]. In Section 4, we take a closer look at the singularities of the threefolds listed above. It turns out that number 1,3,5,7 and 9 are singular with only canonical singularities and all of them admit a crepant resolution. Number 6 and 8 are singular with non-canonical singularities but admit a smooth relative minimal model. Number two is singular with only canonical singularities, one of them of type $\mathbf{cA_1}$, and it admits only a singular relative minimal model. Moreover, in all cases, we determine the Cox rings of the resolutions.

The authors would like to thank Ivan Arzhantsev for helpful comments and discussions and also the referee for valuable remarks and many references.

1 UFDs with complexity one multigrading

As mentioned before, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. In Theorem 1.9, we describe all factorial finitely generated $\mathbb{K}$-algebras R with an effective complexity one grading and $R_0 = \mathbb{K}$. Moreover, we characterize the possible Cox rings among these algebras, see Proposition 1.11. First we recall the construction sketched in the introduction.

Construction 1.1. Consider a sequence $A = (a_0, \ldots, a_r)$ of vectors $a_i = (b_i, c_i)$ in $\mathbb{K}^2$ such that any pair (a_i, a_k) with $k \neq i$ is linearly independent, a sequence $\mathfrak{n} = (n_0, \ldots, n_r)$ of positive integers and a family $L = (l_{ij})$ of positive integers, where $0 \le i \le r$ and $1 \le j \le n_i$. For every $0 \le i \le r$, define a monomial

$$f_i \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ \in \ \mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i],$$

for any two indices $0 \le i, j \le r$, set $\alpha_{ij} := \det(a_i, a_j) = b_i c_j - b_j c_i$ and for any three indices $0 \le i < j < k \le r$ define a trinomial

$$g_{i,j,k} \ := \ \alpha_{jk} f_i \ + \ \alpha_{ki} f_j \ + \ \alpha_{ij} f_k \ \in \ \mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i].$$

Note that the coefficients of this trinomial are all nonzero. The triple $(A, \mathfrak{n}, L)$ then defines a ring

$$R(A, \mathfrak{n}, L) \ := \ \mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i] \ / \ \langle g_{i,i+1,i+2};\ 0 \le i \le r-2 \rangle.$$

Proposition 1.2. *For every triple $(A, \mathfrak{n}, L)$ as in 1.1, the ring $R(A, \mathfrak{n}, L)$ is a normal complete intersection of dimension*

$$\dim R(A, \mathfrak{n}, L) \ = \ n - r + 1, \qquad\qquad n \ := \ n_0 + \ldots + n_r.$$

Lemma 1.3. *In the setting of 1.1, one has for any $0 \le i < j < k < l \le r$ the identities*

$$g_{i,k,l} \ = \ \alpha_{kl} \cdot g_{i,j,k} + \alpha_{ik} \cdot g_{j,k,l}, \qquad\qquad g_{i,j,l} \ = \ \alpha_{jl} \cdot g_{i,j,k} + \alpha_{ij} \cdot g_{j,k,l}.$$

In particular, every trinomial $g_{i,j,k}$, where $0 \le i < j < k \le r$ is contained in the ideal $\langle g_{i,i+1,i+2};\ 0 \le i \le r-2 \rangle$.

Proof. The identities are easily obtained by direct computation; note that for this one may assume $a_j = (1,0)$ and $a_k = (0,1)$. The supplement then follows by repeated application of the identities. □

Lemma 1.4. *In the notation of 1.1 and 1.2, set $X := V(\mathbb{K}^n, g_0, \ldots, g_{r-2})$, and let $z \in X$. If we have $f_i(z) = f_j(z) = 0$ for two $0 \le i < j \le r$, then $f_k(z) = 0$ holds for all $0 \le k \le r$.*

Proof. If $i < k < j$ holds, then, according to Lemma 1.3, we have $g_{i,k,j}(z) = 0$, which implies $f_k(z) = 0$. The cases $k < i$ and $j < k$ are obtained similarly. □

Proof of Proposition 1.2. Set $X := V(\mathbb{K}^n; g_0, \ldots, g_{r-2})$, where $g_i := g_{i,i+1,i+2}$. Then we have to show that X is a connected complete intersection with at most normal singularities. In order to see that X is connected, set $\ell := \prod n_i \prod l_{ij}$ and $\zeta_{ij} := \ell n_i^{-1} l_{ij}^{-1}$. Then $X \subseteq \mathbb{K}^n$ is invariant under the $\mathbb{K}^*$-action given by

$$t \cdot z \quad := \quad (t^{\zeta_{ij}} z_{ij})$$

and the point $0 \in \mathbb{K}^n$ lies in the closure of any orbit $\mathbb{K}^* \cdot x \subseteq X$, which implies connectedness. To proceed, consider the Jacobian J_g of $g := (g_0, \ldots, g_{r-2})$. According to Serre's criterion, we have to show that the set of points of $z \in X$ with $J_g(z)$ not of full rank is of codimension at least two in X. Note that the Jacobian J_g is of the shape

$$J_g \quad = \quad \begin{pmatrix} \delta_{0\,0} & \delta_{0\,1} & \delta_{0\,2} & 0 & & & & & & 0 \\ 0 & \delta_{1\,1} & \delta_{1\,2} & \delta_{1\,3} & 0 & & & & & \\ & & & & & \vdots & & & & \\ & & & & & & & & & \\ & & & & & 0 & \delta_{r-3\,r-3} & \delta_{r-3\,r-2} & \delta_{r-3\,r-1} & 0 \\ 0 & & & & & & 0 & \delta_{r-2\,r-2} & \delta_{r-2\,r-1} & \delta_{r-2\,r} \end{pmatrix}$$

where δ_{ti} is a nonzero multiple of the gradient $\delta_i := \operatorname{grad} f_i$. Consider $z \in X$ with $J_g(z)$ not of full rank. Then $\delta_i(z) = 0 = \delta_k(z)$ holds with some $0 \le i < k \le r$. This implies $z_{ij} = 0 = z_{kl}$ for some $1 \le j \le n_i$ and $1 \le l \le n_k$. Thus, we have $f_i(z) = 0 = f_k(z)$. Lemma 1.4 gives $f_s(z) = 0$, for all $0 \le s \le r$. Thus, some coordinate z_{st} must vanish for every $0 \le s \le r$. This shows that z belongs to a closed subset of X having codimension at least two in X. $\square$

Lemma 1.5. *Notation as in 1.1. Then the variable T_{ij} defines a prime ideal in $R(A, \mathfrak{n}, L)$ if and only if the numbers $\gcd(l_{k1}, \ldots, l_{kn_k})$, where $k \neq i$, are pairwise coprime.*

Proof. We treat exemplarily T_{01}. Using Lemma 1.3, we see that the ideal of relations of $R(A, \mathfrak{n}, L)$ can be presented as follows

$$\langle g_{s,s+1,s+2};\ 0 \le s \le r-2 \rangle \quad = \quad \langle g_{0,s,s+1};\ 1 \le s \le r-1 \rangle.$$

Thus, the ideal $\langle T_{01} \rangle \subseteq R(A, \mathfrak{n}, L)$ is prime if and only if the following binomial ideal is prime

$$\mathfrak{a} \ := \ \langle \alpha_{s+1\,0} f_s + \alpha_{0s} f_{s+1};\ 1 \le s \le r-1 \rangle \ \subseteq \ \mathbb{K}[T_{ij};\ (i,j) \neq (0,1)].$$

Set $l_i := (l_{i1}, \ldots, l_{in_i})$. Then the ideal $\mathfrak{a}$ is prime if and only if the following family can be complemented to a lattice basis

$$(l_1, -l_2, 0, \ldots, 0),\ \ldots,\ (0, \ldots, 0, l_{r-1}, -l_r).$$

This in turn is equivalent to the statement that the numbers $\gcd(l_{k1}, \ldots, l_{kn_k})$, where $1 \le k \le r$, are pairwise coprime. $\square$

DEFINITION 1.6. We say that a triple $(A, \mathfrak{n}, L)$ as in 1.1 is *admissible* if the numbers $\gcd(l_{i1}, \ldots, l_{in_i})$, where $0 \le i \le r$, are pairwise coprime.

CONSTRUCTION 1.7. Let $(A, \mathfrak{n}, L)$ be an admissible triple and consider the following free abelian groups

$$E \quad := \quad \bigoplus_{i=0}^{r} \bigoplus_{j=1}^{n_i} \mathbb{Z}\cdot e_{ij}, \qquad\qquad K \quad := \quad \bigoplus_{j=1}^{n_0} \mathbb{Z}\cdot u_{0j} \,\oplus\, \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n_i - 1} \mathbb{Z}\cdot u_{ij}$$

and define vectors $u_{in_i} := u_{01} + \ldots + u_{0r} - u_{i1} - \ldots - u_{in_i-1} \in K$. Then there is an epimorphism $\lambda\colon E \to K$ fitting into a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, "e_{ij}\mapsto l_{ij}e_{ij}", "\alpha"'] \arrow[d, "\eta", "e_{ij}\mapsto u_{ij}"'] & E \arrow[r, "e_{ij}\mapsto \overline{e}_{ij}"] \arrow[d, "\lambda"] & \bigoplus_{i,j} \mathbb{Z}/l_{ij}\mathbb{Z} \arrow[r] \arrow[d, leftrightarrow, "\cong"] & 0 \\
0 \arrow[r] & K \arrow[r, "\beta"'] & K \arrow[r] & \bigoplus_{i,j} \mathbb{Z}/l_{ij}\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Define a K-grading of $\mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i]$ by setting $\deg T_{ij} := \lambda(e_{ij})$. Then every $f_i = T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}$ is K-homogeneous of degree

$$\deg f_i \;=\; l_{i1}\lambda(e_{i1}) + \ldots + l_{in_i}\lambda(e_{in_i}) \;=\; l_{01}\lambda(e_{01}) + \ldots + l_{0n_0}\lambda(e_{0n_0}) \;\in\; K.$$

Thus, the polynomials $g_{i,j,k}$ of 1.1 are all K-homogeneous of the same degree and we obtain an effective K-grading of complexity one of $R(A, \mathfrak{n}, L)$.

Proof. Only for the existence of the commutative diagram there is something to show. Write for short $l_i := (l_{i1}, \ldots, l_{in_i})$. By the admissibility condition, the vectors $v_i := (0, \ldots, 0, l_i, -l_{i+1}, 0, \ldots, 0)$, where $0 \le i \le r-1$, can be completed to a lattice basis for E. Consequently, we find an epimorphism $\lambda\colon E \to K$ having precisely $\operatorname{lin}(v_0, \ldots, v_{r-1})$ as its kernel. By construction, $\ker(\lambda)$ equals $\alpha(\ker(\eta))$. Using this, we obtain the induced morphism $\beta\colon K \to K$ and the desired properties. $\square$

LEMMA 1.8. *Notation as in 1.7. Then* $R(A, \mathfrak{n}, L)_0 = \mathbb{K}$ *and* $R(A, \mathfrak{n}, L)^* = \mathbb{K}^*$ *hold. Moreover, the* T_{ij} *define pairwise nonassociated prime elements in* $R(A, \mathfrak{n}, L)$.

Proof. The fact that all elements of degree zero are constant is due to the fact that all degrees $\deg T_{ij} = u_{ij} \in K$ are non-zero and generate a pointed convex cone in $K_{\mathbb{Q}}$. As a consequence, we obtain that all units in $R(A, \mathfrak{n}, L)$ are constant. The T_{ij} are prime by the admissibility condition and Lemma 1.5, and they are pairwise nonassociated because they have pairwise different degrees and all units are constant. $\square$

Theorem 1.9. *Up to isomorphy, the finitely generated factorial $\mathbb{K}$-algebras with an effective complexity one grading $R = \oplus_M R_u$ and $R_0 = \mathbb{K}$ are*

(i) *the polynomial algebras $\mathbb{K}[T_1, \ldots, T_d]$ with a grading $\deg(T_i) = u_i \in \mathbb{Z}^{d-1}$ such that $u_1, \ldots, u_d$ generate $\mathbb{Z}^{d-1}$ as a lattice and the convex cone on $\mathbb{Q}^{d-1}$ generated by $u_1, \ldots, u_d$ is pointed,*

(ii) *the $(K \times \mathbb{Z}^m)$-graded algebras $R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$, where $R(A, \mathfrak{n}, L)$ is the K-graded algebra defined by an admissible triple $(A, \mathfrak{n}, L)$ as in 1.1 and 1.7 and $\deg S_j \in \mathbb{Z}^m$ is the j-th canonical base vector.*

Proof. We first show that for any admissible triple $(A, \mathfrak{n}, L)$ the ring $R(A, \mathfrak{n}, L)$ is a unique factorization domain. If $l_{ij} = 1$ holds for any two i, j, then, by [15, Prop. 2.4], the ring $R(A, \mathfrak{n}, L)$ is the Cox ring of a space $\mathbb{P}_1(A, \mathfrak{n})$ and hence is a unique factorization domain.

Now, let $(A, \mathfrak{n}, L)$ be arbitrary admissible data and let $\lambda\colon E \to K$ be an epimorphism as in 1.7. Set $n := n_0 + \ldots + n_r$ and consider the diagonalizable groups

$$\mathbb{T}^n \ := \ \operatorname{Spec} \mathbb{K}[E], \qquad H \ := \ \operatorname{Spec} \mathbb{K}[K], \qquad H_0 \ := \ \operatorname{Spec} \mathbb{K}[\oplus_{i,j} \mathbb{Z}/l_{ij}\mathbb{Z}].$$

Then $\mathbb{T}^n = (\mathbb{K}^*)^n$ is the standard n-torus and H_0 is the direct product of the cyclic subgroups $H_{ij} := \operatorname{Spec} \mathbb{K}[\mathbb{Z}/l_{ij}\mathbb{Z}]$. Moreover, the diagram in 1.7 gives rise to a commutative diagram with exact rows

$$\begin{tikzcd}
0 & \mathbb{T}^n \arrow[l] & \mathbb{T}^n \arrow[l, "(t_{ij}^{l_{ij}}) \mapsfrom (t_{ij})"'] & H_0 \arrow[l] & 0 \arrow[l] \\
0 & H \arrow[l] \arrow[u, "\imath"] & H \arrow[l] \arrow[u, "\jmath"] & H_0 \arrow[l] \arrow[u, leftrightarrow, "\cong"'] & 0 \arrow[l]
\end{tikzcd}$$

where $t_{ij} = \chi^{e_{ij}}$ are the coordinates of $\mathbb{T}^n$ corresponding to the characters $e_{ij} \in E$ and the maps $\imath$, $\jmath$ are the closed embeddings corresponding to the epimorphisms η, λ respectively.

Setting $\deg T_{ij} := e_{ij}$ defines an action of $\mathbb{T}^n$ on $\mathbb{K}^n = \operatorname{Spec} \mathbb{K}[T_{ij}]$; in terms of the coordinates z_{ij} corresponding to T_{ij} this action is given by $t \cdot z = (t_{ij} z_{ij})$. The torus H acts effectively on $\mathbb{K}^n$ via the embedding $\jmath\colon H \to \mathbb{T}^n$. The generic isotropy group of H along $V(\mathbb{K}^n, T_{ij})$ is the subgroup $H_{ij} \subseteq H$ corresponding to $K \to K/\lambda(E_{ij})$, where $E_{ij} \subseteq E$ denotes the sublattice generated by all e_{kl} with $(k, l) \neq (i, j)$; recall that we have $K/\lambda(E_{ij}) \cong \mathbb{Z}/l_{ij}\mathbb{Z}$.

Now, set $l'_{ij} := 1$ for any two i, j and consider the spectra $X := \operatorname{Spec} R(A, \mathfrak{n}, L)$ and $X' := \operatorname{Spec} R(A, \mathfrak{n}, L')$. Then the canonical surjections $\mathbb{K}[T_{ij}] \to R(A, \mathfrak{n}, L)$ and $\mathbb{K}[T_{ij}] \to R(A, \mathfrak{n}, L')$ define embeddings $X \to \mathbb{K}^n$ and $X' \to \mathbb{K}^n$. These embeddings fit into the following commutative diagram

$$\begin{tikzcd} \mathbb{K}^n & \mathbb{K}^n \arrow[l, "(z_{ij}^{l_{ij}}) \leftarrow\!\shortmid (z_{ij})"', "\pi"] \\ X' \arrow[u] & X \arrow[l] \arrow[u] \end{tikzcd}$$

The action of H leaves X invariant and the induced H-action on X is the one given by the K-grading of $R(A, \mathfrak{n}, L)$. Moreover, $\pi\colon \mathbb{K}^n \to \mathbb{K}^n$ is the quotient map for the induced action of $H_0 \subseteq H$ on $\mathbb{K}^n$, we have $X = \pi^{-1}(X')$, and hence the restriction $\pi\colon X \to X'$ is a quotient map for the induced action of H_0 on X.

Removing all subsets $V(X; T_{ij}, T_{kl})$, where $(i,j) \neq (k,l)$ from X, we obtain an open subset $U \subseteq X$. By Lemma 1.8, the complement $X \setminus U$ is of codimension at least two and each $V(U, T_{ij})$ is irreducible. By construction, the only isotropy groups of the H-action on U are the groups H_{ij} of the points of $V(U, T_{ij})$. The image $U' := \pi(U)$ is open in X', the complement $X' \setminus U'$ is as well of codimension at least two and H/H_0 acts freely on U'. According to [22, Cor. 5.3], we have two exact sequences fitting into the following diagram

$$\begin{tikzcd} & & 1 \arrow[d] & \\ & & \mathrm{Pic}(U') \arrow[d, "\pi^*"] & \\ 1 \arrow[r] & \mathbb{X}(H_0) \arrow[r, "\alpha"] & \mathrm{Pic}_{H_0}(U) \arrow[r, "\beta"] \arrow[d, "\delta"] & \mathrm{Pic}(U) \\ & & \prod_{i,j} \mathbb{X}(H_{ij}) & \end{tikzcd}$$

Since X' is factorial, the Picard group $\mathrm{Pic}(U')$ is trivial and we obtain that δ is injective. Since H_0 is the direct product of the isotropy groups H_{ij} of the Luna strata $V(U, T_{ij})$, we see that $\delta \circ \alpha$ is an isomorphism. It follows that δ is surjective and hence an isomorphism. This in turn shows that α is an isomorphism. Now, every bundle on U is H-linearizable. Since H_0 acts as a subgroup of H, we obtain that every bundle is H_0-linearizable. It follows that β is surjective and hence $\mathrm{Pic}(U)$ is trivial. We conclude $\mathrm{Cl}(X) = \mathrm{Pic}(U) = 0$, which means that $R(A, \mathfrak{n}, L)$ admits unique factorization.

The second thing we have to show is that any finitely generated factorial $\mathbb{K}$-algebra R with an effective complexity one multigrading satisfying $R_0 = \mathbb{K}$ is as claimed. Consider the action of the torus G on $X = \operatorname{Spec} R$ defined by the multigrading, and let $X_0 \subseteq X$ be the set of points having finite isotropy G_x.

Then [15, Prop 3.3] provides a graded splitting

$$R \quad \cong \quad R'[S_1, \ldots, S_m],$$

where the variables S_j are identified with the homogeneous functions defining the prime divisors E_j inside the boundary $X \setminus X_0$ and R' is the ring of functions of X_0, which are invariant under the subtorus $G_0 \subseteq G$ generated by the generic isotropy groups G_j of E_j.

Since $R'_0 = R_0 = \mathbb{K}$ holds, the orbit space X_0/G has only constant functions and thus is a space $\mathbb{P}_1(A, \mathfrak{n})$ as constructed in [15, Section 2]. This allows us to proceed exactly as in the proof of Theorem [15, Thm 1.3] and gives $R' = R(A, \mathfrak{n}, L)$. The admissibility condition follows from Lemma 1.5 and the fact that each T_{ij} defines a prime element in R'. □

Remark 1.10. Let $(A, \mathfrak{n}, L)$ be an admissible triple with $\mathfrak{n} = (1, \ldots, 1)$. Then $K = \mathbb{Z}$ holds, the admissibility condition just means that the numbers l_{ij} are pairwise coprime and we have

$$\dim R(A, \mathfrak{n}, L) \; = \; n_0 + \ldots + n_r - r + 1 \; = \; 2.$$

Consequently, for two-dimensional rings, Theorem 1.9 specializes to Mori's description of almost geometrically graded two-dimensional unique factorization domains provided in [23].

Proposition 1.11. *Let $(A, \mathfrak{n}, L)$ be an admissible triple, consider the associated $(K \times \mathbb{Z}^m)$-graded ring $R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$ as in Theorem 1.9 and let $\mu\colon K \times \mathbb{Z}^m \to K'$ be a surjection onto an abelian group K'. Then the following statements are equivalent.*

(i) *The K'-graded ring $R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$ is the Cox ring of a projective variety X' with $\mathrm{Cl}(X') \cong K'$.*

(ii) *For every pair i, j with $0 \le i \le r$ and $1 \le j \le n_i$, the group K' is generated by the elements $\mu(\lambda(e_{kl}))$ and $\mu(e_s)$, where $(i,j) \neq (k,l)$ and $1 \le s \le m$, for every $1 \le t \le m$, the group K' is generated by the elements $\mu(\lambda(e_{ij}))$ and $\mu(e_s)$, where $0 \le i \le r$, $1 \le j \le n_i$ and $s \neq t$, and, finally the following cone is of full dimension in $K'_{\mathbb{Q}}$:*

$$\bigcap_{(k,l)} \mathrm{cone}(\mu(\lambda(e_{ij})), \mu(e_s);\ (i,j) \neq (k,l)) \ \cap \ \bigcap_{t} \mathrm{cone}(\mu(\lambda(e_{ij})), \mu(e_s);\ s \neq t).$$

Proof. Suppose that (i) holds, let $p\colon \widehat{X}' \to X'$ denote the universal torsor and let $X'' \subseteq X'$ be the set of smooth points. According to [14, Prop. 2.2], the group $H' = \mathrm{Spec}\, \mathbb{K}[K']$ acts freely on $p^{-1}(X'')$, which is a big open subset of the total coordinate space $\mathrm{Spec}\, R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$. This implies the first condition of (ii). Moreover, by [14, Prop. 4.1], the displayed cone is the moving cone of X' and hence of full dimension. Conversely, if (ii) holds, then the K'-graded ring $R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$ can be made into a bunched ring and hence is the Cox ring of a projective variety, use [14, Thm. 3.6]. □

2 Bounds for Fano varieties

We consider d-dimensional Fano varieties X that come with a complexity one torus action and have divisor class group $\mathrm{Cl}(X) \cong \mathbb{Z}$. Then the Cox ring $\mathcal{R}(X)$ of X is factorial [5, Prop. 8.4] and has an effective complexity one grading, which refines the $\mathrm{Cl}(X)$-grading, see [15, Prop. 2.6]. Thus, according to Theorem 1.9, it is of the form

$$\begin{aligned}\mathcal{R}(X) \quad &\cong \quad \mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i][S_1, \dots, S_m] \ / \ \langle g_{i,i+1,i+2};\ 0 \le i \le r-2 \rangle, \\ g_{i,j,k} \quad &:= \quad \alpha_{jk} T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ + \ \alpha_{ki} T_{j1}^{l_{j1}} \cdots T_{jn_j}^{l_{jn_j}} \ + \ \alpha_{ij} T_{k1}^{l_{k1}} \cdots T_{kn_k}^{l_{kn_k}}.\end{aligned}$$

Here, we may (and will) assume $n_0 \ge \ldots \ge n_r \ge 1$. With $n := n_0 + \ldots + n_r$, we have $n + m = d + r$. For the degrees of the variables in $\mathrm{Cl}(X) \cong \mathbb{Z}$, we write $w_{ij} := \deg T_{ij}$ for $0 \le i \le r$, $1 \le j \le n_i$ and $u_k = \deg S_k$ for $1 \le k \le m$. Moreover, for $\mu \in \mathbb{Z}_{>0}$, we denote by $\xi(\mu)$ the number of primes in $\{2, \dots, \mu\}$. The following result provides bounds for the discrete data of the Cox ring.

Theorem 2.1. *In the above situation, fix the dimension $d = \dim(X)$ and the Picard index $\mu = [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$. Then we have*

$$u_k \ \le \ \mu \quad \text{for } 1 \le k \le m.$$

Moreover, for the degree γ of the relations, the weights w_{ij} and the exponents l_{ij}, where $0 \le i \le r$ and $1 \le j \le n_i$ one obtains the following.

(i) *Suppose that $r = 0, 1$ holds. Then $n + m \le d + 1$ holds and one has the bounds*

$$w_{ij} \ \le \ \mu \quad \text{for } 0 \le i \le r \text{ and } 1 \le j \le n_i,$$

and the Picard index is given by

$$\mu \ = \ \mathrm{lcm}(w_{ij}, u_k;\ 0 \le i \le r, 1 \le j \le n_i, 1 \le k \le m).$$

(ii) *Suppose that $r \ge 2$ and $n_0 = 1$ hold. Then $r \le \xi(\mu) - 1$ and $n = r + 1$ and $m = d - 1$ hold and one has*

$$w_{i1} \ \le \ \mu^r \quad \text{for } 0 \le i \le r, \qquad l_{01} \cdots l_{r1} \ | \ \mu, \qquad l_{01} \cdots l_{r1} \ | \ \gamma \ \le \ \mu^{r+1},$$

and the Picard index is given by

$$\mu \ = \ \mathrm{lcm}(\gcd(w_{j1};\ j \ne i), u_k;\ 0 \le i \le r, 1 \le k \le m).$$

(iii) *Suppose that $r \ge 2$ and $n_0 > n_1 = 1$ hold. Then we may assume $l_{11} > \ldots > l_{r1} \ge 2$, we have $r \le \xi(3d\mu) - 1$ and $n_0 + m = d$ and the bounds*

$$w_{01}, \dots, w_{0n_0} \ \le \ \mu, \qquad l_{01}, \dots, l_{0n_0} \ < \ 6d\mu,$$

$$w_{11}, l_{21} \ < \ 2d\mu, \qquad w_{21}, l_{11} \ < \ 3d\mu,$$

$$w_{i1} \;<\; 6d\mu, \quad l_{i1} \;<\; 2d\mu \quad \textit{for } 2 \le i \le r,$$

$$l_{11} \cdots l_{r1} \;\mid\; \gamma \;<\; 6d\mu,$$

and the Picard index is given by

$$\mu \;=\; \mathrm{lcm}(w_{0j}, \gcd(w_{11}, \ldots, w_{r1}), u_k;\; 1 \le j \le n_0, 1 \le k \le m).$$

(iv) *Suppose that $n_1 > n_2 = 1$ holds. Then we may assume $l_{21} > \ldots > l_{r1} \ge 2$, we have $r \le \xi(2(d+1)\mu) - 1$ and $n_0 + n_1 + m = d + 1$ and the bounds*

$$w_{ij} \;\le\; \mu \quad \textit{for } i = 0,1 \textit{ and } 1 \le j \le n_i, \qquad w_{21} \;<\; (d+1)\mu,$$

$$w_{ij}, l_{ij} \;<\; 2(d+1)\mu \quad \textit{for } 0 \le i \le r \textit{ and } 1 \le j \le n_i,$$

$$l_{21} \cdots l_{r1} \;\mid\; \gamma \;<\; 2(d+1)\mu,$$

and the Picard index is given by

$$\mu \;=\; \mathrm{lcm}(w_{ij}, u_k;\; 0 \le i \le 1, 1 \le j \le n_i, 1 \le k \le m).$$

(v) *Suppose that $n_2 > 1$ holds and let s be the maximal number with $n_s > 1$. Then one may assume $l_{s+1,1} > \ldots > l_{r1} \ge 2$, we have $r \le \xi((d+2)\mu) - 1$ and $n_0 + \ldots + n_s + m = d + s$ and the bounds*

$$w_{ij} \;\le\; \mu, \quad \textit{for } 0 \le i \le s,$$

$$w_{ij}, l_{ij} \;<\; (d+2)\mu \quad \textit{for } 0 \le i \le r \textit{ and } 1 \le j \le n_i,$$

$$l_{s+1,1} \cdots l_{r1} \;\mid\; \gamma \;<\; (d+2)\mu,$$

and the Picard index is given by

$$\mu \;=\; \mathrm{lcm}(w_{ij}, u_k;\; 0 \le i \le s, 1 \le j \le n_i, 1 \le k \le m).$$

Putting all the bounds of the theorem together, we obtain the following (raw) bound for the number of deformation types.

Corollary 2.2. *For any pair $(d, \mu) \in \mathbb{Z}^2_{>0}$, the number $\delta(d, \mu)$ of different deformation types of d-dimensional Fano varieties with a complexity one torus action such that $\mathrm{Cl}(X) \cong \mathbb{Z}$ and $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = \mu$ hold is bounded by*

$$\delta(d, \mu) \quad \le \quad (6d\mu)^{2\xi(3d\mu)+d-2} \mu^{\xi(\mu)^2 + 2\xi((d+2)\mu) + 2d + 2}.$$

Proof. By Theorem 2.1 the discrete data r, $\mathfrak{n}$, L and m occuring in $\mathcal{R}(X)$ are bounded as in the assertion. The continuous data in $\mathcal{R}(X)$ are the coefficients α_{ij}; they stem from the family $A = (a_0, \ldots, a_r)$ of points $a_i \in \mathbb{K}^2$. Varying the a_i provides flat families of Cox rings and hence, by passing to the homogeneous spectra, flat families of the resulting Fano varieties X. □

Corollary 2.3. *Fix $d \in \mathbb{Z}_{>0}$. Then the number $\delta(\mu)$ of different deformation types of d-dimensional Fano varieties with a complexity one torus action, $\mathrm{Cl}(X) \cong \mathbb{Z}$ and Picard index $\mu := [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ is asymptotically bounded by $\mu^{A\mu^2/\log^2 \mu}$ with a constant A depending only on d.*

Corollary 2.4. *Fix $\mu \in \mathbb{Z}_{>0}$. Then the number $\delta(d)$ of different deformation types of d-dimensional Fano varieties with a complexity one torus action, $\mathrm{Cl}(X) \cong \mathbb{Z}$ and Picard index $\mu := [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ is asymptotically bounded by d^{Ad} with a constant A depending only on μ.*

We first recall the necessary facts on Cox rings, for details, we refer to [14]. Let X be a complete d-dimensional variety with divisor class group $\mathrm{Cl}(X) \cong \mathbb{Z}$. Then the Cox ring $\mathcal{R}(X)$ is finitely generated and the total coordinate space $\overline{X} := \mathrm{Spec}\,\mathcal{R}(X)$ is a factorial affine variety coming with an action of $\mathbb{K}^*$ defined by the $\mathrm{Cl}(X)$-grading of $\mathcal{R}(X)$. Choose a system $f_1, \ldots, f_\nu$ of homogeneous pairwise nonassociated prime generators for $\mathcal{R}(X)$. This provides an $\mathbb{K}^*$-equivariant embedding

$$\overline{X} \to \mathbb{K}^\nu, \qquad \overline{x} \mapsto (f_1(\overline{x}), \ldots, f_\nu(\overline{x})).$$

where $\mathbb{K}^*$ acts diagonally with the weights $w_i = \deg(f_i) \in \mathrm{Cl}(X) \cong \mathbb{Z}$ on $\mathbb{K}^\nu$. Moreover, X is the geometric $\mathbb{K}^*$-quotient of $\widehat{X} := \overline{X} \setminus \{0\}$, and the quotient map $p\colon \widehat{X} \to X$ is a universal torsor. By the local divisor class group $\mathrm{Cl}(X, x)$ of a point $x \in X$, we mean the group of Weil divisors $\mathrm{WDiv}(X)$ modulo those that are principal near x.

Proposition 2.5. *For any $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_\nu) \in \widehat{X}$ the local divisor class group $\mathrm{Cl}(X, x)$ of $x := p(\overline{x})$ is finite of order $\gcd(w_i;\ \overline{x}_i \neq 0)$. The index of the Picard group $\mathrm{Pic}(X)$ in $\mathrm{Cl}(X)$ is given by*

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \quad = \quad \mathrm{lcm}_{x \in X}(|\,\mathrm{Cl}(X, x)|).$$

Suppose that the ideal of $\overline{X} \subseteq \mathbb{K}^\nu$ is generated by $\mathrm{Cl}(X)$-homogeneous polynomials $g_1, \ldots, g_{\nu-d-1}$ of degree $\gamma_j := \deg(g_j)$. Then one obtains

$$-\mathcal{K}_X \;=\; \sum_{i=1}^{\nu} w_i - \sum_{j=1}^{\nu-d-1} \gamma_j, \quad (-\mathcal{K}_X)^d \;=\; \left(\sum_{i=1}^{\nu} w_i - \sum_{j=1}^{\nu-d-1} \gamma_j\right)^d \frac{\gamma_1 \cdots \gamma_{\nu-d-1}}{w_1 \cdots w_\nu}$$

for the anticanonical class $-\mathcal{K}_X \in \mathrm{Cl}(X) \cong \mathbb{Z}$. In particular, X is a Fano variety if and only if the following inequality holds

$$\sum_{j=1}^{\nu-d-1} \gamma_j \;<\; \sum_{i=1}^{\nu} w_i.$$

Proof. Using [14, Prop. 2.2, Thm. 4.19], we observe that X arises from the bunched ring $(R, \mathfrak{F}, \Phi)$, where $R = \mathcal{R}(X)$, $\mathfrak{F} = (f_1, \ldots, f_\nu)$ and $\Phi = \{\mathbb{Q}_{\geq 0}\}$.

The descriptions of local class groups, the Picard index and the anticanonical class are then special cases of [14, Prop. 4.7, Cor. 4.9 and Cor. 4.16]. The anticanonical self-intersection number is easily computed in the ambient weighted projective space $\mathbb{P}(w_1, \ldots, w_\nu)$, use [14, Constr. 3.13, Cor. 4.13]. □

Remark 2.6. If the ideal of $\overline{X} \subseteq \mathbb{K}^\nu$ is generated by $\mathrm{Cl}(X)$-homogeneous polynomials $g_1, \ldots, g_{\nu-d-1}$, then [14, Constr. 3.13, Cor. 4.13] show that X is a well formed complete intersection in the weighted projective space $\mathbb{P}(w_1, \ldots, w_\nu)$ in the sense of [16, Def. 6.9].

We turn back to the case that X comes with a complexity one torus action as at the beginning of this section. We consider the case $n_0 = \ldots = n_r = 1$, that means that each relation $g_{i,j,k}$ of the Cox ring $\mathcal{R}(X)$ depends only on three variables. Then we may write T_i instead of T_{i1} and w_i instead of w_{i1}, etc.. In this setting, we obtain the following bounds for the numbers of possible varieties X (Fano or not).

Proposition 2.7. *For any pair $(d, \mu) \in \mathbb{Z}^2_{>0}$ there is, up to deformation, only a finite number of complete d-dimensional varieties with divisor class group $\mathbb{Z}$, Picard index $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = \mu$ and Cox ring*

$$\mathbb{K}[T_0, \ldots, T_r, S_1, \ldots, S_m] \ / \ \langle \alpha_{i+1,i+2} T_i^{l_i} + \alpha_{i+2,i} T_{i+1}^{l_{i+1}} + \alpha_{i,i+1} T_{i+2}^{l_{i+2}};\ 0 \leq i \leq r-2 \rangle.$$

In this situation we have $r \leq \xi(\mu) - 1$. Moreover, for the weights $w_i := \deg\, T_i$, where $0 \leq i \leq r$ and $u_k := \deg\, S_k$, where $1 \leq k \leq m$, the exponents l_i and the degree $\gamma := l_0 w_0$ of the relation one has

$$l_0 \cdots l_r \ \mid \ \gamma, \qquad l_0 \cdots l_r \ \mid \ \mu, \qquad w_i \ \leq \ \mu^{\xi(\mu)-1}, \qquad u_k \ \leq \ \mu.$$

Proof. Consider the total coordinate space $\overline{X} \subseteq \mathbb{K}^{r+1+n}$ and the universal torsor $p\colon \widehat{X} \to X$ as discussed before. For each $0 \leq i \leq r$ fix a point $\overline{x}(i) = (\overline{x}_0, \ldots, \overline{x}_r, 0, \ldots, 0)$ in $\widehat{X}$ such that $\overline{x}_i = 0$ and $\overline{x}_j \neq 0$ for $j \neq i$ hold. Then, denoting $x(i) := p(\overline{x}(i))$, we obtain

$$\gcd(w_j; j \neq i) \ = \ |\,\mathrm{Cl}(X, x(i))| \ \mid \ \mu.$$

Consider i, j with $j \neq i$. Since all relations are homogeneous of the same degree, we have $l_i w_i = l_j w_j$. Moreover, by the admissibility condition, l_i and l_j are coprime. We conclude $l_i | w_j$ for all $j \neq i$ and hence $l_i | \gcd(w_j;\ j \neq i)$. This implies

$$l_0 \cdots l_r \ \mid \ l_0 w_0 \ = \ \gamma, \qquad\qquad l_0 \cdots l_r \ \mid \ \mu.$$

We turn to the bounds for the w_i, and first verify $w_0 \leq \mu^r$. Using the relation $l_i w_i = l_0 w_0$, we obtain for every l_i a presentation

$$l_i \ = \ l_0 \cdot \frac{w_0 \cdots w_{i-1}}{w_1 \cdots w_i} \ = \ \eta_i \cdot \frac{\gcd(w_0, \ldots, w_{i-1})}{\gcd(w_0, \ldots, w_i)}$$

with suitable integers $1 \le \eta_i \le \mu$. In particular, the very last fraction is bounded by μ. This gives the desired estimate:

$$w_0 = \frac{w_0}{\gcd(w_0,w_1)} \cdot \frac{\gcd(w_0,w_1)}{\gcd(w_0,w_1,w_2)} \cdots \frac{\gcd(w_0,\dots,w_{r-2})}{\gcd(w_0,\dots,w_{r-1})} \cdot \gcd(w_0,\dots,w_{r-1}) \le \mu^r.$$

Similarly, we obtain $w_i \le \mu^r$ for $1 \le i \le r$. Then we only have to show that $r+1$ is bounded by $\xi(\mu)$, but this follows immediately from the fact that $l_0,\dots,l_r$ are pairwise coprime.

Finally, to estimate the u_k, consider the points $\overline{x}(k) \in \widehat{X}$ having the $(r+k)$-th coordinate one and all others zero. Set $x(k) := p(\overline{x}(k))$. Then $\mathrm{Cl}(X, x(k))$ is of order u_k, which implies $u_k \le \mu$. □

LEMMA 2.8. *Consider the ring* $\mathbb{K}[T_{ij};\ 0 \le i \le 2,\ 1 \le j \le n_i][S_1,\dots,S_k]/\langle g\rangle$ *where* $n_0 \ge n_1 \ge n_2 \ge 1$ *holds. Suppose that* g *is homogeneous with respect to a* $\mathbb{Z}$*-grading of* $\mathbb{K}[T_{ij}, S_k]$ *given by* $\deg T_{ij} = w_{ij} \in \mathbb{Z}_{>0}$ *and* $\deg S_k = u_k \in \mathbb{Z}_{>0}$*, and assume*

$$\deg g \;<\; \sum_{i=0}^{2}\sum_{j=1}^{n_i} w_{ij} \;+\; \sum_{i=1}^{m} u_i.$$

Let $\mu \in \mathbb{Z}_{>1}$*, assume* $w_{ij} \le \mu$ *whenever* $n_i > 1$, $1 \le j \le n_i$ *and* $u_k \le \mu$ *for* $1 \le k \le m$ *and set* $d := n_0 + n_1 + n_2 + m - 2$*. Depending on the shape of* g*, one obtains the following bounds.*

(i) *Suppose that* $g = \eta_0 T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + \eta_1 T_{11}^{l_{11}} + \eta_2 T_{21}^{l_{21}}$ *with* $n_0 > 1$ *and coefficients* $\eta_i \in \mathbb{K}^*$ *holds, we have* $l_{11} \ge l_{21} \ge 2$ *and* l_{11}, l_{21} *are coprime. Then, one has*

$$w_{11}, l_{21} \;<\; 2d\mu, \qquad w_{21}, l_{11} \;<\; 3d\mu, \qquad \deg g \;<\; 6d\mu.$$

(ii) *Suppose that* $g = \eta_0 T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + \eta_1 T_{11}^{l_{11}} \cdots T_{1n_1}^{l_{1n_1}} + \eta_2 T_{21}^{l_{21}}$ *with* $n_1 > 1$ *and coefficients* $\eta_i \in \mathbb{K}^*$ *holds and we have* $l_{21} \ge 2$*. Then one has*

$$w_{21} \;<\; (d+1)\mu, \qquad \deg g \;<\; 2(d+1)\mu.$$

Proof. We prove (i). Set for short $c := (n_0 + m)\mu = d\mu$. Then, using homogeneity of g and the assumed inequality, we obtain

$$l_{11}w_{11} \;=\; l_{21}w_{21} \;=\; \deg g \;<\; \sum_{i=0}^{2}\sum_{j=1}^{n_i} w_{ij} + \sum_{i=1}^{m} u_i \;\le\; c + w_{11} + w_{21}.$$

Since l_{11} and l_{21} are coprime, we have $l_{11} > l_{21} \ge 2$. Plugging this into the above inequalities, we arrive at $2w_{11} < c + w_{21}$ and $w_{21} < c + w_{11}$. We conclude

$w_{11} < 2c$ and $w_{21} < 3c$. Moreover, $l_{11}w_{11} = l_{21}w_{21}$ and $\gcd(l_{11}, l_{21}) = 1$ imply $l_{11}|w_{21}$ and $l_{21}|w_{11}$. This shows $l_{11} < 3c$ and $l_{21} < 2c$. Finally, we obtain

$$\deg\, g \ < \ c + w_{11} + w_{21} \ < \ 6c.$$

We prove (ii). Here we set $c := (n_0 + n_1 + m)\mu = (d + 1)\mu$. Then the assumed inequality gives

$$l_{21}w_{21} \ = \ \deg g \ < \ \sum_{i=0}^{1}\sum_{j=1}^{n_i} w_{ij} + \sum_{i=1}^{m} u_i + w_{21} \ \le \ c + w_{21}.$$

Since we assumed $l_{21} \ge 2$, we can conclude $w_{21} < c$. This in turn gives us $\deg\, g < 2c$ for the degree of the relation. $\square$

Proof of Theorem 2.1. As before, we denote by $\overline{X} \subseteq \mathbb{K}^{n+m}$ the total coordinate space and by $p\colon \widehat{X} \to X$ the universal torsor.

We first consider the case that X is a toric variety. Then the Cox ring is a polynomial ring, $\mathcal{R}(X) = \mathbb{K}[S_1, \ldots, S_m]$. For each $1 \le k \le m$, consider the point $\overline{x}(k) \in \widehat{X}$ having the k-th coordinate one and all others zero and set $x(k) := p(\overline{x}(k))$. Then, by Proposition 2.5, the local class group $\mathrm{Cl}(X, x(k))$ is of order u_k where $u_k := \deg\, S_k$. This implies $u_k \le \mu$ for $1 \le k \le m$ and settles Assertion (i).

Now we treat the non-toric case, which means $r \ge 2$. Note that we have $n \ge 3$. The case $n_0 = 1$ is done in Proposition 2.7. So, we are left with $n_0 > 1$. For every i with $n_i > 1$ and every $1 \le j \le n_i$, there is the point $\overline{x}(i,j) \in \widehat{X}$ with ij-coordinate T_{ij} equal to one and all others equal to zero, and thus we have the point $x(i,j) := p(\overline{x}(i,j)) \in X$. Moreover, for every $1 \le k \le m$, we have the point $\overline{x}(k) \in \widehat{X}$ having the k-coordinate S_k equal to one and all others zero; we set $x(k) := p(\overline{x}(k))$. Proposition 2.5 provides the bounds

$$w_{ij} \ = \ \deg\, T_{ij} \ = \ |\,\mathrm{Cl}(X, x(i,j))| \ \le \ \mu \qquad \text{for } n_i > 1,\ 1 \le j \le n_i,$$

$$u_k \ = \ \deg\, S_k \ = \ |\,\mathrm{Cl}(X, x(k))| \ \le \ \mu \qquad \text{for } 1 \le k \le m.$$

Let $0 \le s \le r$ be the maximal number with $n_s > 1$. Then $g_{s-2,s-1,s}$ is the last polynomial such that each of its three monomials depends on more than one variable. For any $t \ge s$, we have the "cut ring"

$$R_t \quad := \quad \mathbb{K}[T_{ij};\ 0 \le i \le t,\ 1 \le j \le n_i][S_1, \ldots, S_m] \ / \ \langle g_{i,i+1,i+2};\ 0 \le i \le t-2 \rangle$$

where the relations $g_{i,i+1,i+2}$ depend on only three variables as soon as $i > s$

holds. For the degree γ of the relations we have

$$\begin{aligned}(r-1)\gamma &= (t-1)\gamma \ + \ (r-t)\gamma \\ &= (t-1)\gamma \ + \ l_{t+1,1}w_{t+1,1} + \ldots + l_{r1}w_{r1} \\ &< \sum_{i=0}^{r}\sum_{j=1}^{n_i} w_{ij} \ + \ \sum_{i=1}^{m} u_i \\ &= \sum_{i=0}^{t}\sum_{j=1}^{n_i} w_{ij} \ + \ w_{t+1,1} + \ldots + w_{r1} \ + \ \sum_{i=1}^{m} u_i.\end{aligned}$$

Since $l_{i1}w_{i1} > w_{i1}$ holds in particular for $t+1 \le i \le r$, we derive from this the inequality

$$\gamma \ < \ \frac{1}{t-1}\left(\sum_{i=0}^{t}\sum_{j=1}^{n_i} w_{ij} \ + \ \sum_{i=1}^{m} u_i\right).$$

To obtain the bounds in Assertions (iii) and (iv), we consider the cut ring R_t with $t = 2$ and apply Lemma 2.8; note that we have $d = n_0 + n_1 + n_2 + m - 2$ for the dimension $d = \dim(X)$ and that $l_{22} \ge 0$ is due to the fact that X is non-toric. The bounds $w_{ij}, l_{0j} < 6d\mu$ in Assertion (iii) follow from $l_{ij}w_{ij} = \gamma < 6d\mu$ and $l_{i1} < 2d\mu$ follows from $l_{i1} \mid w_{21}$ for $3 \le i \le r$. Moreover, $l_{i1} \mid w_{11}$ for $2 \le i \le r$ implies $l_{11} \cdots l_{r1} \mid \gamma = l_{11}w_{11}$. Similarly $w_{ij}, l_{ij} < 2(d+1)\mu$ in Assertion (iv) follow from $l_{ij}w_{ij} = \gamma < 2(d+1)d\mu$ and $l_{21} \cdots l_{r1} \mid \gamma = l_{21}w_{21}$ follows from $l_{i1} \mid w_{21}$ for $3 \le i \le r$. The bounds on r in (iii) in (iv) are as well consequences of the admissibility condition.

To obtain the bounds in Assertion (v), we consider the cut ring R_t with $t = s$. Using $n_i = 1$ for $i \ge t+1$, we can estimate the degree of the relation as follows:

$$\gamma \ \le \ \frac{(n_0 + \ldots + n_t + m)\mu}{t-1} \ = \ \frac{(d+t)\mu}{t-1} \ \le \ (d+2)\mu.$$

Since we have $w_{ij}l_{ij} \le \deg g_0$ for any $0 \le i \le r$ and any $1 \le j \le n_i$, we see that all w_{ij} and l_{ij} are bounded by $(d+2)\mu$. As before, $l_{s+1,1} \cdots l_{r1} \mid \gamma$ is a consequence of $l_{i1} \mid \gamma$ for $i = s+2, \ldots, r$ and also the bound on r follows from the admissibility condition.

Finally, we have to express the Picard index μ in terms of the weights w_{ij} and u_k as claimed in the Assertions. This is a direct application of the formula of Proposition 2.5. Observe that it suffices to work with the p-images of the following points: For every $0 \le i \le r$ with $n_i > 1$ take a point $\overline{x}(i,j) \in \widehat{X}$ with ij-coordinate T_{ij} equal to one and all others equal to zero, for every $0 \le i \le r$ with $n_i = 1$ whenever $n_i = 1$ take $\overline{x}(i,j) \in \widehat{X}$ with ij-coordinate T_{ij} equal to zero, all other T_{st} equal to one and coordinates S_k equal to zero, and, for every $1 \le k \le m$, take a point $\overline{x}(k) \in \widehat{X}$ having the k-coordinate S_k equal to one and all others zero. □

We conclude the section with discussing some aspects of the not necessarily Fano varieties of Proposition 2.7. Recall that we considered admissible triples $(A, \mathfrak{n}, L)$ with $n_0 = \ldots = n_r = 1$ and thus rings R of the form

$$\mathbb{K}[T_0, \ldots, T_r, S_1, \ldots, S_m] \ / \ \langle \alpha_{i+1,i+2} T_i^{l_i} + \alpha_{i+2,i} T_{i+1}^{l_{i+1}} + \alpha_{i,i+1} T_{i+2}^{l_{i+2}};\ 0 \le i \le r-2 \rangle.$$

Proposition 2.9. *Suppose that the ring R as above is the Cox ring of a non-toric variety X with $\mathrm{Cl}(X) = \mathbb{Z}$. Then we have $m \ge 1$ and $\mu := [\mathrm{Cl}(X) : \mathrm{Pic}(X)] \ge 30$. Moreover, if X is a surface, then we have $m = 1$ and $w_i = l_i^{-1} l_0 \cdots l_r$.*

Proof. The homogeneity condition $l_i w_i = l_j w_j$ together with the admissibility condition $\gcd(l_i, l_j) = 1$ for $0 \le i \ne j \le r$ gives us $l_i \mid \gcd(w_j; j \ne i)$. Moreover, by Proposition 1.11, every set of $m + r$ weights w_i has to generate the class group $\mathbb{Z}$, so they must have greatest common divisor one. Since X is non-toric, $l_i \ge 2$ holds and we obtain $m \ge 1$. To proceed, we infer $l_0 \cdots l_r \mid \mu$ and $l_0 \cdots l_r \mid \deg g_{ijk}$ from Proposition 2.5. As a consequence, the minimal value for μ and $\deg g_{ijk}$ is obviously $2 \cdot 3 \cdot 5 = 30$. what really can be received as the following example shows. Note that if X is a surface we have $m = 1$ and $\gcd(w_i; 0 \le i \le r) = 1$. Thus, $l_i w_i = l_j w_j$ gives us $\deg g_{ijk} = l_0 \cdots l_r$ and $w_i = l_i^{-1} l_0 \cdots l_r$. □

The bound $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \ge 30$ given in the above proposition is even sharp; the surface discussed below realizes it.

Example 2.10. Consider X with $\mathcal{R}(X) = \mathbb{K}[T_0, T_1, T_2, T_3]/\langle g \rangle$ with $g = T_0^2 + T_1^3 + T_2^5$ and the grading

$$\deg T_0 \ = \ 15, \quad \deg T_1 \ = \ 10, \quad \deg T_2 \ = \ 6, \quad \deg T_3 \ = \ 1.$$

Then we have $\gcd(15, 10) = 5$, $\gcd(15, 6) = 3$ and $\gcd(10, 6) = 2$ and therefore $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 30$. Further X is Fano because of

$$\deg g \ = \ 30 \ < \ 32 \ = \ \deg T_0 + \ldots + \deg T_3.$$

Let us have a look at the geometric meaning of the condition $n_0 = \ldots = n_r = 1$. For a variety X with an action of a torus T, we denote by $X_0 \subseteq X$ the union of all orbits with at most finite isotropy. Then there is a possibly non-separated orbit space X_0/T; we call it the maximal orbit space. From [15], we infer that $n_0 = \ldots = n_r = 1$ holds if and only if X_0/T is separated. Combining this with Propositions 2.7 and 2.9 gives the following.

Corollary 2.11. *For any pair $(d, \mu) \in \mathbb{Z}^2_{>0}$ there is, up to deformation, only a finite number of d-dimensional complete varieties X with a complexity one torus action having divisor class group $\mathbb{Z}$, Picard index $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = \mu$ and maximal orbit space $\mathbb{P}_1$ and for each of these varieties the complement $X \setminus X_0$ contains divisors.*

Finally, we present a couple of examples showing that there are also non-Fano varieties with a complexity one torus action having divisor class group $\mathbb{Z}$ and maximal orbit space $\mathbb{P}_1$.

EXAMPLE 2.12. Consider X with $\mathcal{R}(X) = \mathbb{K}[T_0, T_1, T_2, T_3]/\langle g \rangle$ with $g = T_0^2 + T_1^3 + T_2^7$ and the grading

$$\deg T_0 \; = \; 21, \quad \deg T_1 \; = \; 14, \quad \deg T_2 \; = \; 6, \quad \deg T_3 \; = \; 1.$$

Then we have $\gcd(21, 14) = 7$, $\gcd(21, 6) = 3$ and $\gcd(14, 6) = 2$ and therefore $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 42$. Moreover, X is not Fano, because its canonical class $\mathcal{K}_X$ is trivial

$$\mathcal{K}_X \; = \; \deg g - \deg T_0 - \ldots - \deg T_3 \; = \; 0.$$

EXAMPLE 2.13. Consider X with $\mathcal{R}(X) = \mathbb{K}[T_0, T_1, T_2, T_3]/\langle g \rangle$ with $g = T_0^2 + T_1^3 + T_2^{11}$ and the grading

$$\deg T_0 \; = \; 33, \quad \deg T_1 \; = \; 22, \quad \deg T_2 \; = \; 6, \quad \deg T_3 \; = \; 1.$$

Then we have $\gcd(22, 33) = 11$, $\gcd(33, 6) = 3$ and $\gcd(22, 6) = 2$ and therefore $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 66$. The canonical class $\mathcal{K}_X$ of X is even ample:

$$\mathcal{K}_X \; = \; \deg g - \deg T_0 - \ldots - \deg T_3 \; = \; 4.$$

The following example shows that the Fano assumption is essential for the finiteness results in Theorem 2.1.

Remark 2.14. For any pair p, q of coprime positive integers, we obtain a locally factorial $\mathbb{K}^*$-surface $X(p, q)$ with $\mathrm{Cl}(X) = \mathbb{Z}$ and Cox ring

$$\mathcal{R}(X(p,q)) \; = \; \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}] \; / \; \langle g \rangle, \qquad\qquad g \; = \; T_{01}T_{02}^{pq-1} + T_{11}^q + T_{21}^p;$$

the $\mathrm{Cl}(X)$-grading is given by $\deg T_{01} = \deg T_{02} = 1$, $\deg T_{11} = p$ and $\deg T_{21} = q$. Note that $\deg g = pq$ holds and for $p, q \geq 3$, the canonical class $\mathcal{K}_X$ satisfies

$$\mathcal{K}_X \; = \; \deg g - \deg T_{01} - \deg T_{02} - \deg T_{11} - \deg T_{21} \; = \; pq - 2 - p - q \; \geq \; 0.$$

3 CLASSIFICATION RESULTS

In this section, we give classification results for Fano varieties X with $\mathrm{Cl}(X) \cong \mathbb{Z}$ that come with a complexity one torus action; note that they are necessarily rational. The procedure to obtain classification lists for prescribed dimension $d = \dim X$ and Picard index $\mu = [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ is always the following. By Theorem 1.9, we know that their Cox rings are of the form $\mathcal{R}(X) \cong R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$ with admissible triples $(A, \mathfrak{n}, L)$. Note that for the family $A = (a_0, \ldots, a_r)$ of points $a_i \in \mathbb{K}^2$, we may assume

$$a_0 \; = \; (1,0), \qquad a_1 \; = \; (1,1), \qquad a_2 \; = \; (0,1).$$

The bounds on the input data of $(A, \mathfrak{n}, L)$ provided by Theorem 2.1 as well as the criteria of Propositions 1.11 and 2.5 allow us to generate all the possible Cox rings $\mathcal{R}(X)$ of the Fano varieties X in question for fixed dimension d and Picard index μ. Note that X can be reconstructed from $\mathcal{R}(X) = R(A, \mathfrak{n}, L)[S_1, \ldots, S_n]$ as the homogeneous spectrum with respect to the $\mathrm{Cl}(X)$-grading. Thus X is classified by its Cox ring $\mathcal{R}(X)$.

In the following tables, we present the Cox rings as $\mathbb{K}[T_1, \ldots, T_s]$ modulo relations and fix the $\mathbb{Z}$-gradings by giving the weight vector $(w_1, \ldots, w_s)$, where $w_i := \deg T_i$. The first classification result concerns surfaces.

Theorem 3.1. *Let X be a non-toric Fano surface with an effective $\mathbb{K}^*$-action such that $\mathrm{Cl}(X) = \mathbb{Z}$ and $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \le 6$ hold. Then its Cox ring is precisely one of the following.*

$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 1$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	$(-K_X)^2$
1	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1T_2^5 + T_3^3 + T_4^2 \rangle$	$(1, 1, 2, 3)$	1

$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 2$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	$(-K_X)^2$
2	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^4T_2 + T_3^3 + T_4^2 \rangle$	$(1, 2, 2, 3)$	2

$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 3$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	$(-K_X)^2$
3	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^3T_2 + T_3^3 + T_4^2 \rangle$	$(1, 3, 2, 3)$	3
4	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1T_2^3 + T_3^5 + T_4^2 \rangle$	$(1, 3, 2, 5)$	1/3
5	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^7T_2 + T_3^5 + T_4^2 \rangle$	$(1, 3, 2, 5)$	1/3

$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 4$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	$(-K_X)^2$
6	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^2T_2 + T_3^3 + T_4^2 \rangle$	$(1, 4, 2, 3)$	4
7	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^6T_2 + T_3^5 + T_4^2 \rangle$	$(1, 4, 2, 5)$	1

$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 5$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	$(-K_X)^2$

8	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1T_2+T_3^3+T_4^2\rangle$	$(1,5,2,3)$	5
9	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^5T_2+T_3^5+T_4^2\rangle$	$(1,5,2,5)$	$9/5$
10	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^9T_2+T_3^7+T_4^2\rangle$	$(1,5,2,7)$	$1/5$
11	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^7T_2+T_3^4+T_4^3\rangle$	$(1,5,3,4)$	$1/5$

$[\mathrm{Cl}(X):\mathrm{Pic}(X)]=6$

No.	$\mathcal{R}(X)$	$(w_1,\ldots,w_4)$	$(-K_X)^2$
12	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^4T_2+T_3^5+T_4^2\rangle$	$(1,6,2,5)$	$8/3$
13	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^8T_2+T_3^7+T_4^2\rangle$	$(1,6,2,7)$	$2/3$
14	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^6T_2+T_3^4+T_4^3\rangle$	$(1,6,3,4)$	$2/3$
15	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^9T_2+T_3^3+T_4^2\rangle$	$(1,3,4,6)$	$2/3$

Proof. As mentioned, Theorems 1.9, 2.1 and Propositions 1.11, 2.5 produce a list of all Cox rings of surfaces with the prescribed data. Doing this computation, we obtain the list of the assertion. Note that none of the Cox rings listed is a polynomial ring and hence none of the resulting surfaces X is a toric variety. To show that different members of the list are not isomorphic to each other, we use the following two facts. Firstly, observe that any two minimal systems of homogeneous generators of the Cox ring have (up to reordering) the same list of degrees, and thus the list of generator degrees is invariant under isomorphism (up to reordering). Secondly, by Construction 1.7, the exponents $l_{ij}>1$ are precisely the orders of the non-trivial isotropy groups of one-codimensional orbits of the action of the torus T on X. Using both principles and going through the list, we see that different members X cannot be T-equivariantly isomorphic to each other. Since all listed X are non-toric, the effective complexity one torus action on each X corresponds to a maximal torus in the linear algebraic group $\mathrm{Aut}(X)$. Any two maximal tori in the automorphism group are conjugate, and thus we can conclude that two members are isomorphic if and only if they are T-equivariantly isomorphic. $\square$

We remark that in [28, Section 4], log del Pezzo surfaces with an effective $\mathbb{K}^*$-action and Picard number 1 and Gorenstein index less than 4 were classified. The above list contains six such surfaces, namely no. 1-4, 6 and 8; these are exactly the ones where the maximal exponents of the monomials form a platonic triple, i.e., are of the form $(1,k,l)$, $(2,2,k)$, $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$. The remaining ones, i.e., no. 5, 7, and 9-15 have non-log-terminal and thus non-rational singularities; to check this one may compute the resolutions via resolution of the ambient weighted projective space as in [14, Ex. 7.5].

With the same scheme of proof as in the surface case, one establishes the following classification results on Fano threefolds.

Theorem 3.2. *Let X be a three-dimensional locally factorial non-toric Fano variety with an effective two torus action such that $\mathrm{Cl}(X) = \mathbb{Z}$ holds. Then its Cox ring is precisely one of the following.*

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_5)$	$(-K_X)^3$
1	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^5 + T_3^3 + T_4^2 \rangle$	$(1, 1, 2, 3, 1)$	8
2	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2T_3^4 + T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 2, 3)$	8
3	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^2T_3^3 + T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 2, 3)$	8
4	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2 + T_3T_4 + T_5^2 \rangle$	$(1, 1, 1, 1, 1)$	54
5	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^2 + T_3T_4^2 + T_5^3 \rangle$	$(1, 1, 1, 1, 1)$	24
6	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^3 + T_3T_4^3 + T_5^4 \rangle$	$(1, 1, 1, 1, 1)$	4
7	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^3 + T_3T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 1, 2)$	16
8	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^5 + T_3T_4^5 + T_5^2 \rangle$	$(1, 1, 1, 1, 3)$	2
9	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^5 + T_3^3T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 1, 3)$	2

The singular threefolds listed in this theorem are rational degenerations of smooth Fano threefolds from [18]. The (smooth) general Fano threefolds of the corresponding families are non-rational see [12] for no. 1-3, [8] for no. 5, [20] for no. 6, [30, 29] for no. 7 and [19] for no. 8-9. Even if one allows certain mild singularities, one still has non-rationality in some cases, see [13], [9, 25], [10], [6].

Theorem 3.3. *Let X be a three-dimensional non-toric Fano variety with an effective two torus action such that $\mathrm{Cl}(X) = \mathbb{Z}$ and $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 2$ hold. Then its Cox ring is precisely one of the following.*

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_5)$	$(-K_X)^3$
1	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1^4T_2 + T_3^3 + T_4^2 \rangle$	$(1, 2, 2, 3, 1)$	27/2
2	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1^4T_2^3 + T_3^5 + T_4^2 \rangle$	$(1, 2, 2, 5, 1)$	1/2
3	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1^8T_2 + T_3^5 + T_4^2 \rangle$	$(1, 2, 2, 5, 1)$	1/2
4	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1^4T_2 + T_3^3 + T_4^2 \rangle$	$(1, 2, 2, 3, 2)$	16

5	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^4T_2^3+T_3^5+T_4^2\rangle$	$(1,2,2,5,2)$	2
6	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^8T_2+T_3^5+T_4^2\rangle$	$(1,2,2,5,2)$	2
7	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5+T_3^3+T_4^2\rangle$	$(1,1,2,3,2)$	27/2
8	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^9+T_3^5+T_4^2\rangle$	$(1,1,2,5,2)$	1/2
9	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^3T_2^7+T_3^5+T_4^2\rangle$	$(1,1,2,5,2)$	1/2
10	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^{11}+T_3^3+T_4^2\rangle$	$(1,1,4,6,1)$	1/2
11	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^5T_2^7+T_3^3+T_4^2\rangle$	$(1,1,4,6,1)$	1/2
12	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^{11}+T_3^3+T_4^2\rangle$	$(1,1,4,6,2)$	2
13	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^5T_2^7+T_3^3+T_4^2\rangle$	$(1,1,4,6,2)$	2
14	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^2T_2^5+T_3^3+T_4^2\rangle$	$(1,2,4,6,1)$	2
15	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^{10}T_2+T_3^3+T_4^2\rangle$	$(1,2,4,6,1)$	2
16	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^2+T_3^3+T_4^2\rangle$	$(2,2,2,3,1)$	16
17	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^4+T_3^5+T_4^2\rangle$	$(2,2,2,5,1)$	2
18	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^2T_2^3+T_3^5+T_4^2\rangle$	$(2,2,2,5,1)$	2
19	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^2+T_3T_4+T_5^3\rangle$	$(1,1,1,2,1)$	81/2
20	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^4+T_3T_4^2+T_5^5\rangle$	$(1,1,1,2,1)$	5/2
21	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^2T_2^3+T_3T_4^2+T_5^5\rangle$	$(1,1,1,2,1)$	5/2
22	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3+T_3^2T_4+T_5^4\rangle$	$(1,1,1,2,1)$	16
23	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^4+T_3^3T_4+T_5^5\rangle$	$(1,1,1,2,1)$	5/2
24	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^2T_2^3+T_3^3T_4+T_5^5\rangle$	$(1,1,1,2,1)$	5/2
25	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3+T_3^2T_4+T_5^2\rangle$	$(1,1,1,2,2)$	27
26	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5+T_3^2T_4^2+T_5^3\rangle$	$(1,1,1,2,2)$	3/2
27	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5+T_3^4T_4+T_5^3\rangle$	$(1,1,1,2,2)$	3/2
28	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^2T_2^4+T_3^4T_4+T_5^3\rangle$	$(1,1,1,2,2)$	3/2
29	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5+T_3^4T_4+T_5^2\rangle$	$(1,1,1,2,3)$	8
30	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^3T_2^3+T_3^4T_4+T_5^2\rangle$	$(1,1,1,2,3)$	8
31	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^7+T_3^2T_4^3+T_5^2\rangle$	$(1,1,1,2,4)$	1
32	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^3T_2^5+T_3^2T_4^3+T_5^2\rangle$	$(1,1,1,2,4)$	1
33	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^7+T_3^6T_4+T_5^2\rangle$	$(1,1,1,2,4)$	1
34	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^3T_2^5+T_3^6T_4+T_5^2\rangle$	$(1,1,1,2,4)$	1
35	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3+T_3T_4+T_5^4\rangle$	$(1,1,2,2,1)$	27

36	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3T_4^2+T_5^6\rangle$	$(1,1,2,2,1)$	$3/2$
37	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3T_4+T_5^2\rangle$	$(1,1,2,2,2)$	16
38	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3T_4^2+T_5^3\rangle$	$(1,1,2,2,2)$	6
39	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4+T_3T_4^2+T_5^3\rangle$	$(1,1,2,2,2)$	6
40	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^3+T_3T_4^2+T_5^2\rangle$	$(1,1,2,2,2)$	$27/2$
41	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^5+T_3T_4^3+T_5^2\rangle$	$(1,1,2,2,2)$	32
42	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3T_4^2+T_5^2\rangle$	$(1,1,2,2,3)$	4
43	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^7+T_3T_4^3+T_5^2\rangle$	$(1,1,2,2,4)$	32
44	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^9+T_3T_4^4+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
45	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^9+T_3^2T_4^3+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
46	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^7+T_3T_4^4+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
47	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^7+T_3^2T_4^3+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
48	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^5T_2^5+T_3T_4^4+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
49	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^5T_2^5+T_3^2T_4^3+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
50	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3T_4+T_5^3\rangle$	$(1,2,1,2,1)$	48
51	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2+T_3^2T_4+T_5^4\rangle$	$(1,2,1,2,1)$	27
52	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3T_4^2+T_5^5\rangle$	$(1,2,1,2,1)$	10
53	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3^3T_4+T_5^5\rangle$	$(1,2,1,2,1)$	10
54	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2+T_3^3T_4+T_5^5\rangle$	$(1,2,1,2,1)$	10
55	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3^4T_4+T_5^6\rangle$	$(1,2,1,2,1)$	$3/2$
56	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2+T_3^2T_4+T_5^2\rangle$	$(1,2,1,2,2)$	32
57	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^2+T_3^4T_4+T_5^3\rangle$	$(1,2,1,2,2)$	6
58	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3^4T_4+T_5^3\rangle$	$(1,2,1,2,2)$	6
59	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3^4T_4+T_5^2\rangle$	$(1,2,1,2,3)$	$27/2$
60	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3+T_3^2T_4^3+T_5^2\rangle$	$(1,2,1,2,4)$	4
61	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3+T_3^6T_4+T_5^2\rangle$	$(1,2,1,2,4)$	4
62	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^6T_2+T_3^6T_4+T_5^2\rangle$	$(1,2,1,2,4)$	4
63	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^3+T_3^4T_4^3+T_5^2\rangle$	$(1,2,1,2,5)$	$1/2$
64	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3^4T_4^3+T_5^2\rangle$	$(1,2,1,2,5)$	$1/2$
65	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3^8T_4+T_5^2\rangle$	$(1,2,1,2,5)$	$1/2$
66	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2+T_3T_4+T_5^4\rangle$	$(1,2,2,2,1)$	32
67	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3T_4^2+T_5^6\rangle$	$(1,2,2,2,1)$	6

68	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3T_4^2+T_5^2\rangle$	$(1,2,2,2,3)$	16
69	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^3+T_3T_4^4+T_5^2\rangle$	$(1,2,2,2,5)$	2
70	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^3+T_3^2T_4^3+T_5^2\rangle$	$(1,2,2,2,5)$	2
71	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3T_4^4+T_5^2\rangle$	$(1,2,2,2,5)$	2
72	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3^2T_4^3+T_5^2\rangle$	$(1,2,2,2,5)$	2
73	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^{10}+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$
74	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2T_3^9+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$
75	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3T_3^8+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$
76	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^4T_3^7+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$
77	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5T_3^6+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$
78	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3T_3^7+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$
79	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^5T_3^5+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$
80	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^4T_3^5+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$
81	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^2+T_4^3+T_5^2\rangle$	$(1,1,2,2,3)$	$27/2$
82	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3T_3+T_4^3+T_5^2\rangle$	$(1,1,2,2,3)$	$27/2$
83	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^2T_3+T_4^3+T_5^2\rangle$	$(1,1,2,2,3)$	$27/2$
84	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^4+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
85	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3T_3^3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
86	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5T_3^2+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
87	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^7T_3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
88	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^2T_3^3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
89	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^6T_3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
90	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^3T_3^2+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
91	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^5T_3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
92	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^4T_3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
93	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^5+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
94	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3T_3^4+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
95	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5T_3^3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
96	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^7T_3^2+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
97	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^9T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
98	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4T_3^3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2

99	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^8T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
100	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^5T_3^2+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
101	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^7T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
102	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^6T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
103	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^5T_2^5T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
104	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2T_3+T_4^3+T_5^2\rangle$	$(1,2,2,2,3)$	16
105	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2T_3^3+T_4^5+T_5^2\rangle$	$(1,2,2,2,5)$	2
106	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2T_3^2+T_4^5+T_5^2\rangle$	$(1,2,2,2,5)$	2
107	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^6T_2T_3+T_4^5+T_5^2\rangle$	$(1,2,2,2,5)$	2

The varieties no. 2,3 and 25, 26 are rational degenerations of quasismooth varieties from the list in [16]. In [11] the non-rationality of a general (quasismooth) element of the corresponding family was proved.

The varieties listed so far might suggest that we always obtain only one relation in the Cox ring. We discuss now some examples, showing that for a Picard index big enough, we need in general more than one relation, where this refers always to a presentation as in Theorem 1.9 (ii).

Example 3.4. A Fano $\mathbb{K}^*$-surface X with $\mathrm{Cl}(X)=\mathbb{Z}$ such that the Cox ring $\mathcal{R}(X)$ needs two relations. Consider the $\mathbb{Z}$-graded ring

$$R \quad = \quad \mathbb{K}[T_{01},T_{02},T_{11},T_{21},T_{31}]/\langle g_0,g_1\rangle,$$

where the degrees of $T_{01},T_{02},T_{11},T_{21},T_{31}$ are $29,1,6,10,15$, respectively, and the relations g_0,g_1 are given by

$$g_0 \;:=\; T_{01}T_{02}+T_{11}^5+T_{21}^3, \qquad\qquad g_1 \;:=\; \alpha_{23}T_{11}^5+\alpha_{31}T_{21}^3+\alpha_{12}T_{31}^2$$

Then R is the Cox ring of a Fano $\mathbb{K}^*$-surface. Note that the Picard index is given by $[\mathrm{Cl}(X):\mathrm{Pic}(X)]=\mathrm{lcm}(29,1)=29$.

Proposition 3.5. *Let X be a non-toric Fano surface with an effective $\mathbb{K}^*$-action such that $\mathrm{Cl}(X)\cong\mathbb{Z}$ and $[\mathrm{Cl}(X):\mathrm{Pic}(X)]<29$ hold. Then the Cox ring of X is of the form*

$$\mathcal{R}(X) \quad\cong\quad \mathbb{K}[T_1,\dots,T_4]/\langle T_1^{l_1}T_2^{l_2}+T_3^{l_3}+T_4^{l_4}\rangle.$$

Proof. The Cox ring $\mathcal{R}(X)$ is as in Theorem 1.9, and, in the notation used there, we have $n_0+\ldots+n_r+m=2+r$. This leaves us with the possibilities $n_0=m=1$ and $n_0=2$, $m=0$. In the first case, Proposition 2.9 tells us that the Picard index of X is at least 30.

So, consider the case $n_0 = 2$ and $m = 0$. Then, according to Theorem 1.9, the Cox ring $\mathcal{R}(X)$ is $\mathbb{K}[T_{01}, T_{02}, T_1 \ldots, T_r]$ divided by relations

$$g_{0,1,2} = T_{01}^{l_{01}} T_{02}^{l_{02}} + T_1^{l_1} + T_2^{l_2}, \; g_{i,i+1,i+2} = \alpha_{i+1,i+2} T_i^{l_i} + \alpha_{i+2,i} T_{i+1}^{l_{i+1}} + \alpha_{i,i+1} T_{i+2}^{l_{i+2}},$$

where $1 \leq i \leq r-2$. We have to show that $r = 2$ holds. Set $\mu := [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ and let $\gamma \in \mathbb{Z}$ denote the degree of the relations. Then we have $\gamma = w_i l_i$ for $1 \leq i \leq r$, where $w_i := \deg T_i$. With $w_{0i} := \deg T_{0i}$, Proposition 2.5 gives us

$$(r-1)\gamma \quad < \quad w_{01} + w_{02} + w_1 + \ldots + w_r.$$

We claim that w_{01} and w_{02} are coprime. Otherwise they had a common prime divisor p. This p divides $\gamma = l_i w_i$. Since $l_1, \ldots, l_r$ are pairwise coprime, p divides at least $r-1$ of the weights $w_1, \ldots, w_r$. This contradicts the Cox ring condition that any $r+1$ of the $r+2$ weights generate the class group $\mathbb{Z}$. Thus, w_{01} and w_{02} are coprime and we obtain

$$\mu \;\geq\; \mathrm{lcm}(\mathrm{w}_{01}, \mathrm{w}_{02}) \;=\; \mathrm{w}_{01} \cdot \mathrm{w}_{02} \;\geq\; \mathrm{w}_{01} + \mathrm{w}_{02} - 1.$$

Now assume that $r \geq 3$ holds. Then we can conclude

$$2\gamma \;<\; w_{01} + w_{02} + w_1 + w_2 + w_3 \;\leq\; \mu + 1 + \gamma\left(\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3}\right)$$

Since the numbers l_i are pairwise coprime, we obtain $l_1 \geq 5$, $l_2 \geq 3$ and $l_3 \geq 2$. Moreover, $l_i w_i = l_j w_j$ implies $l_i \mid w_j$ and hence $l_1 l_2 l_3 \mid \gamma$. Thus, we have $\gamma \geq 30$. Plugging this in the above inequality gives

$$\mu \;\geq\; \gamma\left(2 - \frac{1}{l_1} - \frac{1}{l_2} - \frac{1}{l_3}\right) - 1 \;=\; 29.$$

□

The Fano assumption is essential in this result; if we omit it, then we may even construct locally factorial surfaces with a Cox ring that needs more then one relation.

Example 3.6. A locally factorial $\mathbb{K}^*$-surface X with $\mathrm{Cl}(X) = \mathbb{Z}$ such that the Cox ring $\mathcal{R}(X)$ needs two relations. Consider the $\mathbb{Z}$-graded ring

$$R \quad = \quad \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}, T_{31}]/\langle g_0, g_1 \rangle,$$

where the degrees of $T_{01}, T_{02}, T_{11}, T_{21}, T_{31}$ are $1, 1, 6, 10, 15$, respectively, and the relations g_0, g_1 are given by

$$g_0 \;:=\; T_{01}^7 T_{02}^{23} + T_{11}^5 + T_{21}^3, \qquad\qquad g_1 \;:=\; \alpha_{23} T_{11}^5 + \alpha_{31} T_{21}^3 + \alpha_{12} T_{31}^2$$

Then R is the Cox ring of a non Fano $\mathbb{K}^*$-surface X of Picard index one, i.e, X is locally factorial.

For non-toric Fano threefolds X with an effective 2-torus action $\mathrm{Cl}(X) \cong \mathbb{Z}$, the classifications 3.2 and 3.3 show that for Picard indices one and two we only obtain hypersurfaces as Cox rings. The following example shows that this stops at Picard index three.

Example 3.7. A Fano threefold X with $\mathrm{Cl}(X) = \mathbb{Z}$ and a 2-torus action such that the Cox ring $\mathcal{R}(X)$ needs two relations. Consider

$$R \quad = \quad \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, T_{31}]/\langle g_0, g_1\rangle$$

where the degrees of $T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, T_{31}$ are $1, 1, 3, 3, 2, 3$, respectively, and the relations are given by

$$g_0 \ = \ T_{01}^5 T_{02} + T_{11}T_{12} + T_{21}^3, \qquad g_1 \ = \ \alpha_{23}T_{11}T_{12} + \alpha_{31}T_{21}^3 + \alpha_{12}T_{31}^2.$$

Then R is the Cox ring of a Fano threefold with a 2-torus action. Note that the Picard index is given by

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \ = \ \mathrm{lcm}(1, 1, 3, 3) \ = \ 3.$$

Finally, we turn to locally factorial Fano fourfolds. Here we observe more than one relation in the Cox ring even in the locally factorial case.

Theorem 3.8. *Let X be a four-dimensional locally factorial non-toric Fano variety with an effective three torus action such that $\mathrm{Cl}(X) = \mathbb{Z}$ holds. Then its Cox ring is precisely one of the following.*

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_6)$	$(-K_X)^4$
1	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^5 + T_3^3 + T_4^2\rangle$	$(1, 1, 2, 3, 1, 1)$	81
2	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^9 + T_3^2 + T_4^5\rangle$	$(1, 1, 2, 5, 1, 1)$	1
3	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1^3T_2^7 + T_3^2 + T_4^5\rangle$	$(1, 1, 2, 5, 1, 1)$	1
4	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2T_3^4 + T_4^3 + T_5^2\rangle$	$(1, 1, 1, 2, 3, 1)$	81
5	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^2T_3^3 + T_4^3 + T_5^2\rangle$	$(1, 1, 1, 2, 3, 1)$	81
6	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2T_3^8 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
7	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^2T_3^7 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
8	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^3T_3^6 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
9	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^4T_3^5 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
10	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1^2T_2^3T_3^5 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
11	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1^3T_2^3T_3^4 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
12	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle$	$(1, 1, 1, 1, 1, 1)$	512
13	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^2 + T_3T_4^2 + T_5^3\rangle$	$(1, 1, 1, 1, 1, 1)$	243

14	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^3+T_3T_4^3+T_5^4\rangle$	$(1,1,1,1,1,1)$	64
15	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^4+T_3T_4^4+T_5^5\rangle$	$(1,1,1,1,1,1)$	5
16	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^4+T_3^2T_4^3+T_5^5\rangle$	$(1,1,1,1,1,1)$	5
17	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1^2T_2^3+T_3^2T_4^3+T_5^5\rangle$	$(1,1,1,1,1,1)$	5
18	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^3+T_3T_4^3+T_5^2\rangle$	$(1,1,1,1,2,1)$	162
19	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^5+T_3T_4^5+T_5^3\rangle$	$(1,1,1,1,2,1)$	3
20	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^5+T_3^2T_4^4+T_5^3\rangle$	$(1,1,1,1,2,1)$	3
21	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^5+T_3T_4^5+T_5^2\rangle$	$(1,1,1,1,3,1)$	32
22	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^5+T_3^3T_4^3+T_5^2\rangle$	$(1,1,1,1,3,1)$	32
23	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^7+T_3T_4^7+T_5^2\rangle$	$(1,1,1,1,4,1)$	2
24	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^7+T_3^3T_4^5+T_5^2\rangle$	$(1,1,1,1,4,1)$	2
25	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1^3T_2^5+T_3^3T_4^5+T_5^2\rangle$	$(1,1,1,1,4,1)$	2
26	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3T_4^3+T_5^3+T_6^2\rangle$	$(1,1,1,1,2,3)$	81
27	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^2T_4^2+T_5^3+T_6^2\rangle$	$(1,1,1,1,2,3)$	81
28	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3T_4^7+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
29	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^2T_4^6+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
30	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^3T_4^5+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
31	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^4T_4^4+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
32	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^2T_4^5+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
33	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^3T_4^4+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
34	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^3T_3^3T_4^3+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
35	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1^2T_2^2T_3^3T_4^3+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
36	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3+T_4T_5^2+T_6^3\rangle$	$(1,1,1,1,1,1)$	243
37	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^2+T_4T_5^3+T_6^4\rangle$	$(1,1,1,1,1,1)$	64
38	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^3+T_4T_5^4+T_6^5\rangle$	$(1,1,1,1,1,1)$	5
39	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^3+T_4^2T_5^3+T_6^5\rangle$	$(1,1,1,1,1,1)$	5
40	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^2+T_4T_5^4+T_6^5\rangle$	$(1,1,1,1,1,1)$	5
41	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^2+T_4^2T_5^3+T_6^5\rangle$	$(1,1,1,1,1,1)$	5
42	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^2+T_4T_5^3+T_6^2\rangle$	$(1,1,1,1,1,2)$	162
43	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^4+T_4T_5^5+T_6^3\rangle$	$(1,1,1,1,1,2)$	3
44	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^4+T_4^2T_5^4+T_6^3\rangle$	$(1,1,1,1,1,2)$	3

45	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^3+T_4T_5^5+T_6^3\rangle$	$(1,1,1,1,1,2)$	3
46	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^3+T_4^2T_5^4+T_6^3\rangle$	$(1,1,1,1,1,2)$	3
47	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1^2T_2^2T_3^2+T_4T_5^5+T_6^3\rangle$	$(1,1,1,1,1,2)$	3
48	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^3+T_4^3T_5^3+T_6^2\rangle$	$(1,1,1,1,1,3)$	32
49	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^3+T_4T_5^5+T_6^2\rangle$	$(1,1,1,1,1,3)$	32
50	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^4+T_4^3T_5^3+T_6^2\rangle$	$(1,1,1,1,1,3)$	32
51	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^4+T_4T_5^5+T_6^2\rangle$	$(1,1,1,1,1,3)$	32
52	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^6+T_4T_5^7+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
53	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^6+T_4^3T_5^5+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
54	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^5+T_4T_5^7+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
55	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^5+T_4^3T_5^5+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
56	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^3T_3^4+T_4T_5^7+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
57	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^3T_3^4+T_4^3T_5^5+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
58	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1^2T_2^3T_3^3+T_4T_5^7+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
59	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1^2T_2^3T_3^3+T_4^3T_5^5+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
60	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2+T_3T_4+T_5T_6\rangle$	$(1,1,1,1,1,1)$	512
61	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle$	$(1,1,1,1,1,1)$	243
62	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^3+T_3T_4^3+T_5T_6^3\rangle$	$(1,1,1,1,1,1)$	64
63	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^3+T_3T_4^3+T_5^2T_6^2\rangle$	$(1,1,1,1,1,1)$	64
64	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^4+T_3T_4^4+T_5T_6^4\rangle$	$(1,1,1,1,1,1)$	5
65	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^4+T_3T_4^4+T_5^2T_6^3\rangle$	$(1,1,1,1,1,1)$	5
66	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^4+T_3^2T_4^3+T_5^2T_6^3\rangle$	$(1,1,1,1,1,1)$	5
67	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1^2T_2^3+T_3^2T_4^3+T_5^2T_6^3\rangle$	$(1,1,1,1,1,1)$	5
68	$\mathbb{K}[T_1,\dots,T_7]/\left\langle \begin{smallmatrix} T_1T_2+T_3T_4+T_5T_6, \\ \alpha T_3T_4+T_5T_6+T_7^2 \end{smallmatrix}\right\rangle$	$(1,1,1,1,1,1,1)$	324
69	$\mathbb{K}[T_1,\dots,T_7]/\left\langle \begin{smallmatrix} T_1T_2^2+T_3T_4^2+T_5T_6^2, \\ \alpha T_3T_4^2+T_5T_6^2+T_7^3 \end{smallmatrix}\right\rangle$	$(1,1,1,1,1,1,1)$	9

where in the last two rows of the table the parameter α can be any element from $\mathbb{K}^* \setminus \{1\}$.

By the result of [26], the singular quintics of this list are rational degenerations of smooth non-rational Fano fourfolds.

4 Geometry of the locally factorial threefolds

In this section, we take a closer look at the (factorial) singularities of the Fano varieties X listed in Theorem 3.2. Recall that the discrepancies of a resolution $\varphi\colon \widetilde{X} \to X$ of a singularity are the coefficients of $K_{\widetilde{X}} - \varphi^* K_X$, where K_X and $K_{\widetilde{X}}$ are canonical divisors such that $K_{\widetilde{X}} - \varphi^* K_X$ is supported on the exceptional locus of φ. A resolution is called crepant, if its discrepancies vanish and a singularity is called canonical (terminal), if it admits a resolution with nonnegative (positive) discrepancies. By a relative minimal model we mean a projective morphism $\widetilde{X} \to X$ such that $\widetilde{X}$ has at most terminal singularities and its relative canonical divisor is relatively nef.

Theorem 4.1. *For the nine 3-dimensional Fano varieties listed in Theorem 3.2, we have the following statements.*

(i) *No. 4 is a smooth quadric in* $\mathbb{P}^4$.

(ii) *Nos. 1,3,5,7 and 9 are singular with only canonical singularities and all admit a crepant resolution.*

(iii) *Nos. 6 and 8 are singular with non-canonical singularities but admit a smooth relative minimal model.*

(iv) *No. 2 is singular with only canonical singularities, one of them of type* $\mathbf{cA_1}$*, and admits only a singular relative minimal model.*

The Cox ring of the relative minimal model $\widetilde{X}$ *as well as the the Fano degree of* X *itself are given in the following table.*

No.	$\mathcal{R}(\widetilde{X})$	$(-K_X)^3$
1	$\mathbb{K}[T_1,\dots,T_{14}]/\langle T_1T_2T_3^2T_4^3T_5^4T_6^5 + T_7^3T_8^2T_9 + T_{10}^2T_{11}\rangle$	8
2	$\mathbb{K}[T_1,\dots,T_9]/\langle T_1T_2T_3^2T_4^4 + T_5T_6^2T_7^3 + T_8^2\rangle$	8
3	$\mathbb{K}[T_1,\dots,T_8]/\langle T_1T_2^2T_3^3 + T_4T_5^3 + T_6T_7^2\rangle$	8
4	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle$	54
5	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2 + T_3T_4^2 + T_5^3T_6\rangle$	24
6	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^3 + T_3T_4^3 + T_5^4T_6\rangle$	4
7	$\mathbb{K}[T_1,\dots,T_7]/\langle T_1T_2^3 + T_3T_4^3 + T_5^2T_6\rangle$	16
8	$\mathbb{K}[T_1,\dots,T_7]/\langle T_1T_2^5 + T_3T_4^5 + T_5^2T_6\rangle$	2
9	$\mathbb{K}[T_1,\dots,T_{46}]/\left\langle \begin{array}{c} T_1T_2T_3T_4^2T_5^2T_6^3T_7^3T_8^4T_9^4T_{10}^5 + \\ + T_{11}\cdots T_{18}T_{19}^2\cdots T_{24}^2T_{25}^3T_{26}^3 + T_{27}\cdots T_{32}T_{33}^2 \end{array} \right\rangle$	2

For the proof, it is convenient to work in the language of polyhedral divisors introduced in [1] and [2]. As we are interested in rational varieties with a complexity one torus action, we only have to consider polyhedral divisors on the projective line $Y = \mathbb{P}^1$. This considerably simplifies the general definitions and allows us to give a short summary. In the sequel, $N \cong \mathbb{Z}^n$ denotes a lattice and $M = \mathrm{Hom}(N, \mathbb{Z})$ its dual. For the associated rational vector spaces we write $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$. A *polyhedral divisor* on the projective line $Y := \mathbb{P}^1$ is a formal sum

$$\mathcal{D} \quad = \quad \sum_{y \in Y} \mathcal{D}_y \cdot y,$$

where the coefficients $\mathcal{D}_y \subseteq N_{\mathbb{Q}}$ are (possibly empty) convex polyhedra all sharing the same tail (i.e. recession) cone $\mathcal{D}_Y = \sigma \subseteq N_{\mathbb{Q}}$, and only finitely many $\mathcal{D}_y$ differ from σ. The *locus* of $\mathcal{D}$ is the open subset $Y(\mathcal{D}) \subseteq Y$ obtained by removing all points $y \subseteq Y$ with $\mathcal{D}_y = \emptyset$. For every $u \in \sigma^\vee \cap M$ we have the *evaluation*

$$\mathcal{D}(u) \quad := \quad \sum_{y \in Y} \min_{v \in \mathcal{D}_y} \langle u, v \rangle \cdot y,$$

which is a usual rational divisor on $Y(\mathcal{D})$. We call the polyhedral divisor $\mathcal{D}$ on Y *proper* if $\deg \mathcal{D} \subsetneq \sigma$ holds, where the *polyhedral degree* is defined by

$$\deg \mathcal{D} \quad := \quad \sum_{y \in Y} \mathcal{D}_y.$$

Every proper polyhedral divisor $\mathcal{D}$ on Y defines a normal affine variety $X(\mathcal{D})$ of dimension $\mathrm{rk}\,(N) + 1$ coming with an effective action of the torus $T = \mathrm{Spec}\, \mathbb{K}[M]$: set $X(\mathcal{D}) := \mathrm{Spec}\, A(\mathcal{D})$, where

$$A(\mathcal{D}) \;:=\; \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y(\mathcal{D}), \mathcal{O}(\mathcal{D}(u))) \;\subseteq\; \bigoplus_{u \in M} \mathbb{K}(Y) \cdot \chi^u.$$

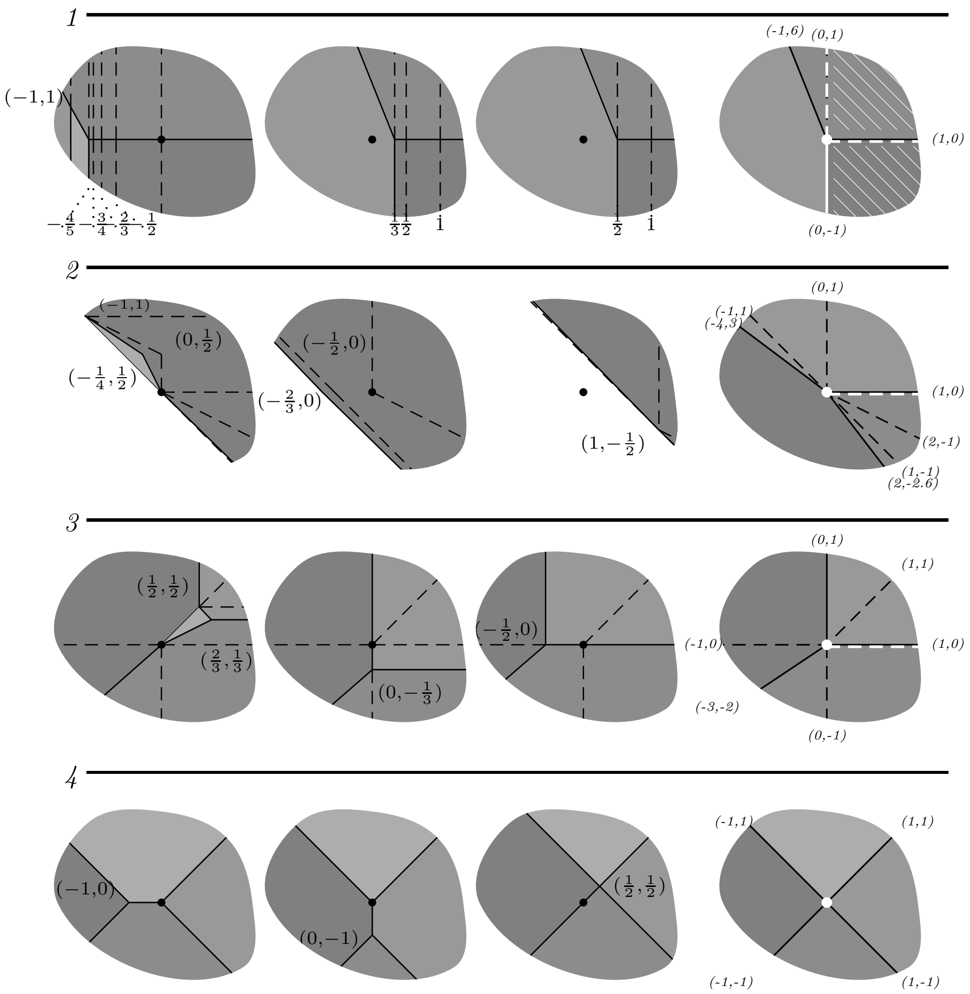
A *divisorial fan*, is a finite set Ξ of polyhedral divisors $\mathcal{D}$ on Y, all having their polyhedral coefficients $\mathcal{D}_y$ in the same $N_{\mathbb{Q}}$ and fulfilling certain compatibility conditions, see [2]. In particular, for every point $y \in Y$, the *slice*

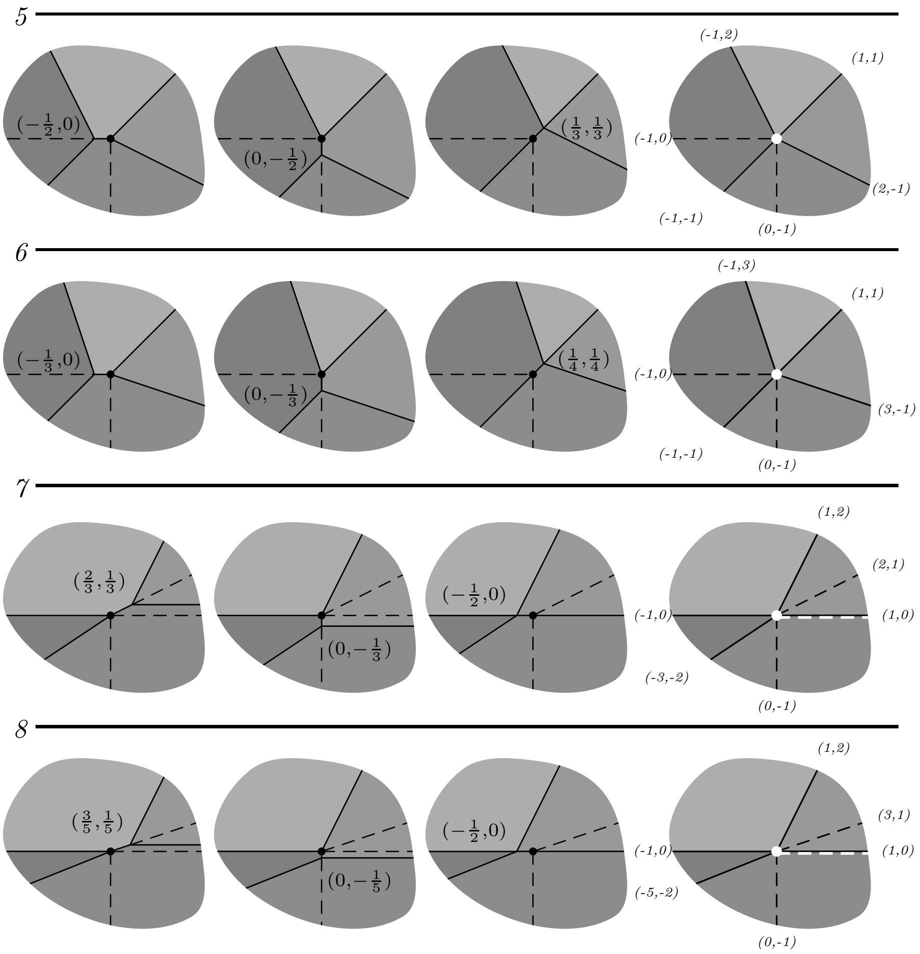
$$\Xi_y \quad := \quad \{\mathcal{D}_y;\ \mathcal{D} \in \Xi\}$$

must be a polyhedral subdivision. The *tail fan* is the set Ξ_Y of the tail cones $\mathcal{D}_Y$ of the $\mathcal{D} \in \Xi$; it is a fan in the usual sense. Given a divisorial fan Ξ, the affine varieties $X(\mathcal{D})$, where $\mathcal{D} \in \Xi$, glue equivariantly together to a normal variety $X(\Xi)$, and we obtain every rational normal variety with a complexity one torus action this way.

Smoothness of $X = X(\Xi)$ is checked locally. For a proper polyhedral divisor $\mathcal{D}$ on Y, we infer the following from [28, Theorem 3.3]. If $Y(\mathcal{D})$ is affine, then $X(\mathcal{D})$ is smooth if and only if $\operatorname{cone}(\{1\} \times \mathcal{D}_y) \subseteq \mathbb{Q} \times N_{\mathbb{Q}}$, the convex, polyhedral cone generated by $\{1\} \times \mathcal{D}_y$, is regular for every $y \in Y(\mathcal{D})$. If $Y(\mathcal{D}) = Y$ holds, then $X(\mathcal{D})$ is smooth if and only if there are $y, z \in Y$ such that $\mathcal{D} = \mathcal{D}_y y + \mathcal{D}_z z$ holds and $\operatorname{cone}(\{1\} \times \mathcal{D}_y) + \operatorname{cone}(\{-1\} \times \mathcal{D}_z)$ is a regular cone in $\mathbb{Q} \times N_{\mathbb{Q}}$. Similarly to toric geometry, singularities of $X(\mathcal{D})$ are resolved by means of subdividing $\mathcal{D}$. This means to consider divisorial fans Ξ such that for any $y \in Y$, the slice Ξ_y is a subdivision of $\mathcal{D}_y$. Such a Ξ defines a dominant morphism $X(\Xi) \to X(\mathcal{D})$ and a slight generalization of [2, Thm. 7.5.] yields that this morphism is proper.

Proposition 4.2. *The 3-dimensional Fano varieties No. 1-8 listed in Theorem 3.2 and their relative minimal models arise from divisorial fans having the following slices and tail cones.*





The above table should be interpreted as follows. The first three pictures in each row are the slices at 0, 1 and ∞ and the last one is the tail fan. The divisorial fan of the fano variety itself is given by the solid polyhedra in the pictures. Here, all polyhedra of the same gray scale belong to the same polyhedral divisor. The subdivisions for the relative minimal models are sketched with dashed lines. In general, polyhedra with the same tail cone belong all to a unique polyhedral divisor with complete locus. For the white cones inside the tail fan we have another rule: for every polyhedron $\Delta \in \Xi_y$ with the given white cone as its tail there is a polyhedral divisor $\Delta \cdot y + \emptyset \cdot z \in \Xi$, with $z \in \{0, 1, \infty\} \setminus \{y\}$. Here, different choices of z lead to isomorphic varieties, only the affine covering given by the $X(\mathcal{D})$ changes.

In order to prove Theorem 4.1, we also have to understand invariant divisors on $X = X(\Xi)$ in terms of Ξ, see [15, Prop. 4.11 and 4.12] for details. A first type of invariant prime divisors, is in bijection $D_{y,v} \leftrightarrow (y, v)$ with the vertices (y, v), where $y \in Y$ and $v \in \Xi_y$ is of dimension zero. The order of the generic isotropy group along $D_{y,v}$ equals the minimal positive integer $\mu(v)$ with $\mu(v)v \in N$. A second type of invariant prime divisors, is in $D_\varrho \leftrightarrow \varrho$ with the extremal rays $\varrho \in \Xi_Y$, where a ray $\varrho \in \Xi_Y$ is called extremal if there is a $\mathcal{D} \in \Xi$ such that $\varrho \subseteq \mathcal{D}_Y$ and $\deg \mathcal{D} \cap \varrho = \emptyset$ holds. The set of extremal rays is denoted by $\Xi_Y^\times$. The divisor of a semi-invariant function $f \cdot \chi^u \in \mathbb{K}(X)$ is then given by

$$\operatorname{div}(f \cdot \chi^u) \;=\; -\sum_{y \in Y} \sum_{v \in \Xi_y^{(0)}} \mu(v) \cdot (\langle v, u\rangle + \operatorname{ord}_y f) \cdot D_{y,v} \;-\; \sum_{\varrho \in \Xi_Y^\times} \langle n_\varrho, u\rangle \cdot D_\varrho.$$

Next we describe the canonical divisor. Choose a point $y_0 \in Y$ such that $\Xi_{y_0} = \Xi_Y$ holds. Then a canonical divisor on $X = X(\Xi)$ is given by

$$K_X \;=\; (s-2) \cdot y_0 \;-\; \sum_{\Xi_y \neq \Xi_Y} \sum_{v \in \Xi_i^{(0)}} D_{y,v} \;-\; \sum_{\varrho \in \Xi_Y^\times} E_\varrho.$$

Proposition 4.3. *Let $\mathcal{D}$ be a proper polyhedral divisor with $Y(\mathcal{D}) = \mathbb{P}_1$, let Ξ be a refinement of $\mathcal{D}$ and denote by $y_1, \ldots, y_s \in Y$ the points with $\Xi_{y_i} \neq \Xi_Y$. Then the associated morphism $\varphi\colon X(\Xi) \to X(\mathcal{D})$ satisfies the following.*

(i) *The prime divisors in the exceptional locus of φ are the divisors $D_{y_i,v}$ and D_ϱ corresponding to $v \in \Xi_{y_i}^{(0)} \setminus \mathcal{D}_{y_i}^{(0)}$ and $\varrho \in \Xi_Y^\times \setminus \mathcal{D}^\times$ respectively.*

(ii) *Then the discrepancies along the prime divisors $D_{y_i,v}$ and D_ϱ of (i) are computed as*

$$d_{y_i,v} \;=\; -\mu(v) \cdot (\langle v, u'\rangle + \alpha_y) - 1, \qquad\qquad d_\varrho \;=\; -\langle v_\varrho, u'\rangle - 1,$$

where the numbers α_i are determined by

$$\left(\begin{array}{ccccc}
-1 & -1 & \ldots & -1 & 0 \\
\hline
\mu(v_1^1) & 0 & \ldots & 0 & \mu(v_1^1)v_1^1 \\
\vdots & \vdots & & \vdots & \vdots \\
\mu(v_1^{r_1}) & 0 & \ldots & 0 & \mu(v_1^{r_1})v_1^{r_1} \\
 & & \ddots & & \\
0 & 0 & \ldots & \mu(v_s^1) & \mu(v_s^1)v_s^1 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & \mu(v_s^{r_s}) & \mu(v_s^{r_s})v_s^{r_s} \\
\hline
0 & 0 & \ldots & 0 & n_{\varrho_1} \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & 0 & n_{\varrho_r}
\end{array}\right) \cdot \begin{pmatrix} \alpha_{y_1} \\ \vdots \\ \alpha_{y_s} \\ u \end{pmatrix} = \left(\begin{array}{c} 2-s \\ 1 \\ \vdots \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{array}\right)$$

Proof. The first claim is obvious by the characterization of invariant prime divisors. For the second claim note that by [24, Theorem 3.1] every Cartier divisor on $X(\mathcal{D})$ is principal. Hence, we may assume

$$\ell \cdot K_X \;=\; \operatorname{div}(f \cdot \chi^u), \qquad\qquad \operatorname{div}(f) \;=\; \sum_y \alpha_y \cdot y.$$

Then our formulæ for $\operatorname{div}(f \cdot \chi^u)$ and K_X provide a row for every vertex $v_i^j \in \Xi_{y_i}$, $i = 0, \dots, s$, and for every extremal ray $\varrho_i \in \Xi^\times$, and $\ell^{-1}(\alpha, u)$ is the (unique) solution of the above system. ☐

Note, that in the above Proposition, the variety $X(\mathcal{D})$ is $\mathbb{Q}$-Gorenstein if and only if the linear system of equations has a solution.

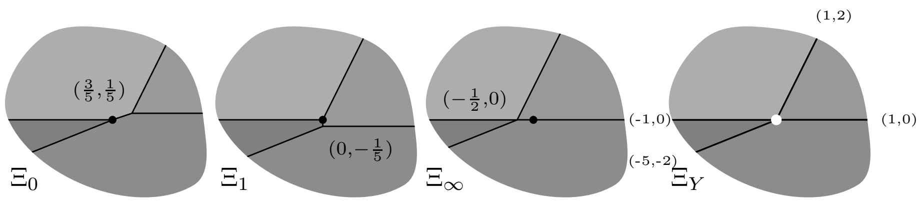
Proof of Theorem 4.1 and Proposition 4.2. We exemplarily discuss variety number eight. Recall that its Cox ring is given as

$$\mathcal{R}(X) \quad = \quad \mathbb{K}[T_1, \dots, T_5]/(T_1T_2^5 + T_3T_4^5 + T_5^2)$$

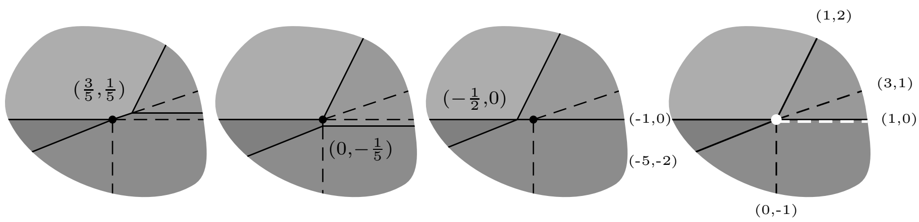
with the degrees $1, 1, 1, 1, 3$. In particular, X is a hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 1, 3)$, and the self-intersection of the anti-canonical divisor can be calculated as

$$(-K_X^3) \ = \ 6 \cdot \frac{(1+1+1+1+3-6)^3}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 3} \ = \ 2.$$

The embedding $X \subseteq \mathbb{P}(1, 1, 1, 1, 3)$ is equivariant, and thus we can use the technique described in [1, Sec. 11] to calculate a divisorial fan Ξ for X. The result is the following divisorial fan; we draw its slices and indicate the polyhedral divisors with affine locus by colouring their tail cones $\mathcal{D}_Y \in \Xi_Y$ white:



One may also use [15, Cor. 4.9.] to verify that Ξ is the right divisorial fan: it computes the Cox ring in terms of Ξ, and, indeed, we obtain again $\mathcal{R}(X)$. Now we subdivide and obtain a divisorial fan having the refined slices as indicated in the following picture.



Here, the white ray $\mathbb{Q}_{\geq 0} \cdot (1, 0)$ indicates that the polyhedral divisors with that tail have affine loci. According to [15, Cor. 4.9.], the corresponding Cox ring is given by

$$\mathcal{R}(\widetilde{X}) \quad = \quad \mathbb{K}[T_1, \dots, T_7]/\langle T_1T_2^5 + T_3T_4^5 + T_5^2T_6 \rangle.$$

We have to check that $\widetilde{X}$ is smooth. Let us do this explicitly for the affine chart defined by the polyhedral divisor $\mathcal{D}$ with tail cone $\mathcal{D}_Y = \operatorname{cone}((1,2),(3,1))$. Then $\mathcal{D}$ is given by

$$\mathcal{D} \quad = \quad \left(\left(\frac{3}{5},\frac{1}{5}\right)+\sigma\right)\cdot\{0\} \;+\; \left(\left[-\frac{1}{2},0\right]\times 0+\sigma\right)\cdot\{\infty\}.$$

Thus, $\operatorname{cone}(\{1\}\times\mathcal{D}_0)+\operatorname{cone}(\{-1\}\times\mathcal{D}_\infty)$ is generated by $(5,3,1)$, $(-2,-1,0)$ and $(-1,0,0)$; in particular, it is a regular cone. This implies smoothness of the affine chart $X(\mathcal{D})$. Furthermore, we look at the affine charts defined by the polyhedral divisors $\mathcal{D}$ with tail cone $\mathcal{D}_Y = \operatorname{cone}(1,0)$. Since they have affine locus, we have to check $\operatorname{cone}(\{1\}\times\mathcal{D}_y)$, where $y \in Y$. For $y \neq 0,1$, we have $\mathcal{D}_y = \mathcal{D}_Y$. In this case, $\operatorname{cone}(\{1\}\times\mathcal{D}_y)$ is generated by $(1,1,0)$, $(0,1,0)$ and thus is regular. For $y = 0$, we obtain that $\operatorname{cone}(\{1\}\times\mathcal{D}_y)$ is generated by $(5,3,1)$, $(1,0,0)$, $(0,1,0)$ and this is regular. For $y = 1$ we get the same result. Hence, the polyhedral divisors with tail cone $\mathcal{D}_y = \operatorname{cone}(1,0)$ give rise to smooth affine charts.

Now we compute the discrepancies according to Proposition 4.3. The resolution has two exceptional divisors $D_{\infty,\mathbf{0}}$ and $E_{(1,0)}$. We work in the chart defined by the divisor $\mathcal{D} \in \Xi$ with tail cone $\mathcal{D}_Y = \operatorname{cone}((1,2),(1,0))$. The resulting system of linear equations and its unique solution are given by

$$\left(\begin{array}{ccccc|c} -1 & -1 & -1 & 0 & 0 & -1 \\ 5 & 0 & 0 & 3 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right), \qquad \left(\begin{array}{c} \alpha_0 \\ \alpha_1 \\ \alpha_\infty \\ \hline u \end{array}\right) = \left(\begin{array}{c} 0 \\ 1 \\ 0 \\ \hline -1 \\ 4 \end{array}\right).$$

The formula for the discrepancies yields $d_{\infty,\mathbf{0}} = -1$ and $d_{(1,0)} = -2$. In particular, X has non-canonical singularities. By a criterion from [24, Sec. 3.4.], we know that $D_{\infty,\mathbf{0}} + 2\cdot E_{(1,0)}$ is a nef divisor. It follows that $\widetilde{X}$ is a minimal model over X. $\square$

References

[1] K. Altmann, J. Hausen: Polyhedral divisors and algebraic torus actions. Math. Ann. 334 (2006), no. 3, 557–607.

[2] K. Altmann, J. Hausen, H. Süß: Gluing Affine Torus Actions Via Divisorial Fans. Transformation Groups 13 (2008), no. 2, 215–242.

[3] V.V. Batyrev: Toric Fano threefolds. Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981), no. 4, 704–717, 927.

[4] V.V. Batyrev: On the classification of toric Fano 4-folds. J. Math. Sci. New York 94, 1021–1050 (1999).

[5] F. Berchtold, J. Hausen: Homogeneous coordinates for algebraic varieties. J. Algebra 266 (2003), no. 2, 636–670.

[6] Cheltsov, Ivan; Park, Jihun Sextic double solids. In. Bogomolov, Fedor (ed.) et al.: Cohomological and geometric approaches to rationality problems. New Perspectives. Boston, MA: Birkhäuser. Progress in Mathematics 282, 75–132 (2010).

[7] J.J. Chen, J.A. Chen, M. Chen: On quasismooth weighted complete intersections. Preprint, arXiv:0908.1439.

[8] C.H. Clemens, P.A. Griffiths: The intermediate Jacobian of the cubic threefold. Ann. Math. (2) 95 (1972), 281–356.

[9] A. Corti: Singularities of linear systems and 3-fold birational geometry. Explicit birational geometry of 3-folds, 259 -312, London Math. Soc. Lecture Note Ser., 281, Cambridge Univ. Press, Cambridge, 2000.

[10] A. Corti, M. Mella: Birational geometry of terminal quartic 3-folds I. Am. J. Math. 126 (2004), No. 4, 739–761.

[11] A. Corti, A. Pukhlikov, M. Reid: Fano 3-fold hypersurfaces. Explicit birational geometry of 3-folds, 175–258, London Math. Soc. Lecture Note Ser., 281, Cambridge Univ. Press, Cambridge, 2000.

[12] M.M. Grinenko: Mori structures on a Fano threefold of index 2 and degree 1. Tr. Mat. Inst. Steklova 246 (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 116–141; translation in Proc. Steklov Inst. Math. 2004, no. 3 (246), 103–128.

[13] M.M. Grinenko: Birational automorphisms of a three-dimensional double cone. Mat. Sb. 189 (1998), no. 7, 37–52; translation in Sb. Math. 189 (1998), no. 7-8, 991–1007.

[14] J. Hausen: Cox rings and combinatorics II. Mosc. Math. J., 8 (2008), 711–757.

[15] J. Hausen, H. Süß: The Cox ring of an algebraic variety with torus action. Adv. Math. 225 (2010), no. 2, 977 -1012.

[16] A.R. Iano-Fletcher: Working with weighted complete intersections. Explicit birational geometry of 3-folds, 101 -173, London Math. Soc. Lecture Note Ser., 281, Cambridge Univ. Press, Cambridge, 2000.

[17] M.-N. Ishida: Graded factorial rings of dimension 3 of a restricted type. J. Math. Kyoto Univ. 17 (1977), no. 3, 441–456.

[18] V.A. Iskovskih: Fano threefolds. II. Izv. Akad. Nauk SSSR Ser. Mat. 42 (1978), no. 3, 506- 549.

[19] V.A. Iskovskih: Birational automorphisms of three-dimensional algebraic varieties. Current problems in mathematics, Vol. 12 (Russian), pp. 159–236, 239 (loose errata), VINITI, Moscow, 1979.

[20] V.A. Iskovskih, Yu.I. Manin: Three-dimensional quartics and counterexamples to the Lüroth problem. Mat. Sb. (N.S.) 86 (128) (1971), 140–166.

[21] J.M. Johnson, J. Kollár: Fano hypersurfaces in weighted projective 4-spaces. Experiment. Math. 10 (2001), no. 1, 151–158.

[22] F. Knop, H Kraft, T. Vust: The Picard group of a G-variety. In: Algebraic Transformation Groups and Invariant Theory, DMV Seminar, Band 3, Birkhäuser.

[23] S. Mori: Graded factorial domains. Japan J. Math. 3 (1977), no. 2, 223–238.

[24] L. Petersen, H. Süß: Torus invariant divisors. Preprint, arXiv:0811.0517 (2008), to appear in Israel J. Math.

[25] A.V. Pukhlikov: Birational automorphisms of a three-dimensional quartic with a simple singularity. Mat. Sb. (N.S.) 135(177) (1988), no. 4, 472–496, 559; translation in Math. USSR-Sb. 63 (1989), no. 2, 457–482.

[26] A.V. Pukhlikov: Birational isomorphisms of four-dimensional quintics. Invent. Math. 87, no. 2 (1987), 303–329.

[27] K. Watanabe, M. Watanabe: The classification of Fano 3-folds with torus embeddings. Tokyo J. Math. 5, 37–48 (1982).

[28] H. Süß: Canonical divisors on T-varieties. Preprint, arXiv:0811.0626v1.

[29] A.S. Tikhomirov: The intermediate Jacobian of double $\mathbf{P}^3$ that is branched in a quartic. Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), no. 6, 1329–1377, 1439.

[30] Voisin, Claire: Sur la jacobienne intermédiaire du double solide d'indice deux. Duke Math. J. 57 (1988), no. 2, 629–646.

Jürgen Hausen
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
Germany

Elaine Herppich
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
Germany

Hendrik Süß
Institut für Mathematik
LS Algebra und Geometrie
BTU Cottbus
PF 10 13 44
03013 Cottbus
Germany

On the Structure of Covers of Sofic Shifts

Rune Johansen

Received: June 6, 2010
Revised: March 1, 2011

Communicated by Joachim Cuntz

Abstract. A canonical cover generalizing the left Fischer cover to arbitrary sofic shifts is introduced and used to prove that the left Krieger cover and the past set cover of a sofic shift can be divided into natural layers. These results are used to find the range of a flow-invariant and to investigate the ideal structure of the universal C^*-algebra associated to a sofic shift space.

2010 Mathematics Subject Classification: Primary: 37B10; Secondary: 46L05
Keywords and Phrases: shift space, sofic shift, Krieger cover, Fischer cover

1 Introduction

Shifts of finite type have been completely classified up to flow equivalence by Boyle and Huang [4, 6, 15], but very little is known about the classification of the class of sofic shift spaces introduced by Weiss [38], even though they are a natural first generalization of shifts of finite type. The purpose of this paper is to investigate the structure of - and relationships between - various standard presentations (the Fischer cover, the Krieger cover, and the past set cover) of sofic shift spaces. These results are used to find the range of the flow-invariant introduced in [1], and to investigate the ideal structure of the C^*-algebras associated to sofic shifts. In this way, the present paper can be seen as a continuation of the strategy applied in [10, 11, 30], where invariants for shift spaces are extracted from the associated C^*-algebras.
Section 2 recalls the definitions of shift spaces, labelled graphs, and covers to make the paper self contained. Section 3 introduces a canonical and flow-invariant cover generalizing the left Fischer cover to arbitrary sofic shifts.

Section 4 introduces the concept of a foundation of a cover, which is used to prove that the left Krieger cover and the past set cover can be divided into natural layers and to show that the left Krieger cover of an arbitrary sofic shift can be identified with a subgraph of the past set cover.

In Section 5, the structure of the layers of the left Krieger cover of an irreducible sofic shift is used to find the range of the flow-invariant introduced in [1]. Section 6 uses the results about the structure of covers of sofic shifts to investigate ideal lattices of the associated C^*-algebras. Additionally, it is proved that Condition $(*)$ introduced by Carlsen and Matsumoto [12] holds if and only if the left Krieger cover is the maximal essential subgraph of the past set cover.

Acknowledgements. This work was supported by the Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation. The author would like to thank David Pask, Toke Meier Carlsen, and Søren Eilers for interesting discussions and helpful comments. The author would also like to thank the anonymous referee for useful comments improving the exposition and to thank the University of Wollongong, Australia and the University of Tokyo, Japan where parts of the research for this paper were carried out during visits funded by *Rejselegat for Matematikere*.

2 Background

Shift spaces. Here, a short introduction to the definition and properties of shift spaces is given to make the present paper self-contained; for a thorough treatment of shift spaces see [21]. Let $\mathcal{A}$ be a finite set with the discrete topology. The *full shift* over $\mathcal{A}$ consists of the space $\mathcal{A}^{\mathbb{Z}}$ endowed with the product topology and the *shift map* $\sigma \colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$ for all $i \in \mathbb{Z}$. Let $\mathcal{A}^*$ be the collection of finite words (also known as blocks) over $\mathcal{A}$. A subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is called a *shift space* if it is invariant under the shift map and closed. For each $\mathcal{F} \subseteq \mathcal{A}^*$, define $\mathsf{X}_{\mathcal{F}}$ to be the set of bi-infinite sequences in $\mathcal{A}^{\mathbb{Z}}$ which do not contain any of the *forbidden words* from $\mathcal{F}$. A subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a shift space if and only if there exists $\mathcal{F} \subseteq \mathcal{A}^*$ such that $X = \mathsf{X}_{\mathcal{F}}$ (cf. [21, Proposition 1.3.4]). X is said to be a *shift of finite type* (SFT) if this is possible for a finite set $\mathcal{F}$.

The *language* of a shift space X is defined to be the set of all words which occur in at least one $x \in X$, and it is denoted $\mathcal{B}(X)$. X is said to be *irreducible* if there for every $u, w \in \mathcal{B}(X)$ exists $v \in \mathcal{B}(X)$ such that $uvw \in \mathcal{B}(X)$. For each $x \in X$, define the *left-ray* of x to be $x^- = \cdots x_{-2}x_{-1}$ and define the *right-ray* of x to be $x^+ = x_0x_1x_2\cdots$. The sets of all left-rays and all right-rays are, respectively, denoted X^- and X^+.

A bijective, continuous, and shift commuting map between two shift spaces is called a *conjugacy*, and when such a map exists, the two shift spaces are said to be *conjugate*. Shift spaces (X, σ_X) and (Y, σ_Y) are said to be *flow equivalent* if the corresponding suspension flows SX and SY are topologically equivalent. Flow equivalence is generated by conjugacy and *symbol expansion* [35].

Graphs. For countable sets E^0 and E^1, and maps $r, s\colon E^1 \to E^0$ the quadruple $E = (E^0, E^1, r, s)$ is called a *directed graph*. The elements of E^0 and E^1 are, respectively, the vertices and the edges of the graph. For each edge $e \in E^1$, $s(e)$ is the vertex where e starts, and $r(e)$ is the vertex where e ends. A *path* $\lambda = e_1 \cdots e_n$ is a sequence of edges such that $r(e_i) = s(e_{i+1})$ for all $i \in \{1, \dots n-1\}$. For each $n \in \mathbb{N}_0$, the set of paths of length n is denoted E^n, and the set of all finite paths is denoted E^*. Extend the maps r and s to E^* by defining $s(e_1 \cdots e_n) = s(e_1)$ and $r(e_1 \cdots e_n) = r(e_n)$. A *circuit* is a path λ with $r(\lambda) = s(\lambda)$ and $|\lambda| > 0$. For $u, v \in E^0$, u is said to be *connected* to v if there is a path $\lambda \in E^*$ such that $s(\lambda) = u$ and $r(\lambda) = v$, and this is denoted by $u \geq v$ [21, Section 4.4]. A vertex is said to be *maximal*, if it is connected to all other vertices. E is said to be irreducible if all vertices are maximal. If E has a unique maximal vertex, this vertex is said to be the *root* of E. E is said to be *essential* if every vertex emits and receives an edge. For a finite essential directed graph E, the *edge shift* (X_E, σ_E) is defined by

$$\mathsf{X}_E = \left\{ x \in (E^1)^{\mathbb{Z}} \mid r(x_i) = s(x_{i+1}) \text{ for all } i \in \mathbb{Z} \right\}.$$

A *labelled graph* $(E, \mathcal{L})$ over an alphabet $\mathcal{A}$ consists of a directed graph E and a surjective labelling map $\mathcal{L}\colon E^1 \to \mathcal{A}$. Extend the labelling map to $\mathcal{L}\colon E^* \to \mathcal{A}^*$ by defining $\mathcal{L}(e_1 \cdots e_n) = \mathcal{L}(e_1) \cdots \mathcal{L}(e_n) \in \mathcal{A}^*$. For a finite essential labelled graph $(E, \mathcal{L})$, define the shift space $(\mathsf{X}_{(E,\mathcal{L})}, \sigma)$ by

$$\mathsf{X}_{(E,\mathcal{L})} = \left\{ \left(\mathcal{L}(x_i)\right)_i \in \mathcal{A}^{\mathbb{Z}} \mid x \in \mathsf{X}_E \right\}.$$

The labelled graph $(E, \mathcal{L})$ is said to be a *presentation* of the shift space $\mathsf{X}_{(E,\mathcal{L})}$, and a *representative* of a word $w \in \mathcal{B}(\mathsf{X}_{(E,\mathcal{L})})$ is a path $\lambda \in E^*$ such that $\mathcal{L}(\lambda) = w$. Representatives of rays are defined analogously. If $H \subseteq E^0$ then the *subgraph of* $(E, \mathcal{L})$ *induced by* H is the labelled subgraph of $(E, \mathcal{L})$ with vertices H and edges $\{e \in E^1 \mid s(e), r(e) \in H\}$.

Sofic shifts. A function $\pi\colon X_1 \to X_2$ between shift spaces X_1 and X_2 is said to be a *factor map* if it is continuous, surjective, and shift commuting. A shift space is called *sofic* [38] if it is the image of an SFT under a factor map. A shift space is sofic if and only if it can be presented by a finite labelled graph [14]. A sofic shift space is irreducible if and only if it can be presented by an irreducible labelled graph (see [21, Section 3.1]). Let $(E, \mathcal{L})$ be a finite labelled graph and let $\pi_{\mathcal{L}}\colon \mathsf{X}_E \to \mathsf{X}_{(E,\mathcal{L})}$ be the factor map induced by the labelling map $\mathcal{L}\colon E^1 \to \mathcal{A}$ then the SFT X_E is called a *cover* of the sofic shift $\mathsf{X}_{(E,\mathcal{L})}$, and $\pi_{\mathcal{L}}$ is called the covering map.

A presentation $(E, \mathcal{L})$ of a sofic shift space X is said to be *left-resolving* if no vertex in E^0 receives two edges with the same label. Fischer proved [14] that, up to labelled graph isomorphism, every irreducible sofic shift has a unique left-resolving presentation with fewer vertices than any other left-resolving presentation. This is called the *left Fischer cover* of X, and it is denoted $(F, \mathcal{L}_F)$. An irreducible sofic shift is said to have *almost finite type* (AFT) [22, 33] if the left Fischer cover is right-closing (see e.g. [21, Definition 5.1.4]).

For $x \in \mathcal{B}(X) \cup X^+$, define the *predecessor set* of x to be the set of left-rays which may precede x in X (see [17, Sections I and III] and [21, Exercise 3.2.8]). The *follower set* of a left-ray or word is defined analogously. Let $(E, \mathcal{L})$ be a labelled graph presenting X and let $v \in E^0$. Define the *predecessor set* of v to be the set of left-rays in X which have a presentation terminating at v. This is denoted $P_\infty^E(v)$, or just $P_\infty(v)$ when $(E, \mathcal{L})$ is understood from the context. The presentation $(E, \mathcal{L})$ is said to be *predecessor-separated* if $P_\infty^E(u) \neq P_\infty^E(v)$ when $u, v \in E^0$ and $u \neq v$.

The *left Krieger cover* of the shift space X is the labelled graph $(K, \mathcal{L}_K)$ where $K^0 = \{P_\infty(x^+) \mid x^+ \in X^+\}$, and where there is an edge labelled $a \in \mathcal{A}$ from $P \in K^0$ to $P' \in K^0$ if and only if there exists $x^+ \in X^+$ such that $P = P_\infty(ax^+)$ and $P' = P_\infty(x^+)$. The *past set cover* of the shift space X is the labelled graph $(W, \mathcal{L}_W)$ where $W^0 = \{P_\infty(w) \mid w \in \mathcal{B}(X)\}$ and where the edges and labels are constructed as in the Krieger cover. A shift space is sofic if and only if the number of predecessor sets is finite [19, §2], so the left Krieger cover is finite exactly when the shift space is sofic. The left Fischer cover, the left Krieger cover, and the past set cover are left-resolving and predecessor-separated presentations of X.

The right Krieger cover and the future set cover are right-resolving and follower-separated covers defined analogously to the left Krieger cover and the past set cover, respectively. Every result developed for left-resolving covers in the following has an analogue for the corresponding right-resolving cover. These results can easily be obtained by considering the transposed shift space X^{T} (see e.g. [21, p. 39]).

3 Generalizing the Fischer cover

Jonoska [16] proved that a reducible sofic shift does not necessarily have a unique minimal left-resolving presentation. The aim of this section is to define a generalization of the left Fischer cover as the subgraph of the left Krieger cover induced by a certain subset of vertices. Let X be a sofic shift space, and let $(K, \mathcal{L}_K)$ be the left Krieger cover of X. A predecessor set $P \in K^0$ is said to be *non-decomposable* if $V \subseteq K^0$ and $P = \bigcup_{Q \in V} Q$ implies that $P \in V$.

Lemma 3.1. *If $P \in K^0$ is non-decomposable then the subgraph of $(K, \mathcal{L}_K)$ induced by $K^0 \setminus \{P\}$ is not a presentation of X.*

Proof. Let E be the subgraph of K induced by $K^0 \setminus \{P\}$. Choose $x^+ \in X^+$ such that $P = P_\infty(x^+)$. Let $V \subseteq K^0 \setminus \{P\}$ be the set of vertices where a presentation of x^+ can start. Then $Q \subseteq P_\infty(x^+) = P$ for each $Q \in V$, and by assumption, there exists $y^- \in P \setminus \bigcup_{Q \in V} Q$. Hence, there is no presentation of $y^- x^+$ in $(E, \mathcal{L}_K|_E)$. □

Lemma 3.1 shows that a subgraph of the left Krieger cover which presents the same shift must contain all the non-decomposable vertices. The next example shows that this subgraph is not always large enough.

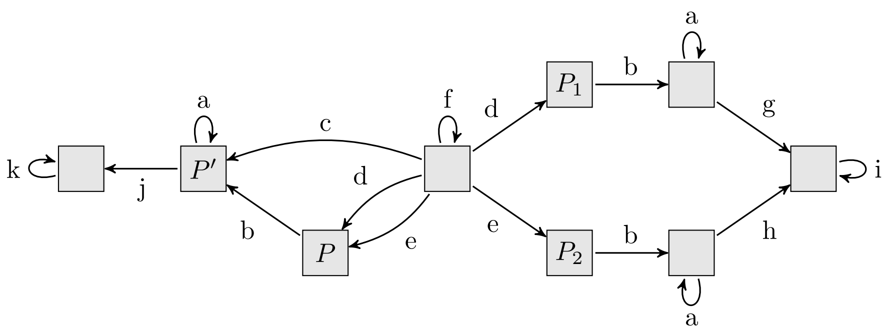


Figure 1: Left Krieger cover of the shift considered in Example 3.2. Note that the labelled graph is no longer a presentation of the same shift if the decomposable predecessor set $P = P_1 \cup P_2$ is removed.

Example 3.2. It is easy to check that the labelled graph in Figure 1 is the left Krieger cover of a reducible sofic shift X. Note that the predecessor set P is decomposable since $P = P_1 \cup P_2$, and that the graph obtained by removing the vertex P and all edges starting at or terminating at P is not a presentation of the same sofic shift since there is no presentation of $f^\infty dbjk^\infty$ in this graph. Note that there is a path from P to the vertex P' which is non-decomposable.

Together with Lemma 3.1, this example motivates the following definition.

Definition 3.3. *The* generalized left Fischer cover $(G, \mathcal{L}_G)$ *of a sofic shift X is defined to be the subgraph of the left Krieger cover induced by $G^0 = \{P \in K^0 \mid P \geq P', P' \text{ non-decomposable}\}$.*

The following proposition justifies the term generalized left Fischer cover.

Proposition 3.4.

(i) The generalized left Fischer cover of a sofic shift X is a left-resolving and predecessor-separated presentation of X.

(ii) If X is an irreducible sofic shift then the generalized left Fischer cover is isomorphic to the left Fischer cover.

(iii) If X_1, X_2 are sofic shifts with disjoint alphabets then the generalized left Fischer cover of $X_1 \cup X_2$ is the disjoint union of the generalized left Fischer covers of X_1 and X_2.

Proof. Given $y^- \in X^-$, choose $x^+ \in X^+$ such that $y^- \in P_\infty(x^+) = P$. By definition of the generalized left Fischer cover, there exist vertices $P_1, \dots, P_n \in G^0$ such that $P = \bigcup_{i=1}^n P_i$. Choose i such that $y^- \in P_i$. By construction, the left Krieger cover contains a path labelled y^- terminating at P_i. Since

$P_i \in G^0$, this is also a path in the generalized left Fischer cover. This proves that the generalized left Fischer cover is a presentation of X^-, and hence also a presentation of X. Since the left Krieger cover is left-resolving and predecessor-separated, so is the generalized left Fischer cover.

Let X be an irreducible sofic shift, and identify the left Fischer cover $(F, \mathcal{L}_F)$ with the top irreducible component of the left Krieger cover $(K, \mathcal{L}_K)$ [19, Lemma 2.7]. By the construction of the generalized left Fischer cover, it follows that the left Fischer cover is a subgraph of the generalized left Fischer cover. Let $x^+ \in X^+$ such that $P = P_\infty(x^+)$ is non-decomposable. Let $S \subseteq F^0$ be the set of vertices where a presentation of x^+ in $(F, \mathcal{L}_F)$ can start. Then $P = \bigcup_{v \in S} P_\infty(v)$, so $P \in S \subseteq F^0$ by assumption.

Since X_1 and X_2 have no letters in common, the left Krieger cover of $X_1 \cup X_2$ is just the disjoint union of the left Krieger covers of X_1 and X_2. The generalized left Fischer cover inherits this property from the left Krieger cover. $\square$

The shift consisting of two non-interacting copies of the even shift is a simple example where the generalized left Fischer cover is a proper subgraph of the left Krieger cover.

Lemma 3.5. *Let X be a sofic shift with left Krieger cover $(K, \mathcal{L}_K)$. If there is an edge labelled a from a non-decomposable $P \in K^0$ to a decomposable $Q \in K^0$ then there exists a non-decomposable $Q' \in K^0$ and an edge labelled a from P to Q'.*

Proof. Choose $x^+ \in X^+$ such that $P = P_\infty(ax^+)$ and $Q = P_\infty(x^+)$. Since Q is decomposable, there exist $n > 1$ and non-decomposable $Q_1, \dots, Q_n \in K^0 \setminus \{Q\}$ such that $Q = Q_1 \cup \dots \cup Q_n$. Let S be the set of predecessor sets $P' \in K^0$ for which there is an edge labelled a from P' to Q_j for some $1 \leq j \leq n$. Given $y^- \in P$, $y^- a x^+ \in X$, so $y^- a \in Q$. Choose $1 \leq i \leq n$ such that $y^- a \in Q_i$. By construction, there exists $P' \in S$ such that $y^- \in P'$. Reversely, if $y^- \in P' \in S$ then there is an edge labelled a from P' to Q_i for some $1 \leq i \leq n$, so $y^- a \in Q_i \subseteq Q$. This implies that $y^- a x^+ \in X$, so $y^- \in P$. Thus $P = \bigcup_{P' \in S} P'$, but P is non-decomposable, so this means that $P \in S$. Hence, there is an edge labelled a from P to Q_i for som i, and Q_i is non-decomposable. $\square$

The following proposition is an immediate consequence of this result and the definition of the generalized left Fischer cover.

Proposition 3.6. *The generalized left Fischer cover is essential.*

The left Fischer cover of an irreducible sofic shift X is minimal in the sense that no other left-resolving presentation of X has fewer vertices. This is not always the case for the generalized left Fischer cover.

Canonical. Krieger proved that a conjugacy $\Phi \colon X_1 \to X_2$ between sofic shifts with left Krieger covers $(K_1, \mathcal{L}_1)$ and $(K_2, \mathcal{L}_2)$, respectively, induces a conjugacy $\varphi \colon \mathsf{X}_{K_1} \to \mathsf{X}_{K_2}$ such that $\Phi \circ \pi_1 = \pi_2 \circ \varphi$ when $\pi_i \colon \mathsf{X}_{K_i} \to X_i$ is the covering map of the left Krieger cover of X_i [19]. A cover with this property is said to be *canonical.* The next goal is to prove that the generalized left Fischer cover is canonical. This will be done by using results and methods used by Nasu [34] to prove that the left Krieger cover is canonical.

Definition 3.7 (Bipartite code). *When $\mathcal{A}, \mathcal{C}, \mathcal{D}$ are alphabets, an injective map $f \colon \mathcal{A} \to \mathcal{C}\mathcal{D}$ is called a* bipartite expression. *If X_1, X_2 are shift spaces with alphabets $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, and if $f_1 \colon \mathcal{A}_1 \to \mathcal{C}\mathcal{D}$ is a bipartite expression then a map $\Phi \colon X_1 \to X_2$ is said to be a* bipartite code induced by f_1 *if there exists a bipartite expression $f_2 \colon \mathcal{A}_2 \to \mathcal{D}\mathcal{C}$ such that one of the following two conditions is satisfied:*

(i) If $x \in X_1$, $y = \Phi(x)$, and $f_1(x_i) = c_i d_i$ with $c_i \in \mathcal{C}$ and $d_i \in \mathcal{D}$ for all $i \in \mathbb{Z}$ then $f_2(y_i) = d_i c_{i+1}$ for all $i \in \mathbb{Z}$.

(ii) If $x \in X_1$, $y = \Phi(x)$, and $f_1(x_i) = c_i d_i$ with $c_i \in \mathcal{C}$ and $d_i \in \mathcal{D}$ for all $i \in \mathbb{Z}$ then $f_2(y_i) = d_{i-1} c_i$ for all $i \in \mathbb{Z}$.

A mapping $\Phi \colon X_1 \to X_2$ is called a bipartite code*, if it is the bipartite code induced by some bipartite expression.*

It is clear that a bipartite code is a conjugacy and that the inverse of a bipartite code is a bipartite code.

Theorem 3.8 (Nasu [34, Thm. 2.4]). *Any conjugacy between shift spaces can be decomposed into a product of bipartite codes.*

Let $\Phi \colon X_1 \to X_2$ be a bipartite code corresponding to bipartite expressions $f_1 \colon \mathcal{A}_1 \to \mathcal{C}\mathcal{D}$ and $f_2 \colon \mathcal{A}_2 \to \mathcal{D}\mathcal{C}$, and use the bipartite expressions to recode X_1 and X_2 to

$$\hat{X}_1 = \{(f_1(x_i))_i \mid x \in X_1\} \subseteq (\mathcal{C}\mathcal{D})^{\mathbb{Z}}$$
$$\hat{X}_2 = \{(f_2(x_i))_i \mid x \in X_2\} \subseteq (\mathcal{D}\mathcal{C})^{\mathbb{Z}}.$$

For $i \in \{1,2\}$, f_i induces a one-block conjugacy from X_i to $\hat{X}_i$, and Φ induces a bipartite code $\hat{\Phi} \colon \hat{X}_1 \to \hat{X}_2$ which commutes with these conjugacies. If Φ satisfies condition (i) in the definition of a bipartite code then $(\hat{\Phi}(\hat{x}))_i = d_i c_{i+1}$ when $\hat{x} = (c_i d_i)_{i \in \mathbb{Z}} \in \hat{X}_1$. If it satisfies condition (ii) then $(\hat{\Phi}(\hat{x}))_i = d_{i-1} c_i$ when $\hat{x} = (c_i d_i)_{i \in \mathbb{Z}} \in \hat{X}_1$. The shifts $\hat{X}_1$ and $\hat{X}_2$ will be called the *recoded shifts* of the bipartite code, and $\hat{\Phi}$ will be called the *recoded bipartite code*.

A labelled graph $(G, \mathcal{L})$ is said to be *bipartite* if G is a bipartite graph (i.e. the vertex set can be partitioned into two sets $(G^0)_1$ and $(G^0)_2$ such that no edge has its range and source in the same set). When $(G, \mathcal{L})$ is a bipartite labelled graph over an alphabet $\mathcal{A}$, define two graphs G_1 and G_2 as follows: For $i \in \{1,2\}$, the vertex set of G_i is $(G^0)_i$, the edge set is the set of paths of length 2 in $(G, \mathcal{L})$ for which both range and source are in $(G^0)_i$, and the range and source maps are inherited from G. For $i \in \{1,2\}$, define $\mathcal{L}_i \colon G_i^1 \to \mathcal{A}^2$

by $\mathcal{L}_i(ef) = \mathcal{L}(e)\mathcal{L}(f)$. The pair $(G_1, \mathcal{L}_1)$, $(G_2, \mathcal{L}_2)$ is called the *induced pair of labelled graphs* of $(G, \mathcal{L})$. This decomposition is not necessarily unique, but whenever a bipartite labelled graph is mentioned, it will be assumed that the induced graphs are specified.

Remark 3.9 (Nasu [34, Remark 4.2]). Let $(G, \mathcal{L})$ be a bipartite labelled graph for which the induced pair of labelled graphs is $(G_1, \mathcal{L}_1)$, $(G_2, \mathcal{L}_2)$. Let X_1 and X_2 be the sofic shifts presented by these graphs, and let $\mathsf{X}_{G_1}, \mathsf{X}_{G_2}$ be the edge shifts generated by G_1, G_2. The natural embedding $f\colon G_1^1 \to (G^1)^2$ is a bipartite expression which induces two bipartite codes $\varphi_\pm\colon \mathsf{X}_{G_1} \to \mathsf{X}_{G_2}$ such that $(\varphi_+(x))_i = f_i e_{i+1}$ and $(\varphi_-(x))_i = f_{i-1}e_i$ when $x = (e_i f_i)_{i\in\mathbb{Z}} \in \mathsf{X}_{G_1}$. Similarly, the embedding $F\colon \mathcal{L}_1(G_1^1) \to (\mathcal{L}(G^1))^2$ is a bipartite expression which induces bipartite codes $\Phi_\pm\colon X_1 \to X_2$ such that $(\Phi_+(x))_i = b_i a_{i+1}$ and $(\Phi_-(x))_i = b_{i-1}a_i$ when $x = (a_i b_i)_{i\in\mathbb{Z}} \in X_1$. By definition, $\Phi_\pm \circ \pi_1 = \pi_2 \circ \varphi_\pm$ when $\pi_1\colon \mathsf{X}_{G_1} \to X_1$, $\pi_2\colon \mathsf{X}_{G_2} \to X_2$ are the covering maps. The bipartite codes $\varphi_\pm$ and $\Phi_\pm$ are called the *standard bipartite codes induced by* $(G, \mathcal{L})$.

Lemma 3.10 (Nasu [34, Cor. 4.6 (1)]). *Let $\Phi\colon X_1 \to X_2$ be a bipartite code between sofic shifts X_1 and X_2. Let $\hat{X}_1$ and $\hat{X}_2$ be the recoded shifts of X_1 and X_2 respectively, and let $(K_1, \mathcal{L}_1)$ and $(K_2, \mathcal{L}_2)$ be the left Krieger covers of $\hat{X}_1$ and $\hat{X}_2$ respectively. Then there exists a sofic shift $\hat{X}$ for which the left Krieger cover is a bipartite labelled graph such that the induced pair of labelled graphs is $(K_1, \mathcal{L}_1)$, $(K_2, \mathcal{L}_2)$ and such that the recoded bipartite code $\hat{\Phi}\colon \hat{X}_1 \to \hat{X}_2$ of Φ is one of the standard bipartite codes $\Phi_\pm$ induced by the left Krieger cover of $\hat{X}$ as defined in Remark 3.9.*

The proof of the following theorem is very similar to the proof of the corresponding result by Nasu [34, Thm. 3.3] for the left Krieger cover.

Theorem 3.11. *The generalized left Fischer cover is canonical.*

Proof. Let $\Phi\colon X_1 \to X_2$ be a bipartite code. Let $\hat{X}_1, \hat{X}_2$ be the recoded shifts, let $(K_1, \mathcal{L}_1)$, $(K_2, \mathcal{L}_2)$ be the corresponding left Krieger covers, and let $\hat{\Phi}\colon \hat{X}_1 \to \hat{X}_2$ be the recoded bipartite code. Use Lemma 3.10 to find a sofic shift $\hat{X}$ such that the left Krieger cover $(K, \mathcal{L})$ of $\hat{X}$ is a bipartite labelled graph for which the induced pair of labelled graphs is $(K_1, \mathcal{L}_1)$, $(K_2, \mathcal{L}_2)$. Let $(G_1, \mathcal{L}_1)$, $(G_2, \mathcal{L}_2)$, and $(G, \mathcal{L})$ be the generalized left Fischer covers of respectively $\hat{X}_1$, $\hat{X}_2$, and $\hat{X}$.

The labelled graph $(G, \mathcal{L})$ is bipartite since G is a subgraph of K. Note that a predecessor set P in K_1^0 or K_2^0 is decomposable if and only if the corresponding predecessor set in K^0 is decomposable. If $i \in \{1, 2\}$ and $Q \in G_i^0 \subseteq K_i^0$ then there is a path in K_i from Q to a non-decomposable $P \in K_i^0$. By considering the corresponding path in K, it is clear that the vertex in K^0 corresponding to Q is in G^0. Conversely, if $Q \in G^0$ then there is a path in K from Q to a non-decomposable $P \in K^0$. If P and Q belong to the same partition K_i^0 then the vertex in K_i corresponding to Q is in G_i^0 by definition. On the other hand, if Q corresponds to a vertex in K_i and if P belongs to the other partition

then Lemma 3.5 shows that there exists a non-decomposable P' in the same partition as Q and an edge from P to P' in K. Hence, there is also a path in K_i from the vertex corresponding to Q to the vertex corresponding to P', so $Q \in G_i^0$. This proves that the pair of induced labelled graphs of $(G, \mathcal{L})$ is $(G_1, \mathcal{L}_1)$, $(G_2, \mathcal{L}_2)$.
Let $\hat{\Psi}_\pm \colon \hat{X}_1 \to \hat{X}_2$ be the standard bipartite codes induced by $(G, \mathcal{L})$. Remark 3.9 shows that there exist bipartite codes $\hat{\psi}_\pm \colon \mathsf{X}_{G_1} \to \mathsf{X}_{G_2}$ such that $\hat{\Psi}_\pm \circ \hat{\pi}_1|_{\mathsf{X}_{G_1}} = \hat{\pi}_2|_{\mathsf{X}_{G_2}} \circ \hat{\psi}_\pm$. The labelled graph $(G, \mathcal{L})$ presents the same sofic shift as $(K, \mathcal{L})$, so they both induce the same standard bipartite codes from $\hat{X}_1$ to $\hat{X}_2$, and by Lemma 3.10, $\hat{\Phi}$ is one of these standard bipartite codes, so $\hat{\Phi} = \hat{\Psi}_+$ or $\hat{\Phi} = \hat{\Psi}_-$. In particular, there exists a bipartite code $\hat{\psi} \colon \mathsf{X}_{G_1} \to \mathsf{X}_{G_2}$ such that $\hat{\Phi} \circ \hat{\pi}_1|_{\mathsf{X}_{G_1}} = \hat{\pi}_2|_{\mathsf{X}_{G_2}} \circ \hat{\psi}$.
By recoding $\hat{X}_1$ to X_1 and $\hat{X}_2$ to X_2 via the bipartite expressions inducing Φ, this gives a bipartite code ψ such that $\Phi \circ \pi_1 = \pi_2 \circ \psi$ when π_1, π_2 are the covering maps of the generalized left Fischer covers of X_1 and X_2 respectively. By Theorem 3.8, any conjugacy can be decomposed as a product of bipartite codes, so this proves that the generalized left Fischer cover is canonical. □

Theorem 3.12. *For $i \in \{1, 2\}$, let X_i be a sofic shift with generalized left Fischer cover $(G_i, \mathcal{L}_i)$ and covering map $\pi_i \colon \mathsf{X}_{G_i} \to X_i$. If $\Phi \colon SX_1 \to SX_2$ is a flow equivalence then there exists a unique flow equivalence $\varphi \colon S\mathsf{X}_{G_1} \to S\mathsf{X}_{G_2}$ such that $\Phi \circ S\pi_1 = S\pi_2 \circ \varphi$.*

Proof. In [5] it is proved that the left Krieger cover respects symbol expansion: If X is a sofic shift with alphabet $\mathcal{A}$, $a \in \mathcal{A}$, $\bullet$ is some symbol not in $\mathcal{A}$, and if $\hat{X}$ is obtained from X via a symbol expansion which inserts a $\bullet$ after each a then the left Krieger cover of $\hat{X}$ is obtained by replacing each edge labelled a in the left Krieger cover of X by two edges in sequence labelled a and $\bullet$ respectively. Clearly, the generalized left Fischer cover inherits this property. By [5], any canonical cover which respects flow equivalence has the desired property, so the result follows from Theorem 3.11. □

4 Foundations and layers of covers

Let $\mathcal{E} = (E, \mathcal{L})$ be a finite left-resolving and predecessor-separated labelled graph. For each $V \subseteq E^0$ and each word w over the alphabet $\mathcal{A}$ of $\mathcal{L}$ define

$$wV = \{u \in E^0 \mid u \text{ is the source of a path labelled } w \text{ terminating in } V\}.$$

Definition 4.1. *Let S be a subset of the power set $\mathcal{P}(E^0)$, and let $\sim$ be an equivalence relation on S. The pair $(S, \sim)$ is said to be* past closed *if*

- *$\{v\} \in S$,*
- *$\{u\} \sim \{v\}$ implies $u = v$,*
- *$aV \neq \emptyset$ implies $aV \in S$, and*

- *$U \sim V$ and $aU \neq \emptyset$ implies $aV \neq \emptyset$ and $aU \sim aV$*

for all $u, v \in E^0$, $U, V \in S$, and $a \in \mathcal{A}$.

Let $(S, \sim)$ be past closed. For each $V \in S$, let $[V]$ denote the equivalence class of V with respect to $\sim$. When $a \in \mathcal{A}$ and $V \in S$, $[V]$ is said to receive a if $aV \neq \emptyset$. For each $[V] \in S/{\sim}$, define $|[V]| = \min_{V \in [V]} |V|$.

Definition 4.2. *Define $\mathcal{G}(\mathcal{E}, S, \sim)$ to be the labelled graph with vertex set $S/{\sim}$ for which there is an edge labelled a from $[aV]$ to $[V]$ whenever $[V]$ receives a. For each $n \in \mathbb{N}$, the* nth layer *of $\mathcal{G}(\mathcal{E}, S, \sim)$ is the labelled subgraph induced by $S_n = \{[V] \in S/{\sim} \mid n = |[V]|\}$. $\mathcal{E}$ is said to be a* foundation *of any labelled graph isomorphic to $\mathcal{G}(\mathcal{E}, S, \sim)$.*

If a labelled graph $\mathcal{H}$ is isomorphic to $\mathcal{G}(\mathcal{E}, S, \sim)$ then the subgraph of $\mathcal{H}$ corresponding to the nth layer of $\mathcal{G}(\mathcal{E}, S, \sim)$ is be said to be the *nth layer of $\mathcal{H}$ with respect to $\mathcal{E}$*, or simply the *nth layer* if $\mathcal{E}$ is understood from the context.

Proposition 4.3. *$\mathcal{E}$ and $\mathcal{G}(\mathcal{E}, S, \sim)$ present the same sofic shift, and $\mathcal{E}$ is labelled graph isomorphic to the first layer of $\mathcal{G}(\mathcal{E}, S, \sim)$.*

Proof. By assumption, there is a bijection between E^0 and the set of vertices in the first layer of $\mathcal{G}(\mathcal{E}, S, \sim)$. By construction, there is an edge labelled a from u to v in $\mathcal{E}$ if and only if there is an edge labelled a from $[\{u\}]$ to $[\{v\}]$ in $\mathcal{G}(\mathcal{E}, S, \sim)$. Every finite word presented by $\mathcal{G}(\mathcal{E}, S, \sim)$ is also presented by $\mathcal{E}$, so they present the same sofic shift. $\square$

The following proposition motivates the use of the term layer by showing that edges can never go from higher to lower layers.

Proposition 4.4. *If $[V] \in S/{\sim}$ receives $a \in \mathcal{A}$ then $|[aV]| \leq |[V]|$. If $\mathcal{G}(\mathcal{E}, S, \sim)$ has an edge from a vertex in the mth layer to a vertex in the nth layer then $m \leq n$.*

Proof. Choose $V \in [V]$ such that $|V| = |[V]|$. Each $u \in aV$ emits at least one edge labelled a terminating in V, and $\mathcal{E}$ is left-resolving, so $|[aV]| \leq |aV| \leq |V| = |[V]|$. The second statement follows from the definition of $\mathcal{G}(\mathcal{E}, S, \sim)$. $\square$

Example 4.5. Let $(F, \mathcal{L}_F)$ be the left Fischer cover of an irreducible sofic shift X. For each $x^+ \in X^+$, define $s(x^+) \subseteq F^0$ to be the set of vertices where a presentation of x^+ can start. $S = \{s(x^+) \mid x^+ \in X^+\} \subseteq \mathcal{P}(F^0)$ is past closed since each vertex in the left Fischer cover is the predecessor set of an intrinsically synchronizing right-ray, so the *multiplicity set cover* of X can be defined to be $\mathcal{G}((F, \mathcal{L}_F), S, =)$. An analogous cover can be defined by considering the vertices where presentations of finite words can start. Thomsen [37] constructs the derived shift space ∂X of X using right-resolving graphs, but an analogous construction works for left-resolving graphs. The procedure from [37, Example 6.10] shows that this ∂X is presented by the labelled graph obtained by removing the left Fischer cover from the multiplicity set cover.

Let X be a sofic shift, and let $(K, \mathcal{L}_K)$ be the left Krieger cover of X. In order to use the preceding results to investigate the structure of the left Krieger cover and the past set cover, define an equivalence relation on $\mathcal{P}(K^0)$ by $U \sim_\cup V$ if and only if $\bigcup_{P \in U} P = \bigcup_{Q \in V} Q$. Clearly, $\{P\} \sim_\cup \{Q\}$ if and only if $P = Q$. If $U, V \subseteq K^0$, $a \in \mathcal{A}$, $aV \neq \emptyset$, and $U \sim_\cup V$ then $aU \sim_\cup aV$ by the definition of the left Krieger cover.

Theorem 4.6. *For a sofic shift X, the generalized left Fischer cover $(G, \mathcal{L}_G)$ is a foundation of the left Krieger cover $(K, \mathcal{L}_K)$, and no smaller subgraph is a foundation.*

Proof. Define $S = \{V \subseteq G^0 \mid \exists x^+ \in X^+ \text{ such that } P_\infty(x^+) = \bigcup_{P \in V} P\}$. Note that $\{P\} \in S$ for every $P \in G^0$. If $x^+ \in X^+$ with $P_\infty(x^+) = \bigcup_{P \in V} P$ and if $aV \neq \emptyset$ for some $a \in \mathcal{A}$ then $ax^+ \in X^+$ and $P_\infty(ax^+) = \bigcup_{P \in aV} P$. This proves that the pair $(S, \sim_\cup)$ is past closed, so $\mathcal{G}((G, \mathcal{L}_G), S, \sim_\cup)$ is well defined. Since $(G, \mathcal{L}_G)$ is a presentation of X, there is a bijection $\varphi\colon S/{\sim_\cup} \to K^0$ defined by $\varphi([V]) = \bigcup_{P \in V} P$. By construction, there is an edge labelled a from $[U]$ to $[V]$ in $\mathcal{G}((G, \mathcal{L}_G), S, \sim_\cup)$ if and only if there exists $x^+ \in X^+$ such that $P_\infty(ax^+) = \bigcup_{P \in U} P$ and $P_\infty(x^+) = \bigcup_{Q \in V} Q$, so $\mathcal{G}((G, \mathcal{L}_G), S, \sim_\cup)$ is isomorphic to $(K, \mathcal{L}_K)$. It follows from Lemma 3.1 that no proper subgraph of $(G, \mathcal{L}_G)$ can be a foundation of the left Krieger cover. □

The example from [12, Section 4] shows that the left Krieger cover can be a proper subgraph of the past set cover. The following lemma will be used to further investigate this relationship.

Lemma 4.7. *Let X be a sofic shift. For every right-ray $x^+ = x_1 x_2 x_3 \ldots \in X^+$ there exists $n \in \mathbb{N}$ such that $P_\infty(x^+) = P_\infty(x_1 x_2 \ldots x_k)$ for all $k \geq n$.*

Proof. It is clear that $P_\infty(x_1) \supseteq P_\infty(x_1 x_2) \supseteq \cdots \supseteq P_\infty(x^+)$. Since X is sofic, there are only finitely many different predecessor sets of words, so there must exist $n \in \mathbb{N}$ such that $P_\infty(x_1 x_2 \ldots x_k) = P_\infty(x_1 x_2 \ldots x_n)$ for all $k \geq n$. If $y^- \in P_\infty(x_1 x_2 \ldots x_n)$ is given, then $y^- x_1 x_2 \ldots x_k \in X$ for all $k \geq n$, so $y^- x^+$ contains no forbidden words, and therefore $y^- \in P_\infty(x^+)$. Since y^- was arbitrary, $P_\infty(x^+) = P_\infty(x_1 x_2 \ldots x_n)$. □

Theorem 4.8. *For a sofic shift X, the generalized left Fischer cover $(G, \mathcal{L}_G)$ and the left Krieger cover $(K, \mathcal{L}_K)$ are both foundations of the past set cover $(W, \mathcal{L}_W)$.*

Proof. Define $S = \{V \subseteq G^0 \mid \exists w \in \mathcal{B}(X) \text{ such that } P_\infty(w) = \bigcup_{P \in V} P\}$, and use Lemma 4.7 to conclude that S contains $\{P\}$ for every $P \in G^0$. By arguments analogous to the ones used in the proof of Theorem 4.6, it follows that $\mathcal{G}((G, \mathcal{L}_G), S, \sim_\cup)$ is isomorphic to $(W, \mathcal{L}_W)$. To see that $(K, \mathcal{L}_K)$ is also a foundation, define $T = \{V \subseteq K^0 \mid \exists w \in \mathcal{B}(X) \text{ such that } P_\infty(w) = \bigcup_{P \in V} P\}$, and apply arguments analogous to the ones used above to prove that $(W, \mathcal{L}_W)$ is isomorphic to $\mathcal{G}((K, \mathcal{L}_K), T, \sim_\cup)$. □

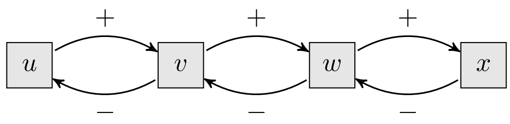


Figure 2: Left Fischer cover of the 3-charge constrained shift.

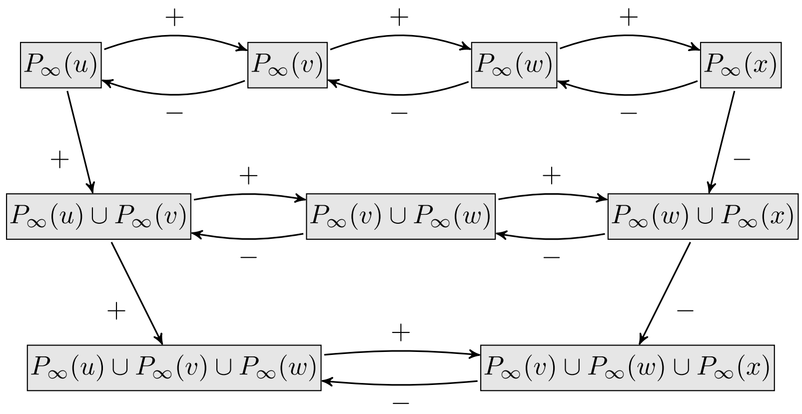


Figure 3: Left Krieger cover of the 3-charge constrained shift.

In the following, the nth layer of the left Krieger cover (past set cover) will always refer to the nth layer with respect to the generalized left Fischer cover $(G, \mathcal{L}_G)$. For a right-ray (word) x, $P_\infty(x)$ is a vertex in the nth layer of the left Kriger cover (past set cover) for some $n \in \mathbb{N}$, and such an x is said to be $1/n$-*synchronizing*. Note that x is $1/n$-synchronizing if and only if n is the smallest number such that there exist $P_1, \ldots, P_n \in G^0$ with $\bigcup_{i=1}^n P_i = P_\infty(x)$. In an irreducible sofic shift with left Fischer cover $(F, \mathcal{L}_F)$, this happens if and only if n is the smallest number such that there exist $u_1, \ldots, u_n \in F^0$ with $\bigcup_{i=1}^n P_\infty(u_i) = P_\infty(x)$.

Example 4.9. Figures 2 and 3 show, respectively, the left Fischer and the left Krieger cover of the 3-charge constrained shift (see e.g. [21, Example 1.2.7] for the definition of charge constrained shifts). There are 3 vertices in the second layer of the left Krieger cover and two in the third. Note how the left Fischer cover can be identified with the first layer of the left Krieger cover.

Corollary 4.10. *If the left Krieger cover of a sofic shift is reducible then so is the past set cover.*

Proof. This follows from Proposition 4.4 and Theorem 4.8. $\square$

5 THE RANGE OF A FLOW INVARIANT

Let E be a directed graph. Vertices $u, v \in E^0$ *properly communicate* [1] if there are paths $\mu, \lambda \in E^*$ of length greater than or equal to 1 such that $s(\mu) = u$, $r(\mu) = v$, $s(\lambda) = v$, and $r(\lambda) = u$. This relation is used to construct maximal disjoint subsets of E^0, called *proper communication sets of vertices*, such that $u, v \in E^0$ properly communicate if and only if they belong to the same subset. The *proper communication graph* $PC(E)$ is defined to be the directed graph for which the vertices are the proper communication sets of vertices of E and for which there is an edge from one proper communication set to another if and only if there is a path from a vertex in the first set to a vertex in the second. The proper communication graph of the left Krieger cover of a sofic shift space is a flow-invariant [1].

Let X be an irreducible sofic shift with left Fischer cover $(F, \mathcal{L}_F)$ and left Krieger cover $(K, \mathcal{L}_K)$, and let E be the proper communication graph of K. By construction, E is finite and contains no circuit. The left Fischer cover is isomorphic to an irreducible subgraph of $(K, \mathcal{L}_K)$ corresponding to a root $r \in E^0$ [19, Lemma 2.7], and by definition, there is an edge from $u \in E^0$ to $v \in E^0$ whenever $u > v$. The following proposition gives the range of the flow-invariant by proving that all such graphs can occur.

PROPOSITION 5.1. *Let E be a finite directed graph with a root and without circuits. E is the proper communication graph of the left Krieger cover of an AFT shift if there is an edge from $u \in E^0$ to $v \in E^0$ whenever $u > v$.*

Proof. Let E be an arbitrary finite directed graph which contains no circuit and which has a root r, and let $\tilde{E}$ be the directed graph obtained from E by adding an edge from $u \in E^0$ to $v \in E^0$ whenever $u > v$. The goal is to construct a labelled graph $(F, \mathcal{L}_F)$ which is the left Fischer cover of an irreducible sofic shift with the desired properties. For each $v \in E^0$, let $l(v)$ be the length of the longest path from r to v. This is well-defined since E does not contain any circuits. For each $v \in E^0$, define $n(v) = 2^{l(v)}$ vertices $v_1, \dots, v_{n(v)} \in F^0$. The single vertex corresponding to the root $r \in E^0$ is denoted r_1. For each $v \in E^0$, draw a loop of length 1 labelled a_v at each of the vertices $v_1, \dots, v_{n(v)} \in F^0$. If there is an edge from $u \in E^0$ to $v \in E^0$ then $l(v) > l(u)$. From each vertex $u_1, \dots, u_{n(u)}$ draw $n(u,v) = n(v)/n(u) = 2^{l(v)-l(u)} \geq 2$ edges labelled $a^1_{u,v}, \dots, a^{n(u,v)}_{u,v}$ such that every vertex $v_1, \dots, v_{n(v)}$ receives exactly one of these edges. For each sink $v \in E^0$ draw a uniquely labelled edge from each vertex $v_1, \dots, v_{n(v)}$ to r_1. This finishes the construction of $(F, \mathcal{L}_F)$.

By construction, F is irreducible, right-resolving, and left-resolving. Additionally, it is predecessor-separated because there is a uniquely labelled path to every vertex in F^0 from r_1. Thus, $(F, \mathcal{L}_F)$ is the left Fischer cover of an AFT shift X. Let $(K, \mathcal{L}_K)$ be the left Krieger cover of X.

For every $v \in E^0$, $P_\infty(a_v^\infty) = \bigcup_{i=1}^{n(v)} P_\infty(v_i)$ and no smaller set of vertices has this property, so $P_\infty(a_v^\infty)$ is a vertex in the $n(v)$th layer of the left Krieger cover. There is clearly a loop labelled a_v at the vertex $P_\infty(a_v^\infty)$, so it belongs

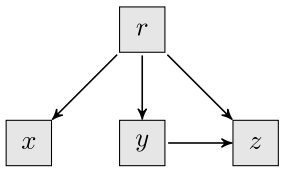


Figure 4: A directed graph with root r and without circuits.

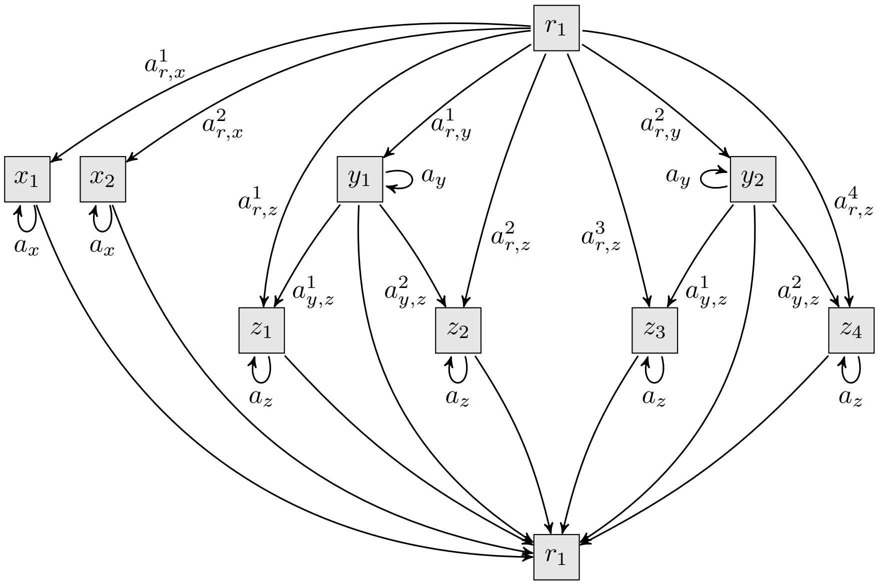


Figure 5: Left Fischer cover of the sofic shift X considered in Example 5.2.

to a proper communication set of vertices. Furthermore, $ba_v^\infty \in X^+$ if and only if $b = a_v$ or $b = a^i_{u,v}$ for some $u \in E^0$ and $1 \leq i \leq n(u,v)$. By construction, $P_\infty(a^i_{u,v}a_v^\infty) = \bigcup_{i=1}^{n(u)} P_\infty(u_i) = P_\infty(a_u^\infty)$, so there is an edge from $P_\infty(a_u^\infty)$ to $P_\infty(a_v^\infty)$ if and only if there is an edge from u to v in E. This proves that E, and hence also $\tilde{E}$, are subgraphs of the proper communication graph of K. Since the edges which terminate at r_1 are uniquely labelled, any $x^+ \in X^+$ which contains one of these letters must be intrinsically synchronizing. If $x^+ \in X^+$ does not contain any of these letters then x^+ must be eventually periodic with $x^+ = wa_v^\infty$ for some $v \in E^0$ and $w \in \mathcal{B}(X)$. Thus, K only has the vertices described above, and therefore the proper communication graph of K is $\tilde{E}$. □

EXAMPLE 5.2. To illustrate the construction used in the proof of Proposition 5.1, let E be the directed graph drawn in Figure 4. E has a unique maximal vertex r and contains no circuit, so it is the proper communication graph of

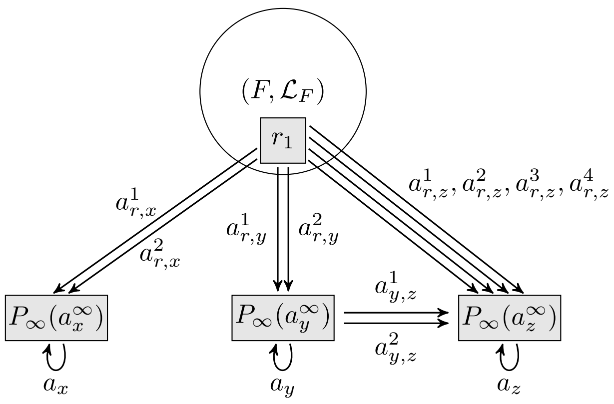


Figure 6: Left Krieger cover of the shift space X considered in Example 5.2. The structure of the irreducible component corresponding to the left Fischer cover has been suppressed.

the left Krieger cover of an irreducible sofic shift. Note that $l(x) = l(y) = 1$ and that $l(z) = 2$. Figure 5 shows the left Fischer cover of a sofic shift X constructed using the method from the proof of Proposition 5.1. Note that the top and bottom vertices should be identified, and that the labelling of the edges terminating at r_1 has been suppressed. Figure 6 shows the left Krieger cover of X, but the structure of the irreducible component corresponding to the left Fischer cover has been suppressed to emphasize the structure of the higher layers.

In [1], it was also remarked that an invariant analogous to the one discussed in Proposition 5.1 is obtained by considering the proper communication graph of the right Krieger cover. The following example shows that the two invariants may carry different information.

Example 5.3. The labelled graph in Figure 7 is left-resolving, irreducible, and predecessor-separated, so it is the left Fischer cover of an irreducible sofic shift. Similarly, the labelled graph in Figure 8 is irreducible, right-resolving and follower-separated, so it is the right Fischer cover of an irreducible sofic shift. By considering the edges labelled d, it is easy to see that the two graphs present the same sofic shift space X.

Every right-ray which contains a letter different from a or a' is intrinsically synchronizing, so consider a right-ray $x^+ \in X^+$ such that $(x^+)_i \in \{a, a'\}$ for all $i \in \mathbb{N}$. By considering Figure 7, it is clear that $P_\infty(x^+) = P_\infty(u) \cup P_\infty(v) \cup P_\infty(y) = P_\infty(y)$, so $P(x^+)$ is also in the first layer of the left Krieger cover. Hence, the proper communication graph has only one vertex and no edges.

Every left-ray containing a letter different from a or a' is intrinsically synchronizing, so consider the left-ray $a^\infty \in X^-$. Figure 8 shows that $F_\infty(a^\infty) =$

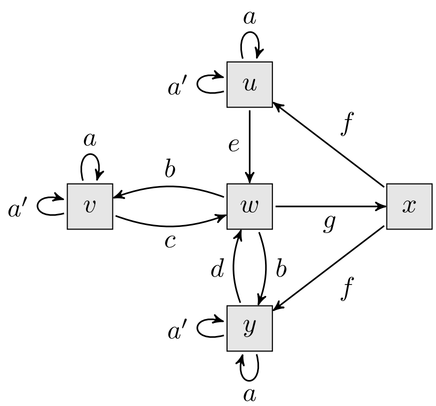


Figure 7: Left Fischer cover of the irreducible sofic shift X discussed in Example 5.3.

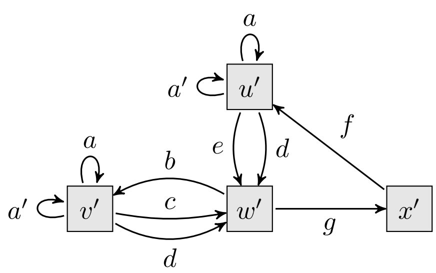


Figure 8: Right Fischer cover of the irreducible sofic shift X discussed in Example 5.3.

$F_\infty(u') \cup F_\infty(v')$ and that no single vertex y' in the right Fischer cover has $F_\infty(y') = F_\infty(a^\infty)$, so there is a vertex in the second layer of the right Krieger cover. In particular, the corresponding proper communication graph is non-trivial.

6 C^*-Algebras associated to sofic shift spaces

Cuntz and Krieger [13] introduced a class of C^*-algebras which can naturally be viewed as the universal C^*-algebras associated to shifts of finite type. This was generalized by Matsumoto [23] who associated two C^*-algebras $\mathcal{O}_X$ and $\mathcal{O}_{X^*}$ to every shift space X, and these Matsumoto algebras have been studied intensely [8, 18, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. The two Matsumoto algebras $\mathcal{O}_X$ and $\mathcal{O}_{X^*}$ are generated by elements satisfying the same relations, but they are not isomorphic in general [12]. This paper will follow the approach of Carlsen in [9] where a universal C^*-algebra $\mathcal{O}_{\tilde{X}}$ is associated to every one-sided shift space $\tilde{X}$. This also gives a way to associate C^*-algebras to every two-sided shift since a two-sided shift X corresponds to two one-sided shifts X^+ and X^-.

Ideal lattices. Let X be a sofic shift space and let $\mathcal{O}_{X^+}$ be the universal C^*-algebra associated to the one-sided shift X^+ as defined in [9]. Carlsen proved that $\mathcal{O}_{X^+}$ is isomorphic to the Cuntz-Krieger algebra of the left Krieger cover of X [8], so the lattice of gauge invariant ideals in $\mathcal{O}_{X^+}$ is given by the proper communication graph of the left Krieger cover of X [3, 20], and all ideals are given in this way if the left Krieger cover satisfies Condition (K) [36, Theorem 4.9]. Hence, Proposition 4.4 and Theorem 4.6 can be used to investigate the ideal lattice of $\mathcal{O}_{X^+}$. For a reducible sofic shift, a part of the ideal lattice is given by the structure of the generalized left Fischer cover, which is reducible, but if X is an irreducible sofic shift, and the left Krieger cover of X satisfies Condition (K) then the fact that the left Krieger cover has a unique top component implies that $\mathcal{O}_{X^+}$ will always have a unique maximal ideal. The following proposition shows that all these lattices can be realized.

Proposition 6.1. *Any finite lattice of ideals with a unique maximal ideal is the ideal lattice of the universal C^*-algebra $\mathcal{O}_{X^+}$ associated to an AFT shift X.*

Proof. Let E be a finite directed graph whitout circuits and with a unique maximal vertex. Consider the following slight modification of the algorithm from the proof of Proposition 5.1. For each $v \in E$, draw two loops of length 1 at each vertex $v_1, \dots, v_{n(v)}$ associated to v: One labelled a_v and one labelled a'_v. The rest of the construction is as before. Let $(K, \mathcal{L}_K)$ be the left Krieger cover of the corresponding sofic shift. As before, the proper communication graph of K is given by E, and now $(K, \mathcal{L}_K)$ satisfies Condition (K), so there is a bijective correspondence between the hereditary subsets of E^0 and the ideals of $C^*(K) \cong \mathcal{O}_{X^+}$. Since E was arbitrary, any finite ideal lattice with a unique maximal ideal can be obtained in this way. □

The C^*-algebras $\mathcal{O}_{X^+}$ and $\mathcal{O}_{X^-}$. Every two-sided shift space X corresponds to two one-sided shift spaces X^+ and X^-, and this gives two natural ways to associate a universal C^*-algebra to X. The next goal is to show that these two C^*-algebras may carry different information about the shift space. Let $\mathcal{O}_{X^-}$ be the universal C^*-algebra associated to the one-sided shift space $(X^{\mathrm{T}})^+$ as defined in [9]. The left Krieger cover of X^{T} is the transpose of the right Krieger cover of X, so by [8], $\mathcal{O}_{X^-}$ is isomorphic to the Cuntz-Krieger algebra of the transpose of the right Krieger cover of X.

Example 6.2. Let X be the sofic shift from Example 5.3. Note that the left and right Krieger covers of X both satisfy Condition (K) from [36], so the corresponding proper communication graphs completely determine the ideal lattices of $\mathcal{O}_{X^+}$ and $\mathcal{O}_{X^-}$. The proper communication graph of the left Krieger cover $(K, \mathcal{L}_K)$ of X is trivial, so $\mathcal{O}_{X^+}$ is simple, while there are precisely two vertices in the proper communication graph of the right Krieger cover of X, so there is exactly one non-trivial ideal in $\mathcal{O}_{X^-}$. In particular, $\mathcal{O}_{X^+}$ and $\mathcal{O}_{X^-}$ are not isomorphic.

Consider the edge shift $Y = \mathsf{X}_K$. This is an SFT, and the left and right Krieger covers of Y are both $(K, \mathcal{L}_{\mathrm{Id}})$, where $\mathcal{L}_{\mathrm{Id}}$ is the identity map on the edge set K^1. By [8], $\mathcal{O}_{X^+}$ and $\mathcal{O}_{Y^+}$ are isomophic to $C^*(K)$. Similarly, $\mathcal{O}_{Y^-}$ is isomorphic to $C^*(K^{\mathrm{T}})$ and K^{T} is an irreducible graph satisfying Condition (K), so $\mathcal{O}_{Y^-}$ is simple. In particular, $\mathcal{O}_{Y^-}$ is not isomophic to $\mathcal{O}_{X^-}$. This shows that the C^*-algebras associated to X^+ and X^- are not always isomorphic, and that there can exist a shift space Y such that $\mathcal{O}_{Y^+}$ is isomorphic to $\mathcal{O}_{X^+}$ while $\mathcal{O}_{Y^-}$ is not isomorphic to $\mathcal{O}_{X^-}$.

AN INVESTIGATION OF CONDITION $(*)$. In [12], two C^*-algebras $\mathcal{O}_X$ and $\mathcal{O}_{X^*}$ are associated to every two-sided shift space X. The C^*-algebras $\mathcal{O}_X$, $\mathcal{O}_{X^*}$, and $\mathcal{O}_{X^+}$ are generated by partial isometries satisfying the same relations, but $\mathcal{O}_{X^+}$ is always universal unlike $\mathcal{O}_X$ [9]. In [12], it is proved that $\mathcal{O}_X$ and $\mathcal{O}_{X^*}$ are isomorphic when X satisifies a condition called Condition $(*)$. The example from [12, Section 4] shows that not all sofic shift spaces satisfy this condition by constructing a sofic shift where the left Krieger cover and the past set cover are not isomorphic. The relationship between Condition $(*)$ and the structure of the left Krieger cover and the past set cover is further clarified by the final main result. For each $l \in \mathbb{N}$ and $w \in \mathcal{B}(X)$ define $P_l(w) = \{v \in \mathcal{B}(X) \mid vw \in \mathcal{B}(X), |v| \leq l\}$. Two words $v, w \in \mathcal{B}(X)$ are said to be l-past equivalent if $P_l(v) = P_l(w)$. For $x^+ \in X^+$, $P_l(x^+)$ and l-past equivalence are defined analogously.

CONDITION $(*)$. For every $l \in \mathbb{N}$ and every infinite $F \subseteq \mathcal{B}(X)$ such that $P_l(u) = P_l(v)$ for all $u, v \in F$ there exists $x^+ \in X^+$ such that $P_l(w) = P_l(x^+)$ for all $w \in F$.

LEMMA 6.3. *A vertex P in the past set cover of a sofic shift X is in an essential subgraph if and only if there exist infinitely many $w \in \mathcal{B}(X)$ such that $P_\infty(w) = P$.*

Proof. Let P be a vertex in an essential subgraph of the past set cover of X, and let $x^+ \in X^+$ be a right ray with a presentation starting at P. Given $n \in \mathbb{N}$, there exists $w_n \in \mathcal{B}(X)$ such that $P = P_\infty(x_1 x_2 \dots x_n w_n)$. To prove the converse, let P be a vertex in the past set cover for which there exist infinitely many $w \in \mathcal{B}(X)$ such that $P = P_\infty(w)$. For each w, there is a path labelled $w_{[1,|w|-1]}$ starting at P. There are no sources in the past set cover, so this implies that P is not stranded. $\square$

PROPOSITION 6.4. *A sofic shift X satisfies Condition $(*)$ if and only if the left Krieger cover is the maximal essential subgraph of the past set cover.*

Proof. Assume that X satisfies Condition $(*)$. Let P be a vertex in an essential subgraph of the past set cover and define $F = \{w \in \mathcal{B}(X) \mid P_\infty(w) = P\}$. Choose $m \in \mathbb{N}$ such that for all $x, y \in \mathcal{B}(X) \cup X^+$, $P_\infty(x) = P_\infty(y)$ if and only if $P_m(x) = P_m(y)$. By Lemma 6.3, F is an infinite set, so Condition $(*)$ can be used to choose $x^+ \in X^+$ such that $P_m(x^+) = P_m(w)$ for all $w \in F$. By the choice of m, this means that $P_\infty(x^+) = P_\infty(w) = P$ for all $w \in F$, so P is a vertex in the left Krieger cover.

To prove the other implication, assume that the left Krieger cover is the maximal essential subgraph of the past set cover. Let $l \in \mathbb{N}$ be given, and consider an infinite set $F \subseteq \mathcal{B}(X)$ for which $P_l(u) = P_l(v)$ for all $u, v \in F$. Since X is sofic, there are only finitely many different predecessor sets, so there must exist $w \in F$ such that $P_\infty(w) = P_\infty(v)$ for infinitely many $v \in F$. By Lemma 6.3, this proves that $P = P_\infty(w)$ is a vertex in the maximal essential subgraph of the past set cover. By assumption, this means that it is a vertex in the left Krieger cover, so there exists $x^+ \in X^+$ such that $P_\infty(w) = P_\infty(x^+)$. In particular, $P_l(x^+) = P_l(w) = P_l(v)$ for all $v \in F$, so Condition $(*)$ is satisfied. □

In [2], it was proved that $\mathcal{O}_{X^*}$ is isomorphic to the Cuntz-Krieger algebra of the past set cover of X when X satisfies a condition called Condition (I). According to Carlsen [7], a proof similar to the proof which shows that $\mathcal{O}_{X^+}$ is isomorphic to the Cuntz-Krieger algebra of the left Krieger cover of X should prove that $\mathcal{O}_{X^*}$ is isomorphic to the Cuntz-Krieger algebra of the subgraph of the past set cover of X induced by the vertices P for which there exist infinitely many words w such that $P_\infty(w) = P$. Using Lemma 6.3, this shows that $\mathcal{O}_{X^*}$ is always isomorphic to the Cuntz-Krieger algebra of the maximal essential subgraph of the past set cover of X.

References

[1] T. Bates, S. Eilers, and D. Pask. Reducibility of covers of AFT shifts. *Israel J. Math.*, 2011 (in press).

[2] T. Bates and D. Pask. C^*-algebras of labelled graphs. *J. Operator Theory*, 57(1):207–226, 2007.

[3] T. Bates, D. Pask, I. Raeburn, and W. Szymański. The C^*-algebras of row-finite graphs. *New York J. Math.*, 6:307–324 (electronic), 2000.

[4] M. Boyle. Flow equivalence of shifts of finite type via positive factorizations. *Pacific J. Math.*, 204(2):273–317, 2002.

[5] M. Boyle, T. M. Carlsen, and S. Eilers. Flow equivalence of sofic shifts. *In preparation.*

[6] M. Boyle and D. Huang. Poset block equivalence of integral matrices. *Trans. Amer. Math. Soc.*, 355(10):3861–3886 (electronic), 2003.

[7] T. M. Carlsen. Personal communication.

[8] T. M. Carlsen. On C^*-algebras associated with sofic shifts. *J. Operator Theory*, 49(1):203–212, 2003.

[9] T. M. Carlsen. Cuntz-Pimsner C^*-algebras associated with subshifts. *Internat. J. Math.*, 19(1):47–70, 2008.

[10] T. M. Carlsen and S. Eilers. Augmenting dimension group invariants for substitution dynamics. *Ergodic Theory Dynam. Systems*, 24(4):1015–1039, 2004.

[11] T. M. Carlsen and S. Eilers. Matsumoto K-groups associated to certain shift spaces. *Doc. Math.*, 9:639–671 (electronic), 2004.

[12] T. M. Carlsen and K. Matsumoto. Some remarks on the C^*-algebras associated with subshifts. *Math. Scand.*, 95(1):145–160, 2004.

[13] J. Cuntz and W. Krieger. A class of C^*-algebras and topological Markov chains. *Invent. Math.*, 56(3):251–268, 1980.

[14] R. Fischer. Sofic systems and graphs. *Monatsh. Math.*, 80(3):179–186, 1975.

[15] D. Huang. Flow equivalence of reducible shifts of finite type. *Ergodic Theory Dynam. Systems*, 14(4):695–720, 1994.

[16] N. Jonoska. Sofic shifts with synchronizing presentations. *Theoret. Comput. Sci.*, 158(1-2):81–115, 1996.

[17] N. Jonoska and B. Marcus. Minimal presentations for irreducible sofic shifts. *IEEE Trans. Inform. Theory*, 40(6):1818–1825, 1994.

[18] Y. Katayama, K. Matsumoto, and Y. Watatani. Simple C^*-algebras arising from β-expansion of real numbers. *Ergodic Theory Dynam. Systems*, 18(4):937–962, 1998.

[19] W. Krieger. On sofic systems. I. *Israel J. Math.*, 48(4):305–330, 1984.

[20] A. Kumjian, D. Pask, I. Raeburn, and J. Renault. Graphs, groupoids, and Cuntz-Krieger algebras. *J. Funct. Anal.*, 144(2):505–541, 1997.

[21] D. Lind and B. Marcus. *An introduction to symbolic dynamics and coding.* Cambridge University Press, Cambridge, 1995.

[22] B. Marcus. Sofic systems and encoding data. *IEEE Trans. Inform. Theory*, 31(3):366–377, 1985.

[23] K. Matsumoto. On C^*-algebras associated with subshifts. *Internat. J. Math.*, 8(3):357–374, 1997.

[24] K. Matsumoto. K-theory for C^*-algebras associated with subshifts. *Math. Scand.*, 82(2):237–255, 1998.

[25] K. Matsumoto. Dimension groups for subshifts and simplicity of the associated C^*-algebras. *J. Math. Soc. Japan*, 51(3):679–698, 1999.

[26] K. Matsumoto. Relations among generators of C^*-algebras associated with subshifts. *Internat. J. Math.*, 10(3):385–405, 1999.

[27] K. Matsumoto. A simple C^*-algebra arising from a certain subshift. *J. Operator Theory*, 42(2):351–370, 1999.

[28] K. Matsumoto. On automorphisms of C^*-algebras associated with subshifts. *J. Operator Theory*, 44(1):91–112, 2000.

[29] K. Matsumoto. Stabilized C^*-algebras constructed from symbolic dynamical systems. *Ergodic Theory Dynam. Systems*, 20(3):821–841, 2000.

[30] K. Matsumoto. Bowen-Franks groups for subshifts and Ext-groups for C^*-algebras. *K-Theory*, 23(1):67–104, 2001.

[31] K. Matsumoto. C^*-algebras associated with presentations of subshifts. *Doc. Math.*, 7:1–30 (electronic), 2002.

[32] K. Matsumoto, Y. Watatani, and M. Yoshida. KMS states for gauge actions on C^*-algebras associated with subshifts. *Math. Z.*, 228(3):489–509, 1998.

[33] M. Nasu. An invariant for bounded-to-one factor maps between transitive sofic subshifts. *Ergodic Theory Dynam. Systems*, 5(1):89–105, 1985.

[34] M. Nasu. Topological conjugacy for sofic systems. *Ergodic Theory Dynam. Systems*, 6(2):265–280, 1986.

[35] B. Parry and D. Sullivan. A topological invariant of flows on 1-dimensional spaces. *Topology*, 14(4):297–299, 1975.

[36] I. Raeburn. *Graph algebras*, volume 103 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2005.

[37] K. Thomsen. On the structure of a sofic shift space. *Trans. Amer. Math. Soc.*, 356(9):3557–3619 (electronic), 2004.

[38] B. Weiss. Subshifts of finite type and sofic systems. *Monatsh. Math.*, 77:462–474, 1973.

Rune Johansen
Department of
Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 København Ø
Denmark

Ergodic Properties and KMS Conditions on C^*-Symbolic Dynamical Systems

Kengo Matsumoto

Received: March 3, 2010
Revised: March 12, 2011

Communicated by Joachim Cuntz

Abstract. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ consists of a unital C^*-algebra $\mathcal{A}$ and a finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms ρ_α of $\mathcal{A}$ indexed by symbols α of Σ satisfying some conditions. The endomorphisms $\rho_\alpha, \alpha \in \Sigma$ yield both a subshift Λ_ρ and a C^*-algebra $\mathcal{O}_\rho$. We will study ergodic properties of the positive operator $\lambda_\rho = \sum_{\alpha\in\Sigma} \rho_\alpha$ on $\mathcal{A}$. We will next introduce KMS conditions for continuous linear functionals on $\mathcal{O}_\rho$ under gauge action at inverse temperature taking its value in complex numbers. We will study relationships among the eigenvectors of λ_ρ in $\mathcal{A}^*$, the continuous linear functionals on $\mathcal{O}_\rho$ satisfying KMS conditions and the invariant measures on the associated one-sided shifts. We will finally present several examples of continuous linear functionals satisfying KMS conditions.

2000 Mathematics Subject Classification: Primary 46L55, 37B10; Secondary 37D35.
Keywords and Phrases: C^*-algebra, symbolic dynamics, subshift, ergodic, KMS condition, invariant measure.

1. Introduction

D. Olesen and G. K. Pedersen [37] have shown that the C^*-dynamical system $(\mathcal{O}_N, \alpha, \mathbf{R})$ for the Cuntz algebra $\mathcal{O}_N$ with gauge action α admits a KMS state at the inverse temperature γ if and only if $\gamma = \log N$, and the admitted KMS state is unique. By Enomoto-Fujii-Watatani [9], the result has been generalized to the Cuntz-Krieger algebras $\mathcal{O}_A$ as $\gamma = \log r_A$, where r_A is the Perron-Frobenius eigenvalue for the irreducible matrix A with entries in $\{0, 1\}$. These results are generalized to several classes of C^*-algebras having gauge actions (cf. [7], [10], [11], [15], [17], [18], [27], [35], [36], [41], etc.).

Cuntz-Krieger algebras are considered to be constructed by finite directed graphs which yield an important class of symbolic dynamics called shifts of finite type. In [29], the author has generalized the notion of finite directed graphs to a notion of labeled Bratteli diagrams having shift like maps, which we call λ-graph systems. A λ-graph system $\mathfrak{L}$ gives rise to both a subshift $\Lambda_{\mathfrak{L}}$ and a C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ with gauge action. Some topological conjugacy invariants of subshifts have been studied through the C^*-algebras constructed from λ-graph systems ([30]).

A C^**-symbolic dynamical system* is a generalization of both a λ-graph system and an automorphism of a unital C^*-algebra ([31]). It is a finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms indexed by a finite set Σ of a unital C^*-algebra $\mathcal{A}$ such that $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}$ for $\alpha \in \Sigma$ and $\sum_{\alpha\in\Sigma}\rho_\alpha(1) \geq 1$ where $Z_{\mathcal{A}}$ denotes the center of $\mathcal{A}$. A finite directed labeled graph $\mathcal{G}$ gives rise to a C^*-symbolic dynamical system $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$ such that $\mathcal{A}_{\mathcal{G}} = \mathbf{C}^N$ for some $N \in \mathbf{N}$. A λ-graph system $\mathfrak{L}$ also gives rise to a C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ such that $\mathcal{A}_{\mathfrak{L}}$ is $C(\Omega_{\mathfrak{L}})$ for some compact Hausdorff space $\Omega_{\mathfrak{L}}$ with $\dim\Omega_{\mathfrak{L}} = 0$. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a subshift denoted by Λ_ρ over Σ and a Hilbert C^*-bimodule $(\phi_\rho, \mathcal{H}^\rho_{\mathcal{A}})$ over $\mathcal{A}$. By using general construction of C^*-algebras from Hilbert C^*-bimodules established by M. Pimsner [40], a C^*-algebra denoted by $\mathcal{O}_\rho$ from $(\phi_\rho, \mathcal{H}^\rho_{\mathcal{A}})$ has been introduced in [31]. The C^*-algebra $\mathcal{O}_\rho$ is realized as the universal C^*-algebra generated by partial isometries $S_\alpha, \alpha \in \Sigma$ and $x \in \mathcal{A}$ subject to the relations:

$$\sum_{\gamma\in\Sigma} S_\gamma S_\gamma^* = 1, \qquad S_\alpha S_\alpha^* x = x S_\alpha S_\alpha^*, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$. We call the algebra $\mathcal{O}_\rho$ *the* C^**-symbolic crossed product of* $\mathcal{A}$ *by the subshift* Λ_ρ. The gauge action on $\mathcal{O}_\rho$ denoted by $\hat\rho$ is defined by

$$\hat\rho_z(x) = x, \quad x \in \mathcal{A} \quad \text{and} \quad \hat\rho_z(S_\alpha) = zS_\alpha, \quad \alpha \in \Sigma$$

for $z \in \mathbf{C}, |z| = 1$. If $\mathcal{A} = C(X)$ with $\dim X = 0$, there exists a λ-graph system $\mathfrak{L}$ such that Λ_ρ is the subshift presented by $\mathfrak{L}$ and $\mathcal{O}_\rho$ is the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with $\mathfrak{L}$. If in particular, $\mathcal{A} = \mathbf{C}^N$, the subshift Λ_ρ is a sofic shift and $\mathcal{O}_\rho$ is a Cuntz-Krieger algebra. If $\Sigma = \{\alpha\}$ an automorphism α of a unital C^*-algebra $\mathcal{A}$, the C^*-algebra $\mathcal{O}_\rho$ is the ordinary C^*-crossed product $\mathcal{A} \times_\alpha \mathbf{Z}$. Throughout the paper, we will assume that the C^*-algebra $\mathcal{A}$ is commutative. For a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, define the positive operator λ_ρ on $\mathcal{A}$ by

$$\lambda_\rho(x) = \sum_{\alpha\in\Sigma}\rho_\alpha(x), \qquad x \in \mathcal{A}.$$

We set for a complex number $\beta \in \mathbf{C}$ the eigenvector space of λ_ρ

$$\mathcal{E}_\beta(\rho) = \{\varphi \in \mathcal{A}^* \mid \varphi \circ \lambda_\rho = \beta\varphi\}. \tag{1.1}$$

Let $Sp(\rho)$ be the set of eigenvalues of λ_ρ defined by

$$Sp(\rho) = \{\beta \in \mathbf{C} \mid \mathcal{E}_\beta(\rho) \neq \{0\}\}. \tag{1.2}$$

Let r_ρ denote the spectral radius of λ_ρ on $\mathcal{A}$. We set $T_\rho = \frac{1}{r_\rho}\lambda_\rho$. $(\mathcal{A},\rho,\Sigma)$ is said to be *power-bounded* if the sequence $\|T_\rho^k\|, k \in \mathbf{N}$ is bounded. A state φ on $\mathcal{A}$ is said to be invariant if $\varphi \circ T_\rho = \varphi$. If an invariant state is unique, $(\mathcal{A},\rho,\Sigma)$ is said to be *uniquely ergodic*. If $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}T_\rho^k(a)$ exists in $\mathcal{A}$ for $a \in \mathcal{A}$, $(\mathcal{A},\rho,\Sigma)$ is said to be *mean ergodic*. If there exists no nontrivial ideal of $\mathcal{A}$ invariant under λ_ρ, $(\mathcal{A},\rho,\Sigma)$ is said to be *irreducible*. It will be proved that a mean ergodic and irreducible $(\mathcal{A},\rho,\Sigma)$ is uniquely ergodic and power-bounded (Theorem 3.12).

Let $A = [A(i,j)]_{i,j=1}^N$ be an irreducible matrix with entries in $\{0,1\}$, and $S_i, i = 1,\dots,N$ be the canonical generating family of partial isometries of the Cuntz-Krieger algebra $\mathcal{O}_A$. Let $\mathcal{A}_A$ be the C^*-subalgebra of $\mathcal{O}_A$ generated by the projections $S_jS_j^*, j = 1,\dots,N$. Put $\Sigma = \{1,\dots,N\}$ and $\rho_i^A(x) = S_i^*xS_i, x \in \mathcal{A}_A, i \in \Sigma$. Then the triplet $(\mathcal{A}_A,\rho^A,\Sigma)$ yields an example of C^*-symbolic dynamical system such that its C^*-symbolic crossed product $\mathcal{O}_{\rho^A}$ is the Cuntz-Krieger algebra $\mathcal{O}_A$. The above space $\mathcal{E}_\beta(\rho)$ is identified with the eigenvector space of the matrix A for an eigenvalue β. By Enomoto-Fujii-Watatani [9], a tracial state $\varphi \in \mathcal{E}_\beta(\rho^A)$ on $\mathcal{A}_A$ extends to a KMS state for gauge action on $\mathcal{O}_A$ if and only if $\beta = r_A$ the Perron-Frobenius eigenvalue, and its inverse temperature is $\log r_A$. The admitted KMS state is unique.

In this paper, we will study the space $\mathcal{E}_\beta(\rho)$ of a general C^*-symbolic dynamical system $(\mathcal{A},\rho,\Sigma)$ for a general eigenvalue β in $\mathbf{C}$ not necessarily maximum eigenvalue and then introduce KMS condition for inverse temperature taking its value in complex numbers. In this generalization, we will study possibility of extension of a continuous linear functional on $\mathcal{A}$ belonging to the eigenvector space $\mathcal{E}_\beta(\rho)$ to the whole algebra $\mathcal{O}_\rho$ as a continuous linear functional satisfying KMS condition. For a C^*-algebra with a continuous action of the one-dimensional torus group $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ and a complex number $\beta \in \mathbf{C}$, we will introduce KMS condition for a continuous linear functional without assuming its positivity at inverse temperature $\mathrm{Log}\beta$. Let $\mathcal{B}$ be a C^*-algebra and $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ be a continuous action of $\mathbf{T}$ to the automorphism group $\mathrm{Aut}(\mathcal{B})$. We write a complex number β with $|\beta| > 1$ as $\beta = re^{i\theta}$ where $r > 1, \theta \in \mathbf{R}$. Denote by $\mathcal{B}^*$ the Banach space of all complex valued continuous linear functionals on $\mathcal{B}$.

Definition. A continuous linear functional $\varphi \in \mathcal{B}^*$ is said to satisfy *KMS condition* at $\mathrm{Log}\beta$ if φ satisfies the condition

$$\varphi(y\alpha_{i\log r}(x)) = \varphi(\alpha_\theta(x)y), \qquad x \in \mathcal{B}^a, y \in \mathcal{B}, \tag{1.3}$$

where $\mathcal{B}^a$ is the set of analytic elements of the action $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ (cf.[3]). We will prove

Theorem 1.1. *Let $(\mathcal{A},\rho,\Sigma)$ be an irreducible and power-bounded C^*-symbolic dynamical system. Let $\beta \in \mathbf{C}$ be a complex number with $|\beta| > 1$.*

(i) *If $\beta \in Sp(\rho)$ and $|\beta| = r_\rho$ the spectral radius of the positive operator $\lambda_\rho : \mathcal{A} \longrightarrow \mathcal{A}$, then there exists a nonzero continuous linear functional* *on $\mathcal{O}_\rho$ satisfying KMS condition at* $\mathrm{Log}\beta$ *under gauge action. The converse implication holds if $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic.*

(ii) *Under the condition $|\beta| = r_\rho$, there exists a linear isomorphism between the space $\mathcal{E}_\beta(\rho)$ of eigenvectors of continuous linear functionals on $\mathcal{A}$ and the space $KMS_\beta(\mathcal{O}_\rho)$ of continuous linear functionals on $\mathcal{O}_\rho$ satisfying KMS condition at* $\mathrm{Log}\beta$.

(iii) *If $(\mathcal{A}, \rho, \Sigma)$ is uniquely ergodic, there uniquely exists a state on $\mathcal{O}_\rho$ satisfying KMS condition at* $\log r_\rho$.

(iv) *If in particular $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic, then* $\dim \mathcal{E}_\beta(\rho) \le 1$ *for all* $\beta \in \mathbf{C}$.

In the proof of the above theorem, a Perron-Frobenius type theorem is proved (Theorem 3.13).

Let $\mathcal{D}_\rho$ be the C^*-subalgebra of $\mathcal{O}_\rho$ generated by all elements of the form: $S_{\alpha_1} \cdots S_{\alpha_k} x S^*_{\alpha_k} \cdots S^*_{\alpha_1}$ for $x \in \mathcal{A}, \alpha_1, \dots, \alpha_k \in \Sigma$. Let ϕ_ρ be the endomorphism on $\mathcal{D}_\rho$ defined by $\phi_\rho(y) = \sum_{\alpha\in\Sigma} S_\alpha y S^*_\alpha, y \in \mathcal{D}_\rho$, which comes from the left-shift on the underlying shift spase Λ_ρ. Suppose that $(\mathcal{A}, \rho, \Sigma)$ is uniquely ergodic. The restriction of the unique KMS state on $\mathcal{O}_\rho$ is not necessarily a ϕ_ρ-invariant state. We will clarify a relationship between KMS states on $\mathcal{O}_\rho$ and ϕ_ρ-invariant states on $\mathcal{D}_\rho$ as in the following way:

Theorem 1.2. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and mean ergodic. Let τ be the restriction to $\mathcal{D}_\rho$ of the unique KMS state on $\mathcal{O}_\rho$ at* $\log r_\rho$ *and x_ρ be a positive element of $\mathcal{A}$ defined by the limit of the mean*

$$\lim_{n\to\infty} \frac{1}{n}(1 + T_\rho(1) + \cdots + T_\rho^{n-1}(1)).$$

Let μ_ρ be a linear functional on $\mathcal{D}_\rho$ defined by

$$\mu_\rho(y) = \tau(y x_\rho), \qquad y \in \mathcal{D}_\rho.$$

(i) *μ_ρ is a faithful, ϕ_ρ-invariant and ergodic state on $\mathcal{D}_\rho$ in the sense that the formula*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu_\rho(\phi_\rho^k(y)x) = \mu_\rho(y)\mu_\rho(x), \qquad x, y \in \mathcal{D}_\rho$$

holds.

(ii) *μ_ρ gives rise to a unique ϕ_ρ-invariant probability measure absolutely continuous with respect to the probability measure for the state τ.*

(iii) *μ_ρ is equivalent to the state τ as a measure on $\mathcal{D}_\rho$.*

For a C^*-symbolic dynamical system $(\mathcal{A}_A, \rho^A, \Sigma)$ coming from an irreducible matrix $A = [A(i,j)]_{i,j=1}^N$ with entries in $\{0, 1\}$, the subalgebra $\mathcal{D}_{\rho^A}$ is nothing but the commutative C^*-algebra $C(X_A)$ of all continuous functions on the right one-sided topological Markov shift X_A. As ϕ_{ρ^A} corresponds to the left-shift σ_A on X_A, the above unique ϕ_{ρ^A} -invariant state τ is the Parry measure on X_A. The positive element x_{ρ^A} is given by the positive Perron eigenvector

$x_{\rho^A} = [x_j]_{j=1}^N$ of the transpose A^t of A satisfying $\sum_{j=1}^N \tau(S_j S_j^*) x_j = 1$, where $[\tau(S_j S_j^*)]_{j=1}^N$ is the normalized Perron eigenvector of A.
This paper is organized as follows: In Section 2, we will briefly review C^*-symbolic dynamical systems and its C^*-algebras $\mathcal{O}_\rho$. In Section 3, we will study ergodic properties of the operator $T_\rho : \mathcal{A} \longrightarrow \mathcal{A}$ and the eigenspace $\mathcal{E}_\beta(\rho)$. In Section 4, we will study extendability of a linear functional belonging to $\mathcal{E}_\beta(\rho)$ to the subalgebra $\mathcal{D}_\rho$ of $\mathcal{O}_\rho$, which will extend to $\mathcal{O}_\rho$. In Section 5, we will prove Theorem 1.1. In Section 6, we will study a relationship between KMS states and ϕ_ρ-invariant states on $\mathcal{D}_\rho$ to prove Theorem 1.2. In Section 7, we will present several examples of continuous linear functionals on $\mathcal{O}_\rho$ satisfying KMS conditions.

2. C^*-symbolic dynamical systems and their crossed products

Let $\mathcal{A}$ be a unital C^*-algebra. In what follows, an endomorphism of $\mathcal{A}$ means a $*$-endomorphism of $\mathcal{A}$ that does not necessarily preserve the unit 1 of $\mathcal{A}$. Denote by $Z_\mathcal{A}$ the center $\{x \in \mathcal{A} \mid ax = xa \text{ for all } a \in \mathcal{A}\}$ of $\mathcal{A}$. Let Σ be a finite set. A finite family of nonzero endomorphisms $\rho_\alpha, \alpha \in \Sigma$ of $\mathcal{A}$ indexed by elements of Σ is said to be *essential* if $\rho_\alpha(Z_\mathcal{A}) \subset Z_\mathcal{A}$ for $\alpha \in \Sigma$ and $\sum_{\alpha\in\Sigma} \rho_\alpha(1) \geq 1$. If in particular, $\mathcal{A}$ is commutative, the family $\rho_\alpha, \alpha \in \Sigma$ is essential if and only if $\sum_{\alpha\in\Sigma} \rho_\alpha(1) \geq 1$. We remark that the definition in [31] of "essential" for $\rho_\alpha, \alpha \in \Sigma$ is weaker than the above dfinition. It is said to be *faithful* if for any nonzero $x \in \mathcal{A}$ there exists a symbol $\alpha \in \Sigma$ such that $\rho_\alpha(x) \neq 0$.
Definition ([31]). A *C^*-symbolic dynamical system* is a triplet $(\mathcal{A}, \rho, \Sigma)$ consisting of a unital C^*-algebra $\mathcal{A}$ and an essential, faithful finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms of $\mathcal{A}$.
Two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are said to be isomorphic if there exist an isomorphism $\Phi : \mathcal{A} \to \mathcal{A}'$ and a bijection $\pi : \Sigma \to \Sigma'$ such that $\Phi \circ \rho_\alpha = \rho'_{\pi(\alpha)} \circ \Phi$ for all $\alpha \in \Sigma$. For an automorphism α of a unital C^*-algebra $\mathcal{A}$, by setting $\Sigma = \{\alpha\}, \rho_\alpha = \alpha$ the triplet $(\mathcal{A}, \rho, \Sigma)$ becomes a C^*-symbolic dynamical system. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a subshift Λ_ρ over Σ such that a word $\alpha_1 \cdots \alpha_k$ of Σ is admissible for Λ_ρ if and only if $\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1} \neq 0$ ([31, Proposition 2.1]). We say that *a subshift Λ acts on a C^*-algebra $\mathcal{A}$* if there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift Λ_ρ is Λ.
For a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ the C^*-algebra $\mathcal{O}_\rho$ has been originally constructed in [31] as a C^*-algebra from a Hilbert C^*-bimodule by using a Pimsner's general construction of Hilbert C^*-bimodule algebras [40] (cf. [16] etc.). It is called *the C^*-symbolic crossed product of $\mathcal{A}$ by the subshift* Λ_ρ, and realized as the universal C^*-algebra $C^*(x, S_\alpha; x \in \mathcal{A}, \alpha \in \Sigma)$ generated by $x \in \mathcal{A}$ and partial isometries $S_\alpha, \alpha \in \Sigma$ subject to the following relations called (ρ):

$$\sum_{\gamma\in\Sigma} S_\gamma S_\gamma^* = 1, \qquad S_\alpha S_\alpha^* x = x S_\alpha S_\alpha^*, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$.

Let $\mathcal{G} = (G, \lambda)$ be a left-resolving finite labeled graph with underlying finite directed graph $G = (V, E)$ and labeling map $\lambda : E \to \Sigma$ (see [28, p.76]). Denote by $v_1, \dots, v_N$ the vertex set V. Assume that every vertex has both an incoming edge and an outgoing edge. Consider the N-dimensional commutative C^*-algebra $\mathcal{A}_{\mathcal{G}} = \mathbf{C}E_1 \oplus \cdots \oplus \mathbf{C}E_N$ where each minimal projection E_i corresponds to the vertex v_i for $i = 1, \dots, N$. Define an $N \times N$-matrix for $\alpha \in \Sigma$ by

$$A^{\mathcal{G}}(i, \alpha, j) = \begin{cases} 1 & \text{if there exists an edge } e \text{ from } v_i \text{ to } v_j \text{ with } \lambda(e) = \alpha, \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

for $i, j = 1, \dots, N$. We set $\rho^{\mathcal{G}}_\alpha(E_i) = \sum_{j=1}^N A^{\mathcal{G}}(i, \alpha, j) E_j$ for $i = 1, \dots, N$. Then $\rho^{\mathcal{G}}_\alpha, \alpha \in \Sigma$ define endomorphisms of $\mathcal{A}_{\mathcal{G}}$ such that $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$ is a C^*-symbolic dynamical system for which the subshift $\Lambda_{\rho^{\mathcal{G}}}$ is the sofic shift $\Lambda_{\mathcal{G}}$ presented by $\mathcal{G}$. Conversely, for a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, if $\mathcal{A}$ is $\mathbf{C}^N$, there exists a left-resolving labeled graph $\mathcal{G}$ such that $\mathcal{A} = \mathcal{A}_{\mathcal{G}}$ and $\Lambda_\rho = \Lambda_{\mathcal{G}}$ the sofic shift presented by $\mathcal{G}$ ([31, Proposition 2.2]). Put $A_{\mathcal{G}}(i, j) = \sum_{\alpha \in \Sigma} A^{\mathcal{G}}(i, \alpha, j), i, j = 1, \dots, N$. The $N \times N$ matrix $A_{\mathcal{G}} = [A_{\mathcal{G}}(i, j)]_{i,j=1,\dots,N}$ is called the underlying nonnegative matrix for $\mathcal{G}$. Consider the matrix $A^{[2]}_{\mathcal{G}} = [A^{[2]}_{\mathcal{G}}(e, f)]_{e,f \in E}$ indexed by edges E whose entries are in $\{0, 1\}$ by setting

$$A^{[2]}_{\mathcal{G}}(e, f) = \begin{cases} 1 & \text{if } f \text{ follows } e, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The C^*-algebra $\mathcal{O}_{\rho^{\mathcal{G}}}$ for the C^*-symbolic dynamical system $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$ is the Cuntz-Krieger algebra $\mathcal{O}_{A^{[2]}_{\mathcal{G}}}$ (cf. [30, Proposition 7.1], [1]).

More generally let $\mathfrak{L}$ be a λ-graph system (V, E, λ, ι) over Σ. We equip each vertex set V_l with discrete topology. We denote by $\Omega_{\mathfrak{L}}$ the compact Hausdorff space with $\dim \Omega_{\mathfrak{L}} = 0$ of the projective limit $V_0 \xleftarrow{\iota} V_1 \xleftarrow{\iota} V_2 \xleftarrow{\iota} \cdots$ as in [30, Section 2]. Since the algebra $C(V_l)$ denoted by $\mathcal{A}_{\mathfrak{L},l}$ of all continuous functions on V_l is the commutative finite dimensional algebra, the commutative C^*-algebra $C(\Omega_{\mathfrak{L}})$ is an AF-algebra, that is denoted by $\mathcal{A}_{\mathfrak{L}}$. We then have a C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ such that the subshift $\Lambda_{\rho^{\mathfrak{L}}}$ coincides with the subshift $\Lambda_{\mathfrak{L}}$ presented by $\mathfrak{L}$. Conversely, for a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, if the algebra $\mathcal{A}$ is $C(X)$ with $\dim X = 0$, there exists a λ-graph system $\mathfrak{L}$ over Σ such that the associated C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ is isomorphic to $(\mathcal{A}, \rho, \Sigma)$ ([31, Theorem 2.4]). The C^*-algebra $\mathcal{O}_{\rho^{\mathfrak{L}}}$ is the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph system $\mathfrak{L}$.

Let α be an automorphism of a unital C^*-algebra $\mathcal{A}$. Put $\Sigma = \{\alpha\}$ and $\rho_\alpha = \alpha$. The C^*-algebra $\mathcal{O}_\rho$ for the C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is the ordinary C^*-crossed product $\mathcal{A} \times_\alpha \mathbf{Z}$.

In what follows, for a subset F of a C^*-algebra $\mathcal{B}$, we will denote by $C^*(F)$ the C^*-subalgebra of $\mathcal{B}$ generated by F.

Let $(\mathcal{A}, \rho, \Sigma)$ be a C^*-symbolic dynamical system over Σ and Λ the associated subshift Λ_ρ. We denote by $B_k(\Lambda)$ the set of admissible words μ of Λ with length

$|\mu| = k$. Put $B_*(\Lambda) = \cup_{k=0}^{\infty} B_k(\Lambda)$, where $B_0(\Lambda)$ consists of the empty word. Let $S_\alpha, \alpha \in \Sigma$ be the partial isometries in $\mathcal{O}_\rho$ satisfying the relation (ρ). For $\mu = (\mu_1, \dots, \mu_k) \in B_k(\Lambda)$, we put $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$ and $\rho_\mu = \rho_{\mu_k} \circ \cdots \circ \rho_{\mu_1}$. In the algebra $\mathcal{O}_\rho$, we set for $k \in \mathbf{Z}_+$,

$$
\begin{aligned}
\mathcal{D}_\rho^k &= C^*(S_\mu x S_\mu^* : \mu \in B_k(\Lambda), x \in \mathcal{A}),\\
\mathcal{D}_\rho &= C^*(S_\mu x S_\mu^* : \mu \in B_*(\Lambda), x \in \mathcal{A}),\\
\mathcal{F}_\rho^k &= C^*(S_\mu x S_\nu^* : \mu, \nu \in B_k(\Lambda), x \in \mathcal{A}) \quad \text{and}\\
\mathcal{F}_\rho &= C^*(S_\mu x S_\nu^* : \mu, \nu \in B_*(\Lambda), |\mu| = |\nu|, x \in \mathcal{A}).
\end{aligned}
$$

The identity $S_\mu x S_\nu^* = \sum_{\alpha\in\Sigma} S_{\mu\alpha}\rho_\alpha(x) S_{\nu\alpha}^*$ for $x \in \mathcal{A}$, $\mu, \nu \in B_k(\Lambda)$ holds so that the algebra $\mathcal{F}_\rho^k$ is embedded into the algebra $\mathcal{F}_\rho^{k+1}$ such that $\cup_{k\in\mathbf{Z}_+}\mathcal{F}_\rho^k$ is dense in $\mathcal{F}_\rho$. Similarly $\mathcal{D}_\rho^k$ is embedded into the algebra $\mathcal{D}_\rho^{k+1}$ such that $\cup_{k\in\mathbf{Z}_+}\mathcal{D}_\rho^k$ is dense in $\mathcal{D}_\rho$. The gauge action $\hat{\rho}$ of the one-dimensional torus group $\mathbf{T} = \{z \in \mathbf{C} \mid |z| = 1\}$ on $\mathcal{O}_\rho$ is defined by $\hat{\rho}_z(x) = x$ for $x \in \mathcal{A}$ and $\hat{\rho}_z(S_\alpha) = zS_\alpha$ for $\alpha \in \Sigma$. The fixed point algebra of $\mathcal{O}_\rho$ under $\hat{\rho}$ is denoted by $(\mathcal{O}_\rho)^{\hat{\rho}}$. Let $E_\rho : \mathcal{O}_\rho \longrightarrow (\mathcal{O}_\rho)^{\hat{\rho}}$ be the conditional expectaton defined by

$$
E_\rho(X) = \int_{z\in\mathbf{T}} \hat{\rho}_z(X)dz, \qquad X \in \mathcal{O}_\rho. \tag{2.3}
$$

It is routine to check that $(\mathcal{O}_\rho)^{\hat{\rho}} = \mathcal{F}_\rho$.

Definition ([33]). A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ satisfies *condition* (I) if there exists a unital increasing sequence

$$
\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}
$$

of C^*-subalgebras of $\mathcal{A}$ such that $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$ for all $l \in \mathbf{Z}_+, \alpha \in \Sigma$, the union $\cup_{l\in\mathbf{Z}_+}\mathcal{A}_l$ is dense in $\mathcal{A}$ and for $\epsilon > 0$, $k, l \in \mathbf{N}$ with $k \le l$ and $X_0 \in \mathcal{F}_{\rho,l}^k = C^*(S_\mu x S_\nu^* : \mu, \nu \in B_k(\Lambda), x \in \mathcal{A}_l)$, there exists an element $g \in \mathcal{D}_\rho \cap {\mathcal{A}_l}'(= \{y \in \mathcal{D}_\rho \mid ya = ay \text{ for } a \in \mathcal{A}_l\})$ with $0 \le g \le 1$ such that

(i) $\|X_0\phi_\rho^k(g)\| \ge \|X_0\| - \epsilon$,

(ii) $g\phi_\rho^m(g) = 0$ for all $m = 1, 2, \dots, k$, where $\phi_\rho^m(X) = \sum_{\mu\in B_m(\Lambda)} S_\mu X S_\mu^*$.

As the element g belongs to the diagonal subalgebra $\mathcal{D}_\rho$ of $\mathcal{F}_\rho$, the condition (I) is intrinsically determined by $(\mathcal{A}, \rho, \Sigma)$ by virtue of [31, Lemma 4.1]. The condition (I) for $(\mathcal{A}, \rho, \Sigma)$ yields the uniqueness of the C^*-algebra $\mathcal{O}_\rho$ under the relations (ρ) ([33]).

If a λ-graph system $\mathfrak{L}$ over Σ satisfies condition (I), then $(\mathcal{A}_\mathfrak{L}, \rho^\mathfrak{L}, \Sigma)$ satisfies condition (I) (cf. [30, Lemma 4.1]).

Recall that the positive operator $\lambda_\rho : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by $\lambda_\rho(x) = \sum_{\alpha\in\Sigma} \rho_\alpha(x)$, $x \in \Sigma$. Then a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *irreducible*, if there exists no nontrivial ideal of $\mathcal{A}$ invariant under λ_ρ. It has been shown in [31] that if $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I) and is irreducible, then the C^*-algebra $\mathcal{O}_\rho$ is simple.

Interesting examples of $(\mathcal{A}, \rho, \Sigma)$ in [31], [34] which we have seen from the view point of symbolic dynamics come from ones for which $\mathcal{A}$ is commutative. Hence we assume that the algebra $\mathcal{A}$ is commutative so that $\mathcal{A}$ is written as

$C(\Omega)$ for some compact Hausdorff space Ω henceforth. For the cases that $\mathcal{A}$ is noncommutative, our discussions in this paper well work by considering tracial states on $\mathcal{A}$ in stead of states on $\mathcal{A}$ under slight modifications.

3. ERGODICITY AND PERRON-FROBENIUS TYPE THEOREM

In this section, we will study ergodic properties of a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ and prove a Perron-Frobenius type theorem.
Let $\mathcal{A}^*$ denote the Banach space of all complex valued continuous linear functionals on $\mathcal{A}$. For $\beta \in \mathbf{C}$ with $\beta \neq 0$, set

$$\mathcal{E}_\beta(\rho) = \{\varphi \in \mathcal{A}^* \mid \varphi \circ \lambda_\rho(a) = \beta\varphi(a) \text{ for all } a \in \mathcal{A}\}.$$

It is possible that $\mathcal{E}_\beta(\rho)$ is $\{0\}$. A nonzero continuous linear functional φ in $\mathcal{E}_\beta(\rho)$ is called an eigenvector of the operator λ_ρ^* with respect to the eigenvalue β. Let r_ρ be the spectral radius of the positive operator $\lambda_\rho : \mathcal{A} \longrightarrow \mathcal{A}$. Since $\lambda_\rho^k(1) \geq 1, k \in \mathbf{N}$, one sees that $r_\rho \geq 1$. As $\mathrm{Sp}(\lambda_\rho) = \mathrm{Sp}(\lambda_\rho^*)$ (cf. [8, VI. 2.7]), we note $r_\rho = r(\lambda_\rho^*)$. Let $\mathcal{S}(\mathcal{A})$ denote the state space of $\mathcal{A}$.

LEMMA 3.1. *$(\mathcal{A}, \rho, \Sigma)$ is irreducible if and only if for a state φ on $\mathcal{A}$ and a nonzero element $x \in \mathcal{A}$, there exists a natural number n such that $\varphi(\lambda_\rho^n(x^*x)) >$ 0.*

Proof. Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. For a state φ on $\mathcal{A}$, put

$$I_\varphi = \{x \in \mathcal{A} | \varphi(\lambda_\rho^n(x^*x)) = 0 \text{ for all } n \in \mathbf{N}\}$$

which is an ideal of $\mathcal{A}$ because $\mathcal{A}$ is commutative. The Schwarz type inequality

$$\lambda_\rho^n(\lambda_\rho(x)^*\lambda_\rho(x)) \leq \|\lambda_\rho\| \lambda_\rho^{n+1}(x^*x) \qquad \text{for} \quad x \in \mathcal{A}$$

implies that I_φ is λ_ρ-invariant. Hence I_φ is trivial.
Conversely, let I be an ideal of $\mathcal{A}$ invariant under λ_ρ. Put $\mathcal{B} = \mathcal{A}/I$. Denote by $q : \mathcal{A} \longrightarrow \mathcal{B}$ the quotient map. Take $\psi \in \mathcal{S}(\mathcal{B})$ a state. Put $\varphi = \psi \circ q$. For $y \in I$, as $\varphi(\lambda_\rho^n(y^*y)) = 0, n \in \mathbf{N}$, one sees that $y = 0$ and hence $I = \{0\}$ by the hypothesis. Hence $(\mathcal{A}, \rho, \Sigma)$ is irreducible. □

We denote by $T_\rho : \mathcal{A} \longrightarrow \mathcal{A}$ the positive operator $\frac{1}{r_\rho}\lambda_\rho$. The spectral radius of T_ρ is 1. A state τ on $\mathcal{A}$ is called an *invariant state* if $\tau \circ T_\rho = \tau$ on $\mathcal{A}$, equivalently $\tau \in \mathcal{E}_{r_\rho}(\rho)$.

COROLLARY 3.2. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. Then any positive eigenvector of λ_ρ^* for a nonzero eigenvalue is faithful.*

Proof. Let $\varphi \in \mathcal{E}_\beta(\rho)$ be a positive linear functional for some nonzero $\beta \in \mathbf{C}$. Since $\varphi(\lambda_\rho(1)) = \beta\varphi(1)$, one has $\beta > 0$. By the preceding lemma, one has $\varphi(x^*x) > 0$ for nonzero $x \in \mathcal{A}$. □

Yasuo Watatani has kindly informed to the author that the lemma below, which is seen from [41, Theorem 2.5], is needed in the proof of Lemma 3.4. In our restrictive situation, we may directly prove it as in the following way.

Lemma 3.3. *The spectral radius r_ρ of the operator λ_ρ is contained in the spectrum $Sp(\lambda_\rho)$ of λ_ρ.*

Proof. The resolvent $R(z) = (z - \lambda_\rho)^{-1}$ for λ_ρ has the expansion $R(z) = \sum_{n=0}^{\infty} \frac{{\lambda_\rho}^n}{z^{n+1}}$ for $z \in \mathbf{C}, |z| > r_\rho$ which converges in norm. We note that the family $\{R(z)\}_{|z|>r_\rho}$ is not uniformly bounded. Otherwise, there exists a constant $M > 0$ such that $\|R(z)\| < M$ for $z \in \mathbf{C}, |z| > r_\rho$. By the compactness of $\mathrm{Sp}(\lambda_\rho)$, we may find $z_\circ \in \mathrm{Sp}(\lambda_\rho)$ with $|z_\circ| = r_\rho$. Take $z_n \notin \mathrm{Sp}(\lambda_\rho)$ satisfying $\lim_{n\to\infty} z_n = z_\circ$ and $|z_n| > r_\rho$. The resolvent equation $R(z_n) - R(z_m) = (z_n - z_m)R(z_n)R(z_m)$ implies the inequality $\|R(z_n) - R(z_m)\| \le |z_n - z_m|M^2$ so that there exists a bounded linear operator $R_\circ = \lim_{n\to\infty} R(z_n)$ on $\mathcal{A}$. The equality $(z_n - \lambda_\rho)R(z_n)x = x, x \in \mathcal{A}$ implies $(z_\circ - \lambda_\rho)R_\circ x = x, x \in \mathcal{A}$ and hence $z_\circ \notin \mathrm{Sp}(\lambda_\rho)$ a contradiction. Thus there exists $r_n \in \mathbf{C}$ such that $|r_n| \notin \mathrm{Sp}(\lambda_\rho)$ and $|r_n| \downarrow r_\rho$ and $\lim_{n\to\infty} \|R(r_n)f\| = \infty$ for some $f \in \mathcal{A}$. We may assume that $f \ge 0$. For a state φ on $\mathcal{A}$, one has

$$|\varphi(R(r_n)f)| \le \sum_{k=0}^{\infty} \frac{\varphi({\lambda_\rho}^k(f))}{|r_n|^{k+1}} = \varphi(R(|r_n|)f).$$

Denote by $w(y)$ the numerical radius of an element $y \in \mathcal{A}$, which is defined by

$$w(y) = \sup\{\varphi(y) \mid \varphi \in \mathcal{S}(\mathcal{A})\}.$$

As the inequalities $\frac{1}{2}\|y\| \le w(y) \le \|y\|$ always hold (cf. [13, p.95]), one sees

$$\frac{1}{2}\|R(r_n)f\| \le w(R(r_n)f) \le w(R(|r_n|)f) \le \|R(|r_n|)f\|$$

so that

$$\lim_{n\to\infty} \|R(|r_n|)f\| = \infty.$$

If $r_\rho \notin \mathrm{Sp}(\lambda_\rho)$, the condition $|r_n| \notin \mathrm{Sp}(\lambda_\rho)$ means that $R(|r_n|) \uparrow R(r_\rho)$ because $R(z)$ increases for $z \downarrow r_\rho$. Hence $R(|r_n|)f \uparrow R(r_\rho)f$ and $\lim_{n\to\infty} \|R(|r_n|)f\| = \|R(r_\rho)f\| < \infty$, a contradiction. Therefore we conclude $r_\rho \in \mathrm{Sp}(\lambda_\rho)$. □

The following lemma is crucial.

Lemma 3.4. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. Then there exists a faithful invariant state on $\mathcal{A}$.*

Proof. We denote by $R^*(t)$ the resolvent of $\lambda_\rho^* : \mathcal{A}^* \to \mathcal{A}^*$ defined by $R^*(t)\varphi = (t - \lambda_\rho^*)^{-1}\varphi$ for $\varphi \in \mathcal{A}^*, t > r(\lambda_\rho^*)$. As $r_\rho = r(\lambda_\rho^*)$, there exists $\varphi_0 \in \mathcal{A}^*$ such that $\|R^*(t)\varphi_0\|$ is unbounded for $t \downarrow r_\rho$ by Lemma 3.3. We may assume that φ_0 is a state on $\mathcal{A}$. Put

$$\varphi_n = \frac{R^*(r_\rho + \frac{1}{n})\varphi_0}{\|R^*(r_\rho + \frac{1}{n})\varphi_0\|} \qquad \text{for} \quad n = 1, 2, ...$$

Since $R^*(t)$ is positive for $t > r_\rho$, each φ_n is a state on $\mathcal{A}$ so that there exists a weak* cluster point $\varphi_\infty \in \mathcal{S}(\mathcal{A})$ of the sequence $\{\varphi_n\}$ in $\mathcal{S}(\mathcal{A})$. As we see

$$(r_\rho - \lambda_\rho^*)\varphi_n = -\frac{1}{n}\varphi_n + \frac{\varphi_0}{\|R^*(r_\rho + \frac{1}{n})\varphi_0\|},$$

we get $r_\rho \varphi_\infty = \lambda_\rho^* \varphi_\infty$ so that $\varphi_\infty \in \mathcal{E}_{r_\rho}(\rho)$. By Corollary 3.2, one knows that φ_∞ is faithful on $\mathcal{A}$. □

Definition. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *uniquely ergodic* if there exists a unique invariant state on $\mathcal{A}$. Denote by τ the unique invariant state.

If $(\mathcal{A}, \rho, \Sigma)$ is irreducible and uniquely ergodic, the unique invariant state τ is automatically faithful because any invariant state is faithful.

There is an example of a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ for which a unique invariant state is not faithful, unless $(\mathcal{A}, \rho, \Sigma)$ is irreducible. Let $\mathcal{A} = \mathbf{C} \oplus \mathbf{C}$, $\Sigma = \{1, 2\}$ and $\rho_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \rho_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\lambda_\rho = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $r_\rho = 2$ and $T_\rho = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a unique invariant state on $\mathcal{A}$, that is not faithful.

We will see, in Section 7, that the C^*-symbolic dynamical system $(\mathcal{A}_\mathcal{G}, \rho^\mathcal{G}, \Sigma)$ for a finite labeled graph $\mathcal{G}$ is uniquely ergodic if and only if the underlying nonnegative matrix $A_\mathcal{G}$ is irreducible.

We will next consider the eigenvector space of the operator λ_ρ on $\mathcal{A}$. We are assuming that the algebra $\mathcal{A}$ is commutative so that $\mathcal{A}$ is written as $C(\Omega)$ for some compact Hausdorff space Ω.

Lemma 3.5. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible.*

(i) *If T_ρ has a nonzero fixed element in $\mathcal{A}$, then T_ρ has a nonzero positive fixed element in $\mathcal{A}$,*

(ii) *A nonzero positive fixed element by T_ρ in $\mathcal{A}$ must be strictly positive.*

(iii) *If there exist two nonzero positive fixed elements by T_ρ in $\mathcal{A}$, then one is a scalar multiple of the other.*

(iv) *The dimension of the space consisting of the fixed elements by T_ρ is at most one.*

Proof. (i) Let $y \in \mathcal{A}$ be a nonzero fixed element by T_ρ. Since y^* is also fixed by T_ρ, we may assume that $y = y^*$. Denote by $y = y^+ - y^-$ with $y^+, y^- \geq 0$ the Jordan decomposition of y. We have $y^+ \geq y$ and hence $T_\rho(y^+) \geq T_\rho(y) = y$. As $T_\rho(y^+) \geq 0$, one sees that $T_\rho(y^+) \geq y^+$. Now $(\mathcal{A}, \rho, \Sigma)$ is irreducible so that there exists a faithful invariant state τ on $\mathcal{A}$. Since $\tau(T_\rho(y^+) - y^+) = 0$, one has $T_\rho(y^+) = y^+$. Similarly we have $T_\rho(y^-) = y^-$. As $y \neq 0$, either y^+ or y^- is not zero.

(ii) Let $y \in \mathcal{A}$ be a nonzero fixed positive element by T_ρ. Suppose that there exists $\omega_0 \in \Omega$ such that $y(\omega_0) = 0$. Let I_y be the closed ideal of $\mathcal{A}$ generated by y. For a nonzero positive element $f \in \mathcal{A}$ we have

$$T_\rho(fy) \leq \|f\| T_\rho(y) = \|f\| y$$

so that $T_\rho(fy)$ belongs to I_y. As the ideal I_y is approximated by linear combinations of the elements of the form $fy, f \in \mathcal{A}, f \geq 0$, the ideal I_y is invariant under T_ρ. Now $(\mathcal{A}, \rho, \Sigma)$ is irreducible so that $I_y = \mathcal{A}$. As any element of I_y vanishes at ω_0, a contradiction.

(iii) Let $x, y \in \mathcal{A}$ be nonzero positive fixed elements by T_ρ. By the above discussions, they are strictly positive. Set $c_0 = \min\{\frac{x(\omega)}{y(\omega)} \mid \omega \in \Omega\}$. The function $x - c_0 y$ is positive element but not strictly positive, so that it must be zero.
(iv) Let $y \in \mathcal{A}$ be a fixed element under T_ρ, which is written as the Jordan decomposition $y = y_1 - y_2 + i(y_3 - y_4)$ for some positive elements $y_i, i = 1, 2, 3, 4$ in $\mathcal{A}$. By the above discussions, all the elements $y_i, i = 1, 2, 3, 4$ are fixed under T_ρ and they are strictly positive if it is nonzero. Hence (iii) implies the desired assertion. □

Definition. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to satisfy *(FP)* if there exists a nonzero fixed element in $\mathcal{A}$ under T_ρ.
If in particular, $(\mathcal{A}, \rho, \Sigma)$ is irreducible, a nonzero fixed element can be taken as a strictly positive element in $\mathcal{A}$ by the previous lemma.

Lemma 3.6. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and satisfies (FP).*

(i) *If there exists a state in $\mathcal{E}_\beta(\rho)$ for some $\beta \in \mathbf{C}$ with $\beta \neq 0$, then we have $\beta = r_\rho$.*
(ii) *If in particular, $(\mathcal{A}, \rho, \Sigma)$ is uniquely ergodic, the eigenspace $\mathcal{E}_{r_\rho}(\rho)$ is of one-dimensional.*

Proof. (i) Suppose that there exists a state ψ in $\mathcal{E}_\beta(\rho)$ for some $\beta \in \mathbf{C}$ with $\beta \neq 0$. Let $x_0 \in \mathcal{A}$ be a nonzero fixed element by T_ρ. One may take it to be strictly positive by the preceding lemma. Since $\lambda_\rho(x_0) = r_\rho x_0$, one has

$$\beta\psi(x_0) = \psi(\lambda_\rho(x_0)) = r_\rho \psi(x_0).$$

By Corollary 3.2, one has $\psi(x_0) > 0$ so that $\beta = r_\rho$.
(ii) Take an arbitrary $\varphi \in \mathcal{E}_{r_\rho}(\rho)$. Put $\varphi^*(x) = \overline{\varphi(x^*)}, x \in \mathcal{A}$ and hence $\varphi^* \in \mathcal{E}_{r_\rho}(\rho)$. Both of the continuous linear functionals $\varphi_{Re} = \frac{1}{2}(\varphi + \varphi^*)$ and $\varphi_{Im} = \frac{1}{2i}(\varphi - \varphi^*)$ belong to $\mathcal{E}_{r_\rho}(\rho)$ which come from real valued measures on Ω. Put $\psi = \varphi_{Re}$. Let $\psi = \psi_+ - \psi_-$ be the Jordan decomposition of ψ, where ψ_+, ψ_- are positive linear functionals on $\mathcal{A}$. Since $\psi_+ \geq \psi$, one has $T_\rho^* \psi_+ \geq T_\rho^* \psi = \psi$. As $T_\rho^* \psi_+$ is positive, one has $T_\rho^* \psi_+ \geq \psi_+$. Now $(\mathcal{A}, \rho, \Sigma)$ is irreducible and satisfies (FP) so that one finds a strictly positive element $x_0 \in \mathcal{A}$ fixed by T_ρ. Then $\tilde{\psi} = T_\rho^* \psi_+ - \psi_+$ is a positive linear functional satisfying $\tilde{\psi}(x_0) = 0$. It follows that $\tilde{\psi} = 0$ so that $T_\rho^* \psi_+ = \psi_+$. Similarly we have $T_\rho^* \psi_- = \psi_-$. As both ψ_+, ψ_- are positive linear functionals on $\mathcal{A}$, the unique ergodicity asserts that there exist $0 \leq c_+, c_- \in \mathbf{R}$ such that $\psi_+ = c_+ \tau, \psi_- = c_- \tau$. By putting $c_{Re} = c_+ - c_-$, one has $\varphi_{Re} = c_{Re}\tau$ and similarly $\varphi_{Im} = c_{Im}\tau$ for some real number c_{Im}. Therefore we have

$$\varphi = (c_{Re} + i c_{Im})\tau.$$

Hence any continuous linear functional fixed by T_ρ is a scalar multiple of τ, so that

$$\dim \mathcal{E}_{r_\rho}(\rho) = 1.$$

□

A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *power-bounded* if the sequence $\{\|T_\rho^k\| \mid k \in \mathbf{N}\}$ is bounded. As $T_\rho^k : \mathcal{A} \longrightarrow \mathcal{A}$ is completely positive, the equalities $\|T_\rho^k\| = \|T_\rho^k(1)\| = \|\frac{1}{r_\rho{}^k} \sum_{\mu \in B_k(\Lambda)} \rho_\mu(1)\|$ hold. We remark that for an irreducible matrix $A = [A(i,j)]_{i,j=1}^N$ with entries in $\{0,1\}$, the associated C^*-symbolic dynamical system $(\mathcal{A}_A, \rho^A, \Sigma)$ defined in the Cuntz-Krieger algebra $\mathcal{O}_A$ is power-bounded. One indeed sees that there is a constant $d > 0$ such that

$$\sum_{i,j=1}^N A^k(i,j) \le d \cdot r_A^k \qquad \text{(cf. [28, Proposition 4.2.1]).}$$

Hence

$$\|\lambda_A^k(1)\| = \max_i \sum_{j=1}^N A^k(i,j) \le d \cdot r_A^k.$$

LEMMA 3.7. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. If $(\mathcal{A}, \rho, \Sigma)$ satisfies (FP), then $(\mathcal{A}, \rho, \Sigma)$ is power-bounded.*

Proof. As $(\mathcal{A}, \rho, \Sigma)$ is irreducible and satisfies (FP), there exists a strictly positive fixed element x_0 of $\mathcal{A}$ under T_ρ. Since Ω is compact, one finds positive constants c_1, c_2 such that $0 < c_1 < x_0(\omega) < c_2$ for all $\omega \in \Omega$. It follows that

$$c_1 T_\rho^n(1) = T_\rho^n(c_1 1) \le T_\rho^n(x_0) = x_0 \le c_2, \qquad n \in \mathbf{N}.$$

Thus we have $\|T_\rho^n\| = \|T_\rho^n(1)\| \le \frac{c_2}{c_1}$ for $n \in \mathbf{N}$. $\square$

We define the mean operator $M_n : \mathcal{A} \longrightarrow \mathcal{A}$ for $n \in \mathbf{N}$ by setting

$$M_n(a) = \frac{a + T_\rho(a) + T_\rho^2(a) + \cdots + T_\rho^{n-1}(a)}{n}, \qquad a \in \mathcal{A}. \tag{3.1}$$

DEFINITION. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *mean ergodic* if for $a \in \mathcal{A}$ the limit $\lim_{n\to\infty} M_n(a)$ exists in $\mathcal{A}$ under norm-topology. For a mean ergodic $(\mathcal{A}, \rho, \Sigma)$, the limit $\lim_{n\to\infty} M_n(1)$ exists in $\mathcal{A}$ under norm-topology, which we denote by $x_\rho \in \mathcal{A}$

LEMMA 3.8. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. For a mean ergodic $(\mathcal{A}, \rho, \Sigma)$, we have for $a \in \mathcal{A}$,*

$$\lim_{n\to\infty} M_n(a) = \lim_{n\to\infty} M_n(T_\rho(a)) = \lim_{n\to\infty} T_\rho(M_n(a)). \tag{3.2}$$

In particular x_ρ is a nonzero positive element which satisfies $x_\rho = T_\rho(x_\rho)$ and $\tau(x_\rho) = 1$ for an invariant state $\tau \in \mathcal{E}_{r_\rho}(\rho)$.

Proof. For $a \in \mathcal{A}$, the equality $T_\rho(M_n(a)) = M_n(T_\rho(a))$ is clear. As

$$(n+1)M_{n+1}(a) - nM_n(a) = T_\rho^n(a),$$

one has

$$\frac{1}{n} T_\rho^n(a) = M_{n+1}(a) - M_n(a) + \frac{1}{n} M_{n+1}(a)$$

so that $\lim_{n\to\infty}\frac{1}{n}T_\rho^n(a)=0$. By the equality

$$T_\rho(M_n(a))-M_n(a)=\frac{1}{n}(T_\rho^n(a)-a)$$

we have

$$\lim_{n\to\infty}(T_\rho(M_n(a))-M_n(a))=\lim_{n\to\infty}\frac{1}{n}(T_\rho^n(a)-a)=0.$$

Take a faithful invariant state τ on $\mathcal{A}$, we have

$$\tau(x_\rho)=\lim_{n\to\infty}\tau(M_n(1))=\tau(1)=1.$$

$\square$

Proposition 3.9. *Assume that $(\mathcal{A},\rho,\Sigma)$ is irreducible. If $(\mathcal{A},\rho,\Sigma)$ is mean ergodic, there exists a faithful invariant state τ on $\mathcal{A}$ such taht*

$$\lim_{n\to\infty}M_n(a)=\tau(a)x_\rho,\qquad a\in\mathcal{A}. \tag{3.3}$$

Proof. For $a\in\mathcal{A}$, the limit $\varPhi(a)=\lim_{n\to\infty}M_n(a)$ is fixed by T_ρ so that it is a scalar multiple of x_ρ by Lemma 3.5 (iv). One may put

$$\varPhi(a)=\tau(a)x_\rho\quad\text{ for some }\tau(a)\in\mathbf{C}.$$

It is easy to see that $\tau:\mathcal{A}\longrightarrow\mathbf{C}$ is a state. As $\varPhi(T_\rho(a))=\varPhi(a)$, one sees $\tau(T_\rho(a))=\tau(a)$ for $a\in\mathcal{A}$. Hence τ is an invariant state on $\mathcal{A}$. Now $(\mathcal{A},\rho,\Sigma)$ is irreducible, the invariant state is faithful. $\square$

Hence the following corollary is clear.

Corollary 3.10. *Assume that $(\mathcal{A},\rho,\Sigma)$ is irreducible. Then the following two assertions are equivalent:*

(i) *$(\mathcal{A},\rho,\Sigma)$ is mean ergodic.*
(ii) *There exist an invariant state τ on $\mathcal{A}$ and a positive element $x_0\in\mathcal{A}$ with $\tau(x_0)=1$ such that $\lim_{n\to\infty}M_n(a)=\tau(a)x_0$ for $a\in\mathcal{A}$.*

In this case x_0 is given by $\lim_{n\to\infty}M_n(1)(=x_\rho)$, and τ is faithful.

Theorem 3.11. *Assume that $(\mathcal{A},\rho,\Sigma)$ is irreducible. Then the following two assertions are equivalent:*

(i) *$(\mathcal{A},\rho,\Sigma)$ is mean ergodic.*
(ii) *$(\mathcal{A},\rho,\Sigma)$ is uniquely ergodic and satisfies (FP).*

Proof. (i) $\Rightarrow$ (ii): Suppose that $(\mathcal{A},\rho,\Sigma)$ is mean ergodic. Put $\varPhi(a)=\lim_{n\to\infty}M_n(a)$ for $a\in\mathcal{A}$. The element $x_\rho=\varPhi(1)$ is a nonzero fixed element of $\mathcal{A}$ under T_ρ. By the previous corollary, there exists an invariant state τ on $\mathcal{A}$ satisfying $\varPhi(a)=\tau(a)x_\rho$ for $a\in\mathcal{A}$. For any invariant state ψ on $\mathcal{A}$, we have $\psi\circ M_n(a)=\psi(a)$ for $a\in\mathcal{A}$. Hence $\psi(\varPhi(a))=\psi(a)$ so that $\psi(a)=\psi(\tau(a)x_\rho)=\tau(a)\psi(x_\rho)$. Since $\psi(x_\rho)=1$, we obtain $\psi(a)=\tau(a)$. Therefore $\psi=\tau$ so that $(\mathcal{A},\rho,\Sigma)$ is uniquely ergodic.
(ii) $\Rightarrow$ (i): Suppose that $(\mathcal{A},\rho,\Sigma)$ is uniquely ergodic and satisfies (FP). By Lemma 3.7, $(\mathcal{A},\rho,\Sigma)$ is power-bounded. Hence the sequence $\{\frac{1}{n}\sum_{k=0}^{n-1}T_\rho^k\}_{n\in\mathbf{N}}$

is uniformly bounded. This means that $T_\rho : \mathcal{A} \longrightarrow \mathcal{A}$ is Cesàro bounded (cf. [22, p.72]). As $\lim_{n\to\infty} \frac{T_\rho^{n-1}(a)}{n} = 0$ for $a \in \mathcal{A}$, the operator $T_\rho : \mathcal{A} \to \mathcal{A}$ satisfies the assumption of [22, p.74 Theorem 1.4]. To prove mean ergodicity, it suffices to show that $F = \{x \in \mathcal{A} \mid T_\rho x = x\}$ separates $F^* = \{\varphi \in \mathcal{A}^* \mid \varphi \circ T_\rho = \varphi\}$. By Lemma 3.6, one knows that $F^* = \mathbf{C}\tau$, where τ is a unique faithful invariant state on $\mathcal{A}$. Hence if $\varphi = c\tau \in F^*$ is nonzero, then $c \neq 0$ and $\varphi(x_\rho) = c\tau(x_\rho) = c \neq 0$. This implies that F separates F^*. Thus by [22, p.74 Theorem 1.4], $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic. □

REMARK. In [22, p.179], it is shown that a mean ergodic irreducible "Markov operator "is uniquely ergodic. In our situation, the operator T_ρ does not necessarily satisfy $T_\rho(1) = 1$. Hence the operator T_ρ is not necessarily a Markov operator.

We summarize results obtained in this section as in the following way:

THEOREM 3.12. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. Then the following implications hold:*

$$\begin{array}{c} \textit{(ME)} \Longleftrightarrow \textit{(UE)} + \textit{(FP)} \Longrightarrow \textit{(FP)} \Longrightarrow \textit{(PB)} \\ \Downarrow \\ \dim \mathcal{E}_{r_\rho}(\rho) = 1 \Longrightarrow \textit{(UE)}, \end{array}$$

where (ME) means mean ergodic, (UE) means uniquely ergodic, and (PB) means power-bounded.

If in particular $(\mathcal{A}, \rho, \Sigma)$ is irreducible and mean ergodic, the following Perron-Frobenius type theorem holds.

THEOREM 3.13. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and mean ergodic.*

(i) *There exists a unique pair of a faithful state τ on $\mathcal{A}$ and a strictly positive element x_ρ in $\mathcal{A}$ satisfying the conditions:*

$$\tau \circ \lambda_\rho = r_\rho \tau, \qquad \lambda_\rho(x_\rho) = r_\rho x_\rho \quad \textit{and} \quad \tau(x_\rho) = 1,$$

where r_ρ is the spectral radius of the positive operator λ_ρ on $\mathcal{A}$.

(ii) *If there exists a continuous linear functional ψ on $\mathcal{A}$ satisfying*

$$\psi \circ \lambda_\rho = r_\rho \psi,$$

then $\psi = c\tau$ for some complex number $c \in \mathbf{C}$.

(iii) *If there exists a state φ on $\mathcal{A}$ and a complex number $\beta \in \mathbf{C}$ with $\beta \neq 0$ satisfying*

$$\varphi \circ \lambda_\rho = \beta \varphi,$$

then $\varphi = \tau$ and $\beta = r_\rho$.

(iv) *For any* $a \in \mathcal{A}$*, the limit* $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \frac{\lambda_\rho^k(a)}{r_\rho^k}$ *exists in* $\mathcal{A}$ *in the norm topology such that*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \frac{\lambda_\rho^k(a)}{r_\rho^k} = \tau(a)x_\rho.$$

Proof. Under the assumption that $(\mathcal{A}, \rho, \Sigma)$ is irreducible, mean ergodicity is equivalent to unique ergodicity with (FP). (i) and (iv) follows from Corollary 3.10 and unique ergodicity. (ii) follows from Lemma 3.6 (ii). (iii) follows from Lemma 3.6 (i) and unique ergodicity. □

4. Extension of eigenvectors to $\mathcal{F}_\rho$

In this section, we will study extendability of an eigenvector in $\mathcal{E}_\beta(\rho)$ to the subalgebra $\mathcal{F}_\rho$. We fix a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ satisfying condition (I) henceforth.

Lemma 4.1. *Fix a nonnegative integer* $k \in \mathbf{Z}_+$*. For any element* $x \in \mathcal{F}_\rho^k$ *there uniquely exists* $x_{\mu,\nu}$ *in* $\mathcal{A}$ *for each* $\mu, \nu \in B_k(\Lambda)$ *such that*

$$x = \sum_{\mu,\nu\in B_k(\Lambda)} S_\mu x_{\mu,\nu} S_\nu^* \qquad \textit{and} \qquad x_{\mu,\nu} = \rho_\mu(1)x_{\mu,\nu}\rho_\nu(1). \tag{4.1}$$

If in particular x *belongs to* $\mathcal{D}_\rho^k$*, there uniquely exists* x_μ *in* $\mathcal{A}$ *for each* $\mu \in B_k(\Lambda)$ *such that*

$$x = \sum_{\mu\in B_k(\Lambda)} S_\mu x_\mu S_\mu^* \qquad \textit{and} \qquad x_\mu = \rho_\mu(1)x_\mu\rho_\mu(1). \tag{4.2}$$

Proof. For an element x in $\mathcal{F}_\rho^k$ and $\mu, \nu \in B_k(\Lambda)$, put $x_{\mu,\nu} = S_\mu^* x S_\nu$ that belongs to $\mathcal{A}$ and satisfies the equalities (4.1). □

We set

$${\mathcal{D}_\rho}^{\mathrm{alg}} = \text{the algebraic linear span of } S_\mu a S_\mu^* \text{ for } \mu \in B_*(\Lambda), a \in \mathcal{A}, \text{ and}$$
$${\mathcal{F}_\rho}^{\mathrm{alg}} = \text{the algebraic linear span of } S_\mu a S_\nu^* \text{ for } \mu,\nu \in B_*(\Lambda), |\mu| = |\nu|, a \in \mathcal{A}.$$

Hence ${\mathcal{D}_\rho}^{\mathrm{alg}} = \cup_{k=0}^\infty \mathcal{D}_\rho^k$ and ${\mathcal{F}_\rho}^{\mathrm{alg}} = \cup_{k=0}^\infty \mathcal{F}_\rho^k$. They are dense $*$-subalgebras of $\mathcal{D}_\rho$ and $\mathcal{F}_\rho$ respectively.

Lemma 4.2. *For* $\beta \in \mathbf{C}$ *with* $|\beta| > 1$ *and* $\varphi \in \mathcal{E}_\beta(\rho)$ *on* $\mathcal{A}$*, put*

$$\tilde\varphi(S_\mu a S_\mu^*) = \frac{1}{\beta^{|\mu|}}\varphi(a\rho_\mu(1)), \qquad a \in \mathcal{A}, \mu \in B_*(\Lambda). \tag{4.3}$$

Then $\tilde\varphi$ *is a well-defined (not necessarily continuous) linear functional on* ${\mathcal{D}_\rho}^{alg}$*, that is an extension of* φ*.*

Proof. By the expansion (4.2) for an element $x \in \mathcal{D}_\rho^k$, the following definition of $\varphi_k(x)$ yields a linear functional φ_k on $\mathcal{D}_\rho^k$

$$\varphi_k(x) = \sum_{\mu \in B_k(\Lambda)} \frac{1}{\beta^k} \varphi(x_\mu). \tag{4.4}$$

We will show that $\varphi_k = \varphi_{k+1}$ on $\mathcal{D}_\rho^k$. As $S_\mu x_\mu S_\mu^* = \sum_{\alpha \in \Sigma} S_{\mu\alpha} \rho_\alpha(x_\mu) S_{\mu\alpha}^*$ and $\rho_{\mu\alpha}(1)\rho_\alpha(x_\mu)\rho_{\mu\alpha}(1) = S_\alpha^* \rho_\mu(1) x_\mu \rho_\mu(1) S_\alpha = \rho_\alpha(x_\mu)$, the following expression of x in $\mathcal{D}_\rho^{k+1}$

$$x = \sum_{\mu \in B_k(\Lambda), \alpha \in \Sigma} S_{\mu\alpha} \rho_\alpha(x_\mu) S_{\mu\alpha}^*$$

is the unique expression of (4.2). Hence we obtain

$$\varphi_{k+1}(x) = \sum_{\mu \in B_k(\Lambda), \alpha \in \Sigma} \frac{1}{\beta^{k+1}} \varphi(\rho_\alpha(x_\mu)) = \frac{1}{\beta^k} \sum_{\mu \in B_k(\Lambda)} \varphi(x_\mu) = \varphi_k(x).$$

The family $\{\varphi_k\}_{k \in \mathbf{Z}_+}$ of linear functionals on the subalgebras $\{\mathcal{D}_\rho^k\}_{k \in \mathbf{Z}_+}$ yields a linear functional on the algebra ${\mathcal{D}_\rho}^{\mathrm{alg}}$. We denote it by $\tilde{\varphi}$. As the expansion $a = \sum_{\alpha \in \Sigma} S_\alpha \rho_\alpha(a) S_\alpha^*$ for $a \in \mathcal{A}$ is the unique expansion of a in (4.2) as an element of $\mathcal{D}_\rho^1$, we have $\tilde{\varphi}(a) = \frac{1}{\beta} \sum_{\alpha \in \Sigma} \varphi(\rho_\alpha(a)) = \varphi(a)$ so that $\tilde{\varphi} = \varphi$ on $\mathcal{A}$. $\square$

We will extend λ_ρ on $\mathcal{A}$ to $\mathcal{F}_\rho$ such as

$$\lambda_\rho(x) = \sum_{\alpha \in \Sigma} S_\alpha^* x S_\alpha \qquad \text{for } x \in \mathcal{F}_\rho.$$

LEMMA 4.3. *Let ψ be a linear functional on ${\mathcal{F}_\rho}^{alg}$ such that its restriction to $\mathcal{A}$ is continuous. Then the following three conditions are equivalent:*

(i) *ψ is tracial and $\psi \circ \lambda_\rho(x) = \beta\psi(x)$ for $x \in {\mathcal{F}_\rho}^{alg}$.*

(ii) *$\psi(S_\mu x S_\nu^*) = \delta_{\mu,\nu} \frac{1}{\beta^{|\mu|}} \psi(x S_\mu^* S_\mu)$ for $x \in {\mathcal{F}_\rho}^{alg}, \mu, \nu \in B_*(\Lambda)$ with $|\mu| = |\nu|$.*

(iii) *There exists $\varphi \in \mathcal{E}_\beta(\rho)$ such that*

$$\psi(S_\mu a S_\nu^*) = \delta_{\mu,\nu} \frac{1}{\beta^{|\mu|}} \varphi(a \rho_\mu(1)) \textit{ for } a \in \mathcal{A}, \mu, \nu \in B_*(\Lambda) \textit{with } |\mu| = |\nu|.$$

Proof. (i) $\Rightarrow$ (ii): The equation (i) implies that for $k \in \mathbf{N}$,

$$\psi(x) = \frac{1}{\beta^k} \sum_{\gamma \in B_k(\Lambda)} \psi(S_\gamma^* x S_\gamma), \quad x \in {\mathcal{F}_\rho}^{\mathrm{alg}}.$$

It then follows that for $\mu, \nu \in B_k(\Lambda)$

$$\psi(S_\mu x S_\nu^*) = \frac{1}{\beta^k} \sum_{\gamma \in B_k(\Lambda)} \psi(S_\gamma^* S_\mu x S_\nu^* S_\gamma) = \delta_{\mu,\nu} \frac{1}{\beta^k} \psi(x S_\mu^* S_\mu).$$

(ii) $\Rightarrow$ (iii): Define a linear functional φ on $\mathcal{A}$ by the restriction of ψ to the subalgebra $\mathcal{A}$. By the equation (ii) for $a \in \mathcal{A}$ and hence $S_\alpha^* a S_\alpha \in \mathcal{A}$, we see

$$\psi(S_\alpha S_\alpha^* a) = \psi(S_\alpha S_\alpha^* a S_\alpha S_\alpha^*) = \frac{1}{\beta}\psi(S_\alpha^* a S_\alpha S_\alpha^* S_\alpha) = \frac{1}{\beta}\psi(S_\alpha^* a S_\alpha)$$

so that $\varphi \in \mathcal{E}_\beta(\rho)$. The equation (iii) is clear.
(iii) $\Rightarrow$ (i): We will see that ψ is tracial. Let $x, y \in \mathcal{F}_\rho^k$ be expanded as in (4.1) so that $x = \sum_{\mu,\nu \in B_k(\Lambda)} S_\mu x_{\mu,\nu} S_\nu^*$, $y = \sum_{\mu,\nu \in B_k(\Lambda)} S_\mu y_{\mu,\nu} S_\nu^*$. We have

$$xy = \sum_{\mu,\nu,\gamma \in B_k(\Lambda)} S_\mu x_{\mu,\nu} \rho_\nu(1) y_{\nu,\gamma} S_\gamma^* = \sum_{\mu,\gamma \in B_k(\Lambda)} S_\mu \Big(\sum_{\nu \in B_k(\Lambda)} x_{\mu,\nu} y_{\nu,\gamma} \Big) S_\gamma^*$$

and $\sum_{\nu \in B_k(\Lambda)} x_{\mu,\nu} y_{\nu,\gamma} = \rho_\mu(1)(\sum_{\nu \in B_k(\Lambda)} x_{\mu,\nu} y_{\nu,\gamma})\rho_\gamma(1)$, similarly

$$yx = \sum_{\eta,\nu \in B_k(\Lambda)} S_\eta \Big(\sum_{\gamma \in B_k(\Lambda)} y_{\eta,\gamma} x_{\gamma,\nu} \Big) S_\nu^*$$

and $\sum_{\gamma \in B_k(\Lambda)} y_{\eta,\gamma} x_{\gamma,\nu} = \rho_\eta(1)(\sum_{\nu \in B_k(\Lambda)} y_{\eta,\gamma} x_{\gamma,\nu})\rho_\nu(1)$. It follows that

$$\psi(xy) = \sum_{\mu,\nu \in B_k(\Lambda)} \frac{1}{\beta^k}\varphi(x_{\mu,\nu} y_{\nu,\mu}) = \sum_{\gamma,\eta \in B_k(\Lambda)} \frac{1}{\beta^k}\varphi(y_{\eta,\gamma} x_{\gamma,\eta}) = \psi(yx).$$

Hence ψ is tracial on $\mathcal{F}_\rho^k$.
We will finally show that the equality in (i) holds. For $S_\mu a S_\nu^* \in \mathcal{F}_\rho^k$ with $a \in \mathcal{A}, \mu = (\mu_1, \dots, \mu_k), \nu = (\nu_1, \dots, \nu_k) \in B_k(\Lambda)$, put $\mu_{[2,k]} = (\mu_2, \dots, \mu_k), \nu_{[2,k]} = (\nu_2, \dots, \nu_k) \in B_{k-1}(\Lambda)$. One has

$$\begin{aligned}
&\sum_{\alpha \in \Sigma} \psi(S_\alpha^*(S_\mu a S_\nu^*) S_\alpha) \\
=&\delta_{\mu_1,\nu_1}\psi(\rho_{\mu_1}(1) S_{\mu_{[2,k]}} a S_{\nu_{[2,k]}}^* \rho_{\nu_1}(1)) \\
=&\delta_{\mu_1,\nu_1}\psi(S_{\mu_{[2,k]}} S_{\mu_{[2,k]}}^* \rho_{\mu_1}(1) S_{\mu_{[2,k]}} a S_{\nu_{[2,k]}}^* \rho_{\nu_1}(1) S_{\nu_{[2,k]}} S_{\nu_{[2,k]}}^*) \\
=&\delta_{\mu_1,\nu_1}\psi(S_{\mu_{[2,k]}} \rho_\mu(1) a \rho_\nu(1) S_{\nu_{[2,k]}}^*) \\
=&\delta_{\mu_1,\nu_1}\delta_{\mu_{[2,k]},\nu_{[2,k]}} \frac{1}{\beta^{k-1}}\varphi(\rho_\mu(1) a \rho_\nu(1) \rho_{\nu_{[2,k]}}(1)) \\
=&\delta_{\mu,\nu}\frac{1}{\beta^{k-1}}\varphi(\rho_\mu(1) a \rho_\nu(1)) \\
=&\beta\psi(S_\mu a S_\nu^*).
\end{aligned}$$

□

Let $E_{\mathcal{D}} : \mathcal{F}_\rho \longrightarrow \mathcal{D}_\rho$ denote the expectation satisfying

$$E_{\mathcal{D}}(S_\mu a S_\nu^*) = \delta_{\mu,\nu} S_\mu a S_\mu^*, \qquad a \in \mathcal{A}, \quad \mu, \nu \in B_*(\Lambda), |\mu| = |\nu|.$$

Once we have an extension $\tilde{\varphi}$ to $\mathcal{D}_\rho$ of $\varphi \in \mathcal{E}_\beta(\rho)$, $\tilde{\varphi}$ has a further extension to $\mathcal{F}_\rho$ by $\tilde{\varphi} \circ E_{\mathcal{D}}$. The extension $\tilde{\varphi} \circ E_{\mathcal{D}}$ on $\mathcal{F}_\rho$ is continuous if $\tilde{\varphi}$ is so on $\mathcal{D}_\rho$. It satisfies

$$\tilde{\varphi} \circ E_{\mathcal{D}}(S_\mu a S_\nu^*) = \delta_{\mu,\nu}\frac{1}{\beta^{|\mu|}}\varphi(a\rho_\mu(1)) \tag{4.5}$$

for $a \in \mathcal{A}, \mu, \nu \in B_*(\Lambda)$ with $|\mu| = |\nu|$. Hence the extension of a continuous linear functional on $\mathcal{D}_\rho$ to $\mathcal{F}_\rho$ is automatic. We have only to study extension of a linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ on $\mathcal{A}$ to $\mathcal{D}_\rho$. The condition (iii) of Lemma 4.3 is equivalent to $\psi = \tilde{\varphi} \circ E_{\mathcal{D}}$ where $\tilde{\varphi}$ is a linear functional on ${\mathcal{D}_\rho}^{\mathrm{alg}}$ obtained from $\varphi \in \mathcal{E}_\beta(\rho)$ as in Lemma 4.2, and so thst ψ is continuous if and only if $\tilde{\varphi}$ is continuous. We call the extension $\tilde{\varphi}$ on ${\mathcal{D}_\rho}^{\mathrm{alg}}$ of $\varphi \in \mathcal{E}_\beta(\rho)$ the *canonical extension* of φ.

LEMMA 4.4. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and power-bounded. For $\beta \in \mathbf{C}$ with $|\beta| = r_\rho > 1$, a (not necessarily positive) continuous linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ on $\mathcal{A}$ extends to a continuous linear functional $\tilde{\varphi}$ on $\mathcal{D}_\rho$ satisfying* (4.3).

Proof. As $(\mathcal{A}, \rho, \Sigma)$ is irreducible, we may take a faithful invariant state τ on $\mathcal{A}$, which we will fix. By the hypothesis that $(\mathcal{A}, \rho, \Sigma)$ is power-bounded, there exists a positive number M such that $\frac{\|\lambda_\rho^k(1)\|}{r_\rho^k} < M$ for all $k \in \mathbf{N}$. By [43, Theorem 4.2], there exists a partial isometry $v \in \mathcal{A}^{**}$ and a positive linear functional $\psi \in \mathcal{A}^*$ such that

$$\varphi(a) = \psi(av), \qquad a \in \mathcal{A}.$$

For $x = \sum_{\mu \in B_k(\Lambda)} S_\mu x_\mu S_\mu^* \in \mathcal{D}_\rho^k$ as in (4.2). Define a linear functional φ_k on $\mathcal{D}_\rho^k$ by (4.4). As in Lemma 4.2, $\varphi_{k+1}|_{\mathcal{D}_\rho^k} = \varphi_k$ and hence $\{\varphi_k\}_{k \in \mathbf{N}}$ defines a linear functional on ${\mathcal{D}_\rho}^{\mathrm{alg}}$. It then follows that

$$|\varphi(x_\mu)| = |\psi(\rho_\mu(1)x_\mu\rho_\mu(1)v)| \le \psi(\rho_\mu(1))^{\frac{1}{2}} \|x_\mu^* x_\mu\|^{\frac{1}{2}} \psi(v^* \rho_\mu(1) v)^{\frac{1}{2}}.$$

Since $\rho_\mu(1)$ commutes with v and

$$\|x_\mu\| = \|S_\mu x_\mu S_\mu^*\| \le \max_{\nu \in B_k(\Lambda)} \|S_\nu x_\nu S_\nu^*\| = \|x\|, \tag{4.6}$$

we have

$$|\varphi(x_\mu)| \le \|x\|\psi(\rho_\mu(1))$$

and hence

$$|\varphi_k(x)| \le \frac{1}{|\beta|^k} \sum_{\mu \in B_k(\Lambda)} |\varphi(x_\mu)| \le \frac{1}{|\beta|^k} \|x\|\psi(\lambda_\rho^k(1)) = \frac{\|\lambda_\rho^k(1)\|}{r_\rho^k} \psi(1)\|x\|.$$

Therefore we have

$$|\varphi_k(x)| \le M\psi(1)\|x\|, \qquad x \in \mathcal{D}_\rho^k$$

and hence $\{\varphi_k\}_{k \in \mathbf{N}}$ extends to a continuous linear functional on the closure $\mathcal{D}_\rho$ of ${\mathcal{D}_\rho}^{\mathrm{alg}}$. □

If in particular a linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ is positive on $\mathcal{A}$, it always extends to a continuous linear functionl on $\mathcal{D}_\rho$ as in the following way:

Lemma 4.5. *Let $\beta \in \mathbf{C}$ be $|\beta| > 1$. If $\varphi \in \mathcal{E}_\beta(\rho)$ is a positive linear functional on $\mathcal{A}$, then β becomes a positive real number and the canonical extension $\tilde{\varphi}$ to $\mathcal{D}_\rho$ is continuous on $\mathcal{D}_\rho$.*

Proof. One may assume that $\varphi \neq 0$ and $\varphi(1) = 1$. We have $\beta = \beta\varphi(1) = \varphi(\lambda_\rho(1)) \geq 1$. For $k \in \mathbf{N}$, define a linear functional φ_k on $\mathcal{D}_\rho^k$ by (4.4). Since for $x = \sum_{\mu \in B_k(\Lambda)} S_\mu x_\mu S_\mu^* \in \mathcal{D}_\rho^k$ we have by (4.6),

$$|\varphi(\rho_\mu(1)x_\mu\rho_\mu(1))| \leq \varphi(\rho_\mu(1))^{\frac{1}{2}}\varphi(\rho_\mu(1)x_\mu^*x_\mu\rho_\mu(1))^{\frac{1}{2}} \leq \|x\|\varphi(\rho_\mu(1)),$$

it follows that

$$|\varphi_k(x)| \leq \frac{1}{|\beta|^k}\sum_{\mu \in B_k(\Lambda)}|\varphi(\rho_\mu(1)x_\mu\rho_\mu(1))| \leq \frac{1}{|\beta|^k}\|x\|\varphi(\lambda_\rho^k(1)) = \|x\|.$$

Therefore $\{\varphi_k\}_{k\in\mathbf{N}}$ extends to a state on $\mathcal{D}_\rho$. □

We are now assuming that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. By Lemma 3.4, there exists a faithful invariant state $\tau \in \mathcal{E}_{r_\rho}(\rho)$ on $\mathcal{A}$. By the previous lemma, the canonical extension $\tilde{\tau}$ is continuous on $\mathcal{D}_\rho$ which satisfies

$$\tilde{\tau}(S_\mu a S_\mu^*) = \frac{1}{r_\rho^{|\mu|}}\tau(a\rho_\mu(1)), \qquad a \in \mathcal{A}, \mu \in B_*(\Lambda). \tag{4.7}$$

Lemma 4.6. *For a faithful invariant state $\tau \in \mathcal{E}_{r_\rho}(\rho)$ on $\mathcal{A}$, the canonical extension $\tilde{\tau}$ is faithful on $\mathcal{D}_\rho$.*

Proof. Suppose that $\tilde{\tau}$ is not faithful on $\mathcal{D}_\rho$. Put

$$I_{\tilde{\tau}} = \{x \in \mathcal{D}_\rho \mid \tilde{\tau}(x^*x) = 0\}.$$

Since $\tilde{\tau}$ is tracial on $\mathcal{D}_\rho$, $I_{\tilde{\tau}}$ is a nonzero ideal of $\mathcal{D}_\rho$. By Lemma 4.3, the equality $\tilde{\tau} \circ \lambda_\rho = r_\rho\tilde{\tau}$ holds on $\mathcal{D}_\rho$ so that $I_{\tilde{\tau}}$ is λ_ρ-invariant. The sequence $\mathcal{D}_\rho^k, k \in \mathbf{N}$ of algebras is increasing such that $\cup_{k\in\mathbf{N}}\mathcal{D}_\rho^k$ is dense in $\mathcal{D}_\rho$. We may find $k \in \mathbf{N}$ such that $I_{\tilde{\tau}} \cap \mathcal{D}_\rho^k \neq 0$. It is easy to see that $\lambda_\rho^k(\mathcal{D}_\rho^k) \subset \mathcal{A}$ so that there exists a nonzero positive element $x \in I_{\tilde{\tau}} \cap \mathcal{D}_\rho^k$ such that $\lambda_\rho^k(x) \in I_{\tilde{\tau}} \cap \mathcal{A}$. Hence $I_{\tilde{\tau}} \cap \mathcal{A}$ is a nonzero λ_ρ-invariant ideal of $\mathcal{A}$. By the hypothesis that $(\mathcal{A}, \rho, \Sigma)$ is irreducible, we have a contradiction. □

For a faithful invariant state τ on $\mathcal{A}$, we will write the canonical extension $\tilde{\tau}$ of τ to $\mathcal{D}_\rho$ as still τ. Define a unital endomorphism $\phi_\rho : \mathcal{D}_\rho \longrightarrow \mathcal{D}_\rho$ by setting

$$\phi_\rho(y) = \sum_{\alpha\in\Sigma} S_\alpha y S_\alpha^*, \qquad y \in \mathcal{D}_\rho. \tag{4.8}$$

It induces a unital endomorphism on the enveloping von Neumann algebra ${\mathcal{D}_\rho}^{**}$ of $\mathcal{D}_\rho$, which we still denote by ϕ_ρ. The restriction of the positive map λ_ρ on $\mathcal{F}_\rho$ to $\mathcal{D}_\rho$ similarly induces a positive map on ${\mathcal{D}_\rho}^{**}$. We then need the following lemma for further discussions.

Lemma 4.7. *The equality*

$$\lambda_\rho(x\phi_\rho(y)) = \lambda_\rho(x)y, \qquad x, y \in {\mathcal{D}_\rho}^{**} \tag{4.9}$$

holds.

Proof. Since $\mathcal{D}_\rho$ is dense in ${\mathcal{D}_\rho}^{**}$ under $\sigma({\mathcal{D}_\rho}^{**}, {\mathcal{D}_\rho}^{*})$-topology, it suffices to show the equality (4.9) for $x, y \in \mathcal{D}_\rho$. One has

$$\begin{aligned}\lambda_\rho(x\phi_\rho(y)) &= \sum_{\alpha,\gamma\in\Sigma} S_\alpha^* x S_\gamma y S_\gamma^* S_\alpha \\ &= \sum_{\alpha\in\Sigma} S_\alpha^* x S_\alpha y S_\alpha^* S_\alpha = \sum_{\alpha\in\Sigma} S_\alpha^* x S_\alpha y = \lambda_\rho(x)y.\end{aligned}$$

□

Recall that for a continuous linear functional ψ on a C^*-algebra $\mathcal{B}$ there exist a partial isometry $v \in \mathcal{B}^{**}$ and a positive linear functional $|\psi| \in \mathcal{B}^*$ in a unique way such that

$$v^*v = s(|\psi|), \qquad \psi(x) = |\psi|(xv) \quad \text{for } x \in \mathcal{B}, \tag{4.10}$$

where $s(|\psi|)$ denotes the support projection of $|\psi|$ (cf. [43, Theorem 4.2]). The decomposition (4.10) is called the polar decomposition of ψ. The linear functional $\psi : x \longrightarrow |\psi|(xv)$ is denoted by $v|\psi|$.

Lemma 4.8. *Let $\beta = re^{i\theta} \in \mathbf{C}$ be $r, \theta \in \mathbf{R}$ with $r > 1$. For a (not necessarily positive) linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ on $\mathcal{A}$, let $\tilde\varphi$ be the extension on ${\mathcal{D}_\rho}^{alg}$ satisfying* (4.3)*. Suppose that the linear functional $\tilde\varphi$ extends to a continuous linear functional on $\mathcal{D}_\rho$. Denote by $\tilde\varphi = v|\tilde\varphi|$ its polar decomposition for a partial isometry $v \in {\mathcal{D}_\rho}^{**}$ and a positive linear functional $|\tilde\varphi|$ on $\mathcal{D}_\rho$ such that $v^*v = s(|\tilde\varphi|)$. Then we have*

$$\phi_\rho(v) = e^{i\theta}v, \qquad |\tilde\varphi|(\lambda_\rho(x)) = r|\tilde\varphi|(x) \quad \text{for } x \in \mathcal{D}_\rho.$$

Hence the restriction of $|\tilde\varphi|$ to $\mathcal{A}$ belongs to $\mathcal{E}_r(\rho)$ and $|\tilde\varphi|$ satisfies

$$|\tilde\varphi|(S_\mu a S_\mu^*) = \frac{1}{r^{|\mu|}}|\tilde\varphi|(a\rho_\mu(1)), \qquad a \in \mathcal{A}, \mu \in B_*(\Lambda).$$

Proof. Put a positive linear functional ψ on $\mathcal{D}_\rho$ and a partial isometry u in ${\mathcal{D}_\rho}^{**}$ by setting

$$\psi(x) = \frac{1}{r}|\tilde\varphi|(\lambda_\rho(x)) \quad \text{for } x \in \mathcal{D}_\rho \quad \text{and} \quad u = e^{-i\theta}\phi_\rho(v).$$

As $\lambda_\rho(xu) = e^{-i\theta}\lambda_\rho(x)v$ for $x \in \mathcal{D}_\rho$ by Lemma 4.7. It follows that for $x \in \mathcal{D}_\rho$

$$(u\psi)(x) = \frac{1}{r}|\tilde\varphi|(\lambda_\rho(xu)) = \frac{1}{\beta}|\tilde\varphi|(\lambda_\rho(x)v) = \tilde\varphi(x).$$

Hence we have

$$\tilde\varphi = u\psi \qquad \text{on } \mathcal{D}_\rho.$$

We will next show that $s(\psi) = u^*u$. For $y \in \mathcal{D}_\rho$, we have by Lemma 4.7

$$\psi(yu^*u) = \frac{1}{r}|\tilde{\varphi}|(\lambda_\rho(yu^*u)) = \frac{1}{r}|\tilde{\varphi}|(\lambda_\rho(y\phi_\rho(v^*v))) = \frac{1}{r}|\tilde{\varphi}|(\lambda_\rho(y)v^*v) = \psi(y).$$

Hence we have $u^*u \geq s(\psi)$. On the other hand, suppose that a projection $p \in \mathcal{D}_\rho{}^{**}$ satisfies

$$\psi(yp) = \psi(y) \qquad \text{for } y \in \mathcal{D}_\rho.$$

We then have $|\tilde{\varphi}|(\lambda_\rho(y(1-p))) = 0$ for all $y \in \mathcal{D}_\rho$. For $y = S_\alpha S_\alpha^*, \alpha \in \Sigma$, one has $|\tilde{\varphi}|(S_\alpha^*(1-p)S_\alpha)) = 0$. As $S_\alpha^*(1-p)S_\alpha$ is a projection in $\mathcal{D}_\rho$, one obtains that $S_\alpha^*(1-p)S_\alpha \leq 1 - v^*v$ so that $1 - p \leq 1 - \phi_\rho(v^*v)$. This implies that $u^*u \leq p$. Therefore we have $u^*u \leq s(\psi)$ and hence

$$u^*u = s(\psi).$$

By the uniqueness of the polar decomposition, we conclude that

$$v = u \qquad \text{and} \qquad |\tilde{\varphi}| = \psi \quad \text{on } \mathcal{D}_\rho$$

so that

$$\phi_\rho(v) = e^{i\theta}v, \qquad |\tilde{\varphi}|(\lambda_\rho(x)) = r|\tilde{\varphi}|(x) \quad \text{for} \quad x \in \mathcal{D}_\rho.$$

□

Therefore we have

Theorem 4.9. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and power-bounded. For $\beta \in \mathbf{C}$ with $|\beta| > 1$, a (not necessarily positive) linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ on $\mathcal{A}$ extends to $\mathcal{D}_\rho$ as a continuous linear functional $\tilde{\varphi}$ satisfying*

$$\tilde{\varphi}(S_\mu a S_\mu^*) = \frac{1}{\beta^{|\mu|}}\varphi(a\rho_\mu(1)), \qquad a \in \mathcal{A}, \mu \in B_*(\Lambda)$$

if $|\beta| = r_\rho$. If in particular, $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic, the converse implication holds.

Proof. The first part of the assertions is direct from Lemma 4.4. Under the condition that $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic, assume that the canonical extension $\tilde{\varphi}$ is continuous on $\mathcal{D}_\rho$. The preceding lemma says that the positive linear functional $|\tilde{\varphi}|$ belongs to $\mathcal{E}_{|\beta|}(\rho)$. Since the mean ergodicity implies (FP), by Lemma 3.6 (i) we see that $|\beta| = r_\rho$. □

Let us now assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and satisfies $\dim \mathcal{E}_{r_\rho}(\rho) = 1$, and hence it is uniquely ergodic. Take a unique invariant state τ on $\mathcal{A}$ and denote still by τ its canonical extension on $\mathcal{D}_\rho$. Denote by $p_\tau \in \mathcal{D}_\rho{}^{**}$ its support projection.

Lemma 4.10. *Let $w \in \mathcal{D}_\rho{}^{**}$ be a partial isometry satisfying*

$$w^*w = p_\tau \qquad \text{and} \qquad \phi_\rho(w) = w. \tag{4.11}$$

Then w is a scalar multiple of the projection p_τ.

Proof. Put $w\tau(x) = \tau(xw)$ for $x \in \mathcal{D}_\rho$ and hence $w\tau \in {\mathcal{D}_\rho}^*$. Since $\lambda_\rho(x)w = \lambda_\rho(x\phi_\rho(w)) = \lambda_\rho(xw)$ by Lemma 4.7, it follows that for $x \in \mathcal{D}_\rho$

$$w\tau(\lambda_\rho(x)) = \tau(\lambda_\rho(xw)) = r_\rho\tau(xw) = r_\rho w\tau(x).$$

In particular, we have $w\tau \in \mathcal{E}_{r_\rho}(\rho)$. As $\dim \mathcal{E}_{r_\rho}(\rho) = 1$ by hypothesis, $w\tau$ is a scalar multiple of τ. Hence there exists $c \in \mathbf{C}$ such that $\tau(xw) = c\tau(x)$ for $x \in \mathcal{A}$. Since $w\tau$ is the canonical extension of $\tau(\cdot\, w) = w\tau$ on $\mathcal{A}$ to $\mathcal{D}_\rho$ and the canonical extension is unique, one has $\tau(xw) = c\tau(x)$ for $x \in \mathcal{D}_\rho$ so that

$$\tau(xw) = \tau(xcp_\tau) \qquad \text{for } x \in \mathcal{D}_\rho. \tag{4.12}$$

As $c = c\tau(1) = \tau(w)$, one has

$$1 = \tau(p_\tau) = \tau(w^*w) = c\tau(w^*) = c\overline{\tau(w)} = c\bar{c}$$

so that

$$(cp_\tau)^*(cp_\tau) = p_\tau = w^*w.$$

By the uniqueness of the polar decomposition, we have by (4.12) $w = cp_\tau$. $\square$

Proposition 4.11. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and satisfies $\dim \mathcal{E}_{r_\rho}(\rho) = 1$. Then $\dim \mathcal{E}_\beta(\rho) \le 1$ for $\beta \in \mathbf{C}$ with $|\beta| = r_\rho > 1$.*

Proof. Let $|\beta| = r_\rho > 1$. Take an arbitrary linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ with $\varphi \ne 0$. Its canonical extension $\tilde{\varphi}$ to $\mathcal{D}_\rho$ is continuous. Denote by $\tilde{\varphi} = v_{\tilde{\varphi}}|\tilde{\varphi}|$ the polar decomposition in ${\mathcal{D}_\rho}^*$ where $v_{\tilde{\varphi}}$ is a partial isometry in ${\mathcal{D}_\rho}^{**}$. By Lemma 4.7, the restriction of $|\tilde{\varphi}|$ to $\mathcal{A}$ is a positive linear functional belonging to $\mathcal{E}_{r_\rho}(\rho)$. Since $(\mathcal{A}, \rho, \Sigma)$ is uniquely ergodic, by putting $c_{\tilde{\varphi}} = |\tilde{\varphi}|(1)$ one has $|\tilde{\varphi}| = c_{\tilde{\varphi}}\tau$ as a positive linear functional on $\mathcal{A}$. The canonical extension to $\mathcal{D}_\rho$ which satisfies (4.3) is unique and determined by its behavior on $\mathcal{A}$. Hence the equalty $|\tilde{\varphi}| = c_{\tilde{\varphi}}\tau$ holds as a positive linear functional on $\mathcal{D}_\rho$ so that we have $\operatorname{supp}(|\tilde{\varphi}|) = \operatorname{supp}(\tau)$ and hence $v_{\tilde{\varphi}}^* v_{\tilde{\varphi}} = p_\tau$. For another linear functional $\psi \in \mathcal{E}_\beta(\rho)$ with $\psi \ne 0$, we have similar decompositions

$$\tilde{\psi} = v_{\tilde{\psi}}|\tilde{\psi}|, \qquad |\tilde{\psi}| = c_{\tilde{\psi}}\tau, \qquad v_{\tilde{\psi}}^* v_{\tilde{\psi}} = p_\tau.$$

Put a partial isometry $w = v_{\tilde{\varphi}}^* v_{\tilde{\psi}} \in {\mathcal{D}_\rho}^{**}$ so that $w^*w = p_\tau$. By Lemma 4.8, one has $\phi_\rho(w) = w$. Lemma 4.10 implies $w = cp_\tau$ for some $c \in \mathbf{C}$ with $|c| = 1$ so that $v_{\tilde{\psi}} = cv_{\tilde{\varphi}}$. Therefore we have

$$\tilde{\psi} = v_{\tilde{\psi}}|\tilde{\psi}| = cv_{\tilde{\varphi}}c_{\tilde{\psi}}\tau = c\frac{c_{\tilde{\psi}}}{c_{\tilde{\varphi}}}\tilde{\varphi}$$

on $\mathcal{D}_\rho$. In particular we have $\psi = c\frac{c_{\tilde{\psi}}}{c_{\tilde{\varphi}}}\varphi$ on $\mathcal{A}$ so that $\dim \mathcal{E}_\beta(\rho) \le 1$. $\square$

Corollary 4.12. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and mean ergodic. Then for $\beta \in \mathbf{C}$ with $|\beta| > 1$, we have $\dim \mathcal{E}_\beta(\rho) \le 1$ if $|\beta| = r_\rho$, otherwise $\mathcal{E}_\beta(\rho) = \{0\}$.*

Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and mean ergodic. Hence $(\mathcal{A}, \rho, \Sigma)$ is uniquely ergodic with a unique faithful invariant state $\tau \in \mathcal{E}_{r_\rho}(\rho)$. Denote by $p_\tau \in {\mathcal{D}_\rho}^{**}$ the support projection of the canonical extension of τ to $\mathcal{D}_\rho$, where the extension is still denoted by τ. For $\beta = re^{i\theta} \in \mathbf{C}$ with $r = r_\rho > 1$, we set

$$P_\beta(\mathcal{D}_\rho, \tau) = \{ v \in {\mathcal{D}_\rho}^{**} \mid \phi_\rho(v) = e^{i\theta} v, \ v^* v = p_\tau \}.$$

Denote by $\mathbf{R}_+$ the set of all nonnegative real numbers. For $\varphi \in \mathcal{E}_\beta(\rho)$ denote by $\tilde{\varphi}$ its canonical extension to $\mathcal{D}_\rho$. As $|\beta| = r_\rho$, $\tilde{\varphi}$ is continuous and has a unique polar decomposition $\tilde{\varphi} = v_{\tilde{\varphi}} |\tilde{\varphi}|$ for some $v_{\tilde{\varphi}} \in {\mathcal{D}_\rho}^{**}$ and positive linear functional $|\tilde{\varphi}| \in {\mathcal{D}_\rho}^{*}$. By Lemma 4.8, we know the structure of the eigenspace $\mathcal{E}_\beta(\rho)$ as in the following way:

Proposition 4.13. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and mean ergodic. There exists a bijective correspondence between the eigenspace $\mathcal{E}_\beta(\rho)$ and the product set $P_\beta(\mathcal{D}_\rho, \tau) \times \mathbf{R}_+$ through the correspondences*

$$\varphi \in \mathcal{E}_\beta(\rho) \longrightarrow (v_{\tilde{\varphi}}, |\tilde{\varphi}|(1)) \in P_\beta(\mathcal{D}_\rho, \tau) \times \mathbf{R}_+,$$
$$c\tau(\cdot\ v) \in \mathcal{E}_\beta(\rho) \longleftarrow (v, c) \in P_\beta(\mathcal{D}_\rho, \tau) \times \mathbf{R}_+.$$

5. Extension to $\mathcal{O}_\rho$ and KMS condition

In [9], Enomoto-Fujii-Watatani have proved that KMS states for gauge action on the Cuntz-Krieger algebra $\mathcal{O}_A$ exist if and only if its inverse temperature is $\log r_A$, where r_A is the Perron-Frobenius eigenvalue for the irreducible matrix A. They have showed that the KMS states bijectively correspond to the normalized positive eigenvectors of A for the eigenvalue r_A.

In this section, we will study KMS conditions for linear functionals without assuming its positivity at inverse temperature taking complex numbers. The extended notation is needed to study eigenvector spaces for C^*-symbolic dynamical systems.

Following after [3], KMS states for one-parameter group action α on a C^*-algebra $\mathcal{B}$ is defined as follows: For a positive real number $\gamma \in \mathbf{R}$, a state ψ on $\mathcal{B}$ is a KMS state at inverse temperature γ if ψ satisfies

$$\psi(y \alpha_{i\gamma}(x)) = \psi(xy), \qquad x \in \mathcal{B}^a, y \in \mathcal{B} \tag{5.1}$$

where $\mathcal{B}^a$ is the set of analytic elements of the action $\alpha : \mathbf{R} \longrightarrow \mathrm{Aut}(\mathcal{B})$ (cf.[3]). The equation (5.1) for ψ is called the KMS condition with respect to the action α.

In what follows, we restrict our interest to periodic actions so as to extend KMS condition to (not necessarily positive) linear functionals at inverse temperature taking complex numbers. We assume that an action α of $\mathbf{R}$ has its period 2π so that α is regarded as an action of one-dimensional torus group $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. Let $\mathcal{B}$ be a C^*-algebra and $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ a continuous action of $\mathbf{T}$ to the automorphism group $\mathrm{Aut}(\mathcal{B})$. We write a complex number $\beta \in \mathbf{C}$ as $\beta = re^{i\theta}$ where $r, \theta \in \mathbf{R}$ with $r > 1$.

Definition. A continuous linear functional $\varphi \in \mathcal{B}^*$ on $\mathcal{B}$ is said to satisfy *KMS condition* at Logβ if φ satisfies the following condition

$$\varphi(y\alpha_{i\log r}(x)) = \varphi(\alpha_\theta(x)y), \qquad x \in \mathcal{B}^a, y \in \mathcal{B}. \tag{5.2}$$

Remark.

(i) As $\alpha_\theta(x) = \alpha_{\theta+2\pi}(x)$, the right hand side $\varphi(\alpha_\theta(x)y)$ of (5.2) does not depend on the choice of $\theta \in \mathbf{R}$ as long as $\beta = re^{i\theta}$.

(ii) The above KMS condition (5.2) is equivalent to the following condition:

$$\varphi(y\alpha_{\zeta+i\log r}(x)) = \varphi(\alpha_{\zeta+\theta}(x)y), \qquad x \in \mathcal{B}^a, y \in \mathcal{B}, \quad \zeta \in \mathbf{C} \tag{5.3}$$

(iii) In case of $\theta = 0$, the above definition of KMS condition coincides with the original definition of KMS condition for states.

(iv) The above equality (5.2) can be written formally as

$$\varphi(y\alpha_{i\mathrm{Log}\beta}(x)) = \varphi(xy), \qquad x \in \mathcal{B}^a, y \in \mathcal{B}, \tag{5.4}$$

if we denote $\mathrm{Log}\beta = \log r + i\theta$.

We will present some examples of linear functionals satisfying the extended KMS conditions.

Examples.

(i) Let $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ be an action of $\mathbf{T}$ to a C^*-algebra $\mathcal{B}$ such that there exists a projection $H \in \mathcal{B}$ satisfying $\alpha_t(a) = e^{itH}ae^{-itH}, a \in \mathcal{B}, t \in \mathbf{T}$. Assume that there exists an α-invariant tracial state tr on $\mathcal{B}$. Put

$$\varphi(x) = \frac{tr(e^{-\mathrm{Log}\beta H}x)}{tr(e^{-\mathrm{Log}\beta H})}, \qquad x \in \mathcal{B},$$

where $\mathrm{Log}\beta = \log r + i\theta$. Then φ satisfies KMS condition at Logβ.

(ii) Let $\mathcal{B} = \otimes_{k=1}^\infty M_2$ be the UHF-algebra of type 2^∞ and $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ an action of $\mathbf{T}$ to $\mathcal{B}$ defined by

$$\alpha_t = \otimes_{k=1}^\infty Ad \begin{bmatrix} 1 & 0 \\ 0 & e^{it} \end{bmatrix}, \qquad t \in \mathbf{T}.$$

Put

$$\begin{aligned}
\mathcal{B}_n &= \otimes_{k=1}^n M_2 = M_2 \otimes \cdots \otimes M_2, \\
u_t^n &= \otimes_{k=1}^n \begin{bmatrix} 1 & 0 \\ 0 & e^{it} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{it} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & 0 \\ 0 & e^{it} \end{bmatrix} \in \mathcal{B}_n, \\
\alpha_t^n &= Ad(u_t^n) \in \mathrm{Aut}(\mathcal{B}_n), \quad t \in \mathbf{T}.
\end{aligned}$$

Let $\beta = re^{i\theta} \in \mathbf{C}$ be $r > 1$. Put

$$H = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in M_2, \qquad h_n = \otimes_{k=1}^n \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\beta} \end{bmatrix} \in \mathcal{B}_n,$$

and hence $h_n = \otimes_{k=1}^n e^{-\mathrm{Log}\beta H}$, $\alpha_t^n = \otimes_{k=1}^n Ad(e^{itH}), t \in \mathbf{T}$. It is straightforward to see that

$$tr(e^{-\mathrm{Log}\beta H} b\alpha_{i\log r}(a)) = tr(e^{-\mathrm{Log}\beta H}\alpha_\theta(a)b), \qquad a, b \in M_2.$$

Put

$$\varphi_n(x) = \otimes_{k=1}^n tr(xh_n) \qquad \text{for } x \in \mathcal{B}_n$$

so that we have

$$\varphi_n(y\alpha_{i\log r}(x)) = \varphi_n(\alpha_\theta(x)y), \qquad x, y \in \mathcal{B}_n.$$

As $\|h_n\| = 1$, φ_n extends to a continuous linear functional on $\mathcal{B}$, which we denote by φ. Then φ satisfies KMS condition at $\mathrm{Log}\beta$:

$$\varphi(y\alpha_{i\log r}(x)) = \varphi_n(\alpha_\theta(x)y), \qquad x \in \mathcal{B}^a, y \in \mathcal{B}.$$

We see the following two propositions whose proofs are similar to the case of usual KMS states.

Proposition 5.1 (cf. [39, 8.12.3]). *Let $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ be a continuous action of $\mathbf{T}$ to the automorphism group $\mathrm{Aut}(\mathcal{B})$ of a C^*-algebra $\mathcal{B}$ and β a complex number with $\beta = re^{i\theta}$, $r > 1$. The following conditions for a continuous linear functional φ on $\mathcal{B}$ are equivalent:*

(i) *φ satisfies the KMS condition at $\mathrm{Log}\beta$.*

(ii) *φ satisfies the equality (5.2) for just a dense set of elements in $\mathcal{B}^a$.*

(iii) *For all $x, y \in \mathcal{B}$, there is a bounded continuous function f on the strip*

$$\Omega_{\log r} = \{\zeta \in \mathbf{C} \mid 0 \le \mathrm{Im}\zeta \le \log r\}$$

such that f is holomorphic in the interior of $\Omega_{\log r}$ and

$$f(t) = \varphi(y\alpha_t(x)), \qquad f(t + i\log r) = \varphi(\alpha_{t+\theta}(x)y), \qquad t \in \mathbf{R}.$$

Proposition 5.2 (cf. [39, 8.12.4]). *Let $\mathcal{B}$ be a C^*-algebra and $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ be a continuous action of $\mathbf{T}$ to the automorphism group $\mathrm{Aut}(\mathcal{B})$. Let φ be a continuous linear functional on $\mathcal{B}$. If φ satisfies KMS condition at $\mathrm{Log}\beta$ for some complex number β with $\beta = re^{i\theta}$ with $r > 1$, then φ is α-invariant, that is,*

$$\varphi \circ \alpha_t = \varphi, \qquad t \in \mathbf{T}.$$

We henceforth go back to our previous situations. Let $(\mathcal{A}, \rho, \Sigma)$ be a C^*-symbolic dynamical system. Recall that the positive operator λ_ρ on $\mathcal{A}$ extends to $\mathcal{F}_\rho$ by setting $\lambda_\rho(x) = \sum_{\alpha\in\Sigma} S_\alpha^* x S_\alpha, x \in \mathcal{F}_\rho$. For $\beta \in \mathbf{C}$ with $\beta \ne 0$, we set

$$\mathcal{E}_\beta^{\mathcal{D}}(\rho) = \{\varphi \in {\mathcal{D}_\rho}^* \mid \varphi(\lambda_\rho(x)) = \beta\varphi(x), x \in \mathcal{D}_\rho\}, \tag{5.5}$$

$$\mathcal{E}_\beta^{\mathcal{F}}(\rho) = \{\phi \in {\mathcal{F}_\rho}^* \mid \phi(\lambda_\rho(x)) = \beta\phi(x), x \in \mathcal{F}_\rho,\ \phi \text{ is tracial on } \mathcal{F}_\rho\}. \tag{5.6}$$

It is possible that both $\mathcal{E}_\beta^{\mathcal{D}}(\rho)$ and $\mathcal{E}_\beta^{\mathcal{F}}(\rho)$ are $\{0\}$. Recall that $E_{\mathcal{D}} : \mathcal{F}_\rho \longrightarrow \mathcal{D}_\rho$ is the canonical expectation satisfying by $E_{\mathcal{D}}(S_\mu a S_\nu^*) = \delta_{\mu,\nu} S_\mu a S_\nu^*$ for $a \in \mathcal{A}$ with $\mu, \nu \in B_*(\Lambda)$, $|\mu| = |\nu|$. By composing it to a given linear functional $\varphi \in \mathcal{E}_\beta^{\mathcal{D}}(\rho)$ on $\mathcal{D}_\rho$, φ extends to $\mathcal{F}_\rho$.

LEMMA 5.3. *Let $\beta \in \mathbf{C}$ with $|\beta| > 1$. A (not necessarily positive) continuous linear functional $\varphi \in \mathcal{E}_\beta^{\mathcal{D}}(\rho)$ on $\mathcal{D}_\rho$ uniquely extends to $\mathcal{F}_\rho$ as a tracial continuous linear functional $\phi = \varphi \circ E_{\mathcal{D}}$ such that*

$$\phi(S_\mu x S_\nu^*) = \delta_{\mu,\nu} \frac{1}{\beta^{|\mu|}} \phi(x S_\mu^* S_\nu), \qquad x \in \mathcal{F}_\rho, \mu, \nu \in B_*(\Lambda) \text{ with } |\mu| = |\nu|. \tag{5.7}$$

Hence the sets $\mathcal{E}_\beta^{\mathcal{D}}(\rho)$ and $\mathcal{E}_\beta^{\mathcal{F}}(\rho)$ bijectively correspond to each other.

Proof. For $\varphi \in \mathcal{E}_\beta^{\mathcal{D}}(\rho)$, as in the proof of Lemma 4.3 (i) $\Rightarrow$ (ii), the equality

$$\varphi(S_\mu a S_\mu^*) = \frac{1}{\beta^{|\mu|}} \varphi(a \rho_\mu(1)), \qquad a \in \mathcal{A}, \mu \in B_*(\Lambda)$$

holds so that

$$\phi(S_\mu a S_\nu^*) = \delta_{\mu,\nu} \frac{1}{\beta^{|\mu|}} \varphi(a \rho_\mu(1)), \qquad a \in \mathcal{A},\ \mu, \nu \in B_*(\Lambda) \text{ with } |\mu| = |\nu|.$$

By Lemma 4.3 (iii) $\Rightarrow$ (i), ϕ belongs to $\mathcal{E}_\beta^{\mathcal{F}}(\rho)$. $\square$

Recall that $E_\rho : \mathcal{O}_\rho \longrightarrow {\mathcal{O}_\rho}^{\hat\rho} = \mathcal{F}_\rho$ denotes the conditional expectation defined by (2.3).

PROPOSITION 5.4. *For any tracial continuous linear functional $\phi \in \mathcal{E}_\beta^{\mathcal{F}}(\rho)$, the composition $\psi = \phi \circ E_\rho$ is a continuous linear functional on $\mathcal{O}_\rho$ which satisfies KMS condition at $\mathrm{Log}\beta$ for gauge action $\hat\rho$ of $\mathbf{T}$.*

Proof. Let $\mathcal{P}_\rho$ be the dense $*$-subalgebra of $\mathcal{O}_\rho$ generated algebraically by $S_\alpha, \alpha \in \Sigma$ and $a \in \mathcal{A}$. It is clear that for each element $x \in \mathcal{P}_\rho$ the function $t \in \mathbf{T} = \mathbf{R}/2\pi\mathbf{R} \to \hat\rho_t(x) \in \mathcal{O}_\rho$ extends to an entire analytic function on $\mathbf{C}$. Put $\psi = \phi \circ E_\rho$. We will show that the equality (5.2) holds for ψ. Elements $x, y \in \mathcal{P}_\rho$ can be expanded as finite linear combinations

$$x = \sum x_{-\nu} S_\nu^* + x_0 + \sum S_\mu x_\mu, \quad y = \sum y_{-\nu} S_\nu^* + y_0 + \sum S_\mu y_\mu \tag{5.8}$$

for some $x_{-\nu}, x_0, x_\mu, y_{-\nu}, y_0, y_\mu \in {\mathcal{F}_\rho}^{\mathrm{alg}}$. As ψ is a tracial linear functional on $\mathcal{F}_\rho$, it suffices to check the equality (5.2) for the following two cases

$$(1) \quad x = S_\nu x_\nu, \quad y = y_{-\nu} S_\nu^*, \qquad (2) \quad x = x_{-\mu} S_\mu^*, \quad y = S_\mu y_\mu.$$

Case (1):

$$\begin{aligned}
\psi(y \hat\rho_{i \log r}(x)) &= \psi(y_{-\nu} S_\nu^* e^{-|\nu| \log r} S_\nu x_\nu) \\
&= \frac{1}{\beta^{|\nu|}} \psi(e^{i|\nu|\theta} x_\nu y_{-\nu} S_\nu^* S_\nu) \\
&= \psi(e^{i|\nu|\theta} S_\nu x_\nu y_{-\nu} S_\nu^*) \\
&= \psi(\hat\rho_\theta(x) y).
\end{aligned}$$

Case (2):

$$\begin{aligned}\psi(y\hat\rho_{i\log r}(x)) &= \psi(S_\mu y_\mu e^{|\mu|\log r} x_{-\mu} S_\mu^*)\\ &= \frac{r^{|\mu|}}{\beta^{|\mu|}}\psi(y_\mu x_{-\mu} S_\mu^* S_\mu)\\ &= \psi(e^{-i|\mu|\theta} x_{-\mu} S_\mu^* S_\mu y_\mu)\\ &= \psi(\hat\rho_\theta(x) y).\end{aligned}$$

This completes the proof. □

Conversely we have

LEMMA 5.5. *If a continuous linear functional ψ on $\mathcal{O}_\rho$ satisfies KMS condition at* Logβ *for some $\beta \in \mathbf{C}$ with $|\beta| > 1$, then the restriction $\phi = \psi|_{\mathcal{F}_\rho}$ to $\mathcal{F}_\rho$ belongs to $\mathcal{E}_\beta^{\mathcal{F}}(\rho)$ and satisfies the equality $\psi = \phi \circ E_\rho$.*

Proof. Let $\beta = re^{i\theta}$ with $r > 1$. For any $x \in \mathcal{F}_\rho, \mu \in B_*(\Lambda)$, we see

$$\psi(S_\mu x) = \frac{1}{\beta^{|\mu|}}\psi(xS_\mu) = \frac{1}{\beta^{|\mu|}}\psi(S_\mu\hat\rho_{i\log r}(\alpha_{-\theta}(x))) = \frac{1}{\beta^{|\mu|}}\psi(S_\mu x)$$

so that $\psi(S_\mu x) = 0$ because $|\beta| > 1$. We similarly have $\psi(xS_\mu^*) = 0$. Since any element of $\mathcal{P}_\rho$ can be expanded as in (5.8), we get $\psi(y) = \phi \circ E_\rho(y)$ for $y \in \mathcal{P}_\rho$. We will next show that ϕ belongs to $\mathcal{E}_\beta^{\mathcal{F}}(\rho)$. For $x, y \in \mathcal{F}_\rho$, one sees $\hat\rho_{i\log r}(x) = \hat\rho_{-\theta}(x) = x$ so that $\psi(yx) = \psi(xy)$. Hence ψ gives rise to a tracial linear functional ϕ on $\mathcal{F}_\rho$. By KMS condition, we get for any $x \in \mathcal{F}_\rho, \mu \in B_*(\Lambda)$,

$$\psi(S_\mu \cdot xS_\mu^*) = \psi(xS_\mu^*\hat\rho_{i\log r}(\hat\rho_{-\theta}(S_\mu)) = \frac{1}{\beta^{|\mu|}}\psi(xS_\mu^* S_\mu).$$

Thus by Lemma 4.3, we know $\phi \in \mathcal{E}_\beta^{\mathcal{F}}(\rho)$. □

We set for $\beta \in \mathbf{C}$ with $|\beta| > 1$,

$$\begin{aligned}&KMS_\beta(\mathcal{O}_\rho)\\ =&\{\psi \in {\mathcal{O}_\rho}^* \mid \psi \text{ satisfies KMS condition at Log}\beta \text{ for gauge action}\}\end{aligned}$$

and

$$Sp(\rho) = \{\beta \in \mathbf{C} \mid \varphi \circ \lambda_\rho = \beta\varphi \text{ for some } \varphi \in \mathcal{A}^* \text{ with } \varphi \neq 0\}.$$

By Proposition 5.4 and Lemma 5.5, we have

PROPOSITION 5.6. *Let $(\mathcal{A}, \rho, \Sigma)$ be an irreducible C^*-symbolic dynamical system. Assume that $(\mathcal{A}, \rho, \Sigma)$ is power-bounded. Let $\beta \in \mathbf{C}$ be a complex number with $|\beta| > 1$. If $|\beta| = r_\rho$ and $\beta \in Sp(\rho)$, we have $KMS_\beta(\mathcal{O}_\rho) \neq \{0\}$. If in particular, $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic, $KMS_\beta(\mathcal{O}_\rho) \neq \{0\}$ if and only if $|\beta| = r_\rho$ and $\beta \in Sp(\rho)$.*

Proof. Under the assumption that $(\mathcal{A}, \rho, \Sigma)$ is power-bounded, any continuous linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ on $\mathcal{A}$ can uniquely extend to a continuous linear functional $\tilde\varphi$ on $\mathcal{D}_\rho$, that belongs to $\mathcal{E}_\beta^{\mathcal{D}}(\rho)$ if $|\beta| = r_\rho$. By Proposition 5.4,

$\tilde{\varphi} \circ E_{\mathcal{D}} \in \mathcal{E}_\beta^{\mathcal{F}}(\rho)$ has an extension on $\mathcal{O}_\rho$ as a continuous linear functional that satisfies KMS condition at $\mathrm{Log}\beta$.
Conversely, the restriction of a continuous linear functional $KMS_\beta(\mathcal{O}_\rho)$ to the subalgebra $\mathcal{A}$ yields a nonzero element of $\mathcal{E}_\beta(\rho)$ which has continuous extension to $\mathcal{D}_\rho$. If in particular, $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic, $|\beta|$ must be r_ρ by Theorem 4.9. □

Therefore we conclude

Theorem 5.7. *Let $(\mathcal{A}, \rho, \Sigma)$ be an irreducible C^*-symbolic dynamical system. Let $\beta \in \mathbf{C}$ be a complex number with $|\beta| = r_\rho > 1$.*

(i) *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is power-bounded. Then there exist linear isomorphisms among the four spaces $\mathcal{E}_\beta(\rho)$, $\mathcal{E}_\beta^{\mathcal{D}}(\rho)$, $\mathcal{E}_\beta^{\mathcal{F}}(\rho)$ and $KMS_\beta(\mathcal{O}_\rho)$ through the correspondences $\varphi \in \mathcal{E}_\beta(\rho)$, $\tilde{\varphi} \in \mathcal{E}_\beta^{\mathcal{D}}(\rho)$, $\tilde{\varphi} \circ E_{\mathcal{D}} \in \mathcal{E}_\beta^{\mathcal{F}}(\rho)$, $\tilde{\varphi} \circ E_{\mathcal{D}} \circ E_\rho \in KMS_\beta(\mathcal{O}_\rho)$ respectively. In particular, there exists a bijective correspondence between the set $\mathcal{E}_\beta(\rho)$ of eigenvectors of λ_ρ^* for eigenvalue β consisting of continuous linear functionals on $\mathcal{A}$ and the set $KMS_\beta(\mathcal{O}_\rho)$ of continuous linear functionals on $\mathcal{O}_\rho$ satisfying KMS condition at* $\mathrm{Log}\beta$.

(ii) *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic. Then the dimension $\dim KMS_\beta(\mathcal{O}_\rho)$ of the space of continuous linear functionals on $\mathcal{O}_\rho$ satisfying KMS condition at* $\mathrm{Log}\beta$ *is one if there exists a nonzero eigenvector of λ_ρ^* on $\mathcal{A}^*$ for the eigenvalue β. In particular there uniquely exists a faithful KMS state on $\mathcal{O}_\rho$ at* $\log r_\rho$.

The following corollary is a generalization of [9, Theorem 6].

Corollary 5.8. *Suppose that A is an irreducible matrix with entries in* $\{0, 1\}$ *with its period p_A. Let β be a complex number with $|\beta| > 1$.*

(i) *There exists a nonzero continuous linear functional on the Cuntz-Krieger algebra $\mathcal{O}_A$ satisfying KMS condition for gauge action at* $\mathrm{Log}\beta$ *if and only if β is a p_A-th root of the Perron-Frobenius eigenvalue r_A of A.*

(ii) *The space of admitted continuous linear functionals on $\mathcal{O}_A$ satisfying KMS condition for gauge action at* $\mathrm{Log}\beta$ *is of one-dimensional.*

(iii) *If in particular $\beta = r_A$, the space of admitted continuous linear functionals on $\mathcal{O}_A$ satisfying KMS condition for gauge action at* $\log r_A$ *is the scalar multiples of a unique KMS state.*

6. KMS states and invariant measures

In this section, we will study a relationship between KMS states on $\mathcal{O}_\rho$ and invariant measures on $\mathcal{D}_\rho$ under ϕ_ρ. In what follows we assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and fix a faithful invariant state τ on $\mathcal{A}$.

We denote by $\|a\|_2$ the L^2-norm $\tau(a^*a)^{\frac{1}{2}}$ for $a \in \mathcal{A}$, and by $\mathcal{H}_\tau$ the completion of $\mathcal{A}$ by the norm $\|\cdot\|_2$. By the inequalities for $n \in \mathbf{N}$, $a \in \mathcal{A}$

$$\tau(\lambda_\rho^n(a)^*\lambda_\rho^n(a)) \leq \|\lambda_\rho^n\|\tau(\lambda_\rho^n(a^*a)) = \|\lambda_\rho^n\|r_\rho^n\tau(a^*a) \leq \|\lambda_\rho^n\|^2\|a\|_2^2, \tag{6.1}$$

the operators $T_\rho^n, n \in \mathbf{N}$ induce bounded linear operators on $\mathcal{H}_\tau$. The induced operators on $\mathcal{H}_\tau$, which we also denote by $T_\rho^n, n \in \mathbf{N}$, are uniformly bounded in the operator norm on $\mathcal{H}_\tau$, if $(\mathcal{A}, \rho, \Sigma)$ is power-bonded. We provide the following lemma, which shows power-boundedness of $(\mathcal{A}, \rho, \Sigma)$ induces an ordinary mean ergodicity on $\mathcal{H}_\tau$, is a direct consequence from [22, p.73,Theorem 1.2]. We give a proof for the sake of completeness.

Lemma 6.1. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and power-bounded. Then*

$$\lim_{n\to\infty} \frac{1 + T_\rho + T_\rho^2 + \cdots + T_\rho^{n-1}}{n}$$

converges to an idempotent P_ρ on $\mathcal{H}_\tau$ under strong operator topology in $B(\mathcal{H}_\tau)$. The subspace $P_\rho\mathcal{H}_\tau$ consists of the vectors of $\mathcal{H}_\tau$ fixed under T_ρ.

Proof. The mean operators $M_n, n \in \mathbf{N}$ on $\mathcal{A}$ defined by (3.1) naturally act on $\mathcal{H}_\tau$. Since $(\mathcal{A}, \rho, \Sigma)$ is power-bounded, there exists a positive number $c > 0$ such that $\|T_\rho^n\| < c$ for all $n \in \mathbf{N}$. As $\|M_n\| < 1 + c, n \in \mathbf{N}$, the sequence $M_n v \in \mathcal{H}_\tau, n \in \mathbf{N}$ for a vector $v \in \mathcal{H}_\tau$ has a cluster point v_0 under the weak topology of $\mathcal{H}_\tau$. The identites

$$(I - T_\rho)M_n = M_n(I - T_\rho) = \frac{1}{n}(I - T_\rho^n)$$

imply the inequalites

$$\|(I - T_\rho)M_n\| = \|M_n(I - T_\rho)\| = \frac{1}{n}\|I - T_\rho^n\| < \frac{1}{n}(1 + c). \tag{6.2}$$

Hence we have $T_\rho v_0 = v_0$. Put

$$Q_n = \frac{1}{n}\{(I + T_\rho) + (I + T_\rho + T_\rho^2) + \cdots + (I + T_\rho + \cdots + T_\rho^{n-2})\}.$$

Then we have $v - M_n v = (I - T_\rho)Q_n v, n \in \mathbf{N}$. Hence $v - v_0$ belongs to the weak closure $\mathcal{K}_\tau$ of the subspace $(I - T_\rho)\mathcal{H}_\tau$. The weak closure $\mathcal{K}_\tau$ is also the norm closure of the subspace $(I - T_\rho)\mathcal{H}_\tau$. For $w \in \mathcal{K}_\tau$, take $w_j \in (I - T_\rho)\mathcal{H}_\tau$ such that $\|w - w_j\|_2 \to 0$ and $w_j = (I - T_\rho)x_j$ for some $x_j \in \mathcal{H}_\tau$. We then have by (6.2)

$$\begin{aligned}\|M_n w\|_2 &\leq \|M_n\|\|w - w_j\|_2 + \|M_n(I - T_\rho)x_j\|_2 \\ &\leq (1 + c)\|w - w_j\|_2 + \frac{1}{n}(1 + c)\|x_j\|_2\end{aligned}$$

so that $\lim_{n\to\infty} \|M_n w\|_2 = 0$. Since $M_n v - v_0 = M_n(v - v_0)$ and $v - v_0 \in \mathcal{K}_\tau$, one has

$$\lim_{n\to\infty} \|M_n v - v_0\|_2 = 0.$$

Put $P_\rho v = v_0$. The inequality

$$\|M_n v - T_\rho M_n v\|_2 = \|(I - T_\rho)M_n v\|_2 < \frac{1}{n}(1+c)\|v\|_2$$

implies that $P_\rho = T_\rho P_\rho$ that is equal to $P_\rho T_\rho$. Therefore $P_\rho = M_n P_\rho = P_\rho M_n$ and hence $P_\rho = P_\rho^2$. $\square$

Remark. Under the same assumption above, one may prove that the limit

$$\lim_{r\downarrow r_\rho}(r - r_\rho)R(r)$$

for the resolvent $R(r) = (r - \lambda_\rho)^{-1}$ with $r > r_\rho$ converges to the idempotent P_ρ on $\mathcal{H}_\tau$ under strong operator topology in $B(\mathcal{H}_\tau)$. Hence the equality

$$\lim_{r\downarrow r_\rho}(r - r_\rho)R(r) = \lim_{n\to\infty}\frac{1 + T_\rho + T_\rho^2 + \cdots + T_\rho^{n-1}}{n} \tag{6.3}$$

holds. We will give a proof of the equality (6.3). It is enough to consider the limit $\lim_{n\to\infty}\frac{1}{n}R(r_\rho + \frac{1}{n})$ instead of $\lim_{r\downarrow r_\rho}(r - r_\rho)R(r)$. As in the above proof, there exists $c > 0$ such that $\|T_\rho^k(a)\|_2 \le c\|a\|_2$ for $a \in \mathcal{A}, k \in \mathbf{N}$. Put $R_n = \frac{1}{n}R(r_\rho + \frac{1}{n})$. Since for $y \in \mathcal{A}$

$$R(r_\rho + \frac{1}{n})y = \sum_{k=0}^{\infty}\frac{\lambda_\rho^k(y)}{(r_\rho + \frac{1}{n})^{k+1}}$$

one has

$$\|R(r_\rho + \frac{1}{n})y\|_2 \le \sum_{k=0}^{\infty}\|T_\rho^k(y)\|\frac{r_\rho^k}{(r_\rho + \frac{1}{n})^{k+1}} \le nc\|y\|_2$$

and hence $\|R_n\| \le c$ for $n \in \mathbf{N}$. The identites

$$(I - T_\rho)R_n = R_n(I - T_\rho) = \frac{1}{n}\frac{1}{r_\rho}(R_n - I)$$

hold so that we have

$$\|(I - T_\rho)R_n\| = \|R_n(I - T_\rho)\| \le \frac{1}{n}\frac{1}{r_\rho}(1+c).$$

A similar argument to the proof of Lemma 6.1 works so that for $u \in \mathcal{H}_\tau$ by taking a cluster point u_0 of the sequence $R_n u, n \in \mathbb{N}$ under the weak topology of $\mathcal{H}_\tau$ we have

$$\lim_{n\to\infty}\|R_n u - u_0\|_2 = 0.$$

Put $\widehat{P}_\rho u = u_0$. The inequality $\|R_n u - T_\rho R_n u\|_2 \le \frac{1}{n}\frac{1}{r_\rho}(1+c)\|u\|_2$ implies that $\widehat{P}_\rho = T_\rho\widehat{P}_\rho$ that is equal to $\widehat{P}_\rho T_\rho$. Hence $\widehat{P}_\rho = R_n\widehat{P}_\rho$ and $\widehat{P}_\rho = \widehat{P}_\rho^2$. The equality $\widehat{P}_\rho = T_\rho\widehat{P}_\rho$ implies $\widehat{P}_\rho = M_n\widehat{P}_\rho$ for all $n \in \mathbf{N}$ so that $\widehat{P}_\rho = P_\rho\widehat{P}_\rho$. Similarly the equalities $P_\rho = T_\rho P_\rho$ and $R_n = \sum_{k=0}^{\infty}T_\rho^k\frac{r_\rho^k}{(r_\rho + \frac{1}{n})^{k+1}}$ imply $P_\rho = R_n P_\rho$ for all $n \in \mathbf{N}$ so that $P_\rho = \widehat{P}_\rho P_\rho$. As $P_\rho\widehat{P}_\rho = \widehat{P}_\rho P_\rho$, one has $P_\rho = \widehat{P}_\rho$.

We denote by $\|a\|_1$ the L^1-norm $\tau(|a|)$ of $a \in \mathcal{A}$, and by $L^1(\mathcal{A},\tau)$ the completion of $\mathcal{A}$ by the norm $\|\cdot\|_1$. The positive operators $\lambda_\rho, T_\rho : \mathcal{A} \longrightarrow \mathcal{A}$ and the state $\tau : \mathcal{A} \longrightarrow \mathbf{C}$ extend to $L^1(\mathcal{A},\tau)$ in natural way, that are also denoted by λ_ρ, T_ρ and τ respectively.

Lemma 6.2. *Suppose that $(\mathcal{A},\rho,\Sigma)$ is uniquely ergodic and power-bounded. Then for $a \in \mathcal{A}$ the limit $\lim_{n\to\infty} M_n(a)$ converges in $L^1(\mathcal{A},\tau)$ under $\|\cdot\|_1$-topology. In particular $\lim_{n\to\infty} M_n(1) = x_\rho$ exists in $L^1(\mathcal{A},\tau)$ and satisfies the equalities*

$$\tau(x_\rho) = 1 \quad \textit{and} \quad \lim_{n\to\infty} M_n(a) = \tau(a)x_\rho \qquad \textit{for } a \in \mathcal{A}. \tag{6.4}$$

Proof. Since $(\mathcal{A},\rho,\Sigma)$ is irreducible and power-bounded, $\lim_{n\to\infty} M_n(a)$ for $a \in \mathcal{A}$ converges in $\mathcal{H}_\tau = L^2(\mathcal{A},\tau)$ under $\|\cdot\|_2$-norm by the previous lemma. By the inequality

$$\|M_n(a) - M_m(a)\|_1 \le \|M_n(a) - M_m(a)\|_2, \qquad a \in \mathcal{A}$$

the limit $\lim_{n\to\infty} M_n(a)$ exists in $L^1(\mathcal{A},\tau)$ under $\|\cdot\|_1$-norm. We denote it by $\Phi_1(a)$. Hence $x_\rho = \Phi_1(1)$. We will show that $\tau(f(\Phi_1(a) - \tau(a)x_\rho)) = 0$ for $f \in \mathcal{A}$. It suffices to show that $\tau(b\Phi_1(a)b^*) = \tau(a)\tau(bx_\rho b^*)$ for $b \in \mathcal{A}$. One may assume that $a \ge 0$. The inequality $a \le \|a\|1$ and hence $M_n(a) \le \|a\|M_n(1)$ implies $b\Phi_1(a)b^* \le \|a\|bx_\rho b^*$ so that we have $0 \le \tau(b\Phi_1(a)b^*) \le \|a\|\tau(bx_\rho b^*)$. Hence $\tau(bx_\rho b^*) = 0$ implies $\tau(b\Phi_1(a)b^*) = 0$. We may assume that $\tau(bx_\rho b^*) \ne 0$. Put $\omega(a) = \frac{\tau(b\Phi_1(a)b^*)}{\tau(bx_\rho b^*)}, a \in \mathcal{A}$. As $\Phi_1 \circ T_\rho(a) = \Phi_1(a)$, one sees that ω is an invariant state on $\mathcal{A}$. Hence we have $\omega = \tau$ by the unique ergodicity of $(\mathcal{A},\rho,\Sigma)$. Therefore we have $\tau(b\Phi_1(a)b^*) = \tau(a)\tau(bx_\rho b^*)$ for $b \in \mathcal{A}$.
The equality $\tau(x_\rho) = 1$ is clear. □

Lemma 6.3. *Keep the above assumptions and notations. The limit $\lim_{n\to\infty} M_n(f)$ for $f \in L^1(\mathcal{A},\tau)$ converges in $L^1(\mathcal{A},\tau)$ under $\|\cdot\|_1$-topology and satisfies the equality*

$$\lim_{n\to\infty} M_n(f) = \tau(f)x_\rho \qquad \textit{for } f \in L^1(\mathcal{A},\tau).$$

Proof. Since for $f \in L^1(\mathcal{A},\tau)$ the inequality $|\lambda_\rho(f)| \le \lambda_\rho(|f|)$ holds, one has $|T_\rho(f)| \le T_\rho(|f|)$ and hence $\|M_n(f)\|_1 \le \|f\|_1$. Take $a_k \in \mathcal{A}$ such as $\|f - a_k\|_1 \to 0$ as $k \to \infty$. It then follows that

$$\begin{aligned}
&\|M_n(f) - \tau(f)x_\rho\|_1 \\
\le&\|M_n(f) - M_n(a_k)\|_1 + \|M_n(a_k) - \tau(a_k)x_\rho\|_1 + \|\tau(a_k)x_\rho - \tau(f)x_\rho\|_1 \\
\le&\|f - a_k\|_1 + \|M_n(a_k) - \tau(a_k)x_\rho\|_1 + |\tau(a_k) - \tau(f)|\|x_\rho\|_1,
\end{aligned}$$

and hence $\lim_{n\to\infty} \|M_n(f) - \tau(f)x_\rho\|_1 = 0$ by the preceding lemma. □

Proposition 6.4. *Keep the above assumptions and notations. If $f \in L^1(\mathcal{A},\tau)$ satisfies $T_\rho(f) = f$ and $\tau(f) = 1$, Then $f = x_\rho$. Namely the space of the fixed elements in $L^1(\mathcal{A},\tau)$ under T_ρ is one-dimensional.*

Proof. By the preceding lemma, we have for $f \in L^1(\mathcal{A}, \tau)$ $\lim_{n\to\infty} M_n(f) = \tau(f)x_\rho$ in $\|\cdot\|_1$-topology. By the condition $T_\rho(f) = f$, we have $M_n(f) = f$ with $\tau(f) = 1$ and hence $f = x_\rho$. $\square$

Let us define the space $L^1(\mathcal{D}_\rho, \tau)$ in a similar way to $L^1(\mathcal{A}, \tau)$. The operators $\lambda_\rho, T_\rho : \mathcal{D}_\rho \longrightarrow \mathcal{D}_\rho$ and the state $\tau : \mathcal{D}_\rho \longrightarrow \mathbf{C}$ naturally act on $L^1(\mathcal{D}_\rho, \tau)$. The inclusion relation $\mathcal{A} \subset \mathcal{D}_\rho$ induces the inclusion relation $L^1(\mathcal{A}, \tau) \subset L^1(\mathcal{D}_\rho, \tau)$.

Lemma 6.5. *Keep the above assumptions and notations. Let x be an element of $L^1(\mathcal{D}_\rho, \tau)$ such that $T_\rho(x) = x$. Then x belongs to $L^1(\mathcal{A}, \tau)$.*

Proof. Take $x_n \in {\mathcal{D}_\rho}^{\mathrm{alg}}$ such that $\|x_n - x\|_1 \to 0$ as $n \to \infty$. As $|\lambda_\rho(y)| \le \lambda_\rho(|y|), y \in \mathcal{D}_\rho$, it then follows that

$$\|\lambda_\rho(x_n) - \lambda_\rho(x)\|_1 = \tau(|\lambda_\rho(x_n - x)|) \le \tau(\lambda_\rho(|x_n - x|) = r_\rho \|x_n - x\|_1$$

so that $\|T_\rho(x_n) - T_\rho(x)\|_1 \le \|x_n - x\|_1$. The element x is fixed by T_ρ so that

$$\|T_\rho^k(x_n) - x\|_1 \le \|x_n - x\|_1, \qquad n \in \mathbf{N}, \quad k \in \mathbf{N}.$$

Since $x_n \in {\mathcal{D}_\rho}^{\mathrm{alg}}$, there exists $k_n \in \mathbf{N}$ such that $T_\rho^{k_n}(x_n) \in \mathcal{A}$. Hence x belongs to $L^1(\mathcal{A}, \tau)$. $\square$

Definition. A state μ on $\mathcal{D}_\rho$ is called a *ϕ_ρ-invariant measure* if it satisfies

$$\mu(y) = \mu(\phi_\rho(y)), \qquad y \in \mathcal{D}_\rho.$$

If the probability measure for a state μ on $\mathcal{D}_\rho$ is absolutely continuous with respect to the probability measure for the state τ on $\mathcal{D}_\rho$, we write it as $\mu \ll \tau$.

Proposition 6.6. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and uniquely ergodic. For a fixed positive element $x \in L^1(\mathcal{A}, \tau)$ by T_ρ satisfying $\tau(x) = 1$, the state μ_x on $\mathcal{D}_\rho$ defined by*

$$\mu_x(y) = \tau(yx), \qquad y \in \mathcal{D}_\rho$$

is a ϕ_ρ-invariant measure on $\mathcal{D}_\rho$ such that $\mu \ll \tau$. Conversely, for any ϕ_ρ-invariant measure μ on $\mathcal{D}_\rho$ such that $\mu \ll \tau$, there exists a fixed positive element $x_\mu \in L^1(\mathcal{A}, \tau)$ by T_ρ satisfying $\tau(x_\mu) = 1$ such that

$$\mu(y) = \tau(yx_\mu), \qquad y \in \mathcal{D}_\rho.$$

Proof. Let $x \in L^1(\mathcal{A}, \tau)$ be a fixed positive element by T_ρ satisfying $\tau(x) = 1$. As $\lambda_\rho(x) = r_\rho x$, it follows that from Lemma 4.7

$$\mu_x(\phi_\rho(y)) = \frac{1}{r_\rho}\tau(\lambda_\rho(\phi_\rho(y)x)) = \frac{1}{r_\rho}\tau(y\lambda_\rho(x)) = \mu_x(y), \qquad y \in \mathcal{D}_\rho$$

so that the state μ_x is a ϕ_ρ-invariant measure on $\mathcal{D}_\rho$ such that $\mu_x \ll \tau$. Conversely for a ϕ_ρ-invariant measure μ on $\mathcal{D}_\rho$ such that $\mu \ll \tau$, there exists a Radon-Nikodym derivative $x_\mu \in L^1(\mathcal{D}_\rho, \tau)$ such that $x_\mu \ge 0, \tau(x_\mu) = 1$ and

$$\mu(y) = \tau(yx_\mu), \qquad y \in \mathcal{D}_\rho.$$

By the equality $\tau(\phi_\rho(y)x_\mu) = \tau(yT_\rho(x_\mu)), y \in \mathcal{D}_\rho$, one sees that $\tau(yx_\mu) = \tau(yT_\rho(x_\mu))$, $y \in \mathcal{D}_\rho$ so that $T_\rho(x_\mu) = x_\mu$, $\tau - a.e.$ Hence x_μ is regarded as an element of $L^1(\mathcal{A},\tau)$ by the preceding lemma. This completes the proof. □

Especially the measure μ_ρ defined by $\mu_\rho(y) = \tau(yx_\rho),\ y \in \mathcal{D}_\rho$ is a ϕ_ρ-invariant measure on $\mathcal{D}_\rho$ such that $\mu_\rho \ll \tau$.
Therefore we have

Theorem 6.7. *Assume that* $(\mathcal{A},\rho,\Sigma)$ *is irreducible, uniquely ergodic and power-bounded. Then a* ϕ_ρ*-invariant measure on* $\mathcal{D}_\rho$ *absolutely continuous with respect to* τ *is unique and is of the form*

$$\mu_\rho(y) = \tau(yx_\rho), \qquad y \in \mathcal{D}_\rho. \tag{6.5}$$

The measure μ_ρ *is faithful, and ergodic in the sense that the formula*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu_\rho(\phi_\rho^k(y)x) = \mu_\rho(y)\mu_\rho(x), \qquad x,y \in \mathcal{D}_\rho$$

holds.

Proof. Let μ be a ϕ_ρ-invariant measure on $\mathcal{D}_\rho$. By the preceding proposition there exists a fixed positive element $x_\mu \in L^1(\mathcal{A},\tau)$ under T_ρ satisfying $\tau(x_\mu) = 1$ such that

$$\mu(y) = \tau(yx_\mu), \qquad y \in \mathcal{D}_\rho.$$

By Proposition 6.4 we have $x_\mu = x_\rho$. For $x, y \in \mathcal{D}_\rho$, the equality

$$\lambda_\rho^k(\phi_\rho^k(y)xx_\rho) = y\lambda_\rho^k(xx_\rho)$$

holds by Lemma 4.7 so that

$$\begin{aligned}
\frac{1}{n}\sum_{k=0}^{n-1}\mu_\rho(\phi_\rho^k(y)x) &= \frac{1}{n}\sum_{k=0}^{n-1}\tau(\phi_\rho^k(y)xx_\rho)\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{r_\rho^k}\tau(\lambda_\rho^k(\phi_\rho^k(y)xx_\rho))\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{r_\rho^k}\tau(y\lambda_\rho^k(xx_\rho))\\
&= \tau(yM_n(xx_\rho)).
\end{aligned}$$

Since

$$\|\cdot\|_1 - \lim_{n\to\infty} M_n(xx_\rho) = \tau(xx_\rho)x_\rho = \mu_\rho(x)x_\rho,$$

we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mu_\rho(\phi_\rho^k(y)x) = \tau(y\mu_\rho(x)x_\rho)) = \mu_\rho(y)\mu_\rho(x).$$

□

Corollary 6.8. *Assume that* $(\mathcal{A},\rho,\Sigma)$ *is irreducible and mean ergodic.*

(i) *The unique ϕ_ρ-invariant probability measure absolutely continuous with respect to τ is obtained by $\mu_\rho(y) = \tau(yx_\rho), y \in \mathcal{D}_\rho$, where τ is the restriction of the unique KMS state on $\mathcal{O}_\rho$ and x_ρ is a positive element of $\mathcal{A}$ defined by the limit of the mean $\lim_{n\to\infty} \frac{1}{n}(1 + T_\rho(1) + \cdots + T_\rho^{n-1}(1))$.*

(ii) *The state μ_ρ is equivalent to the state τ as a measure on $\mathcal{D}_\rho$.*

Proof. (i) Under the assumption that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. Mean ergodicity implies unique ergodicity and (FP), which implies power-boundedness. Therefore the assertion is immediate.
(ii) By the mean ergodicity, the fixed element x_ρ belongs to $\mathcal{A}$ and is strictly positive by Lemma 3.5 (ii). Hence we have $\tau(y) = \mu_\rho(yx_\rho^{-1}), y \in \mathcal{D}_\rho$ so that $\tau \ll \mu_\rho$. □

7. Examples

We will present examples of continuous linear functionals satisfying KMS conditions on some C^*-symbolic dynamical systems.

1. Finite directed graphs

Let $A = [A(i,j)]_{i,j=1,\dots,N}$ be an $N \times N$ matrix with entries in nonnegative integers. Denote by $G_A = (V_A, E_A)$ the associated finite directed graph with vertex set $V = \{v_1, \dots, v_N\}$ and edge set E_A. Let $\mathcal{O}_{A^{[2]}}$ be the Cuntz-Krieger algebra such that the generating partial isometries $S_e, e \in E_A$ indexed by the edges in G_A satisfy

$$\sum_{f\in E_A} S_f S_f^* = 1, \qquad S_e^* S_e = \sum_{f\in E_A} A^{[2]}(e,f) S_f S_f^*, \qquad e \in E_A,$$

where $A^{[2]}(e,f)$ is defined to be one if the edge f follows the edge e, otherwise zero. Put $\mathcal{A}_{G_A}$ the C^*-subalgebra of $\mathcal{O}_{A^{[2]}}$ generated by the projections $S_e^* S_e, e \in E_A$. Denote by ρ_e^A for $e \in E_A$ the endomorphism $\mathcal{A}_{G_A}$ defined by $\rho_e^A(a) = S_e^* a S_e, a \in \mathcal{A}_{G_A}$. Consider the C^*-symbolic dynamical system $(\mathcal{A}_{G_A}, \rho^A, E_A)$. Its associated C^*-algebra $\mathcal{O}_{\rho^A}$ is nothing but the Cuntz-Krieger algebra $\mathcal{O}_{A^{[2]}}$. The finite directed graphs G_A is naturally considered to be a finite labeled graph by regarding an edge itself as its label. Hence this example will be contained in the following examples.

2. Finite labeled graphs

Let $\mathcal{G} = (G, \lambda)$ be a left-resolving finite labeled graph over Σ with underlying finite directed graph $G = (V, E)$ and labeling map $\lambda : E \to \Sigma$. Suppose that the graph G is irreducible. Let $\{v_1, \dots, v_N\}$ be the vertex set V. As in Section 2, we have a C^*-symbolic dynamical system $(\mathcal{A}_\mathcal{G}, \rho^\mathcal{G}, \Sigma)$ such that $\mathcal{A}_\mathcal{G} = \mathbf{C}E_1 \oplus \cdots \oplus \mathbf{C}E_N$ and $\rho_\alpha^\mathcal{G}(E_i) = \sum_{j=1}^N A^\mathcal{G}(i,\alpha,j)E_j$ for $i = 1, \dots, N, \alpha \in \Sigma$, where the $N \times N$-matrix $[A^\mathcal{G}(i,\alpha,j)]_{i,j=1,\dots,N}$ for $\alpha \in \Sigma$ is defined by (2.1). Put $A_\mathcal{G}(i,j) = \sum_{\alpha\in\Sigma} A^\mathcal{G}(i,\alpha,j)$ for $i,j = 1, \dots, N$. Then the matrix $A_\mathcal{G} = [A_\mathcal{G}(i,j)]_{i,j=1}^N$ is irreducible. Let $r_\mathcal{G}$ denote the Perron-Frobenius eigenvalue of the matrix $A_\mathcal{G}$. It is easy to see that $r_\mathcal{G}$ is equal to the spectral radius $r_{\rho^\mathcal{G}}$ of the positive operator $\lambda_{\rho^{\mathcal{G}}}(x) = \sum_{\alpha\in\Sigma} \rho^{\mathcal{G}}_\alpha(x), x \in \mathcal{A}_{\mathcal{G}}$. As

$$\lambda_{\rho^{\mathcal{G}}}(E_i) = \sum_{j=1}^N A_{\mathcal{G}}(i,j)E_j, \qquad i = 1,\dots,N,$$

by identifying $x = \sum_{i=1}^N x_iE_i \in \mathcal{A}_{\mathcal{G}}$ with the vector $[x_i]_{i=1}^N \in \mathbf{C}^N$, one may regard the operator $\lambda_{\rho^{\mathcal{G}}}$ as the transposed matrix $A^t_{\mathcal{G}}$ of $A_{\mathcal{G}}$. For a complex number $\beta \in \mathbf{C}$ with $|\beta| > 1$, let $\varphi \in \mathcal{A}^*_{\mathcal{G}}$ be a continuous linear functional belonging to $\mathcal{E}_\beta(\rho^{\mathcal{G}})$. The equality $\varphi\circ\lambda_{\rho^{\mathcal{G}}}(E_i) = \beta\varphi(E_i)$ implies

$$\sum_{j=1}^N A_{\mathcal{G}}(i,j)\varphi(E_j) = \beta\varphi(E_i), \qquad i = 1,\dots,N$$

so that the vector $[\varphi(E_j)]_{j=1}^N$ is an eigenvector of $A_{\mathcal{G}}$ for eigenvalue β. Conversely an eigenvector $[u_i]_{i=1}^N \in \mathbf{C}$ of the matrix $A_{\mathcal{G}}$ for an eigenvalue β gives rise to a continuous linear functional φ on $\mathcal{A}_{\mathcal{G}}$ by setting $\varphi(E_i) = u_i, i = 1,\dots,N$ so that $\varphi \in \mathcal{E}_\beta(\rho^{\mathcal{G}})$. Hence the space $\mathcal{E}_\beta(\rho^{\mathcal{G}})$ is identified with the eigenvector space of the matrix $A_{\mathcal{G}}$ for eigenvalue β. Especially a faithful invariant state τ on $\mathcal{A}_{\mathcal{G}}$ is the positive normalized eigenvector of $A_{\mathcal{G}}$ for eigenvalue $r_{\mathcal{G}}$. Similarly an element $x = \sum_{j=1}^N x_jE_j \in \mathcal{A}_{\mathcal{G}}$ is fixed by $T_{\rho^{\mathcal{G}}}$ if and only if the vector $[x_j]_{j=1}^N$ is an eigenvector of $A^t_{\mathcal{G}}$ for the eigenvalue $r_{\mathcal{G}}$. The ordinary Perron-Frobenius theorem for nonnegative matrices asserts that $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$ is mean ergodic if $A_{\mathcal{G}}$ is irreducible. The following proposition comes from the ordinary Perron-Frobenius theorem for irreducible nonnegative matrices, which is a special case of Theorem 3.13, and Corollary 6.8.

Proposition 7.1. *Suppose that the adjacency matrix $A_{\mathcal{G}} = [A_{\mathcal{G}}(i,j)]_{i,j=1}^N$ is irreducible. Let $[\tau_i]_{i=1}^N$ and $[x_i]_{i=1}^N$ be right and left Perron eigenvector of $A_{\mathcal{G}}$ respectively, that is,*

$$A_{\mathcal{G}}[\tau_i]_{i=1}^N = r_{\mathcal{G}}[\tau_i]_{i=1}^N, \qquad A^t_{\mathcal{G}}[x_i]_{i=1}^N = r_{\mathcal{G}}[x_i]_{i=1}^N,$$

such that $\sum_{i=1}^N \tau_i = 1$ and $\sum_{i=1}^N \tau_ix_i = 1$. Put $x_{\rho^{\mathcal{G}}} = \sum_{i=1}^N x_iE_i \in \mathcal{A}_{\mathcal{G}}$ and $\tau(a) = \sum_{i=1}^N \tau_ia_i$ for $a = \sum_{i=1}^N a_iE_i \in \mathcal{A}_{\mathcal{G}}$. Then τ is a unique faithful invariant state on $\mathcal{A}_{\mathcal{G}}$ such that the following equalities hold:

$$\lim_{n\to\infty} M_n(a) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} T^k_{\rho^{\mathcal{G}}}(a) = \tau(a)x_{\rho^{\mathcal{G}}}.$$

Furthermore the measure $\mu_{\rho^{\mathcal{G}}}$ on $\mathcal{D}_{\rho^{\mathcal{G}}}$ defined $\mu_{\rho^{\mathcal{G}}}(y) = \tau(yx_{\rho^{\mathcal{G}}})$ for $y \in \mathcal{D}_{\rho^{\mathcal{G}}}$ is a unique $\phi_{\rho^{\mathcal{G}}}$-invariant measure equivalent to the measure τ on $\mathcal{D}_{\rho^{\mathcal{G}}}$.

Remark. Let $X_{\mathcal{G}}$ be the right one-sided sofic shift presented by $\mathcal{G}$. The commutative C^*-algebra $C(X_{\mathcal{G}})$ on $X_{\mathcal{G}}$ is naturally regarded as a C^*-subalgebra of $\mathcal{D}_{\rho^{\mathcal{G}}}$ through the correspondence

$$\chi_\nu \in C(X_{\mathcal{G}}) \longrightarrow S_\nu S^*_\nu \in \mathcal{D}_{\rho^{\mathcal{G}}}, \qquad \nu \in B_k(\Lambda_{\mathcal{G}})$$

where χ_ν is the characteristic function for the cylinder

$$U_\nu = \{(x_i)_{i\in\mathbf{N}} \in X_\mathcal{G} \mid x_1 = \nu_1, \dots, \nu_k = x_k\}.$$

The restriction of the $\phi_{\rho^\mathcal{G}}$-invariant measure $\mu_{\rho^\mathcal{G}}$ on $\mathcal{D}_{\rho^\mathcal{G}}$ to the subalgebra $C(X_\mathcal{G})$ is nothing but a shift-invariant measure on $X_\mathcal{G}$ (cf. [21]).
We will next find continuous linear functionals on $\mathcal{O}_{\rho^\mathcal{G}}$ satisfying KMS conditions in concrete way. Now suppose that the irreducible matrix $A_\mathcal{G}$ has its period $p_\mathcal{G}$ and put

$$N_\mathcal{G}(i,j) = \{n \in \mathbf{Z}_+ \mid A^n_\mathcal{G}(i,j) > 0\}.$$

It is well-known that for $n, m \in N_\mathcal{G}(i,j)$ one has $n \equiv m \pmod{p_\mathcal{G}}$. Then for an eigenvalue $\beta \in \mathbf{C}$ of $A_\mathcal{G}$ with $|\beta| = r_\mathcal{G}$, $\frac{\beta}{r_\mathcal{G}}$ is a $p_\mathcal{G}$-th root of unity. We fix a vertex v_1 and for $k \in \{1, 2, \dots, N\}$ take $n_k \in N_\mathcal{G}(1,k)$. We set

$$u_k = (\frac{\beta}{r_\mathcal{G}})^{n_k} \tau(E_k).$$

Then u_k does not depend on the choice of n_k as long as $n_k \in N_\mathcal{G}(1,k)$.

LEMMA 7.2. $\sum_{j=1}^N A_\mathcal{G}(i,j) u_j = \beta u_i, \quad i = 1, \dots, N.$

Proof. If $A_\mathcal{G}(i,j) \neq 0$, one sees $n_i + 1 \in N(1,j)$ so that

$$A_\mathcal{G}(i,j) u_j = \frac{\beta}{r_\mathcal{G}} (\frac{\beta}{r_\mathcal{G}})^{n_i} A_\mathcal{G}(i,j) \tau(E_j) = \frac{\beta}{r_\mathcal{G}} \frac{u_i}{\tau(E_i)} A_\mathcal{G}(i,j) \tau(E_j).$$

It follows that

$$\sum_{j=1}^N A_\mathcal{G}(i,j) u_j = \frac{\beta}{r_\mathcal{G}} \frac{u_i}{\tau(E_i)} \sum_{j=1}^N A_\mathcal{G}(i,j) \tau(E_j) = \frac{\beta}{r_\mathcal{G}} \frac{u_i}{\tau(E_i)} r_\mathcal{G} \tau(E_i) = \beta u_i.$$

□

Hence $u = [u_k]_{k=1}^N$ yields a nonzero eigenvector of $A_\mathcal{G}$. Define a nonzero continuous linear functional φ on $\mathcal{A}_\mathcal{G}$ by setting

$$\varphi(E_k) = u_k, \qquad k = 1, \dots, N$$

so that the equality $\varphi \circ \lambda_\mathcal{G} = \beta \varphi$ on $\mathcal{A}_\mathcal{G}$ holds. Put $v_\varphi = \sum_{i=1}^N \frac{u_i}{\tau(E_i)} E_i \in \mathcal{A}_\mathcal{G}$. It is easy to see that v_φ is a partial isometry such that $\varphi(E_j) = \tau(E_j v_\varphi), j = 1, \dots, N$ so that

$$\varphi(x) = \tau(x v_\varphi), \qquad x \in \mathcal{A}_\mathcal{G}$$

holds. Therefore we have the following proposition.

PROPOSITION 7.3. *Let $\mathcal{G} = (G, \lambda)$ be a left-resolving finite labeled graph with underlying finite directed graph $G = (V, E)$ and labeling map $\lambda : E \to \Sigma$. Denote by $\{v_1, \dots, v_N\}$ the vertex set V. Assume that G is irreducible. Consider the N-dimensional commutative C^*-algebra $\mathcal{A}_\mathcal{G} = \mathbf{C}E_1 \oplus \dots \oplus \mathbf{C}E_N$ where each minimal projection E_i corresponds to the vertex v_i for $i = 1, \dots, N$. Define an*

$N\times N$- nonnegative matrix $A_{\mathcal{G}} = [A_{\mathcal{G}}(i,j)]_{i,j=1}^N$ by $A_{\mathcal{G}}(i,j) = \sum_{\alpha\in\Sigma} A^{\mathcal{G}}(i,\alpha,j)$ where for $\alpha \in \Sigma$ and $i,j = 1,\ldots,N$

$$A^{\mathcal{G}}(i,\alpha,j) = \begin{cases} 1 & \textit{if there exists an edge } e \textit{ from } v_i \textit{ to } v_j \textit{ with } \lambda(e) = \alpha, \\ 0 & \textit{otherwise.} \end{cases}$$

Let $\mathcal{O}_{A_{\mathcal{G}}}$ be the associated Cuntz-Krieger algebra and τ be the unique KMS state on $\mathcal{O}_{A_{\mathcal{G}}}$ for gauge action. Let $\beta \in \mathbf{C}$ be an eigenvalue of $A_{\mathcal{G}}$ such that $|\beta| = r_{\mathcal{G}}$ the Perron-Frobenius eigenvalue of the matrix $A_{\mathcal{G}}$. Then a continuous linear functional on $\mathcal{O}_{A_{\mathcal{G}}}$ satisfying KMS condition at $\mathrm{Log}\beta$ is a scalar multiple of $\varphi \in \mathcal{O}_{A_{\mathcal{G}}}^*$ giving by for $k = 1,\ldots,N$

$$\varphi(E_k) = (\frac{\beta}{r_{\mathcal{G}}})^{n_k}\tau(E_k) \qquad \textit{where } n_k \textit{ satisfies } A_{\mathcal{G}}^{n_k}(1,k) \neq 0.$$

Consider a finite labeled graph $\mathcal{G}$ whose adjacency matrix A is

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

As

$$A^3 = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

the period of the matrix is 3. The characteristic polynomial of A is $\det(t-A) = t^2(t^3-4)$ so that $Sp(A) = \{\sqrt[3]{4}, \sqrt[3]{4}e^{\frac{2\pi}{3}i}, \sqrt[3]{4}e^{\frac{4\pi}{3}i}, 0\}$ and $r_A = \sqrt[3]{4}$. Hence $\beta \in Sp(A)$ satisfying $|\beta| = \sqrt[3]{4}$ are

$$\sqrt[3]{4}, \quad \sqrt[3]{4}e^{\frac{2\pi}{3}i}, \quad \sqrt[3]{4}e^{\frac{4\pi}{3}i}.$$

Therefore the Cuntz-Krieger algebra $\mathcal{O}_A$ has three continuous linear functionals satisfying KMS conditions for gauge action at inverse temperatures

$$\frac{1}{3}\log 4, \quad \frac{1}{3}\log 4 + \frac{2\pi}{3}i, \quad \frac{1}{3}\log 4 + \frac{4\pi}{3}i$$

respectively.

3. Dyck shifts

We consider the Dyck shift D_N for a fixed natural number $N > 1$ with alphabet $\Sigma = \Sigma^- \cup \Sigma^+$ where $\Sigma^- = \{\alpha_1,\ldots,\alpha_N\}, \Sigma^+ = \{\beta_1,\ldots,\beta_N\}$. The symbols α_i, β_i correspond to the brackets $(_i,)_i$ respectively. The Dyck inverse monoid has the relations

$$\alpha_i\beta_j = \begin{cases} \mathbf{1} & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \tag{7.1}$$

for $i, j = 1, \dots, N$ (cf. [23],[26]). A word $\omega_1 \cdots \omega_n$ of Σ is admissible for D_N precisely if $\prod_{m=1}^n \omega_m \neq 0$. For a word $\omega = \omega_1 \cdots \omega_n$ of Σ, we denote by $\tilde{\omega}$ its reduced form. Namely $\tilde{\omega}$ is a word of $\Sigma \cup \{0, \mathbf{1}\}$ obtained after the operations (7.1). Hence a word ω of Σ is forbidden for D_N if and only if $\tilde{\omega} = 0$.
In [26], an irreducible λ-graph system presenting D_N called the Cantor horizon λ-graph system has been introduced. It is a minimal irreducible component of the canonical λ-graph system $\mathfrak{L}^{C(D_N)}$ and written as $\mathfrak{L}^{Ch(D_N)}$. Let us describe the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_N)}$ of D_N. Let Σ_N be the full N-shift $\{1, \dots, N\}^{\mathbf{Z}}$. We denote by $B_l(D_N)$ and $B_l(\Sigma_N)$ the set of admissible words of length l of D_N and that of Σ_N respectively. The vertices V_l of $\mathfrak{L}^{Ch(D_N)}$ at level l are given by the words of length l consisting of the symbols of Σ^+. That is,

$$V_l = \{(\beta_{\mu_1} \cdots \beta_{\mu_l}) \in B_l(D_N) \mid \mu_1 \cdots \mu_l \in B_l(\Sigma_N)\}.$$

Hence the cardinal number of V_l is N^l. The mapping $\iota (= \iota_{l,l+1}) : V_{l+1} \to V_l$ deletes the rightmost symbol of a word in $B_l(\Sigma_N)$ such as

$$\iota((\beta_{\mu_1} \cdots \beta_{\mu_{l+1}})) = (\beta_{\mu_1} \cdots \beta_{\mu_l}), \qquad (\beta_{\mu_1} \cdots \beta_{\mu_{l+1}}) \in V_{l+1}.$$

There exists an edge labeled α_j from $(\beta_{\mu_1} \cdots \beta_{\mu_l}) \in V_l$ to $(\beta_{\mu_0}\beta_{\mu_1} \cdots \beta_{\mu_l}) \in V_{l+1}$ precisely if $\mu_0 = j$, and there exists an edge labeled β_j from $(\beta_j \beta_{\mu_1} \cdots \beta_{\mu_{l-1}}) \in V_l$ to $(\beta_{\mu_1} \cdots \beta_{\mu_{l+1}}) \in V_{l+1}$. The resulting labeled Bratteli diagram with ι-map becomes a λ-graph system over Σ, denoted by $\mathfrak{L}^{Ch(D_N)}$, that presents the Dyck shift D_N ([26]). It gives rise to a purely infinite simple C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ ([32]) such that

$$K_0(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong \mathbf{Z}/N\mathbf{Z} \oplus C(\mathfrak{K}, \mathbf{Z}), \qquad K_1(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong 0.$$

Let us denote by $(\mathcal{A}^{D_N}, \rho^{D_N}, \Sigma)$ the C^*-symbolic dynamical system associated to the λ-graph system $\mathfrak{L}^{Ch(D_N)}$ as in Section 2. Since the vertex set V_l is indexed by the set $B_l(\Sigma_N)$ of words, the family of projections denoted by $E_{\mu_1 \dots \mu_l}$ for $\mu_1 \cdots \mu_l \in B_l(\Sigma_N)$ in the C^*-algebra $\mathcal{A}^{D_N}$ forms the minimal projectins of $\mathcal{A}_l = C(V_l)$ such that

$$\sum_{\mu_1 \cdots \mu_l \in B_l(\Sigma_N)} E_{\mu_1 \dots \mu_l} = 1, \qquad E_{\mu_1 \dots \mu_l} = \sum_{\mu_{l+1}=1}^{N} E_{\mu_1 \dots \mu_{l+1}}.$$

As the algebra $\mathcal{A}_l$ is embedded into $\mathcal{A}_{l+1}$, the C^*-algebra $\mathcal{A}^{D_N}$ is a commutative AF-algebra generated by the subalgebras $\mathcal{A}_l, l \in \mathbf{N}$. The endomorphisms $\rho_\gamma^{D_N}$: $\mathcal{A}^{D_N} \longrightarrow \mathcal{A}^{D_N}$ for $\gamma \in \Sigma$ are defined by

$$\rho_{\alpha_j}^{D_N}(E_{\mu_1 \dots \mu_l}) = E_{j\mu_1 \dots \mu_l}, \qquad \rho_{\beta_j}^{D_N}(E_{j\mu_1 \dots \mu_{l-1}}) = \sum_{\mu_l, \mu_{l+1}=1}^{N} E_{\mu_1 \dots \mu_{l+1}}$$

for $\mu_1 \dots \mu_l \in B_l(\Sigma_N)$ and $j = 1, \dots, N$. It then follows that

$$\begin{aligned}
\lambda_{\rho^{D_N}}(1) &= \sum_{j=1}^{N} \rho_{\alpha_j}^{D_N}(1) + \sum_{j=1}^{N} \rho_{\beta_j}^{D_N}(1) \\
&= \sum_{j=1}^{N} \sum_{\mu_1 \cdots \mu_l \in B_l(\Sigma_N)} E_{j\mu_1 \dots \mu_l} + \sum_{j=1}^{N} \sum_{\mu_2 \cdots \mu_l \in B_{l-1}(\Sigma_N)} \sum_{\mu_{l+1}, \mu_{l+2}=1}^{N} E_{\mu_2 \dots \mu_{l+2}} \\
&= 1 + N
\end{aligned}$$

so that we have $\|\lambda_{\rho^{D_N}}\| = \|\lambda_{\rho^{D_N}}(1)\| = 1 + N$. Hence we obtain

$$r_{\rho^{D_N}} = 1 + N, \qquad T_{\rho^{D_N}}(1) = 1.$$

This implies that 1 is a fixed element by $T_{\rho^{D_N}}$ and hence $(\mathcal{A}^{D_N}, \rho^{D_N}, \Sigma)$ satisfies (FP). As in [32], $(\mathcal{A}^{D_N}, \rho^{D_N}, \Sigma)$ is irreducible and uniquely ergodic, so that it is mean ergodic. One then sees that there exists a KMS state at inverse temperature $\log \beta$ if and only if $\beta = 1 + N$. The admitted KMS state is unique ([32, Theorem 1.2]).

4. β-shifts

Let $\beta > 1$ be an arbitrary real number. Take a natural number N with $N - 1 < \beta \le N$. Put $\Sigma = \{0, 1, ..., N-1\}$. For a nonnegative real number t, we denote by $[t]$ the integer part of t. Let $f_\beta : [0,1] \to [0,1]$ be the mapping defined by

$$f_\beta(x) = \beta x - [\beta x], \qquad x \in [0,1]$$

that is called the β-transformation ([38], [42]). The β-expansion of $x \in [0,1]$ is a sequence $\{d_i(x,\beta)\}_{i \in \mathbf{N}}$ of integers of Σ determined by

$$d_i(x,\beta) = [\beta f_\beta^{i-1}(x)], \qquad i \in \mathbf{N}.$$

By this sequence, we can write x as

$$x = \sum_{i=1}^{\infty} \frac{d_i(x,\beta)}{\beta^i}.$$

We endow the infinite product $\Sigma^{\mathbf{N}}$ with the product topology and the lexicographical order. Put $\zeta_\beta = \sup_{x \in [0,1)} (d_i(x,\beta))_{i \in \Sigma^{\mathbf{N}}}$. We define the shift-invariant compact subset X_β of $\Sigma^{\mathbf{N}}$ by

$$X_\beta = \{\omega \in \Sigma^{\mathbf{N}} | \sigma^i(\omega) \le \zeta_\beta, i = 0, 1, 2, ...\},$$

where σ denotes the shift $\sigma((\omega_i)_{i \in \mathbf{N}}) = (\omega_{i+1})_{i \in \mathbf{N}}$. The one-sided subshift (X_β, σ) is called the right one-sided β-shift (cf. [38], [42]). Its (two-sided) subshift

$$\Lambda_\beta = \{(\omega_i)_{i \in \mathbf{Z}} \in \Sigma^{\mathbf{Z}} \mid (\omega_{i-k})_{i \in \mathbf{N}} \in X_\beta, k = 0, 1, 2, \dots\}$$

is called the β-shift. In [17], the C^*-algbera $\mathcal{O}_\beta$ associated with the β-shift has been introduced and studied. It is simple and purely infinite for every $\beta > 1$ and generated by $N - 1$ isometries $S_0, S_1, \dots, S_{N-2}$ and one partial isometry S_{N-1} having certain operator relations (see [17]). The family $\mathcal{O}_\beta,\ 1 < \beta \in \mathbf{R}$

interpolates the Cuntz algebras $\mathcal{O}_n$, $1 < n \in \mathbf{N}$. Denote by $\mathcal{A}_\beta$ the C^*-subalgebra of $\mathcal{O}_\beta$ genertaed by the family of the projections $S_\mu^* S_\mu, \mu \in B_*(\Lambda_\beta)$. The algebra is commutative and is of infinite dimensional unless Λ_β is sofic, where Λ_β is sofic if and only if the sequence $(d_i(1,\beta))_{i\in\mathbf{N}}$ is ultimately periodic. Define a family $\{\rho_j^\beta\}_{j=0,1,\dots,N-1}$ of endomorphisms on $\mathcal{A}_\beta$ by

$$\rho_j^\beta(x) = S_j^* x S_j, \qquad x \in \mathcal{A}_\beta,\ j = 0,1,\dots,N-1$$

so that we have a C^*-symbolic dynamical system $(\mathcal{A}_\beta, \rho^\beta, \Sigma)$. It is direct to see that the C^*-algebra $\mathcal{O}_{\rho^\beta}$ is canonically isomorphic to the C^*-algebra $\mathcal{O}_\beta$. We set the positive operator λ_β on $\mathcal{A}_\beta$ by

$$\lambda_\beta(x) = \sum_{j=0}^{N-1} \rho_j^\beta(x), \qquad x \in \mathcal{A}_\beta.$$

LEMMA 7.4. *The spectral radius r_β of the positive operator λ_β on $\mathcal{A}_\beta$ is β.*

Proof. Denote by θ_k the cardinal number of the admissible words $B_k(\Lambda_\beta)$ of length k. Then we have

$$\|\lambda_\beta^k\| = \|\lambda_\beta^k(1)\| \le \sum_{\mu\in B_k(\Lambda_\beta)} \|S_\mu^* S_\mu\| = \theta_k.$$

As in [44, p. 179], $\lim_{k\to\infty}\frac{\theta_k}{\beta^k}$ converges to a positive real number so that there exists a positive constant $M > 0$ such that $\frac{\|\lambda_\beta^k\|}{\beta^k} < M$ for all $k \in \mathbf{N}$. Hence $\lim_{k\to\infty}\|\lambda_\beta^k\|^{\frac{1}{k}} \le \beta$ so that $r_\beta \le \beta$. As in [17], there exists a state τ on $\mathcal{A}_\beta$ satisfying $\tau\circ\lambda_\beta = \beta\tau$. This implies $\beta \in Sp(\lambda_\beta)$ so that $r_\beta = \beta$. □

PROPOSITION 7.5. *$(\mathcal{A}_\beta, \rho^\beta, \Sigma)$ is irreducible, uniquely ergodic and power-bounded.*

Proof. It has been proved in [17] that there is no nontrivial ideal of $\mathcal{A}_\beta$ invariant under λ_β and there exists a unique state τ on $\mathcal{A}_\beta$ satisfying $\tau\circ\lambda_\beta = r_\beta\tau$. Hence $(\mathcal{A}_\beta, \rho^\beta, \Sigma)$ is irreducible, uniquely ergodic. As in the proof of the above lemma, there exists a positive constant $M > 0$ such that $\frac{\|\lambda_\beta^k\|}{r_\beta^k} < M$ for all $k \in \mathbf{N}$. This means that $(\mathcal{A}_\beta, \rho^\beta, \Sigma)$ is power-bounded. □

By the above proposition, one knows that $(\mathcal{A}_\beta, \rho^\beta, \Sigma)$ satisfies the hypothesis of Theorem 6.7 so that there uniquely exists a ϕ_{ρ^β}-invariant measure on $\mathcal{D}_{\rho^\beta}$ absolutely continuous with respect to the restriction of the unique KMS-state τ to $\mathcal{D}_{\rho^\beta}$. We note that $C(X_\beta)$ is a C^*-subalgebra of $\mathcal{D}_{\rho^\beta}$ and the restriction of ϕ_{ρ^β} to $C(X_\beta)$ comes from the shift transformation σ. As in [17], the restriction of the KMS-state τ to $\mathcal{D}_{\rho^\beta}$ corresponds to the Lebesgue measure on $[0,1]$ in translating the β-shift to the β-transformation. Hence the uniqueness of the ϕ_{ρ^β}-invariant measure on $\mathcal{D}_{\rho^\beta}$ absolutely continuous with respect to τ exactly corresponds to the uniqueness of the invariant measure on $[0,1]$ under the β-transformation absolutely continuous with respect to the Lebesgue measure studied in [14], [38] and [42]. In fact, the density function h_β appeared in [14], [38] and [42] of the invariant measure for the β-transformation with respect to the Lebesgue measure is the element x_{ρ^β} realized as the mean $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{\lambda_\beta^k(1)}{\beta^k}$ in Theorem 6.7.

Acknowledgments: The author would like to thank Yasuo Watatani who pointed out inaccuracy in the proof of Lemma 3.4 in the first draft of the paper. The author is also deeply indebted to the referee whose many useful suggestions, guidences and comments improved the paper. This work was supported by JSPS Grant-in-Aid for Scientific Reserch ((C), No 20540215).

References

[1] T. Bates and D. Pask, *The C^*-algebras of labelled graphs*, J. Operator Theory 57(2007), pp. 207–226.

[2] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Math. 470, Springer-Verlag, New York, Berlin, Heidelberg (1975).

[3] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics II*, Springer-Verlag, New York, Berlin, Heidelberg (1981).

[4] J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. 57(1977), pp. 173–185.

[5] J. Cuntz, *A class of C^*-algebras and topological Markov chains II: reducible chains and the Ext-functor for C^*-algebras*, Invent. Math. 63(1980), pp. 25–40.

[6] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. 56(1980), pp. 251–268.

[7] J. Cuntz and G. K. Pedersen, *Equivalence and KMS states on periodic C^*-dynamical systems*, J. Funct. Anal. 34(1979), pp. 79–86.

[8] N. Dunford and J. T. Schwartz, *Linear operators, Part I : General Theory*, Interscience Publishers, New York (1966).

[9] M. Enomoto, M. Fujii and Y. Watatani, *KMS states for gauge action on $\mathcal{O}_A$*, Math. Japon. 29(1984), pp. 607–619.

[10] R. Exel, *KMS states for generalized gauge actions on Cuntz–Krieger algebras, An application of the Ruelle–Perron–Frobenius Theorem*, Bull. Braz. Math. Soc., New Series 35(2004), pp. 1–12.

[11] R. Exel and M. Laca, *Partial dynamical systems and the KMS condition*, Commun. Math. Phys. 232(2002), pp. 223–277.

[12] R. Fischer, *Sofic systems and graphs*, Monats. für Math. 80(1975), pp. 179–186.

[13] T. Furuta, *Invitation to Linear Operators, from matrices to bounded linear opeartors on a Hilbert space*, Taylor & Francis, London (2001).

[14] F. Hofbauer, *β-shifts have unique maximal measure*, Monatsh. Math. 85(1978), pp. 189–198.

[15] M. Izumi, T. Kajiwara and Y. Watatani, *KMS states and branched points*, Ergodic Theory Dynam. Systems 27(2007), pp. 1887–1918.

[16] T. Kajiwara, C. Pinzari and Y. Watatani, *Ideal structure and simplicity of the C^*-algebras generated by Hilbert modules*, J. Funct. Anal. 159(1998), pp. 295–322.

[17] Y. Katayama, K. Matsumoto and Y. Watatani, *Simple C^*-algebras arising from β-expansion of real numbers*, Ergodic Theory Dynam. Systems 18(1998), pp. 937–962.

[18] D. Kerr and C. Pinzari, *Noncommutative pressure and the variational principle in Cuntz-Krieger-type C^*-algebras*, J. Funct. Anal. 188(2002), pp. 156–215.

[19] M. Kessebohmer, M. Stadlbauer and B. O. Stratmann, *Lyapunov spectra for KMS states on Cuntz-Krieger algebras*, Math. Z. 256(2007), pp. 871–893.

[20] B. P. Kitchens, *Symbolic dynamics*, Springer-Verlag, New York, Berlin, Heidelberg (1998).

[21] B. P. Kitchens and S. Tuncel, *Finitary measures for subshifts of finite type and sofic systems*, Memoirs of Amer. Math. Soc. 58(1985), No 338.

[22] U. Krengel, *Ergodic Theorems*, de Gruyter Studies in Mathematics 6, Walter de Gruyter, Berlin, New York (1985).

[23] W. Krieger, *On the uniqueness of the equilibrium state*, Math. Systems Theory 8(1974), pp. 97–104.

[24] W. Krieger, *On sofic systems I*, Israel J. Math. 48(1984), pp. 305–330.

[25] W. Krieger, *On sofic systems II*, Israel J. Math. 60(1987), pp. 167–176.

[26] W. Krieger and K. Matsumoto, *A lambda-graph system for the Dyck shift and its K-groups*, Documenta Math. 8(2003), pp. 79–96.

[27] M. Laca and S. Neshveyev, *KMS-states of quasi-free dynamics on Pimsner algebras*, J. Funct. Anal. 211(2004), pp. 457–482.

[28] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge (1995).

[29] K. Matsumoto, *Presentations of subshifts and their topological conjugacy invariants*, Documenta Math. 4(1999), pp. 285–340.

[30] K. Matsumoto, *C^*-algebras associated with presentations of subshifts*, Documenta Math. 7(2002), pp. 1–30.

[31] K. Matsumoto, *Actions of symbolic dynamical systems on C^*-algebras*, J. Reine Angew. Math. 605(2007), pp. 23–49.

[32] K. Matsumoto, *On the simple C^*-algebras arising from Dyck systems*, J. Operator Theory 58(2007), pp. 205–226.

[33] K. Matsumoto, *Orbit equivalence in C^*-algebras defined by actions of symbolic dynamical systems*, Contemporary Math. 503(2009), pp. 121–140.

[34] K. Matsumoto, *Actions of symbolic dynamical systems on C^*-algebras II, Simplicity and some examples*, Math. Z. 265(2010), pp. 735-760.

[35] K. Matsumoto, Y. Watatani and M. Yoshida, *KMS-states for gauge actions on C^*-algebras associated with subshifts*, Math. Z. 228(1998), pp. 489–509.

[36] R. Okayasu, *Type III factors arising from Cuntz-Krieger algebras*, Proc. Amer. Math. Soc. 131(2003), pp. 2145–2153.

[37] D. Olesen and G. K. Pedersen, *Some C^*-dynamical system with a single KMS state*, Math. Scand. 42(1978), pp. 111–118.

[38] W. Parry, *On the β-expansion of real numbers*, Acta Math. Acad. Sci. Hungar. 11(1960), pp. 401–416.

[39] G. K. Pedersen, *C^*-algebras and their automorphism groups*, Acadenic Press, London, New York, San Francisco (1979).

[40] M. V. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by* **Z**, in Free Probability Theory, Fields Institute Communications 12(1996), pp. 189–212.

[41] C. Pinzari, Y. Watatani and K. Yonetani, *KMS-states, entropy and the variational principle in full C^*-dynamical systems*, Comm. Math. Phys. 213(2000), pp. 231–379.

[42] A. Renyi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. 8(1957), pp. 477–493.

[43] M. Takesaki, *Theory of Operator Alegebras I*, Springer-Verlag, New York, Berlin, Heidelberg (1979).

[44] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics 79, Springer-Verlag, New York, Berlin, Heidelberg (1975).

[45] B. Weiss, *Subshifts of finite type and sofic systems*, Monats. Math. 77, (1973), pp. 462–474.

Kengo Matsumoto
Department of Mathematics,
Joetsu University of Education,
Joetsu 943-8512, Japan
kengo@juen.ac.jp

An Infinite Level Atom coupled to a Heat Bath

Martin Koenenberg[1]

Received: January 13, 2011

Communicated by Heinz Siedentop

Abstract. We consider a W^*-dynamical system $(\mathfrak{M}_\beta, \tau)$, which models finitely many particles coupled to an infinitely extended heat bath. The energy of the particles can be described by an unbounded operator, which has infinitely many energy levels. We show existence of the dynamics τ and existence of a (β, τ) -KMS state under very explicit conditions on the strength of the interaction and on the inverse temperature β.

2010 Mathematics Subject Classification: Primary 81V10; Secondary 37N20, 47N50.
Keywords and Phrases: KMS-state, thermal equilibrium, W^*-dynamical system, open quantum system.

1 Introduction

In this paper, we study a W^*-dynamical system $(\mathfrak{M}_\beta, \tau)$ which describes a system of finitely many particles interacting with an infinitely extended bosonic reservoir or heat bath at inverse temperature β. Here, $\mathfrak{M}_\beta$ denotes the W^*-algebra of observables and τ is an automorphism-group on $\mathfrak{M}_\beta$, which is defined by

$$\tau_t(X) := e^{it\mathcal{L}_Q}\, X\, e^{-it\mathcal{L}_Q},\ X \in \mathfrak{M}_\beta,\ t \in \mathbb{R}. \tag{1}$$

In this context, t is the time parameter. $\mathcal{L}_Q$ is the Liouvillean of the dynamical system at inverse temperature β, Q describes the interaction between particles and heat bath. On the one hand the choice of $\mathcal{L}_Q$ is motivated by heuristic arguments, which allow to derive the Liouvillean $\mathcal{L}_Q$ from the Hamiltonian H of the joint system of particles and bosons at temperature zero. On the other hand we ensure that $\mathcal{L}_Q$ anti-commutes with a certain anti-linear conjugation $\mathcal{J}$, that will be introduced later on. The Hamiltonian, which represents the interaction

[1]Supported by the DFG (SFB/TR 12)

with a bosonic gas at temperature zero, can be the Standard Hamiltonian of the non-relativistic QED, (see or instance [2]), or the Pauli-Fierz operator, which is defined in [7, 2], or the Hamiltonian of Nelson's Model. We give the definition of these Hamiltonians in the sequel of Definition 11.
Our first result is the following:

Theorem 1.1. *$\mathcal{L}_Q$, defined in* (16)*, has a unique self-adjoint realization and $\tau_t(X) \in \mathfrak{M}_\beta$ for all $t \in \mathbb{R}$ and all $X \in \mathfrak{M}_\beta$.*

The proof follows from Theorem 4.2 and Lemma 5.2. The main difficulty in the proof is, that $\mathcal{L}_Q$ is not semi-bounded, and that one has to define a suitable auxiliary operator in order to apply Nelson's commutator theorem.
Partly, we assume that the isolated system of finitely many particles is confined in space. This is reflected in Hypothesis 1, where we assume that the particle Hamiltonian H_{el} possesses a Gibbs state. In the case where H_{el} is a Schrödinger-operator, we give in Remark 2.1 a sufficient condition on the external potential V to ensure the existence of a Gibbs state for H_{el}. Our second theorem is

Theorem 1.2. *Assume Hypothesis 1 and that $\Omega_0^\beta \in \mathrm{dom}(e^{-(\beta/2)(\mathcal{L}_0+Q)})$. Then there exists a (β,τ)-KMS state ω^β on $\mathfrak{M}_\beta$.*

This theorem ensures the existence of an equilibrium state on $\mathfrak{M}_\beta$ for the dynamical system $(\mathfrak{M}_\beta, \tau)$. Its proof is part of Theorem 5.3 below. Here, $\mathcal{L}_0$ denotes the Liouvillean for the joint system of particles and bosons, where the interaction part is omitted. Ω_0^β is the vector representative of the (β,τ)-KMS state for the system without interaction. In a third theorem we study the condition $\Omega_0^\beta \in \mathrm{dom}(e^{-(\beta/2)(\mathcal{L}_0+Q)})$:

Theorem 1.3. *Assume Hypothesis 1 is fulfilled. Then there are two cases,*

1. *If $0 \leqslant \gamma < 1/2$ and $\underline{\eta}_1 (1+\beta) \ll 1$, then $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0+Q)})$.*
2. *If $\gamma = 1/2$ and $(1+\beta)(\underline{\eta}_1 + \underline{\eta}_2) \ll 1$, then $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0+Q)})$.*

Here, $\gamma \in [0, 1/2)$ is a parameter of the model, see (32) and $\underline{\eta}_1, \underline{\eta}_2$ are parameters, which describe the strength of the interaction, see (32). In a last theorem we consider the case where $H_{el} = -\Delta_q + \Theta^2 q^2$ and the interaction Hamiltonian is $\lambda\, q\, \Phi(f)$ at temperature zero for $\lambda \neq 0$. Then,

Theorem 1.4. *Ω_0^β is in $\mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0\,+\,Q)})$ for all $\beta \in (0,\infty)$, whenever*

$$|2\Theta^{-1}\,\lambda|\; \| |k|^{-1/2}\, f\|_{\mathcal{H}_{ph}} < 1.$$

Furthermore, we show that our strategy can not be improved to obtain a result, which ensures existence for all values of λ, see (60).
In the last decade there appeared a large number of mathematical contributions to the theory of open quantum system. Here we only want to mention

some of them [3, 6, 8, 9, 10, 13, 14, 15], which consider a related model, in which the particle Hamilton H_{el} is represented as a finite symmetric matrix and the interaction part of the Hamiltonian is linear in annihilation and creation operators. In this case one can prove existence of a β- KMS without any restriction to the strength of the coupling. (In this case we can apply Theorem 1.3 with $\gamma = 0$ and $\underline{\eta}_1 = 0$). We can show existence of KMS-states for an infinite level atom coupled to a heat bath. Furthermore, in [6] there is a general theorem, which ensures existence of a (β, τ)-KMS state under the assumption, that $\Omega_0^\beta \in \mathrm{dom}(e^{-(\beta/2)Q})$, which implies $\Omega_0^\beta \in \mathrm{dom}(e^{-(\beta/2)(\mathcal{L}_0+Q)})$. In Remark 7.3 we verify that this condition implies the existence of a (β, τ)-KMS state in the case of a harmonic oscillator with dipole interaction $\lambda\, q \cdot \Phi(f)$, whenever $(1+\beta)\lambda\|(1+|k|^{-1/2})f\| \ll 1$.

2 Mathematical Preliminaries

2.1 Fock Space, Field- Operators and Second Quantization

We start our mathematical introduction with the description of the joint system of particles and bosons at temperature zero. The Hilbert space describing bosons at temperature zero is the *bosonic Fock space* $\mathcal{F}_b$, where

$$\mathcal{F}_b := \mathcal{F}_b[\mathcal{H}_{ph}] := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_{ph}^{(n)}, \qquad \mathcal{H}_{ph}^{(n)} := \bigotimes_{sym}^{n} \mathcal{H}_{ph}.$$

$\mathcal{H}_{ph}$ is either a closed subspace of $L^2(\mathbb{R}^3)$ or $L^2(\mathbb{R}^3 \times \{\pm\})$, being invariant under complex conjugation. If phonons are considered we choose $\mathcal{H}_{ph} = L^2(\mathbb{R}^3)$, if photons are considered we choose $\mathcal{H}_{ph} = L^2(\mathbb{R}^3 \times \{\pm\})$. In the latter case "+" or "-" labels the polarization of the photon. However, we will write $\langle f \,|\, g \rangle_{\mathcal{H}_{ph}} := \int \overline{f(k)}\, g(k)\, dk$ for the scalar product in both cases. This is an abbreviation for $\sum_{p=\pm} \int \overline{f(k,p)}\, g(k,p)\, dk$ in the case of photons.

$\mathcal{H}_{ph}^{(n)}$ is the n-fold symmetric tensor product of $\mathcal{H}_{ph}$, that is, it contains all square integrable functions f_n being invariant under permutations π of the variables, i.e., $f_n(k_1, \ldots, k_n) = f_n(k_{\pi(1)}, \ldots, k_{\pi(n)})$. For phonons we have $k_j \in \mathbb{R}^3$ and $k_j \in \mathbb{R}^3 \times \{\pm\}$ for photons. The wave functions in $\mathcal{H}_{ph}^n$ are states of n bosons.

The vector $\Omega := (1, 0, \ldots) \in \mathcal{F}_b$ is called the *vacuum*. Furthermore we denote the subspace $\mathcal{F}_b$ of finite sequences with ${\mathcal{F}_b}^{fin}$. On ${\mathcal{F}_b}^{fin}$ the *creation and* *annihilation operators*, $a^*(h)$ and $a(h)$, are defined for $h \in \mathcal{H}_{ph}$ by

$$\begin{aligned}
&(a^*(h)\, f_n)(k_1, \ldots, k_{n+1}) && (2)\\
&\quad = \; (n+1)^{-1/2} \sum_{i=1}^{n+1} h(k_i)\, f_n(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_{n+1}),\\
&(a(h)\, f_{n+1})(k_1, \ldots, k_n) && (3)\\
&\quad = \; (n+1)^{1/2} \int \overline{h(k_{n+1})}\, f_{n+1}(k_1, \ldots, k_{n+1})\, dk_{n+1},
\end{aligned}$$

and $a^*(h)\,\Omega \,=\, h,\; a(h)\,\Omega \,=\, 0$. Since $a^*(h) \,\subset\, (a(h))^*$ and $a(h) \,\subset\, (a^*(h))^*$, the operators $a^*(h)$ and $a(h)$ are closable. Moreover, the canonical commutation relations (CCR) hold true, i.e.,

$$[a(h)\,,\, a(\widetilde{h})] \,=\, [a^*(h)\,,\, a^*(\widetilde{h})] \,=\, 0, \qquad [a(h)\,,\, a^*(\widetilde{h})] \,=\, \langle\, h \,|\, \widetilde{h}\,\rangle_{\mathcal{H}_{ph}}.$$

Furthermore we define field operator by

$$\Phi(h) \;:=\; 2^{-1/2}\,(a(h) \,+\, a^*(h)), \qquad h \in \mathcal{H}_{ph}.$$

It is a straightforward calculation to check that the vectors in ${\mathcal{F}_b}^{fin}$ are analytic for $\Phi(h)$. Thus, $\Phi(h)$ is essentially self-adjoint on ${\mathcal{F}_b}^{fin}$. In the sequel, we will identify $a^*(h)$, $a(h)$ and $\Phi(h)$ with their closures. The Weyl operators $W(h)$ are given by $W(h) \,=\, \exp(i\,\Phi(h))$. They fulfill the CCR-relation for the Weyl operators, i.e.,

$$W(h)\,W(g) \;=\; \exp(i/2\,\mathrm{Im}\,\langle\, h \,|\, g\,\rangle_{\mathcal{H}_{ph}})\,W(g+h),$$

which follows from explicit calculations on ${\mathcal{F}_b}^{fin}$. The Weyl algebra $W(\mathfrak{f})$ over a subspace $\mathfrak{f}$ of $\mathcal{H}_{ph}$ is defined by

$$W(\mathfrak{f}) \;:=\; \mathrm{cl}\,\mathrm{LH}\{W(g) \in \mathcal{B}(\mathcal{F}_b) \,:\, g \in \mathfrak{f}\}. \qquad (4)$$

Here, cl denotes the closure with respect to the norm of $\mathcal{B}(\mathcal{F}_b)$, and "LH" denotes the linear hull.

Let $\alpha \,:\, \mathbb{R}^3 \,\to\, [0,\,\infty)$ be a locally bounded Borel function and $\mathrm{dom}(\alpha) \,:=\, \{f \in \mathcal{H}_{ph} \,:\, \alpha f \,\in\, \mathcal{H}_{ph}\}$. Note, that $(\alpha f)(k)$ is given by $\alpha(k)\,f(k,p)$ for photons. If $\mathrm{dom}(\alpha)$ is dense subspace of $\mathcal{H}_{ph}$, α defines a self-adjoint multiplication operator on $\mathcal{H}_{ph}$. In this case, the second quantization $d\Gamma(\alpha)$ of α is defined by

$$(d\Gamma(\alpha)\, f_n)(k_1, \ldots, k_n) \;:=\; (\alpha(k_1) + \alpha(k_2) + \ldots + \alpha(k_n))\, f_n(k_1, \ldots, k_n)$$

and $d\Gamma(\alpha)\,\Omega \,=\, 0$ on its maximal domain.

2.2 Hilbert space and Hamiltonian for the particles

Let $\mathcal{H}_{el}$ be a closed, separable subspace of $L^2(X, d\mu)$, that is invariant under complex conjugation. The Hamiltonian H_{el} for the particle is a self-adjoint operator on $\mathcal{H}_{el}$ being bounded from below. We set $H_{el,+} := H_{el} - \inf \sigma(H_{el}) + 1$. Partly, we need the assumption

Hypothesis 1. *Let $\beta > 0$. There exists a small positive constant $\epsilon > 0$, and*

$$\mathrm{Tr}_{\mathcal{H}_{el}}\{e^{-(\beta - \epsilon)\, H_{el}}\} < \infty.$$

The condition implies the existence of a Gibbs state

$$\omega_{el}^{\beta}(A) = \mathcal{Z}^{-1}\, \mathrm{Tr}_{\mathcal{H}_{el}}\{e^{-\beta\, H_{el}} A\}, \quad A \in \mathcal{B}(\mathcal{H}_{el}),$$

for $\mathcal{Z} = \mathrm{Tr}_{\mathcal{H}_{el}}\{e^{-\beta\, H_{el}}\}$.

Remark 2.1. *Let $\mathcal{H}_{el} = L^2(\mathbb{R}^n, d^n x)$ and $H_{el} = -\Delta_x + V_1 + V_2$, where V_1 is a $-\Delta_x$-bounded potential with relative bound $a < 1$ and V_2 is in $L^2_{loc}(\mathbb{R}^n, d^n x)$. Thus H_{el} is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^n)$. Moreover, if additionally*

$$\int e^{-(\beta - \epsilon)\, V_2(x)}\, d^n x < \infty, \tag{5}$$

then one can show, using the Golden-Thompson-inequality, that Hypothesis 1 is satisfied.

2.3 Hilbert space and Hamiltonian for the interacting system

The Hilbert space for the joint system is $\mathcal{H} := \mathcal{H}_{el} \otimes \mathcal{F}_b$. The vectors in $\mathcal{H}$ are sequences $f = (f_n)_{n \in \mathbb{N}_0}$ of wave functions, $f_n \in \mathcal{H}_{el} \otimes \mathcal{H}_{ph}^{(n)}$, obeying

$$\begin{aligned} \underline{k}_n &\mapsto f_n(x, \underline{k}_n) \in \mathcal{H}_{ph}^{(n)} && \text{for } \mu\text{- almost every x} \\ x &\mapsto f_n(x, \underline{k}_n) \in \mathcal{H}_{el} && \text{for Lebesgue - almost every } \underline{k}_n, \end{aligned}$$

where $\underline{k}_n = (k_1, \ldots, k_n)$. The complex conjugate vector is $\overline{f} := (\overline{f}_n)_{n \in \mathbb{N}_0}$. Let $G^j := \{G_k^j\}_{k \in \mathbb{R}^3}$, $H^j := \{H_k^j\}_{k \in \mathbb{R}^3}$ and $F := \{F_k\}_{k \in \mathbb{R}^3}$ be families of closed operators on $\mathcal{H}_{el}$ for $j = 1, \ldots, r$. We assume, that $\mathrm{dom}(F_k^*)$, $\mathrm{dom}(F_k) \supset \mathrm{dom}(H_{el,+}^{1/2})$ and that

$$k \mapsto G_k^j,\ (H_k^j),\ F_k H_{el,+}^{-1/2},\ (F_k)^* H_{el,+}^{-1/2} \in \mathcal{B}(\mathcal{H}_{el})$$

are weakly (Lebesgue-)measurable. For $\phi \in \mathrm{dom}(H_{el,+}^{1/2})$ we assume that

$$k \mapsto (G_k^j\, \phi)(x),\ (H_k^j\, \phi)(x),\ (F_k\, \phi)(x) \in \mathcal{H}_{ph}, \tag{6}$$

$$k \mapsto ((G_k^j)^*\, \phi)(x),\ ((H_k^j)^*\, \phi)(x),\ ((F_k)^*\, \phi)(x) \in \mathcal{H}_{ph}, \text{ for } x \in X. \tag{7}$$

Moreover we assume for $\vec{G} = (G^1, \ldots, G^r)$, $\vec{H} := (H^1, \ldots, H^r)$ and F, that

$$\|\vec{G}\|_w < \infty, \ \|\vec{H}\|_w < \infty, \ \|F\|_{w,1/2} < \infty,$$

where

$$\|G_j\|_w^2 := \int (\alpha(k) + \alpha(k)^{-1}) \left(\|(G_k^j)^*\|^2_{\mathcal{B}(H_{el})} + \|G_k^j\|^2_{\mathcal{B}(H_{el})}\right) dk$$

$$\|\vec{G}\|_w^2 := \sum_{j=1}^r \|G_j\|_w^2, \qquad \|F\|^2_{w,1/2} := \|F H_{el,+}^{-1/2}\|_w^2 + \|F^* H_{el,+}^{-1/2}\|_w^2.$$

We define for $f = (f_n)_{n=0}^\infty \in \mathrm{dom}(H_{el,+}^{1/2}) \otimes \mathcal{F}_b^{fin}$ the (generalized) creation operator

$$\begin{aligned}(a^*(F)\, f_n)(x, k_1, \ldots, k_{n+1}) \\ := (n+1)^{-1/2} \sum_{i=1}^{n+1} (F_{k_i}\, f_n)(x, k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_{n+1})\end{aligned} \tag{8}$$

and $a(F)\, f_0(x) = 0$. The (generalized) annihilation operator is

$$\begin{aligned}(a(F)\, f_{n+1})(x, k_1, \ldots, k_n) \\ := (n+1)^{1/2} \int (F^*_{k_{n+1}}\, f_{n+1})(x, k_1, \ldots, k_n, k_{n+1})\, dk_{n+1}.\end{aligned} \tag{9}$$

Moreover, the corresponding (generalized) field operator is $\Phi(F) := 2^{-1/2}\,(a(F) + a^*(F))$. $\Phi(F)$ is symmetric on $\mathrm{dom}(H_{el,+}^{1/2}) \otimes \mathcal{F}_b^{fin}$. The bounds follow directly from Equations (8) and (9).

$$\begin{aligned}\|\, a(F) H_{el,+}^{-1/2}\, f\|^2_{\mathcal{H}} &\leqslant \int |\alpha(k)|^{-1} \|F_k^*\, H_{el,+}^{-1/2}\|^2_{\mathcal{B}(\mathcal{H}_{el})}\, dk \cdot \|d\Gamma(|\alpha|)^{1/2} f\|^2_{\mathcal{H}} \\ \|a^*(F) H_{el,+}^{-1/2}\, f\|^2_{\mathcal{H}} &\leqslant \int |\alpha(k)|^{-1} \|F_k\, H_{el,+}^{-1/2}\|^2_{\mathcal{B}(\mathcal{H}_{el})}\, dk \cdot \|d\Gamma(|\alpha|)^{1/2}\, f\|^2_{\mathcal{H}} \\ &\quad + \int \|F_k\, H_{el,+}^{-1/2}\|^2_{\mathcal{B}(\mathcal{H}_{el})}\, dk \cdot \|f\|^2_{\mathcal{H}}.\end{aligned} \tag{10}$$

For $(G_k)^j$, $(H_k)^j \in \mathcal{B}(\mathcal{H}_{el})$, the factor $H_{el,+}^{-1/2}$ can be omitted. The Hamiltonians for the non-interacting, resp. interacting model are

Definition 2.2. *On* $\mathrm{dom}(H_{el}) \otimes \mathrm{dom}(d\Gamma(\alpha)) \cap \mathcal{F}_b^{fin}$ *we define*

$$H_0 := H_{el} \otimes \mathbf{1} + \mathbf{1} \otimes d\Gamma(\alpha), \qquad H := H_0 + W, \tag{11}$$

where $W := \Phi(\vec{G})\, \Phi(\vec{H}) + \mathrm{h.c.} + \Phi(F)$ *and* $\Phi(\vec{G})\, \Phi(\vec{H}) := \sum_{j=1}^r \Phi(G^j)\, \Phi(H^j)$. *The abbreviation "h.c." means the formal adjoint operator of* $\Phi(\vec{G})\, \Phi(\vec{H})$.

We give examples for possible configurations:
Let $\gamma \in \mathbb{R}$ be a small coupling parameter.

▶ The Nelson Model:
$\mathcal{H}_{el} \subset L^2(\mathbb{R}^{3N})$, $H_{el} := -\Delta + V$, $\mathcal{H}_{ph} = L^2(\mathbb{R}^3)$ and $\alpha(k) = |k|$. The form factor is $F_k = \gamma \sum_{\nu=1}^{N} e^{-i\,kx_\nu}\, |k|^{-1/2}\, \mathbf{1}[\,|k| \leqslant \kappa]$, $x_\nu \in \mathbb{R}^3$ and H^j, $G^j = 0$.

▶ The Standard Model of Nonrelativistic QED:
$\mathcal{H}_{el} \subset L^2(\mathbb{R}^{3N})$, $H_{el} := -\Delta + V$, $\mathcal{H}_{ph} = L^2(\mathbb{R}^3 \times \{\pm\})$ and $\alpha(k) = |k|$. The form factors are

$$F_{\mathbf{k}} = 4\gamma^{3/2}\,\pi^{-1/2} \sum_{\nu=1}^{N} (-i\nabla_{x_\nu} \cdot \epsilon(k,p)) e^{-i\,\gamma^{1/2} kx_\nu}\, (2|k|)^{-1/2}\, \mathbf{1}[\,|k| \leqslant \kappa] + \text{h. c.},$$

$$G_{\mathbf{k}}^{i,\,\nu} = H_{\mathbf{k}}^{i,\,\nu} = 2\gamma^{3/2}\,\pi^{-1/2}\,\epsilon_i(k,\,p)\, e^{-i\,\gamma^{1/2}\,kx_\nu}\, (2|k|)^{-1/2}\, \mathbf{1}[\,|k| \leqslant \kappa]$$

for $i = 1, 2, 3$, $\nu = 1, \ldots, N$, $x_\nu \in \mathbb{R}^3$ and $\mathbf{k} = (k,p) \in \mathbb{R}^3 \times \{\pm\}$. $\epsilon_i(k,\,\pm) \in \mathbb{R}^3$ are polarization vectors.

▶ The Pauli-Fierz-Model:
$\mathcal{H}_{el} \subset L^2(\mathbb{R}^{3N})$, $H_{el} := -\Delta + V$, $\mathcal{H}_{ph} = L^2(\mathbb{R}^3)$ or $\mathcal{H}_{ph} = L^2(\mathbb{R}^3 \times \{\pm\})$, and $\alpha(k) = |k|$. The form factor is $F_k = \gamma \sum_{\nu=1}^{N} \mathbf{1}[\,|k| \leqslant \kappa]\, k \cdot x_\nu$ and $G_k^j = H_k^j = 0$

3 The Representation π

In order to describe the particle system at inverse temperature β we introduce the algebraic setting. For $\mathfrak{f} = \{f \in \mathcal{H}_{ph} \,:\, \alpha^{-1/2} f \in \mathcal{H}_{ph}\}$ we define the algebra of observables by

$$\mathfrak{A} = \mathcal{B}(\mathcal{H}_{el}) \otimes \mathcal{W}(\mathfrak{f}).$$

For elements $A \in \mathfrak{A}$ we define $\tilde{\tau}_t^0(A) := e^{i\,t\,H_0}\, A\, e^{-i\,t\,H_0}$ and $\tilde{\tau}_t^g(A) := e^{i\,t\,H} A e^{-i\,t\,H}$. We first discuss the model without interaction.

3.1 The Representation π_f

The time-evolution for the Weyl operators is given by

$$e^{i\,t\,\check{H}}\, \mathcal{W}(f)\, e^{-i\,t\,\check{H}} = \mathcal{W}(e^{i\,t\,\alpha}\, f).$$

For this time-evolution an equilibrium state ω_f^β is defined by

$$\omega_f^\beta(\mathcal{W}(f)) = \langle\, f \,|\, (1 + 2\,\varrho_\beta)\, f \,\rangle_{\mathcal{H}_{ph}},$$

where $\varrho_\beta(k) = \big(\exp(\beta\,\alpha(k)) - 1\big)^{-1}$. It describes an infinitely extended gas of bosons with momentum density ϱ_β at temperature β. Since ω_f^β is a quasi-free state on the Weyl algebra, the definition of ω_f^β extends to polynomials of creation and annihilation operators. One has

$$\omega_f^\beta(a(f)) = \omega_f^\beta(a^*(f)) = \omega_f^\beta(a(f)\,a(g)) = \omega_f^\beta(a^*(f)\,a^*(g)) = 0,$$
$$\omega_f^\beta(a^*(f)\,a(g)) = \langle\, g \,|\, \varrho_\beta\, f\,\rangle_{\mathcal{H}_{ph}}.$$

For polynomials of higher degree one can apply Wick's theorem for quasi-free states, i.e.,

$$\omega_f^\beta(a^{\sigma_{2m}}(f_{2m})\cdots a^{\sigma_1}(f_1)) = \sum_{P\in\mathcal{Z}_2}\prod_{\substack{\{i,j\}\in P\\ i>j}} \omega_f^\beta\big(a^{\sigma_i}(f_i)\,a^{\sigma_j}(f_j)\big), \tag{12}$$

where $a^{\sigma_k} = a^*$ or $a^{\sigma_k} = a$ for $k = 1, \ldots, 2m$. $\mathcal{Z}_2$ are the pairings, that is

$$P\in\mathcal{Z}_2, \text{ iff } P = \{Q_1,\ldots,Q_m\},\ \#Q_i = 2 \text{ and } \bigcup_{i=1}^m Q_i = \{1,\ldots,2m\}.$$

The Araki-Woods isomorphism $\pi_f : \mathcal{W}(\mathfrak{f}) \to \mathcal{B}(\mathcal{F}_b\otimes\mathcal{F}_b)$ is defined by

$$\pi_f[\mathcal{W}(f)] := \mathcal{W}_\beta(f) := \exp(i\,\Phi_\beta(f)),$$
$$\Phi_\beta(f) := \Phi((1+\varrho_\beta)^{1/2} f)\otimes\mathbf{1} + \mathbf{1}\otimes\Phi(\varrho_\beta^{1/2}\,\overline{f}).$$

The vector $\Omega_f^\beta := \Omega\otimes\Omega$ fulfills

$$\omega_f^\beta(\mathcal{W}(f)) = \langle\,\Omega_f^\beta\,|\,\pi_f[\mathcal{W}(f)]\,\Omega_f^\beta\,\rangle. \tag{13}$$

3.2 The representation π^{el}

The particle system without interaction has the observables $\mathcal{B}(\mathcal{H}_{el})$, the states are defined by density operators ρ, i.e., $\rho\in\mathcal{B}(\mathcal{H}_{el}),\ 0\leqslant\rho,\ \mathrm{Tr}\{\rho\} = 1$. The expectation of $A\in\mathcal{B}(\mathcal{H}_{el})$ in ρ at time t is

$$\mathrm{Tr}\{\,\rho\, e^{i\,t\,H_{el}}\,A\,e^{-i\,t\,H_{el}}\}.$$

Since ρ is a compact, self-adjoint operator, there is an ONB $(\phi_n)_n$ of eigenvectors, with corresponding (positive) eigenvalues $(p_n)_n$. Let

$$\sigma(x,y) = \sum_{n=1}^\infty p_n^{1/2}\,\phi_n(x)\,\overline{\phi_n(y)} \in \mathcal{H}_{el}\otimes\mathcal{H}_{el}. \tag{14}$$

For $\psi\in\mathcal{H}_{el}$ we define $\sigma\,\psi := \int\sigma(x,y)\,\psi(y)\,d\mu(y)$. Obviously, σ is an operator of Hilbert-Schmidt class. Note, $\overline{\sigma}\,\psi := \overline{\sigma\,\overline{\psi}}$ has the integral kernel $\overline{\sigma(x,y)}$. It is a straightforward calculation to verify that

$$\mathrm{Tr}\{\rho\,e^{i\,t\,H_{el}}\,A\,e^{-i\,t\,H_{el}}\} = \langle\, e^{-i\,t\,\mathcal{L}_{el}}\,\sigma\,|\,(A\otimes 1)\,e^{-i\,t\,\mathcal{L}_{el}}\sigma\,\rangle_{\mathcal{H}_{el}\otimes\mathcal{H}_{el}},$$

where $\mathcal{L}_{el} = H_{el}\otimes\mathbf{1}-\mathbf{1}\otimes\overline{H}_{el}$. This suggests the definition of the representation

$$\pi^{el} : \mathcal{B}(\mathcal{H}_{el})\to\mathcal{B}(\mathcal{H}_{el}\otimes\mathcal{H}_{el}),\quad A\mapsto A\otimes\mathbf{1}.$$

Now, we define the representation map for the joint system by

$$\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K}), \qquad \pi := \pi_{el} \otimes \pi_f,$$

where $\mathcal{K} := \mathcal{H}_{el} \otimes \mathcal{H}_{el} \otimes \mathcal{F}_b \otimes \mathcal{F}_b$. Let $\mathfrak{M}_\beta := \pi[\mathfrak{A}]''$ be the *enveloping W^*-algebra*, here $\pi[\mathfrak{A}]'$ denotes the commutant of $\pi[\mathfrak{A}]$, and $\pi[\mathfrak{A}]''$ the bicommutant. We set $\mathcal{D} := U_1 \otimes \overline{U_1} \otimes \mathcal{C}$, where $\mathcal{C}$ is a subspace of vectors in $\mathcal{F}_b^{fin} \otimes \mathcal{F}_b^{fin}$, with compact support, and $U_1 := \cup_{n=1}^{\infty} \operatorname{ran} \mathbb{1}[\mathcal{H}_{el} \leqslant n]$. On $\mathcal{D}$ the operator $\mathcal{L}_0$, given by

$$\begin{aligned} \mathcal{L}_0 &:= \mathcal{L}_{el} \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}_f, \quad \text{on } \mathcal{K}, \\ \mathcal{L}_f &:= d\Gamma(\alpha) \otimes \mathbf{1} - \mathbf{1} \otimes d\Gamma(\alpha), \quad \text{on } \mathcal{F}_b \otimes \mathcal{F}_b, \end{aligned}$$

is essentially self-adjoint and we can define

$$\tau_t^0(X) := e^{it\mathcal{L}_0} X e^{-it\mathcal{L}_0} \in \mathfrak{M}_\beta, \quad X \in \mathfrak{M}_\beta, \quad t \in \mathbb{R},$$

It is not hard to see, that

$$\pi[\tilde{\tau}_t^0(A)] = \tau_t^0(\pi[A]), \quad A \in \mathfrak{A}, \quad t \in \mathbb{R}$$

On $\mathcal{K}$ a we introduce a conjugation by

$$\mathcal{J}(\phi_1 \otimes \phi_2 \otimes \psi_1 \otimes \psi_2) = \overline{\phi_2} \otimes \overline{\phi_1} \otimes \overline{\psi_2} \otimes \overline{\psi_1}.$$

It is easily seen, that $\mathcal{J}\mathcal{L}_0 = -\mathcal{L}_0\mathcal{J}$. In this context one has $\mathfrak{M}_\beta' = \mathcal{J}\mathfrak{M}_\beta\mathcal{J}$, see for example [4]. In the case, where H_{el} fulfills Hypothesis 1, we define the vector representative $\Omega_{el}^\beta \in \mathcal{H}_{el} \otimes \mathcal{H}_{el}$ of the Gibbs state ω_{el}^β as in (14) for $\rho = e^{-\beta H_{el}} \mathcal{Z}^{-1}$.

Theorem 3.1. *Assume Hypothesis 1 is fulfilled. Then, $\Omega_0^\beta := \Omega_{el}^\beta \otimes \Omega_f^\beta$ is a* cyclic *and* separating *vector for $\mathfrak{M}_\beta$. $e^{-\beta/2\mathcal{L}_0}$ is a* modular operator *and $\mathcal{J}$ is the* modular conjugation *for Ω_0^β, that is*

$$X\Omega_0^\beta \in \operatorname{dom}(e^{-\beta/2\,\mathcal{L}_0}), \quad \mathcal{J}\,X\,\Omega_0^\beta = e^{-\beta/2\,\mathcal{L}_0}\,X^*\,\Omega_0^\beta \tag{15}$$

for all $X \in \mathfrak{M}_\beta$ and $\mathcal{L}_0\,\Omega_0^\beta = 0$. Moreover,

$$\omega_0^\beta(X) := \langle \Omega_0^\beta \,|\, X\,\Omega_0^\beta \rangle_{\mathcal{K}}, \qquad X \in \mathfrak{M}_\beta$$

is a (τ^0, β)-KMS-state for $\mathfrak{M}_\beta$, i.e., for all $X, Y \in \mathfrak{M}_\beta$ exists $F_\beta(X, Y, \cdot)$, analytic in the strip $S_\beta = \{z \in \mathbb{C} : 0 < Im z < \beta\}$, continuous on the closure and taking the boundary conditions

$$\begin{aligned} F_\beta(X, Y, t) &= \omega_0^\beta(X\,\tau_t^0(Y)) \\ F_\beta(X, Y, t + i\beta) &= \omega_0^\beta(\tau_t^0(Y)\,X) \end{aligned}$$

For a proof see [14].

4 The Liouvillean $\mathcal{L}_Q$

In this and the next section we will introduce the Standard Liouvillean $\mathcal{L}_Q$ for a dynamics τ on $\mathfrak{M}_\beta$, describing the interaction between particles and bosons at inverse temperature β. The label Q denotes the interaction part of the Liouvillean, it can be deduced from the interaction part W of the corresponding Hamiltonian by means of formal arguments, which we will not give here. In a first step we prove self-adjointness of $\mathcal{L}_Q$ and of other Liouvilleans. A main difficulty stems from the fact, that $\mathcal{L}_Q$ and the other Liouvilleans, mentioned before, are not bounded from below. The proof of self-adjointness is given in Theorem 4.2, it uses Nelson's commutator theorem and auxiliary operators which are constructed in Lemma 4.1. The proof, that $\tau_t(X) \in \mathfrak{M}_\beta$ for $X \in \mathfrak{M}_\beta$, is given in Lemma 5.2. Assuming $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2(\mathcal{L}_0+Q)})$ we can ensure existence of a (τ, β)-KMS state $\omega^\beta(X) = \langle \Omega^\beta \,|X\, \Omega^\beta\rangle \cdot \|\Omega^\beta\|^{-2}$ on $\mathfrak{M}_\beta$, where $\Omega^\beta = e^{-\beta/2(\mathcal{L}_0+Q)}\Omega_0^\beta$. Moreover, we can show that $e^{-\beta \mathcal{L}_Q}$ is the modular operator for Ω^β and conjugation $\mathcal{J}$. This is done in Theorem 5.3.

Our proof of 5.3 is inspired by the proof given in [6]. The main difference is that we do not assume, that Q is self-adjoint and that $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta Q})$. For this reason we need to introduce an additional approximation Q_N of Q, which is self-adjoint and affiliated with $\mathfrak{M}_\beta$, see Lemma 5.1.

The interaction on the level of Liouvilleans between particles and bosons is given by Q , where

$$Q \,:=\, \Phi_\beta(\vec{G})\,\Phi_\beta(\vec{H}) \,+\, \mathrm{h.c.} + \Phi_\beta(F), \quad \Phi_\beta(\vec{G})\,\Phi_\beta(\vec{H}) \,:=\, \sum_{j=1}^{r} \Phi_\beta(G^j)\,\Phi_\beta(H^j).$$

For each family $K = \{K_k\}_k$ of closed operators on $\mathcal{H}_{el}$ with $\|K\|_{w,1/2} < \infty$ we set

$$\Phi_\beta(K) \,:=\, \big(a^*((1 \,+\, \varrho_\beta)^{1/2}\, K) \otimes \mathbf{1} \,+\, \mathbf{1} \otimes a^*(\varrho_\beta^{1/2}\, K^*)\big) \,+\, \mathrm{h.c.}.$$

Here, K_k acts as $K_k \otimes \mathbf{1}$ on $\mathcal{H}_{el} \otimes \mathcal{H}_{el}$. A Liouvillean, that describes the dynamics of the joint system of particles and bosons is the so-called *Standard Liouvillean*

$$\mathcal{L}_Q\,\phi \,:=\, (\mathcal{L}_0 \,+\, Q \,-\, Q^{\mathcal{J}})\,\phi, \qquad \phi \in \mathcal{D}, \tag{16}$$

which is distinguished by $\mathcal{J}\,\mathcal{L}_Q = -\mathcal{L}_Q\,\mathcal{J}$. For an operator A, acting on $\mathcal{K}$, the symbol $A^{\mathcal{J}}$ is an abbreviation for $\mathcal{J}\,A\,\mathcal{J}$. An important observation is, that $[Q\,,\, Q^{\mathcal{J}}] \,=\, 0$ on $\mathcal{D}$. Next, we define four auxiliary operators on $\mathcal{D}$

$$\begin{aligned}
\mathcal{L}_a^{(1)} &:= \big(H_{el,+} \otimes \mathbf{1} \,+\, \mathbf{1} \otimes \overline{H}_{el,+}\big) \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}_{f,a} + \mathbf{1} \qquad (17)\\
\mathcal{L}_a^{(2)} &:= H^Q_{el,+} + (H^Q_{el,+})^{\mathcal{J}} + c_1 \mathbf{1} \otimes \mathcal{L}_{f,a} + c_2\\
\mathcal{L}_a^{(3)} &:= H^Q_{el,+} + (H_{el,+})^{\mathcal{J}} + c_1 \mathbf{1} \otimes \mathcal{L}_{f,a} + c_2\\
\mathcal{L}_a^{(4)} &:= H_{el,+} \otimes \mathbf{1} + (H^Q_{el,+})^{\mathcal{J}} + c_1 \mathbf{1} \otimes \mathcal{L}_{f,a} + c_2,
\end{aligned}$$

where $\mathcal{L}_{f,a}$ is an operator on $\mathcal{F}_b \otimes \mathcal{F}_b$ and $H^Q_{el,+}$ acts on $\mathcal{K}$. Furthermore,

$$\begin{aligned}\mathcal{L}_{f,a} &= d\Gamma(1+\alpha)\otimes \mathbf{1} + \mathbf{1}\otimes d\Gamma(1+\alpha) + \mathbf{1},\\ \mathcal{L}_{el,a} &= H_{el,+}\otimes \mathbf{1} \,+\, \mathbf{1}\otimes \overline{H}_{el,+} \quad H^Q_{el,+} := H_{el,+}\otimes \mathbf{1} + Q.\end{aligned}$$

Obviously, $\mathcal{L}^{(i)}_a,\ i = 1, 2, 3, 4$ are symmetric operators on $\mathcal{D}$.

LEMMA 4.1. *For sufficiently large values of* $c_1, c_2 \geqslant 0$ *we have that* $\mathcal{L}^{(i)}_a,\ i = 1, 2, 3, 4$ *are essentially self-adjoint and positive. Moreover, there is a constant* $c_3 > 0$ *such that*

$$c_3^{-1}\,\|\mathcal{L}^{(1)}_a\,\phi\| \leqslant \|\mathcal{L}^{(i)}_a\,\phi\| \leqslant c_3\,\|\mathcal{L}^{(1)}_a\,\phi\|, \qquad \phi\in \mathrm{dom}(\mathcal{L}^{(1)}_a). \tag{18}$$

Proof. Let $a, a' \in \{l, r\}$ and $K_i,\ i = 1, 2$ be families of bounded operators with $\|K_i\|_w < \infty$. Let $\Phi_l(K_i) = \Phi(K_i)\otimes \mathbf{1}$ and $\Phi_r(K_i) := \mathbf{1}\otimes\Phi(K_i)$. We have for $\phi\in\mathcal{D}$

$$\begin{aligned}\|\,\Phi_a(\eta K_1)\,\Phi_{a'}(\eta' K_2)\,\phi\,\| &\leqslant \mathrm{const}\,\|\,\mathcal{L}_{f,a}\,\phi\,\| \\ \|\,\Phi_a(\eta F)\,\phi\,\| &\leqslant \mathrm{const}\,\|(\mathcal{L}_{el,a})^{1/2}(\mathcal{L}_{f,a})^{1/2}\,\phi\,\|,\end{aligned} \tag{19}$$

where $\eta, \eta' \in \{(1+\varrho_\beta)^{1/2}, \varrho_\beta^{1/2}\}$. Note, that the estimates hold true, if $\Phi_a(\eta K_i)$ or $\Phi_a(\eta F)$ are replaced by $\Phi_a(\eta K_i)^{\mathcal{J}}$ or $\Phi_a(\eta F)^{\mathcal{J}}$. Thus, we obtain for sufficiently large $c_1 \gg 1$, depending on the form-factors, that

$$\|Q\,\phi\| + \|Q^{\mathcal{J}}\,\phi\| \leqslant 1/2\,\big\|\big(\mathcal{L}_{el,a} + c_1\,\mathcal{L}_{f,a}\big)\,\phi\big\|. \tag{20}$$

By the Kato-Rellich-Theorem ([17], Thm. X.12) we deduce that $\mathcal{L}^{(i)}_a$ is self-adjoint on $\mathrm{dom}(\mathcal{L}_{el,a} + c_1\,\mathcal{L}_{f,a})$, bounded from below and that $\mathcal{L}_{el,a} + c_1\,\mathcal{L}_{f,a}$ is $\mathcal{L}^{(i)}_a$-bounded for every $c_2 \geqslant 0$ and $i = 2, 3, 4$. In particular, $\mathcal{D}$ is a core of $\mathcal{L}^{(i)}_a$. The proof follows now from $\|\mathcal{L}^{(i)}_a\,\phi\| \leqslant \|(\mathcal{L}_{el,a}+c_1\,\mathcal{L}_{f,a})\,\phi\| \leqslant c_1\,\|\mathcal{L}^{(1)}_a\,\phi\|$ for $\phi\in\mathcal{D}$. □

THEOREM 4.2. *The operators*

$$\mathcal{L}_0, \quad \mathcal{L}_Q = \mathcal{L}_0 + Q - Q^{\mathcal{J}}, \quad \mathcal{L}_0 + Q, \quad \mathcal{L}_0 - Q^{\mathcal{J}}, \tag{21}$$

defined on $\mathcal{D}$*, are essentially self-adjoint. Every core of* $\mathcal{L}^{(1)}_a$ *is a core of the operators in line* (21)*.*

Proof. We restrict ourselves to the case of $\mathcal{L}_Q$. We check the assumptions of Nelson's commutator theorem ([17], Thm. X.37). By Lemma 4.1 it suffices to show $\|\mathcal{L}_Q\phi\| \leqslant \mathrm{const}\,\|\mathcal{L}^{(1)}_a\phi\|$ and $|\langle\,\mathcal{L}_Q\phi\,|\,\mathcal{L}^{(2)}_a\phi\rangle - \langle\,\mathcal{L}^{(2)}_a\phi|\,\mathcal{L}_Q\phi\rangle| \leqslant \mathrm{const}\,\|(\mathcal{L}^{(1)}_a)^{1/2}\phi\|^2$ for $\phi\in\mathcal{D}$. The first inequality follows from Equation (20).

To verify the second inequality we observe

$$
\begin{aligned}
&\big|\big\langle\, \mathcal{L}_Q\,\phi \,\big|\, \mathcal{L}_a^{(2)}\,\phi \,\big\rangle - \big\langle\, \mathcal{L}_a^{(2)}\,\phi \,\big|\, \mathcal{L}_Q\,\phi \,\big\rangle\big| \qquad (22)\\
&\quad \leqslant c_1 \Big|\big\langle\, Q\phi \,\big|\, \mathcal{L}_{f,a}\phi\big\rangle - \big\langle\, \mathcal{L}_{f,a}\phi \,\big|\, Q\phi\big\rangle\Big|\\
&\qquad + c_1 \Big|\big\langle\, Q^{\mathcal{J}}\phi \,\big|\, \mathcal{L}_{f,a}\phi\,\big\rangle - \big\langle\, \mathcal{L}_{f,a}\phi \,\big|\, Q^{\mathcal{J}}\phi\,\big\rangle\Big|\\
&\qquad + \Big|\big\langle\, \mathcal{L}_f\phi \big|\, Q\phi\,\big\rangle - \big\langle\, Q\phi \,\big|\, \mathcal{L}_f\phi\,\big\rangle\Big| + \Big|\big\langle\, \mathcal{L}_f\phi \,\big|\, Q^{\mathcal{J}}\phi\,\big\rangle - \big\langle\, Q^{\mathcal{J}}\phi \,\big|\, \mathcal{L}_f\phi\,\big\rangle\Big|,
\end{aligned}
$$

where we used, that $\big[H_{el,+}^{Q},\, (H_{el,+}^{Q})^{\mathcal{J}}\big] \;=\; 0$. Let $K_i \in \{G_j,\, H_j\}$ and $\eta,\, \eta' \in \{\varrho^{1/2},\, (1\,+\,\varrho)^{1/2}\}$. We remark, that

$$
\begin{aligned}
[\Phi_a(\eta\, K_1)\, \Phi_{a'}(\eta'\, K_2)\,,\, \mathcal{L}_{f,a}] \;&=\; i\,\Phi_a(i\,(1\,+\,\alpha)\,\eta\, K_1)\,\Phi_{a'}v(\eta'\, K_2) \qquad (23)\\
&\quad+\; i\,\Phi_a(\eta\, K_1)\,\Phi_{a'}(i\,(1\,+\,\alpha)\,\eta'\, K_2)\\
[\Phi_a(\eta\, F)\,,\, \mathcal{L}_{f,a}] \;&=\; i\,\Phi_a(i\,(1\,+\,\alpha)\,\eta\, F).
\end{aligned}
$$

Hence, for $\phi \in \mathrm{dom}(\mathcal{L}_a^{(2)})$, we have by means of (10) that

$$
\begin{aligned}
\big|\big\langle\, \phi \,\big|\, [\Phi_a(\eta K_1)\,\Phi_{a'}(\eta' K_2),\; \mathcal{L}_{f,a}]\phi\,\big\rangle\big| \;&\leqslant\; \mathrm{const}\,\|\mathcal{L}_{f,a}^{1/2}\phi\|^2 \qquad (24)\\
\big|\big\langle \phi \,\big|\, [\Phi_a(\eta F),\; \mathcal{L}_{f,a}]\phi\,\big\rangle\big| \;&\leqslant\; \mathrm{const}\,\|\mathcal{L}_{f,a}^{1/2}\phi\|\,\|(\mathcal{L}_{el,a})^{1/2}\phi\|.
\end{aligned}
$$

Thus, (24) is bounded by a constant times $\|(\mathcal{L}_a^{(1)})^{1/2}\phi\|^2$. The essential self-adjointness of $\mathcal{L}_Q$ follows now from estimates analog to (23) and (24), where $\mathcal{L}_{f,a}$ is replaced by $\mathcal{L}_f$ in (23) and in the left side of (24). For $\mathcal{L}_0{+}Q$ and $\mathcal{L}_0{-}Q^{\mathcal{J}}$ one has to consider the commutator with $\mathcal{L}_a^{(3)}$ and $\mathcal{L}_a^{(4)}$, respectively. □

REMARK 4.3. *In the same way one can show, that H is essentially self-adjoint on any core of $H_1 := H_{el} \,+\, d\Gamma(1\,+\,\alpha)$, even if H is not bounded from below.*

5 REGULARIZED INTERACTION AND STANDARD FORM OF $\mathfrak{M}_\beta$

In this subsection a regularized interaction Q_N is introduced:

$$
Q_N \;:=\; \Big\{\Phi_\beta(\vec{G}_N)\,\Phi_\beta(\vec{H}_N) \,+\, \mathrm{h.c.}\Big\} \,+\, \Phi_\beta(F_N). \qquad (25)
$$

The regularized form factors $\vec{G}_N$, $\vec{H}_N$, F_N are obtained by multiplying the finite rank projection $P_N := \mathbf{1}[H_{el} \leqslant N]$ from the left and the right. Moreover, an additional ultraviolet cut-off $\mathbf{1}[\alpha \leqslant N]$, considered as a spectral projection, is added. The regularized form factors are

$$
\begin{aligned}
\vec{G}_N(k) \;:=\; \mathbf{1}[\alpha \,\leqslant\, N]\,P_N\,\vec{G}(k)\,P_N, \qquad \vec{H}_N(k) \;:=\; \mathbf{1}[\alpha \,\leqslant\, N]\,P_N\,\vec{H}(k)\,P_N,\\
F_N(k) \;:=\; \mathbf{1}[\alpha \,\leqslant\, N]\,P_N\,F(k)\,P_N.
\end{aligned}
$$

Lemma 5.1. *i)* Q_N *is essentially self-adjoint on* $\mathcal{D} \subset \mathrm{dom}(Q_N)$. Q_N *is affiliated with* $\mathfrak{M}_\beta$*, i.e,.* Q_N *is closed and*

$$X' \, Q_N \subset Q_N \, X', \quad \forall \, X' \in \mathfrak{M}'_\beta.$$

ii) $\mathcal{L}_0 + Q_N$*,* $\mathcal{L}_0 - \mathcal{J} Q_N \mathcal{J}$ *and* $\mathcal{L}_0 + Q_N - \mathcal{J} Q_N \mathcal{J}$ *converges in the strong resolvent sense to* $\mathcal{L}_0 + Q$*,* $\mathcal{L}_0 - \mathcal{J} Q \mathcal{J}$ *and* $\mathcal{L}_0 + Q - \mathcal{J} Q \mathcal{J}$*, respectively.*

Proof. Let Q_N be defined on $\mathcal{D}$. With the same arguments as in the proof of Theorem 4.2 we obtain

$$\|Q_N \phi\| \leqslant C \|\mathcal{L}_{f,a}\phi\|, \quad \big| \big\langle \, Q_N \phi \, \big| \, \mathcal{L}_{f,a}\phi \big\rangle \, - \, \big\langle \, \mathcal{L}_{f,a}\phi \, \big| \, Q_N \phi \big\rangle \big| \, \leqslant \, C \big\| (\mathcal{L}_{f,a})^{1/2} \phi \big\|^2,$$

for $\phi \in \mathcal{D}$ and some constant $C > 0$, where we have used that $\|F_N\|_w < \infty$. Thus, from Theorem 4.2 and Nelson's commutator theorem we obtain that $\mathcal{D}$ is a common core for Q_N, $\mathcal{L}_0 + Q_N$, $\mathcal{L}_0 - Q_N^{\mathcal{J}}$, $\mathcal{L}_0 + Q_N - Q_N^{\mathcal{J}}$ and for the operators in line (21). A straightforward calculation yields

$$\lim_{N \to \infty} Q_N \phi \, = \, Q\phi, \quad \lim_{N \to \infty} \mathcal{J} Q_N \mathcal{J} \phi \, = \, \mathcal{J} Q \mathcal{J} \phi \qquad \forall \, \phi \in \mathcal{D}.$$

Thus statement ii) follows, since it suffices to check strong convergence on the common core $\mathcal{D}$, see [16, Theorem VIII.25 a)].
Let $N_f := d\Gamma(1) \otimes \mathbf{1} + \mathbf{1} \otimes d\Gamma(\mathbf{1})$ be the number-operator. Since $\mathrm{dom}(N_f) \supset \mathcal{D}$ and $\mathcal{W}_\beta(f)^{\mathcal{J}} : \mathrm{dom}(N_f) \to \mathrm{dom}(N_f)$, see [4], we obtain

$$Q_N (A \otimes \mathbf{1} \otimes \mathcal{W}_\beta(f))^{\mathcal{J}} \phi \, = \, (A \otimes \mathbf{1} \otimes \mathcal{W}_\beta(f))^{\mathcal{J}} Q_N \phi \tag{26}$$

for $A \in \mathcal{B}(\mathcal{H}_{el})$, $f \in \mathfrak{f}$ and $\phi \in \mathcal{D}$. By closedness of Q_N and density arguments the equality holds for $\phi \in \mathrm{dom}(Q_N)$ and $X \in \mathfrak{M}_\beta$ instead of $A \otimes \mathbf{1} \otimes \mathcal{W}_\beta(f)$. Thus Q_N is affiliated with $\mathfrak{M}_\beta$ and therefore $e^{i\,tQ_N} \in \mathfrak{M}_\beta$ for $t \in \mathbb{R}$.

□

Lemma 5.2. *We have for* $X \in \mathfrak{M}_\beta$ *and* $t \in \mathbb{R}$

$$\tau_t(X) = e^{it(\mathcal{L}_0 + Q)} \, X \, e^{it(\mathcal{L}_0 + Q)}, \quad \tau_t^0(X) = e^{it(\mathcal{L}_0 - Q^{\mathcal{J}})} \, X \, e^{it(\mathcal{L}_0 - Q^{\mathcal{J}})} \tag{27}$$

Moreover, $\tau_t(X) \in \mathfrak{M}_\beta$ *for all* $X \in \mathfrak{M}_\beta$ *and* $t \in \mathbb{R}$*, such as*

$$E_Q(t) \, := \, e^{i\,t\,(\mathcal{L}_0\,+\,Q)} \, e^{-i\,t\,\mathcal{L}_0} = e^{i\,t\,\mathcal{L}_Q} \, e^{-i\,t\,(\mathcal{L}_0 - Q^{\mathcal{J}})} \in \mathfrak{M}_\beta.$$

Proof. First, we prove the statement for Q_N, since Q_N is affiliated with $\mathfrak{M}_\beta$ and therefore $e^{itQ_N} \in \mathfrak{M}_\beta$. We set

$$\hat{\tau}_t^N(X) = e^{i\,t\,(\mathcal{L}_0\,+\,Q_N)} \, X \, e^{-i\,t\,(\mathcal{L}_0\,+\,Q_N)}, \quad \hat{\tau}_t(X) = e^{i\,t\,(\mathcal{L}_0\,+\,Q)} \, X \, e^{-i\,t\,(\mathcal{L}_0\,+\,Q)} \tag{28}$$

On account of Lemma 5.1 and Theorem 4.2 we can apply the Trotter product formula to obtain

$$\begin{aligned}\hat{\tau}_t^N(X) &= \text{w-lim}_{n\to\infty} \big(e^{i\frac{t}{n}\mathcal{L}_0}\, e^{i\frac{t}{n}Q_N}\big)^n X \big(e^{-i\frac{t}{n}Q_N}\, e^{-i\frac{t}{n}\mathcal{L}_0}\big)^n \\ &= \text{w-lim}_{n\to\infty}\, \tau^0_{\frac{t}{n}}\big(e^{i\frac{t}{n}Q_N}\cdots \tau^0_{\frac{t}{n}}(e^{i\frac{t}{n}Q_N}\, X\, e^{-i\frac{t}{n}Q_N})\cdots e^{-i\frac{t}{n}Q_N}\big).\end{aligned}$$

Since $e^{i\frac{t}{n}Q_N}, X \in \mathfrak{M}_\beta$ and since τ^0 leaves $\mathfrak{M}_\beta$ invariant, $\hat{\tau}_t^N(X)$ is the weak limit of elements of $\mathfrak{M}_\beta$, and hence $\hat{\tau}_t^N(X) \in \mathfrak{M}_\beta$. Moreover,

$$\hat{\tau}_t(X) \;=\; \text{w-lim}_{N\to\infty}\, \hat{\tau}_t^N(X) \in \mathfrak{M}_\beta.$$

For $E_N(t) := e^{i\,t\,(\mathcal{L}_0+Q_N)} e^{-i\,t\,\mathcal{L}_0} \in \mathcal{B}(\mathcal{K})$ we obtain

$$\begin{aligned}e^{it(\mathcal{L}_0+Q_N)} e^{-i\,t\,\mathcal{L}_0} &= \text{s-lim}_{n\to\infty} \big(e^{i\frac{t}{n}\mathcal{L}_0} e^{i\frac{t}{n}Q_N}\big)^n e^{-i\,t\,\mathcal{L}_0} \\ &= \text{s-lim}_{n\to\infty}\, \tau^0_{\frac{t}{n}}(e^{i\frac{t}{n}Q_N})\tau^0_{\frac{2t}{n}}(e^{i\frac{t}{n}Q_N})\cdots \tau^0_{\frac{nt}{n}}(e^{i\frac{t}{n}Q_N}) \in \mathfrak{M}_\beta.\end{aligned}$$

By virtue of Lemma 5.1 we get $E_Q(t) := e^{i\,t\,(\mathcal{L}_0+Q)} e^{-i\,t\,\mathcal{L}_0} =$ w-lim$_{N\to\infty} E_N(t) \in \mathfrak{M}_\beta$. Since $\mathcal{J}$ leaves $\mathcal{D}$ invariant and thanks to Lemma 5.1, we deduce, that $\mathcal{D}$ is a core of $\mathcal{J}Q_N\mathcal{J}$. Moreover, we have $e^{-itQ_N^{\mathcal{J}}} = \mathcal{J}e^{itQ_N}\mathcal{J} \in \mathfrak{M}'_\beta$. Since we have shown, that $\hat{\tau}^N$ leaves $\mathfrak{M}_\beta$ invariant, we get

$$\begin{aligned}\tau_t^N(X) &= \text{w-lim}_{n\to\infty}(e^{i\frac{t}{n}(\mathcal{L}_0+Q_N)}e^{i\frac{t}{n}(-Q_N^{\mathcal{J}})})^n X\,(e^{-i\frac{t}{n}(-Q_N^{\mathcal{J}})}\, e^{-i\frac{t}{n}(\mathcal{L}_0+Q_N)})^n \\ &= \text{w-lim}_{n\to\infty}\, \hat{\tau}^N_{\frac{t}{n}}\big(e^{-i\frac{t}{n}Q_N^{\mathcal{J}}}\cdots \hat{\tau}^N_{\frac{t}{n}}(e^{-i\frac{t}{n}Q_N^{\mathcal{J}}}\, X\, e^{i\frac{t}{n}Q_N^{\mathcal{J}}})\cdots e^{i\frac{t}{n}Q_N^{\mathcal{J}}}\big) \\ &= \hat{\tau}_t^N(X).\end{aligned}$$

Thanks to Lemma 5.1 we also have

$$\tau_t(X) = \text{w-lim}_{n\to\infty}\, \tau_t^N(X) = \text{w-lim}_{\mathbb{N}\to\infty}\, \hat{\tau}_t^N(X) = \hat{\tau}_t(X). \tag{29}$$

The proof of $\tau_t^0(X) = e^{it(\mathcal{L}_0 - Q^{\mathcal{J}})}\, X\, e^{it(\mathcal{L}_0 - Q^{\mathcal{J}})}$ follows analogously. Using the Trotter product formula we obtain

$$\begin{aligned}e^{it(\mathcal{L}_0+Q_N)}\, e^{-it\mathcal{L}_0} &= \text{s-lim}_{n\to\infty} \big(e^{i\frac{t}{n}\mathcal{L}_0} e^{i\frac{t}{n}Q_N}\big)^n e^{-it\mathcal{L}_0} \\ &= \text{s-lim}_{n\to\infty}\, \tau^0_{\frac{t}{n}}(e^{i\frac{t}{n}Q_N})\tau^0_{\frac{2t}{n}}(e^{i\frac{t}{n}Q_N})\cdots \tau^0_{\frac{nt}{n}}(e^{i\frac{t}{n}Q_N}) \\ &= \text{s-lim}_{n\to\infty} \big(e^{i\frac{t}{n}(\mathcal{L}_0 - Q_N{}^{\mathcal{J}})} e^{i\frac{t}{n}Q_N}\big)^n e^{-it(\mathcal{L}_0 - Q_N{}^{\mathcal{J}})} \\ &= e^{it(\mathcal{L}_0+Q_N-\mathcal{J}Q_N\mathcal{J})}\, e^{-it(\mathcal{L}_0 - Q_N{}^{\mathcal{J}})}.\end{aligned}$$

By strong resolvent convergence we may deduce $E(t) = e^{it\mathcal{L}_Q}\, e^{-it(\mathcal{L}_0 - Q^{\mathcal{J}})}$. □

Let $\mathcal{C}$ be the natural positive cone associated with $\mathcal{J}$ and Ω_0^β and let $\mathfrak{M}_\beta^{ana}$ be the τ-analytic elements of $\mathfrak{M}_\beta$, (see [4]).

Theorem 5.3. *Assume Hypothesis 1 and* $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0 + Q)})$. *Let* $\Omega^\beta := e^{-\beta/2\,(\mathcal{L}_0 + Q)}\,\Omega_0^\beta$. *Then*

$$\begin{aligned} \mathcal{J}\,\Omega^\beta &= \Omega^\beta, \qquad \Omega^\beta = e^{\beta/2\,(\mathcal{L}_0 - Q^{\mathcal{J}})}\,\Omega_0^\beta, \qquad (30)\\ \mathcal{L}_Q\,\Omega^\beta &= 0, \qquad \mathcal{J}\,X^*\,\Omega^\beta = e^{-\beta/2\,\mathcal{L}_Q}X\,\Omega^\beta, \quad \forall X \in \mathfrak{M}_\beta \end{aligned}$$

Furthermore, Ω^β *is separating and cyclic for* $\mathfrak{M}_\beta$*, and* $\Omega^\beta \in \mathcal{C}$*. The state* ω^β *is defined by*

$$\omega^\beta(X) := \|\Omega^\beta\|^{-2}\,\langle \Omega^\beta\,|X\,\Omega^\beta\rangle, \; X \in \mathfrak{M}_\beta$$

is a (τ, β)*-KMS state on* $\mathfrak{M}_\beta$*.*

Proof. First, we define $\Omega(z) = e^{-z\,(\mathcal{L}_0+Q)}\,\Omega_0^\beta$ for $z \in \mathbb{C}$ with $0 \leqslant \mathrm{Re}\, z \leqslant \beta/2$. Since $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0+Q)})$, $\Omega(z)$ is analytic on $\mathcal{S}_{\beta/2} := \{z \in \mathbb{C} : 0 < \mathrm{Re}\,(z) < \alpha\}$ and continuous on the closure of $\mathcal{S}_{\beta/2}$, see Lemma A.2 below.

► Proof of $\mathcal{J}\,\Omega(\beta/2) = \Omega(\beta/2)$:
We pick $\phi \in \bigcup_{n\in\mathbb{N}} \mathrm{ran}\,\mathbf{1}[|\mathcal{L}_0| \leqslant n]$. Let $f(z) := \langle\, \phi\,|\,\mathcal{J}\,\Omega(\overline{z})\,\rangle$ and $g(z) := \langle\, e^{-(\beta/2\,-\,\overline{z})\,\mathcal{L}_0}\,\phi\,|\,e^{-z\,(\mathcal{L}_0\,+\,Q)}\,\Omega_0^\beta\,\rangle$. Both f and g are analytic on $\mathcal{S}_{\beta/2}$ and continuous on its closure. Thanks to Lemma 5.2 we have $E_Q(t) \in \mathfrak{M}_\beta$, and hence

$$f(it) = \langle\, \phi\,|\,\mathcal{J}\,E_Q(t)\,\Omega_0^\beta\,\rangle = \langle\, \phi\,|\,e^{-\beta/2\,\mathcal{L}_0}\,E_Q(t)^*\,\Omega_0^\beta\,\rangle = g(i\,t), \; t \in \mathbb{R}.$$

By Lemma A.1, f and g are equal, in particular in $z = \beta/2$. Note that ϕ is any element of a dense subspace.

► Proof of $\Omega_0^\beta \in \mathrm{dom}(e^{\beta/2\,(\mathcal{L}_0 - Q^{\mathcal{J}})})$ and $\Omega(\beta/2) = e^{\beta/2\,(\mathcal{L}_0 - Q^{\mathcal{J}})}\,\Omega_0^\beta$:
Let $\phi \in \bigcup_{n\in\mathbb{N}} \mathrm{ran}\,\mathbf{1}[|\mathcal{L}_0 - Q^{\mathcal{J}}| \leqslant n]$. We set $g(z) := \langle\, e^{\overline{z}(\mathcal{L}_0\,-Q^{\mathcal{J}})}\,\phi\,|\,e^{-z\,\mathcal{L}_0}\,\Omega_0^\beta\,\rangle$. Since $E_Q(t)^{\mathcal{J}} = e^{i\,t\,(\mathcal{L}_0 - Q^{\mathcal{J}})}\,e^{-i\,t\,\mathcal{L}_0}$, g coincides for $z = i\,t$ with $f(z) := \langle\, \phi\,|\,\mathcal{J}\,\Omega(\overline{z})\,\rangle$. Hence they are equal in $z = \beta/2$. The rest follows since $e^{\beta/2\,(\mathcal{L}_0 - Q^{\mathcal{J}})}$ is self-adjoint.

► Proof of $\mathcal{L}_Q\,\Omega(\beta/2) = 0$:
Choose $\phi \in \bigcup_{n\in\mathbb{N}} \mathrm{ran}\,\mathbf{1}[|\mathcal{L}_Q| \leqslant n]$. We define $g(z) := \langle\, e^{-\overline{z}\mathcal{L}_Q}\phi\,|\,e^{z\,(\mathcal{L}_0 - Q^{\mathcal{J}})}\,\Omega_0^\beta\,\rangle$ and $f(z) := \langle\, \phi\,|\,\Omega(z)\,\rangle$ for z in the closure of $\mathcal{S}_{\beta/2}$. Again both functions are equal on the line $z = i\,t$, $t \in \mathbb{R}$. Hence f and g are identical, and therefore $\Omega(\beta/2) \in \mathrm{dom}(e^{-\beta/2\,\mathcal{L}_Q})$ and $e^{-\beta/2\mathcal{L}_Q}\,\Omega(\beta/2) = \Omega(\beta/2)$. We conclude that $\mathcal{L}_Q\,\Omega(\beta/2) = 0$.

► Proof of $\mathcal{J}\,X^*\,\Omega(\beta/2) = e^{-\beta/2\mathcal{L}_Q}\,X\,\Omega(\beta/2), \; \forall X \in \mathfrak{M}_\beta$:
Fore $A \in \mathfrak{M}_\beta^{ana}$ we have, that

$$\begin{aligned} \mathcal{J}\,A^*\,\Omega(-it) &= \mathcal{J}\,A^*\,E_Q(t)\,\Omega_0^\beta = e^{-\beta/2\mathcal{L}_0}\,E_Q(t)^*A\Omega_0^\beta\\ &= e^{-(\beta/2\,-\,i\,t)\,\mathcal{L}_0}\,e^{-it(\mathcal{L}_0+Q)}A\,\Omega_0^\beta\\ &= e^{-(\beta/2-it)\mathcal{L}_0}\,\tau_{-t}(A)\,e^{-it(\mathcal{L}_0+Q)}\,\Omega_0^\beta. \end{aligned}$$

Let $\phi \in \bigcup_{n\in\mathbb{N}} \operatorname{ran} \mathbf{1}[|\mathcal{L}_0| \leqslant n]$. We define $f(z) = \langle \phi \,|\, \mathcal{J} A^* \,\Omega(\overline{z}) \rangle$ and $g(z) = \langle e^{-(\beta/2-\overline{z})\mathcal{L}_0} \phi \,|\, \tau_{iz}(A)\,\Omega(z) \rangle$. Since f and g are analytic and equal for $z = it$, we have $\mathcal{J} A^* \Omega(\beta/2) = \tau_{i\,\beta/2}(A)\,\Omega(\beta/2)$. To finish the proof we pick $\phi \in \bigcup_{n\in\mathbb{N}} \operatorname{ran} \mathbf{1}[|\mathcal{L}_Q| \leqslant n]$, and set $f(z) := \langle \phi \,|\, \tau_{iz}(A)\,\Omega(\beta/2) \rangle$ and $g(z) := \langle e^{-\overline{z}\mathcal{L}_Q} \phi \,|\, A\Omega(\beta/2) \rangle$. For $z = i\,t$ we see

$$g(it) = \langle \phi \,|\, e^{-it\mathcal{L}_Q} A\, e^{it\mathcal{L}_Q} \Omega(\beta/2) \rangle = \langle \phi \,|\, \tau_{-t}(A)\,\Omega(\beta/2) \rangle = f(i\,t).$$

Hence $A\,\Omega(\beta/2) \in \operatorname{dom}(e^{-\beta/2\,\mathcal{L}_Q})$ and $\mathcal{J} A^* \Omega(\beta/2) = e^{-\beta/2\mathcal{L}_Q} A\Omega(\beta/2)$.
Since $\mathfrak{M}_\beta^{ana}$ is dense in the strong topology, the equality holds for all $X \in \mathfrak{M}_\beta$.

► Proof, that Ω^β is separating for $\mathfrak{M}_\beta$:
Let $A \in \mathfrak{M}_\beta^{ana}$. We choose $\phi \in \bigcup_{n\in\mathbb{N}} \operatorname{ran} \mathbf{1}[|(\mathcal{L}_0 + Q)| \leqslant n]$. First, we have

$$\mathcal{J} A^* \,\Omega(\beta/2) = \tau_{i\beta/2}(A)\,\Omega(\beta/2).$$

Let $f_\phi(z) = \langle \phi | \tau_z(A)\Omega(\beta/2) \rangle$ and $g_\phi(z) = \langle e^{\overline{z}(\mathcal{L}_0+Q)} \phi \,|\, A e^{-(\beta/2+z)(\mathcal{L}_0+Q)} \Omega_0^\beta \rangle$ for $-\beta/2 \leqslant \operatorname{Re} z \leqslant 0$. Both functions are continuous and analytic if $-\beta/2 < \operatorname{Re} z < 0$. Furthermore, $f_\phi(i\,t) = g_\phi(i\,t)$ for $t \in \mathbb{R}$. Hence $f_\phi = g_\phi$ and for $z = -\beta/2$

$$\langle \phi \,|\, \mathcal{J} A^* \Omega(\beta/2) \rangle = \langle e^{-\beta/2\,(\mathcal{L}_0+Q)} \phi \,|\, A\Omega_0^\beta \rangle.$$

This equation extends to all $A \in \mathfrak{M}_\beta$, we obtain $A\,\Omega_0^\beta \in \operatorname{dom}(e^{-\beta/2\,(\mathcal{L}_0+Q)})$, such as $e^{-\beta/2(\mathcal{L}_0+Q)} A\,\Omega_0^\beta = \mathcal{J} A^* \Omega(\beta/2)$ for $A \in \mathfrak{M}_\beta$. Assume $A^*\,\Omega(\beta/2) = 0$, then
$e^{-\beta/2\,(\mathcal{L}_0+Q)} A\Omega_0^\beta = 0$ and hence $A\Omega_0^\beta = 0$. Since Ω_0^β is separating, it follows that $A = 0$ and therefore $A^* = 0$.

► Proof of $\Omega^\beta \in \mathcal{C}$, and that Ω^β is cyclic for $\mathfrak{M}_\beta$:
To prove that $\phi \in \mathcal{C}$ it is sufficient to check that $\langle \phi \,|\, A\mathcal{J}A\Omega_0^\beta \rangle \geqslant 0$ for all $A \in \mathfrak{M}_\beta$. We have

$$\begin{aligned}\langle \Omega(\beta/2) |\, A\mathcal{J}A\,\Omega_0^\beta \rangle &= \overline{\langle \mathcal{J} A^* \Omega(\beta/2) \,| A\Omega_0^\beta \rangle} \\ &= \overline{\langle e^{-\beta/2(\mathcal{L}_0+Q)} A\Omega_0^\beta \,|\, A\Omega_0^\beta \rangle} \geqslant 0.\end{aligned}$$

The proof follows, since every separating element of $\mathcal{C}$ is cyclic.
► Proof, that ω^β is a $(\tau,\,\beta)$-KMS state:
For $A,\,B \in \mathfrak{M}_\beta$ and $z \in S_\beta$ we define

$$F_\beta(A,\,B,\,z) = c\,\langle e^{-i\overline{z}/2\mathcal{L}_Q} A^*\Omega^\beta \,|\, e^{iz/2\mathcal{L}_Q} B\Omega^\beta \rangle,$$

where $c := \|\Omega^\beta\|^{-2}$. First, we observe

$$\begin{aligned}F_\beta(A,\,B,\,t) &= c\,\langle e^{-it/2\mathcal{L}_Q} A^*\Omega^\beta \,|\, e^{it/2\mathcal{L}_Q} B\Omega^\beta \rangle = c\,\langle \Omega^\beta \,| A\tau_t(B)\Omega^\beta \rangle \\ &= \omega^\beta(A\,\tau_t(B))\end{aligned}$$

and

$$\begin{aligned}\omega^\beta(\tau_t(B)A) &= c\,\langle\, \tau_t(B^*)\Omega^\beta \mid A\Omega^\beta\,\rangle \;=\; c\,\langle\, \mathcal{J}A\Omega^\beta \mid \mathcal{J}\tau_t(B^*)\Omega^\beta\,\rangle \\ &= c\,\langle\, e^{-\beta/2\mathcal{L}_Q}A^*\Omega^\beta \mid e^{-\beta/2\,\mathcal{L}_Q}\tau_t(B)\Omega^\beta\,\rangle \\ &= c\,\langle\, e^{-i\overline{(i\beta+t)}/2\mathcal{L}_Q}A^*\Omega^\beta \mid e^{i\,(i\beta+t)/2\mathcal{L}_Q}B\Omega^\beta\,\rangle \\ &= F_\beta(A,\, B, t+i\beta).\end{aligned}$$

The requirements on the analyticity of $F_\beta(A,\, B,\, \cdot)$ follow from Lemma A.2. $\square$

6 Proof of Theorem 1.3

For $\underline{s}_n := (s_n, \dots, s_1) \in \mathbb{R}^n$ we define

$$Q_N(\underline{s}_n) := Q_N(s_n)\cdots Q_N(s_1), \qquad Q_N(s) := e^{-s\mathcal{L}_0}Q_N e^{s\,\mathcal{L}_0},\ s\in\mathbb{R} \tag{31}$$

At this point, we check that $Q_N(\underline{s}_n)\Omega_0^\beta$ is well defined, and that it is an analytic vector of $\mathcal{L}_0$, see Equation (25). The goal of Theorem 1.3 is to give explicit conditions on H_{el} and W, which ensure $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0+Q)})$. Let

$$\underline{\eta}_1 := \int \big(\|\vec{G}(k)\|^2_{\mathcal{B}(\mathcal{H}_{el})} + \|\vec{H}(k)\|^2_{\mathcal{B}(\mathcal{H}_{el})}\big)(2+4\alpha(k)^{-1})\,dk \tag{32}$$

$$\underline{\eta}_2 := \int \big(\|F(k)\,H_{el,+}^{-\gamma}\|^2_{\mathcal{B}(\mathcal{H}_{el})} + \|F(k)^*\,H_{el,+}^{-\gamma}\|_{\mathcal{B}(\mathcal{H}_{el})}\big)(2+4\alpha(k)^{-1})\,dk$$

The idea of the proof is the following. First, we expand $e^{-\beta/2(\mathcal{L}_0+Q_N)}e^{\mathcal{L}_0}$ in a Dyson-series, i.e.,

$$\begin{aligned}&e^{-\beta/2(\mathcal{L}_0+Q_N)}e^{\mathcal{L}_0} \\ &\quad= \mathbf{1} + \sum_{n=1}^\infty(-1)^n\int_{\Delta^n_{\beta/2}} e^{-s_n\mathcal{L}_0}Q_N e^{s_n\,\mathcal{L}_0}\cdots e^{-s_1\mathcal{L}_0}Q_N e^{s_1\,\mathcal{L}_0}\,d\underline{s}_n.\end{aligned} \tag{33}$$

Under the assumptions of Theorem 1.3 we obtain an upper bound, uniform in N, for

$$\begin{aligned}&\langle\Omega_0^\beta \mid e^{-\beta(\mathcal{L}_0+Q_N)}\Omega_0^\beta\rangle \\ &\quad= 1 + \sum_{n=1}^\infty(-1)^n\int_{\Delta^n_\beta}\langle\Omega_0^\beta \mid e^{-s_n\mathcal{L}_0}Q_N e^{s_n\,\mathcal{L}_0}\cdots e^{-s_1\mathcal{L}_0}Q_N e^{s_1\,\mathcal{L}_0}\Omega_0^\beta\rangle\,d\underline{s}_n.\end{aligned} \tag{34}$$

This is proven in Lemma 6.4 below, which is the most important part of this section. In Lemma 6.1 and Lemma 6.2 we deduce from the upper bound for (34) an upper bound for $\|e^{-(\beta/2)(\mathcal{L}_0+Q_N)}\Omega_0^\beta\|$, which is uniform in N. The proof of Theorem 1.3 follows now from Lemma 6.3, where we show that $\Omega_0^\beta \in \mathrm{dom}(e^{-(\beta/2)(\mathcal{L}_0+Q)})$.

Lemma 6.1. *Assume*

$$\limsup_{n\to\infty} \sup_{0\leqslant x\leqslant \beta/2} \Big\| \int_{\Delta_x^n} Q_N(\underline{s}_n)\Omega_0^\beta \, d\underline{s}_n \Big\|^{1/n} < 1.$$

for all $N \in \mathbb{N}$. *Then* $\Omega_0^\beta \in \mathrm{dom}(e^{-x(\mathcal{L}_0+Q_N)})$, $0 < x \leqslant \beta/2$ *and*

$$e^{-x\,(\mathcal{L}_0+Q_N)}\Omega_0^\beta = \Omega_0^\beta + \sum_{n=1}^{\infty}(-1)^n \int_{\Delta_x^n} Q_N(\underline{s}_n)\Omega_0^\beta \, d\underline{s}_n. \tag{35}$$

In this context $\Delta_x^n = \{(s_1,\ldots,s_n)\in\mathbb{R}^n : 0 \leqslant s_n \leqslant \ldots \leqslant s_1 \leqslant x\}$ *is a simplex of dimension* n *and sidelength* x.

Proof. Let $\phi \in \mathrm{ran}\,\mathbf{1}[|\mathcal{L}_0 + Q_N| \leqslant k]$ and $0 \leqslant x \leqslant \beta/2$ be fixed. An m-fold application of the fundamental theorem of calculus yields

$$\begin{aligned} \langle\, e^{-x\,(\mathcal{L}_0\,+\,Q_N)}\,\phi \,|\, e^{x\,\mathcal{L}_0}\,\Omega_0^\beta \,\rangle &= \Big\langle\, \phi \,|\, \Omega_0^\beta + \sum_{n=1}^{m}(-1)^n \int_{\Delta_x^n} Q_N(\underline{s}_n)\,\Omega_0^\beta\, d\underline{s}_n \Big\rangle \\ &+(-1)^{m+1}\int_{\Delta_x^{m+1}} \langle\, e^{-s_{m+1}\,(\mathcal{L}_0\,+\,Q_N)}\phi \,|\, e^{s_{m+1}\,\mathcal{L}_0}\, Q_N(\underline{s}_{m+1})\,\Omega_0^\beta \,\rangle\, d\underline{s}_{m+1} \end{aligned} \tag{36}$$

Since $\mathcal{L}_0\,\Omega_0^\beta = 0$ we have for $r(\underline{s}_{m+1}) := (s_m - s_{m+1},\ldots, s_1 - s_{m+1})$ that

$$e^{s_{m+1}\mathcal{L}_0}\, Q_N(\underline{s}_{m+1})\Omega_0^\beta = Q_N\, Q_N(r(\underline{s}_{m+1}))\Omega_0^\beta,$$

We turn now to the second expression on the right side of Equation (36), after a linear transformation depending on s_{m+1} we get

$$(-1)^{m+1}\int_0^x \Big\langle e^{-s_{m+1}(\mathcal{L}_0+Q_N)}\phi \,|\, Q_N \int_{\Delta^m_{x-s_{m+1}}} Q_N(\underline{r}_m)\Omega_0^\beta\, d\underline{r}_m \Big\rangle\, ds_{m+1}.$$

Since $\|e^{-s_{m+1}\,(\mathcal{L}_0\,+\,Q_N)}\,\phi\| \leqslant e^{\beta/2\,k}\,\|\phi\|$, and using that $Q_N(\underline{r}_m)\,\Omega_0^\beta$ is a state with at most $2m$ bosons, we obtain the upper bound

$$\mathrm{const}\,\|\phi\|\,\sqrt{(2m)\,(2m+1)}\,\sup_{0\leqslant x\leqslant\beta/2}\Big\|\int_{\Delta^m_{x-s_{m+1}}} Q_N(\underline{r}_m)\,\Omega_0^\beta\, d\underline{r}_m\Big\|.$$

Hence, for $m \to \infty$ we get

$$\langle\, e^{-x(\mathcal{L}_0+Q_N)}\phi \mid \Omega_0^\beta \,\rangle = \Big\langle\, \phi \,|\, \Omega_0^\beta + \sum_{n=1}^{\infty}(-1)^n \int_{\Delta_x^n} Q_N(\underline{s}_n)\,\Omega_0^\beta\, d\underline{s}_n \Big\rangle.$$

Since $\bigcup_{k=1}^\infty \mathrm{ran}\,\mathbf{1}[|\mathcal{L}_0 + Q_N| \leqslant k]$ is a core of $e^{-x(\mathcal{L}_0\,+\,Q_N)}$, the proof follows from the self-adjointness of $e^{-x(\mathcal{L}_0\,+\,Q_N)}$. ∎

Lemma 6.2. *Let $0 < x \leqslant \beta/2$. We have the identity*

$$\begin{aligned}&\int_{\Delta^n_{x/2}} \int_{\Delta^m_{x/2}} \langle\, Q_N(\underline{r}_m)\Omega_0^\beta \,|\, Q_N(\underline{s}_n)\Omega_0^\beta \,\rangle \, d\underline{r}_m \, d\underline{s}_n \\ &\quad = \int_{\Delta^{n+m}_\beta} \mathbf{1}[z_m \geqslant \beta - x \geqslant x \geqslant z_{m+1}] \,\big\langle\, \Omega_0^\beta \,\big|\, Q_N(\underline{z}_{n+m})\Omega_0^\beta \,\big\rangle \, d\underline{z}_{n+m}.\end{aligned} \tag{37}$$

For $m = n$ it follows

$$\Big\| \int_{\Delta^n_{x/2}} Q_N(\underline{s}_n)\Omega_0^\beta \, d\underline{s}_n \Big\|^2 \leqslant \int_{\Delta^{2n}_\beta} \big| \big\langle\, \Omega_0^\beta \,|\, Q_N(\underline{s}_{2n})\Omega_0^\beta \,\big\rangle \big| \, d\underline{s}_{2n}. \tag{38}$$

Proof. Recall Theorem 3.1 and Lemma 5.1. Since $\mathcal{J}$ is a conjugation we have $\langle\, \phi \,|\, \psi \,\rangle = \langle\, \mathcal{J}\,\psi \,|\, \mathcal{J}\,\phi \,\rangle$, and for every operator X, that is affiliated with $\mathfrak{M}_\beta$, we have $\mathcal{J}\, X\, \Omega_0^\beta = e^{-\beta/2\, \mathcal{L}_0}\, X^*\, \Omega_0^\beta$. Thus,

$$\begin{aligned}&\int_{\Delta^n_{x/2}} \int_{\Delta^m_{x/2}} \langle\, Q_N(\underline{r}_m)\Omega_0^\beta \,|\, Q_N(\underline{s}_n)\, \Omega_0^\beta \,\rangle \, d\underline{r}_m \, d\underline{s}_n \\ &\quad = \int_{\Delta^n_{x/2}} \int_{\Delta^m_{x/2}} \big\langle\, e^{-\beta/2\mathcal{L}_0}\, Q_N(\underline{s}_n)^*\, \Omega_0^\beta \,\big|\, e^{-\beta/2\mathcal{L}_0}\, Q_N(\underline{r}_m)^*\, \Omega_0^\beta \,\big\rangle \, d\underline{r}_m \, d\underline{s}_n\end{aligned} \tag{39}$$

Since $\mathcal{L}_0\, \Omega_0^\beta = 0$ we have

$$e^{-\beta \mathcal{L}_0}\, Q_N(\underline{r}_m)^*\, \Omega_0^\beta = Q_N(\beta - r_1) \cdots Q_N(\beta - r_m)\, \Omega_0^\beta.$$

Next, we introduce new variables for $\underline{r}$, namely $y_i := \beta - r_{m-i+1}$. Let $D^m_{x/2} := \{\underline{y}_m \in \mathbb{R}^m : \beta - x \leqslant y_m \leqslant \ldots \leqslant y_1 \leqslant \beta\}$. Thus the right side of Equation (39) equals

$$\begin{aligned}&\int_{\Delta^n_{x/2}} \int_{D^m_{x/2}} \big\langle\, \Omega_0^\beta \,\big|\, Q_N(\underline{s}_n)\, Q_N(\underline{y}_m)^*\, \Omega_0^\beta \,\big\rangle \, d\underline{s}_n \, d\underline{y}_m \\ &\quad = \int_{\Delta^{n+m}_\beta} \mathbf{1}[z_m \geqslant \beta - x \geqslant x \geqslant z_{m+1}] \,\big\langle\, \Omega_0^\beta \,\big|\, Q_N(\underline{z}_{n+m})\, \Omega_0^\beta \,\big\rangle \, d\underline{z}_{n+m}.\end{aligned}$$

The second statement of the Lemma follows by choosing $n = m$. ☐

Lemma 6.3. *Assume $\sup_{N \in \mathbb{N}} \| e^{-x(\mathcal{L}_0 + Q_N)} \Omega_0^\beta \| < \infty$ then $\Omega_0^\beta \in \mathrm{dom}(e^{-x(\mathcal{L}_0 + Q)})$ and*

$$\| e^{-x(\mathcal{L}_0 + Q)} \Omega_0^\beta \| \leqslant \sup_{N \in \mathbb{N}} \| e^{-x(\mathcal{L}_0 + Q_N)}\, \Omega_0^\beta \|$$

Proof. For $f \in \mathcal{C}_0^\infty(\mathbb{R})$ and $\phi \in \mathcal{K}$ we define $\psi_N := f(\mathcal{L}_0 + Q_N)\, \phi$. Obviously, for $g(r) = e^{-x\, r}\, f(r) \in \mathcal{C}_0^\infty(\mathbb{R})$ we have $e^{-x\,(\mathcal{L}_0 + Q_N)}\, \psi_N = g(\mathcal{L}_0 + Q_N)\, \phi$.

Since $\mathcal{L}_0 + Q_N$ tends to $\mathcal{L}_0 + Q$ in the strong resolvent sense as $N \to \infty$, we know from [16] that $\lim_{N\to\infty} \psi_N = f(\mathcal{L}_0 + Q)\,\phi =: \psi$ and

$$\lim_{N\to\infty} e^{-x\,(\mathcal{L}_0+Q_N)}\,\psi_N \;=\; \lim_{N\to\infty} g(\mathcal{L}_0+Q_N)\,\phi \;=\; g(\mathcal{L}_0+Q)\,\phi \;=\; e^{-x\,(\mathcal{L}_0+Q)}\,\psi.$$

Thus,

$$\begin{aligned} |\langle\, e^{-x(\mathcal{L}_0+Q)}\,\psi \,|\, \Omega_0^\beta \,\rangle| \;&=\; \lim_{N\to\infty} |\langle\, e^{-x(\mathcal{L}_0+Q_N)}\psi_N \,|\, \Omega_0^\beta \,\rangle| \\ &\leqslant\; \sup_{N\in\mathbb{N}} \|e^{-x\,(\mathcal{L}_0+Q_N)}\Omega_0^\beta\|\,\|\psi\|, \end{aligned}$$

Since $\{f(\mathcal{L}_0+Q)\,\phi \in \mathcal{K} \,:\, \phi\in\mathcal{K},\ f\in\mathcal{C}_0^\infty(\mathbb{R})\}$ is a core of $e^{-x(\mathcal{L}_0+Q)}$, we obtain $\Omega_0^\beta \in \mathrm{dom}(e^{-x(\mathcal{L}_0+Q)})$. ☐

Lemma 6.4. *For some $C>0$ we have*

$$\begin{aligned} &\int_{\Delta_\beta^n} \Big|\big\langle\, \Omega_0^\beta \,|\, Q_N(\underline{s}_n)\Omega_0^\beta \,\big\rangle\Big|\, d\underline{s}_n \\ &\qquad\leqslant \mathrm{const}\,(n+1)^2\,(1+\beta)^n \left(8\underline{\eta}_1 + \frac{(8C\underline{\eta}_2)^{1/2}}{(n+1)^{(1-2\gamma)/2}}\right)^n, \end{aligned}$$

where $\underline{\eta}_1$ and $\underline{\eta}_2$ are defined in (32).

Proof of 6.4. First recall the definition of Q_N and $Q_N(\underline{s}_n)$ in Equation (25) and Equation (31), respectively. Let

$$\int_{\Delta_\beta^n} \big|\big\langle\, \Omega_0^\beta \,|\, Q_N(\underline{s}_n)\,\Omega_0^\beta \,\big\rangle\big|\, d\underline{s}_n \;=:\; \int_{\Delta_1^n} \beta^n\, J_n(\beta,\underline{s})\, d\underline{s}_n,$$

The functions $J_n(\beta,\underline{s})$ clearly depends on N, but since we want to find an upper bound independent of N, we drop this index. Let $W_1 = \Phi(\vec{G})\,\Phi(\vec{H}) + \text{h.c.}$, $W_2 := \Phi(F)$ and $W := W_1 + W_2$. By definition of ω_0^β in (3.1), see also (13), we obtain

$$\begin{aligned} J_n(\beta,\,\underline{s}_n) \;&=\; \omega_0^\beta\Big(\big(e^{-\beta\,s_n\,H_0}\,W\,e^{\beta s_n H_0}\big)\cdots\big(e^{-\beta\,s_1\,H_0}\,W\,e^{\beta\,s_1\,H_0}\big)\Big) \\ &= (\mathcal{Z})^{-1} \sum_{\kappa\in\{1,\,2\}^n} \omega_f^\beta\Big(\,\mathrm{Tr}_{\mathcal{H}_{el}}\big\{e^{-\beta\,H_{el}}\big(e^{-\beta\,s_n\,H_0}\,W_{\kappa(n)}\,e^{\beta\,s_n\,H_0}\big)\cdots \\ &\qquad\qquad \cdots\big(e^{-\beta\,s_1\,H_0}\,W_{\kappa(1)}\,e^{\beta\,s_1\,H_0}\big)\big\}\Big) \end{aligned}$$

By definition of ω_f^β it suffices to consider expressions with an even number of field operators. In the next step we sum over all expression, where n_1 times W_1 occurs and $2n_2$ times W_2. The sum of n_1 and n_2 is denoted by m. For fixed n_1 and n_2 the remaining expressions are all expectations in ω_f^β of $2m$ field operators. In this case the expectations in ω_f^β can be expressed by an integral over $\mathbb{R}^{2m} \times \{\pm\}^{2m}$ with respect to ν, which is defined in Lemma A.4 below. To give a precise formula we define

$$M(m_1, m_2) = \{\kappa \in \{1, 2\}^n : \#\kappa^{-1}(\{i\}) = m_i, \quad i = 1, 2\}.$$

Thus we obtain

$$J_n(\beta, \underline{s}_n) = (\mathcal{Z})^{-1} \sum_{\substack{(n_1, n_2) \in \mathbb{N}^2 \\ n_1 + 2n_2 = n}} \sum_{\substack{\kappa \in M(n_1, 2n_2) \\ m := n_1 + n_2}} \int \nu(d\underline{k}_{2m} \otimes d\underline{\tau}_{2m}) \tag{40}$$
$$\mathrm{Tr}_{\mathcal{H}_{el}} \left\{ e^{-(\beta - \beta(s_1 - s_{2m}))H_{el}} I_{2m} e^{-\beta(s_{2m-1} - s_{2m}) H_{el}} \cdots e^{-\beta(s_1 - s_2) H_{el}} I_1 \right\},$$

Of course I_j depends on $\underline{k}_{2m} \times \underline{\tau}_{2m}$, namely for $\kappa(j) = 1, 2$ we have

$$I_j = \begin{cases} I_j(m, \tau, m', \tau'), & \kappa(j) = 1 \\ I_j(m, \tau), & \kappa(j) = 2, \end{cases}$$

where (m, τ), $(m', \tau') \in \{(k_j, \tau_j) : j = 1, \ldots, m\}$. For $\kappa(j) = 1$ we have that

$$\begin{aligned} I_j(m, +, m', -) &= \vec{G}^*(m)\, \vec{H}(m') + \vec{H}^*(m)\, \vec{G}(m') \\ I_j(m, -, m', +) &= \vec{G}(m)\, \vec{H}^*(m') + \vec{H}(m)\, \vec{G}^*(m') \\ I_j(m, +, m', +) &= \vec{G}^*(m)\, \vec{H}^*(m') + \vec{H}^*(m)\, \vec{G}^*(m') \\ I_j(m, -, m', -) &= \vec{G}(m)\, \vec{H}(m') + \vec{H}(m)\, \vec{G}(m') \end{aligned}$$

and for $\kappa(j) = 2$ we have that

$$\begin{aligned} I_j(m, +) &= F^*(m) \\ I_j(m, -) &= F(m). \end{aligned}$$

In the integral (40) we insert for (m, τ) and (m', τ') in the definition of I_j from left to right $k_{2m}, \tau_{2m}, \ldots, k_1, \tau_1$.
For fixed $(\underline{k}_{2m}, \underline{\tau}_{2m})$ the integrand of (40) is a trace of a product of $4m$ operators in $\mathcal{H}_{el}$. We will apply Hölder's-inequality for the trace, i.e.,

$$|\,\mathrm{Tr}_{\mathcal{H}_{el}}\{A_{2m}\, B_{2m} \cdots A_1\, B_1\}| \leqslant \prod_{j=1}^{2m} \|B_j\|_{\mathcal{B}(\mathcal{H}_{el})} \cdot \prod_{j=1}^{2m} \mathrm{Tr}_{\mathcal{H}_{el}}\{A_i^{p_j}\}^{p_j^{-1}}.$$

In our case $p_i := (s_{i-1} - s_i)^{-1}$ for $i = 2, \ldots, 2m$ and $p_1 := (1 - s_1 + s_{2m})^{-1}$ and

$$(A_j, B_j) := \begin{cases} \left(e^{-\beta\, p_j^{-1}\, H_{el}},\ I_j(m, \tau, m', \tau')\right), & \kappa(j) = 1 \\ \left(e^{-\beta\, p_j^{-1}\, H_{el}}\, H_{el,+}^{\gamma},\ H_{el,+}^{-\gamma}\, I_j(m, \tau,)\right), & \kappa(j) = 2 \end{cases}.$$

We define

$$\eta_1(k) \;=\; \max\big\{\|\vec{G}(k)\|_{\mathcal{B}(\mathcal{H}_{el})^r},\, \|\vec{H}(k)\|_{\mathcal{B}(\mathcal{H}_{el})^r}\big\}$$
$$\eta_2(k) \;=\; \max\big\{\|F(k)\, H_{el,+}^{-\gamma}\|_{\mathcal{B}(\mathcal{H}_{el})},\, \|F^*(k)\, H_{el,+}^{-\gamma}\|_{\mathcal{B}(\mathcal{H}_{el})}\big\}.$$

By definition of B_j we have

$$\|B_j\|_{\mathcal{B}(\mathcal{H}_{el})} \leqslant \begin{cases} \eta_1(m)\eta_1(m'), & \kappa(j)=1 \\ \eta_2(m), & \kappa(j)=2 \end{cases}. \tag{41}$$

Furthermore,

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}_{el}}\{A_i^{p_j}\}^{p_j^{-1}} &= \mathrm{Tr}_{\mathcal{H}_{el}}\big\{e^{-\beta H_{el}}\, H_{el,+}^{p_j\gamma}\big\}^{p_j^{-1}} \\ &\leqslant \|e^{-\epsilon H_{el}}\, H_{el,+}^{p_j\gamma}\|_{\mathcal{H}_{el}}^{p_j^{-1}}\, \mathrm{Tr}_{\mathcal{H}_{el}}\big\{e^{-(\beta-\epsilon)\,H_{el}}\big\}^{p_j^{-1}}, \quad k(j)=2\end{aligned}$$

Let $E_{gs} := \inf \sigma(H_{el})$. The spectral theorem for self-adjoint operators implies

$$\|e^{-\epsilon\, H_{el}}\, H_{el,+}^{p_i\,\gamma}\|_{\mathcal{H}_{el}}^{p_i^{-1}} \;\leqslant\; \sup_{r\,\geqslant\, E_{gs}} e^{-\epsilon\, p_i^{-1}\, r}(r\,-\,E_{gs}\,+1)^\gamma \;\leqslant\; \epsilon^{-\gamma}\, p_i^\gamma\, e^{-\epsilon\, p_i^{-1}\,(E_{gs}\,-\,1)}.$$

Inserting this estimates we get

$$\begin{aligned}&\mathrm{Tr}_{\mathcal{H}_{el}}\big\{e^{-(\beta\,-\,\beta(s_1\,-\,s_{2m}))H_{el}}I_{2m}e^{-\beta\,(s_{2m-1}\,-\,s_{2m})\,H_{el}}\cdots e^{-\beta\,(s_1\,-\,s_2)\,H_{el}}\,I_1\big\} \\ &\qquad\leqslant C_\kappa(\underline{s}_n)\prod_{j=1}^{2m}\|B_j\|_{\mathcal{B}(\mathcal{H}_{el})}\end{aligned}$$

where

$$C_\kappa(\underline{s}_n) := (1-\,s_1\,+\,s_n)^{-\alpha_1}\prod_{i=1}^{n-1}(s_i\,-\,s_{i+1})^{-\alpha_i} \tag{42}$$

and

$$\alpha_i = \begin{cases} 0, & \kappa(i)=1 \\ 1/2, & \kappa(i)=2 \end{cases} \tag{43}$$

Now, we recall the definition of ν. Roughly speaking, one picks a pair of variables (k_i,k_j) and integrates over $\delta_{k_i,k_j}\coth(\beta/2\alpha(k_i))\,dk_idk_j$. Subsequently one picks the next pair and so on. At the end one sums up all $\frac{(2m)!}{2^m\,m!}$ pairings and all 4^m combinations of $\underline{\tau}_{2m}$. Inserting Estimate (41) and that

$$\int \eta_\nu(k)\eta_{\nu'}(k)\coth(\beta/2\alpha(k))\,dk \leqslant (1+\beta^{-1})\underline{\eta}_\nu^{1/2}\,\underline{\eta}_{\nu'}^{1/2},$$

we obtain

$$|J_n(\beta,\,\underline{s})| \leqslant \frac{(1+\beta^{-1})^n}{\mathcal{Z}}\sum_{\substack{(n_1,\,n_2)\in\mathbb{N}_0^2 \\ n_1\,+\,2n_2\,=\,n}}\;\sum_{\substack{\kappa\in M(n_1,\,2n_2) \\ m\,:=\,n_1\,+\,n_2}}(\,\underline{\eta}_1)^{n_1}\,(C\underline{\eta}_2)^{n_2}\,\frac{(2m)!2^m}{m!}C_\kappa(\underline{s})$$

By Lemma A.3 below and since $(2m)!/(m!)^2 \leqslant 4^m$ we have

$$\int_{\Delta_\beta^n} \big| \langle \, \Omega_0^\beta \, | \, Q_N(\underline{s}_n) \, \Omega_0^\beta \, \rangle \big| \, d\underline{s}_n$$

$$\leqslant \text{const}(1+\beta)^n \sum_{\substack{(n_1,\, n_2) \in \mathbb{N}_0^2 \\ n_1 + 2n_2 = n}} \binom{n}{n_1} \frac{(8\underline{\eta}_1)^{n_1} \, (8C'\underline{\eta}_2)^{n_2}}{(n+1)^{(1-2\gamma)\, n_2 - 2}}$$

This completes the proof. □

7 The Harmonic Oscillator

Let $L^2(X, d\mu) = L^2(\mathbb{R})$ and $H_{el} =: H_{osc} := -\Delta_q + \Theta^2 q^2$ be the one dimensional harmonic oscillator and $\mathcal{H}_{ph} = L^2(\mathbb{R}^3)$. We define

$$H \,=\, H_{osc} \,+\, \Phi(F) \,+\, \check{H}, \qquad \check{H} \,:=\, d\Gamma(|k|), \tag{44}$$

where $\Phi(F) \,=\, q \,\cdot\, \Phi(f)$, with $\lambda\,(|k|^{-1/2} \,+\, |k|^{1/2})\, f \in L^2(\mathbb{R}^3)$.
H_{osc} is the harmonic oscillator, the form-factor F comes from the dipole approximation.
The Standard Liouvillean for this model is denoted by $\mathcal{L}_{osc}$. Now we prove Theorem 1.4.

Proof. We define the creation and annihilation operators for the electron.

$$A^* \,=\, \frac{\Theta^{1/2}\, q \,-\, i\, \Theta^{-1/2}\, p}{\sqrt{2}}, \quad A \,=\, \frac{\Theta^{1/2}\, q \,+\, i\, \Theta^{-1/2}\, p}{\sqrt{2}}, \quad p \,=\, -i\, \partial_x, \tag{45}$$

$$\Phi(c) \,=\, c_1\, q \,+\, c_2\, p, \quad \text{for } c \,=\, c_1 \,+\, i\, c_2 \in \mathbb{C},\; c_i \in \mathbb{R}. \tag{46}$$

These operators fulfill the CCR-relations and the harmonic- oscillator is the number-operator up to constants.

$$[A,\, A^*] \,=\, 1, \qquad [A^*,\, A^*] \,=\, [A,\, A] \,=\, 0, \qquad H_{osc} \,=\, \Theta A^*\, A \,+\, \Theta/2, \tag{47}$$

$$[H_{osc},\, A] \,=\, -\Theta\, A, \qquad [H_{osc},\, A^*] \,=\, \Theta\, A^*. \tag{48}$$

The vector $\Omega \,:=\, \big(\frac{\Theta}{\pi}\big)^{1/4}\, e^{-\Theta\, q^2\,/2}$ is called the vacuum vector. Note, that one can identify $\mathcal{F}_b[\mathbb{C}]$ with $L^2(\mathbb{R})$, since $\mathrm{LH}\{(A^*)^n\, \Omega \,|\, n \in \mathbb{N}^0\}$ is dense in $L^2(\mathbb{R})$. It follows, that ω_β^{osc} is quasi-free, as a state over $W(\mathbb{C})$ and

$$\omega_\beta^{osc}(W(c)) \,=\, (\mathcal{Z})^{-1}\, \mathrm{Tr}_{\mathcal{H}_{el}}\{e^{-\beta H_{el}}\, W(c)\} \,=\, \exp\big(-\,1/4\, \coth(\beta\, \Theta/2)\, |c|^2\big), \tag{49}$$

where $\mathcal{Z} \,=\, \mathrm{Tr}_{\mathcal{H}_{el}}\{e^{-\beta\, \mathcal{H}_{el}}\}$ is the partition function for $\mathcal{H}_{el}$.
First, we remark, that Equation (31) is defined for this model without regularization by $P_N \,:=\, \mathbf{1}[H_{el} \,\leqslant\, N]$. Moreover we obtain from Lemma 6.2,

that

$$\Big\| \int_{\Delta^n_{\beta/2}} Q(\underline{s}_n)\Omega_0^\beta \, d\underline{s}_{2n} \Big\|^2 \leqslant \int_{\Delta^{2n}_\beta} \big| \big\langle \, \Omega_0^\beta \, | \, Q(\underline{s}_{2n}) \, \Omega_0^\beta \, \big\rangle \big| \, d\underline{s}_{2n} \, =: \, h_{2n}(\beta, \, \lambda). \tag{50}$$

To show that $\Omega^\beta \in \operatorname{dom}(e^{-\beta/2\,(\mathcal{L}_0 + Q)})$ is suffices to prove, that $\sum_{n=0}^\infty h_{2n}(\beta, \, \lambda)^{1/2} \, < \, \infty$. We have

$$\begin{aligned} h_{2n}(\beta, \, \lambda) \, = \, \frac{(-\beta \, \lambda)^{2n}}{\mathcal{Z}} \int_{\Delta_1^{2n}} & \omega_\beta^{osc}\big(\big(e^{-\beta \, s_{2n} \, H_{el}} \, q \, e^{\beta \, s_{2n} \, H_{el}}\big) \\ & \cdots \big(e^{-\beta \, s_1 \, H_{el}} \, q \, e^{\beta \, s_1 \, H_{el}}\big)\big) \\ \cdot \omega_f^\beta\big((e^{-\beta \, s_{2n} \, \check{H}} \, \Phi(f) \, e^{\beta \, s_{2n} \, \check{H}}) & \cdots (e^{-\beta \, s_1 \, \check{H}} \, \Phi(f) \, e^{\beta \, s_1 \, \check{H}})\big) \, d\underline{s}_{2n}. \end{aligned} \tag{51}$$

Moreover, we have

$$\begin{aligned} e^{-\beta \, s_i \, H_{el}} \, q \, e^{\beta \, s_i \, H_{el}} &= (2\Theta)^{-1/2}\big(e^{-\beta \, \Theta \, s_i} \, A^* \, + \, e^{\beta \, \Theta \, s_i} \, A\big) \\ e^{-\beta \, s_i \, \check{H}} \, \Phi(f) \, e^{\beta \, s_i \, \check{H}} &= 2^{-1/2}\Big(a^*(e^{-\beta \, s_i \, |k|} \, f) + a(e^{\beta \, s_i \, |k|} \, f)\Big). \end{aligned} \tag{52}$$

Inserting the identities of Equation (52) in Equation (51) and applying Wick's theorem [5, p. 40] yields

$$\begin{aligned} h_{2n}(\beta, \, \lambda) = (\beta \, \lambda)^{2n} \int_{\Delta_1^{2n}} \sum_{P \in \mathcal{Z}_2} & \prod_{\{i, \, j\} \in P} K_{osc}(|s_i \, - \, s_j|, \, \beta) \\ & \cdot \sum_{P' \in \mathcal{Z}_2} \prod_{\{k, \, l\} \in P'} K_f(|s_k - s_l|, \, \beta) \, d\underline{s}_{2n} \\ = \frac{(\beta \, \lambda)^{2n}}{(2n)!} \int_{[0, \, 1]^{2n}} & \sum_{P, \, P' \in \mathcal{Z}_2} \prod_{\substack{\{i, \, j\} \in P \\ \{k, \, l\} \in P'}} K_{osc}(|s_i - s_j|, \, \beta) \, K_f(|s_k \, - \, s_l|, \beta) \, d\underline{s}_{2n}, \end{aligned} \tag{53}$$

where for $k \, < \, l$ and $i \, < \, j$, such as

$$\begin{aligned} K_f(|s_k \, - \, s_l|, \, \beta) \quad &:= \quad \omega_f^\beta((e^{-\beta \, s_k \, \check{H}} \, \Phi(f) \, e^{\beta \, s_k \, \check{H}}) \, (e^{-\beta \, s_l \, \check{H}} \, \Phi(f) \, e^{\beta \, s_l \, \check{H}})) \\ K_{osc}(|s_i \, - \, s_j|, \, \beta) \quad &:= \quad \omega_\beta^{osc}(e^{-\beta \, s_i \, H_{el}} \, q \, e^{\beta \, s_i \, H_{el}} \, e^{-\beta \, s_j \, H_{el}} \, q \, e^{\beta \, s_j \, H_{el}}). \end{aligned}$$

The last equality in (53) holds, since the integrand is invariant with respect to a change of the axis of coordinates.

We interpret two pairings P and $P' \in \mathcal{Z}_2$ as an indirected graph $G \, = \, G(P, \, P')$, where $M_{2n} \, = \, \{1, \ldots, \, 2n\}$ is the set of points. Any graph in G has two kinds of lines, namely lines in $L_{osc}(G)$, which belong to elements of P and lines in $L_f(G)$, which belong to elements of P'.

Let $\mathcal{G}(A)$ be the set of undirected graphs with points in $A \subset M_{2n}$, such that for each point "i" in A, there is exact one line in $L_f(G)$, which begins in "i", and

exact one line in $L_{osc}(G)$, which begins with "i". $\mathcal{G}_c(A)$ is the set of connected graphs. We do not distinguish, if points are connected by lines in $L_f(G)$ or by lines in $L_{osc}(G)$.
Let

$$\mathcal{P}_k := \Big\{P : P = \{A_1, \ldots, A_k\},\ \emptyset \neq A_i \subset M_{2n},\ A_i \cap A_j = \emptyset \text{ for } i \neq j,\ \bigcup_{i=1}^{k} A_i = M_{2n}\Big\}$$

be the family of decompositions of M_{2n} in k disjoint set. It follows

$$\begin{aligned} h_{2n}(\beta, \lambda) &= \frac{(\beta\,\lambda)^{2n}}{(2n)!} \sum_{G \in \mathcal{G}(M_{2n})} \int_{M_{2n}} \prod_{\substack{\{i,j\} \in L_{osc}(G) \\ \{k,l\} \in L_f(G)}} K_{osc}(|s_i - s_j|, \beta) \\ &\quad K_f(|s_k - s_l|, \beta)\, d\underline{s}_n \\ &= \frac{(\beta\lambda)^{2n}}{(2n)!} \sum_{k=1}^{2n} \sum_{\{A_1,\ldots,A_k\} \in \mathcal{P}_k} \sum_{\substack{(G_1,\ldots,G_k) \\ G_a \in \mathcal{G}_c(A_a)}} \prod_{a=1}^{k} J(G_a, A_a, \beta) \\ &= \frac{(\beta\lambda)^{2n}}{(2n)!} \sum_{k=1}^{2n} \frac{1}{k!} \sum_{\substack{A_1,\ldots,A_k \subset M_{2n}, \\ \{A_1,\ldots,A_k\} \in \mathcal{P}_k}} \sum_{\substack{(G_1,\ldots,G_k) \\ G_a \in \mathcal{G}_c(A_a)}} \prod_{a=1}^{k} J(G_a, A_a, \beta), \end{aligned} \tag{54}$$

where

$$J(G_a, A_a, \beta) := \int_{A_a} \prod_{\substack{\{i,j\} \in L_{osc}(G_a) \\ \{k,l\} \in L_f(G_a)}} K_{osc}(|s_i - s_j|, \beta) K_f(|s_k - s_l|, \beta)\, d\underline{s}. \tag{55}$$

$\int_{A_a} d\underline{s}$ means, $\int_{-1}^{1} ds_{j_1} \int_{-1}^{1} ds_{j_2} \ldots \int_{-1}^{1} ds_{j_m}$, where $A_a = \{j_1, \ldots, j_m\}$ and $\#A_a = m$.
From the first to the second line we summarize terms with graphs, having connected components containing the same set of points. From the second to the third line the order of the components is respected, hence the correction factor $\frac{1}{k!}$ is introduced. Due to Lemma 7.2 the integral depends only on the number of points in the connected graph, i. e. $J(G, A, \beta) = J(\#A, \beta)$. Moreover, Lemma 7.2 states that $\beta^{\#A} \cdot J(\#A, \beta) \leqslant (2\||k|^{-1/2} f\|_2 (\Theta\,\beta)^{-1})^{\#A} (C\,\beta + 1)$. To ensure that $\mathcal{G}_c(A_a)$ is not empty, $\#A_a$ must be even. For $(m_1, \ldots, m_k) \in \mathbb{N}^k$ with $m_1 + \cdots + m_k = n$ we obtain

$$\sum_{\substack{A_1,\ldots,A_k \subset M_{2n}, \#A_i = 2m_i \\ \{A_1,\ldots,A_k\} \in \mathcal{P}_k}} 1 = \frac{(2n)!}{(2m_1)! \cdots (2m_k)!}. \tag{56}$$

Let now be $A_a \subset M_{2n}$ with $\#A_a = 2m_a > 2$ fixed. In G_a are $\#A_a$ lines in $L_{osc}(G_a)$, since such lines have no points in common, we have $\frac{(2m_a)!}{m_a!\,2^{m_a}}$ choices. Let now be the lines in $L_{osc}(G_a)$ fixed. We have now $\big((2m_a-2)(2m_a-4)\cdots 1\big)$ choices for m_a lines in $L_f(G_a)$, which yield a connected graph. Thus

$$\sum_{G_a \in \mathcal{G}_c(A_a),} 1 = \frac{(2m_a)!}{m_a!\,2^{m_a}}\big((2m_a - 2)\,(2m_a - 4)\cdots 1\big) = \frac{(2m_a)!}{2m_a}. \tag{57}$$

For $\#A_a = 2$ exists only one connected graph. We obtain for h_{2n}

$$\begin{aligned} h_{2n}(\beta,\lambda) &= (\lambda)^{2n}\sum_{k=1}^{2n}\frac{1}{k!}\sum_{\substack{(m_1,\ldots,m_k)\in\mathbb{N}^k\\ m_1+\ldots+m_k=n}}\prod_{a=1}^{k}\frac{J(2m_a,\beta)(\beta^2)^{m_a}}{2m_a} \qquad (58)\\ &\leqslant (2\Theta^{-1}\,\||k|^{-1/2}f\|\,\lambda)^{2n}\sum_{k=1}^{2n}\frac{1}{k!}\sum_{\substack{(m_1,\ldots,m_k)\in\mathbb{N}^k\\ m_1+\ldots+m_k=n}}\prod_{a=1}^{k}\frac{(C\beta+1)}{2m_a}\\ &\leqslant (2\Theta^{-1}\,\||k|^{-1/2}f\|\,\lambda)^{2n}\sum_{k=1}^{2n}\frac{\big((C\beta+1)/2\sum_{m=1}^{n}\frac{1}{m}\big)^k}{k!}. \end{aligned}$$

Since the $\sum_{m=1}^{n}\frac{1}{m}$ can be considered as a lower Riemann sum for the integral $\int_1^{m+1} r^{-1}\,dr$, we have $\sum_{m=1}^{n}\frac{1}{m} \leqslant \ln(n+1)$. Thus,

$$\begin{aligned} h_{2n}(\beta,\lambda) &\leqslant (2\Theta^{-1}\,\||k|^{-1/2}f\|\,\lambda)^{2n}\sum_{k=1}^{2n}\frac{\big((C\beta+1)/2\,\ln(n+1)\big)^k}{k!} \qquad (59)\\ &\leqslant (2\Theta^{-1}\,\||k|^{-1/2}f\|\,\lambda)^{2n}(n+1)^{(C\beta+1)/2}. \end{aligned}$$

Since $2|\lambda|\,\||k|^{1/2}f\| < \Theta$ the series $\sum_{n=0}^{\infty} h_{2n}(\beta,\lambda)^{1/2}$ converges absolutely for all $\beta > 0$. It follows, that

$$e^{-\beta/2\,(\mathcal{L}_0+Q)}\,\Omega_0^\beta = \Omega_0^\beta + \sum_{n=1}^{\infty}\int_{\Delta^n_{\beta/2}} Q(\underline{s}_n)\,\Omega_0^\beta\,d\underline{s}_n$$

exists. □

Conversely, Equation (58) and Lemma 7.2 imply

$$h_{2n}(\beta,\lambda) \geqslant \lambda^{2n}\frac{J(2n,\beta)\,\beta^{2n}}{2n} = \frac{\Big(\Theta^{-1}\int\frac{\beta^2\,\lambda^2\,|f(k)|^2}{\sinh(|k|\,\beta/2)\sinh(\beta\,\Theta/2)}\,dk\Big)^n}{2n}. \tag{60}$$

Hence for every $\beta > 0$ exists a $\lambda \in \mathbb{R}$, such that $h_{2n}(\beta,\lambda) \geqslant \frac{1}{2n}$. Thus $\sum_{n=1}^{\infty} h_{2n}(\beta,\lambda)^{1/2} = \infty$

Remark 7.1. *We can therefore not extended Theorem 1.4 to an existence proof for all $\lambda > 0$.*

Lemma 7.2. *Following statements are true.*

$$\begin{aligned} J(G,\,A,\,\beta) &= J(\#A,\beta),\;\; G\in\mathcal{G}_c(A)\\ J(\#A,\,\beta) &\leqslant (2\||k|^{-1/2}\,f\|_2\,(\Theta\,\beta)^{-1})^{\#A}\cdot(C\,\beta\,+\,1)\\ J(\#A,\,\beta) &\geqslant \Big(\Theta^{-1}\int\frac{|f(k)|^2}{\sinh(|k|\,\beta/2)\sinh(\Theta\,\beta/2)}\,dk\Big)^{\#A/2}, \end{aligned}$$

where $\#A\,=\,2m$ *and* $C\,=\,(1/2)\,\frac{\|f\|^2}{\||k|^{1/2}\,f\|^2}$.

Proof of 7.2. A relabeling of the integration variables yields

$$\begin{aligned} J(G,\,A,\,\beta) \leqslant \overline{K}_f\int_{[0,1]^{2m}} & K_{osc}(|t_1-t_2|,\,\beta)\,K_f(|t_2\,-\,t_3|,\,\beta)\cdots\\ &\cdots K_{osc}(|t_{2m-1}\,-\,t_{2m}|,\,\beta)\,d\underline{t} \end{aligned}$$

for $\overline{K}_f\;:=\;\sup_{s\in[0,1]}K_f(s,\,\beta)$. We transform due to $s_i\;:=\;t_i\,-\,t_{i+1},\;i\;\leqslant\;2m-1$ and $s_{2m}\;=\;t_{2m}$, hence $-1\;\leqslant\;s_i\;\leqslant\;1,\;i\;=\;1,\ldots,\,2m$, since integrating a positive function we obtain

$$\begin{aligned} J(G,\,A,\,\beta) \leqslant & \Big(\int_{-1}^{1}K_{osc}(|s|,\,\beta)\,ds\Big)^m\Big(\int_{-1}^{1}K_f(|s|,\,\beta)\,ds\Big)^{m\,-\,1}\\ &\cdot\sup_{s\in[0,1]}K_f(s,\,\beta). \end{aligned}$$

We recall that

$$\int_{-1}^{1}K_{osc}(|s|,\,\beta)\,ds\;=\;(2\Theta)^{-1}\int_{-1}^{1}\frac{\cosh(\beta\,\Theta\,|s|\,-\,\Theta\,\beta/2)}{\sinh(\Theta\,\beta/2)}\,ds\;=\;2(\Theta^2\,\beta)^{-1}$$

and

$$\begin{aligned} \int_{-1}^{1}K_f(|s|,\,\beta)\,ds\;&=\;\int_{-1}^{1}\int\frac{\cosh(\beta\,|s|\,|k|-\beta|k|/2)\,|f(k)|^2}{2\sinh(\beta\,|k|/2)}\,dk\,ds\\ &=\;2\int\frac{|f(k)|^2}{\beta\,|k|}\,dk. \end{aligned}$$

Using $\coth(x)\leqslant 1+1/x$ and using convexity of cosh, we obtain

$$\sup_{s\in[0,\,1]}K_f(s,\,\beta)\leqslant(1/2)\int|f(k)|^2\,dk\,+\,\frac{1}{\beta}\int\frac{|f(k)|^2}{|k|}\,dk.$$

Due to the fact, that $t\,\mapsto\,K_f(t,\,\beta)$ and $t\,\mapsto\,K_{osc}(t,\,\beta)$ attain their minima at $t\,=\,1/2$, we obtain the lower bound for $J(\#A,\,\beta)$. ☐

Remark 7.3. *In the literature there is one criterion for* $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0+Q)})$*, to our knowledge, that can be applied in this situation [6]. One has to show that* $\|e^{-\beta/2\,Q}\,\Omega_0^\beta\| < \infty$*. If we consider the case, where the criterion holds for* $\pm\lambda$*, then the expansion in* λ *converges,*

$$\begin{aligned}\|e^{-\beta/2\,Q}\,\Omega_0^\beta\|^2 &= \sum_{n=0}^{\infty} \frac{(\lambda\,\beta)^{2n}}{(2n)!}\,\omega_{el}^\beta(q^{2n})\,\omega_f^\beta(\Phi(f)^{2n})\\ &= \sum_{n=0}^{\infty} \frac{(\lambda\,\beta)^{2n}}{(2n)!}\Big(\frac{(2n)!}{n!\,2^n}\Big)^2 K_{osc}(0,\,\beta)^n\,K_f(0,\,\beta)^n\\ &= \sum_{n=0}^{\infty}(\lambda\,\beta)^{2n}\,\Theta^{-n}\binom{2n}{n}2^{-2n}\Big(\coth(\Theta\,\beta/2)\int|f(k)|^2\,\coth(\beta\,|k|/2)\,dk\Big)^n\\ &\geqslant \sum_{n=0}^{\infty}(\lambda\,\beta)^{2n}(4\,\Theta)^{-n}\Big(\int|f(k)|^2\,dk\Big)^n.\end{aligned}$$

Obviously, for any value of $\lambda \neq 0$*, there is a* $\beta > 0$*, for which* $\|e^{-\beta/2\,Q}\,\Omega_0^\beta\| < \infty$ *is not fulfilled.*

Acknowledgments:
This paper is part of the author's PhD requirements. I am grateful to Volker Bach for many useful discussions and helpful advice. The main part of this work was done during the author's stay at the Institut for Mathematics at the University of Mainz. The work has been partially supported by the DFG (SFB/TR 12).

A

Lemma A.1. *Let* $f,\,g\,:\,\{z\in\mathbb{C}\,:\,0\leqslant Re(z)\leqslant\alpha\}\to\mathbb{C}$ *continuous and analytic in the interior. Moreover, assume that* $f(t) = g(t)$ *for* $t\in\mathbb{R}$*. Then* $f = g$*.*

Proof of A.1. Let $h\,:\,\{z\in\mathbb{C}\,:\,|\,\mathrm{Im}\,(z) < \alpha\}\to\mathbb{C}$ defined by

$$h(z) := \begin{cases} f(\,z\,) - g(\,z\,), & \text{on } \{z\in\mathbb{C}\,:\,0\leqslant\mathrm{Im}\,(z) < \alpha\}\\ \overline{f(\,\overline{z}\,)} - \overline{g(\,\overline{z}\,)}, & \text{on } \{z\in\mathbb{C}\,:\,-\alpha < \mathrm{Im}\,(z) < 0\}\end{cases} \tag{61}$$

Thanks to the Schwarz reflection principle h is analytic. Since $h(t) = 0$ for all $t\in\mathbb{R}$, we get $h = 0$. Hence $f = g$ on $\{z\in\mathbb{C}\,:\,0\leqslant\mathrm{Re}\,(z) < \alpha\}$. Since both f and g are continuous, we infer that $f = g$ on the whole domain. □

Lemma A.2. *Let* H *be some self-adjoint operator in* $\mathcal{H}$*,* $\alpha > 0$ *and* $\phi\in\mathrm{dom}(e^{\alpha\,H})$*. Then* $\phi\in\mathrm{dom}(e^{z\,H})$ *for* $z\in\{z\in\mathbb{C}\,:\,0\leqslant Re(z)\leqslant\alpha\}$*.* $z\mapsto e^{z\,H}\phi$ *is continuous on* $\{z\in\mathbb{C}\,:\,0\leqslant Re(z)\leqslant\alpha\}$ *and analytic in the interior.*

Proof of A.2. Due to the spectral calculus we have

$$\int e^{2\mathrm{Re}\ z\,s} d\langle \phi \,|\, \mathbb{E}_s\, \phi\rangle \leqslant \int (1 + e^{2\alpha\, s}) d\langle \phi \,|\, \mathbb{E}_s\, \phi\rangle =: C_1^2 < \infty.$$

Thus $\phi \in \mathrm{dom}(e^{z\,H})$. Let $\psi \in \mathcal{H}$ and $f(z) = \langle \psi \,|\, e^{z\,H}\,\phi\rangle$. There is a sequence $\{\psi_n\}$ with $\psi_n \in \bigcup_{m\in\mathbb{N}} \mathrm{ran}\ \mathbb{1}[|H| \leqslant m]$ and $\lim_{n\to\infty} \psi_n = \psi$. We set $f_n(z) = \langle \psi_n \,|\, e^{z\,H}\,\phi\rangle$. It is not hard to see that f_n is analytic, since ψ_n is an analytic vector for H, and that $|f_n(z)| \leqslant C_1 \|\psi_n\|$ and $\lim_{n\to\infty} f_n(z) = f(z)$. Thus f is analytic and hence $z \mapsto e^{z\,H}\phi$ is analytic. Thanks to the dominated convergence theorem the right side of

$$\|e^{z_n H}\phi - e^{zH}\phi\|^2 \leqslant \int (e^{2\mathrm{Re}\, z_n s} + e^{2\mathrm{Re}\, z s} - e^{\bar{z_n} s + zs} - e^{\bar{z} s + z_n s}) d\langle \phi \,|\, \mathbb{E}_s\, \phi\rangle \quad (62)$$

tends to zero for $\lim_{n\to\infty} z_n = z$. This implies the continuity of $z \mapsto e^{z\,H}\phi$. □

Lemma A.3. *We have for* $n_1 + n_2 \geqslant 1$

$$\int_{\Delta_1^n} C_\kappa(\underline{s})\, d\underline{s}_n \leqslant \frac{\mathrm{const}\ C^{n_2}}{(n_1+n_2)!\,(n+1)^{(1-2\gamma)\,n_2 - 2}} \quad (63)$$

Proof of A.3. We turn now to the integral

$$\int_{\Delta_1^n} C_\kappa(\underline{s})\, d\underline{s}_n = \int_{\Delta_1^n} (1 - s_1 + s_n)^{-\alpha_1} \prod_{i=1}^{n-1} (s_i - s_{i+1})^{-\alpha_i}\, d\underline{s}_n. \quad (64)$$

We define for $k = 1, \ldots, 2n$, a change of coordinates by $s_k = r_1 - \sum_{j=2}^{k} r_j$, the integral transforms to

$$\begin{aligned} &\int_{S^n} (1 - (r_2 + \cdots + r_n))^{-\alpha_1} \prod_{i=2}^{n} r_i^{-\alpha_i}\, d\underline{r}_n \quad (65) \\ &= \int_{T^{n-1}} (1 - (r_2 + \cdots + r_n))^{1-\alpha_1} \prod_{i=2}^{n} r_i^{-\alpha_i}\, d\underline{r}_{n-1} \\ &= \frac{\Gamma(1-\alpha_1)^{-1}\Gamma\big(1-\gamma\big)^{2n_2}}{\Gamma\big(n_1 + 2n_2\,(1-\gamma)\big)} \end{aligned}$$

where $S^{2n} := \{\underline{r} \in \mathbb{R}^{2n} : 0 \leqslant r_i \leqslant 1,\ r_2 + \cdots + r_{2n} \leqslant r_1\}$ and $T^{2n-1} := \{\underline{r} \in \mathbb{R}^{2n-1} : 0 \leqslant r_i \leqslant 1,\ r_2 + \cdots + r_{2n} \leqslant 1\}$. From the first to the second formula we integrate over dr_1. The last equality follows from [11, Formula 4.635 (4)], here Γ denotes the Gamma-function.
From Stirling's formula we obtain

$$(2\pi)^{1/2}\, x^{x-1/2}\, e^{-x} \leqslant \Gamma(x) \leqslant (2\pi)^{1/2}\, x^{x-1/2}\, e^{-x+1}, \quad x \geqslant 1. \quad (66)$$

Since $n_1 + n_2 \geqslant 1$ get

$$\frac{\Gamma(n_1+n_2+1)}{\Gamma(n_1+2(1-\gamma)\,n_2)} \leqslant (n+1)^2\Big(\frac{n_1+2(1-\gamma)n_2}{e}\Big)^{-(1-2\gamma)n_2}. \tag{67}$$

Note that $\Gamma(n_1+n_2+1) = (n_1+n_2)!$. □

Lemma A.4. *Let $(1+\alpha(k)^{-1/2})\,f_1,\ldots,(1+\alpha(k)^{-1/2})\,f_{2m} \in \mathcal{H}_{ph}$ and $\sigma \in \{+,-\}^{2m}$. Let $a^+ = a^*$ and $a^- = a$*

$$\begin{aligned}&\omega_f^\beta\big(a^{\sigma_{2m}}(e^{-\sigma_{2m}\,s_{2m}\,\alpha(k)}\,f_{2m})\cdots a^{\sigma_1}(e^{-\sigma_1\,s_1\,\alpha(k)}\,f_1)\big)\\ &\quad= \int f_{2m}^{\sigma_{2m}}(k_{2m},\,\tau_{2m})\cdots f_1^{\sigma_1}(k_1,\,\tau_1)\,\nu(d\underline{k}_{2m}\otimes d\underline{\tau}_{2m}),\end{aligned}$$

where $\nu(d\underline{k}_{2m}\otimes d\underline{\tau}_{2m})$ is a measure on $(\mathbb{R}^3)^{2m}\times\{+,\,-\}^{2m}$ for phonons, respectively on $(\mathbb{R}^3\times\{\pm\})^{2m}\times\{+,\,-\}^{2m}$ for photons, and

$$\nu(d\underline{k}_{2m}\otimes d\underline{\tau}_{2m}) \leqslant \sum_{P\in\mathcal{Z}_{2m}}\ \sum_{\underline{\tau}\in\{+,-\}^{2m}}\ \prod_{\{i>j\}\in P}\Big(\delta_{k_i,\,k_j}\coth(\beta\,\alpha(k_i)/2)\Big)\,d\underline{k}_{2m}. \tag{68}$$

for $f^+(k,\,\tau) := f(k)\,\mathbf{1}[\tau=+]$ and $f^+(k,\,\tau) := \overline{f(k)}\,\mathbf{1}[\tau=-]$.

Proof of A.4. Since ω_f^β is quasi-free, we obtain with $a^+ := a^*$ and $a^- := a$

$$\begin{aligned}&\omega_f^\beta\big(a^{\sigma_{2m}}(e^{-\sigma_{2m}\,s_{2m}\,\alpha(k)}\,f_{2m})\cdots a^{\sigma_1}(e^{-\sigma_1\,s_1\,\alpha(k)}\,f_1)\big)\\ &\quad= \sum_{P\in\mathcal{Z}_2}\ \prod_{\substack{\{i,\,j\}\in P\\ i>j}}\omega_f^\beta\big(a^{\sigma_i}(e^{-\sigma_i\,s_i\,\alpha(k)}\,f_i)\,a^{\sigma_j}(e^{-\sigma_j\,s_j\,\alpha(k)}\,f_j)\big),\end{aligned}$$

see Equation (12). For the expectation of the so called two point functions we obtain:

$$\omega_f^\beta\big(a^+(e^{s_i\,\alpha(k)}\,f_i)\,a^+(e^{s_j\,\alpha(k)}\,f_j)\big) = 0 = \omega_f^\beta\big(a(e^{-s_i\,\alpha(k)}\,f_i)\,a(e^{-s_j\,\alpha(k)}\,f_j)\big),$$

such as

$$\begin{aligned}\omega_f^\beta\big(a^+(e^{xs_i\alpha(k)}f_i)a^-(e^{-x\,s_j\alpha(k)}f_j)\big) &= \int f_i(k)\,\overline{f_j(k)}\,\frac{e^{x\,(s_i\,-\,s_j)\alpha(k)}}{e^{\beta\,\alpha(k)}-1}\,dk\\ \omega_f^\beta\big(a^-(e^{xs_i\alpha(k)}f_i)\,a^+(e^{-xs_j\alpha(k)}f_j)\big) &= \int f_j(k)\,\overline{f_i(k)}\,\frac{e^{(\beta+xs_j-xs_i)\alpha(k)}}{e^{\beta\alpha(k)}-1}\,dk\end{aligned}$$

Hence it follows

$$\begin{aligned}&\omega_f^\beta\big(a^{\sigma_{2m}}(e^{-\sigma_{2m}\,s_{2m}\,\alpha(k)}\,f_{2m})\cdots a^{\sigma_1}(e^{-\sigma_1\,s_1\,\alpha(k)}\,f_1)\big)\\ &\quad= \int f_{2m}^{\sigma_{2m}}(k_{2m},\,\tau_{2m})\cdots f_1^{\sigma_1}(k_1,\,\tau_1)\,\nu(d\underline{k}_{2m}\otimes d\underline{\tau}_{2m}),\end{aligned}$$

where $f^{+}(k,\,\tau) := f(k)\,\mathbf{1}[\tau = +]$ and $f^{-}(k,\,\tau) := \overline{f(k)}\,\mathbf{1}[\tau = -]$. $\nu(d^{3(2m)}k \otimes d^{2m}\tau)$ is a measure on $(\mathbb{R}^3)^{2m} \times \{+,\,-\}^{2m}$, which is defined by

$$\sum_{P \in \mathcal{Z}_{2m}} \sum_{\underline{\tau} \in \{+,-\}^{2m}} \prod_{\{i>j\} \in P} \delta_{\tau,\,-\tau}\, \delta_{k_i,\,k_j} \left(\delta_{\tau,\,+} \frac{e^{x\,(s_i - s_j)\,\alpha(k_i)}}{e^{\beta\,\alpha(k_i)} - 1} + \delta_{\tau,\,-} \frac{e^{(\beta - x\,(s_i - s_j))\,\alpha(k_i)}}{e^{\beta\,\alpha(k_i)} - 1} \right) d\underline{k}_{2m}. \tag{69}$$

□

References

[1] H. Araki, E. Woods. Representations of the Canonical Commutation Relations describing a non-relativistic infinite free Bose Gas. *J. Math. Phys.*, 4:637–662, 1963.

[2] V. Bach, J. Fröhlich, and I. M. Sigal. Quantum electrodynamics of confined non-relativistic particles. *Adv. in Math.* , 137:299–395, 1998.

[3] V. Bach, J. Fröhlich, I. M.Sigal. Return to Equilibrium. *J. Math. Phys.*, 41:3985–4060, 2000.

[4] O. Bratteli, D. Robinson. Operator Algebras and Quantum Statistical Mechanics 1, Text and Monographs in Physics, Springer-Verlag, 1987.

[5] O. Bratteli, D. Robinson. Operator Algebras and Quantum Statistical Mechanics 2, Text and Monographs in Physics, Springer-Verlag, 1996.

[6] Derezinski J., Jaksic V., Pillet C.-A. Perturbation theory of W*-dynamics, Liouvilleans and KMS-states. *Rev. Math. Phys*, 15: 447-489, 2003.

[7] J. Derezinski, V. Jaksic. Spectral theory of Pauli-Fierz operators. *J. Funct. Anal.*, 180 : 243–327, 2001

[8] J. Derezinski, V. Jaksic. Return to Equilibrium for Pauli-Fierz Operators. *Ann. Henri Poincare*, 4 : 739–793, 2003

[9] J. Fröhlich, M. Merkli. Another Return to Equilibrium *Commun. Math. Phys.*, 251: 235–262, 2004.

[10] J. Fröhlich, M. Merkli, I. M.Sigal. Ionization of Atoms in a Thermal Field. *Journal of Statistical Physics*, 116: 311–359, 2004. DOI 10.1023/B:JOSS.0000037226.16493.5e

[11] I.S. Gradstein, I.M. Ryzhik. Table of Integrals, Series, and Products. Academic Press, 1980, 4

[12] R. Haag, N. Hugenholz, M. Winnink. On the Equilibrium States in Quantum Statistical Mechanics. *Commun. Math. Phys.*, 5: 215–236, 1967.

[13] V. Jakšić, C. A. Pillet. On a Model for Quantum Friction. II: Fermi's Golden Rule and Dynamics at Positive Temperature. *Commun. Math. Phys.*, 176:619–643, 1996

[14] V. Jakšić, C. A. Pillet. On a Model for Quantum Friction III: Ergodic Properties of the Spin-Boson System. *Commun. Math. Phys.*, 178:627–651, 1996

[15] M. Merkli. Positive Commutators in Non-Equilibrium Statistical Quantum Mechanics, *Commun. Math. Phys.*, 223: 327–362, 2001.

[16] M. Reed,B. Simon. Methods of Modern Mathematical Physics: I. Functional Analysis Academic Press, 1980.

[17] M. Reed,B. Simon. Methods of Modern Mathematical Physics: II. Fourier Analysis and Self-Adjointness Academic Press, 1980.

Martin Könenberg
Fakultät für Mathematik
und Informatik
FernUniversität Hagen
Lützowstraße 125
D-58084 Hagen, Germany
martin.koenenberg@fernuni
-hagen.de

Divisorial Cohomology Vanishing on Toric Varieties

Markus Perling

Received: March 10, 2010
Revised: February 24, 2011

Communicated by Thomas Peternell

Abstract. This work discusses combinatorial and arithmetic aspects of cohomology vanishing for divisorial sheaves on toric varieties. We obtain a refined variant of the Kawamata-Viehweg theorem which is slightly stronger. Moreover, we prove a new vanishing theorem related to divisors whose inverse is nef and has small Iitaka dimension. Finally, we give a new criterion for divisorial sheaves for being maximal Cohen-Macaulay.

2010 Mathematics Subject Classification: 13C14, 14B15, 14M25
Keywords and Phrases: toric varieties, cohomology, local cohomology, Weil divisors, Maximal Cohen-Macaualy modules

Contents

1 Introduction

This work is motivated by numerical experiments [Per04] related to the conjecture of King [Kin97] concerning the derived category smooth complete toric varieties. These experiments led to the calculations of [HP06], where a counterexample to King's conjecture was given. Our goal is to develop a more systematic approach to the combinatorial and arithmetic aspects of cohomology vanishing for divisorial sheaves on toric varieties and to better understand from these points of view some phenomena related to this problem.

Based on work of Bondal (see [Rud90], [Bon90]), it was conjectured [Kin97] that on every smooth complete toric variety X there exists a full strongly exceptional collection of line bundles. That is, a collection of line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on X which generates $D^b(X)$ and has the property that $\operatorname{Ext}^k(\mathcal{L}_i, \mathcal{L}_j) = 0$ for all $k > 0$ and all i, j. Such a collection induces an equivalence of categories $\operatorname{RHom}(\bigoplus_i \mathcal{L}_i, \,.\,) : D^b(X) \longrightarrow D^b\big(\operatorname{End}(\bigoplus_i \mathcal{L}_i) - \operatorname{mod}\big)$. This possible generalization of Beilinson's theorem (pending the existence of a full strongly exceptional collection) has attracted much interest, notably also in the context of the homological mirror conjecture [Kon95]. For line bundles, the problem of Ext-vanishing can be reformulated to a problem of cohomology vanishing for line bundles by the isomorphisms

$$\operatorname{Ext}^k(\mathcal{L}_i, \mathcal{L}_j) \cong H^k(X, \mathcal{L}_i\check{} \otimes \mathcal{L}_j) = 0 \text{ for all } k \geq 0 \text{ and all } i, j.$$

So we are facing a quite peculiar cohomology vanishing problem: let n denote the rank of the Grothendieck group of X, then we look for a certain constellation of $n(n-1)$ – not necessarily distinct – line bundles, all of which have vanishing higher cohomology groups. The strongest general vanishing theorems so far are of the Kawamata-Viehweg type (see [Mus02] and [Fuj07], and also [Mat02] for Bott type formulas for cohomologies of line bundles), but it can be seen from very easy examples, such as Hirzebruch surfaces, that these alone in general do not suffice to prove or disprove the existence of strongly exceptional collections by means of cohomology vanishing. In [HP06], on a certain toric surface X, all line bundles $\mathcal{L}$ with the property that $H^i(X, \mathcal{L}) = H^i(X, \mathcal{L}\check{}) = 0$ for all $i > 0$ were completely classified by making use of an explicit toric representation of the cohomology vanishing problem for line bundles. This approach exhibits quite complicated combinatorial as well as number theoretic conditions for cohomology vanishing which we are going to describe in general.

We will consider and partially answer the following more general problem. Let D be a Weil divisor on any toric variety X and $V \subset X$ a torus invariant closed subscheme. Then what are necessary and sufficient conditions for the (global) local cohomology modules $H^i_V\big(X, \mathcal{O}_X(D)\big)$ to vanish? Given this spectrum of cohomology vanishing problems, we have at one extreme the cohomology vanishing problem for line bundles, and at the other extreme the classification problem for maximal Cohen Macaulay (MCM) modules over semigroup rings: on an affine toric variety X, the sheaf $\mathcal{O}_X(D)$ is MCM if and only if the local cohomologies $H^i_x\big(X, \mathcal{O}_X(D)\big)$ vanish for $i \neq \dim X$, where $x \in X$ is the torus fixed point. These local cohomologies have been studied by Stanley [Sta82], [Sta96] and Bruns and Gubeladze [BG03] showed that only finitely many sheaves in this class are MCM. MCM sheaves over affine toric varieties have only been classified for some special cases (see for instance [BGS87] and [Yos90]). Our contribution will be to give a more explicit combinatorial characterization of MCM modules of rank one over normal semigroup rings and their ties to the birational geometry of toric varieties.

One important aspect of our results is that, though we will also make use of $\mathbb{Q}$-divisors, our vanishing results will completely be formulated in the integral setting. We will illustrate the effect of this by the following example. Consider the weighted projective surface $\mathbb{P}(2,3,5)$. Then the divisor class group $A_1\big(\mathbb{P}(2,3,5)\big)$ is isomorphic to $\mathbb{Z}$ and, after fixing the generator $D = 1$ of $A_1\big(\mathbb{P}(2,3,5)\big)$ to be $\mathbb{Q}$-effective, the torus invariant irreducible divisors can be identified with the integers 2, 3, and 5, and the canonical divisor has class -10. By the toric Kawamata-Viehweg theorem we obtain that $H^2\big((\mathbb{P}(2,3,5), \mathcal{O}(kD)\big) = 0$ for $k > -10$. However, as we will explain in more detail below, the set of all divisors kD with nontrivial second cohomology is given by all k with $-k = 2r + 3s + 5t$ with r, s, t positive integers. So, Kawamata-Viehweg misses the divisor $-11D$. The reason is that the toric Kawamata-Viehweg vanishing theorem tells us that the cohomology of some divisor D' vanishes if the rational equivalence class over $\mathbb{Q}$ of $D' - K_{\mathbb{P}(2,3,5)}$ is contained in the interior of the nef cone in $A_1\big(\mathbb{P}(2,3,5)\big)_{\mathbb{Q}}$. Over the integers, the domain of cohomology vanishing thus in general is larger than over $\mathbb{Q}$. Below we will see that this is a general feature of cohomology vanishing, even for smooth toric varieties, as can be seen, for instance, by considering the strict transform of the divisor $-11D$ along some toric blow-up $X \longrightarrow \mathbb{P}(2,3,5)$ such that X is smooth.

The main results. The first main result will be an integral version of the Kawamata-Viehweg vanishing theorem. Consider the nef cone $\operatorname{nef}(X) \subset A_{d-1}(X)_{\mathbb{Q}}$, then the toric Kawamata-Viehweg vanishing theorem (see Theorem 3.29) can be interpreted such that if $D - K_X$ is contained in the interior of $\operatorname{nef}(X)$, then $H^i\big(X, \mathcal{O}_X(D)\big) = 0$ for all $i > 0$. For our version we will define a set $\mathfrak{A}_{\mathrm{nef}} \subset A_{d-1}(X)$, which we call the *arithmetic core* of $\operatorname{nef}(X)$ (see definition 4.11). The set $\mathfrak{A}_{\mathrm{nef}}$ has the property that it contains all integral Weil divisors which map to the interior of the cone $K_X + \operatorname{nef}(X)$ in $A_{d-1}(X)_{\mathbb{Q}}$. But in general it is strictly larger, as in the example above. We can lift the cohomology vanishing theorem for divisors in $\mathrm{nef}(X)$ to $\mathfrak{A}_{\mathrm{nef}}$:

Theorem (4.14): *Let X be a complete toric variety and $D \in \mathfrak{A}_{\mathrm{nef}}$. Then $H^i\big(X, \mathcal{O}_X(D)\big) = 0$ for all $i > 0$.*

One can consider Theorem 4.14 as an "augmentation" of the standard vanishing theorem for nef divisors to the subset $\mathfrak{A}_{\mathrm{nef}}$ of $A_{d-1}(X)$. In general, Theorem 4.14 is slightly stronger than the toric Kawamata-Viehweg vanishing theorem and yields refined arithmetic conditions.
However, the main goal of this paper is to find vanishing results which cannot directly be derived from known vanishing theorems. Let D be a nef Cartier divisor whose Iitaka dimension is positive but smaller than d. This class of divisors is contained in nonzero faces of the nef cone of X which are contained in the intersection of the nef cone with the boundary of the effective cone of X (see Section 4.3). Let F be such a face. Similarly as with $\mathfrak{A}_{\mathrm{nef}}$, we can define for the inverse cone $-F$ an arithmetic core $\mathfrak{A}_{-F}$ (see 4.11) and associate to it a vanishing theorem, which may be considered as the principal result of this article:

Theorem (4.17): *Let X be a complete d-dimensional toric variety. Then $H^i\big(X, \mathcal{O}(D)\big) = 0$ for every i and all D which are contained in some $\mathfrak{A}_{-F}$, where F is a face of $\mathrm{nef}(X)$ which contains nef divisors of Iitaka dimension $0 < \kappa(D) < d$. If $\mathfrak{A}_{-F}$ is nonempty, then it contains infinitely many divisor classes.*

This theorem cannot be an augmentation of a vanishing theorem for $-F$, as it is not true in general that $H^i\big(X, \mathcal{O}_X(-D)\big) = 0$ for all i for D nef of Iitaka dimension smaller than d. In particular, the set of $\mathbb{Q}$-equivalence classes of elements in $\mathfrak{A}_{-F}$ does not intersect $-F$.
For the case of a toric surface X we show that above vanishing theorems combine to a nearly complete vanishing theorem for X. Recall that in the fan associated to a complete toric surface X every pair of opposite rays by projection gives rise to a morphism from X to $\mathbb{P}^1$ (e.g. such a pair does always exist if X is smooth and $X \neq \mathbb{P}^2$). Correspondingly, we obtain a family of nef divisors of Iitaka dimension 1 on X given by the pullbacks of the sheaves $\mathcal{O}_{\mathbb{P}_1}(i)$ for $i > 0$. We get:

Theorem (4.21): *Let X be a complete toric surface. Then there are only finitely many divisors D with $H^i\big(X, \mathcal{O}_X(D)\big) = 0$ for all $i > 0$ which are not contained in $\mathfrak{A}_{\mathrm{nef}} \cup \bigcup_F \mathfrak{A}_{-F}$, where the union ranges over all faces of $\mathrm{nef}(X)$ which correspond to pairs of opposite rays in the fan associated to X.*

Some more precise numerical characterizations on the sets $\mathfrak{A}_{-F}$ will be given in subsection 4.3. The final result is a birational characterization of MCM-sheaves of rank one. This is a test case to see whether point of view of birational geometry might be useful for classifying more general MCM-sheaves. The idea for this comes from the investigation of MCM-sheaves over surface singularities in terms of resolutions in the context of the McKay correspondence (see [GSV83], [AV85], [EK85]). For an affine toric variety X, in general one cannot expect to find a similar nice correspondence. However, there is a set of preferred partial resolutions of singularities $\pi : \tilde{X} \longrightarrow X$ which is parameterized by the secondary fan of X. Our result is a toric analog of a technical criterion of loc. cit.

Theorem (4.36): *Let X be a d-dimensional affine toric variety whose associated cone has simplicial facets and let $D \in A_{d-1}(X)$. If $R^i\pi_*\mathcal{O}_{\tilde{X}}(\pi^*D) = 0$ for every regular triangulation $\pi : \tilde{X} \longrightarrow X$, then $\mathcal{O}_X(D)$ is MCM. For $d = 3$ the converse is also true.*

Note that the facets of a 3-dimensional cone are always simplicial.
To prove our results we will require a lot of bookkeeping, combining various geometric, combinatorial and arithmetic aspects of toric varieties. This has the unfortunate effect that the exposition will be rather technical and incorporate many notions (though not much theory) coming from combinatorics. As this might be cumbersome to follow for the more geometrically inclined reader, we will give an overview of the key structures and explain how they fit together.
From now X denotes an arbitrary d-dimensional toric variety, Δ the fan associated to X, $M \cong \mathbb{Z}^d$ the character group of the torus which acts on X. We denote N the dual module of M, $l_1, \ldots, l_n \in N$ the set of primitive vectors of the 1-dimensional cones in Δ and $D_1, \ldots, D_n$ the corresponding torus invariant prime divisors on X. By abuse of notion, we will often identify the sets $[n] := \{1, \ldots, n\}$ and $\{l_1, \ldots, l_n\}$.

The circuit geometry of a toric variety. In order to compute the cohomology $H^i_V\big(X, \mathcal{O}_X(D)\big)$ of a torus-invariant Weil divisor $D = \sum_{i=1}^n c_iD_i$ with respect to some torus-invariant support $V \subseteq X$, one uses the induced eigenspace decomposition

$$H^i_V\big(X, \mathcal{O}_X(D)\big) \cong \bigoplus_{m \in M} H^i_V\big(X, \mathcal{O}_X(D)\big)_m.$$

By a well-known formula, we can compute every eigenspace by computing the relative cohomology of a certain simplicial complex:

$$H^i_V\big(X, \mathcal{O}_X(D)\big)_m \cong H^{i-1}(\hat{\Delta}_m, \hat{\Delta}_{V,m}; k).$$

Here $\hat{\Delta}$ denotes the simplicial model of Δ, i.e. the abstract simplicial complex on the set $[n]$ such that any subset $I \subset [n]$ is in $\hat{\Delta}$ iff there exists a cone σ in Δ such that elements in I are faces of σ. Similarly, $\hat{\Delta}_V$ is a subcomplex of $\hat{\Delta}$, generated by only those cones in Δ whose associated orbits in X are not contained in V (see also Section 2). For any character $m \in M$, $\hat{\Delta}_m$ and $\hat{\Delta}_{V,m}$ are the full subcomplexes which are supported on those l_i with $l_i(m) < -c_i$ (see Theorem 2.1).

By this, for an invariant divisor $D = \sum_{i=1}^n c_i D_i$, the eigenspaces $H^i_V\big(X, \mathcal{O}_X(D)\big)_m$ depend on the simplicial complexes $\hat{\Delta}$, $\hat{\Delta}_V$ as well as on the position of the characters m with respect to the hyperplanes $H^{\underline{c}}_i = \{m \in M_{\mathbb{Q}} \mid l_i(m) = -c_i\}$, where $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$. The chamber decomposition of $M_{\mathbb{Q}}$ induced by the $H^{\underline{c}}_i$ (or their intersection poset) can be interpreted as the combinatorial type of D. Our strategy will be to consider the variations of combinatorial types depending on $\underline{c} = (c_1, \ldots, c_n) \in \mathbb{Q}^n$. The solution to this discriminantal problem is given by the *discriminantal arrangement* associated to the vectors $l_1, \ldots, l_n$, which has first been considered by Crapo [Cra84] and Manin and Schechtman [MS89]. The discriminantal arrangement is constructed as follows. Consider the standard short exact sequence associated to X:

$$0 \longrightarrow M_{\mathbb{Q}} \xrightarrow{L} \mathbb{Q}^n \xrightarrow{D} A_{\mathbb{Q}} \longrightarrow 0, \tag{1}$$

where L is given by $L(m) = \big(l_1(m), \ldots, l_n(m)\big)$, and $A_{\mathbb{Q}} := A_{d-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the rational divisor class group of X. The matrix D is called the *Gale* transform of L, and its i-th column D_i is the Gale transform of l_i. The most important property of the Gale transform is that the linear dependencies among the l_i and among the D_i are inverted. That is, for any subset among the l_i which forms a basis, the complementary subset of the D_i forms a basis of $A_{\mathbb{Q}}$, and vice versa. Moreover, for every *circuit*, i.e. a minimal linearly dependent subset, $\mathcal{C} \subset [n]$ the complementary set $\{D_i \mid l_i \notin \mathcal{C}\}$ spans a hyperplane $H_{\mathcal{C}}$ in $A_{\mathbb{Q}}$. Then the discriminantal arrangement is given by the hyperplane arrangement

$$\{H_{\mathcal{C}} \mid \mathcal{C} \subset [n] \text{ circuit}\}.$$

The stratification of $A_{\mathbb{Q}}$ by this arrangement then is in bijection with the combinatorial types of the arrangements given by the $H^{\underline{c}}_i$ under variation of $\underline{c}$. As we will see, virtually all properties of X concerning its birational geometry and cohomology vanishing of divisorial sheaves on X depend on the discriminantal arrangement. In particular, (see Proposition 3.19), the discriminantal arrangement coincides with the hyperplane arrangement generated by the facets of the secondary fan. Ubiquitous standard constructions such as the effective cone, nef cone, and the Picard group can easily be identified as its substructures.
Another interesting aspect is that the discriminantal arrangement by itself (or the associated matroid, respectively) represents a combinatorial invariant of the variety X, which one can refer to as its *circuit geometry*. This circuit geometry refines the combinatorial information coming with the toric variety, that is, the fan Δ and the matroid structure underlying the l_i (i.e. their linear dependencies). It depends only on the l_i, and even for two combinatorially equivalent fans Δ, Δ' such that corresponding sets of primitive vectors $l_1, \ldots, l_n$ and $l'_1, \ldots, l'_n$ have the same underlying linear dependencies, their associated circuit geometries are different in general. This already is the case for surfaces, see, for instance, Crapo's example of a plane tetrahedral line configuration ([Cra84], §4). Falk ([Fal94], Example 3.2) gives a 3-dimensional example.

Circuits and the diophantine Frobenius problem. Circuits are also the building blocks for our arithmetic conditions on cohomology vanishing, which can easily be illustrated for the case of weighted projective spaces. Assume, for simplicity, that the first $d+1$ primitive vectors $l_1, \dots, l_{d+1}$ generate N and form a circuit. Then we have a relation

$$\sum_{i=1}^{d+1} \alpha_i l_i = 0 \tag{2}$$

where the α_i are nonzero integers whose largest common divisor is one. This relation is unique up to sign and we assume for simplicity that $\alpha_i > 0$ for at least one i. In the special case that all the α_i are positive, $l_1, \dots, l_{d+1}$ generate the fan of a weighted projective space $\mathbb{P}(\alpha_1, \dots, \alpha_{d+1})$. Denote D the unique $\mathbb{Q}$-effective generator of $A_{d-1}\big(\mathbb{P}(\alpha_1, \dots, \alpha_{d+1})\big)$. Then there is a standard construction for counting global sections

$$\begin{aligned} &\dim H^0\big(\mathbb{P}(\alpha_1, \dots, \alpha_{d+1}), \mathcal{O}_{\mathbb{P}(\alpha_1,\dots,\alpha_{d+1})}(nD)\big) \\ =&\Big|\{(k_1, \dots, k_{d+1}) \in \mathbb{N}^{d+1} \mid \sum_{i=1}^{d+1} k_i \alpha_i = n\}\Big| =: \mathrm{VP}_{\alpha_1,\dots,\alpha_{d+1}}(n), \end{aligned}$$

for any $n \in \mathbb{Z}$. Here, $\mathrm{VP}_{\alpha_1,\dots,\alpha_{d+1}}$ is the so-called vector partition function (or denumerant function) with respect to the α_i. The problem of determining the zero set of $\mathrm{VP}_{\alpha_1,\dots,\alpha_{d+1}}$ (or the maximum of this set) is quite famously known as the diophantine Frobenius problem. This problem is hard in general (though not necessarily so in specific cases) and there does not exist a general closed expression to determine the zero set (for a survey of the diophantine Frobenius problem we refer to the book [Ram05]). Analogously, one can write down similar functions for any circuit among the l_i (see subsection 4.1).

The basic idea now is to transport the discriminantal arrangement from $A_\mathbb{Q}$ to some diophantine analog in $A_{d-1}(X)$. For any circuit $\mathcal{C} \subset [n]$ there is a short exact sequence

$$0 \longrightarrow H_\mathcal{C} \longrightarrow A_\mathbb{Q} \longrightarrow A_{\mathcal{C},\mathbb{Q}} \longrightarrow 0.$$

By lifting the surjection $A_\mathbb{Q} \to A_{\mathcal{C},\mathbb{Q}}$ to its integral counterpart $A_{d-1}(X) \to A_\mathcal{C}$, we lift the zero set of the corresponding vector partition function on $A_\mathcal{C}$ to $A_{d-1}(X)$. By doing this for every circuit $\mathcal{C}$, we construct in $A_{d-1}(X)$ what we call the *Frobenius discriminantal arrangement.* One can consider the Frobenius discriminantal arrangement as an arithmetic thickening of the discriminantal arrangement. This thickening in general is just enough to enlarge the relevant strata in the discriminantal arrangement such that it encompasses the Kawamata-Viehweg-like theorems. To derive other vanishing results, our analysis will mostly be concerned with analyzing the birational geometry of X and its implications on the combinatorics of the discriminantal arrangement, and the transport of this analysis to the Frobenius arrangement.

Overview. Section 2 introduce some general notation and results related to toric varieties. In section 3 we survey discriminantal arrangements, secondary fans, and rational aspects of cohomology vanishing. Several technical facts will be collected which are important for the subsequent sections. Section 4 contains all the essential results of this work. In 4.3 we will prove our main arithmetic vanishing results. These will be applied in 4.4 to give a quite complete characterization of cohomology vanishing for toric surfaces. Section 4.5 is devoted to maximal Cohen-Macaulay modules.

Acknowledgments. Thanks to Laurent Bonavero, Michel Brion, Lutz Hille, Vic Reiner, and Jan Stienstra for discussion and useful hints.

2 Toric Preliminaries

In this section we first introduce notions from toric geometry which will be used throughout the rest of the paper. As general reference for toric varieties we use [Oda88], [Ful93]. We will always work over an algebraically closed field k.

Let Δ be a fan in the rational vector space $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ over a lattice $N \cong \mathbb{Z}^d$. Let M be the lattice dual to N, then the elements of N represent linear forms on M and we write $n(m)$ for the canonical pairing $N \times M \to \mathbb{Z}$, where $n \in N$ and $m \in M$. This pairing extends naturally over $\mathbb{Q}$, $M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$. Elements of M are denoted by m, m', etc. if written additively, and by $\chi(m)$, $\chi(m')$, etc. if written multiplicatively, i.e. $\chi(m + m') = \chi(m)\chi(m')$. The lattice M is identified with the group of characters of the algebraic torus $T = \mathrm{Hom}(M, k^*) \cong (k^*)^d$ which acts on the toric variety $X = X_\Delta$ associated to Δ. Moreover, we will use the following notation:

- cones in Δ are denoted by small greek letters $\rho, \sigma, \tau, \ldots$, their natural partial order by $\prec$, i.e. $\rho \prec \tau$ iff $\rho \subseteq \tau$;
- $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma$ denotes the support of Δ;
- for $0 \leq i \leq d$ we denote $\Delta(i) \subset \Delta$ the set of i-dimensional cones; for $\sigma \in \Delta$, we denote $\sigma(i)$ the set of i-dimensional faces of σ;
- U_σ denotes the associated affine toric variety for any $\sigma \in \Delta$;
- $\check{\sigma} := \{m \in M_{\mathbb{Q}} \mid n(m) \geq 0 \text{ for all } n \in \sigma\}$ is the cone *dual* to σ;
- $\sigma^\perp = \{m \in M_{\mathbb{Q}} \mid n(m) = 0 \text{ for all } n \in \sigma\}$;
- $\sigma_M := \check{\sigma} \cap M$ is the submonoid of M associated to σ.

We will mostly be interested in the structure of Δ as a combinatorial cellular complex. For this, we make a few convenient identifications. We always denote n the cardinality of $\Delta(1)$. i.e. the number of 1-dimensional cones (*rays*) and

$[n] := \{1, \ldots, n\}$. The primitive vectors along rays are denoted $l_1, \ldots, l_n$, and, by abuse of notion, we will usually identify the sets $\Delta(1)$, the set of primitive vectors, and $[n]$. Also, we will often identify $\sigma \in \Delta$ with the set $\sigma(1) \subset [n]$. With these identifications, and using the natural order of $[n]$, we obtain a combinatorial cellular complex with support $[n]$; we may consider this complex as a combinatorial model for Δ. In the case where Δ is simplicial, this complex is just a combinatorial simplicial complex in the usual sense. If Δ is not simplicial, we consider the *simplicial cover* $\hat{\Delta}$ of Δ, modelled on $[n]$: some subset $I \subset [n]$ is in $\hat{\Delta}$ iff there exists some $\sigma \in \Delta$ such that $I \subset \sigma(1)$. The identity on $[n]$ then induces a surjective morphism $\hat{\Delta} \longrightarrow \Delta$ of combinatorial cellular complexes. This morphism has a natural representation in terms of fans. We can identify $\hat{\Delta}$ with the fan in $\mathbb{Q}^n$ which is defined as follows. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Q}^n$, then for any set $I \subset [n]$, the vectors $\{e_i\}_{i \in I}$ span a cone over $\mathbb{Q}_{\geq 0}$ iff there exists $\sigma \in \Delta$ with $I \subset \sigma(1)$. The associated toric variety $\hat{X}$ is open in $\mathbb{A}^n_k$, and the vector space homomorphism defined by mapping $e_i \mapsto l_i$ for $i \in [n]$ induces a map of fans $\hat{\Delta} \to \Delta$. The induced morphism $\hat{X} \to X$ is the quotient presentation due to Cox [Cox95]. We will not make explicit use of this construction, but it may be useful to have it in mind.

An important fact used throughout this work is the following exact sequence which exists for any toric variety X with associated fan Δ:

$$M \xrightarrow{L} \mathbb{Z}^n \longrightarrow A_{d-1}(X) \longrightarrow 0. \tag{3}$$

Here $L(m) = (l_1(m), \ldots, l_n(m))$, i.e. as a matrix, the primitive vectors l_i represent the row vectors of L. Note that L is injective iff Δ is not contained in a proper subspace of $N_{\mathbb{Q}}$. The sequence follows from the fact that every Weil divisor D on X is rationally equivalent to a T-invariant Weil divisor, i.e. $D \sim \sum_{i=1}^n c_i D_i$, where $\underline{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ and $D_1, \ldots, D_n$, the T-invariant irreducible divisors of X. Moreover, any two T-invariant divisors D, D' are rationally equivalent if and only if there exists $m \in M$ such that $D - D' = \sum_{i=1}^n l_i(m) D_i$. To every Weil divisor D, one associates its divisorial sheaf $\mathcal{O}_X(D) = \mathcal{O}(D)$ (we will omit the subscript X whenever there is no ambiguity), which is a reflexive sheaf of rank one and locally free if and only if D is Cartier. Rational equivalence classes of Weil divisors are in bijection with isomorphism classes of divisorial sheaves. If D is T-invariant, the sheaf $\mathcal{O}(D)$ acquires a T-equivariant structure and the equivariant isomorphism classes of sheaves $\mathcal{O}(D)$ are one-to-one with $\mathbb{Z}^n$.

Consider a closed T-invariant subscheme $V \subseteq X$. Then for any T-invariant Weil divisor D there are induced linear representations of T on the local cohomology groups $H^i_V\big(X, \mathcal{O}(D)\big)$. In particular, each such module has a natural eigenspace decomposition

$$H^i_V\big(X, \mathcal{O}(D)\big) \cong \bigoplus_{m \in M} H^i_V\big(X, \mathcal{O}(D)\big)_m.$$

The eigenspaces $H^i_V\big(X, \mathcal{O}(D)\big)_m$ can be characterized by the relative cohomologies of certain simplicial complexes. For any $I \subset [n]$ we denote $\hat{\Delta}_I$ the maximal subcomplex of $\hat{\Delta}$ which is supported on I. We denote $\hat{\Delta}_V$ the simplicial cover of the fan associated to the complement of the reduced subscheme underlying V in X. Correspondingly, for $I \subset [n]$ we denote $\hat{\Delta}_{V,I}$ the maximal subcomplex of $\hat{\Delta}_V$ which is supported on I. If $\underline{c} \in \mathbb{Z}^n$ is fixed, and $D = \sum_{i \in [n]} c_i D_i$, then every $m \in M$ determines a subset $I(m)$ of $[n]$ which is given by

$$I(m) = \{i \in [n] \mid l_i(m) < -c_i\}.$$

Then we will write $\hat{\Delta}_m$ and $\hat{\Delta}_{V,m}$ instead of $\hat{\Delta}_{I(m)}$ and $\hat{\Delta}_{V,I(m)}$, respectively. In the case where Δ is generated by just one cone σ, we will also write $\hat{\sigma}_m$, etc. With respect to these notions we get:

THEOREM 2.1: *Let $D \in \mathbb{Z}^{\Delta(1)}$ be a T-invariant Weil divisor on X. Then for every T-invariant closed subscheme V of X, every $i \geq 0$ and every $m \in M$ there exists an isomorphism of k-vector spaces*

$$H^i_V\big(X, \mathcal{O}(D)\big)_m \cong H^{i-1}(\hat{\Delta}_m, \hat{\Delta}_{V,m}; k).$$

Note that here $H^{i-1}(\hat{\Delta}_m, \hat{\Delta}_{V,m})$ denotes the *reduced* relative cohomology group of the pair $(\hat{\Delta}_m, \hat{\Delta}_{V,m})$.

Proof. For $V = X$ it follows from [EMS00], §2 that $H^i\big(X, \mathcal{O}(D)\big)_m \cong H^{i-1}(\hat{\Delta}_m; k)$ and $H^i\big(X \setminus V, \mathcal{O}(D)\big)_m \cong H^{i-1}(\hat{\Delta}_{V,m}; k)$. Then the assertion follows from comparing the long exact relative cohomology sequence of the pair $(\hat{\Delta}_m, \hat{\Delta}_{V,m})$ with the long exact local cohomology sequence with respect to X and V in degree m. □

We mention a special case of this theorem, which follows from the long exact cohomology sequence.

COROLLARY 2.2: *Let $X = U_\sigma$ and V a T-invariant closed subvariety of X and denote $\hat{\sigma}$ the simplicial model for the fan generated by σ. Then for every $m \in M$ and every $i \in \mathbb{Z}$:*

$$H^i_V\big(X, \mathcal{O}(D)\big)_m = \begin{cases} 0 & \text{if } \hat{\sigma}_m = \emptyset, \\ H^{i-2}(\hat{\sigma}_{V,m}; k) & \text{else.} \end{cases}$$

3 DISCRIMINANTS AND COMBINATORIAL ASPECTS COHOMOLOGY VANISHING

A toric variety X is specified by the set of primitive vectors $l_1, \ldots, l_n \in N$ and the fan Δ supported on these vectors. We can separate three properties which govern the geometry of X and are relevant for cohomology vanishing problems:

(i) the linear algebra given by the vectors $l_1, \ldots, l_n$ and their linear dependencies as $\mathbb{Q}$-vectors;

(ii) arithmetic properties, which are also determined by the l_i, but considered as integral vectors;

(iii) its combinatorics, which is given by the fan Δ.

In this section we will have a closer look into the linear algebraic and combinatorial aspects. In subsection 3.1 we will introduce the notion of oriented and non-oriented circuits associated to the vectors l_i. In subsection 3.2 we consider circuits of the matrix L and the induced stratification of $A_{d-1}(X)_{\mathbb{Q}}$. In subsection 3.3 we will collect some well-known material on secondary fans from [GKZ94], [OP91], and [BFS90] and explain their relation to discriminantal arrangements. Subsection 3.4 then applies this to certain statements about the birational geometry of toric varieties and cohomology vanishing.
For this section and the following sections we will introduce the following conventions.

Convention 3.1: We will denote L the matrix whose rows are given by the l_i. For any subset I of $[n]$ we will denote L_I the submatrix of L consisting of the rows which are given by the l_i with $i \in I$. In general, we will not distinguish between $\{l_i\}_{i \in I}$ and L_I. Similarly, we will usually identify subsets $I \subset [n]$ with the corresponding subsets of $\{l_1, \ldots, l_n\}$. If Δ is a fan in $N_{\mathbb{R}}$ such that $\Delta(1)$ is generated by some subset of the l_i, then we say that Δ is *supported on* L (resp. on $l_1, \ldots, l_n$).
Let $\mathcal{C}$ be a subset of $[n]$ which is minimal with the property that the l_i with $i \in \mathcal{C}$ are linearly dependent. Then the set $\{l_i\}_{i \in \mathcal{C}}$ is called a *circuit*. By abuse of notion we will also call $\mathcal{C}$ itself a circuit.

3.1 Circuits

Let $\mathcal{C} \subseteq [n]$ be a circuit. Then we have a relation

$$\sum_{i \in [n]} \alpha_i l_i = 0,$$

which is unique up to a common multiple of the α_i, and the α_i are nonzero. Without loss of generality, we will assume that the α_i are integral and $\gcd\{|\alpha_i|\}_{i \in [n]} = 1$. To simplify the discussion, we will further assume that $L_{\mathcal{C}}$ generates a submodule $N_{\mathcal{C}}$ of finite index in N. For a fixed choice of the α_i, we have a partition $\mathcal{C} = \mathfrak{C}^+ \coprod \mathfrak{C}^-$, where $\mathfrak{C}^{\pm} = \{i \in [n] \mid \pm\alpha_i > 0\}$. This decomposition depends only on the signs of the α_i; flipping the signs exchanges $\mathfrak{C}^+$ and $\mathfrak{C}^-$. We want to keep track of these two possibilities and call the choice of $\mathfrak{C}^+ \coprod \mathfrak{C}^- =: \mathfrak{C}$ the *oriented circuit* with underlying circuit $\mathcal{C}$ (or simply an *orientation* of $\mathfrak{C}$), and $-\mathfrak{C} := -\mathfrak{C}^+ \coprod -\mathfrak{C}^-$ its inverse, where $-\mathfrak{C}^{\pm} := \mathfrak{C}^{\mp}$.

Definition 3.2: We denote $\mathcal{C}(L)$ the set of circuits of L and $\mathfrak{C}(L)$ the set of oriented circuits of L, i.e. the set of all orientations $\mathfrak{C}, -\mathfrak{C}$ for $\mathcal{C} \in \mathcal{C}(L)$.

For a given circuit $\mathcal{C}$, the primitive vectors $L_\mathcal{C}$ can support at most two simplicial fans, each corresponding to an orientation of $\mathcal{C}$. For fixed orientation $\mathfrak{C}$, we denote $\Delta = \Delta_\mathfrak{C}$ the fan whose maximal cones are generated by $\mathcal{C} \setminus \{i\}$, where i runs over the elements of $\mathfrak{C}^+$. The only exception for this procedure is the case where $\mathfrak{C}^+$ is empty, which we leave undefined. The associated toric variety $X_{\Delta_\mathfrak{C}}$ is simplicial and quasi-projective.

Definition 3.3: We call a toric variety $X = X_{\Delta_\mathfrak{C}}$ associated to an oriented circuit $\mathfrak{C}$ a *toric 1-circuit variety.*

Now let us assume that the sublattice $N_\mathcal{C}$ of N which is generated by $L_\mathcal{C}$ is saturated. Then we have a short exact sequence

$$0 \longrightarrow M \xrightarrow{L_\mathcal{C}} \mathbb{Z}^n \xrightarrow{G_\mathcal{C}} A \longrightarrow 0, \tag{4}$$

such that $A \cong \mathbb{Z}$ and thus torsion free. Here, $L_\mathcal{C}$ is considered as a tuple of linear forms on M, $A \cong \mathbb{Z}$ and $G_\mathcal{C} = (\alpha_1, \ldots, \alpha_n)$ is a $(1 \times n)$-matrix, i.e. we can consider the α_i as the *Gale transform* of the l_i. Conversely, if the α_i are given, then the l_i are determined up to a $\mathbb{Z}$-linear automorphism of N. We will make more extensive use of the Gale transform later on. For generalities we refer to [OP91] and [GKZ94].
If $N_\mathcal{C} \subsetneq N$, we can formally consider the inclusion of $N_\mathcal{C}$ as the image of N via an injective endomorphism ξ of N. The inverse images of the l_i with respect to ξ satisfy the same relation as the l_i. Therefore, a general toric circuit variety is completely specified by ξ and the integers α_i. More precisely, a toric 1-circuit variety is specified by the Gale duals l_i of the α_i and an injective endomorphism ξ of N with the property that $\xi(l_i)$ is primitive in N for every $i \in [n]$.

Definition 3.4: Let $\underline{\alpha} = (\alpha_i \mid i \in \mathcal{C}) \in \mathbb{Z}^\mathcal{C}$ with $\alpha_i \neq 0$ for every i and $\gcd\{|\alpha_i|\}_{i \in [n]} = 1$, $\mathfrak{C}$ the associated oriented circuit with $\mathfrak{C}^+ = \{i \mid \alpha_i > 0\}$, and $\xi : N \longrightarrow N$ an injective endomorphism of N which maps the Gale duals of the α_i to primitive elements p_i in N. Then we denote $\mathbb{P}(\underline{\alpha}, \xi)$ the toric 1-circuit variety associated to the fan $\Delta_\mathfrak{C}$ spanned by the primitive vectors p_i.

The endomorphism ξ translates into an isomorphism

$$\mathbb{P}(\underline{\alpha}, \xi) \cong \mathbb{P}(\underline{\alpha}, \mathrm{id}_N)/H,$$

where $H \cong \operatorname{spec} k[N/N_\mathcal{C}]$. Note that in positive characteristic, H in general is a group scheme rather than a proper algebraic group. Moreover, in sequence (4) we can identify A with the divisor class group $A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \mathrm{id}_N)\big)$. Similarly, we get $A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big) \cong A \oplus N/N_\mathcal{C}$ and the natural surjection from $A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big)$ onto $A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \mathrm{id}_N)\big)$ just projects away the torsion part.

Remarks 3.5: (i) In the case $\alpha_i > 0$ for all i and $\xi = \mathrm{id}_N$, we just recover the usual weighted projective spaces. In many respects, the spaces $\mathbb{P}(\underline{\alpha}, \xi)$ can be treated the same way as has been done in the standard references for weighted

projective spaces, see [Del75], [Dol82], [BR86]. In our setting there is the slight simplification that we naturally can assume that $\gcd\{|\alpha_j|\}_{j \neq i} = 1$ for every $i \in [n]$, which eliminates the need to discuss reduced weights.
(ii) In the case that $L_{\mathcal{C}}$ spans a subspace N' of $N_{\mathbb{Q}}$ of positive codimension r, then for some orientation $\mathfrak{C}$ of $\mathcal{C}$ the variety $X(\Delta_{\mathfrak{C}})$ is isomorphic to $\mathbb{P}(\underline{\alpha}, \xi) \times (k^*)^r$, where $\mathbb{P}(\underline{\alpha}, \xi)$ is defined as before with respect to N'. Note that if $\mathfrak{C}^+ = \mathcal{C}$, then the fan $\Delta_{-\mathfrak{C}}$ is empty. By convention, in that case one can define $X(\Delta_{-\mathfrak{C}}) := (k^*)^r$ as the associated toric variety.
(iii) The spaces $\mathbb{P}(\underline{\alpha}, \xi)$ are building blocks for the birational geometry of general toric varieties. In fact, to every extremal curve $V(\tau)$ in some simplicial toric variety X, there is associated some variety $\mathbb{P}(\underline{\alpha}, \xi)$ whose fan $\Delta_{\mathfrak{C}}$ is a subfan of Δ and which embeds as an open invariant subvariety of X. If $|\mathfrak{C}^+| \notin \{n, n-1\}$, the primitive vectors l_i span a convex polyhedral cone, giving rise to an affine toric variety Y and a canonical morphism $\pi : \mathbb{P}(\underline{\alpha}, \xi) \longrightarrow Y$ which is a partial resolution of singularities. Sign change $\underline{\alpha} \to -\underline{\alpha}$ then encodes the transition from $\mathfrak{C}$ to $-\mathfrak{C}$ and and a birational map from $\mathbb{P}(\underline{\alpha}, \xi)$ to $\mathbb{P}(-\underline{\alpha}, \xi)$, which provides a local model for well-known combinatorial operation which called *bistellar operation* [Rei99] or *modification of a triangulation* [GKZ94]:

$$\begin{tikzcd} \mathbb{P}(\underline{\alpha}, \xi) \arrow[rr, dashed] \arrow[dr, "\pi"'] & & \mathbb{P}(-\underline{\alpha}, \xi) \arrow[dl, "\pi'"] \\ & Y & \end{tikzcd}$$

(for $|\mathfrak{C}^+| = d - 1$, one can identify $\mathbb{P}(-\underline{\alpha}, \xi)$ with Y and one just obtains a blow-down).

3.2 Circuits and discriminantal arrangements

Recall that for any torus invariant divisor $D = \sum_{i \in [n]} c_i D_i$, the isotypical components $H^i_V\big(X, \mathcal{O}(D)\big)_m$ for some cohomology group depend on simplicial complexes $\hat{\Delta}_I$, where $I = I(m) = \{i \in [n] \mid l_i(m) < -c_i\}$. So, the set of all possible subcomplexes $\hat{\Delta}_I$ depends on the chamber decomposition of $M_{\mathbb{Q}}$ which is induced by the hyperplane arrangement which is given by hyperplanes $H_1^{\underline{c}}, \ldots, H_n^{\underline{c}}$, where

$$H_i^{\underline{c}} := \{m \in M_Q \mid l_i(m) = -c_i\}.$$

The set of all relevant $I \subset [n]$ is determined by the map

$$\mathfrak{s}^{\underline{c}} : M_{\mathbb{Q}} \longrightarrow 2^{[n]}, \quad m \mapsto \{i \in [n] \mid l_i(m) < -c_i\}.$$

Definition 3.6: For $m \in M_{\mathbb{Q}}$, we call $\mathfrak{s}^{\underline{c}}$ the *signature* of m. We call the image of $M_{\mathbb{Q}}$ in $2^{[n]}$ the *combinatorial type* of $\underline{c}$.

Remark 3.7: The combinatorial type encodes what in combinatorics is known as *oriented matroid* (see [BLS+93]). We will not make use of this kind of structure, but we will find it sometimes convenient to borrow some notions.

So, given $l_1, \ldots, l_n$, we would like to classify all possible combinatorial types, depending on $\underline{c} \in \mathbb{Q}^n$. The natural parameter space for all hyperplane arrangements up to translation by some element $m \in M_{\mathbb{Q}}$ is given by the set $A_{\mathbb{Q}} \cong \mathbb{Q}^n / M_{\mathbb{Q}}$, which is given by the following short exact sequence:

$$0 \longrightarrow M_{\mathbb{Q}} \xrightarrow{L} \mathbb{Q}^n \xrightarrow{D} A_{d-1}(X)_{\mathbb{Q}} = A_{\mathbb{Q}} \longrightarrow 0.$$

Then the $D_1, \ldots, D_n$ are the images of the standard basis vectors of $\mathbb{Q}^n$. This procedure of constructing the D_i from the l_i is often called *Gale transformation*, and the D_i are the *Gale duals* of the l_i.
Now, a hyperplane arrangement $H_i^{\underline{c}}$ for some $\underline{c} \in \mathbb{Q}^n$, is considered in general position if the hyperplanes $H_i^{\underline{c}}$ intersect in the smallest possible dimension. When varying $\underline{c}$ and passing from one arrangement in general position to another one which has a different combinatorial type, this necessarily implies that has to take place some specialization for some $\underline{c} \in \mathbb{Q}^n$, i.e. where the corresponding hyperplanes $H_i^{\underline{c}}$ do not intersect in the smallest possible dimension. So we see that the combinatorial types of hyperplane arrangements with fixed L and varying induce a stratification of $A_{\mathbb{Q}}$, where the maximal strata correspond to hyperplane arrangements in general position. To determine this stratification is the *discriminant problem* for hyperplane arrangements. To be more precise, let $I \subset [n]$ and denote

$$H_I := \{\underline{c} + M_{\mathbb{Q}} \in A_{\mathbb{Q}} \mid \bigcap_{i \in I} H_i^{\underline{c}} \neq 0\},$$

i.e. H_I represents the set of all hyperplane arrangements (up to translation) such that the hyperplanes $\{H_i\}_{i \in I}$ have nonempty intersection. The sets H_I can be described straightforwardly by the following commutative exact diagram:

$$\begin{tikzcd}
 & & & H_I \arrow[d, hook] & \\
0 \arrow[r] & M_{\mathbb{Q}} \arrow[r, "L"] \arrow[d, equal] & \mathbb{Q}^n \arrow[r, "D"] \arrow[d, two heads] & A_{\mathbb{Q}} \arrow[r] \arrow[d, two heads] & 0 \\
 & M_{\mathbb{Q}} \arrow[r, "L_I"] & \mathbb{Q}^I \arrow[r, "D_I"] & A_{I,\mathbb{Q}} \arrow[r] & 0.
\end{tikzcd} \tag{5}$$

In particular, H_I is a subvector space of $A_{\mathbb{Q}}$. Moreover, we immediately read off diagram (5):

LEMMA 3.8: *(i) H_I is generated by the D_i with $i \in [n] \setminus I$.*

(ii) $\dim H_I = n - |I| - \dim(\ker L_I)$.

(iii) If $J \subseteq I$ then $H_I \subseteq H_J$.

(iv) Let $I, J \subset [n]$, then $H_{I \cup J} \subset H_I \cap H_J$.

Note that in (iv) the reverse inclusion in general is not true. It follows that the hyperplanes among the H_I are determined by the formula:

$$|I| = \operatorname{rk} L_I + 1.$$

By Lemma 3.8 (iii), we can always consider circuits fulfilling this condition. It turns out that the hyperplane $H_{\mathcal{C}}$ suffice to completely describe the discriminants of L:

Lemma 3.9: *Let $I \subset [n]$, then*

$$H_I = \bigcap_{\mathcal{C} \subset I \, circuit} H_{\mathcal{C}},$$

where, by convention, the right hand side equals $A_{\mathbb{Q}}$, if the l_i with $i \in I$ are linearly independent.

Hence, the stratification of $A_{\mathbb{Q}}$ which we were looking for is completely determined by the hyperplanes $H_{\mathcal{C}}$.

Definition 3.10: We denote the set $\{H_{\mathcal{C}} \mid \mathcal{C} \subset [n] \text{ a circuit}\}$ the *discriminantal arrangement* of L.

Remark 3.11: The discriminantal arrangement carries a natural matroid structure. This structure can be considered as another combinatorial invariant of L (or the toric variety X, respectively), its *circuit geometry*. Discriminantal arrangements seem to have been appeared first in [Cra84], where the notion of 'circuit geometry' was coined. The notion of discriminantal arrangements stems from [MS89]. Otherwise, this subject seems to have been studied explicitly only in very few places, see for instance [Fal94], [BB97], [Ath99], [Rei99], [Coh86], [CV03], though it is at least implicit in the whole body of literature on secondary fans. Above references are mostly concerned with genericity properties of discriminantal arrangements. Unfortunately, in toric geometry, the most interesting cases (such as smooth projective toric varieties, for example) virtually never give rise to discriminantal arrangements in general position. Instead, we will focus on certain properties of nongeneric circuit geometries, though we will not undertake a thorough combinatorial study of these.

Virtually all problems related to cohomology vanishing on a toric variety X must depend on the associated discriminantal arrangement and therefore on the circuits of L. In subsection 3.3 we will see that the discriminantal arrangement is tightly tied to the geometry of X.
As we have seen in section 3.1, to every circuit $\mathcal{C} \subset [n]$ we can associate two oriented circuits. These correspond to the signature of the bounded chamber of the subarrangement in $M_{\mathbb{Q}}$ given by the $H_i^{\mathcal{C}}$ with $i \in \mathcal{C}$ (or better to the bounded chamber in $M_{\mathbb{Q}} / \ker L_I$, as we do no longer require that the l_i with $i \in \mathcal{C}$ span $M_{\mathbb{Q}}$). Lifting this to $A_{\mathbb{Q}}$, this corresponds to the half spaces in $A_{\mathbb{Q}}$ which are bounded by $H_{\mathcal{C}}$.

DEFINITION 3.12: Let $\mathcal{C} \subset [n]$ be a circuit, then we denote $H_{\mathfrak{C}}$ the half space in $A_{\mathbb{Q}}$ bounded by $H_{\mathcal{C}}$ corresponding to the orientation $\mathfrak{C}$.

The following is straightforward to check:

LEMMA 3.13: *Let $\mathcal{C}$ be a circuit of L and $\mathfrak{C}$ an orientation of $\mathcal{C}$. Then the hyperplane $H_{\mathcal{C}}$ is* separating, *i.e. for every $i \in [n]$ one of the following holds:*

(i) $i \in [n] \setminus \mathcal{C}$ *iff* $D_i \in H_{\mathcal{C}}$;

(ii) *if* $i \in \mathfrak{C}^+$, *then* $D_i \in H_{\mathfrak{C}} \setminus H_{\mathcal{C}}$;

(iii) *if* $i \in \mathfrak{C}^-$, *then* $D_i \in H_{-\mathfrak{C}} \setminus H_{\mathcal{C}}$.

Now we are going to borrow some terminology from combinatorics. Consider any subvector space U of $A_{\mathbb{Q}}$ which is the intersection of some of the $H_{\mathcal{C}}$. Then the set $\mathcal{F}_U$ of all $\mathcal{C} \in \mathcal{C}(L)$ such that $H_{\mathcal{C}}$ contains U is called a *flat*. The subvector space is uniquely determined by the flat and vice versa. We can do the same for the actual strata rather than for subvector spaces. For this, we just need to consider instead the oriented circuits and their associated half spaces in $A_{\mathbb{Q}}$: any stratum S of the discriminantal arrangement uniquely determines a finite set $\mathfrak{F}_S$ of oriented circuits $\mathfrak{C}$ such that $S \subset H_{\mathfrak{C}}$. From the set $\mathfrak{F}_S$ we can reconstruct the closure of S:

$$\overline{S} = \bigcap_{\mathfrak{C} \in \mathfrak{F}_S} H_{\mathfrak{C}},$$

We give a formal definition:

DEFINITION 3.14: For any subvector space $U \subset A_{\mathbb{Q}}$ which is a union of strata of the discriminantal arrangement, we denote $\mathcal{F}_U := \{\mathcal{C} \in \mathcal{C}(L) \mid U \subset H_{\mathcal{C}}\}$ the associated *flat*. For any single stratum $S \subset A_{\mathbb{Q}}$ of the discriminantal arrangement, we denote $\mathfrak{F}_S := \{\mathfrak{C} \in \mathfrak{C}(L) \mid U \subset H_{\mathfrak{C}}\}$ the associated *oriented flat*.

The notion of flats gives us some flexibility in handling strata. Note that flats reverse inclusions, i.e. $S \subset T$ iff $\mathfrak{F}_T \subset \mathfrak{F}_S$. Moreover, if a stratum S is contained in some $H_{\mathcal{C}}$, then its oriented flat contains both $H_{\mathfrak{C}}$ and $H_{-\mathfrak{C}}$, and vice versa. So from the oriented flat we can reconstruct the subvector space of $A_{\mathbb{Q}}$ generated by S.

DEFINITION 3.15: Let $\mathcal{S} := \{S_1, \dots, S_k\}$ be a collection of strata of the discriminantal arrangement. We call

$$\mathfrak{F}_{\mathcal{S}} := \bigcap_{i=1}^{k} \mathfrak{F}_{S_i}$$

the *discriminantal hull* of $\mathcal{S}$.

The discriminantal hull defines a closed cone in $A_{\mathbb{Q}}$ which is given by the intersection $\bigcap_{\mathfrak{C} \in \mathfrak{F}_{\mathcal{S}}} H_{\mathfrak{C}}$. This cone contains the union of the closures $\overline{S}_i$, but is bigger in general.

Lemma 3.16: *(i) Let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a collection of discriminantal strata whose union is a closed cone in $A_{\mathbb{Q}}$. then $\mathfrak{F}_{\mathcal{S}} = \bigcap_{i=1}^{k} \mathfrak{F}_{S_i}$.*

(ii) Let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a collection of discriminantal strata and U the subvector space of $A_{\mathbb{Q}}$ generated by the S_i. Then the forgetful map $\mathfrak{F}_{\mathcal{S}} \to \mathcal{F}_U$ is surjective iff $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{S_i}$ for some i.

Proof. For (i) just note that because $\bigcup_{i=1}^{k} S_k$ is a closed cone, it must be an intersection of some $H_{\mathfrak{C}}$. For (ii): the set $\bigcap_{\mathfrak{C} \in \mathfrak{F}_{\mathcal{S}}} H_{\mathfrak{C}}$ is a cone which contains the convex hull of all the $\overline{S}_i$. If some $\mathcal{C}$ is not in the image of the forgetful map, then the hyperplane $H_{\mathcal{C}}$ must intersect the relative interior of this cone. So the assertion follows. □

3.3 Secondary Fans

For any $\underline{c} \in \mathbb{Q}^n$ the arrangement $H_i^{\underline{c}}$ induces a chamber decomposition of $M_{\mathbb{Q}}$, where the closures of the chambers are given by

$$P_{\underline{c}}^{I} := \{m \in M_{\mathbb{Q}} \mid l_i(m) \leq -c_i \text{ for } i \in I \text{ and } l_i(m) \geq -c_i \text{ for } i \notin I\}$$

for every $I \subset [n]$ which belongs to the combinatorial type of $\underline{c}$. In particular, $\underline{c}$ represents an element $D \in A_{\mathbb{Q}}$ with

$$D \in \bigcap_{I \in \mathfrak{s}^{\underline{c}}(M_{\mathbb{Q}})} C_I,$$

where C_I is the cone in $A_{\mathbb{Q}}$ which is generated by the $-D_i$ for $i \in I$ and the D_i with $i \notin I$ for some $I \subset [n]$. For an invariant divisor $D = \sum_{i \in [n]} c_i D_i$ we will also write P_D^I instead of $P_{\underline{c}}^I$. If $I = \emptyset$, we will occasionally omit the index I. The faces of the C_I can be read off directly from the signature:

Proposition 3.17: *Let $I \subset [n]$, then C_I is an nonredundant intersection of the $H_{\mathfrak{C}}$ with $\mathfrak{C}^- \subset I$ and $\mathfrak{C}^+ \cap I = \emptyset$.*

Proof. First of all, it is clear that C_I coincides with the intersection of half spaces

$$C_I = \bigcap_{\substack{\mathfrak{C}^+ \subset I \\ \mathfrak{C}^- \cap I = \emptyset}} H_{\mathfrak{C}}.$$

For any $H_{\mathfrak{C}}$ in the intersection let $H_{\mathcal{C}}$ its boundary. Then $H_{\mathcal{C}}$ contains a cone of codimension 1 in $A_{\mathbb{Q}}$ which is spanned by D_i with $i \in [n] \setminus (\mathcal{C} \cup I)$ and by $-D_i$ with $i \in I \setminus \mathcal{C}$ which thus forms a proper facet of C_I. □

Recall that the secondary fan of L is a fan in $A_{\mathbb{Q}}$ whose maximal cones are in one-to-one correspondence with the regular simplicial fans which are supported on the l_i. That is, if $\underline{c}$ is chosen sufficiently general, then the polyhedron $P_{\underline{c}}^{\emptyset}$ is simplicial and its inner normal fan is a simplicial fan which is supported on the l_i. Wall crossing in the secondary fan then corresponds locally to a transition $\Delta_{\mathfrak{C}} \longrightarrow \Delta_{-\mathfrak{C}}$ as in section 3.1. Clearly, the secondary fan is a substructure of the discriminantal arrangement in the sense that its cones are unions of strata of the discriminantal arrangements. However, the secondary fan in general is much coarser than the discriminantal arrangement, as it only keeps track of the particular chamber $P_{\underline{c}}^{\emptyset}$. In particular, the secondary fan is only supported on $C_{\emptyset}$ which in general does not coincide with $A_{\mathbb{Q}}$. Of course, there is no reason to consider only one particular type of chamber — we can consider secondary fans for every $I \subset [n]$ and every type of chamber $P_{\underline{c}}^{I}$. For this, observe first that, if $\mathcal{B}$ is a subset of $[n]$ such that the l_i with $i \in \mathcal{B}$ form a basis of $M_{\mathbb{Q}}$, then the complementary Gale duals $\{D_i\}_{i \notin \mathcal{B}}$ form a basis of $A_{\mathbb{Q}}$. Then we set:

Definition 3.18: Let $I \subset [n]$ and $\mathcal{B} \subset [n]$ such that the l_i with $i \in \mathcal{B}$ form a basis of $M_{\mathbb{Q}}$, then we denote $K_{\mathcal{B}}^{I}$ the cone in $A_{\mathbb{Q}}$ which is generated by $-D_i$ for $i \in I \setminus \mathcal{B}$ and by D_i for $i \in [n] \setminus (I \cup \mathcal{B})$. The secondary fan $\mathrm{SF}(L, I)$ of L with respect to I is the fan whose cones are the intersections of the $K_{\mathcal{B}}^{I}$, where $\mathcal{B}$ runs over all bases of L.

Note that $\mathrm{SF}(L, \emptyset)$ is just the secondary fan as usually defined. Clearly, the chamber structure of the discriminantal arrangement still refines the chamber structure induced by all secondary arrangements. But now we have sufficient data to even get equality:

Proposition 3.19: *The following induce identical chamber decompositions of* $A_{\mathbb{Q}}$*:*

(i) the discriminantal arrangement,

(ii) the intersection of all secondary fans $\mathrm{SF}(L, I)$*,*

(iii) the intersection of the C_I *for all* $I \subset [n]$*.*

Proof. Clearly, the facets of every orthant C_I span a hyperplane which is part of the discriminantal arrangement, so the chamber decomposition induced by the secondary fan is a refinement of the intersection of the C_I's. The C_I induce a refinement of the secondary fans as follows. Without loss of generality, it suffices to show that every $K_{\mathcal{B}}^{\emptyset}$ is the intersection of some C_I. We have

$$K_{\mathcal{B}}^{\emptyset} \subseteq \bigcap_{I \subset \mathcal{B}} C_I.$$

On the other hand, for every facet of $K_{\mathcal{B}}^{\emptyset}$, we choose I such that C_I shares this face and $K_{\mathcal{B}}^{\emptyset}$ is contained in C_I. This can always be achieved by choosing I so

that every generator of C_I is in the same half space as $K_{\mathcal{B}}^{\emptyset}$. The intersection of these C_I then is contained in $K_{\mathcal{B}}^{\emptyset}$.
Now it remains to show that the intersection of the secondary fans refines the discriminantal arrangement. This actually follows from the fact, that for every hyperplane $H_{\mathcal{C}}$, one can choose a minimal generating set which we can complete to a basis of $A_{\mathbb{Q}}$ from the D_i, where $i \notin \mathcal{C}$. By varying the signs of this generating set, we always get a simplicial cone whose generators are contained in some secondary fan, and this way $H_{\mathcal{C}}$ is covered by a set of facets of secondary cones. $\square$

The maximal cones in the secondary fan $\mathrm{SF}(L, \emptyset)$ correspond to regular simplicial fans supported on $l_1, \ldots, l_n$. More precisely, if Δ denotes such a fan, then the corresponding cone is given by $\bigcap_{\mathcal{B}} K_{\mathcal{B}}^{\emptyset}$, where $\mathcal{B}$ runs over all bases among the l_i which span a maximal cone in Δ. With respect to a simplicial model $\hat{\Delta}$ for Δ, we define:

Definition 3.20: Let Δ be a fan supported on L, then we set:

$$\mathrm{nef}(\Delta) := \bigcap_{\substack{\mathcal{B} \in \hat{\Delta} \\ \mathcal{B} \text{ basis in } L}} K_{\mathcal{B}}^{\emptyset}$$

and denote $\mathfrak{F}_{\mathrm{nef}} = \mathfrak{F}_{\mathrm{nef}(\Delta)}$ the discriminantal hull of $\mathrm{nef}(\Delta)$.

Note that by our conventions we identify $\mathcal{B} \in \hat{\Delta}$ with the set of corresponding primitive vectors, or the corresponding rows of L, respectively. Of course, $\mathrm{nef}(\Delta)$ is just the nef cone of the toric variety associated to Δ.

Proposition 3.21: *We have:*

$$\mathrm{nef}(\Delta) = \bigcap_{\hat{\Delta} \cap (\Delta_{\mathfrak{C}})_{\max} \neq \emptyset} H_{\mathfrak{C}}.$$

Proof. For some basis $\mathcal{B} \subset [n]$, the cone $K_{\mathcal{B}}^{\emptyset}$ is simplicial, and for every $i \in [n] \setminus \mathcal{B}$, the facet of $K_{\mathcal{B}}^{\emptyset}$ which is spanned by the D_j with $j \notin \mathcal{B} \cup \{i\}$, spans a hyperplane $H_{\mathcal{C}}$ in P. This hyperplane corresponds to the unique circuit $\mathcal{C} \subset \mathcal{B} \cup \{i\}$. As we have seen before, a maximal cone in $\Delta_{\mathfrak{C}}$ is of the form $\mathcal{C} \setminus \{j\}$ for some $j \in \mathfrak{C}^+$. So we have immediately:

$$K_{\mathcal{B}} = \bigcap_{\substack{\exists F \in (\Delta_{\mathfrak{C}})_{\max} \\ \text{with } F \subset \mathcal{B}}} H_{\mathfrak{C}}$$

and the assertion follows. $\square$

Remark 3.22: If $\Delta = \hat{\Delta}$ is a regular simplicial fan, then $\mathrm{nef}(\Delta)$ is a maximal cone in the secondary fan. Let $\mathfrak{C}$ be an oriented circuit such that Δ is supported on $\Delta_{\mathfrak{C}}$ in the sense of [GKZ94], §7, Def. 2.9, and denote Δ' the fan resulting in the bistellar operation by changing $\Delta_{\mathfrak{C}}$ to $\Delta_{-\mathfrak{C}}$. Then, by [GKZ94], §7, Thm. 2.10, the hyperplane $H_{\mathcal{C}}$ is a proper wall of $\mathrm{nef}(\Delta)$ iff Δ' is regular, too.

3.4 MCM sheaves, $\mathbb{Q}$-Cartier divisors and the toric Kawamata-Viehweg vanishing theorem

Recall that a $\mathbb{Q}$-divisor on X is $\mathbb{Q}$-Cartier if an integral multiple is Cartier in the usual sense. A torus invariant Weil divisor $D = \sum_{i \in [n]} c_i D_i$ is $\mathbb{Q}$-Cartier iff for every $\sigma \in \Delta$ there exists some $m_\sigma \in M_{\mathbb{Q}}$ such that $c_i = l_i(m)$ for all $i \in \sigma(1)$. A result of Bruns and Gubeladze [BG03] states that every toric $\mathbb{Q}$-Cartier divisor is maximal Cohen-Macaulay. The MCM property is useful, as it allows to replace the Ext-groups by cohomologies in Serre duality:

PROPOSITION 3.23: *Let X be a normal variety with dualizing sheaf ω_X and $\mathcal{F}$ a coherent sheaf on X such that for every $x \in X$, the stalk $\mathcal{F}_x$ is MCM over $\mathcal{O}_{X,x}$. Then for every $i \in \mathbb{Z}$ there exists an isomorphism*

$$\operatorname{Ext}^i_X\big(\mathcal{F}, \omega_X\big) \cong H^i\big(X, \mathcal{H}om(\mathcal{F}, \omega_X)\big).$$

Proof. For any two $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ there exists the following spectral sequence

$$E_2^{pq} = H^p\big(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\big) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

We apply this spectral sequence to the case $\mathcal{G} = \omega_X$. For every closed point $x \in X$ we have the following identity of stalks:

$$\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F}, \omega_X)_x \cong \operatorname{Ext}^q_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \omega_{X,x}).$$

As $\mathcal{F}$ is maximal Cohen-Macaulay, the latter vanishes for all $q > 0$, and thus the sheaf $\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$ is the zero sheaf for all $q > 0$. So the above spectral sequence degenerates and we obtain an isomorphism

$$H^p(X, \mathcal{H}om(\mathcal{F}, \omega_X)) \cong \operatorname{Ext}^p_X(\mathcal{F}, \omega_X)$$

for every $p \in \mathbb{Z}$. □

In the case where X a toric variety, we have $\omega_X \cong \mathcal{O}(K_X)$, where $K_X = -\sum_{i \in [n]} D_i$. Then, if $\mathcal{F} = \mathcal{O}(D)$ for some $D \in A$, we can identify $\mathcal{H}om(\mathcal{O}(D), \omega_X)$ with $\mathcal{O}(K_X - D)$:

COROLLARY 3.24: *Let X be a toric variety and D a Weil divisor such that $\mathcal{O}(D)$ is an MCM sheaf. Then there is an isomorphism:*

$$\operatorname{Ext}^i_X\big(\mathcal{O}(D), \omega_X\big) \cong H^i\big(X, \mathcal{O}_X(K_X - D)\big).$$

And by Grothendieck-Serre duality:

COROLLARY 3.25: *If X is a complete toric variety and D a Weil divisor such that $\mathcal{O}(D)$ is an MCM sheaf, then*

$$H^i\big(X, \mathcal{O}(D)\big) \cong H^{d-i}\big(X, \mathcal{O}(K_X - D)\big)\check{}.$$

For any Cartier divisor D on some normal variety X denote $N(X, D) := \{k \in \mathbb{N} \mid H^0\big(X, \mathcal{O}(kD)\big) \neq 0\}$. Then the Iitaka dimension of D is defined as

$$\kappa(D) := \max_{k \in N(X,D)} \{\dim \phi_k(X)\},$$

where $\phi_k : X \dashrightarrow \mathbb{P}|kD|$ is the family of morphisms given by the linear series $|kD|$.
In the case where X is a toric variety and $D = \sum_{i \in [n]} c_i D_i$ invariant, the Iitaka dimension of D is just the dimension of P_{kD} for $k >> 0$. For a $\mathbb{Q}$-Cartier divisor D, we define its Iitaka dimension by $\kappa(D) := \kappa(rD)$ for $r > 0$ such that rD is Cartier.
If D is a nef divisor, then the morphism $\phi : X \longrightarrow \mathbb{P}|D|$ is torus equivariant, its image is a projective toric variety of dimension $\kappa(D)$ whose associated fan is the inner normal fan of P_D. If $\kappa(D) < d$, then necessarily D is contained in some hyperplane $H_{\mathcal{C}}$ such that $\mathfrak{C}^+ = \mathcal{C}$ for some orientation $\mathfrak{C}$ of $\mathcal{C}$. The toric variety $X_{\Delta_{\mathfrak{C}}}$ is isomorphic to a finite cover of a weighted projective space. This kind of circuit will play an important role later on, so that we will give it a distinguished name:

DEFINITION 3.26: We call a circuit $\mathcal{C}$ such that $\mathcal{C} = \mathfrak{C}^+$ for one of its orientations, *fibrational*. For $D \in A_{d-1}(X)_{\mathbb{Q}}$ we denote $\mathrm{fib}(D) \subset \mathcal{C}(L)$ the set of fibrational circuits such that $D \in H_{\mathcal{C}}$.

By Proposition 3.17, such a divisor D is contained in the intersection of $\mathrm{nef}(X)$ with the effective cone of X, which we identify with $C_{\emptyset}$. More precisely, it follows from linear algebra that D is contained in all $H_{\mathcal{C}}$ where $\mathcal{C}$ is fibrational and $l_i(P_D) = 0$ for all $i \in \mathcal{C}$.
The fibrational circuits of a nef divisor D tell us immediately about its Iitaka dimension:

PROPOSITION 3.27: *Let D be a nef $\mathbb{Q}$-Cartier divisor. Then $\kappa(D) = d - \mathrm{rk}\, L_T$, where $T := \bigcup_{\mathcal{C} \in \mathrm{fib}(D)} \mathcal{C}$.*

Proof. We just remark that $\mathrm{rk}\, L_T$ is the dimension of the subvector space of $M_{\mathbb{Q}}$ which is generated by the l_i which are contained in a fibrational circuit. □

PROPOSITION 3.28: *Let X be a complete toric variety and D a nef divisor, then $H^i\big(X, \mathcal{O}(-D)\big) = 0$ for $i \neq \kappa(D)$.*

Proof. Consider the hyperplane arrangement given by the $H_i^{\underline{c}}$ in $M_{\mathbb{Q}}$. Let $m \in M_{\mathbb{Q}}$ and $I = \mathfrak{s}^{\underline{c}}(m)$. Then the simplicial complex $\hat{\Delta}_I$ can be characterized as follows. Consider $Q \subset P_D$ the union of the set faces of P_D which are contained in any $H_i^{\underline{c}}$ with $i \in I$. This is precisely the portion of P_D, which the the point m "sees", and therefore contractible, where the convex hull of Q and m provides the homotopy between Q and m. Therefore, every $\hat{\Delta}_I$ is contractible with an exception for $I = \emptyset$, because $\hat{\Delta}_{\emptyset} = \emptyset$, which is not acyclic with respect to reduced cohomology. Now we pass to the inverse, i.e. we consider the signature of $-m$ with respect to $H_i^{-\underline{c}}$. Then for any such $-m$ which does not sit in the relative interior of the polytope $P_{-\underline{c}}^{[n]}$, there exists $m' \in M_{\mathbb{Q}}$ with signature $\mathfrak{s}^{\underline{c}}(m') =: J$ such that $\hat{\Delta}_J$ is contractible and $\mathfrak{s}^{-\underline{c}}(m) = [n] \setminus J$. As $\hat{\Delta}$ is homotopic to a $d-1$-sphere, we can apply Alexander duality and thus the simplicial complex $\hat{\Delta}_{[n]\setminus J}$ is acyclic. Thus there remain only the elements in the relative interior of $P_{-\underline{c}}^{[n]}$. Let m be such an element with signature I, then $\hat{\Delta}_I$ is isomorphic to a $d-\kappa(D)-1$-sphere, and the assertion follows. □

This proposition implies the toric Kodaira and Kawamata-Viehweg vanishing theorems (see also [Mus02]):

Theorem 3.29 (Kodaira & Kawamata-Viehweg): *Let X be a complete toric variety and D, E $\mathbb{Q}$-divisors with D nef and $E = \sum_{i \in [n]} e_i D_i$ with $-1 < e_i < 0$ for all $i \in [n]$. Then:*

(i) if D is integral, then $H^i\big(X, \mathcal{O}(D + K_X)\big) = 0$ for all $i \neq 0, d - \kappa(D)$;

(ii) if $D + E$ is a Weil divisor, then $H^i\big(X, \mathcal{O}(D+E)\big) = 0$ for all $i > 0$.

Proof. Because a toric $\mathbb{Q}$-Cartier divisor is MCM, we can apply Serre duality (Corollary 3.25) and obtain $H^i\big(X, \mathcal{O}(D + K_X)\big) \cong H^{d-i}\big(X, \mathcal{O}(-D)\big)$ and (i) follows from Proposition 3.28.
For (ii): $D + E$ is contained the interior of every half space $K_X + H_{\mathfrak{C}}$ for $\mathfrak{C} \in \mathfrak{F}_{\mathrm{nef}}$, and the result follows. □

4 Arithmetic aspects of cohomology vanishing

In this section we want to generalize classical vanishing results for integral divisors which cannot directly be derived from the setting of $\mathbb{Q}$-divisors as in section 3.4. From now on we assume that the l_i are integral. Recall that for any integral divisor $D = \sum_{i \in [n]} c_i D_i$ and any torus invariant closed subvariety V of X, vanishing of $H^i_V\big(X, \mathcal{O}(D)\big)$ depends on two things:

(i) whether the set of signatures $\mathfrak{s}^{\underline{c}}(M_{\mathbb{Q}})$ consists of $I \subset [n]$ such that the relative cohomology groups $H^{i-1}(\hat{\Delta}_I, \hat{\Delta}_{V,I}; k)$ vanish, and,

(ii) if $H^{i-1}(\hat{\Delta}_I, \hat{\Delta}_{V,I}; k)$ for one such I, whether the corresponding polytope $P^I_{\underline{c}}$ contain lattice points m with $\mathfrak{s}^{\underline{c}}(m) = I$.

In the Gale dual picture, the signature $\mathfrak{s}^{\underline{c}}(M_{\mathbb{Q}})$ coincides with the set of $I \subset [n]$ such that the class of D in $A_{d-1}(X)_{\mathbb{Q}}$ is contained in C_I. For fixed I, the classes of divisors D in $A_{d-1}(X)$ such that the equation $l_i(m) < -c_i$ for $i \in I$ and $l_i(m) \geq -c_i$ for $i \notin I$ is satisfied, is counted by the *generalized partition function*. That is, by the function

$$D \mapsto \Big|\big\{(k_1, \ldots, k_n) \in \mathbb{N}^n \mid \sum_{i \in [n]\setminus I} k_i D_i - \sum_{i \in I} k_i D_i = D \text{ where } k_i > 0 \text{ for } i \in I\big\}\Big|.$$

So, in the most general picture, we are looking for D lying in the common zero set of the vector partition function for all relevant signatures I of D. In general, this is a difficult problem to determine these zero sets, and it is hardly necessary for practical purposes.

Vector partition functions play an important role in the combinatorial theory of rational polytopes and have been considered, e.g. in [Stu95], [BV97] (see also references therein). In [BV97] closed expressions in terms of residue formulas have been obtained. Moreover it was shown that the vector partition function is a piecewise quasipolynomial function, where the domains of quasipolynomiality are chambers (or possibly unions of chambers) of the secondary fan. In particular, for if $P^{\emptyset}_{\underline{c}}$ is a rational bounded polytope, then the values of the vector partition function for $P^{\emptyset}_{k\cdot\underline{c}}$ for $k \geq 0$, is just the Ehrhart quasipolynomial. A special case which we will consider in subsection 4.1 is where the vectors $l_1, \ldots, l_n$ form circuit. In this form, the computation of generalized partition functions is essentially equivalent to the classically known diophantine Frobenius problem (also known as money change problem or denumerant problem). We refer to the book [Ram05] for a general overview.

4.1 Arithmetic cohomology vanishing for circuits

In this subsection we assume that $n = d + 1$ and $\mathcal{C} = [n]$ forms a circuit. In light of Theorem 2.1, for cohomology vanishing on a toric 1-circuit variety, we have to consider the reduced cohomology of simplicial complexes associated to its fan:

Lemma 4.1: *Let $I \subset [n]$, such that $I \neq \mathfrak{C}^+$, then $H^i((\hat{\Delta}_{\mathfrak{C}})_I; k) = 0$ for all i. Moreover,*

$$(\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^+} \cong S^{|\mathfrak{C}^+|-2} \quad \textit{and} \quad (\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^-} \cong B^{|\mathfrak{C}^-|-1},$$

where B^k is the k-ball, with $B^{-1} := \emptyset$.

Proof. It is easy to see that $(\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^+}$ corresponds to the boundary of the $(|\mathfrak{C}^+| - 1)$-simplex, so it is homeomorphic to $S^{|\mathfrak{C}^+|-2}$. Similarly, $\{l_i\}_{i\in\mathfrak{C}^+}$ span a simplicial cone in $\Delta_{\mathfrak{C}}$ and thus $(\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^-} \cong B^{|\mathfrak{C}^-|-1}$. Now assume there exists $i \in \mathfrak{C}^+ \setminus I$, then I is a face of the cone σ_i and $(\hat{\Delta}_{\mathfrak{C}})_I$ is contractible. On the other hand, if $\mathfrak{C}^+$ is a proper subset of I, the set $I \cap \mathfrak{C}^-$ spans a cone τ in $\Delta_{\mathfrak{C}}$. The simplicial complex $\hat{\Delta}_I$ then is homeomorphic to a simplicial decomposition of the $(|\mathfrak{C}^+| - 1)$-ball with center τ and boundary $(\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^+}$. □

In this special situation, the chamber decomposition of $M_{\mathbb{Q}}$ by hyperplanes $H^{\underline{c}}_i$ as in subsection 3.2 contains at most one bounded chamber. In fact, if D is a rational divisor, all maximal chambers are unbounded. If $D \not\sim 0$, we have precisely one bounded chamber for whose signatures there are precisely two possibilities. Namely, we either have for every m in this chamber that $l_i(m) < -c_i$ for every $i \in \mathfrak{C}^-$ and $l_i(m) \geq -c_i$ for every $i \in \mathfrak{C}^+$, or vice versa. The set of rational divisor classes in $A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big)_{\mathbb{Q}} \cong \mathbb{Q}$ corresponding

to torus invariant divisors whose associated bounded chamber has signature either $\mathfrak{C}^+$ or $\mathfrak{C}^-$ corresponds precisely to the two open intervals $(-\infty, 0)$ and $(0, \infty)$, respectively, in $A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big)_{\mathbb{Q}}$.
To count lattice points in the bounded chamber we can use a special case of the generalized partition function, i.e. the number of lattice points m such that $l_i(m) \geq -c_i$ for $i \in \mathfrak{C}^+$ and $l_i(m) < -c_i$ for $i \in \mathfrak{C}^-$ coincides with the cardinality of the following set:

$$\{(k_1, \ldots, k_{d+1}) \in \mathbb{N}^{d+1} \mid k_i > 0 \text{ for } i \in \mathfrak{C}^- \text{ and } \sum_{i \in \mathfrak{C}^+} k_i D_i - \sum_{i \in \mathfrak{C}^-} k_i D_i = D\}.$$

For the integral case, this leads to *arithmetic thickenings* of the intervals $(-\infty, 0)$ and $(0, \infty)$ as follows:

Definition 4.2: We denote $F_{\mathfrak{C}} \subset A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big)$ the complement of the semigroup of the form $\sum_{i \in \mathfrak{C}^-} c_i D_i - \sum_{i \in \mathfrak{C}^+} c_i D_i$, where $c_i \in \mathbb{N}$ for all i with $c_i > 0$ for $i \in \mathfrak{C}^+$.

The set $F_{\mathfrak{C}}$ is the complement of the set of classes whose associated chamber has signature $\mathfrak{C}^-$ and contains a lattice point. With this we can give a complete characterization of global cohomology vanishing:

Proposition 4.3: *Let $\mathbb{P}(\underline{\alpha}, \xi)$ be as before with associated fan $\Delta_{\mathfrak{C}}$ and $D \in A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big)$, then:*

(i) $H^i\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) = 0$ for $i \neq 0, |\mathfrak{C}^+| - 1$;

(ii) $H^{|\mathfrak{C}^+|-1}\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) = 0$ iff $D \in F_{\mathfrak{C}}$;

(iii) if $\mathfrak{C}^+ \neq \mathcal{C}$, then $H^0\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) \neq 0$;

(iv) if $\mathfrak{C}^+ = \mathcal{C}$, then $H^0\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) = 0$ iff $D \in F_{-\mathfrak{C}}$.

Proof. The proof is immediate. Just observe that the simplicial complex $(\hat{\Delta}_{\mathfrak{C}})_m$, for m an element in the bounded chamber, coincides either with $(\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^+}$ or $(\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^-}$. ☐

Another case of interest is where $\mathfrak{C}^+ \neq \mathcal{C}$ and $V = V(\tau)$, where τ is the cone spanned by the l_i with $i \in \mathfrak{C}^-$, i.e. V is the unique maximal complete torus invariant subvariety of $\mathbb{P}(\underline{\alpha}, \xi)$.

Proposition 4.4: *Consider $\mathbb{P}(\underline{\alpha}, \xi)$ such that $\alpha_i < 0$ for at least one i, $D \in A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big)$ and V the maximal complete torus invariant subvariety of $\mathbb{P}(\underline{\alpha}, \xi)$, then:*

(i) $H^d_V\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) \neq 0$;

(ii) $H^{|\mathfrak{C}^-|}_V\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) = 0$ iff $D \in F_{\mathfrak{C}}$;

(iii) $H^i_V\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) = 0$ *for all* $i \neq d$, $|\mathfrak{C}^-|$.

Proof. Consider first $I = \mathcal{C}$, then $(\hat{\Delta}_{\mathfrak{C}})_I = \hat{\Delta}_{\mathfrak{C}} \cong B^{d-1}$ and $(\hat{\Delta}_{\mathfrak{C}})_{V,I} = (\hat{\Delta}_{\mathfrak{C}})_V \cong S^{d-2}$. It follows that $H^i\big(\hat{\Delta}_{\mathfrak{C}}; k\big) = 0$ for all i and $H^{i-1}\big(\hat{\Delta}_{\mathfrak{C}}, (\hat{\Delta}_{\mathfrak{C}})_V; k\big) \cong H^{i-2}\big((\hat{\Delta}_{\mathfrak{C}})_V; k\big)$. As by assumption, $\mathfrak{C}^+ \neq \mathcal{C}$, so the associated hyperplane arrangement contains an unbounded chamber such that $l_i(m) \geq -c_i$ for all $i \in \mathcal{C}$ and all m in this chamber. Hence (i) follows. As in the proof of lemma 4.1, it follows that $\hat{\Delta}_I$ is contractible whenever $\mathfrak{C}^+ \cap I \neq \emptyset$ and $\mathfrak{C}^- \cap I \neq \emptyset$. So in that case $H^i(\hat{\Delta}_I) = 0$ for all i and $H^{i-1}(\hat{\Delta}_I, \hat{\Delta}_{V,I}; k) = H^{i-2}(\hat{\Delta}_{V,I}; k)$ for all i.
Now let $I = \mathfrak{C}^+$; then $(\hat{\Delta}_{\mathfrak{C}})_I = (\hat{\Delta}\mathfrak{C})_{V,I} \cong S^{\mathfrak{C}^+ - 2}$, so $H^i\big((\hat{\Delta}_{\mathfrak{C}})_I, (\hat{\Delta}\mathfrak{C})_{V,I}; k\big) = 0$ for all i. For $I = \mathfrak{C}^-$, then $(\hat{\Delta}_{\mathfrak{C}})_I \cong B^{|\mathfrak{C}^-|-1}$ and $(\hat{\Delta}_{\mathfrak{C}})_{V,I} \cong S^{|\mathfrak{C}^-|-2}$, the former by Lemma 4.1, the latter by Lemma 4.1 and the fact that $(\hat{\Delta}_{\mathfrak{C}})_{V,I}$ has empty intersection with star(τ). This implies (ii) and consequently (iii). ☐

4.2 Arithmetic Kawamata-Viehweg vanishing

A first — trivial — approximation is given by the observation that the divisors D where the vector partition function takes a nontrivial value map to the cone C_I, shifted by the *offset* $e_I := -\sum_{i \in I} e_i$.

Definition 4.5: We denote $\mathbb{O}'(L, I)$ the saturation of the cone generated the $-D_i$ for $i \in I$ and the D_i for $i \notin I$ and $\mathbb{O}(L, I) := e_I + \mathbb{O}'(L, I)$. Moreover, we denote $\Omega(L, I)$ the zero set in $\mathbb{O}(L, I)$ of the vector partition function as defined above.

In the next step we want to approximate the sets $\Omega(L, I)$ by reducing to the classical diophantine Frobenius problem. For this, fix some $I \subset [n]$ and consider some polytope $P^I_{\underline{c}}$. It follows from Proposition 3.17 that D is contained in the intersection of half spaces $H_{\mathfrak{C}}$ for $\mathfrak{C} \in \mathfrak{C}(L)$ such that $\mathfrak{C}^- = \mathcal{C} \cap I$. In the polytope picture, we can interpret this as follows. For every $\mathfrak{C}$ and its underlying circuit $\mathcal{C}$, we set

$$P^{\mathfrak{C}}_{\underline{c}} := \{m \in M_{\mathbb{Q}} \mid l_i(m) \leq -c_i \text{ for } i \in \mathfrak{C}^- \text{ and } l_i(m) \geq -c_i \text{ for } i \in \mathfrak{C}^+\}.$$

Consequently, we get

$$P^I_{\underline{c}} = \bigcap_{\mathfrak{C}} P^{\mathfrak{C}}_{\underline{c}},$$

where the intersection runs over all $\mathfrak{C} \in \mathfrak{C}(L)$ with $\mathfrak{C}^- = \mathcal{C} \cap I$. It follows that if there exists a compatible oriented circuit $\mathfrak{C}$ such that $P^{\mathfrak{C}}_{\underline{c}}$ does not contain a lattice point, then $P^I_{\underline{c}}$ also does not contain a lattice point. We want to capture this by considering an arithmetic analogue of the discriminantal arrangement in $A_{d-1}(X)$ rather than in $A_{d-1}(X)_{\mathbb{Q}}$. Using the integral pendant to diagram (5):

Definition 4.6: Consider the morphism $\eta_I : A_{d-1}(X) \twoheadrightarrow A_I$. Then we denote Z_I its kernel. For $I = \mathcal{C}$ and $\mathfrak{C}$ some orientation of $\mathcal{C}$ we denote by $F_{\mathfrak{C}}$ the preimage in $A_{d-1}(X)$ of the complement of the semigroup consisting of elements $\sum_{i \in \mathfrak{C}^-} c_i D_i - \sum_{i \in \mathfrak{C}^+} c_i D_i$, where $c_i \geq 0$ for $i \in \mathfrak{C}^-$ and $c_i > 0$ for $i \in \mathfrak{C}^+$. We set $F_{\mathcal{C}} := F_{\mathfrak{C}} \cap F_{-\mathfrak{C}}$.

So, there are two candidates for a discriminantal arrangement in $A_{d-1}(X)$, the $Z_{\mathcal{C}}$ on the one hand, and the $F_{\mathcal{C}}$ on the other.

Definition 4.7: We denote:

- $\{Z_{\mathcal{C}}\}_{\mathcal{C} \in \mathcal{C}(L)}$ the *integral* discriminantal arrangement, and
- $\{F_{\mathcal{C}}\}_{\mathcal{C} \in \mathcal{C}(L)}$ the *Frobenius* discriminantal arrangement.

The integral discriminantal arrangement has similar properties as the H_I, as they give a solution to the integral discriminant problem (compare Lemma 3.9):

Lemma 4.8: *Let $I \subset [n]$, then*

$$Z_I = \bigcap_{\mathcal{C} \in \mathcal{C}(L_I)} Z_{\mathcal{C}}.$$

We can now locate both the rational as well as the integral Picard group in $A_{d-1}(X)_{\mathbb{Q}}$ and $A_{d-1}(X)$, respectively:

Theorem 4.9 (see [Eik92], Theorem 3.2): *Let X be any toric variety, then:*

(i) $\operatorname{Pic}(X)_{\mathbb{Q}} = \bigcap_{\sigma \in \Delta_{\max}} H_{\sigma(1)} = \bigcap_{\substack{\mathcal{C} \in \mathcal{C}(L_{\sigma(1)}) \\ \sigma \in \Delta_{\max}}} H_{\mathcal{C}}$.

(ii) $\operatorname{Pic}(X) = \bigcap_{\sigma \in \Delta_{\max}} Z_{\sigma(1)} = \bigcap_{\substack{\mathcal{C} \in \mathcal{C}(L_{\sigma(1)}) \\ \sigma \in \Delta_{\max}}} Z_{\mathcal{C}}$.

Proof. (i) As remarked in subsection 3.4, a $\mathbb{Q}$-Cartier divisor is specified by a collection $\{m_\sigma\}_{\sigma \in \Delta} \subset M_{\mathbb{Q}}$. In particular, all for every $\sigma \in \Delta$, the hyperplanes $H_i^{\underline{c}}$ with $i \in \sigma(1)$ have nonempty intersection. So the first equality follows. The second equality follows from Lemma 3.9.
(ii) A Cartier divisor is specified by a collection $\{m_\sigma\}_{\sigma \in \Delta} \subset M$ such that the hyperplanes $H_i^{\underline{\mathcal{C}}}$ with $i \in \sigma(1)$ intersect in integral points. So the first equality follows. The second equality follows from Lemma 4.8. □

The Frobenius discriminantal arrangement is not as straightforward. First, we note the following properties:

Lemma 4.10: *Let $\mathcal{C} \in \mathcal{C}(L)$, then:*

(i) $F_{\mathcal{C}}$ is nonempty;

(ii) the saturation of $Z_{\mathcal{C}}$ in $A_{d-1}(X)$ is contained in $F_{\mathcal{C}}$ iff $\mathcal{C}$ is not fibrational.

Proof. The first assertion follows because $F_{\mathcal{C}}$ contains all elements which map to the open interval $(K_{\mathfrak{C}}, K_{-\mathfrak{C}})$ in $A_{\mathcal{C},\mathbb{Q}}$, where $K_{\mathfrak{C}} = -\sum_{i\in\mathfrak{C}^+} D_i$. For the second assertion, note that the set $\{m \in M \mid l_i(m) = 0 \text{ for all } i \in \mathcal{C}\}$ is in $F_{\mathcal{C}}$ iff $\mathfrak{C}^+ \neq \mathcal{C}$ for either orientation $\mathfrak{C}$ of $\mathcal{C}$. □

Lemma 4.10 shows that the $F_{\mathcal{C}}$ are thickenings of the $Z_{\mathcal{C}}$ with one notable exception, where $\mathcal{C}$ is fibrational. In this case, $F_{\mathcal{C}}$ can be considered as parallel to, but slightly shifted away from $Z_{\mathcal{C}}$. In the sequel we will not make any explicit use of the $Z_{\mathcal{C}}$ anymore, but these facts should be kept in mind.
Regarding the Frobenius discriminantal arrangement, we want also to consider integral versions of the discriminantal strata:

DEFINITION 4.11: Let $\mathfrak{C} \in \mathfrak{C}(L)$ and let $\mathfrak{F}_{\mathcal{S}}$ be a discriminantal hull of $\mathcal{S} = \{S_1, \ldots, S_k\}$, then we denote

$$\mathfrak{A}_{\mathcal{S}} := \bigcap_{\mathfrak{C}\in\mathfrak{F}_{\mathcal{S}}} F_{\mathfrak{C}}.$$

the *arithmetic core* of $\mathfrak{F}_{\mathcal{S}}$. In the special case $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\text{nef}}$ we write $\mathfrak{A}_{\text{nef}}$.

REMARK 4.12: The notion *core* refers to the fact that we consider *all* $F_{\mathfrak{C}}$, instead of a non-redundant subset describing the set S as a convex cone.

We will use arithmetic cores to derive arithmetic versions of known vanishing theorems formulated in the setting of $\mathbb{Q}$-divisors and to get refined conditions on cohomology vanishing. This principle is reflected in the following theorem:

THEOREM 4.13: *Let V be a T-invariant closed subscheme of X and S a discriminantal stratum in $A_{d-1}(X)_{\mathbb{Q}}$. If $H^i_V\big(X, \mathcal{O}(D)\big) = 0$ for some i and for all integral divisors $D \in S$, then also $H^i_V\big(X, \mathcal{O}(D)\big) = 0$ for all $D \in \mathfrak{A}_S$.*

Proof. Without loss of generality we can assume that $\dim S > 0$. Consider some nonempty $P^I_{\underline{c}}$ for some $I \subset [n]$. Then for any such I, we can choose some multiple of kD such that $P^I_{k\underline{c}}$ contains a lattice point. But if $H^i_V\big(X, \mathcal{O}(D)\big) = 0$, then also $H^i_V\big(X, \mathcal{O}(kD)\big) = 0$, hence $H^{i-1}(\hat{\Delta}_I, \hat{\Delta}_{V,I}; k) = 0$. Now, any divisor $D' \in \mathfrak{A}_S$ which does not map to S, is contained in $F_{\mathfrak{C}}$ for all $\mathfrak{C} \in \mathfrak{F}_S$ and therefore for any I which is in the signature for D' but not for D, the equations $l_i(m) < -c'_i$ for $i \in I$ and $l_i(m) \geq -c'_i$ for $i \notin I$ cannot have any integral solution. □

We apply Theorem 4.13 to $\mathfrak{A}_{\text{nef}}$:

THEOREM 4.14 (Arithmetic version of Kawamata-Viehweg vanishing): *Let X be a complete toric variety. Then $H^i\big(X, \mathcal{O}(D)\big) = 0$ for all $i > 0$ and all $D \in \mathfrak{A}_{\text{nef}}$.*

Proof. We know that the assertion is true if D is nef. Therefore we can apply Theorem 4.13 to the maximal strata $S_1, \ldots, S_k$ of $\mathrm{nef}(X)$. Therefore the assertion is true for $D \in \bigcap_{i=1^k} \mathfrak{A}_{S_i}$. To prove the theorem, we have to get rid of the $F_{\mathfrak{C}}$, where $H_{\mathcal{C}}$ intersects the relative interior of a face of $\mathrm{nef}(X)$. Let $\mathcal{C}$ be such a circuit and R the face. Without loss of generality, $\dim R > 0$. Then we can choose elements D' in R at an arbitrary distance from $H_{\mathfrak{C}}$, i.e. such that the polytope $P^{\mathfrak{C}}_{\underline{c}}$ becomes arbitrarily big and finally contains a lattice point. Now, if we move outside $\mathrm{nef}(X)$, but stay inside $\mathfrak{A}_{\mathrm{nef}}$, the lattice points of $P^{\mathfrak{C}}_{\underline{c}}$ cannot acquire any cohomology and the assertion follows. □

One can imagine an analog of the set $\mathfrak{A}_S$ in $A_{d-1}(X)_{\mathbb{Q}}$ as the intersection of shifted half spaces

$$\bigcap_{\mathfrak{C} \in \mathfrak{F}_S} \big(- \sum_{i \in \mathfrak{C}^+} D_i + H_{\mathfrak{C}} \big).$$

The main difference here is that one would picture the proper facets of this convex polyhedral set as "smooth", whereas the proper "walls" of $\mathfrak{A}_S$ have "ripples", which arise both from the fact that the groups $A_{\mathcal{C}}$ may have torsion, and that we use Frobenius conditions to determine the augmentations of our half spaces.
In general, the set $\mathfrak{F}_S$ is highly redundant when it comes to determine $\overline{S}$, which implies that above intersection does not yield a cone but rather a polyhedron, whose recession cone corresponds to $\overline{S}$. In the integral situation we do not quite have a recession cone, but a similar property holds:

Proposition 4.15: *Let $V \subset X$ be a closed invariant subscheme and $\mathcal{S} = \{S_1, \ldots, S_k\}$ a collection of discriminantal stata different from zero. Then for any nonzero face of its discriminantal hull $\overline{S}$ there exists the class of an integral divisor $D' \in \overline{S}$ such that the intersection of the half line $D + rD'$ for $0 \leq r \in \mathbb{Q}$ with $\mathfrak{A}_{\mathcal{S}}$ contains infinitely many classes of integral divisors.*

Proof. Let $R \subset \overline{S}$ be any face of S, then the vector space spanned by R is given by an intersection $\bigcap_{\mathcal{C} \text{ with } \mathfrak{C} \in K} H_{\mathcal{C}}$ for a certain subset $K \subset \mathfrak{F}_{\mathcal{S}}$. We assume that K is maximal with this property. The intersection $\bigcap_{\mathfrak{C} \in K} F_{\mathfrak{C}}$ is invariant with respect to translations along certain (though not necessarily all) $D' \in R$. This implies that the line (or any half line, respectively), generated by D' intersects $\bigcap_{\mathfrak{C} \in K} F_{\mathfrak{C}}$ in infinitely many points. As K is maximal, there is no other $\mathfrak{C} \in \mathfrak{F}_{\mathcal{C}}$ parallel to R and the assertion follows. □

The property of Proposition 4.15 is necessary for elements in $\mathfrak{A}_{\mathcal{S}}$, but not sufficient. This leads to the following definition:

Definition 4.16: Let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a collection of discriminantal strata and $D \in A_{d-1}(X)$ such that the property of Proposition 4.15 holds. If D is not contained in $\mathfrak{A}_{\mathcal{S}}$, then we call D $\mathfrak{A}_{\mathcal{S}}$-residual. We call D 0-*residual* if it is in the complement of $\mathfrak{A}_0 = \bigcap_{\mathfrak{C} \in \mathfrak{C}(L)} F_{\mathfrak{C}}$.

In the next subsections we will consider several special cases of interest for cohomology vanishing, which are not directly related to Kawamata-Viehweg vanishing theorems. In subsection 4.3 we will consider global cohomology vanishing for divisors in the inverse nef cone. In subsection 4.4 we will present a more explicit determination of this type of cohomology vanishing for toric surfaces. Finally, in subsection 4.5, we will give a geometric criterion for determing maximally Cohen-Macaulay modules.

4.3 Nonstandard Cohomology Vanishing

In this subsection we want to give a qualitative description of cohomology vanishing which is related to divisors which are *inverse* to nef divisors of Iitaka dimension $0 < \kappa(D) < d$. We show the following theorem:

Theorem 4.17: *Let X be a complete d-dimensional toric variety. Then $H^i\big(X, \mathcal{O}(D)\big) = 0$ for every i and all D which are contained in some $\mathfrak{A}_{-F}$, where F is a face of* $\operatorname{nef}(X)$ *which contains nef divisors of Iitaka dimension $0 < \kappa(D) < d$. If $\mathfrak{A}_{-F}$ is nonempty, then it contains infinitely many divisor classes.*

Proof. Recall that such a divisor, as a $\mathbb{Q}$-divisor, is contained in the intersection $\bigcap_{\mathcal{C} \in \mathrm{fib}(D)} H_{\mathcal{C}}$ and therefore it is in the intersection of the nef cone with the boundary of the effective cone of X by Proposition 3.17. Denote this intersection by F. Then we claim that $H^i\big(X, \mathcal{O}(D')\big) = 0$ for all $D' \in \mathfrak{A}_{-F}$. By Corollary 3.28 we know that $H^i\big(X, \mathcal{O}(E)\big) = 0$ for $0 \leq i < d$ for any divisor E in the interior of the inverse nef cone. This implies that $H^i\big(X, \mathcal{O}(E)\big) = 0$ for any $E \in \mathfrak{A}_{-nef}$ and hence $H^i\big(X, \mathcal{O}(D')\big) = 0$ for any $D' \in \mathfrak{A}_{-F}$, because $\mathfrak{A}_{-F} \subset \mathfrak{A}_{-\mathrm{nef}}$. The latter assertion follows from the fact that the assumption on the Iitaka dimension implies that the face F has positive dimension. $\square$

Note that criterion is not very strong, as it is not clear in general whether the set $\mathfrak{A}_{-F}$ is nonempty. However, this is the case in a few interesting cases, in particular for toric surfaces, as we will see in the next subsection. The following remark shows that our condition indeed is rather weak in general:

Remark 4.18: The inverse of any big and nef divisor D with the property that P_D does not contain any lattice point in its interior has the property that $H^i\big(X, \mathcal{O}(D)\big) = 0$ for all i. This follows directly from the standard fact in toric geometry that the Euler characteristics $\chi(-D)$ counts the inner lattice points of the lattice polytope P_D.

4.4 The case of complete toric surfaces.

Let X be a complete toric surface. We assume that the l_i are cyclically ordered. We consider the integers $[n]$ as system of representatives for $\mathbb{Z}/n\mathbb{Z}$, i.e. for some $i \in [n]$ and $k \in \mathbb{Z}$, the sum $i + k$ denotes the unique element in $[n]$ modulo n.

PROPOSITION 4.19: *Let X be a complete toric surface. Then* $\mathrm{nef}(X) = \overline{S}$, *where S is a single stratum of maximal dimension of the discriminantal arrangement.*

Proof. X is simplicial and projective and therefore $\mathrm{nef}(X)$ is a cone of maximal dimension in $A_1(X)_{\mathbb{Q}}$. We show that no hyperplane $H_{\mathcal{C}}$ intersects the interior of $\mathrm{nef}(X)$. By Proposition 3.17 we can at once exclude fibrational circuits. This leaves us with non-fibrational circuits $\mathcal{C}$ with cardinality three, having orientation $\mathfrak{C}$ with $|\mathfrak{C}^+| = 2$. Assume that D is contained in the interior of $H_{-\mathfrak{C}}$. Then there exists $m \in M_{\mathbb{Q}}$ such that $\mathfrak{C}^+ \subset \mathfrak{s}^D(m)$, which implies that the hyperplane $H_i^{\underline{c}}$ for $\{i\} = \mathfrak{C}^-$ does not intersect P_D, and thus D cannot be nef. It follows that $\mathrm{nef}(X) \subset H_{\mathfrak{C}}$. □

Now assume there exist $p, q \in [n]$ such that $l_q = -l_p$, i.e. l_p and l_q represent a one-dimensional fibrational circuit of L. Then for any nef divisors D which is contained in $H_{p,q}$, the associated polytope P_D is one-dimensional. The only possible variation for P_D is its length in terms of lattice units. So we can conclude that $\mathrm{nef}(X) \cap H_{p,q}$ is a one-dimensional face of $\mathrm{nef}(X)$.

DEFINITION 4.20: Let X be a complete toric surface and $\mathcal{C} = \{p, q\}$ such that $l_p = -l_q$. Then we denote $S_{p,q}$ the relative interior of $-\mathrm{nef}(X) \cap H_{\mathcal{C}}$. Moreover, we denote $\mathfrak{A}_{p,q}$ the arithmetic core of $S_{p,q}$.

Our aim in this subsection is to prove the following:

THEOREM 4.21: *Let X be a complete toric surface. Then there are only finitely many divisors D with $H^i\big(X, \mathcal{O}(D)\big) = 0$ for all $i > 0$ which are not contained in $\mathfrak{A}_{\mathrm{nef}} \cup \bigcup \mathfrak{A}_{p,q}$, where the union ranges over all pairs $\{p, q\}$ such that $l_p = -l_q$.*

We will prove this theorem in several steps. First we show that the interiors of the C_I such that $H^0(\hat{\Delta}_I; k) \neq 0$ cover all of $A_1(X)_{\mathbb{Q}}$ except $\mathrm{nef}(X)$ and $-\mathrm{nef}(X)$.

PROPOSITION 4.22: *Let $D = \sum_{i \in [n]} c_i D_i$ be a Weil divisor which is not contained in* $\mathrm{nef}(X)$ *or* $-\mathrm{nef}(X)$, *then the corresponding arrangement $H_i^{\underline{c}}$ in $M_{\mathbb{Q}}$ has a two-dimensional chamber $P_{\underline{c}}^I$ such that complex $\hat{\Delta}_I$ has at least two components.*

Proof. Recall that $\mathrm{nef}(X) = \bigcap H_{\mathfrak{C}}$, where the intersection runs over all oriented circuits which are associated to extremal curves of X. As the statement is well-known for the case where X is either a 1-circuit toric variety or a Hirzebruch surface, we can assume without loss of generality, that the extremal curves belong to blow-downs, i.e. the associated oriented circuits are of the form $\mathfrak{C}^+ = \{i-1, i+1\}$, $\mathfrak{C}^- = \{i\}$ for any $i \in [n]$. Now assume that D is in the interior of $H_{\mathfrak{C}}$ for such an oriented circuit $\mathfrak{C}$. Then there exists a bounded chamber $P_{\underline{c}}^I$ in $M_{\mathbb{Q}}$ such that $\mathfrak{C}^- = \mathcal{C} \cap \mathfrak{s}^{\underline{c}}(m)$. In order for $\hat{\Delta}_{\mathfrak{s}^{\underline{c}}(m)}$ to be acyclic, it is necessary that $\mathfrak{s}^{\underline{c}}(m) \cap ([n] \setminus \mathcal{C}) = \emptyset$. Let $\{j, k, l\} =: \mathcal{D} \subset [n]$

represent any other circuit such that $\mathfrak{D}^+ = \{j, l\}$ for some orientation $\mathfrak{D}$ of $\mathcal{D}$. The hyperplane arrangement given by the three hyperplanes $H_j^{\underline{c}}, H_k^{\underline{c}}, H_l^{\underline{c}}$ has six unbounded regions, whose signatures contain any subset of $\{j, k, l\}$ except $\{j, l\}$ and $\{k\}$. In the cases $j = i-2, k = i-1, l = 1$ or $j = i, k = i+1, l = i+2$, $P_{\underline{c}}^I$ must be contained in the region with signature $\{i\}$. In every other case $P_{\underline{c}}^I$ must be contained in the region with signature $\emptyset$. In the case, say, $\{j, k, l\} = \{i-2, i-1, i\}$, the hyperplane $H_{i-2}^{\underline{c}}$ should not cross the bounded chamber related to the subarrangement given by the hyperplanes $H_{i-1}^{\underline{c}}$, $H_i^{\underline{c}}$, $H_{i+1}^{\underline{c}}$, as else we obtain a chamber whose signature contains $\{i-1, i+1\}$, but not $\{i-2, i\}$. Then the associated subcomplex of $\hat{\Delta}$ can never be acyclic. This implies that, if D is in the interior of $H_{\mathfrak{C}}$, then $D \in H_{\mathfrak{D}}$, where either $\mathcal{D} = \{i-2, i-1, i\}$ or $\mathcal{D} = \{i, i+1, i+2\}$. By iterating for every extremal (i.e. every invariant) curve, we conclude that $D \in \bigcap_{i \in [n]} H_{\mathfrak{C}} = \operatorname{nef}(X)$. Analogously, we conclude for $D \in H_{-\mathfrak{C}}$ that $D \in -\operatorname{nef}(X)$, and the statement follows. $\square$

Let $\{p, q\} \subset [n]$ such that $l_p = -l_q$. Then these two primitive vectors span a 1-dimensional subvector space of $N_{\mathbb{Q}}$, which naturally separates the set $[n] \setminus \{p, q\}$ into two subsets.

Definition 4.23: Let $\{p, q\} \subset [n]$ such that $l_p = -l_q$. Then we denote $A_{p,q}^1, A_{p,q}^2 \subset [n]$ the two subsets of $[n] \setminus \{p, q\}$ separated by the line spanned by l_p, l_q.

For some fibrational circuit $\{p, q\}$, the closure $\overline{S}_{p,q}$ is a one-dimensional cone in $A_1(X)_{\mathbb{Q}}$ which has a unique primitive vector:

Definition 4.24: Consider$\{p, q\}$ as before. Then the closure $\overline{S}_{p,q}$ is a one-dimensional cone with primitive lattice vector $D_{p,q} := \sum_{i \in A_{p,q}^1} l_i(m) D_i$, where $m \in M$ the unique primitive vector on the ray in $M_{\mathbb{Q}}$ with $l_p(m) = l_q(m) = 0$ and $l_i(m) < 0$ for $i \in A_{p,q}^1$.

Proposition 4.25: *Let X be a complete toric surface. Then every $\mathfrak{A}_{p,q}$-residual divisor on X is either contained in $\mathfrak{A}_{\text{nef}}$, or in some $\mathfrak{A}_{p,q}$, or is $\mathfrak{A}_{\text{nef}}$-residual.*

Proof. For any nef divisor $D \in -S_{p,q}$, the polytope P_D is a line segment such that all $H_i^{\underline{c}}$ intersect this line segment in one of its two end points, depending on whether $i \in A_{p,q}^1$ or $i \in A_{p,q}^2$. This implies that the line spanned by $S_{p,q}$ is the intersection of all $H_{\mathcal{C}}$, where $\mathcal{C} \subset A_{p,q}^1 \cup \{p, q\}$ or $\mathcal{C} \subset A_{p,q}^2 \cup \{p, q\}$. Let D be $\mathfrak{A}_{p,q}$-residual and assume that $H^i\big(X, \mathcal{O}(D + rD_{p,q})\big) = 0$ for all i and for infinitely many r. We first show that $D \in F_{\{p,q\}}$, i.e. that $c_p + c_q = -1$ for any torus invariant representative $D = \sum_{i \in [n]} c_i D_i$. Assume that $c_p + c_q > -1$. Then there exists $m \in M$ such that $p, q \notin \mathfrak{s}^{\underline{c}}(m)$. By adding sufficiently high multiples of $D_{p,q}$ such that $D + rD_{p,q} = \sum c'_i D_i$, we can even find such an m such that $A_1 \cup A_2 \subset \mathfrak{s}^{\underline{c}'}(m)$, hence $H^1\big(X, \mathcal{O}(D + rD_{p,q})\big) \neq 0$ for large r and thus D is not $\mathfrak{A}_{p,q}$-residual. If $c_p + c_q < -1$, there is an $m \in M$ with

$\{p,q\} \subset \mathfrak{s}^{\underline{c}}(m)$, and by the same argument, we get $H^2\big(X, \mathcal{O}(D + rD_{p,q})\big) \neq 0$ for large r. Hence $c_p + c_q = -1$, i.e. $D \in F_{\{p,q\}}$. This implies that for every $m \in M$ either $p \in \mathfrak{s}^{\underline{c}}(m)$ and $q \notin \mathfrak{s}^{\underline{c}}(m)$, or $q \in \mathfrak{s}^{\underline{c}}(m)$ and $p \notin \mathfrak{s}^{\underline{c}}(m)$. Now assume that $D \notin \mathcal{F}_{\mathcal{C}}$ for some $\mathcal{C} = \{i,j,k\} \subset A_1 \cup \{p,q\}$ such that $\mathfrak{C}^+ = \{i,k\}$ for some orientation. Then there exists some $m \in M$ with $\{i,k\} \subset \mathfrak{s}^{\underline{c}}(m)$ or $\{j\} \subset \mathfrak{s}^{\underline{c}}(m)$. In the first case, as before we can simply add some multiple of $D_{p,q}$ such that $i \in \mathfrak{s}^{\underline{c}'}(m)$ and $i \in A_2$, hence $\mathfrak{s}^{\underline{c}'}(m)$ contains at least two $-$-intervals. In the second case, we have either $p \notin \mathfrak{s}^{\underline{c}}(m)$ or $q \notin \mathfrak{s}(m)$, thus at least two $-$-intervals, too. Hence $D \in \mathfrak{A}_{p,q}$ and the assertion follows. □

Proposition 4.26: *Let X be a complete toric surface. Then X has only a finite number of $\mathfrak{A}_{\mathrm{nef}}$-residual divisors.*

Proof. We can assume without loss of generality that X is not $\mathbb{P}^2$ nor a Hirzebruch surface. Assume there is $D \in A_1(X)$ which is not contained in $F_{\mathfrak{C}}$ for some circuit $\mathcal{C} = \{i-1, i, i+1\}$ corresponding to an extremal curve on X. Then there exists a chamber in the corresponding arrangement whose signature contains $\{i-1, i+1\}$. To have this signature to correspond to an acyclic subcomplex of $\hat{\Delta}$, the rest of the signature must contain $[n] \setminus \mathcal{C}$. Now assume we have some integral vector $D_{\mathcal{C}} \in H_{\mathcal{C}}$, then we can add a multiple of $D_{\mathcal{C}}$ to D such that D is parallel translated to $\mathrm{nef}(X)$. In this process necessarily at least one hyperplane passes the critical chamber and thus creates cohomology. Now, D might be outside of $F_{\mathfrak{D}}$ for some $\mathfrak{D} \in \mathfrak{C}(L)$ not corresponding to an extremal curve. If the underlying circuit is not fibrational, then D being outside $F_{\mathfrak{D}}$ implies $F_{\mathfrak{C}}$ for some extremal circuit $\mathfrak{C}$. If $\mathfrak{D}$ is fibrational and $\mathcal{D} = \{p,q\}$, then we argue as in Proposition 4.25 that D has cohomology. If $\mathcal{D}$ is fibrational of cardinality three, the corresponding hypersurface $H_{\mathcal{D}}$ is not parallel to any nonzero face of $\mathrm{nef}(X)$ and there might be a finite number of divisors lying outside $F_{\mathfrak{D}}$ but in the intersection of all $F_{\mathfrak{C}}$, where $\mathcal{C}$ corresponds to an extremal curve. □

Proposition 4.27: *Let X be a complete toric surface. Then X has only a finite number of 0-residual divisors.*

Proof. Let us consider some vector partition function $\mathrm{VP}(L, I) : \mathbb{O}_I \longrightarrow \mathbb{N}$, for I such that C_I does not contain a nonzero subvector space. Let $D = \sum_{i \in [n]} c_i D_i \in \Omega(L, I)$ and let P_D the polytope in $M_{\mathbb{Q}}$ such that $m \in M_{\mathbb{Q}}$ is in P_D iff $l_i(m) < -c_i$ for $i \in I$ and $l_i(m) \geq -c_i$ for $i \in [n] \setminus I$. For any $J \subset [n]$ we denote $P_{D,J}$ the polytope defined by the same inequalities, but only for $i \in J$. Clearly, $P_D \subset P_{D,J}$. Let $J \subset [n]$ be maximal with respect to the property that $P_{D,J}$ does not contain any lattice points. If $J \neq [n]$, then we can freely move the hyperplanes given by $l_i(m) = -c_i$ for $i \in [n] \setminus I$ such that $P_{D,J}$ remains constant and thus lattice point free. This is equivalent to say that there exists a nonzero $D' \in \bigcap_{\mathcal{C} \in \mathcal{C}(L_J)} H_{\mathcal{C}}$ and for every such D' the polytope $P_{D+jD'}$ does not contain any lattice point for any $j \in \mathbb{Q}_{>0}$.

Now assume that $J = [n]$. This implies that the defining inequalities of P_D are irredundant and thus there exists a unique maximal chamber in C_I which contains D (if $I = \emptyset$ this would be the nef cone by 4.19) and thus the combinatorial type of P_D is fixed. Now, clearly, the number of polygons of shape P_D with parallel faces given by integral linear inequalities and which do not contain a lattice point is finite.
By applying this to all (and in fact finitely many) cones $\mathbb{O}_I$ such that C_I does not contain a nontrivial subvector space of $A_{\mathbb{Q}}$, we see that there are only finitely many divisors D which are not contained in $\mathfrak{A}_{\mathrm{nef}}$ or $\mathfrak{A}_{p,q}$. $\square$

Proof of theorem 4.21. By 4.22, $\mathrm{nef}(X)$ and the $S_{p,q}$ are indeed the only relevant strata, which by 4.25 and 4.26 admit only finitely many residual elements. Hence, we are left with the 0-residuals, of which exist only finitely many by 4.27. $\square$

Example 4.28: Figure 1 shows the cohomology free divisors on the Hirzebruch surface $\mathbb{F}_3$ which is given by four rays, say $l_1 = (1,0)$, $l_2 = (0,1)$, $l_3 = (-1,3)$, $l_4 = (0,-1)$ with respect to some choice of coordinates for N. In $\mathrm{Pic}(\mathbb{F}_3) \cong \mathbb{Z}^2$ there are two cones such that $H^1\big(X, \mathcal{O}(D)\big) \neq 0$ for every D which is contained in one of these cones. Moreover, there is one cone such that $H^2\big(X, \mathcal{O}(D)\big) \neq 0$ for every D; its tip is sitting at $K_{\mathbb{F}_3}$. The nef cone is indicated by the dashed lines.

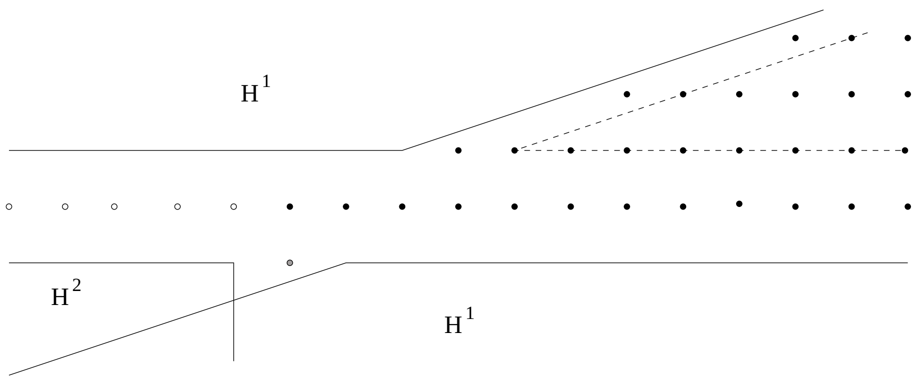


Figure 1: Cohomology free divisors on $\mathbb{F}_3$.

The picture shows the divisors contained in $\mathfrak{A}_{\mathrm{nef}}$ as black dots. The white dots indicate the divisors in $\mathfrak{A}_{2,4}$. There is one 0-residual divisor indicated by the grey dot.
The classification of smooth complete toric surfaces implies that every such surface which is not $\mathbb{P}^2$, has a fibrational circuit of rank one. Thus the theorem implies that on every such surface there exist families of line bundles with vanishing cohomology along the inverse nef cone. For a given toric surface X,

these families can be explicitly computed by checking for every $\mathcal{C} \subset A_1 \cup \{p, q\}$ and every $\mathcal{C} \subset A_2 \cup \{p, q\}$, respectively, whether the inequalities

$$c_i + l_i(m) \begin{cases} \geq 0 & \text{for } i \in \mathfrak{C}^+ \\ < 0 & \text{for } i \in \mathfrak{C}^-, \end{cases} \qquad c_i + l_i(m) \begin{cases} \geq 0 & \text{for } i \in -\mathfrak{C}^+ \\ < 0 & \text{for } i \in -\mathfrak{C}^- \end{cases}$$

have solutions $m \in M$ for at least one of the two orientations $\mathfrak{C}$, $-\mathfrak{C}$ of $\mathcal{C}$. This requires to deal with $\binom{|A_1|+2}{3} + \binom{|A_2|+2}{3}$, i.e. of order $\sim n^3$, linear inequalities. We can reduce this number to order $\sim n^2$ as a corollary from our considerations above:

COROLLARY 4.29: *Let $\mathcal{C} \in A_i$ for $i = 1$ or $i = 2$. Then there exist $\{i, j\} \subset \mathcal{C}$ such that $F_{\{p,q\}} \cap F_{\mathfrak{C}} \supset F_{\{p,q\}} \cap F_{\{i,j,p\}} \cap F_{\{i,j,q\}}$.*

Proof. Assume first that there exists $m \in M$ which for the orientation $\mathfrak{C}$ of $\mathcal{C} = \{i_1, i_2, i_3\}$ with $\mathfrak{C}^+ = \{i_1, i_3\}$ which fulfills the inequalities $l_{i_k}(m) + c_{i_k} \geq 0$ for $k = 1, 3$ and $l_{i_2}(m) + c_{i_2} < 0$. This implies that $H^1\big(X, \mathcal{O}(D)\big) \neq 0$, independent of the configuration of the other hyperplanes, as long as $c_p + c_q = -1$. It is easy to see that we can choose $i, j \in \mathcal{C}$ such that $\{i, j, p\}$ and $\{i, j, q\}$ form circuits. We can choose one of those such that m is contained in the triangle, fulfilling the respective inequalities, and which is not fibrational. For the inverse orientation $-\mathfrak{C}$, we can the same way replace one of the elements of $\mathcal{C}$ by one of p, q. By adding a suitable positive multiple of $D_{p,q}$, we can rearrange the hyperplanes such that $H^1\big(X, \mathcal{O}(D + rD_{p,q})\big) \neq 0$. $\square$

One should read the corollary the way that for any pair i, j in A_1 or in A_2, one has only to check whether a given divisor fulfills certain inequalities for triples $\{i, j, q\}$ and $\{i, j, p\}$. It seems that it is not possible to reduce further the number of equations in general. However, there is a criterion which gives a good reduction of cases for practical purposes:

COROLLARY 4.30: *Let X be a smooth and complete toric surface and $D = \sum_{i \in [n]} c_i D_i \in \mathfrak{A}_{p,q}$, then for every $i \in A_1 \cup A_2$, we have:*

$$c_{i-1} + c_{i+1} - a_i c_i \in [-1, a_i - 1],$$

where the a_i are the self-intersection numbers of the D_i.

Proof. The circuit $\mathcal{C} = \{i-1, i, i+1\}$ comes with the integral relation $l_{i-1} + l_{i+1} + a_i l_i = 0$. So the Frobenius problem for these circuits is trivial and we have only to consider the offset part. $\square$

The following example shows that these equalities are necessary, but not sufficient in general:

EXAMPLE 4.31: We choose some coordinates on $N \cong \mathbb{Z}^2$ and consider the complete toric surface defined by 8 rays $l_1 = (0, -1)$, $l_2 = (1, -2)$, $l_3 = (1, -1)$,

$l_4 = (1,0)$, $l_5 = (1,1)$, $l_6 = (1,2)$, $l_7 = (0,1)$, $l_8 = (-1,0)$. Then any divisor $D = c_1 D_1 + \cdots + c_8 D_8$ with $\underline{c} = (-1,1,1,0,0,1,0,-k)$ for some $k \gg 0$ has nontrivial H^1, though it fulfills the conditions of corollary 4.30.

An interesting and more restricting case is the additional requirement that also $H^i\big(X, \mathcal{O}(-D)\big) = 0$ for all $i > 0$. One may compare the following with the classification of bundles of type B in [HP06].

Corollary 4.32: *Let X be a smooth and complete toric surface and $D \in \mathfrak{A}_{p,q}$ such that $H^i\big(X, \mathcal{O}(D)\big) = H^i\big(X, \mathcal{O}(-D)\big) = 0$ for all $i > 0$. Then for every $i \in A_1 \cup A_2$, we have:*

$$c_{i-1} + c_{i+1} - a_i c_i \in \begin{cases} \{\pm 1, 0\} & \text{if } a_i < -1 \\ \{-1, 0\} & \text{if } a_i = -1, \end{cases}$$

where the a_i are the self-intersection numbers of the D_i.

Proof. For $-D$, we have $c_p + c_q = 1$. Assume that there exists a circuit circuit $\mathcal{C}$ with orientation $\mathfrak{C}$ and $\mathfrak{C}^+ = \{i, j\}$ and $\mathfrak{C}^- = \{k\}$, and morover, some lattice point m such that $\mathfrak{s}^{\underline{c}}(m) \cap \mathcal{C} = \mathfrak{C}^-$. Then we get $\mathfrak{s}^{-\underline{c}}(-m) \cap \mathcal{C} = \mathfrak{C}^+$. this implies that $H^1\big(X, \mathcal{O}(-D)\big) \neq 0$. This implies the restriction $c_{i-1} + c_{i+1} - a_i c_i \in [-1, \min\{1, a_i - 1\}]$. $\square$

Note that example 4.31 also fulfills these more restrictive conditions.

4.5 Maximal Cohen-Macaulay Modules of Rank One

The classification of maximal Cohen-Macaulay modules can sometimes be related to resolution of singularities, the most famous example for this being the McKay correspondence in the case of certain surface singularities ([GSV83], [AV85], see also [EK85]). In the toric case, in general one cannot expect to arrive at such a nice picture, as there does not exist a canonical way to construct resolutions. However, there is a natural set of preferred partial resolutions, which is parameterized by the secondary fan.

Let X be a d-dimensional affine toric variety whose associated convex polyhedral cone σ has dimension d. Denote $x \in X$ torus fixed point. For any Weil divisor D on X, the sheaf $\mathcal{O}_X(D)$ is MCM if and only if $H^i_x\big(X, \mathcal{O}_X(D)\big)$ for all $i < d$. It was shown in [BG03] (see also [BG02]) that there exists only a finite number of such modules.

Now let $\tilde{X}$ be a toric variety given by some triangulation of σ. The natural map $\pi : \tilde{X} \longrightarrow X$ is a partial resolution of the singularities of X which is an isomorphism in codimension two and has at most quotient singularities. In particular, the map of fans is induced by the identity on N and, in turn, induces a bijection on the set of torus invariant Weil divisors. This bijection induces a natural isomorphism $\pi^{-1} : A_{d-1}(X) \longrightarrow A_{d-1}(\tilde{X})$ which can be represented by the identity morphism on the invariant divisor group $\mathbb{Z}^n$. This allows us to identify a torus invariant divisor D on X with its strict transform $\pi^{-1}D$ on $\tilde{X}$. Moreover, there are the natural isomorphisms

$$\pi_* \mathcal{O}_{\tilde{X}}(\pi^{-1}D) \cong \mathcal{O}_X(D) \quad \text{ and } \quad \mathcal{O}_{\tilde{X}}(\pi^{-1}D) \cong \big(\pi^* \mathcal{O}_X(D)\big)^{\check{}\check{}}.$$

Our aim is to compare local cohomology and global cohomology, i.e. $H^i_x\big(X, \mathcal{O}_X(D)\big)$ and $H^i\big(\tilde{X}, \mathcal{O}_{\tilde{X}}(D)\big)$.
Probably the easiest class of cones σ which one can consider is where the primitive vectors $l_1, \dots, l_n$ form a circuit $\mathcal{C} = [n]$. Associated to this data are two small resolutions of singularities $\pi : \mathbb{P}(\underline{\alpha}, \xi) \longrightarrow X$ and $\pi' : \mathbb{P}(-\underline{\alpha}, \xi)) \longrightarrow X$ which are induced by triangulations $\Delta_{\mathfrak{C}}$ and $\Delta_{-\mathfrak{C}}$, respectively.
Now, the question whether $\mathcal{O}(D)$ is a maximal Cohen-Macaulay sheaf can be decided directly on Y or, equivalently, on the resolutions:

Theorem 4.33: *Let X be an affine toric variety whose associated cone σ is spanned by a circuit $\mathcal{C}$ and denote $\mathbb{P}(\underline{\alpha}, \xi)$ and $\mathbb{P}(-\underline{\alpha}, \xi)$ the two canonical small toric resolution of singularities. Then the sheaf $\mathcal{O}(D)$ is maximal Cohen-Macaulay if and only if $R^i\pi_*\mathcal{O}(\pi^{-1}D) = R^i\pi'_*\mathcal{O}((\pi')^{-1}D) = 0$ for all $i > 0$.*

Proof. This toric variety corresponds to the toric subvariety of Y which is the complement of its unique fixed point, which we denote y. We have to show that $H^i_y\big(Y, \mathcal{O}(D)\big) = 0$ for all $i < d$. By Corollary 2.2, we have

$$H^i_y\big(Y, \mathcal{O}(D)\big)_m = H^{i-2}(\hat{\sigma}_{y,m}; k)$$

for every $m \in M$, where $\hat{\sigma}_y$ denotes the simplicial model for the fan associated to $Y \setminus \{y\}$. Denote τ and τ' the cones corresponding to the minimal orbits of $\mathbb{P}(\underline{\alpha}, \xi)$ and $\mathbb{P}(-\underline{\alpha}, \xi)$, respectively. We observe that $(\hat{\Delta}_{\mathfrak{C}})_{V(\tau)} = (\hat{\Delta}_{\mathfrak{C}})_{V(\tau')}$ both coincide with the subfan of σ generated by its facets. It follows that the simplicial complexes relevant for computing the isotypical decomposition of $H^i_y\big(Y, \mathcal{O}(D)\big)$ coincide with the simplicial complexes relevant for computing the $H^i_V\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(\pi^{-1}D)\big)$ and $H^i_{V'}\big(\mathbb{P}(-\underline{\alpha}, \xi), \mathcal{O}((\pi')^{-1}D)\big)$, respectively, where V, V' denote the exceptional sets of the morphisms π and π', respectively. By Proposition 4.4 the corresponding cohomologies vanish for $i < d$ iff $D \in F_{\mathfrak{C}} \cap F_{-\mathfrak{C}}$. Now we observe that $\Gamma\big(Y, R^i\pi_*\mathcal{O}(\pi^{-1}D)\big) = H^i\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(\pi^{-1}D)\big)$ and $\Gamma\big(Y, R^i\pi'_*\mathcal{O}((\pi')^{-1}D)\big) = H^i\big(\mathbb{P}(-\underline{\alpha}, \xi), \mathcal{O}((\pi')^{-1}D)\big)$. By Proposition 4.3, both cohomologies vanish for $i > 0$ iff $D \in F_{\mathfrak{C}} \cap F_{-\mathfrak{C}}$. □

Remark 4.34: The relation between maximal Cohen-Macaulay modules and the diophantine Frobenius problem has also been discussed in [Sta96]. See [Yos90] for a discussion of MCM-finiteness of toric 1-circuit varieties.

More generally, we have the following easy statement about general (i.e. non-regular) triangulations:

Theorem 4.35: *Let X be an affine toric variety of dimension d and $D \in A_{d-1}(X)$. If D is 0-essential, then $R^i\pi_*\mathcal{O}_{\tilde{X}}(\pi^*D) = 0$ for every triangulation $\pi : \tilde{X} \longrightarrow X$.*

Proof. If D is 0-essential, then it is contained in the intersection of all $F_{\mathcal{C}}$, where $\mathcal{C} \in \mathcal{C}(L)$, thus it represents a cohomology-free divisor. $\square$

Note that the statement does hold for any triangulation and not only for regular triangulations. We have a refined statement for affine toric varieties whose associated cone σ has simplicial facets:

Theorem 4.36: *Let X be a d-dimensional affine toric variety whose associated cone σ has simplicial facets and let $D \in A_{d-1}(X)$. If $R^i\pi_*\mathcal{O}_{\tilde{X}}(\pi^*D) = 0$ for every regular triangulation $\pi : \tilde{X} \longrightarrow X$ then $\mathcal{O}_X(D)$ is MCM. For $d = 3$ the converse is also true.*

Proof. Recall that $H^i_x\big(X, \mathcal{O}(D)\big)_m = H^{i-2}(\hat{\sigma}_{V,m}; k)$ for some $m \in M$ and $D \in A$. We are going to show that for every subset $I \subsetneq [n]$ there exists a regular triangulation $\tilde{\Delta}$ of σ such that the simplicial complexes $\hat{\sigma}_{V,I}$ and $\tilde{\Delta}_I$ coincide. This implies that if $H^i_x\big(X, \mathcal{O}_X(D)\big)_m \neq 0$ for some $m \in M$, then also $H^{i+1}\big(\tilde{X}, \mathcal{O}_{\tilde{X}}(D)\big)_m \neq 0$, i.e. if $\mathcal{O}_X(D)$ is not MCM, then $H^i\big(\tilde{X}, \mathcal{O}_{\tilde{X}}(D)\big) \neq 0$ for some $i > 0$.
For given $I \subset [n]$ we get such a triangulation as follows. Let $i \in [n] \setminus I$ and consider the dual cone $\check{\sigma}$. Denote $\rho_i := \mathbb{Q}_{\geq 0} l_i$ and recall that $\check{\rho}_i$ is a halfspace which contains $\check{\sigma}$ and which defines a facet of $\check{\sigma}$ given by $\rho^\perp \cap \check{\sigma}$. Now we move $\check{\rho}_i$ to $m + \check{\rho}$, where $l_i(m) > 0$. So we obtain a new polytope $P := \check{\sigma} \cap (m + \check{\rho})$. As $\rho^\perp$ is not parallel to any face of $\check{\sigma}$, the hyperplane $m + \rho^\perp$ intersects every face of $\check{\sigma}$. This way the inner normal fan of P is a triangulation $\tilde{\Delta}$ of σ which has the property that every maximal cone is spanned by ρ_i and some facet of σ. This implies $\tilde{\Delta}_I = \hat{\sigma}_{V,I}$ and the first assertion follows.
For $d = 3$, a sheaf $\mathcal{O}(D)$ is MCM iff $H^2_x\big(X, \mathcal{O}(D)\big) = 0$, i.e. $H^0(\sigma_{V,m}; k) = 0$ for every $m \in M$. The latter is only possible if $\sigma_{V,m}$ represents an interval on S^1. To compare this with $H^2\big(\tilde{X}, \mathcal{O}(D)\big)$ for some regular triangulation $\tilde{X}$, we must show that $H^1(\tilde{\Delta}_m; k) = 0$ for the corresponding complex $\tilde{\Delta}_m$. To see this, we consider some cross-section $\sigma \cap H$, where $H \subset N \otimes_Z \mathbb{R}$ is some hyperplane which intersects σ nontrivially and is not parallel to any of its faces. Then this cross-section can be considered as a planar polygon and $\sigma_{V,m}$ as some connected sequence of faces of this polygon. Now with respect to the triangulation $\tilde{\Delta}$ of this polygon, we can consider two vertices $p, q \in \sigma_{V,m}$ which are connected by a line belonging to the triangulation and going through the interior of the polygon. We assume that p and q have maximal distance in $\sigma_{V,m}$ with this property. Then it is easy to see that the triangulation of σ induces a triangulation of the convex hull of the line segments connecting p and q. Then $\tilde{\Delta}_m$ is just the union of this convex hulls with respect all such pairs p, q and the remaining line segments and thus has trivial topology. Hence $H^2_x\big(X, \mathcal{O}(D)\big) = 0$ implies $H^2\big(\tilde{X}, \mathcal{O}(D)\big) = 0$ for every triangulation $\tilde{\Delta}$ of σ. $\square$

Example 4.37: Consider the 3-dimensional cone spanned over the primitive vectors $l_1 = (1, 0, 1)$, $l_2 = (0, 1, 1)$, $l_3 = (-1, 0, 1)$, $l_4 = (-1, -1, 1)$, $l_5 =$

$(1, -1, 1)$. The corresponding toric variety X is Gorenstein and its divisor class group is torsion free. For $A_2(X) \cong \mathbb{Z}^2$ we choose the basis $D_1 + D_2 + D_5$, D_5. In this basis, the Gale duals of the l_i are $D_1 = (-1, -1)$, $D_2 = (2, 0)$, $D_3 = (-3, 1)$, $D_4 = (2, -1)$, $D_5 = (0, 1)$. Figure 2 shows the set of MCM modules in $A_2(X)$ which are indicated by circles which are supposed to sit on the lattice $A_2(X) \cong \mathbb{Z}^2$. The picture also indicates the cones C_I with vertices $-e_I$, where $I \in \{\{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}, \{3,5\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{2,3,5\}, \{2,4,5\}\}$. Note that the picture has a reflection symmetry, due to the fact that X is Gorenstein. Altogether, there are 19 MCM modules of rank one, all of which are 0-essential. For $\mathcal{C} = \{l_1, l_3, l_4, l_5\}$, the group $A_2(X)_{\mathcal{C}} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ has torsion. The two white circles indicate modules are contained in the $\mathbb{Q}$-hyperplanes $D_1 + D_4 + H_{\mathcal{C}}$ and $D_2 + D_3 + D_5 + H_{\mathcal{C}}$, respectively, but not in the sets $D_1 + D_4 + Z_{\mathcal{C}}$ and $D_2 + D_3 + D_5 + Z_{\mathcal{C}}$, respectively. Some of the $\mathbb{O}_I$ are not saturated; however, every divisor which is contained in some $(-e_I + C_I) \cap \Omega(L, I)$ is also contained in some $\mathbb{O}_{I'} \setminus \Omega(L, I')$ for some other $I' \neq I$. So for this example, the Frobenius arrangement gives a full description of MCM modules of rank one.

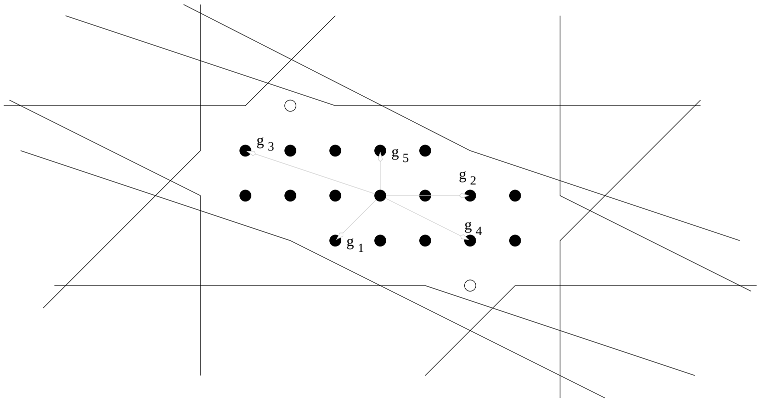


Figure 2: The 19 MCM modules of example 4.37.

Example 4.38: To give a counterexample to the reverse direction of theorem 4.36 for $d > 3$, we consider the four-dimensional cone spanned over the primitive vectors $l_1 = (0, -1, -1, 1)$, $l_2 = (-1, 0, 1, 1)$, $l_3 = (0, 1, 0, 1)$, $l_4 = (-1, 0, 0, 1)$, $l_5 = (-1, -1, 0, 1)$, $l_6 = (1, 0, 0, 1)$. The corresponding variety X has 31 MCM modules of rank one which are shown in figure 3. Here, with basis D_1 and D_6, we have $D_1 = (1, 0)$, $D_2 = (1, 0)$, $D_3 = (-1, -2)$, $D_4 = (3, 1)$, $D_5 = (-2, -2)$, $D_6 = (0, 1)$. There are six cohomology cones corresponding to $I \in \big\{\{1,2\}, \{3,5\}, \{4,6\}, \{1,2,3,5\}, \{1,2,4,6\}, \{3,4,5,6\}\big\}$.
The example features two modules which are not 0-essential, indicated by the grey dots sitting on the boundary of the cones $-e_I + C_I$, where $I \in$

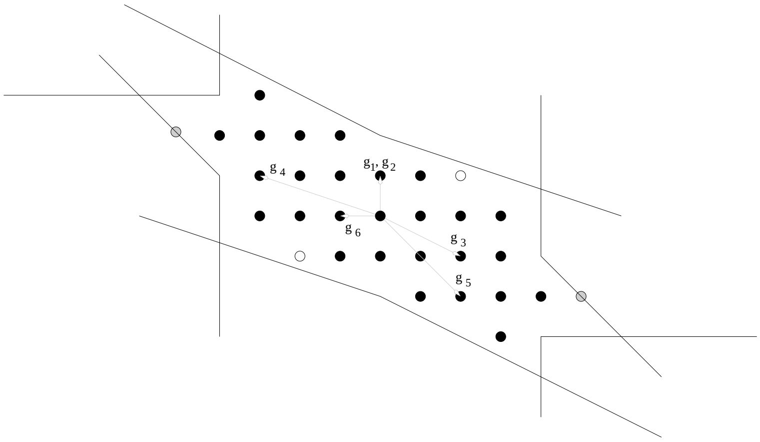


Figure 3: The 31 MCM modules of example 4.38.

$\{\{4,6\},\{1,2,3,5\}\}$. The white dots denote MCM divisors $D, -D$ such that there exists a triangulation of the cone of X such that on the associated variety $\tilde{X}$ we have $H^i(\tilde{X}, \mathcal{O}(\pm D)) \neq 0$ for some $i > 0$. Namely, we consider the triangulation which is given by the maximal cones spanned by $\{l_1, l_2, l_4, l_5\}$, $\{l_1, l_2, l_4, l_6\}$, $\{l_1, l_2, l_5, l_6\}$, $\{l_1, l_3, l_4, l_6\}$, $\{l_2, l_3, l_4, l_6\}$. Figure 4.38 indicates the two-dimensional faces of this triangulation via a three-dimensional cross-section of the cone.

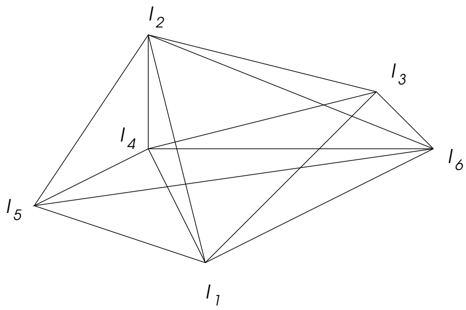


Figure 4: The triangulation for $\tilde{X}$ in example 4.38.

We find that we have six cohomology cones corresponding to $I \in \{\{1,2\},\{3,5\}, \{1,2,3\}, \{4,5,6\}, \{1,2,3,5\}, \{3,4,5,6\}\}$. In particular, we have non-vanishing H^1 for the points $-D_1 - D_2 - D_3$ and for $-D_4 - D_5 - D_6$, which correspond to D and $-D$.

References

[Ath99] A. Athanasiadis. The largest intersection lattice of a discriminantal arrangement. *Beitr. Algebra Geom.*, 40(2):283–289, 1999.

[AV85] M. Artin and J.-L. Verdier. Reflexive modules over rational double points. *Math. Ann.*, 270(1):79–82, 1985.

[BB97] M. Bayer and K. A. Brandt. Discriminantal arrangements, fiber polytopes and formality. *J. Algebraic Combin.*, 6(3):229–246, 1997.

[BFS90] L. J. Billera, P. Filliman, and B. Sturmfels. Constructions and Complexity of Secondary Polytopes. *Advances in Mathematics*, 83:155–179, 1990.

[BG02] W. Bruns and J. Gubeladze. Semigroup algebras and discrete geometry. In *Séminaires et Congrès*, volume 6, pages 43–127. Soc. Math. France, Paris, 2002.

[BG03] W. Bruns and J. Gubeladze. Divisorial Linear Algebra of Normal Semigroup Rings. *Algebras and Representation Theory*, 6:139–168, 2003.

[BGS87] R.-O. Buchweitz, G.-M. Greuel, and F.-O. Schreyer. Cohen-Macaulay modules on hypersurface singularities II. *Invent. Math.*, 88:165–182, 1987.

[BLS+93] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1993.

[Bon90] A. I. Bondal. Representation of associative algebras and coherent sheaves. *Math. USSR Izvestiya*, 34(1):23–42, 1990.

[BR86] M. Beltrametti and L. Robbiano. Introduction to the theory of weighted projective spaces. *Expo. Math.*, 4:111–162, 1986.

[BV97] M. Brion and M. Vergne. Residue formulae, vector partition functions and lattice points in rational polytopes. *J. Am. Math. Soc.*, 10(4):797–833, 1997.

[Coh86] D. C. Cohen. A Conjecture about Compact Quotients by Tori. In *Arrangements — Tokyo 1998*, volume 27 of *Adv. Studies in Pure Mathematics*, pages 59–68. Math. Soc. Japan, 1986.

[Cox95] D. A. Cox. The Homogeneous Coordinate Ring of a Toric Variety. *J. Algebr. Geom. 4*, 1:17–50, 1995.

[Cra84] H. Crapo. Concurrence geometries. *Adv. Math.*, 54:278–301, 1984.

[CV03] D. C. Cohen and A. N. Varchenko. Resonant local systems on complements of discriminantal arrangements and $\mathfrak{sl}_2$ representations. *Geom. Dedicata*, 101:217–233, 2003.

[Del75] C. Delorme. Espaces projectifs anisotropes. *Bull. Soc. Math. France*, 103:203–223, 1975.

[Dol82] I. Dolgachev. Weighted projective varieties. In *Group actions and vector fields, Proc. Pol.-North Am. Semin., Vancouver 1981*, pages 34–71, 1982.

[Eik92] M. Eikelberg. The Picard group of a compact toric variety. *Result. Math.*, 22(1–2):509–527, 1992.

[EK85] H. Esnault and H. Knörrer. Reflexive modules over rational double points. *Math. Ann.*, 272(4):545–548, 1985.

[EMS00] D. Eisenbud, M. Mustaţă, and M. Stillman. Cohomology on Toric Varieties and Local Cohomology with Monomial Supports. *J. Symb. Comput.*, 29(4–5):583–600, 2000.

[Fal94] M. Falk. A note on discriminantal arrangements. *Proc. AMS*, 122(4):1221–1227, 1994.

[Fuj07] O. Fujino. Multiplication maps and vanishing theorems for toric varieties. *Math. Z.*, 257(3):631–641, 2007.

[Ful93] W. Fulton. *Introduction to Toric Varieties*. Princeton University Press, 1993.

[GKZ94] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multidimensional determinants*. Math. Theory Appl. Birkhäuser, 1994.

[GSV83] G. Gonzalez-Sprinberg and J.-L. Verdier. Construction géométrique de la correspondance de McKay. *Ann. Sci. Ecole Norm. Sup.*, 16:409–449, 1983.

[HP06] L. Hille and M. Perling. A Counterexample to King's Conjecture. *Comp. Math*, 142(6):1507–1521, 2006.

[Kin97] A. King. Tilting bundles on some rational surfaces. Unpublished manuscript, 1997.

[Kon95] M. Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich 1994)*, pages 120–139. Birkhäuser, 1995.

[Mat02] E. N. Materov. The Bott formula for toric varieties. *Mosc. Math. J.*, 2(1):161–182, 2002.

[MS89] Y. I. Manin and V. V. Schechtman. Arrangements of hyperplanes, higher braid groups and higher Bruhat orders. In J. Coates et al., editors, *Algebraic number theory. Papers in honor of K. Iwasawa on the occasion of his 70th birthday on September 11, 1987*, volume 17 of *Advances Studies in Pure Mathematics*, pages 67–84. Braunschweig: Vieweg, 1989.

[Mus02] M. Mustaţă. Vanishing Theorems on Toric Varieties. *Tohoku Math. J., II. Ser.*, 54(3):451–470, 2002.

[Oda88] T. Oda. *Convex Bodies and Algebraic Geometry*, volume 15 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1988.

[OP91] T. Oda and H. S. Park. Linear Gale transforms and Gelfand-Kapranov-Zelevinskij decomposition. *Tohoku Math. J., II Ser.*, 43(3):375–399, 1991.

[Per04] M. Perling. `TiltingSheaves` — a program to compute strongly exceptional collections on toric varieties. `http://www.mathematik.uni-kl.de/~perling/ts/ts.html`, 2004.

[Ram05] J. L. Ramírez Alfonsín. *The Diophantine Frobenius Problem*, volume 30 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, 2005.

[Rei99] V. Reiner. The generalized Baues problem. In L. J. Billera et al., editors, *New perspectives in algebraic combinatorics*, pages 293–336. Cambridge University Press, 1999.

[Rud90] A. N. Rudakov. *Helices and vector bundles: seminaire Rudakov*, volume 148 of *London Mathematical Society lecture note series*. Cambridge University Press, 1990.

[Sta82] R. P. Stanley. Linear diophantine equations and local cohomology. *Invent. Math.*, 68:175–193, 1982.

[Sta96] R. P. Stanley. *Combinatorics and commutative algebra*. Birkhäuser, 1996.

[Stu95] B. Sturmfels. On Vector Partition Functions. *Journal of Combinatorial Theory, Series A*, 72:302–309, 1995.

[Yos90] Y. Yoshino. *Cohen-Macaulay modules over Cohen-Macaulay rings*, volume 146 of *London Mathematical Society lecture note series*. Cambridge University Press, 1990.

Markus Perling
Ruhr-Universität Bochum
Fakultät für Mathematik
Universitätsstraße 150
44780 Bochum, Germany
Markus.Perling@rub.de

Algebraic Zip Data

Richard Pink, Torsten Wedhorn, and Paul Ziegler

Received: November 11, 2010

Communicated by Peter Schneider

Abstract. An algebraic zip datum is a tuple $\mathcal{Z} = (G, P, Q, \varphi)$ consisting of a reductive group G together with parabolic subgroups P and Q and an isogeny $\varphi\colon P/R_uP \to Q/R_uQ$. We study the action of the group $E_{\mathcal{Z}} := \big\{(p,q) \in P{\times}Q \bigm| \varphi(\pi_P(p)) = \pi_Q(q)\big\}$ on G given by $((p,q),g) \mapsto pgq^{-1}$. We define certain smooth $E_{\mathcal{Z}}$-invariant subvarieties of G, show that they define a stratification of G. We determine their dimensions and their closures and give a description of the stabilizers of the $E_{\mathcal{Z}}$-action on G. We also generalize all results to non-connected groups.

We show that for special choices of $\mathcal{Z}$ the algebraic quotient stack $[E_{\mathcal{Z}}\backslash G]$ is isomorphic to $[G\backslash Z]$ or to $[G\backslash Z']$, where Z is a G-variety studied by Lusztig and He in the theory of character sheaves on spherical compactifications of G and where Z' has been defined by Moonen and the second author in their classification of F-zips. In these cases the $E_{\mathcal{Z}}$-invariant subvarieties correspond to the so-called "G-stable pieces" of Z defined by Lusztig (resp. the G-orbits of Z').

2010 Mathematics Subject Classification: 14L30 (20G15, 20F55, 20G40)
Keywords and Phrases: Generalized G-stable pieces, zip data, F-zips

1 Introduction

1.1 Background

Let G be a connected reductive linear algebraic group over an algebraically closed field k. Then $G \times G$ acts on G via simultaneous left and right translation $((g_1, g_2), g) \mapsto g_1 g g_2^{-1}$. In a series of papers, Lusztig ([Lus1], [Lus2]), He

([He2], [He1], [He3]), and Springer ([Spr3]) studied a certain spherical $G \times G$-equivariant smooth compactification $\bar{G}$ of G. For G semi-simple adjoint this is the so-called wonderful compactification from [DCP]. In general the $G \times G$-orbits $Z_I \subset \bar{G}$ are in natural bijection to the subsets I of the set of simple reflections in the Weyl group of G. Lusztig and He defined and studied so-called G-stable pieces in Z_I, which are certain subvarieties that are invariant under the diagonally embedded subgroup $G \subset G \times G$. These G-stable pieces play an important role in their study of character sheaves on $\bar{G}$. Lusztig and He also consider non-connected groups, corresponding to twisted group actions. Other generalizations of these varieties have been considered by Lu and Yakimow ([LY2]). A further motivation to study G-stable pieces comes from Poisson geometry: It was proved by Evens and Lu ([EL]), that for certain Poisson structure, each G-orbit on Z_I is a Poisson submanifold.

In [MW] Moonen and the second author studied the De Rham cohomology $H^\bullet_{\mathrm{DR}}(X/k)$ of a smooth proper scheme X with degenerating Hodge spectral sequence over an algebraically closed field k of positive characteristic. They showed that $H^\bullet_{\mathrm{DR}}(X/k)$ carries the structure of a so-called *F-zip*, namely: it is a finite-dimensional k-vector space together with two filtrations (the “Hodge” and the “conjugate” filtration) and a Frobenius linear isomorphism between the associated graded vector spaces (the “Cartier isomorphism”). They showed that the isomorphism classes of F-zips of fixed dimension n and with fixed type of Hodge filtration are in natural bijection with the orbits under $G := \mathrm{GL}_{n,k}$ of a variant Z'_I of the $G\times G$-orbit Z_I studied by Lusztig. They studied the varieties Z'_I for arbitrary reductive groups G and determined the G-orbits in them as analogues of the G-stable pieces in Z_I. By specializing G to classical groups they deduce from this a classification of F-zips with additional structure, e.g., with a non-degenerate symmetric or alternating form. They also consider non-connected groups. Moreover, the automorphism group of an F-zip is isomorphic to the stabilizer in G of any corresponding point in Z'_I.

When X varies in a smooth family over a base S, its relative De Rham cohomology forms a family of F-zips over S. The set of points of S where the F-zip lies in a given isomorphism class is a natural generalization of an Ekedahl-Oort stratum in the Siegel moduli space. Information about the closure of such a stratum corresponds to information about how the isomorphism class of an F-zip can vary in a family, and that in turn is equivalent to determining which G-orbits in Z'_I are contained in the closure of a given G-orbit.

In each of these cases one is interested in the classification of the G-stable pieces, the description of their closures, and the stabilizers of points in G. In this article we give a uniform approach to these questions that generalizes all the above situations.

1.2 Main results

The central definition in this article is the following:

Definition 1.1. A *connected algebraic zip datum* is a tuple $\mathcal{Z} = (G, P, Q, \varphi)$ consisting of a connected reductive linear algebraic group G over k together with parabolic subgroups P and Q and an isogeny $\varphi\colon P/R_uP \to Q/R_uQ$. The group

$$E_{\mathcal{Z}} := \big\{(p,q) \in P{\times}Q \;\big|\; \varphi(\pi_P(p)) = \pi_Q(q)\big\}$$

is called the *zip group associated to* $\mathcal{Z}$. It acts on G through the map $((p,q),g) \mapsto pgq^{-1}$. The union of the $E_{\mathcal{Z}}$-orbits of all elements of a subset $X \subset G$ is denoted by $\mathrm{o}_{\mathcal{Z}}(X)$.

Fix such data $\mathcal{Z} = (G, P, Q, \varphi)$. To apply the machinery of Weyl groups to $\mathcal{Z}$ we choose a Borel subgroup B of G, a maximal torus T of B, and an element g of G such that $B \subset Q$, ${}^gB \subset P$, $\varphi\big(\pi_P({}^gB)\big) = \pi_Q(B)$, and $\varphi\big(\pi_P({}^gT)\big) = \pi_Q(T)$. Let W denote the Weyl group of G with respect to T, and $S \subset W$ the set of simple reflections corresponding to B. Let $I \subset S$ be the type of the parabolic P and $W_I \subset W$ its Weyl group. Let ${}^IW$ be the set of all $w \in W$ that have minimal length in their coset W_Iw. To each $w \in {}^IW$ we associate the $E_{\mathcal{Z}}$-invariant subset

$$G^w = \mathrm{o}_{\mathcal{Z}}(gB\dot{w}B) \tag{1.2}$$

and prove (Theorems 5.10, 5.11 and 5.14):

Theorem 1.3. *The $E_{\mathcal{Z}}$-invariant subsets G^w form a pairwise disjoint decomposition of G into locally closed smooth subvarieties. The dimension of G^w is $\dim P + \ell(w)$.*

Next the isogeny φ induces an isomorphism of Coxeter system $\psi\colon (W_I, I) \overset{\sim}{\to} (W_J, J)$ (see (3.11) for its precise definition), where $W_J \subset W$ and $J \subset S$ are the Weyl group and the type of the parablic subgroup Q. Let $\leq$ denote the Bruhat order on W. We prove (Theorem 6.2):

Theorem 1.4. *The closure of G^w is the union of $G^{w'}$ for all $w' \in {}^IW$ such that there exists $y \in W_I$ with $yw'\psi(y)^{-1} \leq w$.*

We call $\mathcal{Z}$ *orbitally finite* if the number of $E_{\mathcal{Z}}$-orbits in G is finite. We give a necessary and sufficient criterion for this to happen (Proposition 7.1). In particular it happens when the differential of φ at 1 vanishes, for instance if φ is a Frobenius isogeny (Proposition 7.3). We prove (Theorem 7.5):

Theorem 1.5. *If $\mathcal{Z}$ is orbitally finite, then each G^w is a single $E_{\mathcal{Z}}$-orbit, and so the set $\{g\dot{w} \mid w \in {}^IW\}$ is a set of representatives for the $E_{\mathcal{Z}}$-orbits in G.*

One can also consider the $E_{\mathcal{Z}}$-orbit of $g\dot{w}$ for any element $w \in W$ instead of just those in ${}^IW$. It is then natural to ask when two such orbits lie in the same $E_{\mathcal{Z}}$-invariant piece G^w. (For orbitally finite $\mathcal{Z}$ this is equivalent to asking when the orbits are equal.) We give an explicit description of this equivalence relation on W that depends only on the subgroup W_I and the homomorphism ψ (Theorem 9.17). We prove that all equivalence classes have the same cardinality $\#W_I$, although they are in general no cosets of W_I and we

do not know a simple combinatorial description for them. It is intriguing that we obtain analogous results for an *abstract zip datum* based on an arbitrary finitely generated Coxeter group (Theorem 9.11) or even an arbitrary abstract group (Theorem 9.6) in place of W.
Other results include information on point stabilizers and infinitesimal stabilizers (Section 8), the generalization of the main results to non-connected groups (Section 10), a dual parametrization by a set W^J in place of ${}^IW$ (Section 11) and the relation with the varieties Z_I studied by Lusztig and He and their generalizations Z'_I (Section 12).

1.3 Applications

Let us explain why this theory of algebraic zip data is a generalization of the situations described in Subsection 1.1. In Section 12 we consider a connected reductive algebraic group G over k, an isogeny $\varphi\colon G \to G$, a subset I of the set of simple reflections associated to G, and an element x in the Weyl group of G satisfying certain technical conditions. To such data we associate a certain algebraic variety $X_{I,\varphi,x}$ with an action of G, a certain connected algebraic zip datum $\mathcal{Z}$ with underlying group G, and morphisms

$$G \xleftarrow{\lambda} G \times G \xrightarrow{\rho} X_{I,\varphi,x} \tag{1.6}$$

In Theorem 12.8 we show that there is a closure preserving bijection between the $E_{\mathcal{Z}}$-invariant subsets of $A \subset G$ and the G-invariant subsets of $B \subset X_{I,\varphi,x}$ given by $\lambda^{-1}(A) = \rho^{-1}(B)$. We also prove that the stabilizer in $E_{\mathcal{Z}}$ of $g \in G$ is isomorphic to the stabilizer in G of any point of the G-orbits in $X_{I,\varphi,x}$ corresponding to the orbit of g. These results can also be phrased in the language of algebraic stacks, see Theorem 12.7.

In the special case $\varphi = \mathrm{id}_G$ the above $X_{I,\varphi,x}$ is the variety Z_I defined by Lusztig. In Theorem 12.19 we verify that the subsets $G^w \subset G$ correspond to the G-stable pieces defined by him. Thus Theorem 1.4 translates to a description of the closure relation between these G-stable pieces, which had been proved before by He [He2].

If $\mathrm{char}(k)$ is positive and φ is the Frobenius isogeny associated to a model of G over a finite field, the above $X_{I,\varphi,x}$ is the variety Z'_I defined by Moonen and the second author. In this case the zip datum $\mathcal{Z}$ is orbitally finite, and so we obtain the main classification result for the G-orbits in Z'_I from [MW], the closure relation between these G-orbits, and the description of the stabilizers in G of points in Z'_I. In this case the closure relation had been determined in the unpublished note [Wed], the ideas of which are reused in the present article. Meanwhile Viehmann [Vie] has given a different proof of the closure relation in this case using the theory of loop groups. For those cases which pertain to the study of modulo p reductions of F-crystals with additional structure that show up in the study of special fibers of good integral models of Shimura varieties of Hodge type Moonen ([Moo]) and, more generally, Vasiu ([Vas]) have obtained similar classification results. In these cases Vasiu (loc. cit.) has also shown that the connected component of the stabilizers are unipotent.

For $G = \mathrm{GL}_n$ (resp. a classical group) we therefore obtain a new proof of the classification of F-zips (resp. of F-zips with additional structure) from [MW]. We can also deduce how F-zips (possibly with additional structure) behave in families, and can describe their automorphism groups as the stabilizers in $E_{\mathcal{Z}}$ of the corresponding points of G. This is applied in [VW] to the study of Ekedahl-Oort strata for Shimura varieties of PEL type.

1.4 Contents of the paper

In Section 2 we collect some results on algebraic groups and Coxeter systems that are used in the sequel. Algebraic zip data $\mathcal{Z}$ are defined in Section 3, where we also establish basic properties of the triple (B, T, g), called a *frame of* $\mathcal{Z}$.
Section 4 is based on the observation that every $E_{\mathcal{Z}}$-orbit is contained in the double coset $Pg\dot{x}Q$ for some $x \in W$ and meets the subset $g\dot{x}M$, where M is a Levi subgroup of Q. In it we define another zip datum $\mathcal{Z}_{\dot{x}}$ with underlying reductive group M and establish a number of results relating the $E_{\mathcal{Z}}$-orbits in $Pg\dot{x}Q$ to the $E_{\mathcal{Z}_{\dot{x}}}$-orbits in M. This is the main induction step used in most of our results.
In Section 5 we give different descriptions of the $E_{\mathcal{Z}}$-invariant subsets G^w for $w \in {}^I W$ and prove Theorem 1.3. In Section 6 we determine the closure of G^w and prove Theorem 1.4. Orbitally finite zip data are studied in Section 7, proving Theorem 1.5. Section 8 contains some results on point stabilizers and infinitesmial stabilizers. Abstract zip data are defined and studied in Section 9. In Section 10 our main results are generalized to algebraic zip data based on non-connected groups.
In Section 11 we discuss a dual parametrization of the subsets G^w by a subset W^J of W in place of ${}^I W$. Finally, in Section 12 we prove the results described in Subsection 1.3 above.

The paper is based on parts of the unpublished note [Wed] by the second author and the master thesis [Zie] by the third author, but goes beyond both.

After the referee pointed out to us the references [LY1] and [He3], we became aware that there Lu, Yakimov and He study a class of group actions which contains ours when φ is an isomorphism. In this case, Theorems 1.3 and 1.4 were already proven in [loc. cit]. Also, many of the ideas we have used to study the decomposition of G into $E_{\mathcal{Z}}$-stable pieces are already present there.

Acknowledgements. We thank the referee for pointing out some references. The second author was partially supported by the SPP 1388 "Representation theory" of the DFG.

2 Preliminaries on algebraic groups and Coxeter groups

Throughout, the inner automorphism associated to an element h of a group G will be denoted $\mathrm{int}(h)\colon G \to G$, $g \mapsto {}^h g := hgh^{-1}$. Similarly, for any subset $X \subset G$ we set ${}^h X := hXh^{-1}$.

2.1 General facts about linear algebraic groups

Throughout, we use the language of algebraic varieties over a fixed algebraically closed field k. By an algebraic group G we always mean a linear algebraic group over k. We let $\mathcal{R}_u G$ denote the unipotent radical of the identity component of G and $\pi_G\colon G \twoheadrightarrow G/\mathcal{R}_u G$ the canonical projection. An isogeny between two connected algebraic groups is a surjective homomorphism with finite kernel.

Consider an algebraic group G, an algebraic subgroup H of G, and a quasi-projective variety X with a left action of H. Then we denote by $G \times^H X$ the quotient of $G \times X$ under the left action of H defined by $h\cdot(g,x) = (gh^{-1}, h\cdot x)$, which exists by [Ser], Section 3.2. The action of G on $G\times X$ by left multplication on the first factor induces a left action of G on $G \times^H X$. This is the pushout of X with respect to the inclusion $H \hookrightarrow G$.

Lemma 2.1. *For G, H, and X as above, the morphism $X \to G \times^H X$ which sends $x \in X$ to the class of $(1,x)$ induces a closure-preserving bijection between the H-invariant subsets of X and the G-invariant subsets of $G\times^H X$. If $Y \subset X$ is an H-invariant subvariety of X, then the corresponding G-invariant subset of $G \times^H X$ is the subvariety $G \times^H Y$ of $G \times^H X$.*

Proof. The morphism in question is the composite of the inclusion $i\colon X \to G\times X$, $x \mapsto (1,x)$ and the projection $\mathrm{pr}\colon G\times X \to G\times^H X$. Let $(g,h) \in G\times H$ act on $G \times X$ from the left by $(g',x) \mapsto (gg'h^{-1}, h\cdot x)$. Then the $G\times H$-invariant subsets of $G\times X$ are the sets of the form $G\times A$ for H-invariant subsets $A \subset X$. Therefore i induces a closure-preserving bijection between the H-invariant subsets of X and the $G\times H$-invariant subsets of $G\times X$. Furthermore, since $G \times^H X$ carries the quotient topology with respect to pr, the morphism pr induces a closure-preserving bijection between the $G\times H$-invariant subsets of $G\times X$ and the G-invariant of $G\times^H X$. Altogether this proves the claim. ☐

Lemma 2.2 (see [Slo], Lemma 3.7.4). *Let G be an algebraic group with an algebraic subgroup H. Let X be a variety with a left action of G. Let $f\colon X \to G/H$ be a G-equivariant morphism, and let $Y \subset X$ be the fiber $f^{-1}(H)$. Then Y is stabilized by H, and the map $G\times^H Y \to X$ sending the equivalence class of (g,y) to $g\cdot y$ defines an isomorphism of G-varieties.*

Lemma 2.3. *Let G be an algebraic group acting on an algebraic variety Z and let $P \subset G$ be an algebraic subgroup such that G/P is proper. Then for any P-invariant subvariety $Y \subset Z$ one has*

$$G\cdot\overline{Y} = \overline{G\cdot Y}.$$

Proof. Clearly we have
$$G\cdot Y\subset G\cdot\overline{Y}\subset\overline{G\cdot Y}$$
and therefore it suffices to show that $G\cdot\overline{Y}$ is closed in Z. The action $\pi\colon G\times Z\to Z$ of G on Z induces a morphism $\bar\pi\colon G\times^P Z\to Z$ which can be written as the composition
$$G\times^P Z\xrightarrow{\sim} G/P\times Z\longrightarrow Z.$$
Here the first morphism is the isomorphism given by $[g,z]\mapsto(gP,g\cdot z)$ and the second morphism is the projection. As G/P is proper, we deduce that the morphism $\bar\pi$ is closed. Now $\overline{Y}$ is P-invariant and therefore $G\times^P\overline{Y}$ is defined, and it is a closed subscheme of $G\times^P Z$. Therefore $\bar\pi(G\times^P\overline{Y})=G\cdot\overline{Y}$ is closed in Z. □

The following statements concern images under twisted conjugation:

Theorem 2.4 (Lang-Steinberg, see [Ste], Theorem 10.1). *Let G be a connected algebraic group and $\varphi\colon G\to G$ an isogeny with only a finite number of fixed points. Then the morphism $G\to G$, $g\mapsto g\varphi(g)^{-1}$ is surjective.*

Proposition 2.5. *Let G be a connected reductive algebraic group with a Borel subgroup B and a maximal torus $T\subset B$. Let $\varphi\colon G\to G$ be an isogeny with $\varphi(B)=B$. In (b) also assume that $\varphi(T)=T$.*

(a) The morphism $G\times B\to G$, $(g,b)\mapsto gb\varphi(g)^{-1}$ is surjective.

(b) The morphism $G\times T\to G$, $(g,t)\mapsto gt\varphi(g)^{-1}$ has dense image.

If G is simply connected semisimple and φ is an automorphism of G, (b) has been shown by Springer ([Spr2] Lemma 4).

Proof. For (a) see [Ste], Lemma 7.3. Part (b) and its proof are a slight modification of this. Equivalently we may show that for some $t_0\in T$, the image of the morphism $\tilde\alpha\colon G\times T\to G,(g,t)\mapsto gtt_0\varphi(g)^{-1}t_0^{-1}$ is dense. For this it will suffice to show that the differential of $\tilde\alpha$ at 1 is surjective. This differential is the linear map
$$\begin{aligned}\mathrm{Lie}(G)\times\mathrm{Lie}(T)&\to\mathrm{Lie}(G)\\ (X,Y)&\mapsto X+Y-\mathrm{Lie}(\varphi_{t_0})(X),\end{aligned}$$
where $\varphi_{t_0}:=\mathrm{int}(t_0)\circ\varphi$. This linear map has image
$$\mathrm{Lie}(T)+(1-\mathrm{Lie}(\varphi_{t_0}))\,\mathrm{Lie}(G).$$
Let B^- be the Borel subgroup opposite to B with respect to T. Since $\varphi(B)=B$ and $\varphi(T)=T$, the differential of φ_{t_0} at 1 preserves $\mathrm{Lie}(\mathcal{R}_uB)$ and $\mathrm{Lie}(\mathcal{R}_uB^-)$. If we find a t_0 such that $\mathrm{Lie}(\varphi_{t_0})$ has no fixed points on $\mathrm{Lie}(\mathcal{R}_uB)$ and $\mathrm{Lie}(\mathcal{R}_uB^-)$ we will be done.

Let Φ be the set of roots of G with respect to T. For each $\alpha \in \Phi$, let x_α be a basis vector of $\mathrm{Lie}(U_\alpha)$, where U_α is the unipotent root subgroup of G associated to α. As the isogeny φ sends T to itself, it induces a bijection $\tilde{\varphi}\colon \Phi \to \Phi$ such that $\varphi(U_\alpha) = U_{\tilde{\varphi}(\alpha)}$. For each $\alpha \in \Phi$ there exists a $c(\alpha) \in k$ such that $\mathrm{Lie}(\varphi)(x_\alpha) = c(\alpha)x_{\tilde{\varphi}(\alpha)}$. This implies $\mathrm{Lie}(\varphi_{t_0})(x_\alpha) = \alpha(t_0)c(\alpha)x_{\tilde{\varphi}(\alpha)}$. Since φ_{t_0} fixes $\mathcal{R}_uB$ and $\mathcal{R}_uB^-$, its differential permutes Φ^+ and Φ^-, where Φ^+ (resp. Φ^-) is the set of roots that are positive (resp. negative) with respect to B. Hence $\mathrm{Lie}(\varphi_{t_0})$ can only have a fixed point in $\mathrm{Lie}(\mathcal{R}_uB)$ or $\mathrm{Lie}(\mathcal{R}_uB^-)$ if there exists a cycle $(\alpha_1, \cdots, \alpha_n)$ of the permutation $\tilde{\varphi}$ in Φ^+ or Φ^- such that

$$\prod_{i=1}^n \alpha_i(t_0)c(\alpha_i) = 1.$$

This shows that for t_0 in some non-empty open subset of T, the differential $\mathrm{Lie}(\varphi_{t_0})$ has no fixed points on $\mathrm{Lie}(\mathcal{R}_uB)$ and $\mathrm{Lie}(\mathcal{R}_uB^-)$. ☐

2.2 Coset Representatives in Coxeter Groups

Here we collect some facts about Coxeter groups and root systems which we shall need in the sequel. Let W be a Coxeter group and S its generating set of simple reflections. Let ℓ denote the length function on W; thus $\ell(w)$ is the smallest integer $n \geqslant 0$ such that $w = s_1 \cdots s_n$ for suitable $s_i \in S$. Any such product with $\ell(w) = n$ is called a reduced expression for w.

Let I be a subset of S. We denote by W_I the subgroup of W generated by I, which is a Coxeter group with set of simple reflections I. Also, we denote by W^I (respectively ${}^IW$) the set of elements w of W which have minimal length in their coset wW_I (respectively W_Iw). Then every $w \in W$ can be written uniquely as $w = w^I \cdot w_I = \tilde{w}_I \cdot {}^Iw$ with $w_I, \tilde{w}_I \in W_I$ and $w^I \in W^I$ and ${}^Iw \in {}^IW$. Moreover, these decompositions satisfy $\ell(w) = \ell(w_I) + \ell(w^I) = \ell(\tilde{w}_I) + \ell({}^Iw)$ (see [DDPW], Proposition 4.16). In particular, W^I and ${}^IW$ are systems of representatives for the quotients W/W_I and $W_I\backslash W$, respectively. The fact that $\ell(w) = \ell(w^{-1})$ for all $w \in W$ implies that $W^I = ({}^IW)^{-1}$.

Furthermore, if J is a second subset of S, we let ${}^IW^J$ denote the set of $x \in W$ which have minimal length in the double coset W_IxW_J. Then ${}^IW^J = {}^IW \cap W^J$, and it is a system of representatives for $W_I\backslash W/W_J$ (see [DDPW] (4.3.2)).

In the next propositions we take an element $x \in {}^IW^J$, consider the conjugate subset ${}^{x^{-1}}I \subset W$, and abbreviate $I_x := J \cap {}^{x^{-1}}I \subset J$. Then ${}^{I_x}W_J$ is the set of elements w_J of W_J which have minimal length in their coset $W_{I_x}w_J$. Likewise $W_I^{I\cap {}^xJ}$ is the set of elements w_I of W_I which have minimal length in their coset $w_IW_{I\cap {}^xJ}$.

Proposition 2.6 (Kilmoyer, [DDPW], Proposition 4.17). *For $x \in {}^IW^J$ we have*

$$W_I \cap {}^xW_J = W_{I\cap {}^xJ} \qquad \textit{and} \qquad W_J \cap {}^{x^{-1}}W_I = W_{J\cap {}^{x^{-1}}I} = W_{I_x}.$$

Proposition 2.7 (Howlett, [DDPW], Proposition 4.18). *For any $x \in {}^IW^J$, every element w of the double coset W_IxW_J is uniquely expressible in the form $w = w_Ixw_J$ with $w_I \in W_I$ and $w_J \in {}^{I_x}W_J$. Moreover, this decomposition satisfies*
$$\ell(w) = \ell(w_Ixw_J) = \ell(w_I) + \ell(x) + \ell(w_J).$$

Proposition 2.8. *The set ${}^IW$ is the set of all xw_J for $x \in {}^IW^J$ and $w_J \in {}^{I_x}W_J$.*

Proof. Take $x \in {}^IW^J$ and $w_J \in {}^{I_x}W_J$. Then for any $w_I \in {}^IW$, Proposition 2.7 applied to the product w_Ixw_J implies that $\ell(w_Ixw_J) = \ell(w_I)+\ell(x)+\ell(w_J) \geq \ell(x) + \ell(w_J) = \ell(xw_J)$. This proves that $xw_J \in {}^IW$. Conversely take $w \in {}^IW$ and let $w = w_Ixw_J$ be its decomposition from Proposition 2.7. Then by the first part of the proof we have $xw_J \in {}^IW$. Since $W_Iw = W_Ixw_J$, this implies that $w = xw_J$. $\square$

Proposition 2.9. *The set W^J is the set of all w_Ix for $x \in {}^IW^J$ and $w_I \in W_I^{I\cap {}^xJ}$.*

Proof. Apply Proposition 2.8 with I and J interchanged and invert all elements of W^J. $\square$

Next we recall the Bruhat order on W, which we denote by $\leq$ and $<$. This natural partial order is characterized by the following property: For $x, w \in W$ we have $x \leq w$ if and only if for some (or, equivalently, any) reduced expression $w = s_1 \cdots s_n$ as a product of simple reflections $s_i \in S$, one gets a reduced expression for x by removing certain s_i from this product. More information about the Bruhat order can be found in [BB], Chapter 2.
Using this order, the set ${}^IW$ can be described as
$${}^IW = \{w \in W \mid w < sw \text{ for all } s \in I\} \tag{2.10}$$
(see [BB], Definition 2.4.2 and Corollary 2.4.5).
Assume in addition that W is the Weyl group of a root system Φ, with S corresponding to a basis of Φ. Denote the set of positive roots with respect to the given basis by Φ^+ and the set of negative roots by Φ^-. For $I \subset S$, let Φ_I be the root system spanned by the basis elements corresponding to I, and set $\Phi_I^\pm := \Phi_I \cap \Phi^\pm$. Then by [Car], Proposition 2.3.3 we have
$$W^I = \{w \in W \mid w\Phi_I^+ \subset \Phi^+\} = \{w \in W \mid w\Phi_I^- \subset \Phi^-\}. \tag{2.11}$$
Also, by [Car], Proposition 2.2.7, the length of any $w \in W$ is
$$\ell(w) = \#\{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}. \tag{2.12}$$

Lemma 2.13. *Let $w \in {}^IW$ and write $w = xw_J$ with $x \in {}^IW^J$ and $w_J \in W_J$. Then*
$$\ell(x) = \#\{\alpha \in \Phi^+ \smallsetminus \Phi_J \mid w\alpha \in \Phi^- \smallsetminus \Phi_I\}.$$

Proof. First note that $\alpha \in \Phi^+$ and $w\alpha \in \Phi^-$ already imply $w\alpha \notin \Phi_I$, because otherwise we would have $\alpha \in w^{-1}\Phi_I^-$, which by (2.11) is contained in Φ^- because $w^{-1} \in W^I$. Thus the right hand side of the claim is $\#\{\alpha \in \Phi^+ \smallsetminus \Phi_J \mid w\alpha \in \Phi^-\}$. Secondly, if $\alpha \in \Phi_J^+$, using again (2.11) and $x \in W^J$ we find that $w\alpha \in \Phi^-$ if and only if $w_J\alpha \in \Phi_J^-$. Thus with (2.12) we obtain

$$\begin{aligned}\#\{\alpha \in \Phi^+ \smallsetminus \Phi_J \mid w\alpha \in \Phi^-\} &= \#\{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\} - \#\{\alpha \in \Phi_J^+ \mid w_J\alpha \in \Phi_J^-\} \\ &= \ell(w) - \ell(w_J) \;=\; \ell(x).\end{aligned}$$

□

2.3 REDUCTIVE GROUPS, WEYL GROUPS, AND PARABOLICS

Let G be a connected reductive algebraic group, let B be a Borel subgroup of G, and let T be a maximal torus of B. Let $\Phi(G,T)$ denote the root system of G with respect to T, let $W(G,T) := \mathrm{Norm}_G(T)/T$ denote the associated Weyl group, and let $S(G,B,T) \subset W(G,T)$ denote the set of simple reflections defined by B. Then $W(G,T)$ is a Coxeter group with respect to the subset $S(G,B,T)$.

A priori this data depends on the pair (B,T). However, any other such pair (B',T') is obtained on conjugating (B,T) by some element $g \in G$ which is unique up to right multiplication by T. Thus conjugation by g induces isomorphisms $\Phi(G,T) \overset{\sim}{\to} \Phi(G,T')$ and $W(G,T) \overset{\sim}{\to} W(G,T')$ and $S(G,B,T) \overset{\sim}{\to} S(G,B',T')$ that are independent of g. Moreover, the isomorphisms associated to any three such pairs are compatible with each other. Thus $\Phi := \Phi(G,T)$ and $W := W(G,T)$ and $S := S(G,B,T)$ for any choice of (B,T) can be viewed as instances of 'the' root system and 'the' Weyl group and 'the' set of simple reflections of G, in the sense that up to unique isomorphisms they depend only on G. It then also makes sense to say that the result of a construction (as in Subsection 5.2 below) depending on an element of W is independent of (B,T).

For any $w \in W(G,T)$ we fix a representative $\dot{w} \in \mathrm{Norm}_G(T)$. By choosing representatives attached to a Chevalley system (see [DG] Exp. XXIII, §6) for all $w_1, w_2 \in W$ with $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ we obtain

$$\dot{w}_1\dot{w}_2 = (w_1w_2)^{\cdot}. \tag{2.14}$$

In particular the identity element $1 \in W$ is represented by the identity element $1 \in G$.

A parabolic subgroup of G that contains B is called a *standard parabolic of* G. Any standard parabolic possesses a unique Levi decomposition $P = \mathcal{R}_uP \rtimes L$ with $T \subset L$. Any such L is called a standard Levi subgroup of G, and the set I of simple reflections in the Weyl group of L is called the *type of* L or *of* P. In this way there is a unique standard parabolic P_I of type I for every subset $I \subset S$, and vice versa. The type of a general parabolic P is by definition the type of the unique standard parabolic conjugate to P; it is independent of (B,T) in the above sense. Any conjugate of a standard Levi subgroup of G is called a Levi subgroup of G.

For any subset $I \subset S$ let Par_I denote the set of all parabolics of G of type I. Then there is a natural bijection $G/P_I \xrightarrow{\sim} \mathrm{Par}_I$, $gP_I \mapsto {}^gP_I$. For any two subsets I, $J \subset S$ we let G act by simultaneous conjugation on $\mathrm{Par}_I \times \mathrm{Par}_J$. As a consequence of the Bruhat decomposition (see [Spr1] 8.4.6 (3)), the G-orbit of any pair $(P, Q) \in \mathrm{Par}_I \times \mathrm{Par}_J$ contains a unique pair of the form $(P_I, {}^{\dot{x}}P_J)$ with $x \in {}^IW^J$. This element x is called the *relative position of* P *and* Q and is denoted by $\mathrm{relpos}(P, Q)$.

We will also use several standard facts about intersections of parabolics and/or Levi subgroups, for instance (see [Car], Proposition 2.8.9):

Proposition 2.15. *Let L be a Levi subgroup of G and T a maximal torus of L. Let P be a parabolic subgroup of G containing T and $P = \mathcal{R}_uP \rtimes H$ its Levi decomposition with $T \subset H$. Then $L \cap P$ is a parabolic subgroup of L with Levi decomposition*

$$L \cap P = (L \cap \mathcal{R}_uP) \rtimes (L \cap H).$$

If P is a Borel subgroup of G, then $L \cap P$ is a Borel subgroup of L.

3 Connected algebraic zip data

We now define the central technical notions of this article.

Definition 3.1. A *connected algebraic zip datum* is a tuple $\mathcal{Z} = (G, P, Q, \varphi)$ consisting of a connected reductive group G with parabolic subgroups P and Q and an isogeny $\varphi\colon P/R_uP \to Q/R_uQ$. The group

$$E_{\mathcal{Z}} := \big\{(p, q) \in P{\times}Q \;\big|\; \varphi(\pi_P(p)) = \pi_Q(q)\big\} \tag{3.2}$$

is called the *zip group associated to* $\mathcal{Z}$. It acts on G by restriction of the left action

$$(P{\times}Q) \times G \to G, \quad \big((p, q), g\big) \mapsto pgq^{-1}. \tag{3.3}$$

For any subset $X \subset G$ we denote the union of the $E_{\mathcal{Z}}$-orbits of all elements of X by

$$\mathrm{o}_{\mathcal{Z}}(X). \tag{3.4}$$

Note that if X is a constructible subset of G, then so is $\mathrm{o}_{\mathcal{Z}}(X)$.

Throughout the following sections we fix a connected algebraic zip datum $\mathcal{Z} = (G, P, Q, \varphi)$. We also abbreviate $U := \mathcal{R}_uP$ and $V := \mathcal{R}_uQ$, so that φ is an isogeny $P/U \to Q/V$. Our aim is to study the orbit structure of the action of $E_{\mathcal{Z}}$ on G.

Example 3.5. For dimension reasons we have $P = G$ if and only if $Q = G$. In that case the action of $E_{\mathcal{Z}} = \mathrm{graph}(\varphi)$ is equivalent to the action of G on itself by twisted conjugation $(h, g) \mapsto hg\varphi(h)^{-1}$.

In order to work with $\mathcal{Z}$ it is convenient to fix the following data.

DEFINITION 3.6. A *frame of* $\mathcal{Z}$ is a tuple (B,T,g) consisting of a Borel subgroup B of G, a maximal torus T of B, and an element $g \in G$, such that

(a) $B \subset Q$,

(b) ${}^gB \subset P$,

(c) $\varphi\big(\pi_P({}^gB)\big) = \pi_Q(B)$, and

(d) $\varphi\big(\pi_P({}^gT)\big) = \pi_Q(T)$.

PROPOSITION 3.7. *Every connected algebraic zip datum possesses a frame.*

Proof. Choose a Borel subgroup B of Q and a maximal torus T of B. Let $\bar{T}' \subset \bar{B}' \subset P/U$ denote the respective identity components of $\varphi^{-1}(\pi_Q(T)) \subset \varphi^{-1}(\pi_Q(B)) \subset P/U$. Then $\bar{B}'$ is a Borel subgroup of P/U, and $\bar{T}'$ is a maximal torus of $\bar{B}'$. Thus we have $\bar{B}' = \pi_P(B')$ for a Borel subgroup B' of P, and $\bar{T}' = \pi_P(T')$ for some maximal torus T' of B'. Finally take $g \in G$ such that $B' = {}^gB$ and $T' = {}^gT$. Then (B,T,g) is a frame of $\mathcal{Z}$. ☐

PROPOSITION 3.8. *Let* (B,T,g) *be a frame of* $\mathcal{Z}$. *Then every frame of* $\mathcal{Z}$ *has the form* $({}^qB, {}^qT, pgtq^{-1})$ *for* $(p,q) \in E_{\mathcal{Z}}$ *and* $t \in T$, *and every tuple of this form is a frame of* $\mathcal{Z}$.

Proof. Let (B',T',g') be another frame of $\mathcal{Z}$. Since all Borel subgroups of Q are conjugate, we have $B' = {}^qB$ for some element $q \in Q$. Since all maximal tori of B' are conjugate, after multiplying q on the left by an element of B' we may in addition assume that $T' = {}^qT$. Similarly we can find an element $p \in P$ such that ${}^{g'}B' = {}^{pg}B$ and ${}^{g'}T' = {}^{pg}T$. Combining these equations with the defining properties of frames we find that

$$\begin{aligned} {}^{\varphi(\pi_P(p))}\pi_Q(B) \;=\; {}^{\varphi(\pi_P(p))}\varphi\big(\pi_P({}^gB)\big) \;&=\; \varphi\big(\pi_P({}^{pg}B)\big) \;=\; \varphi\big(\pi_P({}^{g'}B')\big) \;= \\ &=\; \pi_Q(B') \;=\; \pi_Q({}^qB) \;=\; {}^{\pi_Q(q)}\pi_Q(B), \end{aligned}$$

and similarly ${}^{\varphi(\pi_P(p))}\pi_Q(T) = {}^{\pi_Q(q)}\pi_Q(T)$. Thus $\varphi(\pi_P(p)) = \pi_Q(q)\cdot\pi_Q(t')$ for some element $t' \in T$. Since we may still replace q by qt' without changing the above equations, we may without loss of generality assume that $\varphi(\pi_P(p)) = \pi_Q(q)$, so that $(p,q) \in E_{\mathcal{Z}}$. On the other hand, the above equations imply that $B = {}^{g^{-1}p^{-1}g'q}B$ and $T = {}^{g^{-1}p^{-1}g'q}T$, so that $t := g^{-1}p^{-1}g'q \in T$ and hence $g' = pgtq^{-1}$. This proves the first assertion. The second involves a straightforward calculation that is left to the conscientious reader. ☐

Throughout the following sections we fix a frame (B,T,g) of $\mathcal{Z}$. It determines unique Levi components ${}^gT \subset L \subset P$ and $T \subset M \subset Q$. Via the isomorphisms $L \xrightarrow{\sim} P/U$ and $M \xrightarrow{\sim} Q/V$ we can then identify φ with an isogeny $\varphi\colon L \to M$. The zip group then becomes

$$E_{\mathcal{Z}} \;=\; \big\{(u\ell, v\varphi(\ell)) \;\big|\; u \in U,\, v \in V,\, \ell \in L\big\} \tag{3.9}$$

and acts on G by $((u\ell, v\varphi(\ell)), g) \mapsto u\ell g\varphi(\ell)^{-1}v^{-1}$. Moreover, conditions 3.6 (c) and (d) are then equivalent to

$$\varphi({}^gB \cap L) = B \cap M, \qquad \text{and} \qquad \varphi({}^gT) = T, \tag{3.10}$$

which are a Borel subgroup and a maximal torus of M, respectively.

Let Φ be the root system, W the Weyl group, and $S \subset W$ the set of simple reflections of G with respect to (B, T). Let $I \subset S$ be the type of ${}^{g^{-1}}P$ and $J \subset S$ the type of Q. Then M has root system Φ_J, Weyl group W_J, and set of simple reflections $J \subset W_J$. Similarly ${}^{g^{-1}}L$ has root system Φ_I, Weyl group W_I, and set of simple reflections $I \subset W_I$, and the inner automorphism $\mathrm{int}(g)$ identifies these with the corresponding objects associated to L. Moreover, the equations (3.10) imply that $\varphi \circ \mathrm{int}(g)$ induces an isomorphism of Coxeter systems

$$\psi\colon (W_I, I) \xrightarrow{\sim} (W_J, J). \tag{3.11}$$

Recall that Φ, W, and S can be viewed as independent of the chosen frame, as explained in Subsection 2.3.

Proposition 3.11. *The subsets I, J and the isomorphism ψ are independent of the frame.*

Proof. Consider another frame $({}^qB, {}^qT, pgtq^{-1})$ with $(p, q) \in E_{\mathcal{Z}}$ and $t \in T$, as in Proposition 3.8. Then we have a commutative diagram

$$\begin{tikzcd}
({}^{g^{-1}}L, B, T) \arrow[r, "\mathrm{int}(g)", "\sim"'] \arrow[d, "\mathrm{int}(qt^{-1})"', "\wr"] & (L, {}^gB, {}^gT) \arrow[r, "\varphi"] \arrow[d, "\mathrm{int}(p)"', "\wr"] & (M, B, T) \arrow[d, "\mathrm{int}(q)"', "\wr"] \\
({}^{qg^{-1}}L, {}^qB, {}^qT) \arrow[r, "\mathrm{int}(pgtq^{-1})", "\sim"'] & ({}^pL, {}^{pg}B, {}^{pg}T) \arrow[r, "\varphi"] & ({}^qM, {}^qB, {}^qT),
\end{tikzcd}$$

whose upper row contains the data inducing ψ for the old frame and whose lower row is the analogue for the new frame. Since the vertical arrows are inner automorphisms, they induce the identity on the abstract Coxeter system (W, S) of G as explained in Subsection 2.3. Everything follows from this. $\square$

4 Induction step

We keep the notations of the preceding section. Since ${}^{g^{-1}}P$ and Q are parabolic subgroups containing the same Borel B, by Bruhat (see [Spr1] 8.4.6 (3)) we have a disjoint decomposition

$$G \;=\; \coprod_{x \in {}^IW^J} {}^{g^{-1}}P\dot{x}Q.$$

Left translation by g turns this into a disjoint decomposition

$$G \;=\; \coprod_{x \in {}^IW^J} Pg\dot{x}Q. \tag{4.1}$$

Here each component $Pg\dot{x}Q$ is an irreducible locally closed subvariety of G that is invariant under the action of $E_{\mathcal{Z}}$. In this section we fix an element $x \in {}^IW^J$ and establish a bijection between the $E_{\mathcal{Z}}$-orbits in $Pg\dot{x}Q$ and the orbits of another zip datum constructed from $\mathcal{Z}$ and $\dot{x}$. This will allow us to prove facts about the orbits inductively. The base case of the induction occurs when the decomposition possesses just one piece, i.e., when $P = Q = G$.

Lemma 4.2. *The stabilizer of $g\dot{x}Q \subset Pg\dot{x}Q$ in $E_{\mathcal{Z}}$ is the subgroup*

$$E_{\mathcal{Z},\dot{x}} := \bigl\{(p,q) \in E_{\mathcal{Z}} \bigm| p \in P \cap {}^{g\dot{x}}Q\bigr\},$$

and the action of $E_{\mathcal{Z}}$ induces an $E_{\mathcal{Z}}$-equivariant isomorphism

$$E_{\mathcal{Z}} \times^{E_{\mathcal{Z},\dot{x}}} g\dot{x}Q \xrightarrow{\sim} Pg\dot{x}Q,\ [((p,q),h)] \mapsto phq^{-1}.$$

Proof. The action (3.3) of $(p,q) \in E_{\mathcal{Z}}$ on $Pg\dot{x}Q$ induces the action on the quotient $Pg\dot{x}Q/Q$ by left multiplication with p. From (3.2) we see that the latter action is transitive, and the stabilizer of the point $g\dot{x}Q$ is $E_{\mathcal{Z},\dot{x}}$; hence there is an $E_{\mathcal{Z}}$-equivariant isomorphism $Pg\dot{x}Q/Q \cong E_{\mathcal{Z}}/E_{\mathcal{Z},\dot{x}}$. Thus everything follows by applying Lemma 2.2 to the projection morphism $Pg\dot{x}Q \twoheadrightarrow Pg\dot{x}Q/Q \cong E_{\mathcal{Z}}/E_{\mathcal{Z},\dot{x}}$. □

Construction 4.3. Consider the following subgroups of the connected reductive algebraic group M (which are independent of the representative $\dot{x}$ of x):

$$\begin{aligned} P_x &:= M \cap {}^{\dot{x}^{-1}g^{-1}}P, & Q_x &:= \varphi(L \cap {}^{g\dot{x}}Q),\\ U_x &:= M \cap {}^{\dot{x}^{-1}g^{-1}}U, & V_x &:= \varphi(L \cap {}^{g\dot{x}}V),\\ L_x &:= M \cap {}^{\dot{x}^{-1}g^{-1}}L, & M_x &:= \varphi(L \cap {}^{g\dot{x}}M). \end{aligned}$$

Proposition 2.15 shows that P_x is a parabolic with unipotent radical U_x and Levi component L_x, and that Q_x is a parabolic with unipotent radical V_x and Levi component M_x. Moreover, $\varphi \circ \mathrm{int}(g\dot{x})$ induces an isogeny $\varphi_{\dot{x}} \colon L_x \to M_x$, or equivalently $P_x/U_x \to Q_x/V_x$. Thus we obtain a connected algebraic zip datum $\mathcal{Z}_{\dot{x}} := (M, P_x, Q_x, \varphi_{\dot{x}})$. By (3.9) its zip group is

$$E_{\mathcal{Z}_{\dot{x}}} \;=\; \bigl\{(u'\ell', v'\varphi_{\dot{x}}(\ell')) \bigm| u' \in U_x,\, v' \in V_x,\, \ell' \in L_x\bigr\}. \tag{4.4}$$

Lemma 4.5. *There is a surjective homomorphism*

$$E_{\mathcal{Z},\dot{x}} \twoheadrightarrow E_{\mathcal{Z}_{\dot{x}}},\ (p,q) \mapsto (m, \varphi(\ell)),$$

where $p = u\ell$ for $u \in U$ and $\ell \in L$, and ${}^{\dot{x}^{-1}g^{-1}}p = vm$ for $v \in V$ and $m \in M$.

Proof. For ease of notation abbreviate $h := g\dot{x}$, so that ${}^hT = {}^gT \subset L$ and therefore $T \subset {}^{h^{-1}}L \subset {}^{h^{-1}}P$. Thus ${}^{h^{-1}}P$ and Q are parabolics of G with T-invariant Levi decompositions ${}^{h^{-1}}P = {}^{h^{-1}}U \rtimes {}^{h^{-1}}L$ and $Q = V \rtimes M$. It follows (see [Car] Thm. 2.8.7) that any element of ${}^{h^{-1}}P \cap Q$ can be written as a product $abu'\ell'$ with unique

$$\begin{aligned} a &\in {}^{h^{-1}}U \cap V, & u' &\in {}^{h^{-1}}U \cap M = U_x, \\ b &\in {}^{h^{-1}}L \cap V, & \ell' &\in {}^{h^{-1}}L \cap M = L_x. \end{aligned}$$

Consider $(p,q) \in E_{\mathcal{Z},\dot{x}}$ with $p = u\ell$ and ${}^{h^{-1}}p = vm$ as in the lemma. Then we can write the element ${}^{h^{-1}}p = abu'\ell' \in {}^{h^{-1}}P \cap Q$ in the indicated fashion. Comparing the different factorizations yields the equations $v = ab$, $m = u'\ell'$, $u = {}^{h}(abu'b^{-1})$, and $\ell = {}^{h}(b\ell')$. Thus $\varphi(\ell) = \varphi({}^{h}b)\varphi({}^{h}\ell') = v'\varphi_{\dot{x}}(\ell')$ with $v' := \varphi({}^{h}b) \in \varphi(L \cap {}^{h}V) = V_x$. In view of (4.4) it follows that $(m, \varphi(\ell)) = (u'\ell', v'\varphi_{\dot{x}}(\ell'))$ lies in $E_{\mathcal{Z}_{\dot{x}}}$, and so the map in question is well-defined. Since m and ℓ are obtained from p by projection to Levi components, the map is a homomorphism. Conversely, every element of $E_{\mathcal{Z}_{\dot{x}}}$ can be obtained in this way from some element $p \in P \cap {}^{h}Q$. By (3.9) we can then also find $q \in Q$ with $(p,q) \in E_{\mathcal{Z},\dot{x}}$. Thus the map is surjective, and we are done. $\square$

LEMMA 4.6. *The surjective morphism*

$$\pi\colon g\dot{x}Q \twoheadrightarrow M,\ g\dot{x}\tilde{m}\tilde{v} \mapsto \tilde{m}$$

for $\tilde{m} \in M$ and $\tilde{v} \in V$ is equivariant under the group $E_{\mathcal{Z},\dot{x}}$, which acts on $g\dot{x}Q$ as in Lemma 4.2 and on M through the homomorphism from Lemma 4.5.

Proof. Take $(p,q) \in E_{\mathcal{Z},\dot{x}}$ with $p = u\ell$ and ${}^{\dot{x}^{-1}g^{-1}}p = vm$ as in Lemma 4.5. Then (3.9) implies that $q = v_1\varphi(\ell)$ for some $v_1 \in V$. Thus the action of (p,q) sends $g\dot{x}\tilde{m}\tilde{v} \in g\dot{x}Q$ to the element

$$pg\dot{x}\cdot\tilde{m}\tilde{v}\cdot q^{-1} = g\dot{x}vm\cdot\tilde{m}\tilde{v}\cdot\varphi(\ell)^{-1}v_1^{-1} = g\dot{x}\cdot m\tilde{m}\varphi(\ell)^{-1}\cdot\big(\text{an element of } V\big).$$

The morphism π maps this element to $m\tilde{m}\varphi(\ell)^{-1} \in M$. But this is also the image of $\tilde{m} = \pi(g\dot{x}\tilde{m}\tilde{v})$ under the action of $(m,\varphi(\ell)) \in E_{\mathcal{Z}_{\dot{x}}}$. Thus the morphism is equivariant. $\square$

PROPOSITION 4.7. *There is a closure-preserving bijection between $E_{\mathcal{Z}_{\dot{x}}}$-invariant subsets $Y \subset M$ and $E_{\mathcal{Z}}$-invariant subsets $X \subset Pg\dot{x}Q$, defined by $Y = M \cap \dot{x}^{-1}g^{-1}X$ and $X = \mathrm{o}_{\mathcal{Z}}(g\dot{x}Y)$. Moreover, Y is a subvariety if and only if X is one, and in that case $X \cong E_{\mathcal{Z}} \times^{E_{\mathcal{Z},\dot{x}}} \pi^{-1}(Y)$.*

Proof. From (3.2) and (3.3) we see that the subgroup $V \cong \{(1,v) \mid v \in V\} \subset E_{\mathcal{Z},\dot{x}}$ acts by right translation on $g\dot{x}Q$. Thus every $E_{\mathcal{Z},\dot{x}}$-invariant subset of $g\dot{x}Q$ is a union of cosets of V and therefore of the form $Z = g\dot{x}YV = \pi^{-1}(Y)$ for a subset $Y \subset M$, which moreover satisfies $Y = M \cap \dot{x}^{-1}g^{-1}Z$. By Lemmas 4.5 and 4.6 this defines a bijection between $E_{\mathcal{Z}_{\dot{x}}}$-invariant subsets $Y \subset M$ and $E_{\mathcal{Z},\dot{x}}$-invariant subsets $Z \subset g\dot{x}Q$. On the other hand, Lemmas 2.1 and 4.2 yield a bijection between $E_{\mathcal{Z},\dot{x}}$-invariant subsets $Z \subset g\dot{x}Q$ and $E_{\mathcal{Z}}$-invariant subsets $X \subset Pg\dot{x}Q$ that is characterized by $Z = g\dot{x}Q \cap X$ and $X = \mathrm{o}_{\mathcal{Z}}(Z)$. Together we

obtain the desired bijection with $Y = M \cap \dot{x}^{-1}g^{-1}(g\dot{x}Q \cap X) = M \cap \dot{x}^{-1}g^{-1}X$ and $X = \mathrm{o}_{\mathcal{Z}}(g\dot{x}YV) = \mathrm{o}_{\mathcal{Z}}(g\dot{x}Y)$.
The equations $Z = \pi^{-1}(Y)$ and $Y = M \cap \dot{x}^{-1}g^{-1}Z$ imply that the bijection between Y and Z preserves closures and maps subvarieties to subvarieties. The corresponding facts for the bijection between Z and X follow from Lemma 2.1, which also implies the last statement. □

Proposition 4.8. *If X and Y in Proposition 4.7 are subvarieties, then*

$$\dim X \;=\; \dim Y + \dim P - \dim P_x + \ell(x).$$

Proof. By the definition of $E_{\mathcal{Z},\dot{x}}$ we have

$$\dim E_{\mathcal{Z}} - \dim E_{\mathcal{Z},\dot{x}} \;=\; \dim P - \dim(P \cap {}^{g\dot{x}}Q) \;=\; \dim P - \dim P_x - \dim(P \cap {}^{g\dot{x}}V).$$

With the last statement of Proposition 4.7 this implies that

$$\dim X \;=\; \dim Y + \dim V + \dim P - \dim P_x - \dim(P \cap {}^{g\dot{x}}V).$$

From the decomposition of V into root subgroups it follows that $\dim V - \dim(P \cap {}^{g\dot{x}}V) = \dim V - \dim(V \cap {}^{\dot{x}^{-1}g^{-1}}P)$ is the cardinality of the set

$$\{\alpha \in \Phi^+ \smallsetminus \Phi_J \mid x\alpha \in \Phi^- \smallsetminus \Phi_I\}.$$

By Lemma 2.13 for $w_J = 1$ this cardinality is $\ell(x)$. □

Lemma 4.9. *For any subset $Y \subset M$ we have $\mathrm{o}_{\mathcal{Z}}(g\dot{x}\,\mathrm{o}_{\mathcal{Z}_{\dot{x}}}(Y)) = \mathrm{o}_{\mathcal{Z}}(g\dot{x}Y)$.*

Proof. It suffices to show that $g\dot{x}\,\mathrm{o}_{\mathcal{Z}_{\dot{x}}}(Y) \subset \mathrm{o}_{\mathcal{Z}}(g\dot{x}Y)$, which follows from a straightforward calculation that is left to the reader. Alternatively the formula can be deduced from the formal properties stated in Proposition 4.7. □

We can also give an inductive description of the stabilizers of points in $Pg\dot{x}Q$. However, this does not give the scheme-theoretic stabilizers, which may in fact be non-reduced. Likewise, the following lemma does not describe the scheme-theoretic kernel:

Lemma 4.10. *The kernel of the homomorphism from Lemma 4.5 is $(U \cap {}^{g\dot{x}}V) \times V$.*

Proof. Let $p = u\ell$ and ${}^{\dot{x}^{-1}g^{-1}}p = vm$ be as in Lemma 4.5. Then (p, q) is in the kernel if and only if $m = 1$ and $\varphi(\ell) = 1$. The first equation is equivalent to $p = {}^{g\dot{x}}v \in {}^{g\dot{x}}V$, which implies that ℓ is unipotent. Being in the kernel of the isogeny φ is then equivalent to $\ell = 1$. Thus the second equation is equivalent to $p \in U$, and the two together are equivalent to $p \in U \cap {}^{g\dot{x}}V$. By (3.9) we then have $q \in V$, and so we are done. □

Proposition 4.11. *For any $m \in M$ there is a short exact sequence*

$$1 \longrightarrow U \cap {}^{g\dot{x}}V \longrightarrow \mathrm{Stab}_{E_{\mathcal{Z}}}(g\dot{x}m) \xrightarrow{4.5} \mathrm{Stab}_{E_{\mathcal{Z}_{\dot{x}}}}(m) \longrightarrow 1.$$

Proof. The second half of Lemma 4.2 and Lemma 4.6 imply that we have an equality, respectively a homomorphism

$$\mathrm{Stab}_{E_{\mathcal{Z}}}(g\dot{x}m) \;=\; \mathrm{Stab}_{E_{\mathcal{Z},\dot{x}}}(g\dot{x}m) \xrightarrow{4.5} \mathrm{Stab}_{E_{\mathcal{Z}_{\dot{x}}}}(m).$$

This homomorphism is surjective, because the subgroup $V \cong \{(1,v) \mid v \in V\}$ contained in the kernel of the surjection $E_{\mathcal{Z},\dot{x}} \twoheadrightarrow E_{\mathcal{Z}_{\dot{x}}}$ acts transitively on the fibers of π. By Lemma 4.10 the kernel is the stabilizer of $g\dot{x}m$ in the group $(U \cap {}^{g\dot{x}}V) \times V$ acting by left and right translation. This stabilizer consists of $(u, {}^{(g\dot{x}m)^{-1}}u)$ for all $u \in U \cap {}^{g\dot{x}}V$, and we are done. $\square$

Finally, the assumption $x \in {}^{I}W^{J}$ allows us to construct a frame of $\mathcal{Z}_{\dot{x}}$:

PROPOSITION 4.12. *The tuple $(M \cap B, T, 1)$ is a frame of $\mathcal{Z}_{\dot{x}}$, and the associated Levi components of P_x and Q_x are L_x and M_x, respectively.*

Proof. First, the assumptions $T \subset M$ and ${}^{g\dot{x}}T = {}^{g}T \subset L$ imply that $T \subset M \cap {}^{\dot{x}^{-1}g^{-1}}L$, the latter being L_x by Construction 4.3. Together with the equation $\varphi({}^{g}T) = T$ from (3.10) they also imply that $T = \varphi({}^{g\dot{x}}T) \subset \varphi(L \cap {}^{g\dot{x}}M)$, the latter being M_x by Construction 4.3. This proves the statement about the Levi components. We can also directly deduce that $\varphi_{\dot{x}}(T) = \varphi({}^{g\dot{x}}T) = T$.
Next, as T is a common maximal torus of M and B, Proposition 2.15 implies that $M \cap B$ is a Borel subgroup of M. Recall that M has the root system Φ_J, so that $M \cap B$ corresponds to the subset $\Phi_J^+ = \Phi_J \cap \Phi^+$. For the same reasons $M \cap {}^{\dot{x}^{-1}}B$ is a Borel subgroup of M corresponding to the subset $\Phi_J \cap x^{-1}\Phi^+$. But with (2.11) the assumption $x \in {}^{I}W^{J} \subset W^{J}$ implies that $x\Phi_J^+ \subset \Phi^+$, and hence $\Phi_J^+ \subset \Phi_J \cap x^{-1}\Phi^+$. Since both subsets correspond to Borel subgroups, they must then coincide, and therefore $M \cap B = M \cap {}^{\dot{x}^{-1}}B$. With the inclusion ${}^{g}B \subset P$ from (3.6) we deduce that

$$M \cap B \;=\; M \cap {}^{\dot{x}^{-1}}B \;\subset\; M \cap {}^{\dot{x}^{-1}g^{-1}}P \;\overset{4.3}{=}\; P_x.$$

In the same way one shows that $L \cap {}^{g}B = L \cap {}^{g\dot{x}}B$, which together with $B \subset Q$ implies that

$$M \cap B \;\overset{(3.10)}{=}\; \varphi(L \cap {}^{g}B) \;=\; \varphi(L \cap {}^{g\dot{x}}B) \;\subset\; \varphi(L \cap {}^{g\dot{x}}Q) \;\overset{4.3}{=}\; Q_x.$$

The equation $M \cap B = \varphi(L \cap {}^{g\dot{x}}B)$ and Construction 4.3 also imply that

$$\begin{aligned}\varphi_{\dot{x}}\big((M \cap B) \cap L_x\big) \;&=\; \varphi\big({}^{g\dot{x}}M \cap {}^{g\dot{x}}B \cap L\big) \;\subset \\ &\subset\; \varphi\big(L \cap {}^{g\dot{x}}M\big) \cap \varphi\big(L \cap {}^{g\dot{x}}B\big) \;=\; (M \cap B) \cap M_x.\end{aligned}$$

As both sides of this inclusion are Borel subgroups of M_x, they must be equal. Thus $(M \cap B, T, 1)$ satisfies Definition 3.6 in the variant (3.10), as desired. $\square$

Recall that M has the Weyl group W_J with the set of simple reflections J, and that $\psi\colon W_I \xrightarrow{\sim} W_J$ is the isomorphism induced by $\varphi \circ \mathrm{int}(g)$.

Proposition 4.13. *(a) The type of the parabolic P_x of M is $I_x := J \cap {}^{x^{-1}}I$.*

(b) The type of the parabolic Q_x of M is $J_x := \psi(I \cap {}^{x}J)$.

(c) The isomomorphism $\psi_x \colon W_{I_x} \xrightarrow{\sim} W_{J_x}$ induced by $\varphi_{\dot{x}}$ is the restriction of $\psi \circ \mathrm{int}(x)$.

Proof. Proposition 2.6 implies that $L_x = M \cap {}^{\dot{x}^{-1}g^{-1}}L$ has the Weyl group $W_J \cap {}^{x^{-1}}W_I = W_{I_x}$, which shows (a). Likewise $M_x = \varphi(L \cap {}^{g\dot{x}}M)$ has the Weyl group $\psi(W_I \cap {}^{x}W_J) = W_{J_x}$, which implies (b). Finally, (c) follows from $\varphi_{\dot{x}} = \varphi \circ \mathrm{int}(g\dot{x})$. □

5 Decomposition of G

In this section we construct a natural decomposition of G into finitely many $E_{\mathcal{Z}}$-invariant subvarieties G^w.

5.1 The Levi subgroup H_w

Fix an element $w \in {}^{I}W$. Note that we can compare any subgroup H of ${}^{\dot{w}^{-1}g^{-1}}L$ with its image $\varphi \circ \mathrm{int}(g\dot{w})(H)$ in M, because both are subgroups of G. Moreover, the collection of all such H satisfying $\varphi \circ \mathrm{int}(g\dot{w})(H) = H$ possesses a unique largest element, namely the subgroup generated by all such subgroups.

Definition 5.1. We let H_w denote the unique largest subgroup of ${}^{\dot{w}^{-1}g^{-1}}L$ satisfying $\varphi \circ \mathrm{int}(g\dot{w})(H_w) = H_w$. We let $\varphi_{\dot{w}} \colon H_w \to H_w$ denote the isogeny induced by $\varphi \circ \mathrm{int}(g\dot{w})$, and let H_w act on itself from the left by the twisted conjugation $(h, h') \mapsto hh'\varphi_{\dot{w}}(h)^{-1}$.

Remark 5.2. Since $\varphi \circ \mathrm{int}(g\dot{w})(T) = \varphi({}^{g}T) = T$ by (3.10), the defining property of H_w implies that $T \subset H_w$. Thus H_w does not depend on the choice of representative $\dot{w}$ of w, justifying the notation H_w. Also, in the case that $w = x \in {}^{I}W^{J}$ observe that the $\varphi_{\dot{w}}$ defined here is the restriction to H_w of the isogeny $\varphi_{\dot{x}}$ from Construction 4.3. Using the same notation for both is therefore only mildly abusive.

Example 5.3. In the case $P = Q = G$ from Example 3.5 we have $M = L = G$ and $I = J = \psi(J) = S$ and hence ${}^{I}W = \{1\}$ and $H_1 = G$.

To analyze H_w in the general case we apply the induction step from Section 4. Let $w = xw_J$ be the decomposition from Proposition 2.8 with $x \in {}^{I}W^{J}$ and $w_J \in {}^{I_x}W_J$ for $I_x = J \cap {}^{x^{-1}}I$. Since W_J is the Weyl group of M, and I_x is the type of the parabolic $P_x \subset M$ by Proposition 4.13 (a), we can also apply Definition 5.1 to the pair $(\mathcal{Z}_{\dot{x}}, w_J)$ in place of $(\mathcal{Z}, w)$.

Lemma 5.4. *The subgroup H_w and the isogeny $\varphi_{\dot{w}}$ associated to $(\mathcal{Z}, w)$ in Definition 5.1 are equal to those associated to $(\mathcal{Z}_{\dot{x}}, w_J)$.*

Proof. Since $\dot{w}_J \in M = \varphi(L)$, Definition 5.1 and Construction 4.3 imply that

$$H_w \;\subset\; M \cap {}^{\dot{w}_J^{-1}\dot{x}^{-1}g^{-1}}L \;=\; {}^{\dot{w}_J^{-1}}\big(M \cap {}^{\dot{x}^{-1}g^{-1}}L\big) \;=\; {}^{\dot{w}_J^{-1}}L_x$$

and that $\varphi_{\dot{x}} \circ \operatorname{int}(\dot{w}_J)(H_w) = \varphi \circ \operatorname{int}(g\dot{w})(H_w) = H_w$. Since H_w is the largest subgroup of ${}^{\dot{w}^{-1}g^{-1}}L$ with this property, it is also the largest in ${}^{\dot{w}_J^{-1}}L_x$. □

REMARK 5.5. The preceding lemma implies that H_w and $\varphi_{\dot{w}}$ also remain the same if we repeat the induction step with $(\mathcal{Z}_{\dot{x}}, w_J)$ in place of $(\mathcal{Z}, w)$, and so on. When the process becomes stationary, we have reached a pair consisting of a zip datum as in Example 5.3 and the Weyl group element 1, whose underlying connected reductive group and isogeny are H_w and $\varphi_{\dot{w}}$. This induction process is the idea underlying many proofs throughout this section.

PROPOSITION 5.6. *The subgroup H_w is the standard Levi subgroup of G containing T whose set of simple reflections is the unique largest subset K_w of ${}^{w^{-1}}I$ satisfying $\psi \circ \operatorname{int}(w)(K_w) = K_w$.*

Proof. For any subset K of ${}^{w^{-1}}I$ the equality $\psi \circ \operatorname{int}(w)(K) = K$ makes sense, because both sides are subsets of W. The collection of all such K satisfying that equality possesses a unique largest element K_w, namely the union of all of them. Then $K_w = \psi \circ \operatorname{int}(w)(K_w) \subset \psi(I) = J \subset S$, and so K_w consists of simple reflections.
Let H denote the standard Levi subgroup of G containing T with the set of simple reflections K_w. Then the isogeny $\varphi \circ \operatorname{int}(g\dot{w})\colon {}^{\dot{w}^{-1}g^{-1}}L \to M$ sends T to itself by Remark 5.2, and the associated isomorphism of Weyl groups $\psi \circ \operatorname{int}(w)\colon {}^{w^{-1}}W_I \to W_J$ sends K_w to itself by construction. Together this implies that $\varphi \circ \operatorname{int}(g\dot{w})(H) = H$ and hence $H \subset H_w$.
We now prove the equality $H_w = H$ by induction on $\dim G$. In the base case $M = G$ we have $I = J = S$ and $w = 1$ and thus $K_1 = S$ and $H = G$, while $H_1 = G$ by Example 5.3; hence we are done. Otherwise write $w = xw_J$ as above. Then Lemma 5.4 and the induction hypothesis show that H_w is a Levi subgroup of M containing T with a set of simple reflections $K \subset {}^{w_J^{-1}}I_x$ satisfying $\psi_x \circ \operatorname{int}(w_J)(K) = K$. But ${}^{w_J^{-1}}I_x = {}^{w_J^{-1}}(J \cap {}^{x^{-1}}I) \subset {}^{w^{-1}}I$ and $\psi_x \circ \operatorname{int}(w_J)$ is the restriction of $\psi \circ \operatorname{int}(x) \circ \operatorname{int}(w_J) = \psi \circ \operatorname{int}(w)$. By the maximality of K_w we thus have $K \subset K_w$ and therefore $H_w \subset H$. Together with the earlier inequality $H \subset H_w$ we deduce that $H_w = H$, as desired. □

5.2 FIRST DESCRIPTION OF G^w

DEFINITION 5.7. For any $w \in {}^{I}W$ we set $G^w := \mathrm{o}_{\mathcal{Z}}(g\dot{w}H_w)$.

PROPOSITION 5.8. *The set G^w does not depend on the representative $\dot{w}$ of w or the frame.*

Proof. The independence of $\dot{w}$ follows from the inclusion $T \subset H_w$. For the rest note first that by Propositions 3.11 and 5.6 the set K_w is independent of the frame. Consider another frame $({}^qB, {}^qT, pgtq^{-1})$ for $(p,q) \in E_{\mathcal{Z}}$ and $t \in T$, as in Proposition 3.8. Recall from Subsection 2.3 that the isomorphism $W(G,T) \overset{\sim}{\to} W(G, {}^qT)$ is induced by $\operatorname{int}(q) : \operatorname{Norm}_G(T) \overset{\sim}{\to} \operatorname{Norm}_G({}^qT)$. It follows that $w \in {}^IW$ as an element of the abstract Weyl group of G is represented by $q\dot{w}q^{-1} \in \operatorname{Norm}_G({}^qT)$, and with Proposition 5.6 it follows that the Levi subgroup associated to w and the new frame is ${}^qH_w$. Thus the right hand side in Definition 5.7 associated to the new frame is

$$\mathrm{o}_{\mathcal{Z}}\big((pgtq^{-1})(q\dot{w}q^{-1})\,{}^qH_w\big) = \mathrm{o}_{\mathcal{Z}}\big(pgt\dot{w}H_wq^{-1}\big) = \mathrm{o}_{\mathcal{Z}}\big(gt\dot{w}H_w\big) = \mathrm{o}_{\mathcal{Z}}\big(g\dot{w}H_w\big),$$

where the second equation follows from $(p,q) \in E_{\mathcal{Z}}$ and the third from $\dot{w}^{-1}t\dot{w} \in T \subset H_w$. Thus G^w is independent of the frame. □

In Example 5.3 we have $H_1 = G$ and hence $G^1 = G$. Otherwise recall from Proposition 4.12 that $\mathcal{Z}_{\dot{x}}$ has the frame $(M \cap B, T, 1)$. Thus by Lemma 5.4, the subset associated to $(\mathcal{Z}_{\dot{x}}, w_J)$ by Definition 5.7 is $M^{w_J} := \mathrm{o}_{\mathcal{Z}_{\dot{x}}}(\dot{w}_J H_w)$.

Lemma 5.9. *Under the bijection of Proposition 4.7, the subset $M^{w_J} \subset M$ corresponds to the subset $G^w \subset Pg\dot{x}Q$. In particular $G^w = \mathrm{o}_{\mathcal{Z}}(g\dot{x}M^{w_J})$. Also, there is a bijection between the $E_{\mathcal{Z}_{\dot{x}}}$-orbits $X' \subset M^{w_J}$ and the $E_{\mathcal{Z}}$-orbits $X \subset G^w$, defined by $X = \mathrm{o}_{\mathcal{Z}}(g\dot{x}X')$.*

Proof. Using, in this order, the definition of G^w, the equation (2.14), Lemma 4.9, and the definition of M^{w_J} we find that

$$G^w = \mathrm{o}_{\mathcal{Z}}(g\dot{w}H_w) = \mathrm{o}_{\mathcal{Z}}(g\dot{x}\dot{w}_J H_w) = \mathrm{o}_{\mathcal{Z}}(g\dot{x}\,\mathrm{o}_{\mathcal{Z}_{\dot{x}}}(\dot{w}_J H_w)) = \mathrm{o}_{\mathcal{Z}}(g\dot{x}M^{w_J}).$$

The other assertions follow from Proposition 4.7. □

5.3 Main properties of G^w

Theorem 5.10. *The G^w for all $w \in {}^IW$ form a disjoint decomposition of G.*

Proof. We show this by induction on $\dim G$. In the base case $M = G$ we have ${}^IW = \{1\}$ and $H_1 = G = G^1$ by Example 5.3; hence the theorem is trivially true. Otherwise take an element $x \in {}^IW^J$. By the induction hypothesis applied to the zip datum $E_{\mathcal{Z}_{\dot{x}}}$ the subsets M^{w_J} for $w_J \in {}^{I_x}W_J$ form a disjoint decomposition of M. Thus by Proposition 4.7 and Lemma 5.9, the subsets G^{xw_J} for $w_J \in {}^{I_x}W_J$ form a disjoint decomposition of $Pg\dot{x}Q$. Combining this with the Bruhat decomposition (4.1) it follows that the subsets G^{xw_J} for all x and w_J form a disjoint decomposition of G. But by Proposition 2.8 these are precisely the subsets G^w for $w \in {}^IW$, as desired. □

Theorem 5.11. *For any $w \in {}^IW$ the subset G^w is a nonsingular subvariety of G of dimension $\dim P + \ell(w)$.*

Proof. Again we proceed by induction on $\dim G$. If $M = G$, there is only one piece $G^1 = G = P$ associated to $w = 1$, and the assertion is clear. Otherwise write $w = xw_J$ as in Proposition 2.8. By the induction hypothesis the subset M^{w_J} is a nonsingular subvariety of M of dimension $\dim P_x + \ell(w_J)$. Thus by Propositions 4.7 and 4.8 and Lemma 5.9 the subset G^w is a nonsingular subvariety of dimension

$$\big[\dim P_x + \ell(w_J)\big] + \dim P - \dim P_x + \ell(x) \;=\; \dim P + \ell(x) + \ell(w_J).$$

By Proposition 2.7 the last expression is equal to $\dim P + \ell(w)$, as desired. $\square$

Theorem 5.12. *For any $w \in {}^IW$, there is a bijection between the H_w-orbits $Y \subset H_w$ and the $E_{\mathcal{Z}}$-orbits $X \subset G^w$, defined by $X = \mathrm{o}_{\mathcal{Z}}(g\dot{w}Y)$ and satisfying*

$$\operatorname{codim}(X \subset G^w) = \operatorname{codim}(Y \subset H_w).$$

Proof. If $M = G$, we have $w = 1$ and $G = G^1 = H_1$, and $E_{\mathcal{Z}} \cong G$ acts on itself by the twisted conjugation $(h, h') \mapsto h \cdot h' \cdot \varphi(h)^{-1}$. Thus the $E_{\mathcal{Z}}$-orbits $X \subset G$ are precisely the cosets gY for H_1-orbits Y according to Definition 5.1, which finishes that case.

If $M \neq G$ write $w = xw_J$ as in Proposition 2.8. Then $\mathcal{Z}_{\dot{x}}$ has the frame $(M \cap B, T, 1)$ by Proposition 4.12, and so by Lemma 5.4 and the induction hypothesis there is a bijection between the H_w-orbits $Y \subset H_w$ and the $E_{\mathcal{Z}_{\dot{x}}}$-orbits $X' \subset M^{w_J}$, defined by $X' = \mathrm{o}_{\mathcal{Z}_{\dot{x}}}(\dot{w}_J Y)$ and satisfying $\operatorname{codim}(X' \subset M^{w_J}) = \operatorname{codim}(Y \subset H_w)$. By Proposition 4.7 and Lemma 5.9 there is a bijection between these X' and the $E_{\mathcal{Z}}$-orbits $X \subset G^w$, defined by $X = \mathrm{o}_{\mathcal{Z}}(g\dot{x}X')$. Moreover, since pushout and flat pullback preserve codimensions, the last statement in Proposition 4.7 implies that

$$\operatorname{codim}(X \subset G^w) = \operatorname{codim}\big(\mathrm{o}_{\mathcal{Z}}(g\dot{x}X') \subset \mathrm{o}_{\mathcal{Z}}(g\dot{x}M^{w_J})\big) = \operatorname{codim}(X' \subset M^{w_J}).$$

Finally, since $\dot{w} = \dot{x}\dot{w}_J$ by (2.14), Lemma 4.9 shows that $X = \mathrm{o}_{\mathcal{Z}}(g\dot{x}\,\mathrm{o}_{\mathcal{Z}_{\dot{x}}}(\dot{w}_J Y)) = \mathrm{o}_{\mathcal{Z}}(g\dot{x}\dot{w}_J Y) = \mathrm{o}_{\mathcal{Z}}(g\dot{w}Y)$, finishing the induction step. $\square$

5.4 Other descriptions of G^w

Lemma 5.13. *For any element $g' \in G$ we have*

$$\mathrm{o}_{\mathcal{Z}}(gBg'B) \;=\; \mathrm{o}_{\mathcal{Z}}(gg'B) \;=\; \mathrm{o}_{\mathcal{Z}}(gBg').$$

Proof. Take any element $b \in B$. Then the condition 3.6 (b) implies that $p := gbg^{-1} \in P$, and so there exists $q \in Q$ such that $(p, q) \in E_{\mathcal{Z}}$. By the condition 3.6 (c) we then have $q \in B$. It follows that $gbg'B = pgg'Bq^{-1} \subset \mathrm{o}_{\mathcal{Z}}(gg'B)$. Since b was arbitrary, this shows that $gBg'B \subset \mathrm{o}_{\mathcal{Z}}(gg'B)$, whence the first equality. A similar argument proves the second equality. $\square$

Theorem 5.14. *For any $w \in {}^IW$ we have*

$$G^w = \mathrm{o}_{\mathcal{Z}}(g\dot{w}H_w) = \mathrm{o}_{\mathcal{Z}}(g\dot{w}(H_w \cap B)) = \mathrm{o}_{\mathcal{Z}}(g\dot{w}B) = \mathrm{o}_{\mathcal{Z}}(gB\dot{w}) = \mathrm{o}_{\mathcal{Z}}(gB\dot{w}B).$$

Proof. The first equation is Definition 5.7 of G^w, and the last two equations are cases of Lemma 5.13. The remaining two equations are proved by induction on $\dim G$. In the base case $M = G$ we have $w = 1$ and $H_1 = G$; hence the second term is $\mathrm{o}_{\mathcal{Z}}(gG) = G$, and the third and fourth terms are both equal to $\mathrm{o}_{\mathcal{Z}}(gB)$. By Proposition 2.5 applied to the isogeny $\varphi \circ \mathrm{int}(g)$ the latter is equal to G, as desired.
In the case $M \neq G$ write $w = xw_J$ as in Proposition 2.8. Then $\mathcal{Z}_{\dot{x}}$ has the frame $(M \cap B, T, 1)$ by Proposition 4.12, and so by Lemma 5.4 and the induction hypothesis we have

$$\mathrm{o}_{\mathcal{Z}_{\dot{x}}}(\dot{w}_J H_w) = \mathrm{o}_{\mathcal{Z}_{\dot{x}}}(\dot{w}_J(H_w \cap B)) = \mathrm{o}_{\mathcal{Z}_{\dot{x}}}(\dot{w}_J(M \cap B)).$$

Using Lemma 4.9 this implies that

$$\mathrm{o}_{\mathcal{Z}}(g\dot{x}\dot{w}_J H_w) = \mathrm{o}_{\mathcal{Z}}(g\dot{x}\dot{w}_J(H_w \cap B)) = \mathrm{o}_{\mathcal{Z}}(g\dot{x}\dot{w}_J(M \cap B)).$$

By (2.14) we may replace $\dot{x}\dot{w}_J$ by $\dot{w}$ in these equations. Moreover, (3.3) and (3.9) show that $g\dot{w}B = g\dot{w}(M \cap B)V \subset \mathrm{o}_{\mathcal{Z}}(g\dot{w}(M \cap B))$ and so $\mathrm{o}_{\mathcal{Z}}(g\dot{w}B) = \mathrm{o}_{\mathcal{Z}}(g\dot{w}(M \cap B))$. Thus both equations follow. ☐

Example 5.15. If P is a Borel subgroup, then so is Q, and we have ${}^I W = W$. The last equation in Theorem 5.14 then implies that $G^w = gB\dot{w}B$ for all $w \in W$.

For a further equivalent description of G^w see Subsection 11.1.

6 Closure relation

In this section, we determine the closure of G^w in G for any $w \in {}^I W$. To formulate a precise result recall that $\leq$ denotes the Bruhat order on W.

Definition 6.1. For w, $w' \in {}^I W$ we write $w' \preccurlyeq w$ if and only if there exists $y \in W_I$ such that $yw'\psi(y)^{-1} \leq w$.

Theorem 6.2. *For any $w \in {}^I W$ we have*

$$\overline{G^w} = \coprod_{\substack{w' \in {}^I W \\ w' \preccurlyeq w}} G^{w'}.$$

A direct consequence of this is:

Corollary 6.3. *The relation $\preccurlyeq$ is a partial order on ${}^I W$.*

Remark 6.4. The relation $\preccurlyeq$ has been introduced by He in [He2] for a somewhat more special class of isomorphisms $\psi\colon W_I \xrightarrow{\sim} W_J$. He gives a direct combinatorial proof that $\preccurlyeq$ is a partial order (Proposition 3.13 of loc. cit.), which can be adapted to our more general setting (see [Wed], Section 4).

The rest of this section is devoted to proving Theorem 6.2. We will exploit the fact that the closure relation for the Bruhat decomposition of G is known. Namely, for any $w \in W$ we have by [Spr1], Proposition 8.5.5:

$$\overline{B\dot{w}B} = \coprod_{\substack{w' \in W \\ w' \leq w}} B\dot{w}'B. \tag{6.5}$$

Lemma 6.6. *For any $w \in W$ we have*

$$\overline{o_{\mathcal{Z}}(gB\dot{w}B)} = \bigcup_{\substack{w' \in W \\ w' \leq w}} o_{\mathcal{Z}}(g\dot{w}'B).$$

Proof. Let $B_{\mathcal{Z}} \subset E_{\mathcal{Z}}$ denote the subgroup of all elements $(u\ell, v\varphi(\ell))$ with $u \in U$, $v \in V$, and $\ell \in L \cap {}^gB$. Then $E_{\mathcal{Z}}/B_{\mathcal{Z}} \cong L/(L \cap {}^gB)$ is proper, and $gB\dot{w}B \subset G$ is a $B_{\mathcal{Z}}$-invariant subvariety. Thus Lemma 2.3 and (6.5) imply that

$$\overline{o_{\mathcal{Z}}(gB\dot{w}B)} \;=\; o_{\mathcal{Z}}\big(\overline{gB\dot{w}B}\big) \;=\; \bigcup_{w' \leq w} o_{\mathcal{Z}}(gB\dot{w}'B).$$

The desired equality then follows from Lemma 5.13. ☐

Lemma 6.7. *For any $w,\, v \in W$ and $b \in B$ there exists $u \in W$ such that $u \leq v$ and $\dot{w}b\dot{v} \in B\dot{w}\dot{u}B$.*

Proof. We prove the statement by induction on $\ell(v)$. If $v = 1$, we may take $u = 1$. For the induction step write $v = v's$ for some simple reflection s such that $\ell(v') = \ell(v) - 1$. By the induction hypothesis there exists $u' \leq v'$ such that $\dot{w}b\dot{v}' \in B\dot{w}\dot{u}'B$. Hence $\dot{w}b\dot{v} \in B\dot{w}\dot{u}'B\dot{s} \subset B\dot{w}\dot{u}'\dot{s}B \cup B\dot{w}\dot{u}'B$, so either $u = u's$ or $u = u'$ will have the required property. ☐

Lemma 6.8. *For any $z \in W$ and $w \in {}^IW$ and $v \in W_I$ such that $z \leq w\psi(v)$, there exists $y \in W_I$ such that $yz\psi(y)^{-1} \leq vw$.*

Proof. Choose reduced expressions for w and v as products of simple reflections. Since $\psi(I) = J$, this also yields a reduced expression for $\psi(v)$. Together this yields an expression for $w\psi(v)$ as a product of simple reflections, which is not necessarily reduced. However, by [BB], Theorem 2.2.2 a reduced expression for $w\psi(v)$ can be obtained from the given one by possibly deleting some factors. By the definition of the Bruhat order, the assumption $z \leq w\psi(v)$ means that a reduced expression for z is obtained from this by deleting further factors, if any. Let y' denote the product of all factors remaining from w. Since all factors in the reduced expression for v lie in I, the product of all factors remaining from $\psi(v)$ is equal to $\psi(y)$ for some $y \in W_I$. By construction we then have $z = y'\psi(y)$, and so $yz\psi(y)^{-1} = yy'$. But the assumptions on w and v imply that $\ell(vw) = \ell(v)+\ell(w)$; hence the product of the given reduced expressions for v and w is a reduced expression for vw. By construction yy' is obtained from that product by possibly deleting some factors, so we deduce that $yy' \leq vw$, as desired. ☐

Lemma 6.9. *For any $w \in {}^IW$ and $w' \in W$ and $b, b' \in B$ such that $\mathrm{o}_{\mathcal{Z}}(g\dot{w}b) = \mathrm{o}_{\mathcal{Z}}(g\dot{w}'b')$ there exists $y \in W_I$ such that $yw\psi(y)^{-1} \leq w'$.*

Proof. We proceed by induction on $\dim G$. In the base case $M = G$ we have $w = 1$ and may take $y = 1$. So assume that $M \neq G$. Write $w = xw_J$ as in Proposition 2.8 with $x \in {}^IW^J$ and $w_J \in {}^{I_x}W_J$. From $\mathrm{o}_{\mathcal{Z}}(g\dot{w}b) = \mathrm{o}_{\mathcal{Z}}(g\dot{w}'b')$ we deduce that $Pg\dot{x}Q = Pg\dot{w}Q = Pg\dot{w}'Q$, which in view of (4.1) implies that $w' \in W_IxW_J$. Write $w' = v'xw'_J$ for $v' \in W_I$ and $w'_J \in {}^{I_x}W_J$, as in Proposition 2.7.

Recall that $\varphi(g\dot{v}'g^{-1}) \in \mathrm{Norm}_M(T)$ is a representative of $\psi(v') \in W_J$. Thus by Lemma 6.7, there exists $u \in W$ such that $u \leq \psi(v')$ and $\dot{x}\dot{w}'_Jb'\varphi(g\dot{v}'g^{-1}) \in B\dot{x}\dot{w}'_J\dot{u}B$. The first condition implies that $u \in W_J$, the Weyl group of M. The action of $E_{\mathcal{Z}}$ and the second condition imply

$$\mathrm{o}_{\mathcal{Z}}(g\dot{w}'b') \;=\; \mathrm{o}_{\mathcal{Z}}(g\dot{v}'\dot{x}\dot{w}'_Jb') \;=\; \mathrm{o}_{\mathcal{Z}}\big(g\dot{x}\dot{w}'_Jb'\varphi(g\dot{v}'g^{-1})\big) \;\subset\; \mathrm{o}_{\mathcal{Z}}(gB\dot{x}\dot{w}'_J\dot{u}B).$$

Here the last term is equal to $\mathrm{o}_{\mathcal{Z}}(g\dot{x}\dot{w}'_J\dot{u}B)$ by Lemma 5.13. Thus there exists $b'' \in B$ such that

$$\mathrm{o}_{\mathcal{Z}}(g\dot{x}\dot{w}_Jb) \;=\; \mathrm{o}_{\mathcal{Z}}(g\dot{w}b) \;=\; \mathrm{o}_{\mathcal{Z}}(g\dot{w}'b') \;=\; \mathrm{o}_{\mathcal{Z}}(g\dot{x}\dot{w}'_J\dot{u}b'').$$

By the action of $E_{\mathcal{Z}}$ we may and do assume that $b, b'' \in M \cap B$. Then $\dot{w}_Jb$ and $\dot{w}'_J\dot{u}b''$ lie in M, and so Proposition 4.7 implies that $\mathrm{o}_{\mathcal{Z}_{\dot{x}}}(\dot{w}_Jb) = \mathrm{o}_{\mathcal{Z}_{\dot{x}}}(\dot{w}'_J\dot{u}b'')$. By the induction hypothesis there therefore exists $y_x \in W_{I_x}$ such that

$$y_xw_J\psi_x(y_x)^{-1} \leq w'_Ju.$$

Now we work our way back up. Since both sides of the last relation lie in W_J, and since $x \in W^J$, we deduce that

$$z := xy_xw_J\psi_x(y_x)^{-1} \leq xw'_Ju.$$

Recall that $u \leq \psi(v')$, which implies that $u = \psi(u')$ for some $u' \in W_I$ satisfying $u' \leq v'$. Also, note that $xw'_J \in {}^IW$ by Proposition 2.8. Thus by Lemma 6.8 there exists $y' \in W_I$ such that

$$y'z\psi(y')^{-1} \leq u'xw'_J.$$

As u' and v' lie in W_I, and $xw'_J \in {}^IW$, we deduce that

$$y'z\psi(y')^{-1} \leq u'xw'_J \leq v'xw'_J = w'.$$

Finally, since $\psi_x = \psi \circ \mathrm{int}(x)$, we have

$$\begin{aligned} y'z\psi(y')^{-1} = y'xy_xw_J\psi(xy_xx^{-1})^{-1}\psi(y')^{-1} = \\ = (y'xy_xx^{-1})xw_J\psi(y'xy_xx^{-1})^{-1} = yw\psi(y)^{-1} \end{aligned}$$

with $y := y'xy_xx^{-1} \in W_I$. Thus $yw\psi(y)^{-1} \leq w'$, as desired. ∎

Lemma 6.10. *For any $w \in {}^I W$, the set $\mathrm{o}_{\mathcal{Z}}(g\dot{w}T)$ is dense in G^w.*

Proof. Theorem 5.12 implies that $\mathrm{o}_{\mathcal{Z}}(g\dot{w}T) = \mathrm{o}_{\mathcal{Z}}(g\dot{w}Y)$, where $Y \subset H_w$ is the orbit of T under twisted conjugation by H_w. But Proposition 2.5 (b) asserts that Y is dense in H_w. Thus $\mathrm{o}_{\mathcal{Z}}(g\dot{w}Y)$ is dense in $\mathrm{o}_{\mathcal{Z}}(\overline{g\dot{w}Y}) = \mathrm{o}_{\mathcal{Z}}(g\dot{w}H_w) = G^w$, as desired. $\square$

Proof of Theorem 6.2. Consider $w' \in {}^I W$ such that $G^{w'} \cap \overline{G^w} \neq \varnothing$. Then by Theorem 5.14 and Lemma 6.6 there exist $b, b' \in B$ and $w'' \in W$ such that $w'' \leq w$ and $\mathrm{o}_{\mathcal{Z}}(gw'b) = \mathrm{o}_{\mathcal{Z}}(gw''b')$. Lemma 6.9 then implies that $yw'\psi(y)^{-1} \leq w''$ for some $y \in W_I$. Together it follows that $yw'\psi(y)^{-1} \leq w$, and hence $w' \preccurlyeq w$, proving "$\subset$".

To prove "$\supset$" consider $w' \in {}^I W$ with $w' \preccurlyeq w$. By definition there exists $y \in W_I$ such that $w'' := yw'\psi(y)^{-1} \leq w$. Lemma 6.6 and Theorem 5.14 then show that $\mathrm{o}_{\mathcal{Z}}(g\dot{w}''T) \subset \overline{G^w}$. Therefore

$$\begin{aligned}\mathrm{o}_{\mathcal{Z}}(g\dot{w}'T) \;=\; \mathrm{o}_{\mathcal{Z}}\big(g\dot{y}\dot{w}'T\varphi(g\dot{y}g^{-1})^{-1}\big) \;=&\\ &= \mathrm{o}_{\mathcal{Z}}\big(g\dot{y}\dot{w}'\varphi(g\dot{y}g^{-1})^{-1}T\big) \;=\; \mathrm{o}_{\mathcal{Z}}(g\dot{w}''T) \;\subset\; \overline{G^w}.\end{aligned}$$

With Lemma 6.10 for $\mathrm{o}_{\mathcal{Z}}(g\dot{w}'T)$ we conclude that $G^{w'} \subset \overline{G^w}$, as desired. $\square$

7 Orbitally finite zip data

Proposition 7.1. *The following assertions are equivalent:*

(a) For any $w \in {}^I W$, the number of fixed points of the endomorphism $\varphi_{\dot{w}} = \varphi \circ \mathrm{int}(g\dot{w})$ of H_w from Definition 5.1 is finite.

(b) For any $w \in {}^I W$ the $E_{\mathcal{Z}}$-invariant subvariety G^w is a single orbit under $E_{\mathcal{Z}}$.

(c) The number of orbits of $E_{\mathcal{Z}}$ on G is finite.

Proof. If (a) holds, the Lang-Steinberg Theorem 2.4 shows that the orbit of $1 \in H_w$ under twisted conjugation is all of H_w, and by Theorem 5.12 this implies (b). The implication (b)$\Rightarrow$(c) is trivial. So assume (c). Then again by Theorem 5.12, the number of orbits in H_w under twisted conjugation by $\varphi_{\dot{w}}$ is finite for any $w \in {}^I W$. In particular there exists an open orbit; let h be an element thereof. Then for dimension reasons its stabilizer is finite. But

$$\mathrm{Stab}_{H_w}(h) \;=\; \big\{h' \in H_w \;\big|\; h'h\varphi_{\dot{w}}(h')^{-1} = h\big\} \;=\; \big\{h' \in H_w \;\big|\; h' = h\varphi_{\dot{w}}(h')h^{-1}\big\}$$

is also the set of fixed points of the endomorphism $\mathrm{int}(h) \circ \varphi_{\dot{w}}$ of H_w. Thus the Lang-Steinberg Theorem 2.4 implies that $\{h'h\varphi_{\dot{w}}(h')^{-1}h^{-1} \mid h' \in H_w\} = H_w$. After right multiplication by h this shows that the orbit of h is all of H_w. We may thus repeat the argument with the identity element in place of h, and deduce that the set of fixed points of $\varphi_{\dot{w}}$ on H_w is finite, proving (a). $\square$

Definition 7.2. We call $\mathcal{Z}$ *orbitally finite* if the conditions in Proposition 7.1 are met.

Proposition 7.3. *If the differential of φ at 1 vanishes, then $\mathcal{Z}$ is orbitally finite.*

Proof. If the differential of φ vanishes, then so does the differential of $\varphi_{\dot{w}} = \varphi \circ \mathrm{int}(g\dot{w})|H_w$ for any $w \in {}^IW$. Let H_w^f denote the fixed point locus of $\varphi_{\dot{w}}$, which is a closed algebraic subgroup. Then the restriction $\varphi_{\dot{w}}|H_w^f$ is the identity and its differential is zero. This is possible only when $\dim H_w^f = 0$, that is, when H_w^f is finite. $\square$

Remark 7.4. In particular Proposition 7.3 applies when the base field has characteristic $p > 0$ and the isogeny φ is a relative Frobenius $L \to L^{(p^r)} \cong M$.

Since $g\dot{w} \in G^w$ by Definition 5.7, we can now rephrase condition 7.1 (b) and Theorems 5.10, 5.11, and 6.2 as follows:

Theorem 7.5. *Assume that $\mathcal{Z}$ is orbitally finite. Then:*

(a) For any $w \in {}^IW$ we have $G^w = \mathrm{o}_{\mathcal{Z}}(g\dot{w})$.

(b) The elements $g\dot{w}$ for $w \in {}^IW$ form a set of representatives for the $E_{\mathcal{Z}}$-orbits in G.

(c) For any $w \in {}^IW$ the orbit $\mathrm{o}_{\mathcal{Z}}(g\dot{w})$ has dimension $\dim P + \ell(w)$.

(d) For any $w \in {}^IW$ the closure of $\mathrm{o}_{\mathcal{Z}}(g\dot{w})$ is the union of $\mathrm{o}_{\mathcal{Z}}(g\dot{w}')$ for all $w' \in {}^IW$ with $w' \preccurlyeq w$.

8 Point stabilizers

In this section we study the stabilizer in $E_{\mathcal{Z}}$ of an arbitrary element $g' \in G$. Take $w \in {}^IW$ such that $g' \in G^w$. Then Theorem 5.12 shows that g' is conjugate to $g\dot{w}h$ for some $h \in H_w$. Thus it suffices to consider the stabilizer of $g\dot{w}h$.
Recall from Definition 5.1 that H_w acts on itself by twisted conjugation with the isogeny $\varphi_{\dot{w}}$, which is defined as the restriction of $\varphi \circ \mathrm{int}(g\dot{w})$.

Theorem 8.1. *For any $w \in {}^IW$ and $h \in H_w$ the stabilizer $\mathrm{Stab}_{E_{\mathcal{Z}}}(g\dot{w}h)$ is the semi-direct product of a connected unipotent normal subgroup with the subgroup*

$$\big\{\big(\mathrm{int}(g\dot{w})(h'), \varphi(\mathrm{int}(g\dot{w})(h'))\big) \;\big|\; h' \in \mathrm{Stab}_{H_w}(h)\big\}. \tag{8.2}$$

Proof. For any $h' \in H_w$ we have $\mathrm{int}(g\dot{w})(h') \cdot g\dot{w}h \cdot \varphi(\mathrm{int}(g\dot{w})(h'))^{-1} = g\dot{w}h$ if and only if $h'h\varphi(\mathrm{int}(g\dot{w})(h'))^{-1} = h$ if and only if $h' \in \mathrm{Stab}_{H_w}(h)$. This implies that (8.2) is a subgroup of $\mathrm{Stab}_{E_{\mathcal{Z}}}(g\dot{w}h)$.
For the rest we proceed by induction on $\dim G$. If $M = G$, we have $w = 1$ and $\dot{w} = 1$ and $G = H_1$, and $(g', \varphi(g')) \in E_{\mathcal{Z}}$ acts on G by the twisted conjugation $g'' \mapsto g'g''\varphi(g')^{-1}$. Under left translation by g this corresponds to the action

of H_w on itself, so that $\mathrm{Stab}_{E_\mathcal{Z}}(g\dot{w}h)$ is precisely the subgroup (8.2) and the normal subgroup is trivial.
If $M \neq G$ write $w = xw_J$ as in Proposition 2.8. Then $\dot{w} = \dot{x}\dot{w}_J$ and $\mathcal{Z}_{\dot{x}}$ has the frame $(M \cap B, T, 1)$, and Proposition 4.11 shows that $\mathrm{Stab}_{E_\mathcal{Z}}(g\dot{w}h)$ is an extension of $\mathrm{Stab}_{E_{\mathcal{Z}_{\dot{x}}}}(\dot{w}_J h)$ by a connected unipotent normal subgroup. Moreover, by the induction hypothesis $\mathrm{Stab}_{E_{\mathcal{Z}_{\dot{x}}}}(\dot{w}_J h)$ is the semi-direct product of a connected unipotent normal subgroup with the subgroup

$$\big\{\big(\mathrm{int}(\dot{w}_J)(h'), \varphi_{\dot{x}}(\mathrm{int}(\dot{w}_J)(h'))\big) \bigm| h' \in \mathrm{Stab}_{H_w}(h)\big\}. \tag{8.3}$$

Furthermore a direct calculation shows that the projection in Proposition 4.11 sends the subgroup (8.2) isomorphically to the subgroup (8.3). Since any extension of connected unipotent groups is again connected unipotent, the theorem follows. □

Remark 8.4. For the stabilizer, a similar result was obtained by Evens and Lu ([EL] Theorem 3.13).

If the differential of φ at 1 vanishes, we can also describe the infinitesimal stabilizer in the Lie algebra. Since in that case the zip datum is orbitally finite by Proposition 7.3, it suffices to consider the stabilizer of $g\dot{w}$.

Theorem 8.5. *Assume that the differential of φ at 1 vanishes. For any $w \in {}^I W$ let $w = xw_J$ be the decomposition from Proposition 2.8. Then the infinitesimal stabilizer of $g\dot{w}$ in the Lie algebra of $E_\mathcal{Z}$ has dimension $\dim V - \ell(x)$.*

Proof. Since $d\varphi = 0$, we have $\mathrm{Lie}\, E_\mathcal{Z} = \mathrm{Lie}\, P \times \mathrm{Lie}\, V \subset \mathrm{Lie}(P \times Q)$. Thus an arbitrary tangent vector of $E_\mathcal{Z}$ at 1 has the form $(1{+}dp, 1{+}dv)$ for $dp \in \mathrm{Lie}\, P$ and $dv \in \mathrm{Lie}\, V$, viewed as infinitesimal elements of P and V in Leibniz's sense. That element stabilizes $g\dot{w}$ if and only if $(1{+}dp)g\dot{w}(1{+}dv)^{-1} = g\dot{w}$. This condition is equivalent to $dp \cdot g\dot{w} - g\dot{w} \cdot dv = 0$, or again to $dp = \mathrm{Ad}_{g\dot{w}}(dv)$. The dimension is therefore $\dim(\mathrm{Lie}\, P \cap \mathrm{Ad}_{g\dot{w}}(\mathrm{Lie}\, V)) = \dim(\mathrm{Lie}\, {}^{g^{-1}}P \cap \mathrm{Lie}\, {}^{\dot{w}}V)$. As both ${}^{g^{-1}}P$ and ${}^{\dot{w}}V$ are normalized by T, the dimension is just the number of root spaces in the last intersection. This number is

$$\begin{aligned} \#\big[(\Phi^+ \cup \Phi_I) \cap w(\Phi^+ \smallsetminus \Phi_J)\big] &= \#\big\{\alpha \in \Phi^+ \smallsetminus \Phi_J \bigm| w\alpha \in \Phi^+ \cup \Phi_I\big\} \\ &= \dim V - \#\big\{\alpha \in \Phi^+ \smallsetminus \Phi_J \bigm| w\alpha \in \Phi^- \smallsetminus \Phi_I\big\}. \end{aligned}$$

By Lemma 2.13 it is therefore $\dim V - \ell(x)$, as desired. □

Remark 8.6. The dimension in Theorem 8.5 depends only on the first factor of $w = xw_J$ and thus only on the Bruhat cell $Pg\dot{w}Q$. Since that Bruhat cell is an irreducible variety and in general composed of more than one $E_\mathcal{Z}$-orbit, these orbits have different dimensions. Thus the corresponding point stabilizers in $E_\mathcal{Z}$ have different dimension, while the dimension of their Lie algebra stabilizer is constant. Therefore the scheme-theoretic stabilizer of $g\dot{w}$ is in general not reduced.

9 Abstract zip data

By Theorem 5.10 the subsets G^w for all $w \in {}^IW$ form a disjoint decomposition of G satisfying $g\dot{w} \in G^w$. It is natural to ask which other elements of the form $g\dot{w}'$ for $w' \in W$ are contained in a given G^w. When $\mathcal{Z}$ is orbitally finite, by Theorem 7.5 this question is equivalent to asking which elements $g\dot{w}'$ for $w' \in W$ lie in the same $E_{\mathcal{Z}}$-orbit. This problem turns out to depend only on the groups $W_I \subset W$ and the homomorphism ψ and can therefore be studied abstractly. We return to this situation at the end of this section.

9.1 Abstract groups

Definition 9.1. An *abstract zip datum* is a tuple $\mathcal{A} = (\Gamma, \Delta, \psi)$ consisting of a group Γ, a subgroup Δ, and a homomorphism $\psi\colon \Delta \to \Gamma$.

Fix such an abstract zip datum $\mathcal{A}$. For any $\gamma \in \Gamma$, the collection of subgroups E of ${}^{\gamma^{-1}}\Delta$ satisfying $\psi \circ \mathrm{int}(\gamma)(E) = E$ possesses a unique largest element, namely the subgroup generated by all such subgroups.

Definition 9.2. For any $\gamma \in \Gamma$ we let E_γ denote the unique largest subgroup of ${}^{\gamma^{-1}}\Delta$ satisfying $\psi \circ \mathrm{int}(\gamma)(E_\gamma) = E_\gamma$.

Lemma 9.3. *For any $\gamma \in \Gamma$ and $\delta \in \Delta$ and $\varepsilon \in E_\gamma$, we have $E_{\delta\gamma\varepsilon\psi(\delta)^{-1}} = {}^{\psi(\delta)}E_\gamma$.*

Proof. Abbreviate $\gamma' := \delta\gamma\varepsilon\psi(\delta)^{-1}$. Then the calculation $\psi\big({}^{\gamma'}({}^{\psi(\delta)}E_\gamma)\big) = \psi({}^{\delta\gamma\varepsilon}E_\gamma) = {}^{\psi(\delta)}\psi({}^{\gamma}E_\gamma) = {}^{\psi(\delta)}E_\gamma$ and the definition of $E_{\gamma'}$ imply that ${}^{\psi(\delta)}E_\gamma \subset E_{\gamma'}$. In particular, $\varepsilon' := {}^{\psi(\delta)}\varepsilon^{-1}$ is an element of $E_{\gamma'}$. Since $\gamma = \delta'\gamma'\varepsilon'\psi(\delta')^{-1}$ with $\delta' := \delta^{-1} \in \Delta$, a calculation like the first shows that ${}^{\psi(\delta')}E_{\gamma'} \subset E_\gamma$. Together it follows that $E_{\gamma'} = {}^{\psi(\delta)}E_\gamma$, as desired. □

Definition 9.4. For any γ, $\gamma' \in \Gamma$ we write $\gamma' \sim \gamma$ if and only if there exist $\delta \in \Delta$ and $\varepsilon \in E_\gamma$ such that $\gamma' = \delta\gamma\varepsilon\psi(\delta)^{-1}$. For any $\gamma \in \Gamma$ we abbreviate $o_{\mathcal{A}}(\gamma) := \{\gamma' \in \Gamma \mid \gamma' \sim \gamma\}$.

Lemma 9.5. *This is an equivalence relation.*

Proof. Reflexivity is clear, and symmetry was shown already in the proof of Lemma 9.3. To prove transitivity, suppose that $\gamma' = \delta\gamma\varepsilon\psi(\delta)^{-1}$ for $\delta \in \Delta$ and $\varepsilon \in E_\gamma$ and $\gamma'' = \delta'\gamma'\varepsilon'\psi(\delta')^{-1}$ for $\delta' \in \Delta$ and $\varepsilon' \in E_{\gamma'}$. Then ${}^{\psi(\delta)^{-1}}\varepsilon' \in E_\gamma$ by Lemma 9.3, and so $\gamma'' = \delta'\delta\gamma\varepsilon\psi(\delta)^{-1}\varepsilon'\psi(\delta')^{-1} = \delta''\gamma\varepsilon''\psi(\delta'')^{-1}$ for $\delta'' := \delta'\delta \in \Delta$ and $\varepsilon'' := \varepsilon\, {}^{\psi(\delta)^{-1}}\varepsilon' \in E_\gamma$, as desired. □

Theorem 9.6. *If Δ is finite, each equivalence class in Γ has cardinality $\#\Delta$ and the number of equivalence classes is $[\Gamma : \Delta]$.*

Proof. Take any $\gamma \in \Gamma$; then the group $E_\gamma \subset {}^{\gamma^{-1}}\Delta$ is finite, too. Consider the surjective map $\Delta \times E_\gamma \twoheadrightarrow \mathrm{o}_{\mathcal{A}}(\gamma)$, $(\delta, \varepsilon) \mapsto \delta\gamma\varepsilon\psi(\delta)^{-1}$. Two elements (δ, ε), $(\delta', \varepsilon') \in \Delta \times E_\gamma$ lie in the same fiber if and only if $\delta\gamma\varepsilon\psi(\delta)^{-1} = \delta'\gamma\varepsilon'\psi(\delta')^{-1}$ if and only if $\varepsilon\psi(\delta^{-1}\delta') = \gamma^{-1}(\delta^{-1}\delta')\gamma\varepsilon'$. With $\varepsilon'' := \gamma^{-1}(\delta^{-1}\delta')\gamma \in {}^{\gamma^{-1}}\Delta$ this is equivalent to $\varepsilon\psi({}^{\gamma}\varepsilon'') = \varepsilon''\varepsilon'$. Since ε, $\varepsilon' \in E_\gamma$, this equation implies that the subgroup generated by E_γ and ε'' is mapped onto itself under $\psi \circ \mathrm{int}(\gamma)$. By maximality it is therefore equal to E_γ, and so $\varepsilon'' \in E_\gamma$. Together we find that the elements in the same fiber as (δ, ε) are precisely the elements (δ', ε') with $\delta' = \delta\,{}^{\gamma}\varepsilon''$ and $\varepsilon' = (\varepsilon'')^{-1}\varepsilon\psi({}^{\gamma}\varepsilon'')$ for some $\varepsilon'' \in E_\gamma$. Thus each fiber has cardinality $\#E_\gamma$, and so the image has cardinality $\#\Delta$, proving the first assertion. The second assertion is a direct consequence of the first. $\square$

We can also perform an induction step as in Section 4 for abstract zip data, obtaining analogues of Lemma 5.4 and Proposition 4.7. For this fix an element $\xi \in \Gamma$, say in a set of representatives for the double quotient $\Delta\backslash\Gamma/\psi(\Delta)$. Then Definitions 9.2 and 9.4 imply that the equivalence class of any $\gamma \in \Delta\xi\psi(\Delta)$ is again contained in $\Delta\xi\psi(\Delta)$.

Construction 9.7. Set $\Gamma_\xi := \psi(\Delta)$ and $\Delta_\xi := \psi(\Delta) \cap {}^{\xi^{-1}}\Delta$, and let $\psi_\xi \colon \Delta_\xi \to \Gamma_\xi$ denote the restriction of $\psi \circ \mathrm{int}(\xi)$. This defines a new, possibly smaller, abstract zip datum

$$\mathcal{A}_\xi := (\Gamma_\xi, \Delta_\xi, \psi_\xi).$$

Lemma 9.8. *For any $\gamma \in \Gamma_\xi$, the group $E_{\xi\gamma}$ associated by Definition 9.2 to the pair $(\mathcal{A}, \xi\gamma)$ is equal to the group associated to the pair $(\mathcal{A}_\xi, \gamma)$.*

Proof. Since $\gamma \in \Gamma_\xi = \psi(\Delta)$, Definition 9.2 implies that

$$E_{\xi\gamma} \;\subset\; \psi(\Delta) \cap {}^{\gamma^{-1}\xi^{-1}}\Delta \;=\; {}^{\gamma^{-1}}\big(\psi(\Delta) \cap {}^{\xi^{-1}}\Delta\big) \;=\; {}^{\gamma^{-1}}\Delta_\xi$$

and that $E_{\xi\gamma} = \psi \circ \mathrm{int}(\xi\gamma)(E_{\xi\gamma}) = \psi_\xi \circ \mathrm{int}(\gamma)(E_{\xi\gamma})$. Since $E_{\xi\gamma}$ is the largest subgroup of ${}^{\gamma^{-1}\xi^{-1}}\Delta$ with this property, it is also the largest in ${}^{\gamma^{-1}}\Delta_\xi$. $\square$

Proposition 9.9. *There is a bijection between $\mathcal{A}_\xi$-equivalence classes in Γ_ξ and $\mathcal{A}$-equivalence classes in $\Delta\xi\psi(\Delta)$, defined by $\mathrm{o}_{\mathcal{A}_\xi}(\gamma) \mapsto \mathrm{o}_{\mathcal{A}}(\xi\gamma)$ and $\mathrm{o}_{\mathcal{A}_\xi}(\gamma) = \Gamma_\xi \cap \xi^{-1}\mathrm{o}_{\mathcal{A}}(\xi\gamma)$.*

Proof. Take any γ, $\gamma' \in \Gamma_\xi$. Then $\gamma' \in \xi^{-1}\mathrm{o}_{\mathcal{A}}(\xi\gamma)$ if and only if $\xi\gamma' = \delta\xi\gamma\varepsilon\psi(\delta)^{-1}$ for some $\delta \in \Delta$ and $\varepsilon \in E_{\xi\gamma}$. Writing $\delta = {}^{\xi}\delta'$ this is equivalent to $\gamma' = \delta'\gamma\varepsilon\psi({}^{\xi}\delta')^{-1}$ for $\delta' \in {}^{\xi^{-1}}\Delta$ and $\varepsilon \in E_{\xi\gamma}$. In this equation γ and γ' and $\psi({}^{\xi}\delta')$ lie in $\Gamma_\xi = \psi(\Delta)$ by assumption, and so does $\varepsilon \in E_{\xi\gamma} \subset \psi(\Delta)$ by Definition 9.2. Thus the equation requires that δ' lies in $\psi(\Delta)$, and so a fortiori in $\psi(\Delta) \cap {}^{\xi^{-1}}\Delta = \Delta_\xi$. In view of Lemma 9.8 the condition is thus equivalent to $\gamma' \in \mathrm{o}_{\mathcal{A}_\xi}(\gamma)$, proving the equation at the end of the proposition.
That equation implies that the map $\mathrm{o}_{\mathcal{A}_\xi}(\gamma) \mapsto \mathrm{o}_{\mathcal{A}}(\xi\gamma)$ from $\mathcal{A}_\xi$-equivalence classes in Γ_ξ to $\mathcal{A}$-equivalence classes in $\Delta\xi\psi(\Delta)$ is well-defined and injective.

But any element of $\Delta\xi\psi(\Delta)$ has the form $\delta\xi\gamma$ for $\delta \in \Delta$ and $\gamma \in \Gamma_\xi$ and is therefore equivalent to $\xi\gamma\psi(\delta) \in \xi\Gamma_\xi$. Thus the map is also surjective, and we are done. □

9.2 Coxeter groups

Definition 9.10. Let W be a Coxeter group with a finite set of simple reflections S. Let $\psi\colon W_I \xrightarrow{\sim} W_J \subset W$ be an isomorphism of Coxeter groups with $\psi(I) = J$ for subsets I, $J \subset S$. Then $\mathcal{A} := (W, W_I, \psi)$ is an abstract zip datum that we call *of Coxeter type*.

Fix such an abstract zip datum of Coxeter type $\mathcal{A}$. Recall that ${}^IW^J$ is a set of representatives for the double quotient $W_I\backslash W/W_J$. We will apply the induction step from Proposition 9.9 to $x \in {}^IW^J$. As in Proposition 4.13 set $I_x := J \cap {}^{x^{-1}}I$ and $J_x := \psi(I \cap {}^xJ)$, which are both subsets of J. Then $W_J = \psi(W_I)$, and $W_{I_x} = \psi(W_I) \cap {}^{x^{-1}}W_I$ by Proposition 2.6, and $\psi_x := \psi \circ \mathrm{int}(x)$ induces an isomorphism $\psi_x\colon W_{I_x} \xrightarrow{\sim} W_{J_x}$ such that $\psi_x(I_x) = J_x$. Thus the new abstract zip datum from Construction 9.7 is $\mathcal{A}_x := (W_J, W_{I_x}, \psi_x)$ and hence again of Coxeter type. Using this we obtain the following analogue of Theorem 5.10, which also has been previously proved by He ([He3] Corollary 2.6).

Theorem 9.11. *For $\mathcal{A}$ of Coxeter type ${}^IW$ is a set of representatives for the equivalence classes in W.*

Proof. We prove this by induction on $\#S$. If $I = S$, we have $W_I = W_J = W$ and so $E_w = W$ for every $w \in W$. Then there is exactly one equivalence class, represented by the unique element of ${}^IW = \{1\}$, and the assertion holds.
Otherwise we have $\#I < \#S$. Take any $x \in {}^IW^J$. Then by the induction hypothesis ${}^{I_x}W_J$ is a set of representatives for the $\mathcal{A}_x$-equivalence classes in W_J. Thus Proposition 9.9 implies that $x\,{}^{I_x}W_J$ is a set of representatives for the $\mathcal{A}$-equivalence classes in W_IxW_J. Varying x, Proposition 2.8 implies that ${}^IW$ is a set of representatives for the equivalence classes in W, as desired. □

For use in Section 11 we include the following results.

Lemma 9.12. *(a) For any $w \in {}^IW$ there exists $y \in W_I$ such that $w' := yw\psi(y)^{-1} \in W^J$.*

(b) The element w' in (a) is independent of y.

Proof. For (a) we use induction on $\#S$. If $I = S$, we have ${}^IW = \{1\}$ and $w = 1$, and so $y = 1$ does the job. Otherwise $\#I < \#S$. Write $w = xw_J$ as in Proposition 2.8 with $x \in {}^IW^J$ and $w_J \in {}^{I_x}W_J$. Then by the induction hypothesis applied to $\mathcal{A}_x$ there exists $y' \in W_{I_x}$ such that

$$w'_J \;:=\; y'w_J\psi_x(y')^{-1} \;\in\; W_J^{J_x} \;=\; W_{\psi(I)}^{\psi(I\cap {}^xJ)} \;=\; \psi\big(W_I^{I\cap {}^xJ}\big).$$

Setting $y := \psi^{-1}(y'w_J) \in W_I$ and using the definition of ψ_x we deduce that

$$\begin{aligned} w' := yw\psi(y)^{-1} &= \psi^{-1}(y'w_J) \cdot xw_J \cdot (y'w_J)^{-1} \\ &= \psi^{-1}(y'w_J) \cdot xy'^{-1}x^{-1} \cdot x \\ &= \psi^{-1}\big(y'w_J\psi_x(y')^{-1}\big) \cdot x \\ &= \psi^{-1}(w'_J) \cdot x \ \in \ W_I^{I \cap {}^xJ} \cdot x. \end{aligned}$$

By Proposition 2.9 the right hand side is contained in W^J, showing (a).
To prove (b) consider another element $y' \in W_I$ such that $w'' := y'w\psi(y')^{-1} \in W^J$. Then with $\tilde{y} := \psi(y'y^{-1}) \in W_J$ we have $w'' = y'y^{-1}w'\psi(y)\psi(y')^{-1} = \psi^{-1}(\tilde{y})w'\tilde{y}^{-1}$ and hence $w''^{-1} = \tilde{y}w'^{-1}\psi^{-1}(\tilde{y})^{-1}$. Now observe that on replacing (I, J, ψ) by (J, I, ψ^{-1}) we obtain another abstract zip datum $\mathcal{A}' := (W, W_J, \psi^{-1})$ dual to $\mathcal{A}$. The last equality then shows that w''^{-1} and w'^{-1} are equivalent according to Definition 9.4 for $\mathcal{A}'$. Since these elements also lie in ${}^JW$, Theorem 9.11 applied to $\mathcal{A}'$ shows that they are equal. Therefore $w'' = w'$, as desired. ☐

Proposition 9.13. *There exists a unique bijection $\sigma\colon {}^IW \to W^J$ with the property that for any $w \in {}^IW$ there exists $y \in W_I$ such that $\sigma(w) = yw\psi(y)^{-1}$.*

Proof. The existence of a unique *map* $\sigma\colon {}^IW \to W^J$ with the stated property is equivalent to Lemma 9.12. By applying the same lemma to the abstract zip datum $\mathcal{A}' := (W, W_J, \psi^{-1})$ in place of $\mathcal{A}$ we find that for any $w' \in {}^JW$ there exists $y' \in W_J$ such that $w := y'w'\psi^{-1}(y')^{-1} \in W^I$, and the element w is independent of y'. After replacing (w', w) by (w'^{-1}, w^{-1}) this means that for any $w' \in W^J$ there exists $y' \in W_J$ such that $w := \psi^{-1}(y')w'y'^{-1} \in {}^IW$, and the element w is independent of y'. But with $y := \psi^{-1}(y')^{-1} \in W_I$ the last equation is equivalent to $w' = yw\psi(y)^{-1}$, and so for any $w' \in W^J$ there exists a unique $w \in {}^IW$ with $w' = \sigma(w)$. In other words the map is bijective, as desired. ☐

Proposition 9.14. *The bijection in Proposition 9.13 satisfies $\ell(w) = \ell(\sigma(w))$ for all $w \in {}^IW$.*

Proof. Write the defining relation in the form $yw = \sigma(w)\psi(y)$. Here $y \in W_I$ and $w \in {}^IW$ imply that $\ell(yw) = \ell(y) + \ell(w)$, and similarly $\sigma(w) \in W^J$ and $\psi(y) \in W_J$ imply that $\ell(\sigma(w)\psi(y)) = \ell(\sigma(w)) + \ell(\psi(y))$. Moreover, since ψ sends simple reflections to simple reflections, it satisfies $\ell(\psi(y)) = \ell(y)$. Together it follows that $\ell(w) = \ell(\sigma(w))$. ☐

Lemma 9.15. *Let $\sigma\colon {}^IW \to W^J$ be the bijection from Proposition 9.13. For any $x \in {}^IW^J$ let $\sigma_x\colon {}^{I_x}W_J \to W_J^{J_x}$ denote the bijection obtained by applying Proposition 9.13 to $\mathcal{A}_x$. Then for all $w_J \in {}^{I_x}W_J$ we have $\sigma(xw_J) = \psi^{-1}(\sigma_x(w_J)) \cdot x$.*

Proof. The proof of Lemma 9.12 (a) shows that $\sigma(w) = w' = \psi^{-1}(w'_J) \cdot x$ where $w'_J = \sigma_x(w_J)$, as desired. ☐

Remark 9.16. Propositions 9.13 and 9.14 can also be deduced from more general results of He ([He3] Proposition 4.3).

9.3 Back to algebraic groups

Now we return to the situation and the notations of the preceding sections. Clearly the connected algebraic zip datum $\mathcal{Z}$ gives rise to an abstract zip datum of Coxeter type $\mathcal{A} := (W, W_I, \psi)$, which by Proposition 3.11 is independent of the frame, up to unique isomorphism. Theorem 5.10 implies that for any $w' \in W$ the element $g\dot{w}'$ lies in G^w for a unique $w \in {}^I W$.

Theorem 9.17. *For any $w' \in W$ and $w \in {}^I W$ we have $g\dot{w}' \in G^w$ if and only if $w' \sim w$ with respect to $\mathcal{A}$.*

Proof. We prove this by induction on $\#S$. If $J = S$, there is exactly one G^w for $w = 1$ and exactly one $\mathcal{A}$-equivalence class in W, so the assertion holds. Otherwise we have $\#J < \#S$. Write $w = xw_J$ with $x \in {}^I W^J$ and $w_J \in {}^{I_x} W_J$, as in Proposition 2.8. Then by (4.1) and Lemma 5.9 the condition $g\dot{w}' \in G^w$ requires that $w' \in W_I x W_J$, and so does the condition $w' \sim w$ by the remarks in Subsection 9.2. It therefore suffices to consider $w' = yxw'_J$ with $y \in W_I$ and $w'_J \in {}^{I_x} W_J$, as in Proposition 2.7. But then $w' \sim xw'_J \psi(y)$ with respect to $\mathcal{A}$, and $g\dot{w}' = g\dot{y}\dot{x}\dot{w}'_J$ is in the same $E_{\mathcal{Z}}$-orbit as $g\dot{x}\dot{w}'_J \varphi({}^g\dot{y})$. After replacing w' by $xw'_J\psi(y)$ we may thus assume that $w' = xw'_J$ for some $w'_J \in W_J$. Then Proposition 4.7 and Lemma 5.9 show that $g\dot{w}' \in G^w$ if and only if $\dot{w}'_J \in M^{w_J}$. By the induction hypothesis this is equivalent to $w'_J \sim w_J$ with respect to $\mathcal{A}_x$. By Proposition 9.9 this in turn is equivalent to $w' \sim w$ with respect to $\mathcal{A}$, as desired. $\square$

Combining Theorems 7.5 and 9.17 we deduce:

Corollary 9.18. *If $\mathcal{Z}$ is orbitally finite, then for any $w, w' \in W$ the elements $g\dot{w}$ and $g\dot{w}'$ lie in the same $E_{\mathcal{Z}}$-orbit if and only if $w \sim w'$ with respect to $\mathcal{A}$.*

10 Non-connected algebraic zip data

In this section we generalize the main results of Sections 5 and 6 to non-connected groups. Throughout we denote a not necessarily connected linear algebraic group by $\hat{G}$, its identity component by G, and its finite group of connected components by $\pi_0(\hat{G}) := \hat{G}/G$; and similarly for other letters of the alphabet. Note that the unipotent radical $R_u G$ is a normal subgroup of $\hat{G}$. Any homomorphism $\hat{\varphi}\colon \hat{G} \to \hat{H}$ restricts to a homomorphism $\varphi\colon G \to H$.

Definition 10.1. An *algebraic zip datum* is a tuple $\hat{\mathcal{Z}} = (\hat{G}, \hat{P}, \hat{Q}, \hat{\varphi})$ consisting of a linear algebraic group $\hat{G}$ with subgroups $\hat{P}$ and $\hat{Q}$ and a homomorphism $\hat{\varphi}\colon \hat{P}/\mathcal{R}_u P \to \hat{Q}/\mathcal{R}_u Q$, such that $\mathcal{Z} := (G, P, Q, \varphi)$ is a connected algebraic zip datum. The zip group $E_{\hat{\mathcal{Z}}} \subset \hat{P} \times \hat{Q}$, its action on $\hat{G}$, and the orbit $\mathrm{o}_{\hat{\mathcal{Z}}}(X)$ of a subset $X \subset \hat{G}$ are defined in exact analogy to (3.2), (3.3), and (3.4).

Throughout this section we fix an algebraic zip datum $\hat{\mathcal{Z}} = (\hat{G}, \hat{P}, \hat{Q}, \hat{\varphi})$ with associated connected algebraic zip datum $\mathcal{Z} = (G, P, Q, \varphi)$. We fix a frame (B, T, g) of $\mathcal{Z}$ and use the other pertaining notations from Sections 3 through 5. We also define

$$\hat{W} := \mathrm{Norm}_{\hat{G}}(T)/T \qquad \text{and} \qquad \Omega := (\mathrm{Norm}_{\hat{G}}(B) \cap \mathrm{Norm}_{\hat{G}}(T))/T,$$

so that $\Omega \cong \pi_0(\hat{G})$ and $\hat{W} = W \rtimes \Omega$. For each $\omega \in \Omega$ we fix a representative $\dot{\omega} \in \mathrm{Norm}_{\hat{G}}(B) \cap \mathrm{Norm}_{\hat{G}}(T)$, and for $\hat{w} = w\omega \in \hat{W}$ with $w \in W$ and $\omega \in \Omega$ we set $\dot{\hat{w}} := \dot{w}\dot{\omega} \in \mathrm{Norm}_{\hat{G}}(T)$.
Note that by definition $E_{\mathcal{Z}}$ is the identity component of $E_{\hat{\mathcal{Z}}}$. Thus to study the $E_{\hat{\mathcal{Z}}}$-orbits in $\hat{G}$, we first study the orbits under $E_{\mathcal{Z}}$ and then the action of $E_{\hat{\mathcal{Z}}}/E_{\mathcal{Z}}$ on them.

Lemma 10.2. *For any $\omega \in \Omega$ the* conjugate *connected algebraic zip datum*

$$^{\dot{\omega}}\mathcal{Z} := (G, P, {}^{\dot{\omega}}Q, \mathrm{int}(\dot{\omega}) \circ \varphi)$$

has zip group $E_{^{\dot{\omega}}\mathcal{Z}} = \{(p, {}^{\dot{\omega}}q) \mid (p,q) \in E_{\mathcal{Z}}\}$ and frame (B, T, g), and the isomorphism of varieties $G \to G\dot{\omega}$, $g' \mapsto g'\dot{\omega}$ induces a bijection from the $E_{^{\dot{\omega}}\mathcal{Z}}$-orbits in G to the $E_{\mathcal{Z}}$-orbits in $G\dot{\omega}$.

Proof. Direct calculation. $\square$

Lemma 10.3. *The subsets $o_{\mathcal{Z}}(gB\dot{\hat{w}}B)$ for all $\hat{w} \in {}^IW\Omega$ form a disjoint decomposition of $\hat{G}$.*

Proof. Take any $\omega \in \Omega$. Then by Theorems 5.10 and 5.14 the subsets $o_{^{\dot{\omega}}\mathcal{Z}}(gB\dot{w}B)$ for all $w \in {}^IW$ form a disjoint decomposition of G. Thus by Lemma 10.2 the subsets $o_{\mathcal{Z}}(gB\dot{w}B\dot{\omega})$ for all $w \in {}^IW$ form a disjoint decomposition of $G\dot{\omega}$. Since $\dot{\omega} \in \mathrm{Norm}_{\hat{G}}(B)$ by assumption, the latter subset is equal to $o_{\mathcal{Z}}(gB\dot{w}\dot{\omega}B)$. By varying ω the proposition follows. $\square$

Next define $\hat{L} := \mathrm{Norm}_{\hat{P}}(L)$ and $\hat{M} := \mathrm{Norm}_{\hat{Q}}(M)$, so that $\hat{P} = U \rtimes \hat{L}$ and $\hat{Q} = V \rtimes \hat{M}$, and $\hat{\varphi}$ can be identified with a homomorphism $\hat{L} \to \hat{M}$. Set

$$\begin{aligned} \hat{W}_I &:= \mathrm{Norm}_{^{g^{-1}}\hat{L}}(T)/T, & \Omega_I &:= (\mathrm{Norm}_{^{g^{-1}}\hat{L}}(B) \cap \mathrm{Norm}_{^{g^{-1}}\hat{L}}(T))/T, \\ \hat{W}_J &:= \mathrm{Norm}_{\hat{M}}(T)/T, & \Omega_J &:= (\mathrm{Norm}_{\hat{M}}(B) \cap \mathrm{Norm}_{\hat{M}}(T))/T. \end{aligned}$$

These groups are subgroups of $\hat{W}$ and satisfy

$$\begin{aligned} \hat{W}_I &= W_I \rtimes \Omega_I, & \Omega_I &\cong \pi_0(\hat{L}) \cong \pi_0(\hat{P}), \\ \hat{W}_J &= W_J \rtimes \Omega_J, & \Omega_J &\cong \pi_0(\hat{M}) \cong \pi_0(\hat{Q}). \end{aligned}$$

Also $\hat{\varphi} \circ \mathrm{int}(g)$ induces a homomorphism $\hat{\psi}\colon \hat{W}_I \to \hat{W}_J$ extending $\psi\colon W_I \to W_J$ and sending Ω_I to Ω_J. Moreover, the elements $({}^g\dot{\omega}, \hat{\varphi}({}^g\dot{\omega}))$ for all $\omega \in \Omega_I$ are representatives of the connected components of $E_{\hat{\mathcal{Z}}}$.

Lemma 10.4. *(a) The map* $(v, \hat{w}) \mapsto v\hat{w}\hat{\psi}(v)^{-1}$ *defines a left action of* Ω_I *on* ${}^I W\Omega$.

(b) Take any $v \in \Omega_I$ *and* $\hat{w} \in {}^I W\Omega$ *and abbreviate* $\hat{w}' := v\hat{w}\hat{\psi}(v)^{-1} \in {}^I W\Omega$. *Then the element* $({}^g\dot{v}, \hat{\varphi}({}^g\dot{v})) \in E_{\hat{\mathcal{Z}}}$ *sends* $\mathrm{o}_{\mathcal{Z}}(gB\dot{\hat{w}}B)$ *to* $\mathrm{o}_{\mathcal{Z}}(gB\dot{\hat{w}}'B)$.

Proof. Conjugation by Ω_I preserves the set of simple reflections I and thus the subset ${}^I W \subset W$. In (a) we therefore have $v\hat{w}\hat{\psi}(v)^{-1} = {}^v\hat{w} \cdot v\hat{\psi}(v)^{-1} \in {}^I W\Omega \cdot \Omega = {}^I W\Omega$, as desired. In (b) the elements $\dot{v}$ and $\hat{\varphi}({}^g\dot{v})$ normalize B; hence the image is

$$ {}^g\dot{v} \cdot \mathrm{o}_{\mathcal{Z}}(gB\dot{\hat{w}}B) \cdot \hat{\varphi}({}^g\dot{v})^{-1} \;=\; \mathrm{o}_{\mathcal{Z}}\big({}^g\dot{v}gB\dot{\hat{w}}B\hat{\varphi}({}^g\dot{v})^{-1}\big) \;=\; \mathrm{o}_{\mathcal{Z}}\big(gB\dot{v}\dot{\hat{w}}\hat{\varphi}({}^g\dot{v})^{-1}B\big). $$

As $\dot{v}\dot{\hat{w}}\hat{\varphi}({}^g\dot{v})^{-1}$ differs from $\dot{\hat{w}}'$ by an element of T, this proves (b). □

For any $\hat{w} \in {}^I W\Omega$ we now define

$$ \hat{G}^{\hat{w}} := \mathrm{o}_{\hat{\mathcal{Z}}}(gB\dot{\hat{w}}B), \tag{10.5} $$

which is independent of the representative $\dot{\hat{w}}$. Lemma 10.4 implies that $\hat{G}^{\hat{w}}$ is the union of $\mathrm{o}_{\mathcal{Z}}(gB\dot{\hat{w}}'B)$ for all $\hat{w}'$ in the Ω_I-orbit of $\hat{w}$ under the action in 10.4 (a). Thus $\hat{G}^{\hat{w}}$ depends only on $\hat{w}$ modulo Ω_I, and with Lemma 10.3 we conclude:

Theorem 10.6. *The subsets* $\hat{G}^{\hat{w}}$ *for all* $\hat{w} \in {}^I W\Omega$ *modulo the action of* Ω_I *from 10.4 (a) form a disjoint decomposition of* $\hat{G}$.

To describe the closure relation between the subsets $\hat{G}^{\hat{w}}$ we define analogues of the Bruhat order $\leq$ on $\hat{W} = W\Omega$ and of the relation $\preccurlyeq$ from Definition 6.1 on ${}^I W\Omega$:

Definition 10.7. For $\hat{w} = w\omega$ and $\hat{w}' = w'\omega'$ with w, $w' \in W$ and ω, $\omega' \in \Omega$ we write $\hat{w}' \leq \hat{w}$ if and only if $w' \leq w$ and $\omega' = \omega$.

Definition 10.8. For $\hat{w}$, $\hat{w}' \in {}^I W\Omega$ we write $\hat{w}' \preccurlyeq \hat{w}$ if and only if there exists $\hat{y} \in \hat{W}_I$ such that $\hat{y}\hat{w}'\hat{\psi}(\hat{y})^{-1} \leq \hat{w}$.

Theorem 10.9. *For any* $\hat{w} \in {}^I W\Omega$ *we have*

$$ \overline{\hat{G}^{\hat{w}}} \;=\; \bigcup_{\substack{\hat{w}' \in {}^I W\Omega \\ \hat{w}' \preccurlyeq \hat{w}}} \hat{G}^{\hat{w}'}. $$

Proof. Write $\hat{w} = w\omega$ with $w \in {}^I W$ and $\omega \in \Omega$. Then the conjugate zip datum ${}^{\dot{\omega}}\mathcal{Z}$ has the isogeny $\mathrm{int}(\dot{\omega}) \circ \varphi\colon L \to {}^{\dot{\omega}}M$ and hence the induced isomorphism of Weyl groups $\mathrm{int}(\omega) \circ \psi\colon W_I \xrightarrow{\sim} {}^{\omega}W_J = W_{{}^\omega J}$. Thus Theorems 5.14 and 6.2 and Definition 6.1 imply that

$$ \overline{\mathrm{o}_{\dot{\omega}\mathcal{Z}}(gB\dot{w}B)} \;=\; \bigcup_{w'} \mathrm{o}_{\dot{\omega}\mathcal{Z}}(gB\dot{w}'B), $$

where the union ranges over all $w' \in {}^IW$ such that $yw'\,{}^{\omega}\psi(y)^{-1} \leq w$ for some $y \in W_I$. Note that this inequality is equivalent to $yw'\omega\psi(y)^{-1} \leq w\omega$ by Definition 10.7. Thus with Lemma 10.2 we deduce that

$$\overline{\mathrm{o}_{\mathcal{Z}}(gB\dot{\hat{w}}B)} \;=\; \overline{\mathrm{o}_{\mathcal{Z}}(gB\dot{w}\dot{\omega}B)} \;=\; \bigcup_{w'} \mathrm{o}_{\mathcal{Z}}(gB\dot{w}'\dot{\omega}B) \;=\; \bigcup_{\hat{w}'} \mathrm{o}_{\mathcal{Z}}(gB\dot{\hat{w}}'B),$$

where the last union ranges over all $\hat{w}' \in {}^IW\Omega$ such that $y\hat{w}'\psi(y)^{-1} \leq \hat{w}$ for some $y \in W_I$. By taking the union of conjugates of this under $({}^g\dot{\upsilon}, \hat{\varphi}({}^g\dot{\upsilon})) \in E_{\hat{\mathcal{Z}}}$ for all $\upsilon \in \Omega_I$ we obtain the closure of $\hat{G}^{\hat{w}}$. By Lemma 10.4 the right hand side then yields the union of $\mathrm{o}_{\mathcal{Z}}(gB\dot{\hat{w}}''B)$ for all $\hat{w}'' = \upsilon\hat{w}'\hat{\psi}(\upsilon)^{-1}$ with $y\hat{w}'\psi(y)^{-1} \leq \hat{w}$ for some $\upsilon \in \Omega_I$ and $y \in W_I$. But here $\hat{y} := y\upsilon^{-1}$ runs through the group $W_I\Omega_I = \hat{W}_I$ and the inequality is equivalent to

$$\hat{y}\hat{w}''\hat{\psi}(\hat{y})^{-1} \;=\; y\upsilon^{-1}\hat{w}'\hat{\psi}(\upsilon)\psi(y)^{-1} \;\leq \hat{w}.$$

By Definition 10.8 these $\hat{w}''$ are precisely the elements of ${}^IW\Omega$ satisfying $\hat{w}'' \preccurlyeq \hat{w}$. □

Finally, let us call $\hat{\mathcal{Z}}$ *orbitally finite* if the conjugates ${}^{\dot{\omega}}\mathcal{Z}$ are orbitally finite for all $\omega \in \Omega$. This holds in particular when the differential of $\hat{\varphi}$ at 1 vanishes, because then we can apply Proposition 7.3 to ${}^{\dot{\omega}}\mathcal{Z}$. Combining Theorem 7.5 with the remarks leading up to Theorem 10.6 we deduce:

Theorem 10.10. *Assume that $\hat{\mathcal{Z}}$ is orbitally finite. Then:*

(a) For any $\hat{w} \in {}^IW\Omega$ we have $\hat{G}^{\hat{w}} = \mathrm{o}_{\hat{\mathcal{Z}}}(g\dot{\hat{w}})$.

(b) If $\hat{w} \in {}^IW\Omega$ runs through a system of representatives for the action of Ω_I from 10.4 (a), then $g\dot{\hat{w}}$ runs through a set of representatives for the $E_{\hat{\mathcal{Z}}}$-orbits in $\hat{G}$.

11 Dual parametrization

The decomposition of G from Theorem 5.10 is parametrized in a natural way by elements of ${}^IW$. In this section we translate that parametrization into an equally natural parametrization by elements of W^J, which was used by Lusztig and He (see Section 12). We also carry out the corresponding translation in the non-connected case.

11.1 The connected case

For any $w \in W^J$ we set

$$G^w := \mathrm{o}_{\mathcal{Z}}\big(gB\dot{w}B\big). \tag{11.1}$$

Note that this does not depend on the representative $\dot{w}$ of w and conforms to Definition 5.7 by Theorem 5.14. In Proposition 9.13 we have already established a natural bijection $\sigma\colon {}^IW \to W^J$.

THEOREM 11.2. *For any $w \in {}^I W$ we have $G^w = G^{\sigma(w)}$.*

Proof. If $I = J = S$, we have ${}^I W = W^J = \{1\}$ and so $w = \sigma(w) = 1$; hence the assertion holds trivially. Otherwise $\#I < \#S$. Write $w = xw_J$ as in Proposition 2.8 with $x \in {}^I W^J$ and $w_J \in {}^{I_x}W_J$, and let $\sigma_x \colon {}^{I_x}W_J \to W_J^{J_x}$ denote the bijection obtained by applying Proposition 9.13 to $\mathcal{A}_x$. Then $\mathcal{Z}_{\dot{x}}$ has the frame $(M \cap B, T, 1)$ by Proposition 4.12, and so the induction hypothesis implies that

$$M^{w_J} \;=\; M^{\sigma_x(w_J)} \;=\; \mathrm{o}_{\mathcal{Z}_{\dot{x}}}\big((M \cap B)\dot{\sigma}_x(w_J)(M \cap B)\big).$$

By Lemma 5.13 this is equal to $\mathrm{o}_{\mathcal{Z}_{\dot{x}}}((M \cap B)\dot{\sigma}_x(w_J))$, and so by Lemmas 5.9 and 4.9 we have

$$G^w \;=\; \mathrm{o}_{\mathcal{Z}}(g\dot{x}M^{w_J}) \;=\; \mathrm{o}_{\mathcal{Z}}\big(g\dot{x}\,\mathrm{o}_{\mathcal{Z}_{\dot{x}}}((M{\cap}B)\dot{\sigma}_x(w_J))\big) \;=\; \mathrm{o}_{\mathcal{Z}}\big(g\dot{x}(M{\cap}B)\dot{\sigma}_x(w_J)\big).$$

Recall from Lemma 9.15 that $\sigma(w) = w_I x$ with $w_I := \psi^{-1}(\sigma_x(w_J)) \in W_I$. It follows that $\sigma_x(w_J) = \psi(w_I)$ and therefore $\dot{\sigma}_x(w_J) \in T \cdot \varphi({}^g\dot{w}_I)$ and $\dot{\sigma}(w) \in T \cdot \dot{w}_I\dot{x}$. Since $T \subset M \cap B$, using the action (3.3) of $E_{\mathcal{Z}}$ we deduce that

$$G^w \;=\; \mathrm{o}_{\mathcal{Z}}\big(g\dot{x}(M \cap B)\varphi({}^g\dot{w}_I)\big) \;=\; \mathrm{o}_{\mathcal{Z}}\big(g\dot{w}_I\dot{x}(M \cap B)\big) \;=\; \mathrm{o}_{\mathcal{Z}}\big(g\dot{\sigma}(w)(M \cap B)\big).$$

Using (3.3) and (3.9) for the action of V, respectively Lemma 5.13, we conclude that

$$G^w \;=\; \mathrm{o}_{\mathcal{Z}}\big(g\dot{\sigma}(w)B\big) \;=\; \mathrm{o}_{\mathcal{Z}}\big(gB\dot{\sigma}(w)B\big) \;=\; G^{\sigma(w)},$$

as desired. □

THEOREM 11.3. *The G^w for all $w \in W^J$ form a disjoint decomposition of G by nonsingular subvarieties of dimension* $\dim P + \ell(w)$.

Proof. Combine Theorems 5.10, 5.11, 11.2 and Proposition 9.14. □

Next, in analogy to Definition 6.1 we define:

DEFINITION 11.4. For w, $w' \in W^J$ we write $w' \preccurlyeq w$ if and only if there exists $y \in W_I$ such that $yw'\psi(y)^{-1} \leq w$.

THEOREM 11.5. *For any $w \in W^J$ we have*

$$\overline{G^w} = \coprod_{\substack{w' \in W^J \\ w' \preccurlyeq w}} G^{w'}.$$

Proof. By combining Theorems 11.2 and 6.2 we already know that $\overline{G^w}$ is the disjoint union of $G^{w'}$ for certain $w' \in W^J$; it only remains to determine which. First consider $w' \in W^J$ with $G^{w'} \subset \overline{G^w}$. Then $g\dot{w}' \in \overline{G^w}$, and so by Lemma 6.6 there exist $b \in B$ and $w'' \in W$ such that $w'' \leq w$ and $\mathrm{o}_{\mathcal{Z}}(g\dot{w}') = \mathrm{o}_{\mathcal{Z}}(g\dot{w}''b)$. Set $\tilde{w}' := \sigma^{-1}(w') \in {}^I W$ and take $y \in W_I$ satisfying $w' = y\tilde{w}'\psi(y)^{-1}$. Then $\dot{w}' = \dot{y}\dot{\tilde{w}}'t\varphi({}^g\dot{y})^{-1}$ for some $t \in T$, and thus $\mathrm{o}_{\mathcal{Z}}(gw') = \mathrm{o}_{\mathcal{Z}}(g\tilde{w}'t)$. Therefore

$o_{\mathcal{Z}}(g\tilde{w}'t) = o_{\mathcal{Z}}(gw''b)$, and so Lemma 6.9 implies that $y'\tilde{w}'\psi(y')^{-1} \leq w''$ for some $y' \in W_I$. Together it follows that

$$(y'y^{-1})w'\psi(y'y^{-1})^{-1} = y'\tilde{w}'\psi(y')^{-1} \leq w'' \leq w$$

and hence $w' \preccurlyeq w$, proving "$\subset$".

Conversely consider $w' \in W^J$ with $w' \preccurlyeq w$, and take $y \in W_I$ such that $w'' := yw'\psi(y)^{-1} \leq w$. Lemma 6.6 then shows that $o_{\mathcal{Z}}(g\dot{w}''T) \subset \overline{G^w}$. Therefore

$$\begin{aligned} o_{\mathcal{Z}}(g\dot{w}'T) \;=\; o_{\mathcal{Z}}\big(g\dot{y}\dot{w}'T\varphi(g\dot{y}g^{-1})^{-1}\big) \;=& \\ =\; o_{\mathcal{Z}}\big(g\dot{y}\dot{w}'\varphi(g\dot{y}g^{-1})^{-1}T\big) \;=\; o_{\mathcal{Z}}(g\dot{w}''T) \;\subset\; \overline{G^w}. \end{aligned}$$

Since also $o_{\mathcal{Z}}(g\dot{w}'T) \subset G^{w'}$, this with the preliminary remark on $\overline{G^w}$ shows that $G^{w'} \subset \overline{G^w}$, proving "$\supset$". ☐

Remark 11.6. In Definitions 5.7 and 11.1 we have introduced the subsets $G^w := o_{\mathcal{Z}}\big(gB\dot{w}B\big)$ only for $w \in {}^IW \cup W^J$, not for arbitrary $w \in W$. Our results do not say anything directly about the latter. Note that in case φ is an isomorphism their closures have been determined in [LY1] Theorem 5.2 and [He3] Proposition 5.8.

11.2 The non-connected case

Now we return to the notations from Section 10. We begin with an analogue of Proposition 9.13:

Proposition 11.7. *There exists a unique bijection $\hat{\sigma}\colon {}^IW\Omega \to \Omega W^J$ with the property that for any $\hat{w} \in {}^IW\Omega$ there exists $y \in W_I$ such that $\hat{\sigma}(\hat{w}) = y\hat{w}\psi(y)^{-1}$.*

Proof. The equation requires that $\hat{\sigma}(\hat{w}) \in \hat{W}$ lie in the same W-coset as $\hat{w}$. Thus for any fixed $\omega \in \Omega$, we need a unique bijection ${}^IW\omega \to \omega W^J$ sending $w\omega$ to an element of the form $yw\omega\psi(y)^{-1}$ for some $y \in W_I$. Multiplying both elements on the right by ω^{-1} this amounts to a unique bijection ${}^IW \to \omega W^J\omega^{-1} = W^{{}^\omega J}$ sending w to an element of the form $yw\omega\psi(y)^{-1}\omega^{-1}$ for some $y \in W_I$. But $\mathrm{int}(\omega) \circ \psi\colon W_I \to \omega W_J\omega^{-1} = W_{{}^\omega J}$ is precisely the isomorphism associated to the conjugate connected algebraic zip datum ${}^{\dot{\omega}}\mathcal{Z}$ from Lemma 10.2. Thus a unique bijection with that property exists by Proposition 9.13 applied to ${}^{\dot{\omega}}\mathcal{Z}$. ☐

For any $\hat{w} \in \Omega W^J$ we now define

$$\hat{G}^{\hat{w}} := o_{\hat{\mathcal{Z}}}(gB\dot{\hat{w}}B). \tag{11.8}$$

Again this does not depend on the representative $\dot{\hat{w}}$ of $\hat{w}$ and conforms to Definition (10.5).

Theorem 11.9. *For any $\hat{w} \in {}^IW\Omega$ we have $\hat{G}^{\hat{w}} = \hat{G}^{\hat{\sigma}(\hat{w})}$.*

Proof. Write $\hat{w} = w\omega$ with $w \in {}^IW$ and $\omega \in \Omega$. In the proof of Proposition 11.7 we have seen that $\hat{\sigma}(\hat{w}) = w'\omega$, where $w' \in W^{{}^\omega J}$ is the image of w under the isomorphism given by Proposition 9.13 applied to ${}^{\dot\omega}\mathcal{Z}$. Thus by Theorem 11.2 we have $\mathrm{o}_{{}^{\dot\omega}\mathcal{Z}}(gB\dot{w}B) = \mathrm{o}_{{}^{\dot\omega}\mathcal{Z}}(gB\dot{w}'B)$ inside G. On multiplying on the right by $\dot\omega$ and applying Lemma 10.2 to both sides we deduce that

$$\mathrm{o}_{\mathcal{Z}}(gB\dot{\hat{w}}B) = \mathrm{o}_{\mathcal{Z}}(gB\dot{w}B\dot\omega) = \mathrm{o}_{\mathcal{Z}}(gB\dot{w}'B\dot\omega) = \mathrm{o}_{\mathcal{Z}}(gB\dot{\hat{\sigma}}(\hat{w})B).$$

The desired equality follows from this by applying $\mathrm{o}_{\hat{\mathcal{Z}}}$. ☐

Lemma 11.10. *(a) The map $(v, \hat{w}) \mapsto v\hat{w}\hat{\psi}(v)^{-1}$ defines a left action of Ω_I on ΩW^J.*

(b) The bijection $\hat{\sigma}\colon {}^IW\Omega \to \Omega W^J$ from Proposition 11.7 is Ω_I-equivariant.

Proof. Take $v \in \Omega_I$ and $\hat{w} \in \Omega W^J$. To prove (a) observe that conjugation by $\hat{\psi}(v) \in \Omega_J$ preserves the set of simple reflections J and thus the subset $W^J \subset W$. We therefore have $v\hat{w}\hat{\psi}(v)^{-1} = v\hat{\psi}(v)^{-1} \cdot {}^{\hat{\psi}(v)}\hat{w} \in \Omega \cdot \Omega W^J = \Omega W^J$, as desired. In (b) write $\hat{\sigma}(\hat{w}) = y\hat{w}\psi(y)^{-1}$ with $y \in W_I$. Then

$$v\hat{\sigma}(\hat{w})\hat{\psi}(v)^{-1} = (vyv^{-1})(v\hat{w}\hat{\psi}(v)^{-1})\hat{\psi}(vyv^{-1})^{-1} = \hat{\sigma}\big(v\hat{w}\hat{\psi}(v)^{-1}\big),$$

because the left hand side is in ΩW^I and $vyv^{-1} \in W_I$. This proves (b). ☐

Theorem 11.11. *The subsets $\hat{G}^{\hat{w}}$ for all $\hat{w} \in \Omega W^J$ modulo the action of Ω_I from 11.10 (a) form a disjoint decomposition of $\hat{G}$.*

Proof. Combine Theorems 10.6 and 11.9 with Lemma 11.10. ☐

Definition 11.12. For $\hat{w}$, $\hat{w}' \in \Omega W^J$ we write $\hat{w}' \preccurlyeq \hat{w}$ if and only if there exists $\hat{y} \in \hat{W}_I$ such that $\hat{y}\hat{w}'\hat{\psi}(\hat{y})^{-1} \leq \hat{w}$.

Theorem 11.13. *For any $\hat{w} \in \Omega W^J$ we have*

$$\overline{\hat{G}^{\hat{w}}} = \bigcup_{\substack{\hat{w}' \in \Omega W^J \\ \hat{w}' \preccurlyeq \hat{w}}} \hat{G}^{\hat{w}'}.$$

Proof. Write $\hat{w} = w\omega$ with $\omega \in \Omega$ and $w \in W^{{}^\omega J}$. Applying Theorem 11.5 to the conjugate zip datum ${}^{\dot\omega}\mathcal{Z}$ shows that $\overline{\mathrm{o}_{{}^{\dot\omega}\mathcal{Z}}(gB\dot{w}B)}$ is the union of the subsets $\mathrm{o}_{{}^{\dot\omega}\mathcal{Z}}(gB\dot{w}'B)$ for all $w' \in W^{{}^\omega J}$ such that $yw'\omega\psi(y)^{-1}\omega^{-1} \leq w$ for some $y \in W_I$. On multiplying on the right by $\dot\omega$ and applying Lemma 10.2 to everything we deduce that $\overline{\mathrm{o}_{\mathcal{Z}}(gB\dot{\hat{w}}B)} = \overline{\mathrm{o}_{\mathcal{Z}}(gB\dot{w}B\dot\omega)}$ is the union of the subsets $\mathrm{o}_{\mathcal{Z}}(gB\dot{w}'B\dot\omega) = \mathrm{o}_{\mathcal{Z}}(gB\dot{w}'\dot\omega B)$ for the same elements w'. Writing $\hat{w}' = w'\omega$ this is equal to the union of the subsets $\mathrm{o}_{\mathcal{Z}}(gB\dot{\hat{w}}'B)$ for all $\hat{w}' \in \Omega W^J$ such that $y\hat{w}'\psi(y)^{-1} \leq w$ for some $y \in W_I$. The theorem follows from this by applying $\mathrm{o}_{\hat{\mathcal{Z}}}$. ☐

12 Generalization of certain varieties of Lusztig

In this section we consider a certain type of algebraic variety with an action of a reductive group G whose orbit structure is closely related to the structure of the $E_{\mathcal{Z}}$-orbits in G for an algebraic zip datum $\mathcal{Z}$. Special cases of such varieties have been defined by Lusztig ([Lus2]) and by Moonen and the second author in [MW].

12.1 The coset variety of an algebraic zip datum

Remark 12.1. To keep notations simple, we restrict ourselves to connected zip data, although everything in this section directly extends to non-connected ones by putting $\hat{}$ in the appropriate places.

In this section we use only the definition of algebraic zip data and the action of the associated zip group from Section 3, but none of the other theory or notations from the preceding sections, not even the concept of a frame. Fix a connected algebraic zip datum $\mathcal{Z} = (G, P, Q, \varphi)$. Recall that $E_{\mathcal{Z}}$ is a subgroup of $P \times Q$ and hence of $G \times G$. We also consider the image of G under the diagonal embedding $\Delta\colon G \hookrightarrow G \times G$, $g \mapsto (g, g)$. We are interested in the left quotient $\Delta(G)\backslash(G \times G)$ and the right quotient $(G \times G)/E_{\mathcal{Z}}$.
The first is isomorphic to G via the projection morphism

$$\lambda\colon G \times G \to G,\ (g, h) \mapsto g^{-1}h. \tag{12.2}$$

Turn the right action of $E_{\mathcal{Z}}$ on $G\times G$ into a left action by letting $(p, q) \in E_{\mathcal{Z}}$ act by right translation with $(p, q)^{-1}$. Then with $E_{\mathcal{Z}}$ acting on G as in the definition of algebraic zip data, a direct calculation shows that λ is $E_{\mathcal{Z}}$-equivariant.
To describe the second quotient recall that φ is a homomorphism $P/U \to Q/V$, where U and V denote the unipotent radicals of P and Q. Consider a left P-coset $X \subset G$ and a left Q-coset $Y \subset G$. Then X/U is a right torsor over P/U, and Y/V is a right torsor over Q/V. By a P/U-equivariant morphism $\Phi\colon X/U \to Y/V$ we mean a morphism satisfying $\Phi(\bar{x}\bar{p}) = \Phi(\bar{x})\varphi(\bar{p})$ for all $\bar{x} \in X/U$ and $\bar{p} \in P/U$.

Definition 12.3. The *coset space of* $\mathcal{Z}$ is the set $C_{\mathcal{Z}}$ of all triples (X, Y, Φ) consisting of a left P-coset $X \subset G$, a left Q-coset $Y \subset G$, and a P/U-equivariant morphism $\Phi\colon X/U \to Y/V$.

For any X, Y as above and any $(g, h) \in G\times G$, left multiplication by g induces an isomorphism $\ell_g\colon X/U \xrightarrow{\sim} gX/U$, and left multiplication by h induces an isomorphism $\ell_h\colon Y/V \xrightarrow{\sim} hY/V$. Therefore $(X, Y, \Phi) \mapsto \big(gX, hY, \ell_h \circ \Phi \circ \ell_g^{-1}\big)$ defines a left action of $G \times G$ on $C_{\mathcal{Z}}$. By applying this action to the canonical base point $(P, Q, \varphi) \in C_{\mathcal{Z}}$ we obtain a morphism

$$\rho\colon G \times G \to C_{\mathcal{Z}},\ (g, h) \mapsto \big(gP, hQ, \ell_h \circ \varphi \circ \ell_g^{-1}\big). \tag{12.4}$$

Clearly this morphism is equivariant under the left action of $G \times G$ and hence under the subgroup $\Delta(G)$.

Lemma 12.5. *There is a unique structure of algebraic variety on $C_{\mathcal{Z}}$ such that ρ identifies $C_{\mathcal{Z}}$ with the quotient variety $(G \times G)/E_{\mathcal{Z}}$.*

Proof. The action of $G \times G$ is obviously transitive on the set of all pairs (X, Y). Moreover, any P/U-equivariant morphism of right torsors $P/U \to Q/V$ has the form $\bar{p} \mapsto \pi_Q(q)\varphi(\bar{p}) = \ell_q \circ \varphi(\bar{p})$ for some $q \in Q$. Thus the subgroup $1 \times Q$ acts transitively on the set of all triples of the form (P, Q, Φ). Together it follows that the action of $G \times G$ on $C_{\mathcal{Z}}$ is transitive.
On the other hand (g, h) lies in the stabilizer of (P, Q, φ) if and only if $g \in P$ and $h \in Q$ and $\ell_h \circ \varphi \circ \ell_g^{-1} = \varphi$. But under the first two of these conditions, we have for all $\bar{p} \in P/U$

$$\ell_h \circ \varphi \circ \ell_g^{-1}(\bar{p}) = \pi_Q(h)\varphi\big(\pi_P(g)^{-1}\bar{p}\big) = \pi_Q(h)\varphi\big(\pi_P(g)\big)^{-1}\varphi(\bar{p}),$$

and so the third condition is equivalent to $\varphi\big(\pi_P(g)\big) = \pi_Q(h)$. Together this means precisely that $(g, h) \in E_{\mathcal{Z}}$, which is therefore the stabilizer of (P, Q, φ). It follows that ρ induces a bijection $(G \times G)/E_{\mathcal{Z}} \overset{\sim}{\to} C_{\mathcal{Z}}$. Since the quotient variety exists by [Ser], Section 3.2, this yields the unique structure of algebraic variety on $C_{\mathcal{Z}}$. $\square$

Following Lemma 12.5 we call $C_{\mathcal{Z}}$ also the *coset variety of $\mathcal{Z}$*. Recall from [Ser] Prop. 2.5.3 that the quotient of an algebraic group by an algebraic subgroup is always a torsor. To summarize we have therefore constructed morphisms with the following properties:

$$\boxed{E_{\mathcal{Z}}\text{-equivariant} \atop \Delta(G)\text{-torsor}\left\{\begin{tikzcd} & G \times G \arrow[dl, "\lambda"'] \arrow[dr, "\rho"] & \\ G & & C_{\mathcal{Z}} \end{tikzcd}\right\}{\Delta(G)\text{-equivariant} \atop E_{\mathcal{Z}}\text{-torsor}} \tag{12.6}$$

Recall that the actions of $\Delta(G)$ and $E_{\mathcal{Z}}$ on $G \times G$ commute and thus combine to an action of $\Delta(G) \times E_{\mathcal{Z}}$. Therefore (12.6) directly implies:

Theorem 12.7. *There are natural isomorphisms of algebraic stacks*

$$[E_{\mathcal{Z}} \backslash G] \xleftarrow[\sim]{[\lambda]} \big[(\Delta(G) \times E_{\mathcal{Z}}) \backslash (G \times G)\big] \xrightarrow[\sim]{[\rho]} [\Delta(G) \backslash C_{\mathcal{Z}}].$$

Even without stacks, we can deduce:

Theorem 12.8. *(a) There is a closure-preserving bijection between $E_{\mathcal{Z}}$-invariant subsets $A \subset G$ and $\Delta(G)$-invariant subsets $B \subset C_{\mathcal{Z}}$, defined by $A = \lambda(\rho^{-1}(B))$ and $B = \rho(\lambda^{-1}(A))$.*

(b) The subset A in (a) is a subvariety, resp. a nonsingular subvariety, if and only if B is one. In that case we also have $\dim A = \dim B$.

(c) *In particular (a) induces a bijection between $E_{\mathcal{Z}}$-orbits in G and $\Delta(G)$-orbits in $C_{\mathcal{Z}}$.*

(d) *For any $g \in G$ and $(X,Y,\Phi) \in C_{\mathcal{Z}}$ whose orbits correspond, there is an isomorphism*

$$\mathrm{Stab}_{E_{\mathcal{Z}}}(g) \;\cong\; \mathrm{Stab}_{\Delta(G)}((X,Y,\Phi)).$$

Proof. By (12.6) any $\Delta(G) \times E_{\mathcal{Z}}$-invariant subset of $G \times G$ must be simultaneously of the form $\lambda^{-1}(A)$ for an $E_{\mathcal{Z}}$-invariant subset $A \subset G$ and of the form $\rho^{-1}(B)$ for a $\Delta(G)$-invariant subset $B \subset C_{\mathcal{Z}}$. Then $A = \lambda(\rho^{-1}(B))$ and $B = \rho(\lambda^{-1}(A))$, giving the bijection in (a). The bijection preserves closures because λ and ρ are smooth. This proves (a), the first sentence in (b), and the special case (c). In (b) it also proves that $\dim A + \dim G = \dim B + \dim E_{\mathcal{Z}}$. But $\dim G = 2\dim U + \dim L = 2\dim V + \dim M$ and $\dim L = \dim M$ imply that $\dim U = \dim V$, and thus using (3.9) that $\dim E_{\mathcal{Z}} = \dim U + \dim L + \dim V = \dim G$. Therefore $\dim A = \dim B$, proving the rest of (b).
In (c) by assumption there exists a point $\underline{x} \in G \times G$ such that $\lambda(\underline{x})$ lies in the $E_{\mathcal{Z}}$-orbit of g and $\rho(\underline{x})$ lies in the $\Delta(G)$-orbit of (X,Y,Φ). Thus after replacing $\underline{x}$ by a suitable translate under $\Delta(G) \times E_{\mathcal{Z}}$ we may assume that $\lambda(\underline{x}) = g$ and $\rho(\underline{x}) = (X,Y,\Phi)$. Then the fact that λ and ρ are torsors implies that the two projection morphisms

$$\mathrm{Stab}_{E_{\mathcal{Z}}}(g) \longleftarrow \mathrm{Stab}_{\Delta(G)\times E_{\mathcal{Z}}}(\underline{x}) \longrightarrow \mathrm{Stab}_{\Delta(G)}((X,Y,\Phi))$$

are isomorphisms, proving (c). (The isomorphism may depend on the choice of $\underline{x}$.) $\square$

With Theorem 12.8 we can translate many results about the $E_{\mathcal{Z}}$-action on G from the preceding sections to the $\Delta(G)$-action on $C_{\mathcal{Z}}$, in particular Theorems 5.10, 5.11, 6.2, 7.5, 8.1, and their counterparts from Sections 10 and 11.

12.2 Algebraic zip data associated to an isogeny of G

In this subsection we consider algebraic zip data whose isogeny extends to an isogeny on all of G. (Not every connected algebraic zip datum has that property, for instance, if L and M have root system A_1 associated to long and short roots, respectively, and the square of the ratio of the root lengths is different from the characteristic of k.)
Fix a connected reductive algebraic group G over k and an isogeny $\varphi\colon G \to G$. Choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$, and let W be the corresponding Weyl group of G and S its set of simple reflections. Choose an element $\gamma \in G$ such that $\varphi({}^{\gamma}B) = B$ and $\varphi({}^{\gamma}T) = T$. Then $\varphi \circ \mathrm{int}(\gamma)\colon \mathrm{Norm}_G(T) \to \mathrm{Norm}_G(T)$ induces an isomorphism of Coxeter systems

$$\bar{\varphi}\colon (W,S) \xrightarrow{\sim} (W,S).$$

For any subset $I \subset S$ recall from Subsection 2.3 that P_I denotes the standard parabolic of type I. Thus the choices imply that $\varphi({}^{\gamma}P_I) = P_{\bar{\varphi}(I)}$. We denote the unipotent radicals of arbitrary parabolics P, Q, P', Q' by U, V, U', V', respectively.

Let $\hat{G}$ be a linear algebraic group over k having identity component G, and let G^1 be an arbitrary connected component of $\hat{G}$. Choose an element $g_1 \in \mathrm{Norm}_{G^1}(B) \cap \mathrm{Norm}_{G^1}(T)$. Then $\mathrm{int}(g_1)$ induces an automorphism of G that we use to twist φ. Let $\delta\colon (W,S) \to (W,S)$ be the isomorphism of Coxeter systems induced by $\mathrm{int}(g_1)$. Then for any subset $I \subset S$ we have ${}^{g_1}P_I = P_{\delta(I)}$.

Fix subsets I, $J \subset S$ and an element $x \in {}^JW^{\delta\bar{\varphi}(I)}$ with $J = {}^x\delta\bar{\varphi}(I)$. Set $y := (\delta\bar{\varphi})^{-1}(x) \in W$.

Lemma 12.9. *(a)* $x\Phi_{\delta\bar{\varphi}(I)} = \Phi_J$.

(b) $x\Phi^+_{\delta\bar{\varphi}(I)} = \Phi^+_J$.

Proof. Part (a) follows from $J = {}^x\delta\bar{\varphi}(I)$. By (2.11) the fact that $x \in {}^JW^{\delta\bar{\varphi}(I)} \subset W^{\delta\bar{\varphi}(I)}$ implies $x\Phi^+_{\delta\bar{\varphi}(I)} \subset \Phi^+$. Together with (a) this implies (b). ☐

Construction 12.10. Set $Q := P_J$ and $P := {}^{\gamma\dot{y}}P_I$ and let L be the Levi component of P containing ${}^{\gamma\dot{y}}T$. Then ${}^{g_1}\varphi(P) = {}^{\dot{x}}({}^{g_1}\varphi({}^{\gamma}P_I)) = {}^{\dot{x}}P_{\delta\bar{\varphi}(I)}$ and $Q = P_J$ have relative position x. Set $M := {}^{g_1}\varphi(L)$; this is a Levi component of ${}^{g_1}\varphi(P)$ containing ${}^{g_1}\varphi({}^{\gamma\dot{y}}T) = {}^{g_1}\varphi({}^{\gamma}T) = {}^{g_1}T = T$. Since the root system of M is $x\Phi_{\delta\bar{\varphi}(I)}$, Lemma 12.9 shows that it is also the Levi component of Q containing T. Let ${}^{g_1}\tilde{\varphi}\colon P/U \to Q/V$ denote the isogeny corresponding to $\mathrm{int}(g_1) \circ \varphi\,|\,L\colon L \to M$. Then we obtain a connected algebraic zip datum $\mathcal{Z} := (G, P, Q, {}^{g_1}\tilde{\varphi})$.

Lemma 12.11. *The triple $(B, T, \gamma\dot{y})$ is a frame of $\mathcal{Z}$, and the Levi components determined by it are $M \subset Q$ and $L \subset P$.*

Proof. The statements about M and L follow from the inclusions $T \subset M$ and ${}^{\gamma\dot{y}}T \subset L$. They also imply that the isogeny $L \to M$ corresponding to ${}^{g_1}\tilde{\varphi}$ is simply the restriction of $\mathrm{int}(g_1) \circ \varphi$. Conditions (a) and (b) in Definition 3.6 assert that $B \subset Q$ and ${}^{\gamma\dot{y}}B \subset P$, which hold by the construction of Q and P. Condition (d) translates to ${}^{g_1}\varphi\big({}^{\gamma\dot{y}}T\big) = T$, which was already shown in 12.10. To prove (c) note first that by Lemma 12.9 we have $x\Phi^+_{\gamma\bar{\varphi}(I)} = \Phi^+_J$ and therefore ${}^{\dot{x}}B \cap M = B \cap M$. The definition of y implies that ${}^{g_1}\varphi(\gamma\dot{y}\gamma^{-1}) \in \dot{x}T$ and hence

$${}^{g_1}\varphi\big({}^{\gamma\dot{y}}B\big) = {}^{{}^{g_1}\varphi(\gamma\dot{y}\gamma^{-1})\cdot g_1}\varphi\big({}^{\gamma}B\big) = {}^{{}^{g_1}\varphi(\gamma\dot{y}\gamma^{-1})}B = {}^{\dot{x}}B.$$

From this we can deduce that

$${}^{g_1}\varphi\big({}^{\gamma\dot{y}}B \cap L\big) = {}^{g_1}\varphi\big({}^{\gamma\dot{y}}B\big) \cap {}^{g_1}\varphi(L) = {}^{\dot{x}}B \cap M = B \cap M,$$

proving the remaining condition (c). ☐

The automorphism ψ defined in (3.11) for the algebraic zip datum $\mathcal{Z}$ is given by

$$\psi := \delta \circ \bar{\varphi} \circ \mathrm{int}(y) = \mathrm{int}(x) \circ \delta \circ \bar{\varphi}\colon (W_I, I) \xrightarrow{\sim} (W_J, J) \tag{12.7}$$

Definition 12.12. Let $X_{I,\varphi,x}$ be the set of all triples $(P',Q',[g'])$ consisting of parabolic subgroups P', Q' of G of type I, J and a double coset $[g'] := V'g'\varphi(U') \subset G^1$ of an element $g' \in G^1$ such that

$$\operatorname{relpos}(Q', {}^{g'}\varphi(P')) = x.$$

One readily verifies that the condition on the relative position depends only on $[g']$, and that $\big((g,h),(P',Q',[g'])\big) \mapsto \big({}^gP', {}^hQ', [hg'\varphi(g)^{-1}]\big)$ defines a left action of $G\times G$ on $X_{I,\varphi,x}$. We also have a standard base point $(P,Q,[g_1]) \in X_{I,\varphi,x}$. One can use the definition of $X_{I,\varphi,x}$ to endow it with the structure of an algebraic variety over k, but in the interest of brevity we define that structure using the following isomorphism:

Proposition 12.13. *There is a natural $G\times G$-equivariant isomorphism*

$$C_{\mathcal{Z}} \xrightarrow{\sim} X_{I,\varphi,x},\ \big(gP, hQ, \ell_h \circ {}^{g_1}\tilde{\varphi} \circ \ell_g^{-1}\big) \mapsto \big({}^gP, {}^hQ, [hg_1\varphi(g)^{-1}]\big).$$

Proof. In view of Lemma 12.5 the assertion is equivalent to saying that the action of $G\times G$ on $X_{I,\varphi,x}$ is transitive and the stabilizer of $(P,Q,[g_1])$ is $E_{\mathcal{Z}}$. The transitivity follows directly from the definition of the action. For the stabilizer note that $\big({}^gP, {}^hQ, [hg_1\varphi(g)^{-1}]\big) = (P,Q,[g_1])$ if and only if $g \in P$ and $h \in Q$ and $Vhg_1\varphi(g)^{-1}\varphi(U) = Vg_1\varphi(U)$. Write $g = u\ell$ for $u\in U$, $\ell \in L$ and $h = vm$ for $v \in V$, $m\in M$. Then the last condition is equivalent to $Vmg_1\varphi(\ell)^{-1}\varphi(U) = Vg_1\varphi(U)$, or again to $m\cdot {}^{g_1}\varphi(\ell)^{-1} \in V\cdot {}^{g_1}\varphi(U)\cap M$. But for any element $v'\cdot {}^{g_1}\varphi(u') = m' \in V\cdot {}^{g_1}\varphi(U)\cap M$ we have ${}^{g_1}\varphi(u') = v'^{-1}m' \in {}^{g_1}\varphi(U)\cap VM$, and since M is also a Levi component of ${}^{g_1}\varphi(P)$, it follows that ${}^{g_1}\varphi(U)\cap VM = {}^{g_1}\varphi(U)\cap V$ and hence $m'=1$. The last condition is therefore equivalent to $m = {}^{g_1}\varphi(\ell)$. Together this shows that the stabilizer is $E_{\mathcal{Z}}$, as desired. $\square$

Lemma 12.14. *For any $w \in {}^IW \cup W^J$ the subset $G^w \subset G$ corresponds via Theorem 12.8 (a) and Proposition 12.13 to the subset*

$$X^w_{I,\varphi,x} := \big\{\big({}^gP_I, {}^{g\dot w}P_J, [g\dot w\dot x g_1 b\varphi(\gamma g^{-1})]\big) \ \big|\ g\in G,\ b\in B\big\} \ \subset\ X_{I,\varphi,x},$$

which is a nonsingular variety of dimension $\dim P + \ell(w)$.

Proof. Since $(B,T,\gamma\dot y)$ is a frame of $\mathcal{Z}$ by Lemma 12.11, Theorem 5.14 for $w\in {}^IW$, respectively (11.1) and Lemma 5.13 for $w \in W^J$, show that $G^w = o_{\mathcal{Z}}(\gamma\dot y B\dot w)$. In other words G^w is the union of the $E_{\mathcal{Z}}$-orbits of $\gamma\dot y b\dot w$ for all $b\in B$. But by (12.2) and (12.4) we have

$$\begin{array}{rcl}\lambda\big((\gamma\dot y b)^{-1},\dot w\big) &=& \gamma\dot y b\dot w, \qquad \text{and}\\ \rho\big((\gamma\dot y b)^{-1},\dot w\big) &=& \big((\gamma\dot y b)^{-1}P, \dot wQ, \ell_{\dot w}\circ\varphi\circ\ell^{-1}_{(\gamma\dot y b)^{-1}}\big),\end{array}$$

and so the $E_{\mathcal{Z}}$-orbit of the former corresponds to the $\Delta(G)$-orbit of the latter under the correspondence from Theorem 12.8. Moreover, under the isomorphism from Proposition 12.13 the latter corresponds to the triple

$$\big({}^{(\gamma\dot y b)^{-1}}P, {}^{\dot w}Q, [\dot w g_1\varphi((\gamma\dot y b)^{-1})^{-1}]\big).$$

The definitions of P and Q show that ${}^{(\gamma\dot{y}b)^{-1}}P = {}^{b^{-1}}P_I = P_I$ and ${}^{\dot{w}}Q = {}^{\dot{w}}P_J$. The definition of y means that ${}^{g_1}\varphi(\gamma\dot{y}\gamma^{-1}) = \dot{x}t$ for some $t \in T$; hence

$$\begin{aligned}\dot{w}g_1\varphi(((\gamma\dot{y}b)^{-1})^{-1} &= \dot{w}g_1\varphi(\gamma\dot{y}b) = \\ &= \dot{w}\cdot\dot{x}t\cdot g_1\cdot\varphi(\gamma b\gamma^{-1})\cdot\varphi(\gamma) = \dot{w}\cdot\dot{x}\cdot g_1\cdot {}^{g_1^{-1}}t\,\varphi(\gamma b\gamma^{-1})\cdot\varphi(\gamma).\end{aligned}$$

Since $\varphi({}^{\gamma}B) = B$, the factor $b' := {}^{g_1^{-1}}t\,\varphi(\gamma b\gamma^{-1})$ runs through B while b runs through B. Thus altogether it follows that G^w corresponds to the union of the $\Delta(G)$-orbits of the triples

$$\big(P_I, {}^{\dot{w}}P_J, [\dot{w}\dot{x}g_1b'\varphi(\gamma)]\big)$$

for all $b' \in B$. This union is just the set $X^w_{I,\varphi,x}$ in the lemma. The rest follows from Theorems 5.11, 11.3, and 12.8. $\square$

Combining this with Theorems 5.10 and 6.2 and 12.8 we conclude:

Theorem 12.15. *(a) The $X^w_{I,\varphi,x}$ for all $w \in {}^IW$ form a disjoint decomposition of $X_{I,\varphi,x}$ by nonsingular subvarieties of dimension* $\dim P + \ell(w)$.

(b) For any $w \in {}^IW$ we have

$$\overline{X^w_{I,\varphi,x}} = \coprod_{\substack{w' \in {}^IW \\ w' \preccurlyeq w}} X^{w'}_{I,\varphi,x}.$$

Analogously, using Theorems 11.3 and 11.5 and 12.8 we obtain:

Theorem 12.16. *(a) The $X^w_{I,\varphi,x}$ for all W^J form a disjoint decomposition of $X_{I,\varphi,x}$ by nonsingular subvarieties of dimension* $\dim P + \ell(w)$.

(b) For any $w \in W^J$ we have

$$\overline{X^w_{I,\varphi,x}} = \coprod_{\substack{w' \in W^J \\ w' \preccurlyeq w}} X^{w'}_{I,\varphi,x}.$$

12.3 Frobenius

Keeping the notations of the preceding subsection, we now assume that k has positive characteristic and that $\varphi\colon G \to G$ is the Frobenius isogeny coming from a model G_0 of G over a finite subfield $\mathbb{F}_q \subset k$ of cardinality q. Then G_0 is quasi-split; hence we may, and do, assume that B and T come from subgroups of G_0 defined over $\mathbb{F}_q$ and therefore satisfy $\varphi(B) = B$ and $\varphi(T) = T$. We can thus take $\gamma := 1$.

In this case, our varieties $X_{I,\varphi,x}$ coincide with the varieties Z_I used in [MW] to study F-zips with additional structures. The isogeny ${}^{g_1}\tilde{\varphi}$ in the connected algebraic zip datum $\mathcal{Z}$ then has vanishing differential; hence $\mathcal{Z}$ is orbitally finite by Proposition 7.3. Thus by Theorem 7.5 each G^w is a single $E_{\mathcal{Z}}$-orbit, and so by Theorem 12.8 and Theorem 12.15 we deduce:

Theorem 12.17. *(a) If φ is the Frobenius isogeny associated to a model of G over a finite field, each $X^w_{I,\varphi,x}$ in Theorem 12.15 is a single $\Delta(G)$-orbit. In particular the set*

$$\big\{(P_I, {}^{\dot w}P_J, [\dot w \dot x g_1]) \bigm| w \in {}^IW\big\}$$

is a system of representatives for the action of $\Delta(G)$ on $X_{I,\varphi,x}$.

(b) For any $w \in {}^IW$, the closure of the orbit of $(P_I, {}^{\dot w}P_J, [\dot w \dot x g_1])$ is the union of the orbits of $(P_I, {}^{\dot w'}P_J, [\dot w' \dot x g_1])$ for those $w' \in {}^IW$ satisfying $w' \preccurlyeq w$.

Theorem 12.17 (a) was proved in [MW], Theorem 3 and (b) answers the question of the closure relation that was left open there.

12.4 Lusztig's varieties

Now we apply the results of Subsection 12.2 to the special case $\varphi = \mathrm{id}$. In this case we can choose $\gamma := 1$ and obtain $\bar\varphi = \mathrm{id}$. Then our varieties $X_{I,\varphi,x}$ coincide with the varieties $Z_{I,x,\delta}$ defined and studied by Lusztig in [Lus2]. There he defines a decomposition of $X_{I,\varphi,x}$ into a certain family of $\Delta(G)$-invariant subvarieties. In [He2], He shows how to parametrize this family by the set $W^{\delta(I)}$. We will denote the piece corresponding to $w \in W^{\delta(I)}$ in this parametrization by $\tilde X^w_{I,\varphi,x}$. (In [He2], He denotes $X_{I,\varphi,x}$ by $\tilde Z_{I,x,\delta}$ and $\tilde X^w_{I,\varphi,x}$ by $\tilde Z^w_{I,x,\delta}$.) We will show that this decomposition is the same as ours from Theorem 12.16 up to a different parametrization.

Lemma 12.18. *The map $w \mapsto wx$ induces a bijection $W^J \xrightarrow{\sim} W^{\delta(I)}$.*

Proof. Take any $w \in W^J$. Using Lemma 12.9 and (2.11) we get $wx\Phi^+_{\delta(I)} = w\Phi^+_J \subset \Phi^+$. By (2.11) this shows that $wx \in W^{\delta(I)}$. A similar argument shows that $wx^{-1} \in W^J$ for any $w \in W^{\delta(I)}$, which finishes the proof. $\square$

Theorem 12.19. *For any $w \in W^J$ we have $X^w_{I,\varphi,x} = \tilde X^{wx}_{I,\varphi,x}$.*

Proof. The statement makes sense by Lemma 12.18. Let $w \in W^J$ and $w' := wx \in W^{\delta(I)}$. In [He2], Proposition 1.7, He shows that

$$\tilde X^{w'}_{I,\varphi,x} = \Delta(G) \cdot \big\{(P_I, {}^{b\dot w'\dot x^{-1}}P_J, [b\dot w' g_1 b']) \bigm| b, b' \in B\big\}.$$

(In [He2], it is assumed that G is semi-simple and adjoint. But this assumption is not needed for the proof of Proposition 1.7 in [loc. cit.].) By acting on such a point $(P_I, {}^{b\dot w'\dot x^{-1}}P_J, [b\dot w' g_1 b'])$ with $\Delta(b^{-1})$ and using $w = w'x^{-1}$ we get

$$\tilde X^{w'}_{I,\varphi,x} = \Delta(G) \cdot \big\{(P_I, {}^{\dot w}P_J, [\dot w \dot x g_1 b']) \bigm| b' \in B\big\}.$$

Since $\gamma = 1$, comparison with Lemma 12.14 proves the claim. $\square$

From Theorem 12.16 we can now deduce the closure relation between the $\tilde X^w_{I,\varphi,x}$:

Theorem 12.20. *For any $w \in W^{\delta(I)}$ we have*

$$\overline{\tilde{X}^{w}_{I,\varphi,x}} = \coprod_{\substack{w' \in W^{\delta(I)} \\ w'x^{-1} \preccurlyeq wx^{-1}}} \tilde{X}^{w'}_{I,\varphi,x}.$$

In the special case $x = 1$ this result is due to He (see [He2], Proposition 4.6).

References

[BB] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics.* Springer, New York, 2005.

[Car] Roger W. Carter. *Finite groups of Lie type.* Wiley Classics Library. John Wiley & Sons Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[DCP] C. De Concini and C. Procesi. Complete symmetric varieties. In *Invariant theory (Montecatini, 1982)*, volume 996 of *Lecture Notes in Math.*, pages 1–44. Springer, Berlin, 1983.

[DDPW] Bangming Deng, Jie Du, Brian Parshall, and Jianpan Wang. *Finite dimensional Algebras and Quantum Groups*, volume 150 of *Mathematical Surveys and Monographs.* AMS, 2008.

[DG] Michel Demazure and Alexandre Grothendieck. *Schémas en groupes. III: Structure des schémas en groupes réductifs.* Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153. Springer-Verlag, Berlin, 1962/1964.

[EL] Sam Evens and Jiang-Hua Lu. On the variety of Lagrangian subalgebras. II. *Ann. Sci. École Norm. Sup. (4)*, 39(2):347–379, 2006.

[He1] Xuhua He. The character sheaves on the group compactification. *Adv. in Math.*, 207:805–827, 2006.

[He2] Xuhua He. The G-stable pieces of the wonderful compactification. *Trans. Amer. Math. Soc.*, 359(7):3005–3024 (electronic), 2007.

[He3] Xuhua He. Minimal length elements in some double cosets of Coxeter groups. *Adv. in Math.*, 215:469–503, 2007.

[Lus1] George Lusztig. Parabolic character sheaves. I. *Mosc. Math. J.*, 4(1):153–179, 311, 2004.

[Lus2] George Lusztig. Parabolic character sheaves. II. *Mosc. Math. J.*, 4(4):869–896, 981, 2004.

[LY1] Jiang-Hua Lu and Milen Yakimov. Partitions of the wonderful group compactification. *Transform. Groups*, 12(4):695–723, 2007.

[LY2] Jiang-Hua Lu and Milen Yakimov. Group orbits and regular partitions of poisson manifolds. *Comm. Math. Phys.*, 283(3):729–748, 2008.

[Moo] Ben Moonen. Group schemes with additional structures and Weyl group cosets. In *Moduli of abelian varieties (Texel Island, 1999)*, volume 195 of *Progr. Math.*, pages 255–298. Birkhäuser, Basel, 2001.

[MW] Ben Moonen and Torsten Wedhorn. Discrete invariants of varieties in positive characteristic. *Int. Math. Res. Not.*, (72):3855–3903, 2004.

[Ser] Jean-Pierre Serre. Espaces fibrés algébriques. *Séminaire Claude Chevalley*, 3:1–37, 1958.

[Slo] Peter Slodowy. *Simple singularities and simple algebraic groups*, volume 815 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[Spr1] T. A. Springer. *Linear algebraic groups*, volume 9 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1998.

[Spr2] T. A. Springer. Twisted conjugacy in simply connected groups. *Transform. Groups*, 11(3):539–545, 2006.

[Spr3] T. A. Springer. An extension of Bruhat's lemma. *J. Algebra*, 313(1):417–427, 2007.

[Ste] Robert Steinberg. *Endomorphisms of linear algebraic groups*. Memoirs of the American Mathematical Society, No. 80. American Mathematical Society, Providence, R.I., 1968.

[Vas] A. Vasiu. Mod p classification of Shimura F-crystals. *Math. Nachr.*, 283(8):1068–1113, 2010.

[Vie] Eva Viehmann. Truncations of level 1 of elements in the loop group of a reductive group. arXiv:math/0907.2331, 2009.

[VW] Eva Viehmann and Torsten Wedhorn. Ekedahl-Oort and Newton strata for Shimura varieties of PEL type. arXiv:1011.3230v1, 2010.

[Wed] Torsten Wedhorn. Specialization of F-Zips. arXiv:math/0507175v1, 2005.

[Zie] Paul Ziegler. Zip data. Master's thesis, ETH Zürich, 2010.

Richard Pink
Dept. of Mathematics
ETH Zürich
CH-8092 Zürich
Switzerland
pink@math.ethz.ch

Torsten Wedhorn
Dept. of Mathematics
University of Paderborn
D-33098 Paderborn
Germany
wedhorn@math.uni-paderborn.de

Paul Ziegler
Dept. of Mathematics
ETH Zürich
CH-8092 Zürich
Switzerland
paul.ziegler@math.ethz.ch

Euler Characteristics of Categories and Homotopy Colimits

Thomas M. Fiore, Wolfgang Lück, and Roman Sauer

Received: October 20, 2010

Communicated by Stephan Schwede

Abstract. In a previous article, we introduced notions of finiteness obstruction, Euler characteristic, and L^2-Euler characteristic for wide classes of categories. In this sequel, we prove the compatibility of those notions with homotopy colimits of $\mathcal{I}$-indexed categories where $\mathcal{I}$ is any small category admitting a finite $\mathcal{I}$-CW-model for its $\mathcal{I}$-classifying space. Special cases of our Homotopy Colimit Formula include formulas for products, homotopy pushouts, homotopy orbits, and transport groupoids. We also apply our formulas to Haefliger complexes of groups, which extend Bass–Serre graphs of groups to higher dimensions. In particular, we obtain necessary conditions for developability of a finite complex of groups from an action of a finite group on a finite category without loops.

2010 Mathematics Subject Classification: Primary: 18F30, 19J05; Secondary: 18G10, 19A49, 55U35, 19A22, 46L10.
Keywords and Phrases: finiteness obstruction, Euler characteristic of a category, L^2-Euler characteristic, projective class group, homotopy colimits of categories, Grothendieck construction, spaces over a category, Grothendieck fibration, complex of groups, small category without loops.

0. Introduction and Statement of Results

In our previous paper [16], we presented a unified conceptual framework for Euler characteristics of categories in terms of finiteness obstructions and projective class groups. Many excellent properties of our invariants stem from the homological origins of our approach: the theory of modules over categories and the dimension theory of modules over von Neumann algebras provide us with an array of tools and techniques. In the present paper, we additionally draw upon the homotopy theory of diagrams to prove the compatibility of our invariants with homotopy colimits.

If $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ is a diagram of categories (or more generally a pseudo functor into the 2-category of small categories), then our invariants of the homotopy colimit can be computed in terms of the invariants of the vertex categories $\mathcal{C}(i)$. In particular, our Homotopy Colimit Formula, Theorem 4.1, states

$$\chi\big(\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{C};R\big) = \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \chi(\mathcal{C}(i_\lambda);R) \tag{0.1}$$

under certain hypotheses. The set Λ_n indexes the $\mathcal{I}$-n-cells of a finite $\mathcal{I}$-CW-model $E\mathcal{I}$ for the $\mathcal{I}$-classifying space of $\mathcal{I}$, that is, we have a functor $E\mathcal{I}\colon \mathcal{I}^{\mathrm{op}} \to \mathsf{SPACES}$ which is inductively built by gluing finitely many cells of the form $\mathrm{mor}_{\mathcal{I}}(-,i_\lambda)\times D^n$ for $\lambda\in\Lambda_n$, and moreover $E\mathcal{I}(i)\simeq *$ for all objects i of $\mathcal{I}$. Similar formulas hold for the finiteness obstruction, the functorial Euler characteristic, the functorial L^2-Euler characteristic, and the L^2-Euler characteristic.

Motivation for such a formula is provided by the classical Inclusion-Exclusion Principle: if A, B, and $A\cap B$ are finite simplicial complexes, then one has

$$\chi(A\cup B) = \chi(A)+\chi(B)-\chi(A\cap B).$$

However, one cannot expect the Euler characteristic to be compatible with pushouts, even in the simplest cases. The pushout in CAT of the discrete categories

$$\{*\} \leftarrow \{y,z\} \to \{*'\}$$

is a point, but $\chi(\text{point}) \neq 1+1-2$. On the other hand, their *homotopy* pushout in CAT is the category whose objects and nontrivial morphisms are pictured below.

$$\begin{tikzcd} y \arrow[r] \arrow[d] & *' \\ * & z \arrow[l] \arrow[u] \end{tikzcd}$$

The classifying space of this category has the homotopy type of S^1, so that

$$\chi(\text{homotopy pushout}) = \chi(\{*\})+\chi(\{*'\})-\chi(\{y,z\})$$

is true. In fact, the formula for homotopy pushouts is a special case of (0.1): the category $\mathcal{I}=\{1\leftarrow 0\to 2\}$ admits a finite model with $\Lambda_0=\{1,2\}$ and $\Lambda_1=\{0\}$, as constructed in Example 2.6. See Example 5.4 for the homotopy pushout formulas of the other invariants.

The Homotopy Colimit Formula in Theorem 4.1 has many applications beyond homotopy pushouts. Other special cases are formulas for Euler characteristics of products, homotopy orbits, and transport groupoids. Our formulas also have ramifications for the developability of Haefliger's *complexes of groups* in geometric group theory. If a group G acts on an M_κ-polyhedral complex by isometries preserving cell structure, and if each $g\in G$ fixes each cell pointwise that g fixes setwise, then the quotient space is also an M_κ-polyhedral complex, see Bridson–Haefliger [11, page 534]. Let us call the quotient M_κ-polyhedral complex Q. To each face $\overline{\sigma}$ of Q, one can assign the stabilizer G_σ of a chosen representative cell σ. This assignment, along with the various conjugated inclusions of groups obtained from face inclusions, is called the *complex of groups associated to the group action.* It is a pseudo functor from the poset of faces of Q into groups. In the finite case, the Euler characteristic and L^2-Euler characteristic of the homotopy colimit can be computed in terms of the original complex and the order of the group. We prove this in Theorem 8.30. Homotopy colimits of complexes of groups play a special role in Haefliger's theory, see the discussion after Definition 8.9.

In Section 1, we review the notions and results from [16] that we need in this sequel. Explanations of the finiteness obstruction, the functorial Euler characteristic, the Euler characteristic, the functorial L^2-Euler characteristic, the L^2-Euler characteristic, and the necessary theorems are all contained in Section 1 in order to make the present paper self-contained. Section 2 is dedicated to an assumption in the Homotopy Colimit Formula, namely the requirement that a finite $\mathcal{I}$-CW-model exists for the $\mathcal{I}$-classifying space of $\mathcal{I}$. We recall the notion of $\mathcal{I}$-CW-complex, present various examples, and prove that finite models are preserved under equivalences of categories. Homotopy colimits of diagrams of categories are recalled in Section 3. The homotopy colimit construction in CAT is the same as the Grothendieck construction, or the category of elements. Thomason proved that the homotopy colimit construction has the expected properties. We prove our main theorem, the Homotopy Colimit Formula, in Section 4, work out various examples in Section 5, and derive the generalized Inclusion-Exclusion Principle in Section 6. We review the groupoid cardinality of Baez–Dolan and the Euler characteristic of Leinster in Section 7, and compare our Homotopy Colimit Formula with Leinster's compatibility with Grothendieck fibrations in terms of weightings. We apply our results to Haefliger complexes of groups in Section 8 to prove Theorems 8.30 and 8.35, which express Euler characteristics of complexes of groups associated to group actions in terms of the initial data.

Acknowledgements. All three authors were supported by the Sonderforschungsbereich 878 – Groups, Geometry and Actions – and the Leibniz-Preis of Wolfgang Lück.

Thomas M. Fiore was supported at the University of Chicago by NSF Grant DMS-0501208. At the Universitat Autònoma de Barcelona he was supported by grant SB2006-0085 of the Spanish Ministerio de Educación y Ciencia under the Programa Nacional de ayudas para la movilidad de profesores de universidad e investigadores españoles y extranjeros. Thomas M. Fiore also thanks the Centre de Recerca Matemàtica in Bellaterra (Barcelona) for its hospitality during the CRM Research Program on Higher Categories and Homotopy Theory in 2007-2008, where he heard Tom Leinster speak about Euler characteristics. He also thanks the Max Planck Insitut für Mathematik for its hospitality and support during his stay in Summer 2010.

Contents

1. The Finiteness Obstruction and Euler Characteristics

We quickly recall the main definitions and results needed from our first paper [16] in order to make this article as self-contained as possible. See [16] for proofs and more detail.

Throughout this paper, let Γ be a category and R an associative, commutative ring with identity. The first ingredient we need is the theory of modules over categories developed by Lück [23], and recalled in [16]. An $R\Gamma$-*module* is a contravariant functor from Γ into the category of left R-modules. For example, if Γ is a group G viewed as a one-object category, then an $R\Gamma$-module is the same as a right module over the group ring RG. An $R\Gamma$-module P is *projective* if it is projective in the usual sense of homological algebra, that is, for every surjective $R\Gamma$-morphism $p\colon M \to N$ and every $R\Gamma$-morphism $f\colon P \to N$ there exists an $R\Gamma$-morphism $\overline{f}\colon P \to M$ such that $p \circ \overline{f} = f$. An $R\Gamma$-module M is *finitely generated* if there is a surjective $R\Gamma$-morphism $B(C) \to M$ from an $R\Gamma$-module $B(C)$ that is free on a collection C of sets indexed by $\mathrm{ob}(\Gamma)$ such that $\coprod_{x\in\mathrm{ob}(\Gamma)} C_x$ is finite. Explicitly, the *free* $R\Gamma$-*module on the* $\mathrm{ob}(\Gamma)$-*set* C is

$$B(C) := \bigoplus_{x\in\mathrm{ob}(\Gamma)} \bigoplus_{C_x} R\,\mathrm{mor}_\Gamma(?,x). \tag{1.1}$$

A contravariant $R\Gamma$-module may be tensored with a covariant $R\Gamma$-module to obtain an R-module: if $M\colon \Gamma^{\mathrm{op}} \to R\text{-}\mathsf{MOD}$ and $N\colon \Gamma \to R\text{-}\mathsf{MOD}$ are functors, then the *tensor product* $M \otimes_{R\Gamma} N$ is the quotient of the R-module

$$\bigoplus_{x\in\mathrm{ob}(\Gamma)} M(x) \otimes_R N(x)$$

by the R-submodule generated by elements of the form

$$(M(f)m) \otimes n - m \otimes (N(f)n)$$

where $f : x \to y$ is a morphism in Γ, $m \in M(y)$, and $n \in N(x)$.
Finite projective resolutions of the constant $R\Gamma$-module $\underline{R}$ play a special role in our theory of Euler characteristic for categories. A resolution P_* of an $R\Gamma$-module M is said to be *finite projective* if it has finite length and each P_n is finitely generated and projective. We say that a category Γ *is of type (FP$_R$)* if the constant $R\Gamma$-module $\underline{R}\colon \Gamma^{\mathrm{op}} \to R\text{-}\mathsf{MOD}$ with value R admits a finite projective resolution. Categories in which every endomorphism is an isomorphism, the so-called *EI-categories*, provide important examples. Finite EI-categories in which $|\operatorname{aut}(x)|$ is invertible in R for each object x are of type (FP$_R$). Further examples of categories of type (FP$_R$) include categories Γ which admit a finite Γ-CW-model for the classifying Γ-space $E\Gamma$ (see Section 2 and Examples 2.4, 2.5, 2.6, and 2.7). In fact, such categories Γ are even *of type (FF$_R$)*: the cellular chains on a finite Γ-CW-model for $E\Gamma$ provide a finite free resolution of $\underline{R}$. In general, if a category is of type (FF$_{\mathbb{Z}}$), then it is of type (FF$_R$) for any associative, commutative ring R with identity.
A home for the finiteness obstruction of a category Γ is provided by the *projective class group* $K_0(R\Gamma)$. The generators of this abelian group are the isomorphism classes of finitely generated projective $R\Gamma$-modules and the relations are given by expressions $[P_0] - [P_1] + [P_2] = 0$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective $R\Gamma$-modules.

Definition 1.2 (Finiteness obstruction of a category). Let Γ be a category of type (FP$_R$) and P_* a finite projective resolution of the constant $R\Gamma$-module $\underline{R}$. The *finiteness obstruction of* Γ *with coefficients in* R is

$$o(\Gamma; R) := \sum_{n \geq 0} (-1)^n \cdot [P_n] \ \in K_0(R\Gamma).$$

We also use the notation $[\underline{R}]$, or simply $[R]$, to denote the finiteness obstruction $o(\Gamma; R)$. The finiteness obstruction, when it exists, does not depend on the choice P_* of finite projective resolution of $\underline{R}$.

The finiteness obstruction is compatible with most everything one could hope for. If $F\colon \Gamma_1 \to \Gamma_2$ is a right adjoint, and Γ_1 is of type (FP$_R$), then Γ_2 is of type (FP$_R$) and $F_* o(\Gamma_1; R) = o(\Gamma_2; R)$ (here the group homomorphism F_* is induced by induction with F). Since an equivalence of categories is a right adjoint (and also a left adjoint), a particular instance of the previous sentence is: if $F\colon \Gamma_1 \to \Gamma_2$ is an equivalence of categories, then Γ_1 is of type (FP$_R$) if and only if Γ_2 is, and in this case $F_* o(\Gamma_1; R) = o(\Gamma_2; R)$. The finiteness obstruction is also compatible with finite coproducts of categories, finite products of categories, restriction along admissable functors, and homotopy colimits, as we prove in Theorem 4.1. If G is a finitely presented group of type (FP$_{\mathbb{Z}}$), then Wall's finiteness obstruction $o(BG)$ is the same as $o(\widehat{G}; \mathbb{Z})$, which is the finiteness obstruction of G viewed as a one-object category $\widehat{G}$ with morphisms G.

The finiteness obstruction in Definition 1.2 is a special case of the finiteness obstruction of a finitely dominated $R\Gamma$-chain complex C, denoted $o(C) \in K_0(R\Gamma)$. The image of $o(C)$ in the reduced K-theory $\tilde{K}_0(R\Gamma)$ vanishes if and only if C is $R\Gamma$-homotopy equivalent to a finite free $R\Gamma$-chain complex, see [23, Chapter 11].

We will occasionally work with directly finite categories. A category is called *directly finite* if for any two objects x and y and morphisms $u\colon x \to y$ and $v\colon y \to x$ the implication $vu = \mathrm{id}_x \implies uv = \mathrm{id}_y$ holds. If Γ_1 and Γ_2 are equivalent categories, then Γ_1 is directly finite if and only if Γ_2 is directly finite. Examples of directly finite categories include groupoids, and more generally EI-categories.

A key result in the theory of modules over an EI-category is Lück's splitting of the projective class group of Γ into the projective class groups of the automorphism groups $\mathrm{aut}_\Gamma(x)$, one ach isomorphism class of objects. We next recall the relevant maps and notation. For $x \in \mathrm{ob}(\Gamma)$, we denote $R\,\mathrm{aut}_\Gamma(x)$ by $R[x]$ for simplicity. The *splitting functor at* $x \in \mathrm{ob}(\Gamma)$

$$S_x\colon \mathsf{MOD}\text{-}R\Gamma \to \mathsf{MOD}\text{-}R[x], \tag{1.3}$$

maps an $R\Gamma$-module M to the quotient of the R-module $M(x)$ by the R-submodule generated by all images of R-module homomorphisms $M(u)\colon M(y) \to M(x)$ induced by all non-invertible morphisms $u\colon x \to y$ in Γ. The right $R[x]$-module structure on $M(x)$ induces a right $R[x]$-module structure on S_xM. Note that S_xM is an $R[x]$-module, not an $R\Gamma$-module. The functor S_x respects direct sums, sends epimorphisms to epimorphisms, and sends free modules to free modules. If Γ is directly finite, then S_x also preserves finitely generated and projective. The *extension functor at* $x \in \mathrm{ob}(\Gamma)$

$$E_x\colon \mathsf{MOD}\text{-}R[x] \to \mathsf{MOD}\text{-}R\Gamma \tag{1.4}$$

maps an $R[x]$-module N to the $R\Gamma$-module $N \otimes_{R[x]} R\,\mathrm{mor}_\Gamma(?, x)$. The functor E_x respects direct sums, sends epimorphisms to epimorphisms, sends free modules to free modules, and preserves finitely generated and projective. If Γ is directly finite, and P is a projective $R[x]$-module, then there is a natural isomorphism $P \cong S_xE_xP$ compatible with direct sums.

Theorem 1.5 (Splitting of $K_0(R\Gamma)$ for EI-categories, Theorem 10.34 on page 196 of Lück [23]). *If Γ is an EI-category, then the group homomorphisms*

$$K_0(R\Gamma) \underset{E}{\overset{S}{\rightleftarrows}} \mathrm{Split}\,K_0(R\Gamma) := \bigoplus_{\overline{x} \in \mathrm{iso}(\Gamma)} K_0(R\,\mathrm{aut}_\Gamma(x))$$

defined by

$$S[P] = \{[S_xP] \mid \overline{x} \in \mathrm{iso}(\Gamma)\}$$

and

$$E\{[Q_x] \mid \overline{x} \in \mathrm{iso}(\Gamma)\} = \sum_{\overline{x} \in \mathrm{iso}(\Gamma)} [E_xQ_x],$$

are isomorphisms and inverse to one another. They are covariantly natural with respect to functors between EI-categories.

Remark 1.6. If Γ is not an EI-category, then the splitting homomorphism $S\colon K_0(R\Gamma) \to \operatorname{Split} K_0(R\Gamma)$ may not be an isomorphism. However, S is covariantly natural with respect to functors between directly finite categories, see [16, Lemma 3.15].

The splitting functors S_x allow us to define the notion of $R\Gamma$-rank $\operatorname{rk}_{R\Gamma}$ for finitely generated $R\Gamma$-modules, which in turn allows the definition of the functorial Euler characteristic, as we explain next. We assume a fixed notion of a rank $\operatorname{rk}_R(N) \in \mathbb{Z}$ for finitely generated R-modules N such that $\operatorname{rk}_R(R) = 1$ and $\operatorname{rk}_R(N_1) = \operatorname{rk}_R(N_0) + \operatorname{rk}_R(N_2)$ for any sequence $0 \to N_0 \to N_1 \to N_2 \to 0$ of finitely generated R-modules. If R is a commutative principal ideal domain, we use $\operatorname{rk}_R(N) := \dim_F(F \otimes_R N)$, where F is the quotient field of R. Let $U(\Gamma)$ be the free abelian group on the set of isomorphism classes of objects in Γ, that is $U(\Gamma) := \mathbb{Z}\operatorname{iso}(\Gamma)$. The augmentation homomorphism $\epsilon\colon U(\Gamma) \to \mathbb{Z}$ adds up the components of an element of $U(\Gamma)$.

Definition 1.7 (Rank of a finitely generated $R\Gamma$-module). If M is a finitely generated $R\Gamma$-module M, then its *$R\Gamma$-rank* is

$$\operatorname{rk}_{R\Gamma}(M) := \left\{\operatorname{rk}_R(S_x M \otimes_{R[x]} R) \mid \overline{x} \in \operatorname{iso}(\Gamma)\right\} \quad \in U(\Gamma).$$

Definition 1.8 (The (functorial) Euler characteristic of a category). Suppose that Γ is of type (FP_R). The *functorial Euler characteristic of* Γ *with coefficients in* R is the image of the finiteness obstruction $o(\Gamma; R) \in K_0(R\Gamma)$ under the homomorphism $\operatorname{rk}_{R\Gamma}\colon K_0(R\Gamma) \to U(\Gamma)$, that is

$$\chi_f(\Gamma; R) := \operatorname{rk}_{R\Gamma} o(\Gamma; R) = \left\{ \sum_{n \geq 0} (-1)^n \operatorname{rk}_R(S_x P_n \otimes_{R[x]} R) \mid \overline{x} \in \operatorname{iso}(\Gamma) \right\} \\ \in U(\Gamma),$$

where P_* is any finite projective $R\Gamma$-resolution of the constant $R\Gamma$-module $\underline{R}$. The *Euler characteristic of* Γ *with coefficients in* R is the sum of the components of the functorial Euler characteristic, that is,

$$\chi(\Gamma; R) := \epsilon(\chi_f(\Gamma; R)) = \sum_{\overline{x} \in \operatorname{iso}(\Gamma)} \sum_{n \geq 0} (-1)^n \operatorname{rk}_R(S_x P_n \otimes_{R[x]} R).$$

For example, if $\mathcal{G}$ is a finite groupoid, then $\chi_f(\mathcal{G}) \in U(\mathcal{G})$ is $(1, 1, \ldots, 1)$, and $\chi(\mathcal{G})$ counts the isomorphism classes of objects, or equivalently the connected components, of $\mathcal{G}$.

Theorem 1.9 (Theorem 4.20 of Fiore–Lück–Sauer [16]). *Let R be a Noetherian ring and Γ a directly finite category of type (FP_R). Then the Euler characteristic and topological Euler characteristic of Γ agree. That is, $H_n(B\Gamma; R)$ is a finitely generated R-module for every $n \geq 0$, there exists a natural number d*

with $H_n(B\Gamma;R)=0$ for all $n>d$, and

$$\chi(\Gamma;R)=\chi(B\Gamma;R)=\sum_{n\geq 0}(-1)^n\cdot \operatorname{rk}_R(H_n(B\Gamma;R))\in\mathbb{Z}.$$

Here $\chi(\Gamma;R)$ is defined in Definition 1.8 and $B\Gamma$ denotes the geometric realization of the nerve of Γ.

The functorial Euler characteristic and Euler characteristic have many desirable properties. They are invariant under equivalence of categories and are compatible with finite products and finite coproducts. As we prove in Theorem 4.1, they are also compatible with homotopy colimits.

The L^2-Euler characteristic, which is in some sense the better invariant, can be defined similarly by taking $R=\mathbb{C}$ and using the L^2-rank $\operatorname{rk}_\Gamma^{(2)}$ rather than the $R\Gamma$-rank. For this we need group von Neumann algebras and their dimension theory from Lück [24] and [25], as recalled in our first paper [16] for the purpose of Euler characteristics. If G is a group, its *group von Neumann algebra*

$$\mathcal{N}(G)=\mathcal{B}(l^2(G))^G$$

is the algebra of bounded operators on $l^2(G)$ that are equivariant with respect to the right G-action. If G is finite, $\mathcal{N}(G)$ is the group ring $\mathbb{C}G$. In any case, the group ring $\mathbb{C}G$ embeds as a subring of $\mathcal{N}(G)$ by sending $g\in G$ to the isometric G-equivariant operator $l^2(G)\to l^2(G)$ given by left multiplication with g. In particular, we can view $\mathcal{N}(G)$ as a $\mathbb{C}G$-$\mathcal{N}(G)$-bimodule and tensor $\mathbb{C}G$-modules on the right with $\mathcal{N}(G)$. If G is the automorphism group of an object in Γ, then we write $\mathcal{N}(x)$ for $\mathcal{N}\big(\operatorname{aut}_\Gamma(x)\big)$.

The *von Neumann dimension*, $\dim_{\mathcal{N}(G)}$, is a function that assigns to *every* right $\mathcal{N}(G)$-module M a non-negative real number of ∞. It is the unique such function which satisfies Hattori-Stallings rank, additivity, cofinality, and continuity. If G is a finite group, then $\mathcal{N}(G)=\mathbb{C}G$ and we get for a $\mathbb{C}G$-module M the von Neumann dimension

$$\dim_{\mathcal{N}(G)}(M)=\frac{1}{|G|}\cdot\dim_{\mathbb{C}}(M),$$

where $\dim_{\mathbb{C}}$ is the dimension of M viewed as a complex vector space. A category Γ is said to be *of type* (L^2) if for one (and hence every) projective $\mathbb{C}\Gamma$-resolution P_* of the constant $\mathbb{C}\Gamma$-module $\underline{\mathbb{C}}$ we have

$$\sum_{\overline{x}\in\operatorname{iso}(\Gamma)}\sum_{n\geq 0}\dim_{\mathcal{N}(x)}H_n\big(S_xP_*\otimes_{\mathbb{C}[x]}\mathcal{N}(x)\big)<\infty.$$

Note that the projective resolution P_* of $\underline{\mathbb{C}}$ is not required to be of finite length, nor finitely generated. Examples of categories of type (L^2) include finite EI-categories, in particular finite posets and finite groupoids. Infinite categories can also be of type (L^2), for example any (small) groupoid with finite automorphism groups such that

$$\sum_{\overline{x}\in\operatorname{iso}(\mathcal{G})}\frac{1}{|\operatorname{aut}_{\mathcal{G}}(x)|}<\infty \tag{1.10}$$

holds is of type (L^2). The condition of type (L^2) is weaker than ($\mathrm{FP}_{\mathbb{C}}$), since any directly finite category of type ($\mathrm{FP}_{\mathbb{C}}$) is also of type (L^2).

DEFINITION 1.11 (The (functorial) L^2-Euler characteristic of a category). Suppose that Γ is of type (L^2). Define

$$U^{(1)}(\Gamma) := \left\{ \sum_{\overline{x} \in \mathrm{iso}(\Gamma)} r_{\overline{x}} \cdot \overline{x} \;\middle|\; r_{\overline{x}} \in \mathbb{R}, \sum_{\overline{x} \in \mathrm{iso}(\Gamma)} |r_{\overline{x}}| < \infty \right\} \subseteq \prod_{\bar{x} \in \mathrm{iso}(\Gamma)} \mathbb{R}.$$

The *functorial L^2-Euler characteristic of* Γ is

$$\chi_f^{(2)}(\Gamma) := \left\{ \sum_{n \geq 0} (-1)^n \dim_{\mathcal{N}(x)} H_n\bigl(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\bigr) \mid \bar{x} \in \mathrm{iso}(\Gamma) \right\} \in U^{(1)}(\Gamma),$$

where P_* is any projective $\mathbb{C}\Gamma$-resolution of the constant $\mathbb{C}\Gamma$-module $\underline{\mathbb{C}}$. The *L^2-Euler characteristic of* Γ is the sum over $\bar{x} \in \mathrm{iso}(\Gamma)$ of the components of the functorial Euler characteristic, that is,

$$\chi^{(2)}(\Gamma) := \sum_{\overline{x} \in \mathrm{iso}(\Gamma)} \sum_{n \geq 0} (-1)^n \dim_{\mathcal{N}(x)} H_n\bigl(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\bigr).$$

If $\mathcal{G}$ is a groupoid such that (1.10) holds, then the functorial L^2-Euler characteristic $\chi_f^{(2)}(\mathcal{G}) \in \prod_{\overline{x} \in \mathrm{iso}(\mathcal{G})} \mathbb{R}$ has at $\overline{x} \in \mathrm{iso}(\mathcal{G})$ the value $1/|\mathrm{aut}_{\mathcal{G}}(x)|$. The L^2-Euler characteristic is

$$\chi^{(2)}(\mathcal{G}) = \sum_{\overline{x} \in \mathrm{iso}(\mathcal{G})} \frac{1}{|\mathrm{aut}_{\mathcal{G}}(x)|}. \tag{1.12}$$

See Lemma 7.5 for an explicit formula for $\chi^{(2)}(\Gamma)$ in the case of a finite, skeletal EI-category Γ in which the left $\mathrm{aut}_\Gamma(y)$-action on $\mathrm{mor}_\Gamma(x, y)$ is free for every two objects $x, y \in \mathrm{ob}(\Gamma)$.

DEFINITION 1.13 (L^2-rank of a finitely generated $\mathbb{C}\Gamma$-module). Let M be a finitely generated $\mathbb{C}\Gamma$-module M. Its *L^2-rank* is

$$\mathrm{rk}_\Gamma^{(2)}(M) := \bigl\{\dim_{\mathcal{N}(x)}(S_x M \otimes_{\mathbb{C}[x]} \mathcal{N}(x)) \mid \bar{x} \in \mathrm{iso}(\Gamma)\bigr\} \in U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{R} = \bigoplus_{\mathrm{iso}(\Gamma)} \mathbb{R}.$$

THEOREM 1.14 (Relating the finiteness obstruction and the L^2-Euler characteristic, Theorem 5.22 of Fiore–Lück–Sauer [16]). *Suppose that Γ is a directly finite category of type ($FP_{\mathbb{C}}$). Then Γ is of type (L^2) and the image of the finiteness obstruction $o(\Gamma; \mathbb{C})$ (see Definition 1.2) under the homomorphism*

$$\mathrm{rk}_\Gamma^{(2)} \colon K_0(\mathbb{C}\Gamma) \to U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{R} = \bigoplus_{\overline{x} \in \mathrm{iso}(\Gamma)} \mathbb{R}$$

is the functorial L^2-Euler characteristic $\chi_f^{(2)}(\Gamma)$.

The L^2-Euler characteristic agrees with the groupoid cardinality of Baez–Dolan [4] and the Euler characteristic of Leinster [21] in certain cases, see Lemma 7.5 and Section 7. In particular, the Baez–Dolan groupoid cardinality of a groupoid satisfying (1.10) is (1.12). However, the Baez–Dolan groupoid cardinality and Leinster's Euler characteristic $\chi_L(\Gamma)$ only depend on the underlying graph of Γ, whereas our invariants truly depend on the category structure. For instance, χ_L is $\frac{1}{2}$ for both the two-element monoid $(\mathbb{Z}/2, \times)$ and the two-element group $(\mathbb{Z}/2, +)$, whereas $\chi^{(2)}$ is 1 respectively $\frac{1}{2}$. The distinction can already be seen on the level of the finiteness obstructions. The Euler characteristic $\chi(-)$ and topological Euler characteristic $\chi(B-)$ can also distinguish categories with the same underlying directed graph as in the following example. For $S = \{1,2,3,4\}$, $G_1 = \langle(1234)\rangle$, $G_2 = \langle(12),(34)\rangle$, and $k = 1,2$, let Γ_k be the EI-category with objects x and y and $\operatorname{mor}(x,y) := S$, $\operatorname{mor}(x,x) := \{\mathrm{id}_x\}$, $\operatorname{mor}(y,y) := G_k$, and $\operatorname{mor}(y,x) = \emptyset$. Composition in Γ_k is the composition in G_k and the left G_k-action on S, that is, Γ_k is the EI-category associated to the respective G_k-$\{1\}$-biset S as in Subsection 6.4 of Fiore–Lück–Sauer [16]. Then Γ_1 and Γ_2 have the same underlying directed graph but $\chi(\Gamma_1;\mathbb{Q}) = \chi(B\Gamma_1;\mathbb{Q}) = 1$ and $\chi(\Gamma_2;\mathbb{Q}) = \chi(B\Gamma_2;\mathbb{Q}) = 0$ by Theorem 6.23 (iii) of Fiore–Lück–Sauer [16]. An infinite example of categories with the same underlying graph but different Euler characteristics is provided by the groups $\mathbb{Z}$ and $\mathbb{Z}*\mathbb{Z}$, each of which admits a finite Γ-CW-model for its respective Γ-classifying space. The categories $\widehat{\mathbb{Z}}$ and $\widehat{\mathbb{Z}*\mathbb{Z}}$ have the same underlying directed graph, but we have $\chi^{(2)}(\widehat{\mathbb{Z}}) = 0 \neq \chi^{(2)}(\widehat{\mathbb{Z}*\mathbb{Z}})$, and similarly for χ. Typically, the Euler characteristic of a category Γ_{free} free on a directed graph (V,E) is the same as the Euler characteristic of the directed graph (V,E). For the topological Euler characteristic this is clearly true, since $B\Gamma_{\mathrm{free}}$ is homotopy equivalent to the topological realization $|(V,E)|$. If Γ_{free} is directly finite and R is Noetherian, then we also have $\chi(\Gamma_{\mathrm{free}}) = \chi(|(V,E)|)$ by Theorem 1.9. For example for the directed graph with one vertex and one arrow we have $\chi(\widehat{\mathbb{N}}) = 0 = \chi(S^1)$.

The functorial L^2-Euler characteristic and the L^2-Euler characteristic have many desirable properties. They are invariant under equivalence of categories and are compatible with finite products, finite coproducts, and isofibrations and coverings between finite groupoids. We prove in Theorem 4.1 the compatibility with homotopy colimits. In the case of a group G, the L^2-Euler characteristic of $\widehat{G}$ coincides with the classical L^2-Euler characteristic of G, which is $1/|G|$ when G is finite. The L^2-Euler characteristic is also closely related to the geometry and topology of the classifying space for proper G-actions, namely the functorial L^2-Euler characteristic of the proper orbit category $\underline{\mathrm{Or}}(G)$ is equal to the equivariant Euler characteristic of the classifying space $\underline{E}G$ for proper G-actions, whenever $\underline{E}G$ admits a finite G-CW-model.

The question arises: what are sufficient conditions for the Euler characteristic and L^2-Euler characteristic to coincide with the Euler characteristic of the classifying space? This is answered in the following Theorem.

Theorem 1.15 (Invariants agree for directly finite and type ($\mathrm{FF}_{\mathbb{Z}}$), Theorem 5.25 of Fiore–Lück–Sauer [16]). *Suppose Γ is directly finite and of type ($FF_{\mathbb{Z}}$). Then the functorial L^2-Euler characteristic of Definition 1.11 coincides with the functorial Euler characteristic of Definition 1.8 for any associative,* *commutative ring R with identity*

$$\chi_f^{(2)}(\Gamma) = \chi_f(\Gamma; R) \in U(\Gamma) \subseteq U^{(1)}(\Gamma),$$

and thus $\chi^{(2)}(\Gamma) = \chi(\Gamma; R)$ in Definition 1.11 and Definition 1.8.
If R is additionally Noetherian, then

$$\chi(B\Gamma; R) = \chi(\Gamma; R) = \chi^{(2)}(\Gamma). \tag{1.16}$$

Moreover, if Γ is merely of type ($FF_{\mathbb{C}}$) rather than ($FF_{\mathbb{Z}}$), then equation (1.16) *holds for any Noetherian ring R containing $\mathbb{C}$.*

Any category Γ which admits a finite Γ-CW-model in the sense of Section 2 is of type (FF_R) for any ring R, by an application of the cellular R-chain functor. Thus, Theorem 1.15 applies to any directly finite category Γ which admits a finite Γ-CW-model. For example, finite categories without loops are directly finite and admit finite models (Lemma 8.4 and Theorem 8.5), so equation (1.16) holds for instance for $\{j \rightrightarrows k\}$, $\{k \leftarrow j \rightarrow \ell\}$, and finite posets. The monoid $\mathbb{N}$ and group $\mathbb{Z}$, viewed as one-object categories $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{Z}}$, are also directly finite and admit finite models (see Example 2.8), so we have

$$0 = \chi(S^1; R) = \chi(B\widehat{\mathbb{N}}; R) = \chi(\widehat{\mathbb{N}}; R) = \chi^{(2)}(\widehat{\mathbb{N}})$$

and

$$0 = \chi(S^1; R) = \chi(B\widehat{\mathbb{Z}}; R) = \chi(\widehat{\mathbb{Z}}; R) = \chi^{(2)}(\widehat{\mathbb{Z}})$$

($B\widehat{\mathbb{N}} \to B\widehat{\mathbb{Z}} \simeq S^1$ is a homotopy equivalence by Quillen's Theorem A, see Rabrenović [35, Proposition 10]). The equations $\chi(\widehat{\mathbb{N}}; R) = 0 = \chi^{(2)}(\widehat{\mathbb{N}})$ and $\chi(\widehat{\mathbb{Z}}; R) = 0 = \chi^{(2)}(\widehat{\mathbb{Z}})$ also follow from Example 5.3, since the finite models for $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{Z}}$ in Example 2.8 each have one $\mathcal{I}$-0-cell and one $\mathcal{I}$-1-cell.

We may use Theorem 1.15 to obtain an explicit formula for Euler characteristics of finite categories without loops as follows. Let Γ be a finite category without loops, and choose a skeleton Γ'. Let $c_n(\Gamma')$ denote the number of paths

$$i_0 \to i_1 \to i_2 \to \cdots \to i_n$$

of n-many non-identity morphisms in Γ'. Then $c_n(\Gamma')$ is the number of n-cells in the CW-complex $B\Gamma'$, and we have

$$\chi(\Gamma; R) = \chi^{(2)}(\Gamma) = \chi(B\Gamma; R) = \chi(B\Gamma'; R) = \sum_{n \geq 0} (-1)^n c_n(\Gamma'). \tag{1.17}$$

See [21, Corollary 1.5] for a different derivation of this formula for Leinster's Euler characteristic $\chi_L(\Gamma)$ in the case Γ was already skeletal. See also Examples 5.3 and 8.7 where skeletality of $\mathcal{I}$ is not required.

REMARK 1.18 (Homotopy Invariance). If $F : \Gamma_1 \to \Gamma_2$ is a functor such that BF is a homotopy equivalence, and both Γ_1 and Γ_2 are of type (FP_R), and if

$$\chi(\Gamma_1; R) = \chi(B\Gamma_1; R) \quad \text{and} \quad \chi(\Gamma_2; R) = \chi(B\Gamma_2; R),$$

then clearly $\chi(\Gamma_1; R) = \chi(\Gamma_2; R)$. However, it is possible for two categories to be homotopy equivalent, one of which is (FP_R) and the other is not, so that one has a notion of Euler characteristic and the other does not. In Section 10 of Fiore–Lück–Sauer [16] such an example is discussed.

2. Spaces over a Category

An important hypothesis in our Homotopy Colimit Formula involves the idea of a space over a category, see Davis–Lück [14]. Namely, we assume that the indexing category $\mathcal{I}$ for the diagram $\mathcal{C}$ of categories admits a finite $\mathcal{I}$-CW-model for its $\mathcal{I}$-classifying space. Essentially this means it is possible to functorially assign a contractible CW-complex $E\mathcal{I}(i)$ to each $i \in \operatorname{ob}(\mathcal{I})$, and moreover, these local CW-complexes are constructed globally by gluing $\mathcal{I}$-n-cells of the form $\operatorname{mor}_{\mathcal{I}}(-,i_\lambda) \times D^n$ onto the already globally constructed $(n-1)$-skeleton $E\mathcal{I}_n$. The Homotopy Colimit Formula then expresses the invariants of the homotopy colimit of $\mathcal{C}$ in terms of the invariants of the categories $\mathcal{C}(i_\lambda)$ at the base objects i_λ for $E\mathcal{I}$.

The gluing described above takes place in the more general category of $\mathcal{I}$-spaces. A *(contravariant) $\mathcal{I}$-space* is a contravariant functor from $\mathcal{I}$ to the category SPACES of (compactly generated) topological spaces. As usual, we will always work in the category of compactly generated spaces (see Steenrod [39]). A *map between $\mathcal{I}$-spaces* is a natural transformation. Given an object $i \in \operatorname{ob}(\mathcal{I})$, we obtain an $\mathcal{I}$-space $\operatorname{mor}_{\mathcal{I}}(?,i)$ which assigns to an object j the discrete space $\operatorname{mor}_{\mathcal{I}}(j,i)$.

The next definition is taken from Davis–Lück [14, Definition 3.2], where an $\mathcal{I}$-CW-complex is called a free $\mathcal{I}$-CW-complex and we will omit the word free here. The more general notion of $\mathcal{I}$-CW-complex was defined by Dror Farjoun [15, 1.16 and 2.1]. See also Piacenza [34].

Definition 2.1 ($\mathcal{I}$-CW-complex). A *(contravariant) $\mathcal{I}$-CW-complex X* is a contravariant $\mathcal{I}$-space X together with a filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots \subset X = \bigcup_{n \geq 0} X_n$$

such that $X = \operatorname{colim}_{n\to\infty} X_n$ and for any $n \geq 0$ the *n-skeleton* X_n is obtained from the $(n-1)$-skeleton X_{n-1} by *attaching $\mathcal{I}$-n-cells*, i.e., there exists a pushout of $\mathcal{I}$-spaces of the form

$$\begin{array}{ccc}
\coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{I}}(-,i_\lambda) \times S^{n-1} & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{I}}(-,i_\lambda) \times D^{n} & \longrightarrow & X_n
\end{array}$$

where the vertical maps are inclusions, Λ_n is an index set, and the i_λ-s are objects of $\mathcal{I}$. In particular, $X_0 = \coprod_{\lambda\in\Lambda_0} \operatorname{mor}_{\mathcal{I}}(-,i_\lambda)$.

We refer to the inclusion functor $\operatorname{mor}_{\mathcal{I}}(-,i_\lambda) \times (D^n - S^{n-1}) \to X$ as an *$\mathcal{I}$-n-cell based at i_λ*.

An $\mathcal{I}$-CW-complex has *dimension* $\leq n$ if $X = X_n$. We call X *finite dimensional* if there exists an integer n with $X = X_n$. It is called *finite* if it is finite dimensional and Λ_n is finite for every $n \geq 0$.
The definition of a *covariant $\mathcal{I}$-CW-complex* is analogous.

Definition 2.2 (Classifying $\mathcal{I}$-space). A model for *the classifying $\mathcal{I}$-space $E\mathcal{I}$* is an $\mathcal{I}$-CW-complex $E\mathcal{I}$ such that $E\mathcal{I}(i)$ is contractible for all objects i.

The universal property of $E\mathcal{I}$ is that for any $\mathcal{I}$-CW-complex X there is up to homotopy precisely one map of $\mathcal{I}$-spaces from X to $E\mathcal{I}$. In particular two models for $E\mathcal{I}$ are $\mathcal{I}$-homotopy equivalent (see Davis–Lück [14, Theorem 3.4]). A model for the usual *classifying space $B\mathcal{I}$* is given by $E\mathcal{I} \otimes_{\mathcal{I}} \{\bullet\}$ (see [14, Definition 3.10]), where $\{\bullet\}$ is the constant covariant $\mathcal{I}$-space with value the one point space and $\otimes_{\mathcal{I}}$ denotes the tensor product of a contravariant and a covariant $\mathcal{I}$-space as follows (see [14, Definition 1.4]).

Definition 2.3 (Tensor product of a contravariant and a covariant $\mathcal{I}$-space). Let X be a contravariant $\mathcal{I}$-space and Y a covariant $\mathcal{I}$-space. The *tensor product of X and Y* is

$$X \otimes_{\mathcal{I}} Y = \left(\coprod_{i \in \mathcal{I}} X(i) \times Y(i) \right) / \sim$$

where $(X(\phi)(x), y) \sim (x, Y(\phi)y)$ for all morphisms $\phi : i \to j$ in $\mathcal{I}$ and points $x \in X(j)$ and $y \in Y(i)$.

We present some examples of classifying $\mathcal{I}$-spaces for various categories $\mathcal{I}$.

Example 2.4. If $\mathcal{I}$ has a terminal object t, then a finite model for the classifying $\mathcal{I}$-space $E\mathcal{I}$ is simply $\mathrm{mor}_{\mathcal{I}}(-, t)$.

Example 2.5. Let $\mathcal{I} = \{j \rightrightarrows k\}$ be the category consisting of two objects and a single pair of parallel arrows between them. All other morphisms are identity morphisms. We obtain a finite model X for the classifying $\mathcal{I}$-space $E\mathcal{I}$ as follows. The $\mathcal{I}$-CW-space X has a single $\mathcal{I}$-0-cell based at k and a single $\mathcal{I}$-1-cell based at j. The gluing map $\mathrm{mor}_{\mathcal{I}}(-, j) \times S^0 \to \mathrm{mor}_{\mathcal{I}}(-, k)$ is induced by the two parallel arrows $j \rightrightarrows k$. Then $X(j) = D^1 \simeq *$ and $X(k) = *$.

Example 2.6. Let $\mathcal{I} = \{k \leftarrow j \to \ell\}$ be the category with objects j, k and ℓ, and precisely one morphism from j to k and one morphism from j to ℓ. All other morphisms are identity morphisms. A finite model for $E\mathcal{I}$ is given by the $\mathcal{I}$-CW-complex with precisely two $\mathcal{I}$-0-cells $\mathrm{mor}_{\mathcal{I}}(?, k)$ and $\mathrm{mor}_{\mathcal{I}}(?, \ell)$ and precisely one $\mathcal{I}$-1-cell $\mathrm{mor}_{\mathcal{I}}(?, j) \times D^1$ whose attaching map $\mathrm{mor}_{\mathcal{I}}(?, j) \times S^0 \to \mathrm{mor}_{\mathcal{I}}(?, k) \amalg \mathrm{mor}_{\mathcal{I}}(?, \ell)$ is the disjoint union of the canonical maps $\mathrm{mor}_{\mathcal{I}}(?, j) \to \mathrm{mor}_{\mathcal{I}}(?, k)$ and $\mathrm{mor}_{\mathcal{I}}(?, j) \to \mathrm{mor}_{\mathcal{I}}(?, \ell)$. The value of this 1-dimensional $\mathcal{I}$-CW-complex at the objects k and ℓ is a point and at the object j is D^1. Hence it is a finite model for $E\mathcal{I}$.

Example 2.7. Let $\mathcal{I}$ be the category with objects the non-empty subsets of $[q] = \{0, 1, \ldots, q\}$ and a unique arrow $J \to K$ if and only if $K \subseteq J$. In other words $\mathcal{I}$ is the *opposite* of the poset of non-empty subsets of $[q]$. Then $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model X for the classifying $\mathcal{I}$-space $E\mathcal{I}$ as follows. The functor $X\colon \mathcal{I}^{\mathrm{op}} \to \mathsf{SPACES}$ assigns to L the space $|\Delta[L]|$, which is the geometric realization of the simplicial set which maps $[m]$ to the set of weakly order preserving maps $[m] \to L$. The space $|\Delta[L]|$ is homeomorphic to the standard simplex with $\mathrm{card}(L)$ vertices. The n-skeleton X_n of X sends each L to the n-skeleton of $|\Delta[L]|$. The $\mathcal{I}$-cells of X are attached globally in the following way. The 0-skeleton is

$$X_0 = \coprod_{J\subseteq[q],|J|=1} \mathrm{mor}_{\mathcal{I}}(-,J).$$

For $n \leq q$, we construct X_n out of X_{n-1} as the pushout

$$\begin{tikzcd} \coprod_J \mathrm{mor}_{\mathcal{I}}(-,J) \times |\partial\Delta[n]| \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod_J \mathrm{mor}_{\mathcal{I}}(-,J) \times |\Delta[n]| \arrow[r] & X_n. \end{tikzcd}$$

The disjoint unions are over all $J \subseteq [q]$ with $|J| = n+1$. The J-component of the gluing map is induced by the $(n-1)$-face inclusion

$$|\Delta[K]| \longrightarrow \partial|\Delta[J]| \cong \partial|\Delta[n]|$$

for all $K \subseteq J$ with $|K| = n$. Clearly X is a finite $\mathcal{I}$-CW-complex. For each object L of $\mathcal{I}$, we have $X(L) = |\Delta[L]| \simeq *$, so that X is a finite model for $E\mathcal{I}$.

Example 2.8. Infinite categories may also admit finite models. Let $\mathcal{I} = \widehat{\mathbb{N}}$ be the monoid of natural numbers $\mathbb{N}$ viewed as a one-object category. A finite model X for the $\widehat{\mathbb{N}}$-classifying space has $X_0(*) = \mathrm{mor}_{\widehat{\mathbb{N}}}(*,*) = \mathbb{N}$ and $X_1(*) = [0,\infty)$. This model has a single $\widehat{\mathbb{N}}$-0-cell $\mathrm{mor}_{\widehat{\mathbb{N}}}(-,*)$ and a single $\widehat{\mathbb{N}}$-1-cell $\mathrm{mor}_{\widehat{\mathbb{N}}}(-,*) \times D^1$. The gluing map $\mathbb{N} \times S^0 \to \mathbb{N}$ sends $(n,-1)$ and $(n,1)$ to n and $n+1$ respectively. Similarly, the group of integers $\mathbb{Z}$ viewed as a one object category admits a finite model Y with one $\widehat{\mathbb{Z}}$-0-cell and one $\widehat{\mathbb{Z}}$-1-cell, so that $Y_0(*) = \mathbb{Z}$ and $Y_1(*) = \mathbb{R}$.

Remark 2.9. Suppose a category $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model for $E\mathcal{I}$. Then the cellular R-chains of a finite model provide a finite free resolution of the constant $R\mathcal{I}$-module $\underline{R}$, so $\mathcal{I}$ is of type (FF_R). If $\mathcal{I}$ is additionally directly finite and R is Noetherian, then $\chi(B\mathcal{I};R) = \chi(\mathcal{I};R) = \chi^{(2)}(\mathcal{I})$ by Theorem 1.15.

Remark 2.10 (Bar construction of classifying $\mathcal{I}$-space). There exists a functorial construction $E^{\mathrm{bar}}\mathcal{I}$ of $E\mathcal{I}$ by a kind of bar construction. Namely, the contravariant functor $E^{\mathrm{bar}}\mathcal{I}\colon \mathcal{I} \to \mathsf{SPACES}$ sends an object i to the space $B^{\mathrm{bar}}(i \downarrow \mathcal{I})$, which is the geometric realization of the nerve of the category of objects under i (see Davis–Lück [14, page 230] and also Bousfield–Kan [10, page 327]). An equivalent definition of the bar construction in terms of the tensor product in Definition 2.3 is

$$E^{\mathrm{bar}}\mathcal{I} = \{*\} \otimes_{\mathcal{I}} B^{\mathrm{bar}}(? \downarrow \mathcal{I} \downarrow ??), \tag{2.11}$$

from which we prove that $E^{\mathrm{bar}}\mathcal{I}$ is an $\mathcal{I}$-CW-complex. The $\mathcal{I}\times\mathcal{I}^{\mathrm{op}}$-space $B^{\mathrm{bar}}(?\downarrow\mathcal{I}\downarrow ??)$ is an $\mathcal{I}\times\mathcal{I}^{\mathrm{op}}$-$CW$-complex (see [14, page 228]). For each path

$$i_0 \to i_1 \to i_2 \to \cdots \to i_n$$

of n-many non-identity morphisms in $\mathcal{I}$, $B^{\mathrm{bar}}(?\downarrow\mathcal{I}\downarrow ??)$ has an n-cell based at (i_0, i_n), that is a cell of the form $\mathrm{mor}_{\mathcal{I}}(?, i_0)\times\mathrm{mor}_{\mathcal{I}}(i_n, ??)\times D^n$. By [14, Lemma 3.19 (2)], the tensor product $E^{\mathrm{bar}}\mathcal{I}$ in (2.11) is an $\mathcal{I}$-CW-complex: an $(m+n)$-cell based at i is an n-cell of $B^{\mathrm{bar}}(?\downarrow\mathcal{I}\downarrow ??)$ based at (i,j) and an m-cell of the CW-complex $*(j)$ (see [14, page 229]). More explicitly, for each path of n-many non-identity morphisms

$$i_0 \to i_1 \to i_2 \to \cdots \to i_n \tag{2.12}$$

the $\mathcal{I}$-CW-complex $E^{\mathrm{bar}}\mathcal{I}$ has an n-cell based at i_0.

Though the bar construction is in general not a finite $\mathcal{I}$-CW-complex, it is in certain cases. For example, if $\mathcal{I}$ has only finitely many morphisms, no nontrivial isomorphisms, and no nontrivial endomorphisms, then there are only finitely many paths as in (2.12), and hence only finitely many $\mathcal{I}$-cells in $E^{\mathrm{bar}}\mathcal{I}$.

The bar construction is also compatible with induction. Given a functor $\alpha\colon\mathcal{I}\to\mathcal{D}$, we obtain a map of $\mathcal{D}$-spaces

$$E^{\mathrm{bar}}\alpha\colon \alpha_* E^{\mathrm{bar}}\mathcal{I}\to E^{\mathrm{bar}}\mathcal{D},$$

where α_* denotes induction with the functor α (see [14, Definition 1.8]). If $T\colon\alpha\to\beta$ is a natural transformation of functors $\mathcal{I}\to\mathcal{D}$, we obtain for any $\mathcal{I}$-space X a natural transformation $T_*\colon\alpha_*X\to\beta_*X$ which comes from the map of $\mathcal{I}$-$\mathcal{D}$-spaces $\mathrm{mor}_{\mathcal{D}}(??,\alpha(?))\to\mathrm{mor}_{\mathcal{D}}(??,\beta(?))$ sending $g\colon ??\to\alpha(?)$ to $T(?)\circ g\colon ??\to\beta(?)$.

Lemma 2.13 (Invariance of finite models under equivalence of categories). *Suppose $\mathcal{I}$ and $\mathcal{J}$ are equivalent categories. Then $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model for $E\mathcal{I}$ if and only if $\mathcal{J}$ admits a finite $\mathcal{J}$-CW-model for $E\mathcal{J}$. More precisely, if $F\colon\mathcal{I}\to\mathcal{J}$ is an equivalence of categories and Y is a finite $\mathcal{J}$-CW-model for $E\mathcal{J}$, then the restriction $\mathrm{res}_F Y$ is a finite $\mathcal{I}$-CW-model for $E\mathcal{I}$.*

Proof. For any functor $F\colon\mathcal{I}\to\mathcal{J}$, we have an adjunction

$$\mathrm{ind}_F\colon \mathcal{I}\text{-}\mathsf{SPACES}\rightleftarrows\mathcal{J}\text{-}\mathsf{SPACES}\colon\ \mathrm{res}_F$$

defined by

$$\mathrm{ind}_F(X) := X(?)\otimes_{\mathcal{I}}\mathrm{mor}_{\mathcal{J}}\big(??,F(?)\big) \qquad \mathrm{res}_F(Y) := Y\circ F(?).$$

The $\mathcal{I}$-space $\mathrm{res}_F(Y)$ is naturally homeomorphic to $Y(?)\otimes_{\mathcal{J}}\mathrm{mor}_{\mathcal{J}}\big(F(??),?\big)$. But since we are assuming F is an equivalence of categories, it is a left adjoint in an adjoint equivalence (F,G), and we have natural homeomorphisms of $\mathcal{I}$-spaces

$$\begin{aligned}\mathrm{res}_F(Y) &\cong Y(?)\otimes_{\mathcal{J}}\mathrm{mor}_{\mathcal{J}}\big(F(??),?\big)\\ &\cong Y(?)\otimes_{\mathcal{J}}\mathrm{mor}_{\mathcal{J}}\big(??,G(?)\big)\\ &\cong \mathrm{ind}_G(Y).\end{aligned}$$

Since ind_G is a left adjoint, so is res_F, and res_F therefore preserves pushouts. Note also

$$\mathrm{res}_F \operatorname{mor}_{\mathcal{J}}(?, j) = \operatorname{mor}_{\mathcal{J}}\big(F(?), j\big) \cong \operatorname{mor}_{\mathcal{I}}\big(?, G(j)\big).$$

If Y is a finite $\mathcal{J}$-CW-model for $E\mathcal{J}$ with n-skeleton

$$\begin{array}{ccc}
\coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{J}}(-, j_\lambda) \times S^{n-1} & \longrightarrow & Y_{n-1} \\
\big\downarrow & & \big\downarrow \\
\coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{J}}(-, j_\lambda) \times D^{n} & \longrightarrow & Y_{n},
\end{array}$$

then $X := \mathrm{res}_F Y$ is a finite $\mathcal{I}$-CW-complex with n-skeleton

$$\begin{array}{ccc}
\coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{I}}\big(-, G(j_\lambda)\big) \times S^{n-1} & \longrightarrow & X_{n-1} \\
\big\downarrow & & \big\downarrow \\
\coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{I}}\big(-, G(j_\lambda)\big) \times D^{n} & \longrightarrow & X_{n}.
\end{array}$$

Clearly, $\mathrm{res}_F Y$ is contractible at each object i, since $\mathrm{res}_F Y(i) = Y(F(i)) \simeq *$. $\square$

3. Homotopy Colimits of Categories

Definition 3.1 (Homotopy colimit for categories). Let $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ be a covariant functor from some (small) index category $\mathcal{I}$ to the category of small categories. Its *homotopy colimit*

$$\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$$

is the following category. Objects are pairs (i, c), where $i \in \mathrm{ob}(\mathcal{I})$ and $c \in \mathrm{ob}\big(\mathcal{C}(i)\big)$. A morphism from (i, c) to (j, d) is a pair (u, f), where $u\colon i \to j$ is a morphism in $\mathcal{I}$ and $f\colon \mathcal{C}(u)(c) \to d$ is a morphism in $\mathcal{C}(j)$. The composition of the morphisms $(u, f)\colon (i, c) \to (j, d)$ and $(v, g)\colon (j, d) \to (k, e)$ is the morphism

$$(v, g) \circ (u, f) = (v \circ u, g \circ \mathcal{C}(v)(f))\colon (i, c) \to (k, e).$$

The identity of (i, c) is given by $(\mathrm{id}_i, \mathrm{id}_c)$.
This homotopy colimit construction for functors is often called the *Grothendieck construction* or the *category of elements*.

In which sense is $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ a homotopy colimit? First, recall from [20] that the nerve functor induces an equivalence of categories $\mathrm{Ho}\ \mathsf{CAT} \to \mathrm{Ho}\ \mathsf{SSET}$, where $\mathrm{Ho}\ \mathsf{CAT}$ denotes the localization of CAT with respect to nerve weak equivalences and $\mathrm{Ho}\ \mathsf{SSET}$ denotes the localization of SSET with respect to the usual weak equivalences. In [40], Thomason proved that $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ in CAT corresponds to the Bousfield–Kan construction in SSET under this equivalence of categories. Consequently, $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ has a universal property in the form of a bijection

$$\mathrm{Ho}\ \mathsf{CAT}(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}, \Gamma) \cong \mathrm{Ho}\ \mathsf{CAT}^{\mathcal{I}}(\mathcal{C}, \underline{\Gamma}), \tag{3.2}$$

for any category Γ. Here $\underline{\Gamma}$ indicates the $\mathcal{I}$-diagram that is constant Γ. In [41], Thomason proved that CAT admits a cofibrantly generated model structure in which the weak equivalences are the nerve weak equivalences, so that the associated projective model structure on $\mathsf{CAT}^{\mathcal{I}}$ exists. The model-theoretic construction of a homotopy colimit of the $\mathcal{I}$-diagram $\mathcal{C}$ in CAT as a colimit of a cofibrant replacement of $\mathcal{C}$ in the projective model structure therefore works. This model-theoretic construction also has the universal property in (3.2), so is isomorphic to $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ in Ho CAT, i.e. weakly equivalent to $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ in CAT.[1] A direct proof that $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ satisfies the universal property (3.2) is in Grothendieck's letter [17], see the article of Maltsiniotis [32, Section 3.1].

Remark 3.3. If $\mathcal{C}$ is merely a pseudo functor, then it of course still has a homotopy colimit. A *pseudo functor* $\mathcal{C} \colon \mathcal{I} \to \mathsf{CAT}$ is like an ordinary functor, but only preserves composition and unit up to specified coherent natural isomorphisms $\mathcal{C}_{v,u} \colon \mathcal{C}(v) \circ \mathcal{C}(u) \Rightarrow \mathcal{C}(v \circ u)$ and $\mathcal{C}_i \colon 1_{\mathcal{C}(i)} \Rightarrow \mathcal{C}(\mathrm{id}_i)$. Moreover, $\mathcal{C}_{v,u}$ is required to be natural in v and u. The objects and morphisms of the *homotopy colimit* $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ are defined as in the strict case of Definition 3.1. The composition in $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ is defined by the modified rule

$$(v,g) \circ (u,f) = (v \circ u, g \circ (\mathcal{C}(v)(f)) \circ \mathcal{C}^{-1}_{v,u}(c))$$

while the identity of the object (i,c) is given by

$$(\mathrm{id}_i, \mathcal{C}_i^{-1}(c)).$$

The homotopy colimit of a pseudo functor $\mathcal{C} \colon \mathcal{I} \to \mathsf{CAT}$ is an ordinary 1-category with strictly associative and strictly unital composition.

Remark 3.4. For a fixed category $\mathcal{I}$, the homotopy colimit construction $\operatorname{hocolim}_{\mathcal{I}} -$ is a strict 2-functor from the strict 2-category of pseudo functors $\mathcal{I} \to \mathsf{CAT}$, pseudo natural transformations, and modifications into the strict 2-category CAT.

Example 3.5 (Homotopy colimit of a constant functor). If $\mathcal{C} \colon \mathcal{I} \to \mathsf{CAT}$ is a constant functor, say constantly a category also called $\mathcal{C}$, then $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C} = \mathcal{I} \times \mathcal{C}$.

Example 3.6 (Homotopy colimit for $\mathcal{I}$ with a terminal object). Suppose $\mathcal{I}$ has a terminal object t and $\mathcal{C} \colon \mathcal{I} \to \mathsf{CAT}$ is a strict covariant functor. Then $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ is homotopy equivalent to $\mathcal{C}(t)$ as follows. This is analogous to the familiar fact that $\mathcal{C}(t)$ is a colimit of $\mathcal{C}$. The components of the universal cocone

$$\pi \colon \mathcal{C} \Rightarrow \Delta_{\mathcal{C}(t)} \tag{3.7}$$

[1] We thank George Maltsiniotis for clarifying these points about homotopy colimits in CAT.

are $\mathcal{C}(i \to t)$. Applying $\operatorname{hocolim}_{\mathcal{I}} -$ to (3.7) and composing with the projection gives us a functor F

$$\begin{tikzcd} \operatorname{hocolim}_{\mathcal{I}} \mathcal{C} \arrow[r, "\operatorname{hocolim}_{\mathcal{I}} \pi"'] \arrow[rr, bend left, "F"] & \mathcal{I} \times \mathcal{C}(t) \arrow[r, "\mathrm{pr}_{\mathcal{C}(t)}"'] & \mathcal{C}(t) \\ (i,c) \arrow[rr, mapsto] & & \mathcal{C}(i \to t)(c). \end{tikzcd}$$

The functor $G\colon \mathcal{C}(t) \to \operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$, $G(c) = (t,c)$ is a homotopy inverse, since $F \circ G = \mathrm{id}_{\mathcal{C}(t)}$ and we have a natural transformation $\mathrm{id}_{\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}} \Rightarrow G \circ F$ with components

$$(i \to t, \mathrm{id}_{\mathcal{C}(i \to t)})\colon (i,c) \longrightarrow (t, \mathcal{C}(i \to t)c).$$

Let $\mathcal{H}$ denote the homotopy colimit of the $\mathcal{I}$-diagram of categories $\mathcal{C}$. We now construct an $\mathcal{I}$-diagram of $\mathcal{H}$-spaces $E^{\mathcal{H}}$ with the property that its tensor product with $E\mathcal{I}$ is $\mathcal{H}$-homotopy equivalent to a classifying $\mathcal{H}$-space for $\mathcal{H}$. This $\mathcal{I}$-diagram of $\mathcal{H}$-spaces $E^{\mathcal{H}}$ will play an important role in the inductive proof of the Homotopy Colimit Formula Theorem 4.1.

Construction 3.8 (Construction of $E^{\mathcal{H}}$). Let $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ be a strict covariant functor, and abbreviate $\mathcal{H} = \operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$. Define a functor

$$E^{\mathcal{H}}\colon \mathcal{I} \to \mathcal{H}\text{-}\mathsf{SPACES} \tag{3.9}$$

as follows. Given an object $i \in \mathcal{I}$, we have the functor

$$\alpha(i)\colon \mathcal{C}(i) \to \mathcal{H} \tag{3.10}$$

sending an object c to the object (i,c) and a morphism $f\colon c \to d$ to the morphism (id_i, f). We define

$$E^{\mathcal{H}}(i) = \alpha(i)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big).$$

Consider a morphism $u\colon i \to j$ in $\mathcal{I}$. It induces a natural transformation $T(u)\colon \alpha(i) \to \alpha(j) \circ \mathcal{C}(u)$ from the functor $\alpha(i)\colon \mathcal{C}(i) \to \mathcal{H}$ to the functor $\alpha(j) \circ \mathcal{C}(u)\colon \mathcal{C}(i) \to \mathcal{H}$ by assigning to an object c in $\mathcal{C}(i)$ the morphism

$$(u, \mathrm{id}_{\mathcal{C}(u)(c)})\colon \alpha(i)(c) = (i,c) \to \alpha(j) \circ \mathcal{C}(u)(c) = (j, \mathcal{C}(u)(c)).$$

From Remark 2.10 we obtain a map of $\mathcal{H}$-spaces

$$T(u)_*\colon \alpha(i)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big) \to \alpha(j)_* \mathcal{C}(u)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big)$$

and a map of $\mathcal{C}(j)$-spaces

$$E^{\mathrm{bar}}\big(\mathcal{C}(u)\big)\colon \mathcal{C}(u)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big) \to E^{\mathrm{bar}}\big(\mathcal{C}(j)\big).$$

Finally, for the morphism u in $\mathcal{I}$, we define $E^{\mathcal{H}}(u)\colon E^{\mathcal{H}}(i) \to E^{\mathcal{H}}(j)$ by the composite of the two maps below.

$$\alpha(i)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big) \xrightarrow{T(u)_*} \alpha(j)_* \mathcal{C}(u)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big)$$

$$\alpha(j)_* \mathcal{C}(u)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big) \xrightarrow{\alpha(j)_*(E^{\mathrm{bar}}\left(\mathcal{C}(u)\right)} \alpha(j)_* E^{\mathrm{bar}}\big(\mathcal{C}(j)\big)$$

Define the homotopy colimit of the covariant functor $E^{\mathcal{H}}$ of (3.9) to be the contravariant $\mathcal{H}$-space

$$\operatorname{hocolim}_{\mathcal{I}} E^{\mathcal{H}} \quad := \quad (i,c) \mapsto E\mathcal{I} \otimes_{\mathcal{I}} \left(E^{\mathcal{H}}(i,c)\right). \tag{3.11}$$

Lemma 3.12. *Consider any model $E\mathcal{I}$ for the classifying $\mathcal{I}$-space of the category $\mathcal{I}$. Then the contravariant $\mathcal{H}$-space $E\mathcal{I} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ of* (3.11) *is $\mathcal{H}$-homotopy equivalent to the classifying $\mathcal{H}$-space $E\mathcal{H}$ of the category $\mathcal{H} := \operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$.*

Proof. We first show that for any object (i,c) in $\mathcal{H}$ the space $E\mathcal{I} \otimes_{\mathcal{I}} \left(E^{\mathcal{H}}(i,c)\right)$ is contractible. The covariant functor $E^{\mathcal{H}}(i,c)\colon \mathcal{I} \to \mathsf{SPACES}$ sends an object j to

$$\begin{aligned}
&\alpha(j)_* \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)(i,c) \\
&\qquad = \quad \alpha(j)_* \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)(?) \otimes_{\mathcal{H}} \operatorname{mor}_{\mathcal{H}}\big((i,c),?\big) \\
&\qquad = \quad \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)(?) \otimes_{\mathcal{C}(j)} \operatorname{mor}_{\mathcal{H}}\big((i,c),(j,?)\big) \\
&\qquad = \quad \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)(?) \otimes_{\mathcal{C}(j)} \left(\coprod_{u \in \operatorname{mor}_{\mathcal{I}}(i,j)} \operatorname{mor}_{\mathcal{C}(j)}\big(\mathcal{C}(u)(c),?\big)\right) \\
&\qquad = \quad \coprod_{u \in \operatorname{mor}_{\mathcal{I}}(i,j)} \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)(?) \otimes_{\mathcal{C}(j)} \operatorname{mor}_{\mathcal{C}(j)}\big(\mathcal{C}(u)(c),?\big) \\
&\qquad = \quad \coprod_{u \in \operatorname{mor}_{\mathcal{I}}(i,j)} \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)\big(\mathcal{C}(u)(c)\big).
\end{aligned}$$

Since $\left(E^{\mathrm{bar}}\mathcal{C}(j)\right)\big(\mathcal{C}(u)(c)\big)$ is contractible, the projection

$$\coprod_{u \in \operatorname{mor}_{\mathcal{I}}(i,j)} \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)\big(\mathcal{C}(u)(c)\big) \to \operatorname{mor}_{\mathcal{I}}(i,j)$$

is a homotopy equivalence. Hence the collection of these projections for $j \in \operatorname{ob}(\mathcal{I})$ induces a map of $\mathcal{I}$-spaces

$$\mathrm{pr}\colon E^{\mathcal{H}}(i,c) \to \operatorname{mor}_{\mathcal{I}}(i,?)$$

whose evaluation at each object j in $\operatorname{ob}(\mathcal{I})$ is a homotopy equivalence. We conclude from Davis–Lück [14, Theorem 3.11] that

$$E\mathcal{I} \otimes_{\mathcal{I}} \mathrm{pr}\colon E\mathcal{I} \otimes_{\mathcal{I}} E^{\mathcal{H}}(i,c) \xrightarrow{\simeq} E\mathcal{I} \otimes_{\mathcal{I}} \operatorname{mor}_{\mathcal{I}}(i,?).$$

is a homotopy equivalence. Since $E\mathcal{I} \otimes_{\mathcal{I}} \operatorname{mor}_{\mathcal{I}}(i,?) = E\mathcal{I}(i)$ is contractible, this implies that for any object (i,c) in $\mathcal{H}$ the space $E\mathcal{I} \otimes_{\mathcal{I}} \left(E^{\mathcal{H}}(i,c)\right)$ is contractible, as we initially claimed.

It remains to show that $E\mathcal{I} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ has the $\mathcal{H}$-homotopy type of an $\mathcal{H}$-CW-complex. It is actually an $\mathcal{H}$-CW-complex. The following argument, that $E\mathcal{I} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ has the homotopy type of an $\mathcal{H}$-CW-complex, will be used again later.[2]

[2]This is a well-known standard argument, which we present only so that the reader easily sees that it works in the setting of $\mathcal{H}$-spaces.

We have a filtration of $E\mathcal{I}$

$$\emptyset = E\mathcal{I}_{-1} \subseteq E\mathcal{I}_0 \subseteq E\mathcal{I}_1 \subseteq \ldots \subseteq E\mathcal{I}_n \subseteq \ldots \subseteq E\mathcal{I} = \bigcup_{n\geq 0} E\mathcal{I}_n$$

such that

$$E\mathcal{I} = \operatorname{colim}_{n\to\infty} E\mathcal{I}_n$$

and for every $n \geq 0$ there exists a pushout of $\mathcal{I}$-spaces

$$\begin{array}{ccc} \coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{I}}(-, i_\lambda) \times S^{n-1} & \longrightarrow & E\mathcal{I}_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{I}}(-, i_\lambda) \times D^n & \longrightarrow & E\mathcal{I}_n. \end{array} \tag{3.13}$$

Since $- \otimes_{\mathcal{I}} Z$ has a right adjoint [14, Lemma 1.9] we get

$$E\mathcal{I} \otimes_{\mathcal{I}} E^{\mathcal{H}} = \operatorname{colim}_{n\to\infty} E\mathcal{I}_n \otimes_{\mathcal{I}} E^{\mathcal{H}}$$

as a colimit of $\mathcal{H}$-spaces. After an application of $- \otimes_{\mathcal{I}} E^{\mathcal{H}}$ to (3.13), we obtain pushouts of $\mathcal{H}$-spaces

$$\begin{array}{ccc} \coprod_{\lambda\in\Lambda_n} E^{\mathcal{H}}(i_\lambda) \times S^{n-1} & \xrightarrow{f_{n-1}} & E\mathcal{I}_{n-1} \otimes_{\mathcal{I}} E^{\mathcal{H}} \\ \downarrow & & \downarrow \\ \coprod_{\lambda\in\Lambda_n} E^{\mathcal{H}}(i_\lambda) \times D^n & \longrightarrow & E\mathcal{I}_n \otimes_{\mathcal{I}} E^{\mathcal{H}} \end{array} \tag{3.14}$$

where the left vertical arrow and hence the right vertical arrow are cofibrations of $\mathcal{H}$-spaces. By induction we may assume that $E\mathcal{I}_{n-1} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ has the homotopy type of an $\mathcal{H}$-CW-complex. Since the vertical maps are cofibrations, by replacing it with a homotopy equivalent $\mathcal{H}$-CW-complex we do not change the homotopy type of the pushout (the usual proof for spaces goes through for $\mathcal{H}$-spaces). Hence we may assume that $E\mathcal{I}_{n-1} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ is a $\mathcal{H}$-CW-complex. We may also assume that f_{n-1} is cellular: since the vertical maps are cofibrations, by replacing f_{n-1} by a homotopic cellular map, which exists by Davis–Lück [14, cf. Theorem 3.7], we also do not change the homotopy type of the pushout. See Selick [38, Theorem 7.1.8] for a proof of this statement for spaces which translates verbatim to the setting of $\mathcal{H}$-spaces. If f_{n-1} is cellular, diagram (3.14) is a cellular pushout. Hence we completed the induction step, showing that $E\mathcal{I}_n \otimes_{\mathcal{I}} E^{\mathcal{H}}$ has the homotopy type of an $\mathcal{H}$-CW-complex.

It remains to show that $E\mathcal{I} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ has the homotopy type of a $\mathcal{H}$-CW-complex: choose $\mathcal{H}$-CW-complexes Z_n and $\mathcal{H}$-homotopy equivalences $g_n \colon Z_n \to E\mathcal{I}_n \otimes_{\mathcal{I}} E^{\mathcal{H}}$. By iteratively replacing Z_n by the mapping cylinder of

$$Z_{n-1} \xrightarrow{g_{n-1}} E\mathcal{I}_{n-1} \otimes_{\mathcal{I}} E^{\mathcal{H}} \to E\mathcal{I}_n \otimes_{\mathcal{I}} E^{\mathcal{H}} \xrightarrow{\bar{g}_n} Z_n,$$

where $\bar{g}_n$ is a homotopy inverse of g_n, one finds a new sequence of homotopy equivalences $g'_n \colon Z_n \to E\mathcal{I}_n \otimes_{\mathcal{I}} E^{\mathcal{H}}$ (with the modified $\mathcal{H}$-CW-complexes Z_n) such that $g'_n|_{Z_{n-1}} = g'_{n-1}$. □

4. Homotopy Colimit Formula for Finiteness Obstructions and Euler Characteristics

In this section we prove the main theorem of this paper: the Homotopy Colimit Formula. It expresses the finiteness obstruction, the Euler characteristic, and the L^2-Euler characteristic of the homotopy colimit of a diagram in CAT in terms of the respective invariants for the diagram entries at the base objects for cells in a finite model for the $\mathcal{I}$-classifying space of $\mathcal{I}$. Analogous formulas for the functorial counterparts of the Euler characteristic and L^2-Euler characteristic are included. The Homotopy Colimit Formula is initially stated and proved for strict functors $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$, but we prove that it also holds for pseudo functors $\mathcal{D}\colon \mathcal{I} \to \mathsf{CAT}$ in Corollary 4.2. The full generality of pseudo functors is needed for the applications to complexes of groups in Section 8.

4.1. Homotopy Colimit Formula.

Theorem 4.1 (Homotopy Colimit Formula). *Let $\mathcal{I}$ be a small category such that there exists a finite $\mathcal{I}$-CW-model for its classifying $\mathcal{I}$-space. Fix such a finite $\mathcal{I}$-CW-model $E\mathcal{I}$. Denote by Λ_n the finite set of n-cells $\lambda = \operatorname{mor}_{\mathcal{I}}(?, i_\lambda) \times D^n$ of $E\mathcal{I}$. Let $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ be a covariant functor. Abbreviate $\mathcal{H} = \operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$. Then:*

(i) *If $\mathcal{I}$ is directly finite, and $\mathcal{C}(i)$ is directly finite for every object $i \in \operatorname{ob}(\mathcal{I})$, then the category $\mathcal{H}$ is directly finite;*

(ii) *If $\mathcal{I}$ is an EI-category, $\mathcal{C}(i)$ is an EI-category for every object $i \in \operatorname{ob}(\mathcal{I})$, and for every automorphism $u\colon i \xrightarrow{\cong} i$ the map $\operatorname{iso}(\mathcal{C}(i)) \to \operatorname{iso}(\mathcal{C}(i))$, $\overline{x} \mapsto \overline{\mathcal{C}(u)(x)}$ is the identity, then the category $\mathcal{H}$ is an EI-category;*

(iii) *If for every object i the category $\mathcal{C}(i)$ is of type (FP$_R$), then the category $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ is of type (FP$_R$);*

(iv) *If for every object i the category $\mathcal{C}(i)$ is of type (FF$_R$), then the category $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ is of type (FF$_R$);*

(v) *If for every object i the category $\mathcal{C}(i)$ is of type (FP$_R$), then we obtain for the finiteness obstruction*

$$o(\mathcal{H}; R) = \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha(i_\lambda)_*(o(\mathcal{C}(i_\lambda); R)),$$

where $\alpha(i_\lambda)_*\colon K_0(R\mathcal{C}(i_\lambda)) \to K_0(R\mathcal{H})$ is the homomorphism induced by the canonical functor $\alpha(i_\lambda)\colon \mathcal{C}(i_\lambda) \to \mathcal{H}$ defined in (3.10)*;*

(vi) *Suppose that $\mathcal{I}$ is directly finite and $\mathcal{C}(i)$ is directly finite for every object $i \in \operatorname{ob}(\mathcal{I})$. If for every object i the category $\mathcal{C}(i)$ is additionally of type (FP$_R$) then we obtain for the functorial Euler characteristic*

$$\chi_f(\mathcal{H}; R) = \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha(i_\lambda)_*(\chi_f(\mathcal{C}(i_\lambda); R)),$$

where $\alpha(i_\lambda)_*\colon U(\mathcal{C}(i_\lambda)) \to U(\mathcal{H})$ is the homomorphism induced by the canonical functor $\alpha(i_\lambda)\colon \mathcal{C}(i_\lambda) \to \mathcal{H}$ defined in (3.10)*. Summing up,* *we also have*

$$\chi\big(\mathcal{H};R\big) = \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \chi(\mathcal{C}(i_\lambda);R).$$

If R is Noetherian, in addition to the direct finiteness and (FP_R) hypotheses, we obtain for the Euler characteristics of the classifying spaces

$$\chi\big(B\mathcal{H};R\big) = \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \chi(B\mathcal{C}(i_\lambda);R);$$

(vii) *Suppose that $\mathcal{I}$ is directly finite and $\mathcal{C}(i)$ is directly finite for every object $i \in \mathrm{ob}(\mathcal{I})$. If for every object i the category $\mathcal{C}(i)$ is additionally of type (L^2), then $\mathcal{H}$ is of type (L^2) and we obtain for the functorial L^2-Euler characteristic*

$$\chi_f^{(2)}(\mathcal{H}) = \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \alpha(i_\lambda)_*\big(\chi_f^{(2)}(\mathcal{C}(i_\lambda))\big),$$

where $\alpha(i_\lambda)_*\colon U^{(1)}(\mathcal{C}(i_\lambda)) \to U^{(1)}(\mathcal{H})$ is the homomorphism induced by the canonical functor $\alpha(i_\lambda)\colon \mathcal{C}(i_\lambda) \to \mathcal{H}$ defined in (3.10)*, and we obtain for the L^2-Euler characteristic*

$$\chi^{(2)}(\mathcal{H}) = \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \chi^{(2)}(\mathcal{C}(i_\lambda)).$$

Proof. (i) Consider morphisms $(u,f)\colon (i,c) \to (j,d)$ and $(v,g)\colon (j,d) \to (i,c)$ in $\mathcal{H}$ with $(v,g)\circ(u,f) = \mathrm{id}_{(i,c)}$. This implies $vu = \mathrm{id}_i$ and $g \circ \mathcal{C}(v)(f) = \mathrm{id}_c$. Since $\mathcal{I}$ and $\mathcal{C}(i)$ are by assumption directly finite, we conclude $uv = \mathrm{id}_j$ and $\mathcal{C}(v)(f)\circ g = \mathrm{id}_{\mathcal{C}(v)(d)}$. Hence

$$\begin{aligned}(u,f)\circ(v,g) &= \big(uv, f\circ\mathcal{C}(u)(g)\big) = \big(uv, \mathcal{C}(uv)(f)\circ\mathcal{C}(u)(g)\big)\\ &= \big(uv, \mathcal{C}(u)(\mathcal{C}(v)(f)\circ g)\big) = \big(uv, \mathcal{C}(u)(\mathrm{id}_{\mathcal{C}(v)(d)})\big)\\ &= \big(\mathrm{id}_j, \mathrm{id}_{\mathcal{C}(u)\left(\mathcal{C}(v)(d)\right)}\big) = (\mathrm{id}_j, \mathrm{id}_d).\end{aligned}$$

(ii) Consider an endomorphism $(u,f)\colon (i,c) \to (i,c)$ in $\mathcal{H}$. Since $\mathcal{I}$ is an EI-category, $u\colon i \to i$ is an automorphism. Since $\overline{\mathcal{C}(u)(c)} = \overline{c}$ by assumption, we can choose an isomorphism $g\colon c \xrightarrow{\cong} \mathcal{C}(u)(c)$. Hence fg is an endomorphism in $\mathcal{C}(i)$. Since $\mathcal{C}(i)$ is an EI-category, and g is an isomorphism, f is also an isomorphism. Since u and f are isomorphisms, (u,f) is an isomorphism.

(iii) and (v). We say that an $R\mathcal{H}$-chain complex C_* is of type (FP_R) if it admits a *finite projective approximation*, i.e., there is a finite length chain complex P_* of finitely generated, projective $R\mathcal{H}$-modules together with an $R\mathcal{H}$-chain map $f_*\colon P_* \to C_*$ such that $H_n(f_*(i,c))$ is bijective for all $n \geq 0$ and $(i,c) \in \mathrm{ob}(\mathcal{H})$. If C_* is of type (FP_R), define its finiteness obstruction

$$o(C_*) := \sum_{n\geq 0}(-1)^n \cdot [P_n] \quad \in K_0(R\mathcal{H})$$

for any choice P_* of finite projective approximation. This is independent of the choice of P_* and the basic properties of it were studied by Lück [23, Chapter 11]. If $0[\underline{R}]$ is the $R\mathcal{H}$-chain complex concentrated in dimension zero and given there by the constant $R\mathcal{H}$-module $\underline{R}$, then $\mathcal{H}$ is of type (FP_R) if and only if $0[\underline{R}]$ is of type (FP_R) and in this case

$$o(\mathcal{H};R) = o(0[\underline{R}]) \in K_0(R\mathcal{H}).$$

Consider a finite $\mathcal{I}$-CW-complex X. We want to show by induction over the dimension of X that the $R\mathcal{H}$-chain complex $C_*(X \otimes_{\mathcal{I}} E^{\mathcal{H}})$ is of type (FP_R) and satisfies

$$o\bigl(C_*(X \otimes_{\mathcal{I}} E^{\mathcal{H}})\bigr) = \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha(i_\lambda)_*(o(\mathcal{C}(i_\lambda);R)),$$

where Λ_n denotes the set of n-cells of X and i_λ is the object at which the n-cell $\lambda = \mathrm{mor}_{\mathcal{I}}(?, i_\lambda) \times D^n$ of X is based.
The induction beginning, where X is the empty set, is obviously true. The induction step is done as follows. Let d be the dimension of X. Then X_d is obtained from X_{d-1} by a pushout of $\mathcal{I}$-spaces

$$\begin{array}{ccc} \coprod_{\lambda \in \Lambda_d} \mathrm{mor}_{\mathcal{C}}(-, i_\lambda) \times S^{d-1} & \longrightarrow & X_{d-1} \\ \downarrow & & \downarrow \\ \coprod_{\lambda \in \Lambda_d} \mathrm{mor}_{\mathcal{C}}(-, i_\lambda) \times D^d & \longrightarrow & X = X_d. \end{array}$$

Applying $- \otimes_{\mathcal{I}} E^{\mathcal{H}}$ to it yields, because $E^{\mathcal{H}}(i) = \alpha(i)_* E^{\mathrm{bar}}\bigl(\mathcal{C}(i)\bigr)$, a pushout of $\mathcal{H}$-spaces with a cofibration as left vertical arrow

$$\begin{array}{ccc} \coprod_{\lambda \in \Lambda_d} \alpha(i_\lambda)_* E^{\mathrm{bar}}\bigl(\mathcal{C}(i_\lambda)\bigr) \times S^{d-1} & \longrightarrow & X_{d-1} \otimes_{\mathcal{I}} E^{\mathcal{H}} \\ \downarrow & & \downarrow \\ \coprod_{\lambda \in \Lambda_d} \alpha(i_\lambda)_* E^{\mathrm{bar}}\bigl(\mathcal{C}(i_\lambda)\bigr) \times D^d & \longrightarrow & X \otimes_{\mathcal{I}} E^{\mathcal{H}}. \end{array}$$

In the sequel we can assume without loss of generality that $X_{d-1} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ and $X \otimes_{\mathcal{I}} E^{\mathcal{H}}$ are $\mathcal{H}$-CW-complexes and the diagram above is a pushout of $\mathcal{H}$-CW-complexes, since this can be arranged by replacing them by homotopy equivalent $\mathcal{H}$-CW-complexes (see the proof of Lemma 3.12). We obtain an exact sequence of $R\mathcal{H}$-chain complexes

$$0 \to C_*(X_{d-1} \otimes_{\mathcal{I}} E^{\mathcal{H}}) \to C_*(X \otimes_{\mathcal{I}} E^{\mathcal{H}}) \to \bigoplus_{\lambda \in \Lambda_d} \Sigma^d C_*\bigl(\alpha(i_\lambda)_* E^{\mathrm{bar}} \mathcal{C}(i_\lambda)\bigr) \to 0.$$

Consider $\lambda \in \Lambda_d$. Since $\mathcal{C}(i_\lambda)$ is of type (FP_R), we can find a finite projective $R\mathcal{C}(i_\lambda)$-chain complex P_* whose homology is concentrated in dimension zero and given there by the constant $R\mathcal{C}(i_\lambda)$-module $\underline{R}$. Since $C_*(E^{\mathrm{bar}}\mathcal{C}(i_\lambda))$ is a projective $R\mathcal{C}(i_\lambda)$-chain complex with the same homology, there is an $R\mathcal{C}(i_\lambda)$-chain homotopy equivalence $f_*\colon P_* \xrightarrow{\simeq} C_*\big(E^{\mathrm{bar}}\mathcal{C}(i_\lambda)\big)$ (see Lück [23, Lemma 11.3 on page 213] and

$$o(\mathcal{C}(i_\lambda);R) = o(P_*) = \sum_{n\geq 0}(-1)^n\cdot[P_n] \in K_0(R\mathcal{C}(i_\lambda)).$$

Obviously

$$\alpha(i_\lambda)_*f_*\colon \alpha(i_\lambda)_*P_* \xrightarrow{\simeq} \alpha(i_\lambda)_*C_*\big(E^{\mathrm{bar}}\big(\mathcal{C}(i_\lambda)\big)\big) = C_*\big(\alpha(i_\lambda)_*E^{\mathrm{bar}}\mathcal{C}(i_\lambda)\big)$$

is an $R\mathcal{H}$-chain homotopy equivalence. Hence $C_*(\alpha(i_\lambda)_*E^{\mathrm{bar}}\mathcal{C}(i_\lambda))$ and, by the induction hypothesis, $C_*(X_{d-1}\otimes_{\mathcal{I}}E^{\mathcal{H}})$ are $R\mathcal{H}$-chain complexes of type (FP_R). We conclude from Lück [23, Lemma 11.3 on page 213] that $C_*(X\otimes_{\mathcal{I}}E^{\mathcal{H}})$ is of type (FP_R) and

$$o\big(C_*(X\otimes_{\mathcal{I}}E^{\mathcal{H}})\big) = o\big(C_*(X_{d-1}\otimes_{\mathcal{I}}E^{\mathcal{H}})\big) + \sum_{\lambda\in\Lambda_d} o\big(\Sigma^d\alpha(i_\lambda)_*C_*(E^{\mathrm{bar}}\mathcal{C}(i_\lambda))\big).$$

This implies together with the induction hypothesis applied to X_{d-1}

$$\begin{aligned} & o\big(C_*(X\otimes_{\mathcal{I}}E^{\mathcal{H}})\big) \\ = {} & \sum_{n=0}^{d-1}(-1)^n\cdot\sum_{\lambda\in\Lambda_n}\alpha(i_\lambda)_*(o(\mathcal{C}(i_\lambda);R)) + \sum_{\lambda\in\Lambda_d}(-1)^d\cdot\alpha(i_\lambda)_*(o(\mathcal{C}(i_\lambda);R)) \\ = {} & \sum_{n=0}^{d}(-1)^n\cdot\sum_{\lambda\in\Lambda_n}\alpha(i_\lambda)_*(o(\mathcal{C}(i_\lambda);R)). \end{aligned}$$

This finishes the induction step.
Assertions (iii) and (v) follow by taking $X = E\mathcal{I}$.

(iv) This proof is analogous to that of assertion (iii).

(vi) By (i) and (iii), the category $\mathcal{H}$ is directly finite and of type (FP_R). Then an application of $\mathrm{rk}_{R\mathcal{H}}$ to the formula for $o(\mathcal{H};R)$ in (v) yields the formula for $\chi_f(\mathcal{H};R)$ in (vi) by the naturality of rk_{R-} with respect to the functors $\alpha(i_\lambda)$ between directly finite categories, see Fiore–Lück–Sauer [16, Lemma 4.9].
An application of the augmentation homomorphism $\epsilon\colon U(\mathcal{H})\to\mathbb{Z}$ to the formula for $\chi_f(\mathcal{H};R)$ yields the formula for $\chi(\mathcal{H};R)$. We also use the naturality of the augmentation homomorphism, that is, the commutativity of diagram (4.5) in [16] for $F = \alpha(i_\lambda)$.
If R is additionally Noetherian, then Theorem 1.9 applies, and the Euler characteristics of the categories agree with the Euler characteristics of the classifying spaces.

(vii) The proofs for the functorial L^2-Euler characteristic and the L^2-Euler characteristic are somewhat more complicated since the property (L^2) is more general than (FP_R), and the L^2-Euler characteristic comes from the finiteness obstruction only in the case (FP_R). The proofs are variations of the proofs for assertions (iii) and (v). Instead of using Lück [23, Lemma 11.3 on page 213], we now use the basic properties of L^2-Euler characteristics for chain complexes of modules over group von Neumann algebras [16, Lemma 5.7]. For example, we use [16, Lemma 5.7 (iv)], which says for any injective group homomorphism $i\colon H \to G$ and $\mathcal{N}(H)$-chain complex C_*, we have $\chi^{(2)}(C_*) = \chi^{(2)}(\mathrm{ind}_{i_*} C_*)$, provided the sum of the L^2-Betti numbers of C_* is finite. The injectivity hypothesis is easily verified: for every object $i \in \mathrm{ob}(\mathcal{I})$ and object $x \in \mathcal{C}(i)$ the functor $\alpha(i)\colon \mathcal{C}(i) \to \mathcal{H}$ clearly induces an injection $\mathrm{aut}_{\mathcal{C}(i)}(x) \to \mathrm{aut}_{\mathcal{H}}(i,x)$. This finishes the proof of Theorem 4.1. $\square$

Corollary 4.2. *Theorem 4.1 on homotopy colimits holds for pseudo functors* $\mathcal{D}\colon \mathcal{I} \to \mathsf{CAT}$.

Proof. We first remark that the pseudo functor $\mathcal{D}\colon \mathcal{I} \to \mathsf{CAT}$ is equivalent to a strict functor $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ in the following sense. As usual, we denote by $\mathrm{Hom}(\mathcal{I}, \mathsf{CAT})$ the strict 2-category of pseudo functors $\mathcal{I} \to \mathsf{CAT}$, pseudo natural transformations between them, and modifications. The pseudo functor $\mathcal{D}$ is equivalent to a strict functor $\mathcal{C}$ *as objects of the 2-category* $\mathrm{Hom}(\mathcal{I}, \mathsf{CAT})$. For example, we may take $\mathcal{C}$ to be the strict functor

$$i \mapsto \mathrm{mor}_{\mathrm{Hom}(\mathcal{I},\mathsf{CAT})}(\mathcal{I}(i,-), \mathcal{D}).$$

The equivalence between $\mathcal{C}$ and $\mathcal{D}$ in $\mathrm{Hom}(\mathcal{I}, \mathsf{CAT})$ has two useful consequences. Since

$$\mathrm{hocolim}_{\mathcal{I}}\colon\ \mathrm{Hom}(\mathcal{I}, \mathsf{CAT}) \to \mathsf{CAT}$$

is a strict 2-functor, it sends any equivalence between $\mathcal{C}$ and $\mathcal{D}$ to an equivalence in CAT between the categories $\mathrm{hocolim}_{\mathcal{I}} \mathcal{C}$ and $\mathrm{hocolim}_{\mathcal{I}} \mathcal{D}$. Another consequence of the equivalence between $\mathcal{C}$ and $\mathcal{D}$ is that for every $i \in \mathcal{I}$, the categories $\mathcal{C}(i)$ and $\mathcal{D}(i)$ are equivalent. With these observations we reduce Corollary 4.2 to Theorem 4.1.

(i) Suppose $\mathcal{D}(i)$ is directly finite for every $i \in \mathrm{ob}(\mathcal{I})$ and $\mathcal{I}$ is directly finite. Since direct finiteness is preserved under equivalence of categories by Fiore–Lück–Sauer [16, Lemma 3.2], and $\mathcal{C}(i)$ is equivalent to $\mathcal{D}(i)$, we see that $\mathcal{C}(i)$ is directly finite for every $i \in \mathrm{ob}(\mathcal{I})$. Hence $\mathrm{hocolim}_{\mathcal{I}} \mathcal{C}$ is directly finite by Theorem 4.1 (i). Since $\mathrm{hocolim}_{\mathcal{I}} \mathcal{D}$ is equivalent to $\mathrm{hocolim}_{\mathcal{I}} \mathcal{C}$, it is also directly finite, again by [16, Lemma 3.2].

(ii) Suppose that $\mathcal{I}$ is an EI-category, $\mathcal{D}(i)$ is an EI-category for every $i \in \mathrm{ob}(\mathcal{I})$, and for every automorphism $u\colon i \xrightarrow{\cong} i$ the map $\mathrm{iso}(\mathcal{D}(i)) \to \mathrm{iso}(\mathcal{D}(i))$, $\overline{y} \mapsto \overline{\mathcal{D}(u)(y)}$ is the identity. Since EI is preserved under equivalence of categories [16, Lemma 3.11], and $\mathcal{C}(i)$ is equivalent to $\mathcal{D}(i)$, we see $\mathcal{D}(i)$ is an EI-category. We claim that for each automorphism u, the functor $\mathcal{C}(u)$ also induces the identity on isomorphism classes of objects of $\mathcal{C}(i)$. Let $\phi\colon \mathcal{D} \to \mathcal{C}$ be a pseudo equivalence, that is, an equivalence in the 2-category $\mathrm{Hom}(\mathcal{I}, \mathsf{CAT})$. For $x \in \mathcal{C}(i)$, there is a $y \in \mathcal{D}(i)$ and an isomorphism $x \cong \phi_i(y)$. We have isomorphisms

$$\mathcal{C}(u)(x) \cong \mathcal{C}(u)\phi_i(y) \cong \phi_i \mathcal{D}(u)(y) \cong \phi_i(y) \cong x,$$

and $\mathcal{C}(u)$ induces the identity on isomorphism classes of objects of $\mathcal{C}(i)$. Then $\mathrm{hocolim}_{\mathcal{I}} \mathcal{C}$ is an EI-category by Theorem 4.1 (ii), and so is $\mathrm{hocolim}_{\mathcal{I}} \mathcal{D}$, again by [16, Lemma 3.11].

(iii) and (iv) similarly follow from Theorem 4.1 (iii) and (iv), since property (FP_R), property (FF_R), and the finiteness obstruction are all invariant under equivalence of categories [16, Theorem 2.8].

(v) Suppose $\mathcal{D}(i)$ is of type (FP_R) for every $i \in \mathrm{ob}(\mathcal{I})$. Then every $\mathcal{C}(i)$ is also of type (FP_R), since property (FP_R) is invariant under equivalence of categories [16, Theorem 2.8]. As in (3.10), we have for each $i \in \mathcal{I}$ the functor

$$\alpha^{\mathcal{D}}(i)\colon \mathcal{D}(i) \to \operatorname{hocolim}_{\mathcal{I}} \mathcal{D}$$

which sends an object d to the object (i,d) and a morphism $g\colon d \to d'$ to the morphism $(\mathrm{id}_i, g \circ \mathcal{D}_i^{-1}(d))$. From a pseudo equivalence $\psi\colon \mathcal{C} \to \mathcal{D}$ we obtain a strictly commutative diagram

$$\begin{array}{ccc} \mathcal{C}(i) & \xrightarrow{\alpha^{\mathcal{C}}(i)} & \operatorname{hocolim}_{\mathcal{I}} \mathcal{C} \\ {\scriptstyle \psi_i}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{hocolim}_{\mathcal{I}} \psi} \\ \mathcal{D}(i) & \xrightarrow[\alpha^{\mathcal{D}}(i)]{} & \operatorname{hocolim}_{\mathcal{I}} \mathcal{D} \end{array} \tag{4.3}$$

for each $i \in \mathrm{ob}(\mathcal{I})$. Since the finiteness obstruction is invariant under equivalence of categories [16, Theorem 2.8], we may use Theorem 4.1 (v) for $\mathcal{C}$ to obtain

$$\begin{aligned} o(\operatorname{hocolim}_{\mathcal{I}} \mathcal{D}; R) &= (\operatorname{hocolim}_{\mathcal{I}} \psi)_*(o(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}; R)) \\ &= (\operatorname{hocolim}_{\mathcal{I}} \psi)_* \left(\sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha^{\mathcal{C}}(i_\lambda)_*(o(\mathcal{C}(i_\lambda); R)) \right) \\ &= \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} (\operatorname{hocolim}_{\mathcal{I}} \psi)_* \circ \alpha^{\mathcal{C}}(i_\lambda)_*(o(\mathcal{C}(i_\lambda); R)) \\ &= \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha^{\mathcal{D}}(i_\lambda)_* \circ (\psi_{i_\lambda})_*(o(\mathcal{C}(i_\lambda); R)) \\ &= \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha^{\mathcal{D}}(i_\lambda)_*(o(\mathcal{D}(i_\lambda); R)). \end{aligned}$$

(vi) follows from (i), (iii), and (v) in the same way that Theorem 4.1 (vi) follows from Theorem 4.1 (ii), (iii), and (v).

(vii) Suppose that $\mathcal{I}$ is directly finite and $\mathcal{D}(i)$ is directly finite for every object $i \in \mathrm{ob}(\mathcal{I})$. Suppose also for every object $i \in \mathcal{I}$ the category $\mathcal{D}(i)$ is of type (L^2). By the proof of Corollary 4.2 (i) above, the values of the strict functor $\mathcal{C}$ are directly finite categories. If Γ_1 and Γ_2 are equivalent categories, then Γ_1 is both directly finite and of type (L^2) if and only if Γ_2 is both directly finite and of type (L^2) [16, Lemma 5.15 (i)]. Since each $\mathcal{D}(i)$ is directly finite, of type (L^2), and equivalent to $\mathcal{C}(i)$, we see that each $\mathcal{C}(i)$ is also directly finite and of type (L^2). So we may now apply Theorem 4.1 (i) and (vii) to $\mathcal{C}$ and conclude that $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ is directly finite and of type (L^2). Again using the preservation of the direct finiteness and (L^2) under equivalence [16, Lemma 5.15 (i)], and the equivalence of $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ with $\operatorname{hocolim}_{\mathcal{I}} \mathcal{D}$, we see $\operatorname{hocolim}_{\mathcal{I}} \mathcal{D}$ is both directly finite and of type (L^2).

To prove the formulas for $\chi_f^{(2)}$ and $\chi^{(2)}$, we use [16, Lemma 5.15 (ii)], which says: if $F\colon \Gamma_1 \to \Gamma_2$ is an equivalence of categories, and both Γ_1 and Γ_2 are both directly finite and of type (L^2), then $U^{(1)}(F)\chi_f^{(2)}(\Gamma_1) = \chi_f^{(2)}(\Gamma_2)$ and $\chi^{(2)}(\Gamma_1) = \chi^{(2)}(\Gamma_2)$. We apply this to the equivalences ψ_i and $\operatorname{hocolim}_{\mathcal{I}} \psi$, and use the commutativity of diagram (4.3). For readability, we write $(\operatorname{hocolim}_{\mathcal{I}} \psi)_*$ for $U(\operatorname{hocolim}_{\mathcal{I}} \psi)$ and $\alpha(i_\lambda)_*$ for $U^{(1)}(\alpha(i_\lambda))$, et cetera.

$$\begin{aligned}
\chi_f^{(2)}(\operatorname{hocolim}_{\mathcal{I}} \mathcal{D}) &= (\operatorname{hocolim}_{\mathcal{I}} \psi)_* \chi_f^{(2)}(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}) \\
&= (\operatorname{hocolim}_{\mathcal{I}} \psi)_* \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha(i_\lambda)_* \big(\chi_f^{(2)}(\mathcal{C}(i_\lambda))\big) \\
&= \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} (\operatorname{hocolim}_{\mathcal{I}} \psi)_* \circ \alpha^{\mathcal{C}}(i_\lambda)_* \big(\chi_f^{(2)}(\mathcal{C}(i_\lambda))\big) \\
&= \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha^{\mathcal{D}}(i_\lambda)_* \circ (\psi_{i_\lambda})_* \big(\chi_f^{(2)}(\mathcal{C}(i_\lambda))\big) \\
&= \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha^{\mathcal{D}}(i_\lambda)_* \big(\chi_f^{(2)}(\mathcal{D}(i_\lambda))\big).
\end{aligned}$$

The formula for $\chi^{(2)}$ follows by summing up the components of the functorial L^2-Euler characteristics. ☐

4.2. The Case of an Indexing Category of Type (FP_R). The Homotopy Colimit Formula of Theorem 4.1 can be extended to the case, where $\mathcal{I}$ is of type (FP_R) and not necessarily of type (FF_R) as follows (recall that the existence of a finite $\mathcal{I}$-CW-model for $E\mathcal{I}$ implies $\mathcal{I}$ is of type (FF_R), since cellular chains then provide a finite free resolution of $\underline{R}$.). The evaluation of the covariant functor

$$E^{\mathcal{H}}\colon \mathcal{I} \to \mathcal{H}\text{-}\mathsf{SPACES}$$

of (3.9) at every object $i \in \mathcal{I}$ is an $\mathcal{H}$-CW-complex. Composing it with the cellular chain complex functor yields a covariant functor

$$C_*(E^{\mathcal{H}})\colon \mathcal{I} \to R\mathcal{H}\text{-}\mathsf{CHCOM}$$

whose evaluation at every object in $\mathcal{I}$ is a free $R\mathcal{H}$-chain complexes. Since by assumption $\mathcal{C}(i)$ is of type (FP_R), $C_*(E^{\mathcal{H}})(i)$ is $R\mathcal{H}$-chain homotopy equivalent to a finite projective $R\mathcal{H}$-chain complex for every object $i \in \mathcal{I}$. Since $R\operatorname{mor}_{\mathcal{I}}(?, i) \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})$ is $R\mathcal{H}$-isomorphic to $C_*(E^{\mathcal{H}})$, we conclude for every finitely generated projective $R\Gamma$-module P that $P \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})$ is $R\mathcal{H}$-chain homotopy equivalent to finite projective $R\mathcal{H}$-chain complex and in particular possesses a finiteness obstruction $o\big(P \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}}\big) \in K_0(R\mathcal{H})$ (see Lück [23, Theorem 11.2 on page 212]). Because of Lück [23, Theorem 11.2 on page 212] we obtain a homomorphism

$$\alpha_{\mathcal{C}}\colon K_0(R\mathcal{I}) \to K_0(R\mathcal{H}), \quad [P] \mapsto o\big(P \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})\big).$$

The chain complex version of the proof of Lemma 3.12 shows that the $R\mathcal{H}$-chain complex $C_*(\mathcal{I}) \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})$ is a projective $R\mathcal{H}$-resolution of the constant $R\Gamma$-module $\underline{R}$. Choose a finite projective $R\mathcal{I}$-chain complex P_* and an $R\mathcal{I}$-chain homotopy equivalence $f_*\colon P_* \xrightarrow{\simeq} C_*(\mathcal{I})$. Then $f_* \otimes_{R\mathcal{I}} \mathrm{id}\colon P_* \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}}) \to C_*(\mathcal{I}) \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})$ is an $R\Gamma$-chain homotopy equivalence of $R\Gamma$-chain complexes and $P_* \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})$ is is $R\mathcal{H}$-chain homotopy equivalent to finite projective $R\mathcal{H}$-chain complex by Lück [23, Theorem 11.2 on page 212]. This implies

$$o(\Gamma; R) = o\big(P_* \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})\big).$$

We conclude from [23, Theorem 11.2 on page 212]

$$o\big(P_* \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})\big) = \sum_{n \geq p} (-1)^n \cdot o\big(P_n \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})\big)$$

Since $o(\mathcal{I}; R)$ is $\sum_{n \geq p} (-1)^n \cdot [P_n]$, this implies

Theorem 4.4 (The Homotopy Colimit Formula for an indexing category of type (FP_R)). *We obtain under the conditions above*

$$\alpha_{\mathcal{C}}\big(o(\mathcal{I}; R)\big) = o(\mathcal{H}; R).$$

Remark 4.5. See Section 7 for a comparison with Leinster's Euler characteristic and his results.

5. Examples of the Homotopy Colimit Formula

We now present several examples of the Homotopy Colimit Formula Theorem 4.1. These include the cases: $\mathcal{I}$ with a terminal object, the constant functor, the trivial functor, homotopy pushouts, homotopy orbits, and the transport groupoid. For the transport groupoid in the finite case, see also Example 8.33.

Example 5.1 (Homotopy Colimit Formula for $\mathcal{I}$ with a terminal object). Suppose that $\mathcal{I}$ has a terminal object t and $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ is a functor. Then $\mathrm{mor}_{\mathcal{I}}(-, t)$ is a finite $\mathcal{I}$-CW model for $E\mathcal{I}$. If every category $\mathcal{C}(i)$ is of type (FP_R), then $o(\mathcal{H}; R) = \alpha(t)_*(o(\mathcal{C}(t); R)$. If $\mathcal{I}$ and $\mathcal{C}$ additionally satisfy the hypotheses of Theorem 4.1 (vi), then $\chi_f(\mathcal{H}; R) = \chi_f(\mathcal{C}(t); R)$ and $\chi(\mathcal{H}; R) = \chi(\mathcal{C}(t); R)$, as we anticipated in Example 3.6. Similar statements hold for $\chi_f^{(2)}$ and $\chi^{(2)}$ in the L^2 case.

Example 5.2 (Homotopy Colimit Formula for a constant functor). Consider the situation of Theorem 4.1 in the special case where the covariant functor $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ is constant $\mathcal{C} \in \mathsf{CAT}$. Suppose that $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model for $E\mathcal{I}$. Then we may draw various conclusions about the homotopy colimit $\mathcal{H} = \mathcal{I} \times \mathcal{C}$. If $\mathcal{I}$ and $\mathcal{C}$ are of type (FP_R), then so is $\mathcal{I} \times \mathcal{C}$. If $\mathcal{I}$ and $\mathcal{C}$ are of type (FF_R), then so is $\mathcal{I} \times \mathcal{C}$. The statements in Theorem 4.1 provide us with formulas in terms of $\mathcal{C}$ for $o(\mathcal{I} \times \mathcal{C}; R)$, $\chi_f(\mathcal{I} \times \mathcal{C}; R)$, $\chi(\mathcal{I} \times \mathcal{C}; R)$, $\chi_f^{(2)}(\mathcal{I} \times \mathcal{C})$, and $\chi^{(2)}(\mathcal{I} \times \mathcal{C})$. We recall that the invariants o, χ_f, χ, $\chi_f^{(2)}$, and $\chi^{(2)}$ are multiplicative, see Fiore–Lück–Sauer [16, Theorems 2.17, 4.22, and 5.17].

Example 5.3 (Homotopy Colimit Formula for the trivial functor). Consider the situation of Theorem 4.1 in the special case where the covariant functor $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ is constantly the terminal category, which consists of a single object and its identity morphism. Then $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ agrees with $\mathcal{I}$, as we see from Example 3.5. Obviously $\mathcal{C}(i)$ is of type (FF_R), its finiteness obstruction is $[R] \in K_0(R) = K_0(R\mathcal{C}(i))$ and both its Euler characteristic and L^2-Euler characteristic equals 1. We obtain from Theorem 4.1

$$\begin{array}{lcll}
o(\mathcal{I};R) & = & \sum_{n\ge 0}(-1)^n\cdot\sum_{\lambda\in\Lambda_n}[R\operatorname{mor}_{\mathcal{I}}(?,i_\lambda)] & \in K_0(R\mathcal{I});\\
\chi_f(\mathcal{I};R) & = & \sum_{n\ge 0}(-1)^n\cdot\sum_{\lambda\in\Lambda_n}\overline{i_\lambda} & \in U(\Gamma);\\
\chi(\mathcal{I};R) & = & \sum_{n\ge 0}(-1)^n\cdot|\Lambda_n| & \in \mathbb{Z};\\
\chi_f^{(2)}(\mathcal{I}) & = & \sum_{n\ge 0}(-1)^n\cdot\sum_{\lambda\in\Lambda_n}\overline{i_\lambda} & \in U^{(1)}(\mathcal{I});\\
\chi^{(2)}(\mathcal{I}) & = & \sum_{n\ge 0}(-1)^n\cdot|\Lambda_n| & \in \mathbb{R}.
\end{array}$$

Example 5.4 (Homotopy pushout formula). Let $\mathcal{I}$ be the category with objects j, k and ℓ such that there is precisely one morphism from j to k and from j to ℓ and all other morphisms are identity morphisms.

$$\mathcal{I} = \{\ k \xleftarrow{g} j \xrightarrow{h} \ell\ \}$$

By Example 2.6, the category $\mathcal{I}$ admits a finite model for the classifying $\mathcal{I}$-space $E\mathcal{I}$.
A covariant functor $\mathcal{C}\colon\mathcal{I}\to\mathsf{CAT}$ is the same as specifying three categories $\mathcal{C}(j)$, $\mathcal{C}(k)$ and $\mathcal{C}(\ell)$ and two functors $\mathcal{C}(g)\colon\mathcal{C}(j)\to\mathcal{C}(k)$ and $\mathcal{C}(h)\colon\mathcal{C}(j)\to\mathcal{C}(\ell)$. Let $\mathcal{H} = \operatorname{hocolim}_{\mathcal{I}}\mathcal{C}$ be the homotopy colimit. Let $\alpha(i)\colon\mathcal{C}(i)\to\mathcal{H}$ be the canonical functor for $i = j,k,\ell$. Then we obtain a square of functors which commutes up to natural transformations

$$\begin{tikzcd}
\mathcal{C}(j) \arrow[r, "\mathcal{C}(g)"] \arrow[d, "\mathcal{C}(h)"'] \arrow[dr, "\alpha(j)"] & \mathcal{C}(k) \arrow[d, "\alpha(k)"] \\
\mathcal{C}(\ell) \arrow[r, "\alpha(\ell)"'] & \mathcal{H}.
\end{tikzcd}$$

It induces diagrams which do not commute in general

$$\begin{tikzcd}
K_0(R\mathcal{C}(j)) \arrow[r, "\mathcal{C}(g)_*"] \arrow[d, "\mathcal{C}(h)_*"'] \arrow[dr, "\alpha(j)_*"] & K_0(R\mathcal{C}(k)) \arrow[d, "\alpha(k)_*"] \\
K_0(R\mathcal{C}(\ell)) \arrow[r, "\alpha(\ell)_*"'] & K_0(\mathcal{H})
\end{tikzcd}$$

and

$$\begin{tikzcd} U(\mathcal{C}(j)) \arrow[r, "\mathcal{C}(g)_*"] \arrow[d, "\mathcal{C}(h)_*"'] \arrow[dr, "\alpha(j)_*"] & U(R\mathcal{C}(k)) \arrow[d, "\alpha(k)_*"] \\ U(R\mathcal{C}(\ell)) \arrow[r, "\alpha(\ell)_*"'] & U(\mathcal{H}). \end{tikzcd}$$

Suppose that $\mathcal{C}(i)$ is of type (FP_R) for $i = j, k, \ell$. We conclude from Theorem 4.1 (iii) that $\mathcal{H}$ is of type (FP_R) and

$$\begin{array}{lcl} o(\mathcal{H};R) & = & \alpha(k)_*\big(o(\mathcal{C}(k);R)) + \alpha(\ell)_*\big(o(\mathcal{C}(\ell);R)) - \alpha(j)_*\big(o(\mathcal{C}(j;R)) \\ & & \in K_0(R\mathcal{H}); \\ \chi_f(\mathcal{H};R) & = & \alpha(k)_*\big(\chi_f(\mathcal{C}(k);R)\big) + \alpha(\ell)_*\big(\chi_f(\mathcal{C}(\ell);R)\big) - \alpha(j)_*\big(\chi_f(\mathcal{C}(j);R)\big) \\ & & \in U(\mathcal{H}); \\ \chi(\mathcal{H};R) & = & \chi(\mathcal{C}(k);R) + \chi(\mathcal{C}(\ell);R) - \chi(\mathcal{C}(j);R) \\ & & \in \mathbb{Z}; \\ \chi_f^{(2)}(\mathcal{H}) & = & \alpha(k)_*\big(\chi_f^{(2)}(\mathcal{C}(k)\big) + \alpha(\ell)_*\big(\chi_f^{(2)}(\mathcal{C}(\ell))\big) - \alpha(j)_*\big(\chi_f^{(2)}(\mathcal{C}(j))\big) \\ & & \in U^{(1)}(\mathcal{H}); \\ \chi^{(2)}(\mathcal{H}) & = & \chi^{(2)}(\mathcal{C}(k)) + \chi^{(2)}(\mathcal{C}(\ell)) - \chi^{(2)}(\mathcal{C}(j)) \\ & & \in \mathbb{R}. \end{array}$$

Example 5.5 (Homotopy orbit formula). Suppose that a group G acts on a category $\mathcal{C}$ from the left. This can be viewed as a covariant functor $\widehat{G} \to \mathsf{CAT}$ whose source is the groupoid $\widehat{G}$ with one object and G as its automorphism group. Let $\mathcal{H} = \operatorname{hocolim}_{\widehat{G}} \mathcal{C}$ be its homotopy colimit, also called the *homotopy orbit*. Notice that $\mathcal{H}$ and $\mathcal{C}$ have the same set of objects.

Suppose there is a finite model for BG of the group G, or equivalently, a finite model for the $\widehat{G}$-classifying space $E\widehat{G}$ of the category $\widehat{G}$. Let $\chi(BG) \in \mathbb{Z}$ be its Euler characteristic. Let $\alpha\colon \mathcal{C} \to \mathcal{H}$ be the canonical inclusion. Suppose that $\mathcal{C}$ is of type (FP_R). Then we conclude from Theorem 4.1 (iii) that $\mathcal{H}$ is of type (FP_R) and we have

$$\begin{array}{lcll} o(\mathcal{H};R) & = & \chi(BG)\cdot\alpha_*\big(o(\mathcal{C};R)\big) & \in K_0(R\mathcal{H}); \\ \chi_f(\mathcal{H};R) & = & \chi(BG)\cdot\alpha_*\big(\chi_f(\mathcal{C};R)\big) & \in U(\mathcal{H}); \\ \chi(\mathcal{H};R) & = & \chi(BG)\cdot\chi(\mathcal{C};R) & \in \mathbb{Z}; \\ \chi_f^{(2)}(\mathcal{H};R) & = & \chi(BG)\cdot\alpha_*\big(\chi_f^{(2)}(\mathcal{C};R)\big) & \in U^{(1)}(\mathcal{H}); \\ \chi^{(2)}(\mathcal{H};R) & = & \chi(BG)\cdot\chi^{(2)}(\mathcal{C};R) & \in \mathbb{R}. \end{array}$$

Example 5.6 (Transport groupoid). Let G be a group and let S be a left G-set. Its *transport groupoid* $\mathcal{G}^G(S)$ has S as its set of objects. The set of morphisms from s_1 to s_2 is $\{g \in G \mid gs_1 = s_2\}$. The composition is given by the multiplication in G. Denote by $\underline{S}$ the category whose set of objects is S and which has no morphisms besides the identity morphisms. The group G acts from the left on $\underline{S}$. One easily checks that $\mathcal{G}^G(S)$ is the homotopy orbit of $\underline{S}$ defined in Example 5.5.

Recall from Fiore–Lück–Sauer [16, Lemma 6.15 (iv)]: if Γ is a quasi-finite EI-category and for any morphism $f\colon x \to y$ in Γ, the order of the finite group $\{g \in \operatorname{aut}(x) \mid f \circ g = f\}$ is invertible in R, then Γ is of type (FP_R) if and only

if $\mathrm{iso}(\Gamma)$ is finite and for every object $x \in \mathrm{ob}(\Gamma)$ the trivial $R[x]$-module R is of type (FP_R). Thus, category $\underline{S}$ is of type (FP_R) if and only if S is finite. Suppose that $\underline{S}$ is of type (FP_R) and there is a finite model for BG. Obviously $o(\underline{S}; R)$ is given in $K_0(R\underline{S}) = \oplus_S K_0(R)$ by the collection $\{[R] \in K_0(R) \mid s \in S\}$.
Suppose for simplicity that G acts transitively on S. Fix an element $s \in S$. Let G_s be its isotropy group. Since S is finite, G_s is a subgroup of G of finite index, namely $[G : G_s] = |S|$. The transport groupoid $\mathcal{G}^G(S)$ is connected and the automorphism group of s is G_s. Hence evaluation at s induces an isomorphism

$$\mathrm{ev}\colon K_0(R\mathcal{G}^G(S)) \xrightarrow{\cong} K_0(R[G_s]).$$

The composition

$$K_0(R\underline{S}) \xrightarrow{\alpha_*} K_0(R\mathcal{G}^G(S)) \xrightarrow{\cong} K_0(R[G_s])$$

sends $o(\underline{S}; R)$ to $|S| \cdot [RG_s]$, where $\alpha\colon \underline{S} \to \mathcal{G}^G(S)$ is the obvious inclusion. Hence Example 5.5 implies

$$\mathrm{ev}\big(o(\mathcal{G}^G(S); R)\big) = \chi(BG) \cdot |S| \cdot [RG_S] \quad \in K_0(RG_s).$$

Since BG has a finite model, BG_s as a finite covering of BG has a finite model. The cellular RG_s-chain complex of EG_s yields a finite free resolution of the trivial RG_s-module R. This implies

$$\mathrm{ev}\big(o(\mathcal{G}^G(S); R)\big) = \chi(BG_s) \cdot [RG_s] \quad \in K_0(RG_s).$$

Hence we obtain the equality in $K_0(RG_s)$

$$\chi(BG_s) \cdot [RG_s] = \chi(BG) \cdot |S| \cdot [RG_S] = \chi(BG) \cdot [G : G_s] \cdot [RG_s].$$

This is equivalent to the equality of integers

$$\chi(BG_s) = \chi(BG) \cdot [G : G_s].$$

This equation is compatible with the well-know fact that for a d-sheeted covering $\overline{X} \to X$ of a finite CW-complex X the total space $\overline{X}$ is again a finite CW-complex and we have $\chi(\overline{X}) = d \cdot \chi(X)$.
For the transport groupoid in the finite case, see also Example 8.33.

6. Combinatorial Illustrations of the Homotopy Colimit Formula

The classical Inclusion-Exclusion Principle follows from the Homotopy Colimit Formula Theorem 4.1. We can also easily calculate the cardinality of a coequalizer in SETS in certain cases. These are different proofs of Examples 3.4.d and 3.4.b of Leinster's paper [21].

Example 6.1 (Inclusion-Exclusion Principle). Let X be a set and $S_0, \dots, S_q$ finite subsets of X. Then

$$|S_0 \cup S_1 \cup \cdots \cup S_q| = \sum_{\emptyset \neq J \subseteq [q]} (-1)^{|J|-1} \cdot \left| \bigcap_{j \in J} S_j \right|.$$

Proof. Let $\mathcal{I}$ be the category in Example 2.7 and consider the finite $\mathcal{I}$-CW-model for its classifying $\mathcal{I}$-space constructed there. We define a functor $\mathcal{C} : \mathcal{I} \to \mathsf{SETS}$ by $\mathcal{C}(J) := \bigcap_{j \in J} S_j$. The functor

$$\operatorname{hocolim}_{\mathcal{I}} \mathcal{C} \longrightarrow \operatorname{colim}_{\mathcal{I}} \mathcal{C} = S_0 \cup S_1 \cup \cdots \cup S_q$$

is an equivalence of categories, since it is surjective on objects and fully faithful. We have

$$\begin{aligned}
|S_0 \cup S_1 \cup \cdots \cup S_q| &= \chi(S_0 \cup S_1 \cup \cdots \cup S_q) \\
&= \chi(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}) \\
&= \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \chi(\mathcal{C}(i_\lambda)) \\
&= \sum_{n \geq 0} (-1)^n \cdot \sum_{J \subseteq [q] \text{ and } |J| = n+1} \chi(\mathcal{C}(J)) \\
&= \sum_{n \geq 0} (-1)^n \left(\sum_{J \subseteq [q] \text{ and } |J| = n+1} \left| \bigcap_{j \in J} S_j \right| \right) \\
&= \sum_{\emptyset \neq J \subseteq [q]} \left((-1)^{|J|-1} \left| \bigcap_{j \in J} S_j \right| \right).
\end{aligned}$$

□

EXAMPLE 6.2 (Cardinality of a Coequalizer). Let $\mathcal{I}$ be the category

$$a \underset{g}{\overset{f}{\rightrightarrows}} b$$

and $\mathcal{C} : \mathcal{I} \to \mathsf{SETS}$ a functor such that:

(i) the maps $\mathcal{C}f$ and $\mathcal{C}g$ are injective,
(ii) the images of the maps $\mathcal{C}f$ and $\mathcal{C}g$ are disjoint, and
(iii) the sets $\mathcal{C}a$ and $\mathcal{C}b$ are finite.

Then the coequalizer $\operatorname{colim} \mathcal{C}$ has cardinality $|\mathcal{C}b| - |\mathcal{C}a|$.

Proof. The assumptions that $\mathcal{C}f$ and $\mathcal{C}g$ are injective and have disjoint images imply that the functor

$$\operatorname{hocolim}_{\mathcal{I}} \mathcal{C} \longrightarrow \operatorname{colim}_{\mathcal{I}} \mathcal{C}$$

is fully faithful. Clearly it is also surjective on objects, and hence an equivalence of categories. The category $\mathcal{I}$ has a finite $\mathcal{I}$-CW-model for its classifying $\mathcal{I}$-space, constructed explicitly in Example 2.5. By Theorem 4.1, we have

$$\begin{aligned}
\chi(\operatorname{colim}_{\mathcal{I}} \mathcal{C}) &= \chi(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}) \\
&= \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \chi(\mathcal{C}(i_\lambda)) \\
&= \chi(\mathcal{C}b) - \chi(\mathcal{C}a) \\
&= |\mathcal{C}b| - |\mathcal{C}a|.
\end{aligned}$$

□

7. Comparison with Results of Baez–Dolan and Leinster

We recall Baez–Dolan's groupoid cardinality [4] and Leinster's Euler characteristic for certain finite categories [21], compare our Homotopy Colimit Formula with his result on compatibility with Grothendieck fibrations, prove an analogue for indexing categories $\mathcal{I}$ that admit finite $\mathcal{I}$-CW-models for their classifying $\mathcal{I}$-spaces, and finally mention a Homotopy Colimit Formula for Leinster's invariant in a restricted case.

7.1. Review of Leinster's Euler Characteristic. Let Γ be a category with finitely many objects and finitely many morphisms. A *weighting* on Γ is a function $q^\bullet\colon \operatorname{ob}(\Gamma) \to \mathbb{Q}$ such that for all objects $x \in \operatorname{ob}(\Gamma)$, we have

$$\sum_{y \in \operatorname{ob}(\Gamma)} |\operatorname{mor}_\Gamma(x,y)| \cdot q^y = 1.$$

A *coweighting* $q_\bullet$ on Γ is a weighting on Γ^{op}. If a finite category admits both a weighting $q^\bullet$ and a coweighting $q_\bullet$, then $\sum_{y\in\operatorname{ob}(\Gamma)} q^y = \sum_{x\in\operatorname{ob}(\Gamma)} q_x$. For a discusion of which matrices have the form $(|\operatorname{mor}_\Gamma(x,y)|)_{x,y\in\operatorname{ob}(\Gamma)}$ for some finite category Γ, see Allouch [2] and [3].
As proved in [16], free resolutions of the constant $R\Gamma$-module $\underline{R}$ give rise to weightings on Γ.

Theorem 7.1 (Weighting from a free resolution, Theorem 7.6 of Fiore–Lück–Sauer [16]). *Let Γ be a finite category. Suppose that the constant $R\Gamma$-module $\underline{R}$ admits a finite free resolution P_*. If P_n is free on the finite $\operatorname{ob}(\Gamma)$-set C_n, that is*

$$P_n = B(C_n) = \bigoplus_{y\in\operatorname{ob}(\Gamma)} \bigoplus_{C_n^y} R\operatorname{mor}_\Gamma(?,y), \tag{7.2}$$

then the function $q^\bullet\colon \operatorname{ob}(\Gamma)\to\mathbb{Q}$ defined by

$$q^y := \sum_{n\geq 0}(-1)^n \cdot |C_n^y|$$

is a weighting on Γ.

Corollary 7.3 (Construction of a weighting from a finite $\mathcal{I}$-CW-model for the classifying $\mathcal{I}$-space, Corollary 7.8 of Fiore–Lück–Sauer [16]). *Let $\mathcal{I}$ be a finite category. Suppose that $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model X for the classifying $\mathcal{I}$-space. Then the function $q^\bullet\colon \operatorname{ob}(\mathcal{I})\to\mathbb{Q}$ defined by*

$$q^y := \sum_{n\geq 0}(-1)^n(\textit{number of } n\textit{-cells of } X \textit{ based at } y)$$

is a weighting on $\mathcal{I}$.

As explained in Section 7.5 of [16], we use this Corollary to obtain several of Leinster's weightings in [21] from $\mathcal{I}$-CW-models for $\mathcal{I}$-classifying spaces. If $\mathcal{I}$ has a terminal object, then we obtain from the finite model in Example 2.4 the weighting which is 1 on the terminal object and 0 otherwise. The category $\mathcal{I} = \{j \rightrightarrows k\}$ in Example 2.5 has weighting $(q^j, q^k) = (-1, 1)$. The category $\mathcal{I} = \{k \leftarrow j \rightarrow \ell\}$ in Example 2.6 has weighting $(q^j, q^k, q^\ell) = (-1, 1, 1)$. Lastly, the category in Example 2.7 has weighting $q^J := (-1)^{|J|-1}$.
Weightings and coweightings play a key role in Leinster's notion of Euler characteristic. See also Berger–Leinster [9].

Definition 7.4 (Definition 2.2 of Leinster [21]). A finite category Γ *has an Euler characteristic in the sense of Leinster* if it admits both a weighting and a coweighting. In this case, its *Euler characteristic in the sense of Leinster* is defined as

$$\chi_L(\Gamma) := \sum_{y \in \mathrm{ob}(\Gamma)} q^y = \sum_{x \in \mathrm{ob}(\Gamma)} q_x$$

for any choice of weighting $q^\bullet$ or coweighting $q_\bullet$.

The Euler characteristic of Leinster agrees with the *groupoid cardinality* of Baez–Dolan [4] in the case of a finite groupoid $\mathcal{G}$, namely they are both

$$\sum_{\overline{x} \in \mathrm{iso}(\mathcal{G})} \frac{1}{|\,\mathrm{aut}_{\mathcal{G}}(x)|}.$$

The Euler characteristic of Leinster agrees with our L^2-Euler characteristic in some cases, as in the following Lemma.

Lemma 7.5 (Lemma 7.3 of Fiore–Lück–Sauer [16]). *Let Γ be a finite EI-category which is skeletal, i.e., if two objects are isomorphic, then they are equal. Suppose that the left $\mathrm{aut}_\Gamma(y)$-action on $\mathrm{mor}_\Gamma(x, y)$ is free for every two objects $x, y \in \mathrm{ob}(\Gamma)$.*
Then Γ is of type ($FP_{\mathbb{C}}$) and of type (L^2), and has an Euler characteristic in the sense of Leinster. Furthermore, the L^2-Euler characteristic $\chi^{(2)}(\Gamma)$ of Definition 1.11 coincides with Leinster's Euler characteristic $\chi_L(\Gamma)$ of Definition 7.4:

$$\chi^{(2)}(\Gamma) = \chi_L(\Gamma).$$

Moreover, these are both equal to

$$\sum_{l \geq 0} (-1)^l \cdot \sum_{x_0, x_l \in \mathrm{ob}(\Gamma)} \sum \frac{1}{|\,\mathrm{aut}(x_l)| \cdot |\,\mathrm{aut}(x_{l-1})| \cdot \cdots \cdot |\,\mathrm{aut}(x_0)|},$$

where the inner sum is over all paths $x_0 \to x_1 \to \cdots \to x_l$ from x_0 to x_l such that $x_0, \ldots, x_l$ are all distinct [16, Example 6.33].

This concludes the review of Leinster's and Baez–Dolan's invariants and how they relate to our L^2-Euler characteristic. Next we turn to a comparison of homotopy colimit results.

7.2. Comparison with Leinster's Proposition 2.8. Leinster's result on homotopy colimits, rephrased in our notation to make the comparison more apparent, is below.

Theorem 7.6 (Proposition 2.8 of Leinster [21]). *Let $\mathcal{I}$ be a category with finitely many objects and finitely many morphisms, and $\mathcal{C} : \mathcal{I} \to \mathsf{CAT}$ a pseudo functor. Assume that* $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ *has finitely many objects and finitely many morphisms. Let $q^\bullet$ be a weighting on $\mathcal{I}$ and suppose that* $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ *and all $\mathcal{C}(i)$ have Euler characteristics. Then*

$$\chi_L(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}) = \sum_{i \in \operatorname{ob}(\mathcal{I})} q^i \chi_L(\mathcal{C}(i)).$$

For example, if $\mathcal{I} = \{k \leftarrow j \rightarrow \ell\}$, then $\mathcal{I}$ admits the weighting $(q^j, q^k, q^\ell) = (-1, 1, 1)$ as discussed above. If $\mathcal{C} : \mathcal{I} \to \mathsf{CAT}$ is a pseudo functor, and the homotopy pushout has finitely many objects and finitely many morphisms, and $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ and all $\mathcal{C}(i)$ have Euler characteristics, then Leinster's result says that the homotopy pushout has the Euler characteristic $\chi_L(\mathcal{C}(k)) + \chi_L(\mathcal{C}(\ell)) - \chi_L(\mathcal{C}(j))$.

Leinster's Proposition 2.8 tells us how the Euler characteristic is compatible with Grothendieck fibrations. We can remove the hypothesis of finite from that Proposition, at the expense of requiring a finite model, as in the following theorem for our invariants.

Theorem 7.7. *Let $\mathcal{I}$ be a finite category. Suppose that $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model X for the classifying $\mathcal{I}$-space of $\mathcal{I}$. Let $q^\bullet \colon \operatorname{ob}(\mathcal{I}) \to \mathbb{Q}$ be the $\mathcal{I}$-Euler characteristic of X, namely*

$$q^i := \sum_{n \geq 0} (-1)^n (\textit{number of } n\textit{-cells of } X \textit{ based at } i).$$

Let $\mathcal{C} : \mathcal{I} \to \mathsf{CAT}$ be a functor such that for every object i the category $\mathcal{C}(i)$ is of type (FP_R). Suppose that $\mathcal{I}$ is directly finite and $\mathcal{C}(i)$ is directly finite for all $i \in \operatorname{ob}(\mathcal{I})$. Then

$$\chi(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}; R) = \sum_{i \in \operatorname{ob}(\mathcal{I})} q^i \chi(\mathcal{C}(i); R).$$

If each $\mathcal{C}(i)$ is of type (L^2) rather than (FP_R), we have

$$\chi^{(2)}(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}) = \sum_{i \in \operatorname{ob}(\mathcal{I})} q^i \chi^{(2)}(\mathcal{C}(i)).$$

Proof. By Theorem 4.1 (vi), we have

$$\begin{aligned}\chi(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}; R) &= \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \chi(\mathcal{C}(i_\lambda); R) \\ &= \sum_{n\geq 0}(-1)^n \cdot \sum_{i\in\operatorname{ob}(\mathcal{I})} (\text{number of } n\text{-cells of } X \text{ at } i)\chi(\mathcal{C}(i); R) \\ &= \sum_{i\in\operatorname{ob}(\mathcal{I})}\sum_{n\geq 0}(-1)^n(\text{number of } n\text{-cells of } X \text{ at } i)\chi(\mathcal{C}(i); R) \\ &= \sum_{i\in\operatorname{ob}(\mathcal{I})} q^i\chi(\mathcal{C}(i); R).\end{aligned}$$

The statement for $\chi^{(2)}$ is proved similarly from Theorem 4.1 (vii). □

Remark 7.8. Whenever $\chi(\operatorname{colim}_{\mathcal{I}} \mathcal{C}; R) = \chi(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}; R)$, Theorem 4.1 and Theorem 7.7 can be used to calculate the Euler characteristic of a colimit. Indeed, the hypotheses of Examples 6.1 and 6.2 guaranteed the equivalence of the colimit and the homotopy colimit, and this equivalence was a crucial ingredient in those proofs. For example, under Leinster's hypothesis of familial representability on $\mathcal{C}$, each connected component of $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ has an initial object, so

$$\chi(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}; R) = \chi(\operatorname{colim}_{\mathcal{I}} \mathcal{C}; R)$$

(recall that $\operatorname{colim}_{\mathcal{I}} \mathcal{C}$ is the set of connected components of $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ whenever $\mathcal{C}$ takes values in SETS). This is the role of familial representability in his Examples 3.4.

As a corollary to our Homotopy Colimit Formula for the L^2-Euler characteristic, we have a Homotopy Colimit Formula for Leinster's Euler characteristic when they agree.

Corollary 7.9 (Homotopy Colimit Formula for Leinster's Euler characteristic). *Let $\mathcal{I}$ be a skeletal, finite, EI-category such that the left $\operatorname{aut}_{\mathcal{I}}(y)$-action on $\operatorname{mor}_{\mathcal{I}}(x,y)$ is free for every two objects $x, y \in \operatorname{ob}(\mathcal{I})$. Assume there exists a finite $\mathcal{I}$-CW-model for the $\mathcal{I}$-classifying space of $\mathcal{I}$. Let $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ be a covariant functor such that for each $i \in \operatorname{ob}(\mathcal{I})$, the category $\mathcal{C}(i)$ is a skeletal, finite, EI and the left $\operatorname{aut}_{\mathcal{C}(i)}(d)$-action on $\operatorname{mor}_{\mathcal{C}(i)}(c,d)$ is free for every two objects $c, d \in \operatorname{ob}(\mathcal{C}(i))$. Assume for every object $i \in \operatorname{ob}(\mathcal{I})$, for each automorphism $u\colon i \to i$ in $\mathcal{I}$, and each $\overline{x} \in \operatorname{iso}(\mathcal{C}(i))$ we have $\overline{\mathcal{C}(u)(x)} = \overline{x}$.*
Then $\mathcal{H} := \operatorname{hocolim}_{i\in I} \mathcal{C}$ is again a skeletal, finite, EI-category such that the left $\operatorname{aut}_{\mathcal{H}}(h)$-action on $\operatorname{mor}_{\mathcal{H}}(g,h)$ is free for every two objects $g, h \in \operatorname{ob}(\operatorname{hocolim}_{i\in I} \mathcal{C})$, and

$$\chi_L(\mathcal{H}) = \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \chi_L(\mathcal{C}(i_\lambda); R).$$

Proof. The category $\mathcal{H}$ is an EI-category by Theorem 4.1 (ii). Skeletality and finiteness of $\mathcal{H}$ follow directly from the skeletality and finiteness of $\mathcal{I}$ and $\mathcal{C}(i)$, and the definition of $\mathcal{H}$. The hypotheses on $\mathcal{C}(i)$ imply that

$\chi^{(2)}(\mathcal{C}(i)) = \chi_L(\mathcal{C}(i))$ by Theorem 7.5, and similarly $\chi^{(2)}(\mathcal{H}) = \chi_L(\mathcal{H})$. Finally, Theorem 4.1 (vii), which is the Homotopy Colimit Formula for the L^2-Euler characteristic $\chi^{(2)}$, implies the formula is also true for Leinster's Euler characteristic χ_L in the special situation of the Corollary. □

8. Scwols and Complexes of Groups

As an illustration of the Homotopy Colimit Formula, we consider Euler characteristics of small categories without loops (*scwols*) and complexes of groups in the sense of Haefliger [18], [19] and Bridson–Haefliger [11]. One-dimensional complexes of groups are the classical Bass–Serre graphs of groups [37]. For finite scwols, the Euler characteristic, L^2-Euler characteristic, and Euler characteristic of the classifying space all coincide, essentially because finite scwols admit finite models for their classifying spaces. The Euler characteristic of a finite scwol is particularly easy to find: one simply chooses a skeleton, counts the paths of non-identity morphisms of length n, and then computes the alternating sum of these numbers.

Scwols and complexes of groups are combinatorial models for polyhedral complexes and group actions on them. The poset of faces of a polyhedral complex is a scwol. Suppose a group G acts on an M_κ-polyhedral complex by isometries preserving cell structure, and suppose each group element $g \in G$ fixes each cell pointwise that g fixes setwise. In this case, the quotient is also an M_κ-polyhedral complex, say Q, and we obtain a pseudo functor from its scwol of faces into groups. Namely, to a face $\overline{\sigma}$ of Q, one associates the stabilizer G_σ for a selected representative σ of $\overline{\sigma}$. Inclusions of subfaces of Q then correspond to inclusions of stabilizer subgroups up to conjugation. This pseudo functor is the complex of groups associated to the group action.

However, it is sometimes easier to work directly with the combinatorial model rather than with the original M_κ-polyhedral complex, and consider instead appropriate group actions on the associated scwol, as in Definition 8.11. Then the quotient category of a scwol is again a scwol, and the associated pseudo functor on the quotient scwol is called the *complex of groups associated to the group action*. Any group-valued pseudo functor on a scwol that arises in this way is called *developable*.

Our main results in this section concern the Euler characteristics of homotopy colimits of complexes of groups associated to group actions in the sense of Definition 8.11. Theorem 8.30, concludes that the Euler characteristic and L^2-Euler characteristic of the homotopy colimit are $\chi(\mathcal{X}/G)$ and $\chi^{(2)}(\mathcal{X})/|G|$ respectively, G and $\mathcal{X}$ are finite. These formulas provide necessary conditions for developability. That is, if F is a pseudo functor from a scwol $\mathcal{Y}$ to groups, one may ask if there are a scwol $\mathcal{X}$ and a group G such that $\mathcal{Y}$ is isomorphic to $\mathcal{X}/G$ and F is the associated complex of groups. To obtain conditions on $\chi(\mathcal{X})$, $\chi^{(2)}(\mathcal{X})$, and $|G|$, one forms the homotopy colimit of F, calculates its Euler characteristic and L^2-Euler characteristic, and then compares with the formulas of Theorem 8.30. A simple case is illustrated in Example 8.31. Another application of the formulas is the computation of the Euler characteristic and L^2-Euler characteristic for the transport groupoid of a finite left G-set, as in Example 8.33. We finish with Theorem 8.35, which extends Haefliger's formula for the Euler characteristic of the classifying space of the homotopy colimit of a complex of groups in terms of Euler characteristics of lower links and groups.

One novel aspect of our approach is that we do not require scwols to be skeletal. We prove in Theorem 8.24 that any scwol with a G-action in the sense of Definition 8.11 can be replaced by a skeletal scwol with a G-action, and this process preserves quotients, stabilizers, complexes of groups, and homotopy colimits. Moreover, if the initial G-action was free on the object set, then so is the G-action on the object set of the skeletal replacement.

We begin by recalling the notions in Chapter III.$\mathcal{C}$ of Bridson–Haefliger [11], rephrased in the conceptual framework of 2-category theory.

Notation 8.1 (2-Category of groups). We denote by GROUPS the 2-category of groups. Objects are groups and morphisms are group homomorphisms. The 2-cells are given by conjugation: a 2-cell (g, a)

$$\begin{tikzcd} H \arrow[r, bend left=50, "a"{name=U}] \arrow[r, bend right=50, "a'"'{name=D}] & G \arrow[Rightarrow, from=U, to=D, "{(g,a)}"] \end{tikzcd}$$

is an element $g \in G$ such that $ga(h)g^{-1} = a'(h)$ for all $h \in H$. The vertical composition is $(g_2, a_2) \odot (g_1, a_1) = (g_2 g_1, a_1)$ and the horizontal composition of

$$\begin{tikzcd} H \arrow[r, bend left=50, "a"{name=U}] \arrow[r, bend right=50, "a'"'{name=D}] & G \arrow[Rightarrow, from=U, to=D, "{(g,a)}"] \arrow[r, bend left=50, "b"{name=U2}] \arrow[r, bend right=50, "b'"'{name=D2}] & K \arrow[Rightarrow, from=U2, to=D2, "{(k,b)}"] \end{tikzcd}$$

is $(kb(g), ba)$.

Definition 8.2 (Scwol). A *scwol*[3] is a small category without loops, that is, a small category in which every endomorphism is trivial.

Example 8.3. The categories $\{j \rightrightarrows k\}$ and $\mathcal{I} = \{k \leftarrow j \rightarrow \ell\}$ of Examples 2.5 and 2.6 are scwols. Every partially ordered set is a scwol, for example, the set of simplices of a simplicial complex, ordered by the face relation, is a scwol. The poset of non-empty subsets of $[q]$, and its opposite category in Example 2.7, are scwols. The opposite category of a scwol is also a scwol.

[3]Bridson–Haefliger additionally require scwols to be skeletal [11, page 574]. However, we do not require scwols to be skeletal, since we prove in Theorem 8.24 that general statements about scwols can be reduced to the skeletal case.

LEMMA 8.4. *Every scwol is an EI-category and consequently also directly finite.*

Proof. Every endomorphism in a scwol is trivial, and therefore an automorphism, so every scwol is an EI-category. By Fiore–Lück–Sauer [16, Lemma 3.13], every EI-category is also directly finite.
For a direct proof of direct finiteness: if $u\colon x \to y$ and $v\colon y \to x$ are morphisms in a scwol, then vu and uv are automorphisms, and hence both $vu = \mathrm{id}_x$ and $uv = \mathrm{id}_y$ hold automatically. □

THEOREM 8.5 (Finite scwols admit finite models). *Suppose $\mathcal{I}$ is a finite scwol. Then $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model for its $\mathcal{I}$-classifying space in the sense of Definition 2.2.*

Proof. By Lemma 2.13, we may assume that $\mathcal{I}$ is skeletal.
Since $\mathcal{I}$ has only finitely many morphisms, no nontrivial isomorphisms, and no nontrivial endomorphisms, there are only finitely many paths of non-identity morphisms. Thus the bar construction of $E^{\mathrm{bar}}\mathcal{I}$ Remark 2.10 has only finitely many $\mathcal{I}$-cells. □

COROLLARY 8.6. *Any finite scwol $\mathcal{I}$ is of types (FF$_R$) and (FP$_R$) for every associative, commutative ring R with identity. Moreover, any finite scwol is also of type (L^2).*

Proof. The cellular R-chains of the finite model in Theorem 8.5 provide a finite, free resolution of the constant module $\underline{R}$. By Theorem 1.14, any directly finite category of type (FP$_{\mathbb{C}}$) is of type (L^2). Scwols are directly finite by Lemma 8.4. □

EXAMPLE 8.7 (Invariants coincide for finite scwols). Let $\mathcal{I}$ be any finite scwol. Then by Corollary 8.6 it is of type (FF$_R$) for any associative, commutative ring with identity, and by Theorems 1.9 and 1.15, we have

$$\chi(\mathcal{I};R) = \chi(B\mathcal{I};R) = \chi^{(2)}(\mathcal{I}).$$

If Γ is any skeleton of $\mathcal{I}$, then by (1.17),

$$\chi(\Gamma;R) = \sum_{n\geq 0}(-1)^n c_n(\Gamma), \tag{8.8}$$

where $c_n(\Gamma)$ is the number of paths of n-many non-identity morphisms in Γ. But by Fiore–Lück–Sauer [16, Theorem 2.8 and Corollary 4.19], type (FF$_R$) and the Euler characteristic are invariant under equivalence of categories between directly finite categories, so $\chi(\mathcal{I};R) = \chi(\Gamma;R)$ and all three invariants $\chi(\mathcal{I};R)$, $\chi(B\mathcal{I};R)$, $\chi^{(2)}(\mathcal{I})$ are given by (8.8).

We now arrive at the main notion of this section: a complex of groups. We will apply our Homotopy Colimit Formula to complexes of groups.

DEFINITION 8.9 (Complex of groups). Let $\mathcal{Y}$ be a scwol. A *complex of groups over* $\mathcal{Y}$ is a pseudo functor $F\colon \mathcal{Y} \to \mathsf{GROUPS}$ such that $F(a)$ is injective for every morphism a in $\mathcal{Y}$. For each object σ of $\mathcal{Y}$, the group $F(\sigma)$ is called the *local group at* σ.

In 2.5 and 2.4 of [18] and [19] respectively, Haefliger denotes by $CG(X)$ the homotopy colimit of a complex of groups $G(X)\colon C(X) \to \mathsf{GROUPS}$. Bridson–Haefliger use the notation $CG(\mathcal{Y})$ in [11, III.$\mathcal{C}$.2.8]. The fundamental group of a complex of groups $G(X)$ in the sense of [11, Definition 3.5 on p. 548] equals the fundamental group of the geometric realization of $CG(X)$ [11, Appendix A.12 on p. 578 and Remark A.14 on p. 579]. Categories which are homotopy colimits of complexes of groups are characterized by Haefliger on page 283 of [19]. From the homotopy colimit $CG(X)$, Haefliger reconstructs the category $C(X)$ and the complex of groups $G(X)$ up to a coboundary on pages 282-283 of [19]. Every aspherical realization [19, Definition 3.3.4] of $G(X)$ has the homotopy type of the geometric realization of the homotopy colimit, denoted $BG(X)$ [19, page 296]. The homotopy colimit also plays a role in the homology and cohomology of complexes of groups [19, Section 4]; a left $G(X)$-module is a functor $CG(X) \to \mathsf{ABELIAN\text{-}GROUPS}$.
We return to our recollection of complexes of groups and examples that arise from group actions.

Definition 8.10 (Morphism from a complex of groups to a group). A *morphism from a complex of groups F to a group G* is a pseudo natural transformation $F \Rightarrow \Delta_G$, where Δ_G indicates the constant 2-functor $\mathcal{Y} \to \mathsf{GROUPS}$ with value G.

A typical example of a complex of groups equipped with a morphism to a group G arises from an action of a group G on a scwol, as we now explain.

Definition 8.11 (Group action on a scwol, 1.11 of Bridson–Haefliger [11]). An *action of a group G on a scwol $\mathcal{X}$* is a group homomorphism from G into the group of strictly invertible functors $\mathcal{X} \to \mathcal{X}$ such that

(i) For every nontrivial morphism a of $\mathcal{X}$ and every $g \in G$, we have $gs(a) \neq t(a)$,
(ii) For every nontrivial morphism a of $\mathcal{X}$ and every $g \in G$, if $gs(a) = s(a)$, then $ga = a$.

Example 8.12. The group $G = \mathbb{Z}_2$ acts in the sense of Definition 8.11 on the scwol $\mathcal{X}$ pictured below.

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, "g"'] & z \\ y & x' \arrow[l, "g'"] \arrow[u, "h'"'] \end{tikzcd}$$

The group $\mathbb{Z}_2$ permutes respectively x and x', g and g', and h and h'. The objects y and z are fixed. This action of $\mathbb{Z}_2$ on $\mathcal{X}$ is a combinatorial model for a reflection action on S^1.

Example 8.13. Consider the scwol $\mathcal{X}$ pictured below. The group $G = \{\pm 1\} \ltimes \mathbb{Z}$ acts on $\mathcal{X}$ in the sense of Definition 8.11 where $-1 \cdot m := -m$ and $n \cdot m := m + 2n$.

$$\cdots \longrightarrow -2 \longleftarrow -1 \longrightarrow 0 \longleftarrow 1 \longrightarrow 2 \longleftarrow \cdots$$

This action of $\{\pm 1\} \ltimes \mathbb{Z}$ on $\mathcal{X}$ is a combinatorial model for the reflection and translation action on $\mathbb{R}$.

Lemma 8.14 (Consequences of group action conditions). *If a group G acts on a scwol $\mathcal{X}$ in the sense of Definition 8.11, then the following statements hold.*

(i) *If σ is an object of $\mathcal{X}$ and $g, h \in G$, then $g\sigma \cong h\sigma$ implies $g\sigma = h\sigma$.*
(ii) *If a is a morphism in $\mathcal{X}$ and $g, h \in G$, then $gs(a) = hs(a)$ implies $ga = ha$.*
(iii) *If $\sigma \cong \tau$, then the stabilizers G_σ and G_τ are equal.*

Proof. For statement (i), $g\sigma \cong h\sigma$ implies $\sigma \cong (g^{-1}h)\sigma$, so $\sigma = (g^{-1}h)\sigma$ by Definition 8.11 part (i), and $g\sigma = h\sigma$.
For statement (ii), $gs(a) = hs(a)$ implies $(h^{-1}g)s(a) = s(a)$ and $(h^{-1}g)a = a$ by Definition 8.11 part (ii), and finally $ga = ha$.
For statement (iii), suppose $\sigma \cong \tau$ and $g\sigma = \sigma$. We have

$$\tau \cong \sigma = g\sigma \cong g\tau.$$

Then $\tau = g\tau$ by (i), and $G_\sigma \subseteq G_\tau$. The proof is symmetric, so we also have $G_\tau \subseteq G_\sigma$. □

Definition 8.15 (Quotient of a scwol by a group action). If a scwol $\mathcal{X}$ is equipped with a G-action as above, then the *quotient scwol* $\mathcal{X}/G$ has objects and morphisms

$$\mathrm{ob}(\mathcal{X}/G) := (\mathrm{ob}(\mathcal{X}))/G$$
$$\mathrm{mor}(\mathcal{X}/G) := (\mathrm{mor}(\mathcal{X}))/G.$$

Composition and identities are induced by those of $\mathcal{X}$.

Remark 8.16 (III.$\mathcal{C}$.1.13 of Bridson–Haefliger [11]). The projection functor $p\colon \mathcal{X} \to \mathcal{X}/G$ induces a bijection

$$\{a \in \mathrm{mor}(\mathcal{X}) | sa = x\} \longrightarrow \{b \in \mathrm{mor}(\mathcal{X}/G) | sb = p(x)\} \tag{8.17}$$

for each $x \in \mathcal{X}$. If $G/\mathcal{X}$ is connected and the action of G on $\mathrm{ob}(\mathcal{X})$ is free, then p is a *covering of scwols*. That is, in addition to the bijection (8.17), p induces a bijection

$$\{a \in \mathrm{mor}(\mathcal{X}) | ta = x\} \longrightarrow \{b \in \mathrm{mor}(\mathcal{X}/G) | tb = p(x)\} \tag{8.18}$$

for each $x \in \mathcal{X}$.

Lemma 8.19 (Quotients of skeletal scwols are skeletal). *If $\mathcal{X}$ is a skeletal scwol, and a group G acts on $\mathcal{X}$ in the sense of Definition 8.11, then the quotient scwol $\mathcal{X}/G$ is also skeletal.*

Proof. Suppose $\overline{\sigma}$ is isomorphic to $\overline{\tau}$ in $\mathcal{X}/G$. We show $\overline{\sigma}$ is actually equal to $\overline{\tau}$. If $\overline{a}\colon \overline{\sigma} \to \overline{\tau}$ is an isomorphism with inverse $\overline{b}$, then there are lifts $a\colon \sigma \to \tau$ and $b\colon \tau \to \sigma'$ in $\mathcal{X}$, and an element $g \in G$ such that $g(ba) = \mathrm{id}_\sigma$. Since g fixes the source of ba, the group element g fixes also ba, so $ba = \mathrm{id}_\sigma$ and $\sigma' = \sigma$. Since ab is an endomorphism of τ, it is therefore id_τ. By the skeletality of $\mathcal{X}$, we have $\sigma = \tau$, and also $\overline{\sigma} = \overline{\tau}$. □

Lemma 8.20 (Quotient of path set is set of paths in quotient). *Suppose $\mathcal{X}$ is a scwol equipped with an action of a group G in the sense of Definition 8.11. Let $\Lambda_n(\mathcal{X})$ respectively $\Lambda_n(\mathcal{X}/G)$ denote the set of paths of n-many non-identity composable morphisms in $\mathcal{X}$ respectively $\mathcal{X}/G$. Give $\Lambda_n(\mathcal{X})$ the induced G-action. Then the function*

$$\Lambda_n(\mathcal{X}) \to \Lambda_n(\mathcal{X}/G)$$

$$(a_1, \ldots, a_n) \mapsto (\overline{a}_1, \ldots, \overline{a}_n)$$

induces a bijection $\Lambda_n(\mathcal{X})/G \to \Lambda_n(\mathcal{X}/G)$.

Proof. Remark 8.16 implies that a path $(a_1, \ldots, a_n)$ in $\mathcal{X}$ consists entirely of non-identity morphisms if and only if the projection $(\overline{a}_1, \ldots, \overline{a}_n)$ in $\mathcal{X}/G$ consists entirely of non-identity morphisms, so from now on we work only with non-identity morphisms. Note

$$(g_1a_1, g_2a_2, \ldots, g_na_n) = (g_1a_1, g_1a_2, \ldots, g_1a_n)$$

by Definition 8.11 (ii). For injectivity, we have $(\overline{a}_1, \ldots, \overline{a}_n) = (\overline{b}_1, \ldots, \overline{b}_n)$ if and only if for some $g_i \in G$

$$(g_1a_1, g_2a_2, \ldots, g_na_n) = (b_1, \ldots, b_n),$$

which happens if and only if for some $g \in G$

$$(ga_1, ga_2, \ldots, ga_n) = (b_1, \ldots, b_n),$$

(take $g = g_1$). For the surjectivity, we can lift any path $(\overline{a}_1, \ldots, \overline{a}_n)$ by first lifting $\overline{a}_1$ to a_1, then $\overline{a}_2$ to a_2, and so on using Remark 8.16. □

Definition 8.21 (Complex of groups from a group action on a scwol, 2.9 of Bridson–Haefliger [11]). Let G be a group and $\mathcal{X}$ a scwol upon which G acts in the sense of Definition 8.11. Let $p\colon \mathcal{X} \to \mathcal{X}/G$ denote the quotient map.
Haefliger and Bridson–Haefliger define a pseudo functor $F\colon \mathcal{X}/G \to \mathsf{GROUPS}$ as follows. In the procedure choices are made, but different choices lead to isomorphic complexes of groups. For each object $\overline{\sigma}$ of $\mathcal{X}/G$, choose an object σ of $\mathcal{X}$ such that $p(\sigma) = \overline{\sigma}$ (our overline convention is the opposite of that in [11]). Then $F(\overline{\sigma})$ is defined to be G_σ, the isotropy group of σ under the G-action.
If $\overline{a}\colon \overline{\sigma} \to \overline{\tau}$ is a morphism in $\mathcal{X}/G$, then there exists a unique morphism a in $\mathcal{X}$ such that $p(a) = \overline{a}$ and $sa = \sigma$, as in (8.17). For $\overline{a}$ we choose an element $h_{\overline{a}} \in G$ such that $h_{\overline{a}} \cdot ta$ is the object τ of $\mathcal{X}$ chosen above so that $p(\tau) = \overline{\tau}$. An injective group homomorphism $F(\overline{a})\colon G_\sigma \to G_\tau$ is defined by

$$F(\overline{a})(g) := h_{\overline{a}} g h_{\overline{a}}^{-1}.$$

Suppose $\overline{a}$ and $\overline{b}$ are composable morphisms of $\mathcal{X}/G$. We define a 2-cell in GROUPS

$$F_{\overline{b},\overline{a}}\colon F(\overline{b}) \circ F(\overline{a}) \Rightarrow F(\overline{b} \circ \overline{a})$$

to be $(h_{\overline{b}\overline{a}} h_{\overline{a}}^{-1} h_{\overline{b}}^{-1}, F(\overline{b}) \circ F(\overline{a}))$ as in Notation 8.1.
The pseudo functor $F\colon \mathcal{X}/G \to \mathsf{GROUPS}$ is called the *complex of groups associated to the group action of G on the scwol $\mathcal{X}$*. This complex of groups

comes equipped with a morphism to the group G, that is, a pseudo natural transformation $F \Rightarrow \Delta_G$. The inclusion of each isotropy group $F(\overline{\sigma}) = G_\sigma$ into G provides the components of the pseudo natural transformation.

EXAMPLE 8.22. The quotient scwols for the actions in Examples 8.12 and 8.13 are both $\{k \leftarrow j \rightarrow \ell\}$, and the associated complexes of groups are both

$$\mathbb{Z}_2 \longleftarrow \{0\} \longrightarrow \mathbb{Z}_2.$$

REMARK 8.23. If a group G acts on a scwol in the sense of Definition 8.11, each object stabilizer is finite, and the quotient scwol is finite, then the associated complex of groups $F\colon \mathcal{X}/G \to \mathsf{GROUPS}$ satisfies all of the hypotheses of the Homotopy Colimit Formula in Theorem 4.1 (vii) and in Corollary 4.2 (vii). If, in addition, R is a ring such that the order $|H|$ of each object stabilizer $H \subset G$ is invertible in R, then $F\colon \mathcal{X}/G \to \mathsf{GROUPS}$ also satisfies all of the hypotheses of the Homotopy Colimit Formula in Theorem 4.1 (vi) and in Corollary 4.2 (vi). See Examples 8.12, 8.13, and 8.22.

Even without finiteness assumptions, it is possible to replace scwols with skeletal scwols and preserve much of the accompanying structure, as Theorem 8.24 explains.

THEOREM 8.24 (Reduction to skeletal case). *Let G be a group acting on a scwol $\mathcal{X}$ in the sense of Definition 8.11. Let Γ be any skeleton of $\mathcal{X}$, $i\colon \Gamma \to \mathcal{X}$ the inclusion, and $r\colon \mathcal{X} \to \Gamma$ a functor equipped with a natural isomorphism $ir \cong \mathrm{id}_{\mathcal{X}}$, and satisfying $ri = \mathrm{id}_\Gamma$. Then there is a G-action on the scwol Γ in the sense of Definition 8.11 such that following hold.*

(i) *The functor r is G-equivariant.*

(ii) *The induced functor $\overline{r}$ on quotient categories is an equivalence of categories compatible with the quotient maps, that is, the diagram below commutes.*

(8.25)
$$\begin{tikzcd} \mathcal{X} \arrow[r, "r"] \arrow[d, "p^{\mathcal{X}}"'] & \Gamma \arrow[d, "p^{\Gamma}"] \\ \mathcal{X}/G \arrow[r, "\overline{r}"'] & \Gamma/G \end{tikzcd}$$

(iii) *The inclusion $i\colon \Gamma \to \mathcal{X}$ preserves stabilizers, that is $G_{i\gamma} = G_\gamma$ for all $\gamma \in \mathrm{ob}(\Gamma)$. Note that the inclusion may not be G-equivariant.*

(iv) *Choices can be made in the definitions of $F^{\mathcal{X}}$ and F^{Γ} (the complexes of groups associated to the G-actions on $\mathcal{X}$ and Γ in Definition 8.21), so that the diagram below strictly commutes.*

(8.26)
$$\begin{tikzcd} \mathcal{X}/G \arrow[rr, "\overline{r}"] \arrow[dr, "F^{\mathcal{X}}"'] & & \Gamma/G \arrow[dl, "F^{\Gamma}"] \\ & \mathsf{GROUPS} & \end{tikzcd}$$

(v) *The functor* $(\overline{r}, \mathrm{id})$ *is an equivalence of categories*

$$(\overline{r}, \mathrm{id})\colon \operatorname{hocolim}_{\mathcal{X}/G} F^{\mathcal{X}} \longrightarrow \operatorname{hocolim}_{\Gamma/G} F^{\Gamma}.$$

(vi) *If* G *acts freely on* $\mathrm{ob}(\mathcal{X})$*, then* G *acts freely on* $\mathrm{ob}(\Gamma)$*.*

Proof. To define the group action, let $\mathrm{Aut}(\mathcal{X})$ and $\mathrm{Aut}(\Gamma)$ denote the strictly invertible endofunctors on $\mathcal{X}$ and Γ respectively, and consider the monoid homomorphism

$$\varphi\colon \mathrm{Aut}(\mathcal{X}) \to \mathrm{End}(\Gamma), \quad F \mapsto r \circ F \circ i. \tag{8.27}$$

This is strictly multiplicatively because the natural isomorphism of functors

$$\begin{aligned} r \circ G \circ F \circ i &= r \circ G \circ \mathrm{id}_{\mathcal{X}} \circ F \circ i \\ &\cong (r \circ G \circ i) \circ (r \circ F \circ i), \end{aligned}$$

and skeletality of Γ imply $\varphi(GF)$ agrees with $\varphi(G)\varphi(F)$ on objects of Γ, so each component $\varphi(GF)(\gamma) \cong \varphi(G)\varphi(F)(\gamma)$ is an endomorphism in the scwol Γ, and is therefore trivial. By naturality, $\varphi(GF)$ and $\varphi(G)\varphi(F)$ agree on morphisms also. Consequently, φ takes values in $\mathrm{Aut}(\Gamma)$ and is a homomorphism $\varphi\colon \mathrm{Aut}(\mathcal{X}) \to \mathrm{Aut}(\Gamma)$.

We define a G-action on Γ as the composite of the action $G \to \mathrm{Aut}(\mathcal{X})$ with φ in (8.27). We indicate the action of g on Γ by $\varphi(g)\gamma$ and the action of g on $\mathcal{X}$ by gx. For simplicity, we suppress i from the notation when indicating the G-action in $\mathcal{X}$ on objects and morphisms of Γ, so for example, if a is morphism in Γ, then $gs(a)$ actually means $gis(a)$ throughout.

To verify Definition 8.11 (i) for Γ, suppose a is a nontrivial morphism in Γ and $\varphi(g)s(a) = t(a)$, that is $rgs(a) = t(a)$. Then $gs(a) \cong t(a)$ in $\mathcal{X}$, but $gs(a) \neq t(a)$ (for if $gs(a) = t(a)$, then a must be trivial by Definition 8.11 (i) for $\mathcal{X}$). Let $b\colon t(a) \to gs(a)$ be an isomorphism in $\mathcal{X}$ and consider the composite $ba\colon s(a) \to t(a) \to gs(a)$. Then $gs(ba) = gs(a) = t(ba)$, so ba must be trivial by Definition 8.11 (i) for $\mathcal{X}$. Consequently $a = b^{-1}$ is a nontrivial *iso*morphism in Γ, and we have a contradiction to either skeletality or the no loops requirement. Thus $\varphi(g)s(a) \neq t(a)$, and Definition 8.11 (i) holds for Γ. The verification of Definition 8.11 (ii) is shorter: if a is a nontrivial morphism in Γ and $\varphi(g)s(a) = s(a)$, that is $rgs(a) = s(a)$, then $gs(a) \cong s(a)$, and $gs(a) = s(a)$ by Lemma 8.14 (i) for $\mathcal{X}$. Finally, $ga = a$ by Definition 8.11 (ii) for $\mathcal{X}$, $rga = a$ as a is in Γ, and $\varphi(g)a = a$. The action of G on Γ satisfies Definition 8.11 and we may form the quotient scwol Γ/G as in Defition 8.15, which is skeletal by Lemma 8.19.

(i) For the G-equivariance of r, let $f\colon x \to y$ be a morphism in $\mathcal{X}$ and consider the naturality diagram.

$$\begin{tikzcd} rgirx \arrow[r, "rgirf=\varphi(g)r(f)"] \arrow[d, "\cong"'] & rgiry \arrow[d, "\cong"] \\ rgx \arrow[r, "rgf"'] & rgy \end{tikzcd}$$

The vertical morphisms must be identities by skeletality of Γ and the no loops condition, so $\varphi(g)r(f) = r(gf)$. Equivariance on objects then follows by taking $f = \mathrm{id}_x$.

(ii) Diagram (8.25) commutes by definition of $\overline{r}$. The functor $\overline{r}$ is surjective on objects because $p^\Gamma r$ and $p^{\mathcal{X}}$ are. The functor $\overline{r}$ is fully faithful since the equivariant bijection $r(x,y)\colon \operatorname{mor}_{\mathcal{X}}(x,y) \to \operatorname{mor}_\Gamma(r(x), r(y))$ induces the equivariant bijection $\overline{r}(p^{\mathcal{X}}x, p^{\mathcal{X}}y)$.

(iii) Let $\gamma \in \operatorname{ob}(\Gamma)$, and suppose $gi\gamma = i\gamma$. Then

$$\begin{aligned} \varphi(g)\gamma &\stackrel{\text{def}}{=} r(gi\gamma) \\ &= r(i\gamma) \\ &= \gamma \end{aligned}$$

and $G_{i\gamma} \subseteq G_\gamma$. Now suppose $\varphi(g)\gamma = \gamma$. Then $r(gi\gamma) = \gamma$ by definition, and $gi\gamma \cong i\gamma$ in $\mathcal{X}$, which says $g \cdot i\gamma = i\gamma$ by Lemma 8.14 (i), and $G_\gamma \subseteq G_{i\gamma}$.

(iv) We claim that choices can be made in the definitions of the associated complexes of groups $F^{\mathcal{X}}$ and F^Γ (see Definition 8.21) so that diagram (8.26) strictly commutes. First choose a skeleton $\mathcal{Q}$ of the quotient $\mathcal{X}/G$, define $F^{\mathcal{X}}$ on object in the skeleton $\mathcal{Q}$, and then extend to all objects in $\mathcal{X}/G$. For every $\overline{q} \in \operatorname{ob}(\mathcal{Q})$, select a $q \in \operatorname{ob}(\mathcal{X})$ such that $p^{\mathcal{X}}(q) = \overline{q}$ and define $F^{\mathcal{X}}(\overline{q}) = G_q$. We remain with the choice of the selected preimage q of $\overline{q}$ throughout. If $\overline{\sigma} \in \operatorname{ob}(\mathcal{X}/G)$ and $\overline{a}\colon \overline{q} \cong \overline{\sigma}$ is an isomorphism in $\mathcal{X}/G$, then also define $F^{\mathcal{X}}(\overline{\sigma}) = G_q$. This is allowed, since $\overline{a}\colon \overline{q} \cong \overline{\sigma}$ implies existence of morphisms $a\colon q \to g_\sigma\sigma$ and $b\colon \sigma \to g_q q$ in $\mathcal{X}$, and the composite

$$q \xrightarrow{a} g_\sigma\sigma \xrightarrow{g_\sigma b} g_\sigma g_q q$$

is trivial by Definition 8.11 (i). The opposite composite is also trivial, as it is a loop, and we have $q \cong g_\sigma\sigma$ in $\mathcal{X}$. Then by Lemma 8.14 (iii), $G_q = G_{g_\sigma\sigma}$ and we may define $F^{\mathcal{X}}(\overline{\sigma}) = G_q$ because $p^{\mathcal{X}}(g_\sigma\sigma) = \overline{\sigma}$. In particular, the selected preimage of $\overline{\sigma}$ in $\mathcal{X}$ is $g_\sigma\sigma$ and we select $h_{\overline{a}} = e_G$ for $\overline{a}\colon \overline{q} \cong \overline{\sigma}$ in Definition 8.21, so $F^{\mathcal{X}}(\overline{a}) = \mathrm{id}_{G_q}$. We remark that the isomorphism $\overline{a}$ is the only morphism $\overline{q} \to \overline{\sigma}$ because there are no loops in $\mathcal{X}/G$, so the element $g_\sigma\sigma$ is uniquely defined as the target of the unique morphism a with source q and $p^{\mathcal{X}}$-image $\overline{a}$. We next define F^Γ on objects of Γ/G using the equivalence $\overline{r}$ and the definition of $F^{\mathcal{X}}$ on objects of $\mathcal{Q}$. For $\overline{q} \in \operatorname{ob}(\mathcal{Q})$, we also define $F^\Gamma(\overline{r}(\overline{q})) = G_q$. This is allowed: for $\overline{r}(\overline{q}) = \overline{r(q)}$ we choose $r(q)$ as the selected preimage in $\operatorname{ob}(\Gamma)$, and $ir(q) \cong q$ in $\mathcal{X}$, so $G_{r(q)} = G_{ir(q)} = G_q$ by (iii) and Lemma 8.14 (iii). Every $\overline{\gamma} \in \operatorname{ob}(\Gamma/G)$ is of the form $\overline{r}(\overline{q})$ for a unique $\overline{q} \in \mathcal{Q}$, so F^Γ is now defined on all objects of Γ/G, and we have $F^\Gamma \circ \overline{r} = F^{\mathcal{X}}$ on all objects of $\mathcal{X}/G$.

We must now define $F^{\mathcal{X}}$ and F^Γ on morphisms so that $F^\Gamma \circ \overline{r} = F^{\mathcal{X}}$ for morphisms also. The idea is to first define $F^{\mathcal{X}}$ on morphisms in the skeleton $\mathcal{Q}$, then extend to all of $\mathcal{X}/G$, and then define F^Γ on morphisms of Γ/G. If $\overline{a}\colon \overline{q}_1 \to \overline{q}_2$ is a morphism in $\mathcal{Q}$, then there is a unique morphism a in $\mathcal{X}$ with source q_1 and $p^{\mathcal{X}}(a) = \overline{a}$. Select any $h_{\overline{a}}$ such that $h_{\overline{a}}ta = q_2$. Then we define an injective group homomorphism $F(\overline{a})\colon G_{q_1} \to G_{q_2}$ by

$$F(\overline{a})(g) := h_{\overline{a}} g h_{\overline{a}}^{-1}.$$

If $\overline{b}\colon \overline{\sigma}_1 \to \overline{\sigma}_2$ is any morphism in $\mathcal{X}/G$, then there exists a unique $\overline{a}$ in $\mathcal{Q}$ and a unique commutative diagram with vertical isomorphisms as below.

$$\begin{tikzcd} \overline{q}_1 \arrow[r, "\overline{a}"] \arrow[d, "\cong"'] & \overline{q}_2 \arrow[d, "\cong"] \\ \overline{\sigma}_1 \arrow[r, "\overline{b}"'] & \overline{\sigma}_2 \end{tikzcd}$$

Then we choose $h_{\overline{b}}$ to be $h_{\overline{a}}$, and we consequently have $F(\overline{a}) = F(\overline{b})$. If $\overline{c}\colon \overline{r}(\overline{q}_1) \to \overline{r}(\overline{q}_2)$ is a morphism in Γ/G, then there is a unique $\overline{a}\colon \overline{q}_1 \to \overline{q}_2$ in $\mathcal{Q}$ with $\overline{r}(\overline{a}) = \overline{c}$ and we choose $h_{\overline{c}}$ to be $h_{\overline{a}}$. Manifestly, we have $F^{\Gamma} \circ \overline{r} = F^{\mathcal{X}}$. The coherences of $F^{\mathcal{X}}$ and F^{Γ} are also compatible, since they are determined by the $h_{\overline{a}}$'s.

(v) From (ii) we know $\overline{r}$ is a surjective-on-objects equivalence of categories and from (iv) we have $F^{\mathcal{X}} = F^{\Gamma} \circ \overline{r}$. From this, one sees

$$(\overline{r}, \mathrm{id})\colon \operatorname{hocolim}_{\mathcal{X}/G} F^{\mathcal{X}} = \operatorname{hocolim}_{\mathcal{X}/G} F^{\Gamma} \circ \overline{r} \longrightarrow \operatorname{hocolim}_{\Gamma/G} F^{\Gamma}$$

is an equivalence of categories.

(vi) If the action of G on $\mathrm{ob}(\mathcal{X})$ is free, then for each $\gamma \in \mathrm{ob}(\Gamma)$, the group $G_{\gamma} = G_{i\gamma}$ (see (iii)) is trivial, and G acts freely on $\mathrm{ob}(\Gamma)$. $\square$

REMARK 8.28. In Theorem 8.24, it is even possible to select a skeleton so that the inclusion is G-equivariant, though we will not need this. See Section 9.

In [16, Theorems 5.30 and 5.37], we proved the compatibility of the L^2-Euler characteristic with coverings and isofibrations of finite connected groupoids. Theorem 8.29 is an analogue for scwols (see Remark 8.16).

THEOREM 8.29 (Compatibility with free actions on finite scwols). *Let G be a finite group acting on a finite scwol $\mathcal{X}$. If G acts freely on* $\mathrm{ob}(\mathcal{X})$*, then*

$$\chi(\mathcal{X}/G;R) = \frac{\chi(\mathcal{X};R)}{|G|} \quad \textit{and} \quad \chi^{(2)}(\mathcal{X}/G) = \frac{\chi^{(2)}(\mathcal{X})}{|G|}.$$

Recall $\chi(-;R)$ and $\chi^{(2)}(-)$ agree for finite scwols by Example 8.7.

Proof. By Theorem 8.24 (i), (ii), and (vi), we may assume $\mathcal{X}$ is skeletal.
A consequence of Definition 8.11 (ii) (independent of skeletality) is that an element $g \in G$ fixes a path $a = (a_1, \ldots, a_n)$ in $\mathcal{X}$ if and only if g fixes sa_1, so $G_{sa_1} = G_a$. Then G acts freely on $\Lambda_n(\mathcal{X})$, since it acts freely on $\mathrm{ob}(\mathcal{X})$.

The scwol $\mathcal{X}/G$ is skeletal by Lemma 8.19, and by Example 8.7 and Lemma 8.20 we have

$$\begin{aligned}
\chi^{(2)}(\mathcal{X}/G) &= \sum_{n\geq 0}(-1)^n c_n(\mathcal{X}/G) \\
&= \sum_{n\geq 0}(-1)^n |\Lambda_n(\mathcal{X}/G)| \\
&= \sum_{n\geq 0}(-1)^n |\Lambda_n(\mathcal{X})/G| \\
&= \sum_{n\geq 0}(-1)^n \frac{|\Lambda_n(\mathcal{X})|}{|G|} \\
&= \frac{1}{|G|}\sum_{n\geq 0}(-1)^n |\Lambda_n(\mathcal{X})| \\
&= \frac{1}{|G|}\sum_{n\geq 0}(-1)^n c_n(\mathcal{X}) \\
&= \frac{\chi^{(2)}(\mathcal{X})}{|G|}.
\end{aligned}$$

□

A complex of groups is called *developable* if it is isomorphic to a complex of groups associated to a group action. A classical theorem of Bass–Serre says that every complex of groups on a scwol with maximal path length 1 is developable. The following gives a necessary condition of developability of a complex of groups from a scwol and group of specified Euler characteristics.

Theorem 8.30 (Euler characteristics of associated complexes of groups). *Let G be a finite group that acts on a finite scwol $\mathcal{X}$ in the sense of Definition 8.11. Let $F\colon \mathcal{X}/G \to \mathsf{GROUPS}$ be the associated complex of groups. Then*

$$\chi^{(2)}(\operatorname{hocolim}_{\mathcal{X}/G} F) = \frac{\chi^{(2)}(\mathcal{X})}{|G|} = \frac{\chi(\mathcal{X};\mathbb{C})}{|G|} = \frac{\chi(B\mathcal{X};\mathbb{C})}{|G|}.$$

If R is a ring such that the orders of subgroups $H \subset G$ are invertible in R, then we also have

$$\chi(\operatorname{hocolim}_{\mathcal{X}/G} F; R) = \chi(\mathcal{X}/G; R).$$

Proof. By Theorem 8.24 (i), (ii), (iv), and (v), we may assume $\mathcal{X}$ is skeletal. Then $\mathcal{X}/G$ is also skeletal by Lemma 8.19.

Let $\Lambda_n(\mathcal{X})$ respectively $\Lambda_n(\mathcal{X}/G)$ denote the set of paths of n-many non-identity composable morphisms in $\mathcal{X}$ respectively $\mathcal{X}/G$. Then by Lemma 8.20, the sets $\Lambda_n(\mathcal{X})/G$ and $\Lambda_n(\mathcal{X}/G)$ are in bijective correspondence.

We will also use the fact that an element $g \in G$ fixes a path $a = (a_1, \dots, a_n)$ in $\mathcal{X}$ if and only if g fixes sa_1, so $G_{sa_1} = G_a$. This is a consequence of Definition 8.11 (ii).

By Theorem 8.5, $E^{\mathrm{bar}}\mathcal{X}$ and $E^{\mathrm{bar}}(\mathcal{X}/G)$ are finite models for the skeletal scwols $\mathcal{X}$ and $\mathcal{X}/G$, and the n-cells are indexed by $\Lambda_n(\mathcal{X})$ and $\Lambda_n(\mathcal{X}/G)$, respectively. For each path $(a_1, \dots, a_n)$ in $\mathcal{X}$, there is an n-cell in $E^{\mathrm{bar}}\mathcal{X}$ based at sa_1. A similar statement holds for $\mathcal{X}/G$ and $E^{\mathrm{bar}}(\mathcal{X}/G)$.
Now we may apply the Homotopy Colimit Formula to the associated complex of groups $F : \mathcal{X}/G \to \mathsf{GROUPS}$ by Remark 8.23. For the Euler characteristic, we have

$$\begin{aligned}
\chi(\operatorname{hocolim}_{\mathcal{X}/G} F; R) &= \sum_{n\geq 0} (-1)^n \cdot \left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X}/G)} \chi(F(s\overline{a}_1); R) \right) \\
&= \sum_{n\geq 0} (-1)^n \cdot \left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X}/G)} 1 \right) \\
&= \sum_{n\geq 0} (-1)^n |\Lambda_n(\mathcal{X}/G)| \\
&= \sum_{n\geq 0} (-1)^n c_n(\mathcal{X}/G) \\
&= \chi(\mathcal{X}/G; R).
\end{aligned}$$

For the L^2-Euler characteristic on the other hand, we have

$$\begin{aligned}
\chi^{(2)}(\operatorname{hocolim}_{\mathcal{X}/G} F) &= \sum_{n\geq 0} (-1)^n \cdot \left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X}/G)} \chi^{(2)}(F(s\overline{a}_1)) \right) \\
&= \sum_{n\geq 0} (-1)^n \cdot \left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X}/G)} \frac{1}{|G_{sa_1}|} \right) \\
&= \sum_{n\geq 0} (-1)^n \cdot \left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X})/G} \frac{1}{|G_a|} \right) \\
&= \sum_{n\geq 0} (-1)^n \cdot \left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X})/G} \frac{|\operatorname{orbit}(a)|}{|G|} \right) \\
&= \frac{1}{|G|} \sum_{n\geq 0} (-1)^n \cdot \left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X})/G} |\operatorname{orbit}(a)| \right) \\
&= \frac{1}{|G|} \sum_{n\geq 0} (-1)^n |\Lambda_n(\mathcal{X})| \\
&= \frac{1}{|G|} \sum_{n\geq 0} (-1)^n c_n(\mathcal{X}) \\
&= \frac{\chi^{(2)}(\mathcal{X})}{|G|}.
\end{aligned}$$

□

Example 8.31. By the classical theorem of Bass–Serre, any injective group homomorphism

$$G_0 \to G_1 \tag{8.32}$$

is a developable complex of groups. The L^2-Euler characteristic of the homotopy colimit of (8.32) is $1/|G_1|$ by Example 5.1. Theorem 8.30 then says we must have

$$\frac{|G|}{|G_1|} = \chi^{(2)}(\mathcal{X}) = \chi(B\mathcal{X}; \mathbb{C})$$

if (8.32) is to be developable from a scwol $\mathcal{X}$ by an action of G in the sense of Definition 8.11. Thus (8.32) is not developable from any scwol $\mathcal{X}$ whose geometric realization has Euler characteristic 0, such as $\{j \rightrightarrows k\}$. Nor can (8.32) be developed from any scwol $\mathcal{X}$ with $\chi(B\mathcal{X}; \mathbb{C})$ negative. The integer $|G|$ must also be divisible by $|G_1|$, since $\chi(B\mathcal{X}; \mathbb{C})$ is always an integer. Moreover, the Euler characteristic of $\mathcal{X}$ must be less than or equal to $|G|$. This trivial example illustrates how one can find necessary conditions on $\mathcal{X}$ and G if a given complex of groups is to be developable from $\mathcal{X}$ and G.

Example 8.33 (Euler characteristics of transport groupoid in finite case). Let X be a finite set and G a finite group acting on X. Let R be a ring such that the orders of subgroups of G are invertible in R. Considering X as a scwol, we clearly have an action in the sense of Definition 8.11. The associated complex of groups $F : X/G \to \mathsf{GROUPS}$ assigns to orbit(σ) the stabilizer G_σ. The homotopy colimit $\operatorname{hocolim}_{X/G} F$ is equivalent to the transport groupoid $\mathcal{G}^G(X)$ of Example 5.6, so

$$\chi\left(\mathcal{G}^G(X); R\right) = \chi(\operatorname{hocolim}_{X/G} F; R) = \chi(X/G; R) = |X/G|.$$

For the L^2-Euler characteristic, on the other hand, we have

$$\chi^{(2)}\left(\mathcal{G}^G(X)\right) = \chi^{(2)}(\operatorname{hocolim}_{X/G} F) = \frac{\chi^{(2)}(X)}{|G|} = \frac{|X|}{|G|},$$

a formula obtained by Baez–Dolan [4].

We also generalize the following formula of Haefliger for the Euler characteristic of the homotopy colimit of a (not necessarily developable) complex of groups.

Theorem 8.34 (Corollary 3.5.3 of Haefliger [19]). *Let $G(X)$ be a complex of groups over a finite ordered simplicial cell complex X. Assume that each G_σ is the fundamental group of a finite aspherical cell complex. Then $BG(X)$ has the homotopy type of a finite complex and its Euler-Poincaré characteristic is* *given by*[4]

$$\chi(BG(X)) = \sum_{\sigma \in \mathrm{ob}(C(X))} (1 - \chi(Lk^\sigma))\chi(G_\sigma).$$

The terms in Haefliger's theorem have the following meanings. An *ordered simplicial cell complex* X is by definition the nerve of a skeletal scwol, denoted $C(X)$. The notation $BG(X)$ signifies the geometric realization of the nerve of the homotopy colimit of the pseudo functor $G(X)\colon C(X) \to \mathsf{GROUPS}$. An *aspherical* cell complex is one for which all homotopy groups beyond the fundamental group vanish. The *lower link* Lk^σ *of the object* σ is the full subcategory of the scwol $\sigma \downarrow C(X)$ on all objects except 1_σ.

Theorem 8.35 (Extension of Corollary 3.5.3 of Haefliger [19]). *Let $\mathcal{I}$ be a finite skeletal scwol and $F\colon \mathcal{I} \to \mathsf{GROUPS}$ a complex of groups such that for each object i of $\mathcal{I}$, the group $F(i)$ is of type ($FF_{\mathbb{Z}}$). Then*

$$\chi(B \operatorname{hocolim}_{\mathcal{I}} F) = \sum_{i \in \mathrm{ob}(\mathcal{I})} (1 - \chi(BLk^i))\chi(BF(i)),$$

where B indicates geometric realization composed with the nerve functor.

Proof. All hypotheses of Theorem 4.1(vi) are satisfied. The skeletal scwol $\mathcal{I}$ is directly finite by Lemma 8.4 and admits a finite $\mathcal{I}$-CW-model for its $\mathcal{I}$-classifying space by Theorem 8.5. Each group $\mathcal{C}(i)$ is automatically directly finite, and assumed to be of type ($\mathrm{FF}_{\mathbb{Z}}$). The bar construction model $E^{\mathrm{bar}}\mathcal{I}$ in Remark 2.10 has an n-cell based at i for each path of n-many non-identity morphisms in $\mathcal{I}$

$$i \to i_1 \to i_2 \to \cdots \to i_n.$$

Each such path in $\mathcal{I}$ corresponds uniquely to a path of $(n-1)$-many non-identity morphisms in the scwol Lk^i beginning at the object $i \to i_1$. Thus

$$\begin{aligned}
1 - \chi(BLk^i) &= 1 - \sum_{m \geq 0} (-1)^m c_m(Lk^i) \\
&= 1 - \sum_{m \geq 0} (-1)^m \mathrm{card}\{(m+1)\text{-paths in } \mathcal{I} \text{ beginning at } i\} \\
&= 1 - \sum_{n \geq 1} (-1)^{n-1} \mathrm{card}\{n\text{-paths in } \mathcal{I} \text{ beginning at } i\} \\
&= 1 + \sum_{n \geq 1} (-1)^{n} \mathrm{card}\{n\text{-paths in } \mathcal{I} \text{ beginning at } i\} \\
&= \sum_{n \geq 0} (-1)^{n} \mathrm{card}\{n\text{-paths in } \mathcal{I} \text{ beginning at } i\}.
\end{aligned}$$

[4]Haefliger's original formula has, instead of the lower link L^σ, the upper link L_σ, which is the full subcategory of the scwol $C(X) \downarrow \sigma$ on all objects except 1_σ. However, this is merely a typo, for if we use the upper link Lk_σ and consider the example $C(X) = \{k \leftarrow j \to \ell\}$ with pseudo functor $G(X)(\ell) := \mathbb{Z}$ and $G(X)(j) := G(X)(k) := \{0\}$, then $\chi(BG(X)) = \chi(S^1) = 0$ but $\sum(1 - \chi(Lk_\sigma))\chi(G_\sigma) = 1$.

Then by Theorem 4.1 (i), Theorem 4.1 (iv), Theorem 1.15, and Theorem 4.1 (vi), we have

$$\begin{aligned}\chi(B\operatorname{hocolim}_{\mathcal{I}} F) &= \chi(\operatorname{hocolim}_{\mathcal{I}} F)\\ &= \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \chi(F(i_\lambda))\\ &= \sum_{i\in \mathrm{ob}(\mathcal{I})} \big(1-\chi(BLk^i)\big)\cdot\chi(F(i))\\ &= \sum_{i\in \mathrm{ob}(\mathcal{I})} \big(1-\chi(BLk^i)\big)\cdot\chi(BF(i)).\end{aligned}$$

□

Remark 8.36. The assumptions in our Theorem 8.35 on the groups $F(i)$ are related to the assumptions in Theorem 8.34 on the groups G_σ in that any finitely presentable group of type ($\mathrm{FF}_{\mathbb{Z}}$) admits a finite model for its classifying space.

9. Appendix

Let G be a group acting on a scwol $\mathcal{X}$ in the sense of Definition 8.11. In connection with Theorem 8.24, we remark that it is possible to choose a skeleton Γ_0 of $\mathcal{X}$, a G-equivariant functor $r\colon \mathcal{X}\to\Gamma_0$, and a natural isomorphism $\eta\colon ir \cong \mathrm{id}_{\mathcal{X}}$ so that

- the inclusion $i_0\colon \Gamma_0\to\mathcal{X}$ is G-equivariant,
- $ri_0 = \mathrm{id}_{\Gamma_0}$, and
- for every object $x\in\mathrm{ob}(\mathcal{X})$ and each $g\in G$, we have $\eta_{gx} = g\eta_x$.

To prove this, we first choose the object set of Γ_0 via an equivariant section of the projection $\pi\colon \mathrm{ob}(\mathcal{X})\to\mathrm{iso}(\mathcal{X})$, which assigns to each object of $\mathcal{X}$ its isomorphism class of objects. Let Θ denote the set of G-orbits of $\mathrm{iso}(\mathcal{X})$. For each G-orbit $\theta\in\Theta$, we use the axiom of choice to select an element $\overline{x}_\theta\in\theta$. For each θ, select then a π-preimage $s(\overline{x}_\theta) := x_\theta$ of $\overline{x}_\theta$. On the orbit of each $\overline{x}_\theta$ we define the section s by $s(g\overline{x}_\theta) := gx_\theta$. This is well defined, for if $g_1\overline{x}_\theta = g_2\overline{x}_\theta$, then $g_1x_\theta \cong g_2x_\theta$, and $g_1x_\theta = g_2x_\theta$ by Lemma 8.14 (i). Define the skeleton Γ_0 to be the full subcategory of $\mathcal{X}$ on the objects in the image of the equivariant section $s\colon \mathrm{iso}(\mathcal{X})\to\mathrm{ob}(\mathcal{X})$.

For each $\overline{x}_\theta$, and each $x\in\overline{x}_\theta$, choose an isomorphism $\eta_x\colon x_\theta\to x$. For gx, we define η_{gx} as $g\eta_x$. Next, we define a functor $r\colon\mathcal{X}\to\Gamma_0$ on objects $x\in\mathrm{ob}(\mathcal{X})$ by $r(x) := s\pi(x)$ and on morphisms $f\colon x\to y$ by $r(f) := \eta_y\circ f\circ\eta_x^{-1}$. Then η is clearly a natural isomorphism, the inclusion $i_0\colon\Gamma_0\to\mathcal{X}$ is G-equivariant, and $ri_0 = \mathrm{id}_{\Gamma_0}$.

References

[1] M. Aigner. *Combinatorial theory.* Springer-Verlag, Berlin, 1979.
[2] S. Allouch. Sur l'existence d'une catégorie ayant une matrice strictement positive donnée. arXiv:0806.0060v1 [math.CT], 2008.

[3] S. Allouch. On the existence of a category with a given matrix. arXiv:1007.2884v1 [math.CT], 2010.

[4] J. Baez and J. Dolan. From finite sets to Feynman diagrams. In *Mathematics Unlimited - 2001 and Beyond*, vol. 1, eds. Björn Engquist and Wilfried Schmid, Springer, Berlin, 2001, pp. 29-50.

[5] A. Bartels and W. Lück. The Borel conjecture for hyperbolic and CAT(0)-groups. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Münster, arXiv:0901.0442v1 [math.GT], 2009.

[6] A. Bartels, W. Lück, and H. Reich. The K-theoretic Farrell-Jones conjecture for hyperbolic groups. *Invent. Math.*, 172(1):29–70, 2008.

[7] A. Bartels, W. Lück, and H. Reich. On the Farrell-Jones Conjecture and its applications. *Journal of Topology*, 1:57–86, 2008.

[8] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and K-theory of group C^*-algebras. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[9] C. Berger and T. Leinster. The Euler characteristic of a category as the sum of a divergent series. *Homology, Homotopy Appl.*, 10(1):41–51, 2008.

[10] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 304.

[11] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*. Springer-Verlag, Berlin, 1999. Die Grundlehren der mathematischen Wissenschaften, Band 319.

[12] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[13] J. Cheeger and M. Gromov. L_2-cohomology and group cohomology. *Topology*, 25(2):189–215, 1986.

[14] J. F. Davis and W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory*, 15(3):201–252, 1998.

[15] E. Dror Farjoun. Homotopy and homology of diagrams of spaces. Algebraic topology, Proc. Workshop, Seattle/Wash. 1985, Lect. Notes Math. 1286, 93-134 (1987).

[16] T. M. Fiore, W. Lück, and R. Sauer. Finiteness obstructions and Euler characteristics of categories. arXiv, 2010. Accepted to *Advances in Mathematics*.

[17] A. Grothendieck. *Pursuing Stacks*. 1983.

[18] A. Haefliger. Complexes of groups and orbihedra. Group theory from a geometrical viewpoint (Trieste, 1990), 504–540, World Sci. Publ., River Edge, NJ, 1991.

[19] A. Haefliger. Extension of complexes of groups. *Ann. Inst. Fourier (Grenoble)*, 42(1-2):275–311, 1992.

[20] L. Illusie. *Complexe cotangent et déformations. II.* Springer-Verlag, Berlin-New York, 1972.

[21] T. Leinster. The Euler characteristic of a category. *Doc. Math.*, 13:21–49, 2008.
[22] W. Lück. Equivariant Eilenberg-Mac Lane spaces $K(G, \mu, 1)$ for possibly non-connected or empty fixed point sets. *Manuscripta Math.*, 58:67–75, 1987.
[23] W. Lück. *Transformation groups and algebraic K-theory*. Springer-Verlag, Berlin, 1989. Mathematica Gottingensis.
[24] W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers. I. Foundations. *J. Reine Angew. Math.*, 495:135–162, 1998.
[25] W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers. II. Applications to Grothendieck groups, L^2-Euler characteristics and Burnside groups. *J. Reine Angew. Math.*, 496:213–236, 1998.
[26] W. Lück. The universal functorial Lefschetz invariant. *Fund. Math.*, 161(1-2):167–215, 1999. Algebraic topology (Kazimierz Dolny, 1997).
[27] W. Lück. *L^2-invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.
[28] W. Lück. L^2-invariants from the algebraic point of view. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 295, Münster, arXiv:math.GT/0310489 v1, 2003.
[29] W. Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005.
[30] W. Lück and D. Meintrup. On the universal space for group actions with compact isotropy. In *Geometry and topology: Aarhus (1998)*, pages 293–305. Amer. Math. Soc., Providence, RI, 2000.
[31] W. Lück and H. Reich. The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory. In *Handbook of K-theory. Vol. 1, 2*, pages 703–842. Springer, Berlin, 2005.
[32] G. Maltsiniotis. La théorie de l'homotopie de Grothendieck. *Astérisque* 301, 2005.
[33] D. Meintrup and T. Schick. A model for the universal space for proper actions of a hyperbolic group. *New York J. Math.*, 8:1–7 (electronic), 2002.
[34] R. J. Piacenza. Homotopy theory of diagrams and CW-complexes over a category. *Can. J. Math.* 43(4):814–824, 1991.
[35] M. D. Rabrenović. Classifying spaces of monoids—applications in geometric combinatorics. *Publ. Inst. Math. (Beograd) (N.S.)*, 77(91):21–27, 2005.
[36] J.-P. Serre. *Linear representations of finite groups*. Springer-Verlag, New York, 1977. Translated from the second French edition by Leonard L. Scott, *Graduate Texts in Mathematics*, Vol. 42.

[37] J.-P. Serre. *Trees.* Springer-Verlag, Berlin, 1980. Translated from the French by J. Stillwell.

[38] P. Selick *Introduction to homotopy theory* AMS, Providence, 1997 *Fields Institute Monographs.*

[39] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967.

[40] R. W. Thomason. Homotopy colimits in the category of small categories. *Math. Proc. Camb. Philos. Soc.* 85:91-109, 1979.

[41] R. W. Thomason. Cat as a closed model category. *Cahiers Topologie Géom. Différentielle* 21(3):305-324, 1980.

[42] T. tom Dieck. *Transformation groups and representation theory.* Springer-Verlag, Berlin, 1979.

[43] T. tom Dieck. *Transformation groups.* volume 8 of *de Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, 1987.

[44] C. T. C. Wall. Finiteness conditions for CW-complexes. *Annals of Math. (2)*, 81:56–69, 1965.

[45] C. T. C. Wall. Finiteness conditions for CW-complexes. II. *Proc. Roy. Soc. Ser. A*, 295:129–139, 1966.

Thomas M. Fiore
Department of Mathematics
and Statistics
University of Michigan-Dearborn
4901 Evergreen Road
Dearborn, MI 48128
U.S.A.
tmfiore@umd.umich.edu
http://www-personal.umd.umich.edu/~tmfiore/

Wolfgang Lück
Mathematisches Institut
der Universität Bonn
Endenicher Allee 60
53115 Bonn
Germany
wolfgang.lueck@him.uni-bonn.de
http://www.him.uni-bonn.de/lueck

Roman Sauer
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg
Germany
roman.sauer@mathematik.uni-regensburg.de
http://www.mathematik.uni-regensburg.de/sauer/

Surfaces de Stein Associées aux Surfaces de Kato Intermédiaires

Laurent Battisti *

Received: August 2, 2010
Revised: November 5, 2010

Communicated by Thomas Peternell

Abstract.
Soient S une surface de Kato intermédiaire, D le diviseur formé des courbes rationnelles de S, $\widetilde{S}$ le revêtement universel de S et $\widetilde{D}$ la préimage de D dans $\widetilde{S}$. On donne deux résultats concernant la surface $\widetilde{S} \setminus \widetilde{D}$, à savoir qu'elle est de Stein (ce qui était connu dans le cas où S est une surface d'Enoki ou d'Inoue-Hirzebruch) et on donne une condition nécessaire et suffisante pour que son fibré tangent holomorphe soit holomorphiquement trivialisable.

2010 Mathematics Subject Classification: 32E10, 32E30, 32J15, 32M25.
Keywords and Phrases: Variétés de Stein, surfaces de Kato.

1. Introduction

Les surfaces de la classe VII de Kodaira sont les surfaces complexes compactes dont le premier nombre de Betti vaut 1 ; on appelle surface de la classe VII_0 une surface de la classe VII qui est minimale. Le cas de ces surfaces dont le second nombre de Betti b_2 est nul est entièrement compris, il s'agit nécessairement d'une surface de Hopf ou d'une surface d'Inoue et le cas $b_2 > 0$ est toujours étudié actuellement ; il a été conjecturé qu'elles contiennent toutes une coquille sphérique globale. La preuve de ce résultat terminerait la classification des surfaces complexes compactes.
Les surfaces à coquille sphérique globale, qui nous intéressent ici, peuvent être obtenues selon un procédé dû à Kato (voir [11]), que l'on rappelle dans la section suivante. Ces surfaces se divisent en trois classes, les surfaces d'Enoki, d'Inoue-Hirzebruch et enfin les surfaces intermédiaires.

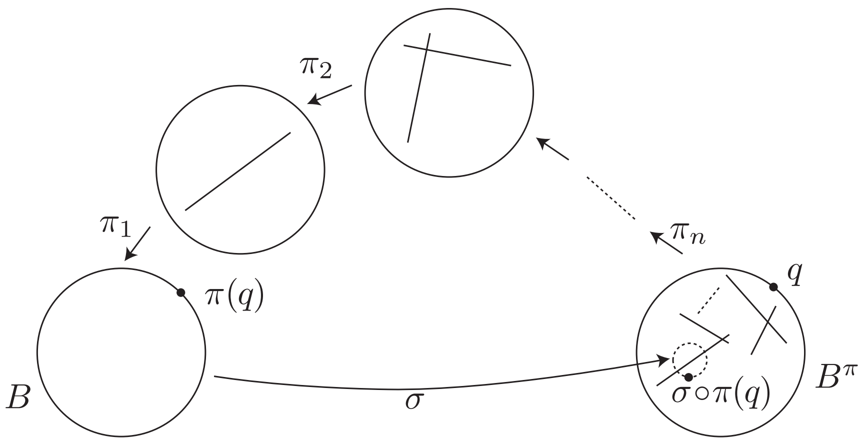
∗. Je tiens à remercier le referee pour ses remarques très importantes. Le financement de cette recherche est assuré par la Région Provence-Alpes-Côte d'Azur dans le cadre d'une bourse doctorale régionale.

Étant donnés une surface minimale S à coquille sphérique globale, D le diviseur maximal de S formé des $b_2(S)$ courbes rationnelles de S et $\varpi : \widetilde{S} \to S$ le revêtement universel de S, nous allons démontrer que $\widetilde{S} \setminus \widetilde{D}$ (où $\widetilde{D} = \varpi^{-1}(D)$) est une variété de Stein. Ce résultat était déjà connu pour les surfaces d'Enoki et d'Inoue-Hirzebruch ; nous allons le montrer dans le cas des surfaces intermédiaires. Dans la dernière partie et toujours dans le cas des surfaces intermédiaires, on donne une condition pour que le fibré tangent holomorphe de la variété $\widetilde{S} \setminus \widetilde{D}$ soit holomorphiquement trivialisable, à savoir que la surface S soit d'indice 1.

2. Préliminaires

On dit qu'une surface compacte S contient une coquille sphérique globale s'il existe une application qui envoie biholomorphiquement un voisinage de la sphère $\mathbb{S}^3 \subset \mathbb{C}^2 \setminus \{0\}$ dans S et telle que le complémentaire dans S de l'image de la sphère par cette application soit connexe.

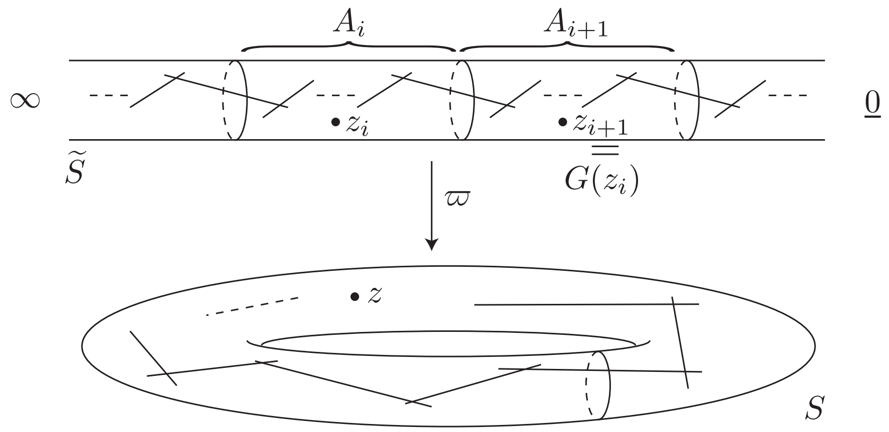
Toute surface contenant une coquille sphérique globale peut être obtenue de la façon suivante : étant données une succession finie d'éclatements $\pi_1, ..., \pi_n$ de la boule unité B de $\mathbb{C}^2$ au-dessus de 0 et $\pi := \pi_1 \circ \cdots \circ \pi_n : B^\pi \to B$ la composée de ces éclatements, ainsi qu'une application $\sigma : \overline{B} \to B^\pi$ biholomorphe sur un voisinage de $\overline{B}$, on recolle les deux bords de $\mathrm{Ann}(\pi, \sigma) := B^\pi \setminus \sigma(\overline{B})$ à l'aide de l'application $\sigma \circ \pi$:



La surface obtenue possède un groupe fondamental isomorphe à $\mathbb{Z}$ et son second nombre de Betti est égal à n (voir [2]). Il s'agit d'une construction due à Kato [11]. Dans la suite, on appellera surface de Kato une surface complexe compacte minimale contenant une coquille sphérique globale, dont le second nombre de Betti est non nul.

Dans [2], Dloussky étudie le germe contractant d'application holomorphe $\varphi = \pi \circ \sigma : B \to B$ associé à la construction précédente. Ce germe détermine à isomorphisme près la surface étudiée (proposition 3.16 loc. cit.).

Soit S une surface de Kato ; on note D le diviseur maximal de S formé des $b_2(S)$ courbes de S, $\widetilde{S}$ le revêtement universel de S et $\widetilde{D}$ la préimage de D dans $\widetilde{S}$.
Suivant les notations de [2], on obtient la surface $\widetilde{S}$ en recollant une infinité d'anneaux A_i $(i \in \mathbb{Z})$ isomorphes à $\mathrm{Ann}(\pi, \sigma)$, en identifiant le bord pseudo-concave de A_i au bord pseudo-convexe de A_{i+1} via l'application $\sigma \circ \pi$. La surface $\widetilde{S}$ possède deux bouts, notés $\underline{0}$ et ∞, le bout $\underline{0}$ possédant une base de voisinages ouverts strictement pseudo-convexes (les $\bigcup_{i \geqslant j} A_i$ pour $j \in \mathbb{Z}$) et le second une base de voisinages strictement pseudo-concaves (les $\bigcup_{i \leqslant j} A_i$ pour $j \in \mathbb{Z}$). Enfin on définit un automorphisme G de $\widetilde{S}$ en posant $G(z_i) := z_{i+1}$ où z_i et z_{i+1} sont les images dans A_i et A_{i+1} respectivement d'un même point $z \in \mathrm{Ann}(\pi, \sigma)$.



Fixons une courbe compacte C de $\widetilde{S}$ avec $C \subset A_0$. On note $(\widehat{S}_C, p_C)$ l'effondrement de $\widetilde{S}$ sur la courbe C, c'est-à-dire la donnée d'une surface $\widehat{S}_C$ n'ayant qu'un bout, d'une application holomorphe p_C de $\widetilde{S}$ dans $\widehat{S}_C$, biholomorphe sur un voisinage du bout ∞ dans $\widetilde{S}$ sur un voisinage du bout de $\widehat{S}_C$, telles que $\widehat{C} = p_C(C)$ soit une courbe d'auto-intersection -1.
La proposition 3.4 de [2] nous assure l'existence d'une telle application p_C pour toute courbe compacte C de $\widetilde{S}$, et d'un point $\widehat{0}_C \in \widehat{C}$ tel que p_C soit également biholomorphe entre $\widetilde{S} \setminus p_C^{-1}(\widehat{0}_C)$ et $\widehat{S}_C \setminus \{\widehat{0}_C\}$.
De plus, la restriction de p_C au complémentaire de $\widetilde{D}$ est un biholomorphisme entre $\widetilde{S} \setminus \widetilde{D}$ et $\widehat{S}_C \setminus p_C(\widetilde{D})$. Enfin, il existe une application holomorphe F_C de $\widehat{S}_C \setminus \{\widehat{0}_C\}$ dans lui-même, contractante en $\widehat{0}_C$, conjuguée à φ et biholomorphe sur $\widehat{S}_C \setminus p_C(\widetilde{D})$.

3. La variété $\widetilde{S} \setminus \widetilde{D}$ est de Stein

Les surfaces de Kato se divisent en trois classes : les surfaces d'Enoki, d'Inoue-Hirzebruch et enfin les surfaces intermédiaires (voir [5]).
Dans le cas des surfaces d'Inoue-Hirzebruch et celles d'Enoki, le fait que $\widetilde{S} \setminus \widetilde{D}$ soit de Stein est déjà connu : pour une surface d'Inoue-Hirzebruch, la variété $\widetilde{S} \setminus \widetilde{D}$ est un domaine de Reinhardt holomorphiquement convexe (voir [13], proposition 2.2) tandis que pour une surface d'Enoki, on a $\widetilde{S} \setminus \widetilde{D} \cong \mathbb{C}^* \times \mathbb{C}$ qui sont bien dans chaque cas des variétés de Stein. Il reste donc à étudier le cas des surfaces intermédiaires.

Favre a donné dans [7] des formes normales pour les germes contractants d'applications holomorphes et on peut en particulier donner la forme du germe associé à une surface intermédiaire, à savoir qu'une telle surface est associée au germe φ de $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ donné par

$$(z, \zeta) \mapsto (\lambda \zeta^s z + P(\zeta) + c_{\frac{sk}{k-1}} \zeta^{\frac{sk}{k-1}}, \zeta^k) \tag{1}$$

où $\lambda \in \mathbb{C}^*$, $k, s \in \mathbb{N}$ avec $k > 1$ et $s > 0$, et $P(\zeta) = c_j \zeta^j + ... + c_s \zeta^s$ avec les conditions suivantes : $0 < j < k$, $j \leqslant s$, $c_j = 1$, $c_{\frac{sk}{k-1}} = 0$ quand $\frac{sk}{k-1} \notin \mathbb{Z}$ ou $\lambda \neq 1$ et enfin $\mathrm{pgcd}\{k, m \mid c_m \neq 0\} = 1$. On trouve dans [12] une condition pour que deux tels germes soient conjugués (et déterminent donc deux surfaces isomorphes).

L'objectif de cette section est de démontrer, dans le cas de surfaces intermédiaires, le

Théorème 3.1. *La surface $\widetilde{S} \setminus \widetilde{D}$ est de Stein.*

Dans un premier temps (section 3.1), on montre qu'il est suffisant de se ramener à la situation du théorème 3.2 énoncé ci-dessous. Pour cela, nous allons écrire notre surface comme réunion croissante d'ouverts et nous verrons que seule une hypothèse manque *a priori* pour pouvoir effectivement appliquer ce théorème, à savoir que chaque paire constituée de deux tels ouverts consécutifs est de Runge. C'est dans la section 3.2 qu'on prouve que cette hypothèse est bien vérifiée.

3.1. Réduction du problème. Reprenons les notations précédentes et donnons-nous un germe de la forme (1). On regarde la surface intermédiaire S associée et on choisit une courbe C de $\widetilde{S}$ donnée par la proposition 3.16 de [2] ; quitte à renuméroter les A_i on suppose que $C \subset A_0$. Notre objectif est de prouver que la variété $\widehat{S}_C \setminus p_C(\widetilde{D})$ est de Stein, en utilisant le théorème suivant (voir [10], théorème 10 p. 215) :

Théorème 3.2. *Soient X un espace analytique complexe et $(X_i)_{i \in \mathbb{N}}$ une suite croissante de sous-espaces de X qui soient de Stein. Supposons que $X = \bigcup X_i$ et que chaque paire (X_{i+1}, X_i) est de Runge, i.e. l'ensemble $\mathcal{O}(X_i)|_{X_{i+1}}$ des* *restrictions à X_i des applications holomorphes sur X_{i+1} est dense dans $\mathcal{O}(X_i)$. Alors X est de Stein.*

Notons :
- $\widehat{A}_i := p_C(A_i)$ pour tout $i \in \mathbb{Z}$ et
- $\mathcal{A}_i := p_C(\bigcup_{j \geqslant i} A_j)$ pour $i \leqslant 0$,

de sorte qu'on a $\mathcal{A}_i \subset \mathcal{A}_{i-1}$ et $\widehat{S}_C \setminus p_C(\widetilde{D}) = \bigcup_{i \leqslant 0} \mathcal{A}_i \setminus p_C(\widetilde{D})$.

Chaque $\mathcal{A}_i \setminus p_C(\widetilde{D})$ est strictement pseudo-convexe, donc de Stein. De plus, on a $F_C(\widehat{A}_i) = \widehat{A}_{i+1}$ pour $i \leqslant -1$, car le diagramme

$$\begin{tikzcd} \widetilde{S} \arrow[r, "G"] \arrow[d, "p_C"'] & \widetilde{S} \arrow[d, "p_C"] \\ \widehat{S}_C \arrow[r, "F_C"'] & \widehat{S}_C \end{tikzcd}$$

est commutatif (c.f. [2], proposition 3.9). Ainsi, on a

$$F_C(\mathcal{A}_{i-1} \setminus p_C(\widetilde{D})) = \mathcal{A}_i \setminus p_C(\widetilde{D}) \tag{2}$$

Supposons établi le fait que la paire $(\mathcal{A}_0 \setminus p_C(\widetilde{D}), F_C(\mathcal{A}_0 \setminus p_C(\widetilde{D})))$ est de Runge. Alors la paire $(\mathcal{A}_{-1} \setminus p_C(\widetilde{D}), \mathcal{A}_0 \setminus p_C(\widetilde{D}))$ est automatiquement de Runge par l'égalité (2) ci-dessus, et par récurrence chaque paire $(\mathcal{A}_{i-1} \setminus p_C(\widetilde{D}), \mathcal{A}_i \setminus p_C(\widetilde{D}))$ est de Runge. Nous sommes alors en mesure d'appliquer le théorème 3.2 qui nous dit que la réunion des $\mathcal{A}_i \setminus p_C(\widetilde{D})$ est de Stein.
Le problème est donc ramené à montrer que le couple $(\mathcal{A}_0 \setminus p_C(\widetilde{D}), F_C(\mathcal{A}_0 \setminus p_C(\widetilde{D})))$ est de Runge.

Remarque 3.3. L'ensemble $\mathcal{A}_0 \setminus p_C(\widetilde{D})$ est biholomorphe à une boule ouverte centrée en 0 privée d'une droite complexe. En effet, on peut écrire $\varphi = \pi \circ \sigma$ où π est une succession d'éclatements de la boule au-dessus de $0 \in \mathbb{C}^2$, $\sigma : \overline{B} \to \pi^{-1}(B)$ est une application définie sur un voisinage de $\overline{B}$ et biholomorphe sur son image, et φ est de la forme normale (1). Par le choix de la courbe C, la proposition 3.16 p. 33 de [2] nous donne l'isomorphisme $\mathcal{A}_0 \setminus p_C(\widetilde{D}) \cong B \setminus \varphi^{-1}(0)$ et en utilisant la forme de φ, on voit que $\varphi^{-1}(0) = \{\zeta = 0\}$.

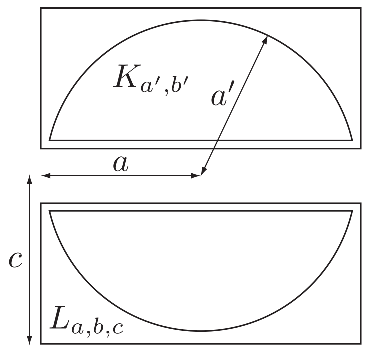
Finalement, démontrer que $(\mathcal{A}_0 \setminus p_C(\widetilde{D}), F_C(\mathcal{A}_0 \setminus p_C(\widetilde{D})))$ est de Runge revient à prouver que c'est le cas de la paire $(B \setminus \{\zeta = 0\}, \varphi(B \setminus \{\zeta = 0\}))$ pour une boule $B \subset \mathbb{C}^2$ centrée en 0 (en notant (z, ζ) les coordonnées de $\mathbb{C}^2$). C'est l'objet de la section suivante.

3.2. La paire $(B \setminus \{\zeta = 0\}, \varphi(B \setminus \{\zeta = 0\}))$ est de Runge. Etant donné un germe φ de la forme (1), introduisons en premier lieu quelques notations :

1. Remarquons tout d'abord que chaque point de $\mathbb{C}\times\Delta^*$ possède exactement k antécédents par φ, où Δ^* est le disque unité ouvert de $\mathbb{C}$ privé de 0. Notons g l'automorphisme de $\mathbb{C}\times\Delta^*$ suivant :
$$g:(z,\zeta)\mapsto\left(\varepsilon^{-s}z+\frac{P(\zeta)-P(\varepsilon\zeta)}{\lambda\varepsilon^s\zeta^s},\varepsilon\zeta\right)$$
où ε est une racine primitive k-ième de l'unité, de sorte que $\varphi\circ g=\varphi$. Pour tout $\ell\in\mathbb{Z}$, on a
$$g^\ell(z,\zeta)=\left((\varepsilon^\ell)^{-s}z+\frac{P(\zeta)-P(\varepsilon^\ell\zeta)}{\lambda(\varepsilon^\ell)^s\zeta^s},\varepsilon^\ell\zeta\right)$$
et $g^{\mathbb{Z}}\cong\mathbb{Z}/k\mathbb{Z}$. L'automorphisme g permute les antécédents d'un même point de l'application φ.
2. On notera également $q(z,\zeta)$ le polynôme $z\displaystyle\prod_{\ell=1}^{k-1}a_\ell(z,\zeta)\zeta^{n_\ell}$ où $a_\ell(z,\zeta)$ est la première composante de $g^\ell(z,\zeta)$ et $n_\ell=s-\min\{n|c_n(1-(\varepsilon^\ell)^n)\neq 0\}$, qui est bien défini et positif ou nul vu la dernière hypothèse sur les coefficients de P, à savoir $\operatorname{pgcd}\{k,m\mid c_m\neq 0\}=1$. Le polynôme $q(z,\zeta)$ est en particulier de la forme $q(z,\zeta)=z(c+\epsilon(z,\zeta))$ où $\epsilon(z,\zeta)\xrightarrow[(z,\zeta)\to(0,0)]{}0$ et $c\neq 0$.
3. Pour $\eta>0$, on note U_η l'ouvert $\{(z,\zeta)\in\mathbb{C}^2\mid |q(z,\zeta)|<\eta\}$. Soient a, b et c trois réels strictement positifs, on définit les ensembles
$$K_{a,b}:=\{(z,\zeta)\in\mathbb{C}^2\mid |z|^2+|\zeta|^2\leqslant a^2,|\zeta|\geqslant b\}=\overline{B(0,a)}\cap\{|\zeta|\geqslant b\}$$
et
$$L_{a,b,c}:=\overline{\mathbb{D}(0,a)}\times\overline{\mathbb{A}_{b,c}}$$
(où $\mathbb{A}_{b,c}$ est l'anneau ouvert centré en 0 de rayons $b<c$). Enfin, on pose
$$\mathcal{K}_{a,b}:=\bigcup_{\ell=0}^{k-1}g^\ell(K_{a,b})$$
et
$$\mathcal{L}_{a,b,c}:=\bigcup_{\ell=0}^{k-1}g^\ell(L_{a,b,c}).$$

Remarque 3.4. Pour a,b et c assez petits, les compacts $g^\ell(L_{a,b,c})$ (resp. $g^\ell(K_{a,b})$) sont disjoints deux à deux : ceci est une conséquence du fait que la fonction φ est localement injective autour de l'origine de $\mathbb{C}^2$, ce qui est démontré, par exemple, dans [5], section 5. En particulier, les ensembles $\mathcal{L}_{a,b,c}$ et $\mathcal{K}_{a,b}$ possèdent chacun k composantes connexes.
D'autre part, on a $K_{a',b'}\subset L_{a,b,c}$ pour $b'\geqslant b$ et $a'\leqslant\min\{a,c\}$, ce qui entraîne notamment $\mathcal{K}_{a',b'}\subset\mathcal{L}_{a,b,c}$.

Enfin, pour $\eta>0$ fixé, il existe $A_\eta>0$ tel que pour tous réels t et δ avec



$0 < t < \delta < A_\eta$, on ait $\mathcal{L}_{\delta,t,\delta} \subset U_\eta$: en calculant $|q(z,\zeta)|$ pour $(z,\zeta) \in \mathcal{L}_{\delta,t,\delta}$ on voit qu'il suffit de choisir δ assez petit pour avoir

$$|\delta| \prod_{\ell=1}^{k-1} (|\delta^{n_\ell+1}| + 2(|c_{s-n_\ell}| + |c_{s-n_\ell+1}|\delta + ... + |c_s|\delta^{n_\ell})/\lambda) < \eta. \tag{3}$$

On appelle $V_{\eta,\delta}$ l'ensemble $U_\eta \cap \{|\zeta| \leqslant \delta\}$.

Proposition 3.5. *Pour $\delta > 0$ assez petit et pour tout $\varepsilon_1 \in]0,\delta[$, le compact $\mathcal{K}_{\delta,\varepsilon_1}$ est holomorphiquement convexe.*

Preuve : En premier lieu, remarquons que l'enveloppe holomorphiquement convexe de $V_{\eta,\delta}$ est l'adhérence $\overline{V}_{\eta,\delta}$ de cet ensemble. On note :
- $\widehat{\mathcal{K}}_{\delta,\varepsilon_1}$ l'enveloppe holomorphiquement convexe de $\mathcal{K}_{\delta,\varepsilon_1}$,
- $\widehat{\mathcal{K}}^\ell_{\delta,\varepsilon_1}$ (resp. $\overline{V}^\ell_{\eta,\delta}$) la composante connexe de $\widehat{\mathcal{K}}_{\delta,\varepsilon_1}$ (resp. $\overline{V}_{\eta,\delta}$) qui contient $g^\ell(K_{\delta,\varepsilon_1})$, pour $\ell \in \{0,...,k-1\}$.

Étape 1 : Montrons tout d'abord que pour η et δ assez petits et pour tout $\varepsilon_1 < \delta$, on a $\overline{V}^0_{\eta,\delta} \cap \mathcal{K}_{\delta,\varepsilon_1} = K_{\delta,\varepsilon_1}$, autrement dit que la composante connexe de $\overline{V}_{\eta,\delta}$ qui contient K_{δ,ε_1} ne rencontre aucune autre composante de $\mathcal{K}_{\delta,\varepsilon_1}$.

Soient $\delta > \varepsilon_1 > 0$. Pour $\ell \in \{1,...,k-1\}$, on a

$$g^\ell(K_{\delta,\varepsilon_1}) \subset g^\ell(L_{\delta,\varepsilon_1,\delta}) = \{(z,\zeta) \in \mathbb{C}^2 \mid |a_{k-\ell}(z,\zeta)| \leqslant \delta, |\zeta| \in [\varepsilon_1,\delta]\}.$$

En particulier, pour $(z,\zeta) \in g^\ell(L_{\delta,\varepsilon_1,\delta})$, on a $z = \dfrac{P(\varepsilon^{k-\ell}\zeta) - P(\zeta)}{\lambda\zeta^s} + w$ où $|w| \leqslant \delta$. En développant, cette égalité devient

$$\begin{aligned} z = \ & \lambda^{-1}\zeta^{-n_\ell}\big(c_{s-n_\ell}((\varepsilon^{-\ell})^{s-n_\ell} - 1) + c_{s-n_\ell+1}((\varepsilon^{-\ell})^{s-n_\ell+1} - 1)\zeta + ... \\ & ... + c_s((\varepsilon^{-\ell})^s - 1)\zeta^{n_\ell}\big) + w. \end{aligned}$$

Autrement dit, z est de la forme

$$\lambda^{-1}\zeta^{-n_\ell}((c_{s-n_\ell}((\varepsilon^{-\ell})^{s-n_\ell}-1)+\zeta R_\ell(\zeta,w))$$

où R_ℓ est un polynôme et par définition de n_ℓ, le terme $c_{s-n_\ell}((\varepsilon^{-\ell})^{s-n_\ell}-1)$ est non nul.

Grâce à cette dernière expression de z, on voit que lorsque $n_\ell > 0$, pour n'importe quelle constante $C > 0$ et lorsque δ est assez petit, tout élément $(z,\zeta) \in g^\ell(L_{\delta,\varepsilon_1,\delta})$ vérifie $|z| > C$.
Dans le cas où $n_\ell = 0$ (donc $c_s \neq 0$), on a $|z| = |\lambda^{-1}(c_s((\varepsilon^{-\ell})^s - 1)) + w|$ est supérieur à une constante non nulle pour δ assez petit.

Posons alors $\alpha := \frac{1}{2}\min_\ell\{|c_s((\varepsilon^\ell)^s-1)| \mid \ell s \not\equiv 0[k]\}$ si $c_s \neq 0$ et $\alpha := 1$ sinon. Par ce qui précède, il existe une constante $A > 0$ telle que pour tous $\delta < A$ et $\varepsilon_1 < \delta$ on ait, pour chaque $\ell \in \{1,...,k-1\}$ et tout élément (z,ζ) de $g^\ell(L_{\delta,\varepsilon_1,\delta})$, l'inégalité

$$|z| \geqslant \alpha. \tag{4}$$

Fixons désormais $\eta > 0$ vérifiant les deux conditions suivantes :
1. $2\eta/|c| < \alpha$ (où $c \neq 0$ est le facteur de z dans le développement limité de q en $(0,0)$, à savoir $q(z,\zeta) = z(c+\epsilon(z,\zeta))$), et
2. $|c+\epsilon(z,\zeta)| > |c|/2$ pour tout $(z,\zeta) \in \mathbb{D}(0,3\eta/|c|) \times \overline{\mathbb{D}(0,3\eta/|c|)}$.

Choisissons maintenant $\delta < \min\{A, A_\eta, 2\eta/|c|\}$ et $\varepsilon_1 \in]0,\delta[$. Alors on a $\mathcal{L}_{\delta,\varepsilon_1,\delta} \subset U_\eta$ (remarque 3.4) et l'inégalité (4) ci-dessus est vérifiée.

Pour tout $|z| \leqslant \delta$ et $|\zeta| \in [\varepsilon_1, \delta]$ on a $(z,\zeta) \in K_{\delta,\varepsilon_1} \subset \overline{V}^0_{\eta,\delta}$. Soit maintenant $\ell \in \{1,...,k-1\}$ et (z,ζ) un point de $g^\ell(K)$, on a $|z| \geqslant \alpha$ et ceci entraîne que $(z,\zeta) \notin \overline{V}^0_{\eta,\delta}$.

En effet, supposons le contraire : la projection de $\overline{V}^0_{\eta,\delta}$ sur la première coordonnée étant connexe, et comme $\delta < 2\eta/|c| < \alpha$, il devrait exister un élément $(z',\zeta') \in \overline{V}^0_{\eta,\delta}$ avec $|z'| = 2\eta/|c|$, ce qui est impossible puisque dans ce cas $|q(z',\zeta')| > \dfrac{2\eta}{|c|}(|c|/2) = \eta$.
Ainsi, la composante connexe $\overline{V}^0_{\eta,\delta}$ de $\overline{V}_{\eta,\delta}$ qui contient K_{δ,ε_1} ne rencontre aucune autre composante de $\mathcal{K}_{\delta,\varepsilon_1}$, ce qu'il fallait démontrer.

À partir de maintenant, on omet les indices δ, ε_1.

Étape 2 : Montrons à présent que $\widehat{\mathcal{K}}^0 = K$. Par l'étape 1, et comme $\widehat{\mathcal{K}}^0 \subset \overline{V}^0$, on sait que $\widehat{\mathcal{K}}^0$ ne rencontre pas d'autre composante de $\mathcal{K}$ que l'ensemble K lui-même.
Soit $(z_0,\zeta_0) \in \widehat{\mathcal{K}}^0 \setminus K$. On suppose que $|\zeta_0| \geqslant \varepsilon_1$ (sinon $(z_0,\zeta_0) \notin \widehat{\mathcal{K}}$), donc

nécessairement $|z_0|^2 + |\zeta_0|^2 > \delta^2$. Comme la boule fermée $\overline{B} := \overline{B(0,\delta)}$ est holomorphiquement convexe dans $\mathbb{C}^2$, il existe une fonction h holomorphe sur $\mathbb{C}^2$ telle que $|h(z_0,\zeta_0)| > \|h\|_{\overline{B}}$. Notons respectivement m_0 et $m_{\overline{B}}$ les quantités $|h(z_0,\zeta_0)|$ et $\|h\|_{\overline{B}}$, ainsi que $m_{\widehat{\mathcal{K}}}$ la quantité $\|h\|_{\widehat{\mathcal{K}}}$, qui est finie puisque $\widehat{\mathcal{K}}$ est compact.

Considérons la fonction $\chi_{\widehat{\mathcal{K}}^0}$ définie sur $\widehat{\mathcal{K}}$ valant 1 sur $\widehat{\mathcal{K}}^0$ (en particulier sur K) et 0 sur $\widehat{\mathcal{K}} \setminus \widehat{\mathcal{K}}^0$ (en particulier sur $g^\ell(K)$ pour $\ell \not\equiv 0[k]$).

Le théorème 6' p. 213 de [10] nous dit que la fonction $\chi_{\widehat{\mathcal{K}}^0}$ est limite uniforme sur $\widehat{\mathcal{K}}$ de fonctions holomorphes sur $\mathbb{C} \times \Delta^*$. Soit donc f une fonction holomorphe vérifiant $\|f - \chi_{\widehat{\mathcal{K}}^0}\|_{\widehat{\mathcal{K}}} < \varepsilon'$ avec $\varepsilon' < \min\left\{\dfrac{m_0}{m_{\widehat{\mathcal{K}}} + m_0}, \dfrac{m_0 - m_{\overline{B}}}{m_0 + m_{\overline{B}}}\right\}$ et appelons F l'application $(z,\zeta) \mapsto h(z,\zeta)f(z,\zeta)$.

Pour $(z_\ell,\zeta_\ell) \in g^\ell(K)$ (avec $\ell \in \{1,...,k-1\}$), on a l'inégalité

$$|F(z_\ell,\zeta_\ell)| \leqslant \|F - h\chi_{\widehat{\mathcal{K}}^0}\|_{\widehat{\mathcal{K}}} + |h(z_\ell,\zeta_\ell)\chi_{\widehat{\mathcal{K}}^0}(z_\ell,\zeta_\ell)|.$$

Le second terme du membre de droite est nul ; quant au premier, il est majoré par $m_{\widehat{\mathcal{K}}}\varepsilon'$. De plus, on a $|F(z_0,\zeta_0)| = m_0|f(z_0,\zeta_0)| > m_0(1-\varepsilon')$ d'une part, et pour tout $(z,\zeta) \in K$ on a

$$|F(z,\zeta)| \leqslant |h(z,\zeta)\left(f(z,\zeta) - \chi_{\widehat{\mathcal{K}}^0}(z,\zeta)\right)| + |h(z,\zeta)\chi_{\widehat{\mathcal{K}}^0}(z,\zeta)|$$

donc $|F(z,\zeta)| \leqslant m_{\overline{B}}\varepsilon' + m_{\overline{B}}$ d'autre part. Le choix de ε' nous assure que $\max\{m_{\widehat{\mathcal{K}}}\varepsilon', m_{\overline{B}}(\varepsilon'+1)\} < m_0(1-\varepsilon')$. Autrement dit, nous avons montré que $(z_0,\zeta_0) \notin \widehat{\mathcal{K}}$, d'où une contradiction. Ainsi, on a bien établi que $\widehat{\mathcal{K}}^0 = K$.

Étape 3 : Il nous reste à conclure. Remarquons que l'enveloppe holomorphe convexe $\widehat{\mathcal{K}}$ de $\mathcal{K}$ est également stable par g et supposons qu'il existe $\ell_0 \in \{1,...,k-1\}$ et un point $(z_{\ell_0},\zeta_{\ell_0}) \in \widehat{\mathcal{K}}^{\ell_0} \setminus g^{\ell_0}(K)$. Alors on a les inclusions suivantes :

$$K \subset g^{-\ell_0}(\widehat{\mathcal{K}}^{\ell_0}) \subset \widehat{\mathcal{K}}^0,$$

la dernière inclusion provenant du fait que la continuité de g entraîne la connexité de $g^{-\ell_0}(\widehat{\mathcal{K}}^{\ell_0})$. On a donc $g^{-\ell_0}(z_{\ell_0},\zeta_{\ell_0}) \in \widehat{\mathcal{K}}^0 = K$, d'où une contradiction.

Finalement, on a établi que $\displaystyle\bigcup_{\ell=0}^{k-1} \widehat{\mathcal{K}}^\ell = \mathcal{K}$. Comme $\mathcal{K}$ est une réunion de composantes connexes de $\widehat{\mathcal{K}}$, c'est un sous-ensemble ouvert et fermé de $\widehat{\mathcal{K}}$, donc holomorphiquement convexe par le corollaire 8 p. 214 de [10]. ■

Notons $\mathcal{O}(\mathbb{C}\times\Delta^*)$ l'algèbre des fonctions holomorphes sur $\mathbb{C}\times\Delta^*$ et $\varphi^*(\mathcal{O}(\mathbb{C}\times\Delta^*))$ l'algèbre des éléments de $\mathcal{O}(\mathbb{C}\times\Delta^*)$ invariants par le groupe $g^{\mathbb{Z}}$. Si A est une algèbre de fonctions holomorphes, on note $\widehat{\mathcal{K}}^A$ l'enveloppe de $\mathcal{K}$ par rapport à l'algèbre A. On a montré que $\widehat{\mathcal{K}}^{\mathcal{O}(\mathbb{C}\times\Delta^*)} = \mathcal{K}$.

Corollaire 3.6. *On a* $\widehat{\mathcal{K}}^{\varphi^*(\mathcal{O}(\mathbb{C}\times\Delta^*))} = \mathcal{K}$.

Preuve : En effet, pour $x \notin \mathcal{K}$, on a :

$$\widehat{(g^{\mathbb{Z}}.x) \cup \mathcal{K}}^{\mathcal{O}(\mathbb{C}\times\Delta^*)} = (g^{\mathbb{Z}}.x) \cup \mathcal{K}. \tag{5}$$

Ceci découle du fait que si $p \notin \mathcal{K}$, pour $q \notin \{p\} \cup \mathcal{K}$, il existe $f_1 \in \mathcal{O}(\mathbb{C} \times \Delta^*)$ telle que $\|f_1\|_{\mathcal{K}} < f_1(q)$. Après avoir éventuellement multiplié f_1 par une constante, on peut supposer que $f_1(q) = 1$. Comme $p \neq q$, il existe également une fonction $f_2 \in \mathcal{O}(\mathbb{C} \times \Delta^*)$ qui vérifie $f_2(p) = 0$, $f_2(q) \neq 0$ et $\|f_2\|_{\mathcal{K}} \leqslant 1/2$; quitte à remplacer f_1 par des puissances d'elle-même, on peut supposer que $\|f_1\|_{\mathcal{K}} \leqslant |f_2(q)|$ et dans ce cas on a $\|f_1 f_2\|_{\mathcal{K}\cup\{p\}} < |f_1(q)f_2(q)|$. Ainsi, on a $\widehat{\{p\} \cup \mathcal{K}}^{\mathcal{O}(\mathbb{C}\times\Delta^*)} = \{p\} \cup \mathcal{K}$; par conséquent, en ajoutant un nombre fini de points à $\mathcal{K}$ l'ensemble obtenu reste holomorphiquement convexe, et on a bien l'égalité (5).

On considère alors la fonction f qui vaut 1 sur $g^{\mathbb{Z}}.x$ et 0 sur $\mathcal{K}$, qui est holomorphe sur $(g^{\mathbb{Z}}.x) \cup \mathcal{K}$. Alors (théorème 6' p. 213 de [10]) il existe une fonction $h \in \mathcal{O}(\mathbb{C} \times \Delta^*)$ telle que $\|f - h\| < 1/2$.

En définissant la fonction holomorphe $H := \dfrac{1}{k}\displaystyle\sum_{j=0}^{k-1}(h \circ g^j)$, il sort que l'on a

$$|H(x) - 1| < 1/2$$

tandis que pour tout $y \in \mathcal{K}$, on a $|H(y)| < 1/2$, donc $x \notin \widehat{\mathcal{K}}^{\varphi^*(\mathcal{O}(\mathbb{C}\times\Delta^*))}$. ■

Corollaire 3.7. *Soit δ un réel positif donné par la proposition 3.5. Alors, la paire*

$$(B(0,\delta) \setminus \{\zeta = 0\}, \varphi(B(0,\delta) \setminus \{\zeta = 0\}))$$

est de Runge.

Preuve : On se donne un compact A de $\varphi(B(0,\delta) \setminus \{\zeta = 0\})$, il est inclus dans un certain $\varphi(\mathcal{K}_{\delta-1/p,1/q})$ (pour p et q assez grands et avec $\delta > 1/p + 1/q$). L'enveloppe de A par rapport à l'algèbre des fonctions holomorphes sur $B(0,\delta) \setminus \{\zeta = 0\}$ est incluse dans $\varphi(\mathcal{K}_{\delta-1/p,1/q})$ par le corollaire 3.6, donc compacte. Ainsi $\varphi(B(0,\delta) \setminus \{\zeta = 0\})$ est holomorphiquement convexe par rapport aux fonctions holomorphes de $B(0,\delta) \setminus \{\zeta = 0\}$, ce qui nous donne la conclusion ([10], corollaire 9 p. 214). ■

3.3. Une généralisation. Soit φ un germe de $(\mathbb{C}^3, 0)$ dans $(\mathbb{C}^3, 0)$ donné par

$$(z, \zeta, \xi) \mapsto (\lambda\zeta^r\xi^s z + P(\zeta, \xi), \zeta^k, \xi^\ell) \tag{6}$$

où $\lambda \in \mathbb{C}^*$, $k, \ell, r, s \in \mathbb{N}$ avec $k, \ell > 1$, $\operatorname{pgcd}(k, \ell) = 1$ et $r, s > 0$, et

$$P(\zeta, \xi) = \sum_{i_1=j_1}^{r} \sum_{i_2=j_2}^{s} c_{i_1,i_2} \zeta^{i_1} \xi^{i_2}$$

avec les conditions suivantes : $0 < j_1 < k$, $0 < j_2 < \ell$, $j_1 \leqslant r$, $j_2 \leqslant s$ et $c_{j_1,j_2} \neq 0$.

Nous ajoutons une hypothèse supplémentaire, à savoir que pour tout $\varepsilon \in \mathbb{U}_k$ (racines k-ièmes de l'unité) et $\tau \in \mathbb{U}_\ell$ avec $\varepsilon\tau \neq 1$ †, il existe des entiers n et m et un polynôme Q avec $Q(0,0) \neq 0$, tels que l'on ait l'égalité :

$$P(\zeta, \xi) - P(\varepsilon\zeta, \tau\xi) = \zeta^n \xi^m Q(\zeta, \xi). \tag{7}$$

Donnons quelques classes d'exemples de polynômes vérifiant cette dernière condition :

1. $P(\zeta, \xi) = \displaystyle\sum_{p=1}^{\min(r,s)} a_p \zeta^p \xi^p$ avec ou bien $\operatorname{pgcd}\{k, p \mid a_p \neq 0\} = 1$, ou bien $\operatorname{pgcd}\{\ell, p \mid a_p \neq 0\} = 1$,
2. $P(\zeta, \xi) = \zeta^{s'} \displaystyle\sum_{p=1}^{r} a_p \xi^p$ avec $\operatorname{pgcd}\{\ell, p \mid a_p \neq 0\} = 1$ et $1 \leqslant s' \leqslant s$,
3. P de la forme précédente, mais en intervertissant les rôles de ζ et ξ.

Etant données ε_k et τ_ℓ deux racines primitives k-ième et ℓ-ième de l'unité respectivement, notons g l'automorphisme de $\mathbb{C} \times (\Delta^*)^2$ qui à (z, ζ, ξ) associe

$$\underbrace{(\varepsilon_k^{-r} \tau_\ell^{-s} z + \frac{P(\zeta, \xi) - P(\varepsilon_k \zeta, \tau_\ell \xi)}{\lambda \varepsilon_k^r \tau_\ell^s \zeta^r \xi^s}}_{a_{k,l}(z,\zeta,\xi)}, \varepsilon_k \zeta, \tau_\ell \xi),$$

et X l'ensemble $B(0,1) \setminus \{\zeta\xi = 0\}$. La condition (7) permet d'adapter le raisonnement de la preuve de la proposition 3.5 et de ses deux corollaires dans cette situation, en posant cette fois-ci

$$q(z, \zeta, \xi) = z \prod_{i=1}^{k-1} \prod_{j=1}^{\ell-1} a_{k,\ell}(z, \zeta, \xi) \zeta^{n_k} \xi^{m_\ell}.$$

Ainsi la paire $(X, \varphi(X))$ est de Runge. On obtient alors une variété de Stein en recollant une infinité dénombrable de copies de $X \setminus \varphi(X)$ grâce à l'application φ. Il est possible de généraliser cette dernière construction en prenant un germe de $(\mathbb{C}^{n+1}, 0)$ dans lui-même, défini cette fois par $(z, \zeta_1, ..., \zeta_n) \mapsto (\lambda \zeta_1^{s_1} ... \zeta_n^{s_n} z + P(\zeta_1, ..., \zeta_n), \zeta_1^{k_1}, ..., \zeta_n^{k_n})$ avec des conditions directement analogues à celles données ci-dessus.

†. Comme k et ℓ sont premiers entre eux, ceci revient à dire que ε et τ ne sont pas simultanément égaux à 1.

4. Invariants

Revenons à présent à notre situation de départ. On note désormais X la variété $\widetilde{S} \setminus \widetilde{D}$. Etant donné un groupe G, on appelle espace $K(G,1)$ tout espace topologique connexe dont le groupe fondamental est isomorphe à G et qui possède un revêtement universel contractile.

Exemple 4.1. Le cercle unité $\mathbb{S}^1$ est un espace $K(\mathbb{Z},1)$.

Remarquons tout d'abord que la variété X est un espace $K(\mathbb{Z}[\frac{1}{k}],1)$. En effet, $\pi_1(X) \cong \mathbb{Z}[\frac{1}{k}]$ et son revêtement universel $\mathbb{C} \times \mathbb{H}$ (c.f. [3] et [8]) est contractile. Le théorème I de [6] (pp. 482-483) nous dit alors que les groupes de cohomologie de X sont isomorphes à ceux du groupe $\mathbb{Z}[\frac{1}{k}]$, c'est-à-dire que pour tout $n \in \mathbb{N}$ et pour tout groupe G, on a un isomorphisme entre $H^n(X,G)$ et $H^n(\mathbb{Z}[\frac{1}{k}],G)$. De plus, on sait (loc. cit. pp. 488-489) que le groupe $H^2(\mathbb{Z}[\frac{1}{k}],G)$ est isomorphe au groupe des extensions centrales de $\mathbb{Z}[\frac{1}{k}]$ par G. Une extension centrale est la donnée d'une extension de groupe

$$0 \to G \xrightarrow{i} E \xrightarrow{p} \mathbb{Z}[\tfrac{1}{k}] \to 0$$

où E est un groupe avec $i(G) \subset Z(E)$, le centre de E.

Nous sommes maintenant en mesure de prouver la

Proposition 4.2. *Le groupe $H^2(X,\mathbb{C})$ est trivial.*

Preuve : Par ce qui précède, il suffit de montrer qu'une extension centrale E de $\mathbb{Z}[\frac{1}{k}]$ par $\mathbb{C}$ est nécessairement triviale, i.e. isomorphe au produit cartésien $\mathbb{C} \times \mathbb{Z}[\frac{1}{k}]$. Soit donc E une telle extension :

$$0 \to \mathbb{C} \xrightarrow{i} E \xrightarrow{p} \mathbb{Z}[\tfrac{1}{k}] \to 0.$$

Montrons que E est abélien. Soient $x, y \in E$ et $a \in \mathbb{N}$ tels que $p(x)$ et $p(y)$ appartiennent tous deux à $\frac{1}{k^a}\mathbb{Z} := \{\frac{n}{k^a}, n \in \mathbb{Z}\}$ qui est un sous-groupe de $\mathbb{Z}[\frac{1}{k}]$ isomorphe à $\mathbb{Z}$.
L'extension E induit une extension $F := p^{-1}(\frac{1}{k^a}\mathbb{Z})$ de $\frac{1}{k^a}\mathbb{Z}$ par $\mathbb{C}$, donc une extension de $\mathbb{Z}$ par $\mathbb{C}$:

$$0 \to \mathbb{C} \xrightarrow{i'} F \xrightarrow{p'} \mathbb{Z} \to 0$$

Il existe une section $s : \mathbb{Z} \to F$ (on choisit $s(1) \in p'^{-1}(1)$ et on pose $s(n) = ns(1)$ pour $n \in \mathbb{Z}$) donc F est produit semi-direct de $\mathbb{Z}$ par $\mathbb{C}$, donné par $\sigma \in \mathrm{Hom}(\mathbb{Z}, \mathrm{Aut}(\mathbb{C}))$. L'extension F étant elle aussi centrale, $\sigma \equiv 1$ est l'unique possibilité, i.e. F est abélien (il est isomorphe à $\mathbb{C} \times \mathbb{Z}$) donc x et y commutent. Ainsi, E est abélien.
Il existe des sections $s : \mathbb{Z}[\frac{1}{k}] \mapsto E$. Pour construire l'une d'elles, fixons $x_0 \in p^{-1}(1)$. Comme $\mathbb{Z}[\frac{1}{k}] \cong E/i(\mathbb{C})$, il existe $x_1' \in p^{-1}(1/k)$ tel que $kx_1' = x_0 + i(w)$ avec $w \in \mathbb{C}$. On pose alors $x_1 := x_1' - i(w/k)$ et on a $kx_1 = x_0$; on définit ainsi par récurrence les $x_i \in p^{-1}(1/k^i)$ vérifiant $kx_{i+1} = x_i$, et notre section est donnée par $s(n/k^a) = nx_a$ pour $n \in \mathbb{Z}$ et $a \in \mathbb{N}$. L'existence d'une telle section nous dit que E est isomorphe au produit

semi-direct $\mathbb{C} \rtimes \mathbb{Z}[\frac{1}{k}]$ donné par $\sigma \in \mathrm{Hom}(\mathbb{Z}[\frac{1}{k}], \mathrm{Aut}(\mathbb{C}))$. Le groupe E étant abélien, on a nécessairement $\sigma \equiv 1$, i.e. E est isomorphe au produit $\mathbb{C} \times \mathbb{Z}[\frac{1}{k}]$. ■

Étant donnés une surface intermédiaire S et son germe associé sous la forme normale (1), on définit l'indice de S comme le plus petit entier m tel que $k-1$ divise ms (voir [12]).
Il existe un feuilletage holomorphe $\mathcal{F}$ sur X défini par la 1-forme holomorphe $\omega = \dfrac{d\zeta}{\zeta}$, qui ne s'annule nulle part (c.f. [4]). De façon équivalente, les feuilles de ce feuilletage sont les ensembles $\{\zeta = const.\}$. Dans le cas où S est d'indice 1, i.e. lorsque $k-1$ divise s, il existe un champ de vecteurs tangent à ce feuilletage qui ne s'annule nulle part, autrement dit on a le

Lemme 4.3. *Lorsque la surface S est d'indice* 1*, le fibré tangent au feuilletage $T\mathcal{F}$ est holomorphiquement trivialisable.*

Preuve : Pour prouver cela, nous montrons qu'il suffit de considérer le champ de vecteurs V sur X induit par le champ de vecteurs $\widetilde{V} = \zeta^{\frac{s}{k-1}} \frac{\partial}{\partial z}$ sur $\mathbb{C} \times \Delta^*$, tangent au feuilletage de $\mathbb{C} \times \Delta^*$ défini par ω.

En effet, d'une part on remarque que X est le quotient de $\mathbb{C} \times \Delta^*$ par G où $G \cong \mathbb{Z}[\frac{1}{k}]/\mathbb{Z}$ est le groupe formé des automorphismes de $\mathbb{C} \times \Delta^*$ de la forme

$$g^{\ell}_{k^n}(z,\zeta) = (z\varepsilon_{k^n}^{-\ell s \frac{k^n-1}{k-1}} + \frac{\displaystyle\sum_{i=0}^{n-1} \lambda^{n-i-1} \zeta^{sk^{i+1}\frac{k^{n-i-1}-1}{k-1}} (P(\zeta^{k^i}) - \varepsilon_{k^n}^{-\ell s k^{i+1}\frac{k^{n-i-1}-1}{k-1}} P((\varepsilon^{\ell}_{k^n}\zeta)^{k^i}))}{\lambda^n (\varepsilon^{\ell}_{k^n}\zeta)^{s\frac{k^n-1}{k-1}}}, \varepsilon^{\ell}_{k^n}\zeta)$$

pour $n \in \mathbb{N}, \ell \in \{0, ..., k^n - 1\}$ et avec $\varepsilon_{k^n} = e^{\frac{2i\pi}{k^n}}$. Ceci provient du fait que X est le quotient de $\mathbb{C} \times \mathbb{H}_g$ par le groupe $\{\gamma^n \gamma_1^{\ell} \gamma^{-n} \mid n, \ell \in \mathbb{Z}\} \cong \mathbb{Z}[\frac{1}{k}]$ où $\mathbb{H}_g = \{w \in \mathbb{C} \mid \Re(w) < 0\}$, $\gamma(z,w) = (\lambda z e^{sw} + P(e^w), kw)$ et $\gamma_1(z,w) = (z, w + 2i\pi)$ (voir [4], proposition 2.3 et section 4). On considère alors le quotient par le sous-groupe $\{\gamma^n \gamma_1^{k^n \ell} \gamma^{-n} \mid n, \ell \in \mathbb{Z}\} \cong \mathbb{Z}$ ce qui nous donne bien $X = (\mathbb{C} \times \Delta^*)/G$.

D'autre part, un champ de vecteurs $\widetilde{V}$ défini sur $\mathbb{C} \times \Delta^*$ induit un champ de vecteurs tangent à X lorsqu'il est invariant par le groupe G, i.e. s'il vérifie :

$$D(g^{\ell}_{k^n})_{z,\zeta}(\widetilde{V}(z,\zeta)) = \widetilde{V}(g^{\ell}_{k^n}(z,\zeta)). \tag{8}$$

Cette condition est bien vérifiée par $\widetilde{V}$ puisque l'on a l'égalité $\varepsilon_{k^n}^{-\ell s \frac{k^n-1}{k-1}} \zeta^{\frac{s}{k-1}} = (\varepsilon^{\ell}_{k^n}\zeta)^{\frac{s}{k-1}}$. Comme $\omega(\widetilde{V}) = 0$, on a bien montré que $V \in H^0(X, T\mathcal{F})$. ■

Remarque 4.4. On peut montrer qu'un champ de vecteurs sur X de la forme $f(z,\zeta)\frac{\partial}{\partial z}$ (où f est une holomorphe ne s'annulant nulle part) existe bien si et seulement si la surface S est d'indice 1, autrement dit on a une équivalence dans le lemme précédent. C'est une conséquence de la condition (8) et le raisonnement est analogue à celui qui sera fait dans le lemme 4.7.

La trivialité du fibré $T\mathcal{F}$ entraine celle du fibré tangent TX, ce que nous voyons à présent.

Lemme 4.5. *Lorsque le fibré $T\mathcal{F}$ est holomorphiquement trivialisable, le fibré tangent holomorphe TX de X l'est aussi.*

Preuve : Étant donné que nous avons une section holomorphe globale V de $T\mathcal{F}$, il nous suffit d'exhiber un deuxième champ de vecteurs global, linéairement indépendant de V en chaque point. Par définition, on peut trouver un recouvrement de X par des ouverts U_i et sur chacun d'eux un champ de vecteurs W_i qui soit linéairement indépendant de V sur U_i. Quitte à remplacer W_i par $W_i/\omega(W_i)$ on peut supposer que $\omega(W_i) \equiv 1$ sur U_i, de sorte que $\omega(W_{i,j}) = 0$ sur $U_{i,j} := U_i \cap U_j$, où l'on a posé $W_{i,j} := W_i - W_j$. La famille $(W_{i,j})$ forme donc un cocyle de $H^1(X, T\mathcal{F})$ qui est aussi un cobord par le théorème B de Cartan. Ainsi il existe un champ de vecteurs Z_i sur chaque U_i tel que $Z_i - Z_j = W_{i,j}$. Posons $\widetilde{Y_i} := W_i - Z_i$, de sorte que $\widetilde{Y_i} = \widetilde{Y_j}$ sur $U_{i,j}$, i.e. les $\widetilde{Y_i}$ se recollent en une section holomorphe globale de TX. Nous avons deux champs de vecteurs V et $\widetilde{Y}$ vérifiant $\omega(V) \equiv 0$ et $\omega(\widetilde{Y}) \equiv 1$, ce qui nous assure qu'ils sont linéairement indépendants en chaque point de la variété étudiée. ■

Nous voulons à présent établir un lien entre le fait que S soit d'indice 1 et la trivialité du fibré canonique de X.
On considère la suite exacte courte $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0$ de faisceaux, qui donne lieu à la suite exacte longue de cohomologie $\cdots \to H^1(X,\mathbb{Z}) \to H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}) \to \cdots$. Toujours d'après le théorème I de [6], on a les isomorphismes

$$H^1(X,\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}\left[\tfrac{1}{k}\right], \mathbb{Z}) = 0,$$
$$H^1(X,\mathbb{C}^*) \cong \mathrm{Hom}(\mathbb{Z}\left[\tfrac{1}{k}\right], \mathbb{C}^*) = \mathbb{C}^*$$

et

$$H^1(X,\mathbb{C}) \cong \mathrm{Hom}(\mathbb{Z}\left[\tfrac{1}{k}\right], \mathbb{C}) \cong \mathbb{C}.$$

D'autre part le groupe $H^2(X,\mathbb{C})$ est trivial, d'où l'on tire finalement la suite exacte courte

$$0 \to H^1(X,\mathbb{C}) \overset{e^{2i\pi\cdot}}{\to} H^1(X,\mathbb{C}^*) \overset{c}{\to} H^2(X,\mathbb{Z}) \to 0.$$

Définition 4.6. *On dira qu'un élément ρ de $H^1(X,\mathbb{C}^*)$ admet un logarithme lorsqu'il existe un morphisme ρ' de $\mathbb{Z}[\frac{1}{k}]$ dans $\mathbb{C}$ tel que $e^{2i\pi\rho'} = \rho$.*

Ainsi, l'image par c d'un élément $\rho \in H^1(X,\mathbb{C}^*)$ est triviale dans $H^2(X,\mathbb{Z})$ si et seulement si ρ admet un logarithme ρ'.

Lemme 4.7. *Si le fibré canonique de X est holomorphiquement trivialisable, alors la surface S est d'indice* 1.

Preuve : Le fibré canonique de X est le fibré des 2-formes holomorphes sur X ; raisonnons par l'absurde et supposons qu'il est holomorphiquement trivialisable et que la surface S n'est pas d'indice 1. Alors il existe une 2-forme holomorphe globale sur X qui ne s'annule nulle part. Une telle forme provient d'une 2-forme holomorphe sur le revêtement $\mathbb{C} \times \Delta^*$ de X donnée par $f(z,\zeta)dz \wedge d\zeta$ (où f est une fonction holomorphe sur $\mathbb{C} \times \Delta^*$ qui ne s'annule nulle part) qui soit stable par le groupe $G = \{g^{\alpha}_{k^n} \mid n \in \mathbb{N}, \alpha \in \{0, ..., k^n - 1\}\}$, i.e. vérifie l'équation

$$(g^{\alpha}_{k^n})^*(f(z,\zeta)dz \wedge d\zeta) = f(z,\zeta)dz \wedge d\zeta$$

(pour tout $n \in \mathbb{N}$ et $\alpha \in \{0, ..., k^n - 1\}$). Ceci donne la condition suivante sur la fonction f :

$$e^{2i\pi \frac{\alpha}{k^n}(s\frac{k^n-1}{k-1}-1)} f(z,\zeta) = f(g^{\alpha}_{k^n}(z,\zeta)). \tag{9}$$

Considérons l'homomorphisme de groupes

$$\begin{array}{rccc} \rho : & \mathbb{Z}[\frac{1}{k}] & \longrightarrow & \mathbb{S}^1. \\ & \frac{\alpha}{k^n} & \longmapsto & e^{2i\pi \frac{\alpha}{k^n}(s\frac{k^n-1}{k-1}-1)} \end{array}$$

Il induit un fibré plat L_ρ au-dessus de X, qui est holomorphiquement trivialisable si et seulement si ρ admet un logarithme, puisque $H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbb{Z})$ car X est de Stein. Étant donné que la fonction f vérifie la condition (9) ci-dessus, elle définit une section holomorphe du fibré plat L_ρ au-dessus de X.
Ainsi pour pouvoir aboutir à une contradiction, il nous reste à voir que ρ n'admet pas de logarithme (et donc qu'une telle fonction f n'existe pas).
Remarquons tout d'abord que l'application $\sigma : \frac{\alpha}{k^n} \mapsto e^{2i\pi \frac{-\alpha}{k^n}(\frac{s}{k-1}+1)}$ est un homomorphisme de $\mathbb{Z}[\frac{1}{k}]$ dans $\mathbb{S}^1$ qui admet un logarithme. Ainsi, ρ admet un logarithme si et seulement si l'homomorphisme $\varphi := \rho/\sigma : \frac{\alpha}{k^n} \mapsto e^{2i\pi \frac{\alpha s}{k-1}}$ admet un logarithme.
Soit m l'indice de la surface S. Comme $k-1$ n'est pas un diviseur de s, le noyau de φ est précisément $m\mathbb{Z}[\frac{1}{k}]$ et cet homomorphisme n'admet donc pas de logarithme. En effet, si un tel morphisme ρ' existait, sa restriction à $m\mathbb{Z}[\frac{1}{k}]$ serait un homomorphisme à valeurs dans $\mathbb{Z}$, nécessairement trivial. On aurait alors $m.\rho\left(\frac{1}{k^n}\right) = 0$, i.e. $\rho\left(\frac{1}{k^n}\right) = 0$ pour tout $n \in \mathbb{N}$. ∎

Remarque 4.8. Le groupe $H^2(X, \mathbb{Z})$ n'est pas trivial ; il contient des éléments de torsion et des éléments qui ne sont pas d'ordre fini. Le morphisme ρ de la preuve du lemme 4.7 fournit un exemple d'élément de torsion, puisqu'on peut voir que ρ^{k-1} admet un logarithme. Pour ce qui est des éléments qui ne sont pas d'ordre fini, donnons-en un exemple. Considérons le groupe $\mathbb{Z}[\frac{1}{6}]$. On a un isomorphisme de groupes $\varphi : \mathbb{Z}[\frac{1}{6}]/\mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{3}]/\mathbb{Z}$ et une injection i de ce groupe (le 3-groupe de Prüfer) dans $\mathbb{S}^1$. Alors on peut voir que $\rho := i \circ \varphi$ n'est pas d'ordre fini dans $H^2(X, \mathbb{Z})$. Ainsi, le groupe $H^2(X, \mathbb{Z})$ possède des éléments d'ordre infini dont l'image est nulle dans $H^2(X, \mathbb{Q})$, ceci est conséquence du fait que le groupe $H_1(X, \mathbb{Z}) \cong \mathbb{Z}[\frac{1}{k}]$ n'est pas finiment engendré (voir [1], théorème 4 p. 144).

Remarque 4.9. On a en fait une équivalence dans le lemme précédent. Lorsque la surface S est d'indice 1, on considère la forme $\zeta^{-(\frac{s}{k-1}+1)}dz \wedge d\zeta$, qui trivialise le fibré canonique.

Les trois lemmes précédents ont en particulier comme conséquence la

Proposition 4.10. *Soient S une surface intermédiaire et $X = \widetilde{S} \setminus \widetilde{D}$. Les trois assertions suivantes sont équivalentes :*
1. La surface S est d'indice 1,
2. Le fibré tangent au feuilletage $T\mathcal{F}$ de X est holomorphiquement trivialisable,
3. Le fibré tangent holomorphe TX de X est holomorphiquement trivialisable.

Preuve : Vu les lemmes 4.3 et 4.5, il suffit de montrer que la troisième assertion entraine la première. C'est une conséquence du lemme 4.7, car si S n'est pas d'indice 1, le fibré canonique de X n'est pas holomorphiquement trivialisable. Dans ce cas, le fibré cotangent de X et donc le fibré tangent TX ne le sont pas non plus. ■

Remarque 4.11. Le problème suivant demeure non résolu actuellement (voir [9]) : une variété de Stein de dimension n dont le fibré tangent holomorphe est holomorphiquement trivialisable est-elle nécessairement un domaine de Riemann au-dessus de $\mathbb{C}^n$? Nous ne connaissons pas la réponse pour les surfaces de Stein que l'on vient de considérer.

Références

[1] R. Bott and J. Heitsch, *A remark on the integral cohomology of $B\Gamma_q$*, Topology 11 (1972), 141–146.

[2] G. Dloussky, *Structure des surfaces de Kato*, Mémoires de la S.M.F. 2e série 14 (1984).

[3] G. Dloussky and K. Oeljeklaus, *Vector fields and foliations on compact surfaces of class VII_0.*, Ann. Inst. Fourier 49 (1999), no. 5, 1503–1545.

[4] G. Dloussky, K. Oeljeklaus, and M. Toma, *Surfaces de la classe* VII_0 *admettant un champ de vecteurs, ii*, Commentarii Mathematici Helvetici 76 (2001), no. 4, 640–664.

[5] ———, *Class* VII_0 *surfaces with b_2 curves*, Tohoku Math. Journal 55 (2003), 283–309.

[6] S. Eilenberg and S. MacLane, *Relations between homology and homotopy groups of spaces*, Annals of Mathematics 46 (1945), no. 3, 480–509.

[7] Ch. Favre, *Classification of 2-dimensional contracting rigid germs*, Journal de Mathématiques Pures et Appliquées 79 (2000), 475–514.

[8] ———, *Dynamique des applications rationnelles*, Ph.D. thesis, Université Paris-Sud, 2002.

[9] F. Forstnerič, *Noncritical holomorphic functions on Stein manifolds*, Acta Mathematica 191 (2003), no. 2, 143–189.

[10] R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, 1965.

[11] Ma. Kato, *Compact complex manifolds containing "global" spherical shells*, Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), Kinokuniya Book Store, Tokyo (1978).

[12] K. Oeljeklaus and M. Toma, *Logarithmic moduli spaces for surfaces of class* VII, Mathematische Annalen 341 (2008), 323–345.

[13] D. Zaffran, *Serre problem and Inoue-Hirzebruch surfaces*, Mathematische Annalen 319 (2001), 395–420.

Laurent Battisti
LATP-UMR(CNRS) 6632
CMI-Université d'Aix-Marseille I
39, rue Joliot-Curie
F-13453 Marseille Cedex 13
France
battisti@cmi.univ-mrs.fr

Asymptotic Behavior of Word Metrics on Coxeter Groups

G. A. Noskov

Received: October 17, 2006
Revised: February 10, 2011

Communicated by Ulf Rehmann

Abstract. We study the geometry of tessellation defined by the walls in the Moussong complex $\mathcal{M}_W$ of a Coxeter group W. It is proved that geodesics in $\mathcal{M}_W$ can be approximated by geodesic galleries of the tessellation. A formula for the translation length of an element of W is given. We prove that the restriction of the word metric on the W to any free abelian subgroup A is Hausdorff equivalent to a regular norm on A.

2000 Mathematics Subject Classification: 20F55, 51F15
Keywords and Phrases: Coxeter group, Moussong complex, translation length

Introduction

For any Coxeter system (W, S), Moussong constructed a certain piecewise Euclidean complex $\mathcal{M}_W$ on which W acts properly and cocompactly by isometries [Mou88]. This complex is complete, contractible, has nonpositive curvature and the Cayley graph $\mathcal{C}_W$ of W (with respect to S) is isomorphic to the 1-skeleton of $\mathcal{M}_W$. A wall in $\mathcal{M}$ is the fixed-point set of a reflection in W. It turns out that the walls are totally geodesic subspaces in $\mathcal{M}_W$ and each wall divides $\mathcal{M}_W$ into two path components. The set of all walls defines a wall tessellation of $\mathcal{M}$. The set of all tiles (=chambers) of this tessellation together with an appropriate adjacency relation is isomorphic to the Cayley graph $\mathcal{C}_W$. We shall prove that geodesics in $\mathcal{M}_W$ can be uniformly approximated by geodesic galleries of the wall tessellation (= geodesic paths in $\mathcal{C}_W$) (Theorem 3.3.2). This approximation result allows us to prove that for any "generic" element $w \in W$ of infinite order there is a conjugate v which is straight i.e., $\ell(v^n) = n\ell(v)$ for all $n \in \mathbb{N}$, where $\ell(v)$ is the word length on W (Theorem 4.1.5). There is a constant $c = c(W)$, such that for any $w \in W$ of infinite order there is a conjugate v of w^c, which is straight (Theorem 4.1.6). The restriction of the word metric on W to any free abelian subgroup A is Hausdorff equivalent to a regular norm on A (Theorem 4.3.2).

Acknowledgment. I would like to thank Mathematisches Institut der Heinrich-Heine-Universitat for the hospitality during 1999. Thanks to H. Abels, B. Brink, M. Davis, M. Gromov, F. Grunewald, R. Howlett and E. B. Vinberg for their help. This research was supported by a DFG grant Gr 627–11 grant and by SFB 343, SFB 701 of the DFG at the University of Bielefeld.

1 Preliminaries on Moussong complexes

To any Coxeter system (W, S) one can canonically associate the Moussong complex $\mathcal{M} = \mathcal{M}_W$, which is a piecewise Euclidean complex with W as the set of vertices. Their cells are Euclidean polyhedra, which are the convex hulls of sets, naturally bijected with the spherical cosets of W. In particular, the 1-cells of $\mathcal{M}$ are in bijective correspondence with the sets $\{w, ws\}$, where $w \in W$ and $s \in S$. Hence the 1-skeleton of $\mathcal{M}$ is nothing but a modified Cayley graph of W with respect to S (the modification consists in identifying an edge $w \xrightarrow{s} ws$ with its inverse $ws \xrightarrow{s} w$). W acts cellularly and isometrically on $\mathcal{M}_W$ and this induces the standard W-action on the Cayley graph of W. In the next subsections we carry out in detail the construction of $\mathcal{M}_W$ following the thesis of D. Krammer [Kra94].

1.1 Coxeter groups

A Coxeter system is a pair (W, S) where W is a group and where S is a finite set of involutions in W such that W has the following presentation:

$$\langle s : s \in S | (ss')^{m_{ss'}} = 1 \text{ when } m_{ss'} < \infty \rangle,$$

where $m_{ss'} \in \{1, 2, 3, \ldots, \infty\}$ is the order of ss', and $m_{ss'} = 1$ if and only if $s = s'$. We refer to W itself as a Coxeter group when the presentation is understood. The number of elements of S is called its rank. The Coxeter system (W, S) is called spherical if W has finite order. A subgroup of W is called special if it is generated by a subset of S. For each $T \subseteq S$, W_T denotes the special subgroup generated by T. Any conjugate of such a subgroup will be called parabolic. A remarkable feature of Coxeter systems is that for any subset $T \subseteq S$ the pair (W_T, T) is a Coxeter system in its right and moreover a presentation of W_T is defined by the numbers $m_{tt'}, t, t' \in T$. If (W_S, S) is a Coxeter system of finite rank then we write V_S for the real vector space with a basis of elements (e_s) for $s \in S$. Put a symmetric bilinear form B on V_S by requiring:

$$B(e_s, e_{s'}) = -\cos(\pi/m_{ss'}).$$

(This expression is interpreted to be -1 in case $m(s,s') = \infty$.) Evidently $B(e_s, e_s) = 1$, while $B(e_s, e_{s'}) \leq 0$ if $s \neq s'$. Since e_s is non-isotropic, the subspace $H_s = e_s^\perp$ orthogonal to e_s is complementary to the line $\mathbb{R}e_s$. Associated to $s \in S$ is an automorphism a_s of B acting as the reflection $v \mapsto v - 2(v, e_s)e_s$ in the hyperplane $e_s^\perp$. The result by Tits asserts that the correspondence $s \mapsto a_s$ extends to a faithful representation of W as a group of automorphisms of the form B. (cf. [Bou], Ch.V, s.4).

1.2 Trading Coxeter cells

The Coxeter group W is finite if and only if the form $B(e_s, e_{s'})$ is positive definite. We call a set $J \subseteq S$ spherical if W_J is finite or, equivalently, the restriction of the form B to the subspace $V_J = \sum_{j \in J} \mathbb{R}\, e_j$ is positive definite. Let $J \subseteq S$ be spherical. Since V_J is non-degenerate, there exists a unique basis $\{f_j^J | j \in J\}$ of V_J dual to $\{e_j : j \in J\}$ with respect to B. A space V_J that comes equipped with a positive definite inner product $B|V_J$ will be denoted by E_J and called the Euclidean space associated to J. Define the Coxeter cell X_J to be the convex hull of the W_J-orbit:

$$X_J = \mathrm{Ch}(W_J x_J)$$

where

$$x_J = \sum_{j \in J} f_j^J \in E_J.$$

For convenience we define $W_\emptyset = \{1\}$ and $X_\emptyset = \{0\}$– the origin of E_J. More generally, for any spherical K and any $J \subseteq K$ we consider the faces of the polyhedron $X_K = \mathrm{Ch}(W_K x_K)$ of the form

$$X_{JK} = \mathrm{Ch}(W_J x_K).$$

We do not exclude the case $J = \emptyset$, where $X_{\emptyset K} = \{x_K\}$. We call the extremal points of the cell X_J the vertices.
For spherical $J \subseteq S$, let $p_J : V_S \to E_J$ denote the orthogonal projection. It is well defined since the quadratic form on E_J is non-degenerate.

Lemma 1.2.1 ([Kra94], B.2.2.) *The dimension of the cell X_J equals the cardinality of J. For spherical subsets $J \subseteq K$ of S we have $p_J x_K = x_J$. Moreover, $p_J|X_{JK} : X_{JK} \to X_J$ is a W_J-equivariant isometry of cells. The nonempty faces of X_K are precisely those of the form wX_{JK} $(J \subseteq K, w \in W_K)$. In particular, the vertex set of X_J is precisely $W_J x_J$.*

Example 1.2.2 1) If $J = \{j\}$ then $f_j^J = e_j$ and $X_J = \mathrm{Ch}(e_j, -e_j)$ is a line segment. 2) Let $J = \{s, s'\}$ be spherical, so $w = ss'$ has finite order $m_{ss'}$. Set $V_{s,s'} = \mathbb{R}e_s + \mathbb{R}e_{s'}$. The restriction of B to $V_{s,s'}$ is positive definite and both s and s' act as orthogonal reflections in the lines $\mathbb{R}f_s, \mathbb{R}f_{s'}$ respectively. Since $B(e_s, e_{s'}) = -\cos(\pi/m_{ss'}) = \cos(\pi - (\pi/m_{ss'}))$, the angle between the rays $\mathbb{R}^+e_s$ and $\mathbb{R}^+e_{s'}$ is equal to $\pi - (\pi/m_{ss'})$, forcing the angle between the reflecting lines $\mathbb{R}f_s, \mathbb{R}f_{s'}$ to be equal $\pi/m_{ss'}$. The vectors $f_s, f_{s'}$ are of the same length, lie in the cone $\mathbb{R}^+e_s + \mathbb{R}^+e_{s'}$; moreover, $f_s + f_{s'}$ is a bisectrix between the reflecting lines $\mathbb{R}f_s, \mathbb{R}f_{s'}$ hence the convex hull of the orbit $W_J(f_s + f_{s'})$ is a regular $2m_{ss'}-$gon.

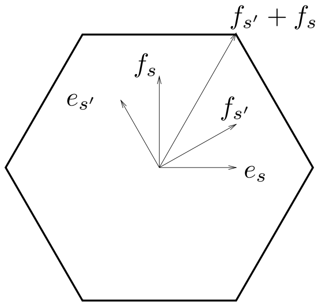


Figure 1: The cell X_J for $J = \{s, s'\}, m_{ss'} = 3$.

1.3 Gluing the Moussong complex

Now we build the Moussong complex of $W = W_S$ as follows. Take the union

$$U = \bigcup\{(w, X_J) : w \in W, J \subseteq S \text{ spherical}\}.$$

Introduce an equivalence relation R on U, generated by the following gluing relations:

1. $(wu, x) \sim (w, u^{-1}x)$, whenever $w \in W, u \in W_J, x \in X_J$,
2. The cells $(w, X_K), (w, X_L)$ are glued along the face $(w, X_J), J = K \cap L$, which is embedded into each of them (by the map p_J) as (w, X_{JK}) and (w, X_{JL}) respectively .

The quotient space of U modulo R is called the Moussong complex of W and is denoted by $\mathcal{M}_W$. The group W acts on U by $u(w, x) = (uw, x)$. This action respects the relation R and hence induces a cellular action of W on $\mathcal{M}_W$. With some abuse in notation we will denote the natural image of $(1, X_J)$ in $\mathcal{M}$ by X_J, so any cell in $\mathcal{M}$ is of the form wX_J for some $w \in W, J \subseteq S$. We call J the type of the cell wX_J. There is a distinguished vertex $x_0 = X_\emptyset$ in $\mathcal{M}$. Note that $x_J = x_0$ for any spherical J (by condition (2)).
It can be shown that the inclusion maps of the cells are injective, see [Kra94]. The canonical metric in each cell allows to measure the lengths of finite polygonal paths in $\mathcal{M}$. The *path metric* d on $\mathcal{M}$ is defined by setting the distance

between $x, y \in \mathcal{M}$ to be the infimum of the lengths of polygonal paths joining x to y.
We summarize the main properties of $\mathcal{M}$ in the following theorem.

Theorem 1.3.1 ([Kra94],[Mou88]) *Relative to the path metric $\mathcal{M}$ is a contractible, complete, proper CAT(0) space. The Coxeter group W acts on $\mathcal{M}$ cellularly and this action is isometric, proper and cocompact. This action is simply transitive on the set of vertices $\mathcal{M}^{(0)}$ of $\mathcal{M}$, in particular $\mathcal{M}^{(0)}$ coincides with Wx_0.*

For the convenience of the reader we repeat the relevant definitions. A geodesic, or geodesic segment, in a metric space (X, d) is a subset isometric to a closed interval of real numbers. Similarly, a loop $S^1 \to X$ is a closed geodesic if it is an isometric embedding. (Here S^1 denotes the standard circle equipped with its arc metric, possibly rescaled so that its length can be arbitrary). We say that X is a geodesic metric space if any two points of X can be connected by a geodesic. We denote by $[x, y]$ any geodesic joining x and y. We will always parameterize $[x, y]$ by $t \mapsto p_t (0 \leq t \leq 1)$, where $d(x, p_t) = td(x, y)$ for all t. Given three points x, y, z in X, the triangle inequality implies that there is a comparison triangle in the Euclidean plane $\mathbb{R}^2$, whose vertices $\overline{x}, \overline{y}, \overline{z}$ have the same pairwise distances as x, y, z. Given a geodesic $[x, y]$ and a point $p = p_t \in [x, y]$, there is a corresponding point $\overline{p} = \overline{p_t}$ on the line segment $[\overline{x}, \overline{y}]$ in $\mathbb{R}^2$. A geodesic metric space X is called a CAT(0) space if for any x, y in X there is a geodesic $[x, y]$ with the following property: For all $p \in [x, y]$ and all $z \in X$, we have

$$d(z, p) \leq d_{\mathbb{R}^2}(\overline{z}, \overline{p}),$$

with $\overline{z}$ and $\overline{p}$ as above. Let X be a CAT(0) space. Then there is a unique geodesic segment joining each pair of points $x, y \in X$ and this geodesic segment varies continuously with its endpoints. Every local geodesic in X is a geodesic. For the proof see [BH99], Chapter II.1 , Prop. 1.4.

Examples 1.3.2 If W is a finite Coxeter group of rank n then $\mathcal{M}_W$ is isometric to an n-dimensional convex polyhedron. If, for example, W is the dihedral group of order $2m$, then $\mathcal{M}$ is a regular $2m$-gon with the usual W-action. If W is an affine Coxeter group of rank n then $\mathcal{M}_W$ is a tessellation of the $n-1$-dimensional Euclidean space E. This tessellation is dual to the tessellation, given by the structure of a Coxeter complex on E. Let, for example, W be an affine Coxeter group generated by the reflections s_1, s_2, s_3 in the sides of an equilateral triangle C in the Euclidean plane. Then $\mathcal{M}_W$ is the tessellation of the plane by hexagons, dual to the tessellation consisting of the images of C under W. If W is a product of $n \geq 2$ copies of $\mathbb{Z}/2$ (that is $m_{ss'} = \infty$ for $s \neq s'$), then $\mathcal{M}_W$ is an infinite n-regular tree with edges of length 2. $\square$

Lemma 1.3.3 *Any cell of a $CAT(0)$ piecewise Euclidean complex X is isometrically embedded into X. In view of uniqueness of geodesics this is equivalent to the convexity of a cell.*

Proof. We have to show is that for any two points a, b of a cell C the Euclidean arc α in C between them is a global geodesic. We may assume that C is of minimal dimension. For any two points x and y in the interior of α the closed subarc $\beta \subset \alpha$ between x and y lies in the interior of C. Clearly there is an $\epsilon > 0$, such that for any cell C', having C as a face, the distance from β to the set $\partial C' - C$ is $\geq \epsilon$. Let us cover β by intervals of radius $\epsilon/2$. Each such an interval is geodesic. Indeed, a geodesic γ connecting the points of the interval can not cross ∂C, hence it lies in the union U of cells, having C as a face. For any cell C', having C as a face, γ can not cross $\partial C' - C$ since it has to pass a distance at least ϵ. Hence it lies in only one such C' and thus coincides with the interval. It follows from the considerations above that β is a local geodesic, and therefore a global geodesic since X is CAT(0).
Now let γ be a path in X joining a to b. For any positive $\epsilon < d_C(a,b)/2$ we may choose points x and y in the interior of α such that $d(a,x) = \epsilon = d(y,b)$ A path from x to y obtained by traveling along α to a then along γ to b has length $length(\gamma) + 2\epsilon$, while a geodesic from x to y has length $d_C(a,b) - 2\epsilon$, so $d_C(a,b) \leq length(\gamma) + 4\epsilon$. Since this is true for any sufficiently small $\epsilon > 0$, we conclude that $d_C(a,b) \leq length(\gamma)$, and so α is a geodesic from a to b. $\square$

1.4 The action of reflections on cells

We refer to the notation of §1.3.

Lemma 1.4.1 *An element $w \in W$ leaves the cell uX_K invariant if and only if $u^{-1}wu \in W_K$. In the latter case w acts on the X_K-coordinate of $ux \in uX_K$ as the element $u^{-1}wu \in W_K$.*

Proof. Indeed, the cell uX_K is uniquely determined by its set of vertices uW_Kx_0 and it is w-invariant if and only if uW_K is w-invariant under left translation. The latter happens if and only if $wuW_K = uW_k \Leftrightarrow u^{-1}wu \in W_K$. The second assertion follows from the equality $w(ux) = u(u^{-1}wux)$. $\square$

Lemma 1.4.2 *(An "overcell" of invariant cell is invariant too.) If $C \subseteq C'$ are cells and $wC = C$ for some $w \in W$, then $wC' = C'$.*

Proof. Writing $C = uX_J$ with $w \in W, J \subseteq S$ we can represent C' in the form $C' = uX_K, J \subseteq K$. By Lemma 1.4.1 $wC = C$ implies $u^{-1}wu \in W_J$ and thus $u^{-1}wu \in W_K$. Again by the same lemma $wC' = C'$. $\square$

Definition 1.4.3 *Let $(W;S)$ be a Coxeter system. A* reflection *in W is an element that is conjugate in W to an element of S.*

Lemma 1.4.4 *For any cell C of $\mathcal{M}$ and any reflection $w \in W$ either $C \cap wC$ is empty or else w acts as a reflection on C.*

Proof. Suppose that the cell $C \cap wC$ is nonempty. Then it is invariant under the action of w. Since it is a face of C, by Lemma 1.4.2 we conclude that $wC = C$. Now by Lemma 1.4.1 $w \in W$ acts as a reflection on C. $\square$

1.5 Angles and geodesics in $\mathcal{M}$

The notion of angle in an arbitrary piecewise Euclidean complex can be defined in terms of the link distance, see e.g. [BB97]. Namely, let X be a piecewise Euclidean complex, $x \in X$ and let A be a Euclidean cell of X containing x. The link $\mathrm{lk}_x A$ of is the set of unit tangent vectors ξ at x such that a nontrivial line segment with the initial direction ξ is contained in A. We define the link $\mathrm{lk}_x X$ by $\mathrm{lk}_x X = \cup_{A \ni x} \mathrm{lk}_x A$, where the union is taken over all closed cells containing x.
Recall that the CAT(0)– condition for X is equivalent to the following (see e.g. [BB97]):

1. X is 1-connected and
2. The length of any geodesic loop in the link of any vertex of X is greater or equal to 2π.

A path $\alpha : [a, b] \to X$ is geodesic if it is an isometric embedding: $d(\alpha(s), \alpha(t)) = |s - t|$, for any $s, t \in [a, b]$. Similarly, a loop $\alpha : S^1 \to X$ is a closed geodesic if it is an isometric embedding. Here S^1 denotes the standard circle equipped with its arc metric (possibly rescaled so that its length can be arbitrary). Angles in $\mathrm{lk}_x A$ induce a natural length metric d_x on $\mathrm{lk}_x S$, which turns $\mathrm{lk}_x S$ into a piecewise spherical complex. For $\xi, \eta \in \mathrm{lk}_x X$ define $\angle(\xi, \eta) = \min(d_x(\xi, \eta), \pi)$. Now any two segments σ_1, σ_2 in X with the same endpoint x have the natural projection image in $\mathrm{lk}_x X$ and we define $\angle_x(\sigma_1, \sigma_2)$ to be the angle between these two projections.
We will use the following criterion of geodesicity:

Lemma 1.5.1 ([BB97]) *Let X be a piecewise Euclidean CAT(0)-complex. If each of the segments σ_1, σ_2 is contained in a cell and $\sigma_1 \cap \sigma_2 = \{x\}$, where x is an endpoint of each of the segments, then the union $\sigma_1 \cup \sigma_2$ is geodesic if and only if $\angle_x(\sigma_1, \sigma_2) = \pi$.*

An m- chain from x to y is an $(m+1)$-tuple $T = (x_0, x_1, \ldots, x_m)$ of points in X such that $x = x_0, y = x_m$ and each consecutive pair of points is contained in a cell. Every m-chain determines a polygonal path in X, given by the concatenation of the line segments $[x_i, x_{i+1}], i = 0, ..., i = m$. An m-taut chain from x to y is an m-chain such that

1. there is no triple of consecutive points contained in a cell and
2. (2) the union of two subsequent segments is geodesic in the union of cells, containing these segments.

(The union is equipped with its path metric). Note that if a chain is taut then only its first and last entries lie in the interior of a top dimensional simplex of X. The result of M. Bridson asserts that if X is a piecewise Euclidean complex then X with its path metric is a geodesic space and the geodesics are the paths determined by taut chains [BH99, Theorem. 7.21].

2 Walls in the Moussong complex

The notion of wall in the Moussong complex (as well as in the Coxeter complex) can be defined as the fixed-point set of reflection from the underlying Coxeter group. On the other hand they can be defined as the equivalence classes of "midplanes" (which are the fixed-point sets of stabilizers of cells). Both points of view are useful. Note that in contrast to the situation with Coxeter complexes, the walls in the Moussong complex are not subcomplexes.

2.1 Midplanes and blocks in cells

Let (W_J, J) be a finite Coxeter group and V_J the Euclidean vector space on which W_J acts. We summarize here the basic properties of a Coxeter complex of $W = W_J$. For more about them see [Hum90] or [Bro96]. We define a reflection in W_J to be a conjugate of element of J. The reflecting hyperplanes H_w of reflection $w \in W_J$ cut V_J into polyhedral pieces, which turn out to be cones over simplices. In this way one obtains a simplicial complex $\mathcal{C} = \mathcal{C}(W)$ which triangulates the unit sphere in V_J. This is called the Coxeter complex associated with W_J. The group W_J acts simplicially on $\mathcal{C}$ and this action is simply transitive on the set of maximal simplices (=chambers). Moreover the closure of any chamber C is a fundamental domain of the action of W on $\mathcal{C}$, i.e., each $x \in V$ is conjugated under W to one and only one point in $\mathcal{C}$. Two chambers are adjacent if they have a common codimension one face. For any two adjacent chambers there is a unique reflection in W_J interchanging these two chambers.

A similar picture we have for the Coxeter cell X_J. By a midplane in X_J we mean the intersection $H_w \cap X_J$, where $w \in W_J$ is a reflection and H_w its reflecting hyperplane. We denote this midplane by $M(J, w)$. By equivariance we define the notion of a midplane in any cell of $\mathcal{M}_W$. Each midplane M defines a unique cell in $\mathcal{M}_W$, the cell of least dimension in $\mathcal{M}_W$ which contains M, and we will denote this by $C(M)$.

Lemma 2.1.1 *Every cell X_J contains an open neighborhood of the origin of V_J. In particular midplanes in X_J have dimension $|J| - 1$ and there is one-to-one correspondence between reflecting hyperplanes and midplanes.*

Proof. Note first that the ray $\mathbb{R}^+ x_J$ lies in the interior of the chamber $C = \{x \in V_J : B(x, e_s) > 0 \ \forall s \in S\}$. Hence in each chamber $wC, w \in W_J$ there is a vertex wx_J of X_J. Now suppose that X_J does not contain the origin in the interior, then there is a hyperplane H through the origin such that X_J is contained in one of the closed half-spaces defined by H, say in H_+. This implies that each chamber has an interior point, lying in H_+. Take an arbitrary closed chamber D. If D lies entirely in H_+ then $-D$ lies in the opposite half-space H_- and hence there is no interior point in it belonging H_+ – contradiction. If D does not lie entirely in H_+ then H separates some codimension one face F of D from the remaining vertex x of D. Let D' be the chamber, adjacent to

D in a face F, then D' lies entirely either in H_+ or in H_- and the previous argument works. □

Definitions 2.1.2 It follows from Lemma 2.1.1 that the midplanes $M(J, w)$ also cut X_J into (relatively open) polyhedral pieces of dimension $|J|$ – blocks. Two blocks are adjacent if they have a common codimension one face. There is a canonical one-to-one correspondence between blocks in X_J, chambers of the Coxeter complex $\mathcal{C}(W_J)$ and vertices of X_J. This correspondence clearly preserves the adjacency relation. Each block contains a unique vertex of X_J since a closed block B is a fundamental domain of the action of W on X_J, i.e., each $x \in X_J$ is conjugated under W to one and only one point in B. The group W_J acts on the set of blocks and this action is simply transitive. For a block B the intersection of the closed block B with a midplane is called by internal face of B.

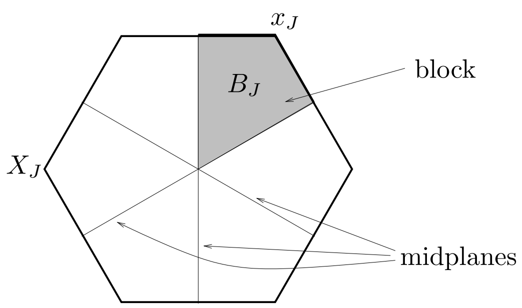


Figure 2: The cell X_J for $J = \{s, s'\}, m_{ss'} = 3$ divided into blocks by midplanes.

Lemma 2.1.3 *The only faces of a cell X_K having nonempty intersection with midplane $M(K, s)$, $s \in S$ are those wX_{JK} with $w^{-1}sw \in W_J$. In particular $M(K, s)$ contains no vertices of X_K. More generally a face of X_K has nonempty intersection with midplane $M(K, usu^{-1})$, $s \in S, u \in W$ iff it is of the form uwX_{JK} with $w^{-1}sw \in W_J$.*

Proof. If $w^{-1}sw \in W_J$ then $swW_J = wW_J$, that is s leaves the vertex set of wX_{JK} invariant and hence it leaves invariant the cell itself and has a nonempty fixed-point set in this cell. Conversely if $M(K, s) \cap wX_{JK}$ is nonempty then there a face F of the cell wX_{JK} such that $M(K, s) \cap F$ contains an interior point of F. But then s leaves F invariant hence by Lemma 1.4.2 it also leaves any "overcell" invariant in particular wX_{JK} and this implies that $w^{-1}sw \in W_J$. To deduce the second statement from the first, one need only to note that $M(K, wsw^{-1}) \cap wX_{JK} = w(M(K, s) \cap X_{JK})$. □

Lemma 2.1.4 *If $w \in W_J$ leaves invariant some midplane M in X_J then it fixes this midplane pointwise.*

Proof. Indeed, w leaves invariant the ambient face C and we can apply Lemma 1.4.1. $\square$

Lemma 2.1.1 *For any cell X_K the following hold:*

1. *The intersection of a midplane of X_K with any of its face is again a midplane.*

2. *Any midplane of any face of X_K is an intersection with this face of a precisely one midplane of X_K.*

Proof. 1) We may assume that a given midplane M is of the form $M(K, s)$ and the face of X_K is $X_{JK}, J \subseteq K$. Since s belongs to W_J, it leaves X_{JK} invariant and its fixed-point set X^s_{JK} bijects onto the fixed-point set X^s_J by a W_J-equivariant isometry $p_J|X_{JK} : X_{JK} \to X_J$. The general assertion follows by equivariance.
2) We may assume that the face is of the form X_{JK} for $J \subseteq K$. Let M_{JK} be a midplane of X_{JK}, then by definition $M_{JK} = (p_J|X_{JK})^{-1}(M(J, w))$ for some $w \in W_J$. Hence, by W_J–equivariance, w is identical on M_{JK} thus $M_{JK} = M(K, w) \cap X_{JK}$. Furthermore, $w \in J$ by Lemma 2.1.3. Hence the segment $\sigma = [wx_J, x_J]$ is an edge of the face X_{JK}, flipped by w. The intersection $M_{JK} \cap \sigma = \{m\}$ is a midpoint of σ and M_{JK} is orthogonal to σ. Now if M any midplane with the same intersection with X_{JK} as M_K, then the reflection in M flips the edge σ and hence this edge is orthogonal to M and thus $M = M_K$. $\square$

Lemma 2.1.5 1)*For every $x \in M(K, s) \cap X_{JK}$ there is a nondegenerate segment of the form $[y, sy], y \in X_{JK}$ with x as a midpoint.* 2) *The segment $[y, sy]$ is orthogonal to midplane $M(K, s)$.* 3) *For any midplane M in X_K there is an edge of X_K, intersected by M in the midpoint.*

Proof. 1) Since $M(K, s) \cap X_{JK}$ is nonempty, it follows from Lemma 2.1.3 that $s \in J$. Let

$$U = \{u \in W_J; x_k \text{ and } ux_K \text{ are on the same side of } M(K, s).\}$$

Clearly $W_J = U \cup sU, U \cap sU = \emptyset$ and the sets Ux_K, sUx_K lie entirely on the different sides of the midplane $M(K, s)$. Since $X_{JK} = \mathrm{Ch}(W_J x_K)$, we have

$$x = \sum_{u \in U} (\lambda_u ux_K + \mu_u sux_K),$$

where $\sum_{u \in U}(\lambda_u + \mu_u) = 1$ and all coefficients are nonnegative. Since x is fixed by s, applying s to both parts of the equality above we get

$$x = \sum_{u \in U} (\mu_u ux_K + \lambda_u sux_K),$$

We conclude from these two equalities that $x = 1/2(y + sy)$, where $y = \sum_{u\in U}(\lambda_u + \mu_u)ux \in X_{JK}$.
2) The segment $[y, sy]$ is orthogonal to $M(K, s)$ since it is flipped by an orthogonal transformation s.
3) If $M = M(K, s)$, then the edge $[sx_K, x_K]$ of X_K is intersected by M in the midpoint. $\square$

We will call the segment $[y, sy]$ from the lemma above to be a perpendicular to $M(K, s)$ in the point x.

Lemma 2.1.6 *Let $x \in M(K, s), z \in X_K, x \neq z$ and let $[y, sy]$ be a perpendicular to $M(K, s)$ in the point x. Then either $[x, z] \subset M(K, s)$ or one of the angles $\angle_x([x, z], [x, y])$, $\angle_x([x, z], [x, y])$ is strictly less than $\pi/2$.*

Proof. It follows from the fact that the tangent space in x is orthogonal sum of a the tangent space of $M(K, s)$ and a tangent space of the segment $[y, sy]$. $\square$

2.2 Walls as equivalence classes of midplanes

We assume that $\mathcal{M} = \mathcal{M}_W$ is the Moussong complex of a Coxeter group W. The following definition mimics the definition of a hyperplane in a cube complex given in [NR98].

Definitions 2.2.1 For midplanes M_1 and M_2 of the cells $C_1 = C(M_1)$ and $C_2 = C(M_2)$ respectively we write $M_1 \sim M_2$ if $M_1 \cap M_2$ is again a midplane (and then of course it is a midplane of $C_1 \cap C_2$). The transitive closure of this symmetric relation is an equivalence relation, and the union of all midplanes in an equivalence class is called a wall in $\mathcal{M}$. Clearly the equivalences above are generated by those of the form $M_1 \sim M_2, C_1 \leq C_2$ or $C_2 \leq C_1$. Thus to prove some property $\mathcal{P}$ for midplanes of a wall H it is enough to prove this property for some midplane in H and then show that the validity of $\mathcal{P}$ is preserved under equivalences just mentioned. If M is a midpoint of a 1-cell(=edge) in $\mathcal{M}$ then the wall spanned by M will be called a dual wall of e and denoted by $H(e)$. We denote by $\mathcal{H}_M$ the union of midplanes in the equivalence class of a midplane M.

It follows immediately from Lemma 2.1.5 that

Lemma 2.2.2 *Any wall H of $\mathcal{M}$ has the form $H(e)$ for some edge e.*

Clearly W acts on the set of midplanes, preserving the equivalence relation and hence acts on the set of walls. For any wall H we denote by $\tilde{H}$ the complex obtained from the disjoint union of midplanes in H by gluing any two midplanes in H along their common submidplane in $\mathcal{M}$ (if such one exists). One can easily see that $\tilde{H}$ is nonpositively curved, i. e. satisfies the link condition. Namely, the link of any cell C of $\mathcal{M}$ is isometric to the product $C \times [-\pi, \pi]$.

Lemma 2.2.3 *Let $p : \tilde{H} \to \mathcal{M}$ be the natural map which sends each midplane in $\tilde{H}$ to its image in $\mathcal{M}$. Then p is an isometry of $\tilde{H}$ onto H. As a consequence of the above walls are convex in $\mathcal{M}$.*

Proof. It is similar to the proof of lemma 2.6 in [NR98]. Clearly, p is an isometry on each midplane. By result of M. Gromov ([Gro87], Section 4) it is enough to show that p is a local isometry, that is if $x \in \tilde{H}$, then there is a neighborhood U of x such that $p|_U$ is an isometry. Clearly p bijects the star $St(x)$ onto the union U of all midplanes, containing $p(x)$. This union is the fixed-point set of some reflection from W (see Lemmas 1.4.1, 1.4.2, 1.4.4). Hence U is convex, and p maps $St(x)$ isometrically onto U. □

Lemma 2.2.4 *Each wall in $\mathcal{M}_W$ is the fixed-point set of a precisely one reflection in W. Conversely, the fixed-point set of a reflection in W is a wall.*

Proof. Let H_M be the wall, spanned by a midplane M of the cell C. From the description of cells and that of the action of W we know that M is the fixed point set of a reflection from the stabilizer S_C of C in W. We will show that H_M coincides with the fixed-point set H_w of w.

Any reflection w fixing a midplane M pointwise fixes also H_M pointwise, i.e., $H_M \subseteq H_w$. We have to show that the claimed property is invariant under equivalence relation of midplanes, see §2.2.1. If $M_1 \sim M_2$ are midplanes in $C_1 = C(M_1), C_2 = C(M_2)$ respectively, $C_1 \leq C_2$, and w fixes M_1 then w leaves C_1 invariant, hence by Lemma 1.4.2 it leaves C_2 invariant and by Lemma 2.1.1 it leaves M_2 invariant and finally by Lemma 2.1.4 it fixes M_2 pointwise. In case $M_1 \sim M_2, C_1 \geq C_2$, and w fixes M_1 pointwise it is clear that w fixes M_2 pointwise.

Every wall H is the fixed-point set of a unique reflection in W. Write H as the dual wall $H = H(e)$ of some edge e of $\mathcal{M}$. If there were two reflections w, w' with the same reflection wall H then their difference $w^{-1}w'$ would fix e pointwise. But W acts simply transitively on the vertices of $\mathcal{M}$ hence $w = w'$. Now any $w \in W$ fixing at least one cell pointwise is an identity. Indeed the set of cells fixing by w pointwise is nonempty and containing with each cell C every its "overcell" $C' \supset C$ because by Lemma 1.4.2 $wC' = C'$ and since the stabilizer of C' acts fixed point free on the cell we conclude that $w = 1$.

H_w coincides with H_M. Suppose, to the contrary, that there is a w-fixed point x outside H_M. Take any $y \in H_M$, then w fixes the endpoints x, y of the geodesic $[x, y]$ hence, by uniqueness, it fixes the whole geodesic. Shortening $[x, y]$ if necessary we may assume that $[x, y)$ is outside H_M. Take $z \in [x, y], z \neq y$ such that the open segment (z, y) is contained entirely in the interior of some cell C. Since w fixes (z, y), it leaves C invariant. As far as $y \in H_M \cap C$, the point y is contained in some midplane $M' \subset H_M$ of C. Because w fixes M' and the segment (z, y), lying entirely outside H_M, we conclude that w fixes C pointwise - contradiction.

For the converse, let w be a reflection in W. Note first that H_w contains at least one midplane. Indeed, since any reflection in W is conjugate to some $s, s \in$

S, we may assume that $w = s$. Take $J = \{s\}$, then the cell $X_J = \mathrm{Ch}(x_J, sx_J)$ is a segment on which s acts as a reflection thereby fixing its midpoint M. We conclude that H_w contains H_M for some midplane M. Therefore, as was proved above, H_w coincides with H_M. □

Lemma 2.2.5 *The edge path in $\mathcal{M}^{(1)}$ is geodesic if and only if it crosses each wall at most once.*

Proof. If an edge path $p = e_1 e_2 \cdots e_k$ crosses a wall H twice, say distinct edges $e_i, e_j, i < j$ cross H, then we delete the subpath $e_i \cdots e_j$ and instead insert the path $w(e_{i+1} \cdots e_{j-1})$, where w is the reflection in the wall H. The resulting path is strictly shorter than p but connects the same vertices. Conversely, suppose that an edge path p from x to y crosses each wall at most once. Let $\mathcal{H}_H$ be the set of all walls crossing by p. Since x and y are at the different sides of each wall from $\mathcal{H}_H$, we conclude that any path from x to y should cross than that of p. □

Any wall in the Moussong complex is "totally geodesic" in the following sense

Lemma 2.2.6 *Any geodesic in $\mathcal{M}$ having nondegenerate piece in a wall H, lies entirely in H.*

Proof. Suppose the lemma is false, then there are nondegenerate segments $\sigma_1 = [x, x_1], \sigma_2 = [x, x_2]$, cells C_1, C_2, and midplanes M_1, M_2 of C_1, C_2 respectively such that
1) $M_1 \sim M_2$,
2) $x \in M_1 \cap M_2$,
3) $\sigma_1 \subset M_1, x_2 \in C_2 - M_2$,
4) $\sigma_1 \cup \sigma_2$ is geodesic.
It follows from Lemma 2.1.5 that there is a reflection $w \in W$ and a segment $[y, wy]$ with x as a midpoint and orthogonal to both M_1 and M_2. Write $\sigma = [x, y], \sigma' = [x, wy]$. Since, by 3), $x_2 \in (C_2 - M_2)$, it follows from Lemma 2.1.6 that one of the angles $\angle_x(\sigma_2, \sigma)$, $\angle_x(\sigma_2, \sigma')$ is strictly less than $\pi/2$ and $\angle_x(\sigma_1, \sigma') = \angle_x(\sigma_2, \sigma) = \pi/2$. Hence the angle between the segments σ_1, σ_2 in the point x is strictly less than π, thus $\sigma_1 \cup \sigma_2$ can not be geodesic by criterion of Lemma 1.5.1.

2.3 Separation properties

Lemma 2.3.1 *Every wall in $\mathcal{M}$ separates $\mathcal{M}$ into exactly two connected components.*

Proof. First, we claim that H separates $\mathcal{M}$ into at least two components. We know from Lemma 2.2.2 that $H = H(e)$ – the dual wall of some edge $e = [x, y]$. We will show that x, y belong to different connectedness components. Suppose, to the contrary, that x, y are in the same connectedness component. Then there is a closed edge path α in $\mathcal{M}^{(1)}$ crossing H only once. (Clearly any edge either intersects H in a midpoint or does not intersects H at all.) Since $\mathcal{M}$ is contractible this path can be contracted to a constant path by a sequence of combinatorial contractions in cells. By Lemma 2.1.1 any cell C either has an empty intersection with H_M or $H_M \cap C$ is a midplane of C. This implies that each combinatorial contraction of the edge path in the cell does not change the number of intersections with H_M modulo 2. Since this number is 0 for the final constant path, it cannot be 1 for the initial path.

To prove that the H cuts out $\mathcal{M}$ into exactly two components, we proceed as in [NR98], lemma 2.3 (preprint version.) Notice first that H is 2-sided, that is there exists a neighborhood of H in $\mathcal{M}$ which is homeomorphic $H \times I, I = [0,1]$. Indeed, by Lemma 2.1.5, in each cell there is a neighborhood which is fibered as $M \times I$: the fibrations can be chosen to agree on face maps so this induces an I-bundle structure on some neighborhood N over H.

Since H itself is CAT(0) it is contractible so the bundle is trivial. It follows that N has two disjoint components, $\{-1/2\} \times H$ and $\{1/2\} \times H$. Any point in the complement of H can be joined to one of these boundary components by a path in the complement of H, and therefore $X - H$ has exactly 2 components as required. □

Lemma 2.3.1 *For any wall H both components of the complement $\mathcal{M} - H$ are convex.*

Proof. Suppose that x_1, x_2 lie on the same side of H, say H^+. We claim that $[x_1, x_2]$ lies entirely in H^+. Suppose the contrary, then by Lemma 2.2.6 the intersection $[x_1, x_2] \cap H$ consists of precisely one point, say x. Similar to the proof of Lemma 2.2.6 there are segments $\sigma_1 \subset [x, x_1], \sigma_2 \subset [x, x_2]$, cells C_1, C_2, and midplanes M_1, M_2 of C_1, C_2 respectively such that

1) $M_1, M_2 \subset H$,

2) $x \in \sigma_1 \cap \sigma_2$,

3) $\sigma_1 \subset C_1, \sigma_2 \subset C_2$,

4) $\sigma_1 \cup \sigma_2$ is geodesic.

5) The interiors of σ_1, σ_2 are contained entirely in H^+.

Then it follows from Lemma 2.1.5 that there exists a reflection $w \in W$ and a segment $[y, wy]$ such that the segment has x as a midpoint and is orthogonal to both M_1 and M_2. By interchanging the roles of y and wy if necessary we may assume that $y \in H^+$. Denote $\sigma = [x, y], \sigma' = [x, wy]$. It follows from Lemma 2.1.6 that the angles $\angle_x(\sigma_2, \sigma)$, $\angle_x(\sigma_2, \sigma')$ are both strictly less than $\pi/2$. But a small nonzero move of x along σ would strictly shorten the length of $\sigma_1 \cup \sigma_2$ contradicting the assumption 4) above. □

3 Chambers and galleries

3.1 Chambers

Since the complex $\mathcal{M}$ is locally finite and there are only finite number of midplanes in each cell, we conclude that the set of all walls $\mathcal{H}$ in $\mathcal{M}$ is locally finite, in the sense that every point of $\mathcal{M}$ has a neighborhood which meets only finitely many $H \in \mathcal{H}$.

Definition 3.1.1 By Lemma 2.3.1 the walls $H \in \mathcal{H}$ yield a partition of $\mathcal{M}$ into open convex sets, which are the connected components of the complement $\mathcal{M} - (\cup_{\mathcal{H}} H)$. We call these sets chambers.

To distinguish chambers from cells, we will denote them by letter D, possibly with indices, dashes, etc.

Lemma 3.1.2 *For any two distinct chambers $D(x), D(y), x, y \in \mathcal{M}^{(0)}$ there is a wall H separating them.*

Proof. Consider a geodesic edge path $p = e_1 e_2 \cdots e_k$ from x to y, then by Lemma 2.2.5 $H(e_1)$ separates x from y and hence separates $D(x)$ from $D(y)$. □

Lemma 3.1.3 *Each chamber contains precisely one vertex of $\mathcal{M}$.*

Proof. Since W acts simply transitively on the set of vertices of $\mathcal{M}$ and each vertex is contained in some chamber we deduce that each chamber contains at least one vertex. Now, if x, y are distinct vertices in a chamber C, we connect them by a geodesic path p in $\mathcal{M}^{(1)}$. Then by criterion of geodesicity any wall crossed by p separates x from y, contradicting the definition of chamber. □

In view of this lemma we will write $D(x)$ for the chamber containing the vertex x of $\mathcal{M}$.

Definitions 3.1.4 Recall from §2.1.2 that midplanes of any cell C in $\mathcal{M}$ yield a partition of C into convex (open) blocks. (Blocks are open in C, not in $\mathcal{M}$.) A maximal block is a block in a maximal cell. Two maximal blocks are adjacent if they are contained in the same maximal cell and share a codimension one face. Two chambers D, D' are adjacent if there are maximal blocks $B \subset D, B' \subset D'$ which are adjacent. A wall H is a wall of a chamber D if there is a maximal cell C such that $H \cap C$ contains a codimension one face F of a maximal block B of D.

Lemma 3.1.5 1) *Every chamber is uniquely determined by any of its maximal blocks.* 2) *Every chamber is a union of maximal blocks, and it contains at most one maximal block from each maximal cell.*

Proof. 1) Indeed, the interior of a maximal block is open in $\mathcal{M}$ and does not intersect any wall, consequently there is only one chamber containing this block.
2) Since $\mathcal{M}$ is a union of maximal cells, any chamber is a union of maximal blocks. Take a chamber D, then

$$D = \cup\{D \cap C : C \text{ is a maximal Moussong cell }\}.$$

The intersection $D \cap C$ is a union of maximal blocks because $D \cap C$ is an intersection of open half-cells in M. Next, if D contains two maximal blocks B_1, B_2 from one cell, then there is a midplane M separating B_1 from B_2 and the ambient wall H also separates B_1 from B_2 contradicting the definition of D. □

Lemma 3.1.6 *Let B, B' be maximal adjacent blocks and let D, D' be corresponding ambient chambers. Let H be a wall separating B from B'. Then H is the only wall that separates D from D'.*

Proof. Let C be a maximal cell containing B, B', then B, B' are adjacent in this cell and clearly there is only one midplane separating them. But the wall is uniquely determined by any of its midplanes, whence the lemma. □

Lemma 3.1.7 *Let D, D' be chambers such that their closures $\overline{D}, \overline{D'}$ have a nonempty intersection. Let H be a wall, separating D from D'. Then H contains the intersection $\overline{D} \cap \overline{D'}$.*

Proof. Suppose, to the contrary, that there is $b \in \overline{D} \cap \overline{D'}$ which is not contained in H. Since H is closed a small neighborhood of b does not intersect H. But this neighborhood contains points both from D and D', which thus belong to one halfspace of H, contradicting the separation hypothesis. □

Lemma 3.1.8 *Two distinct chambers $D(x), D(y)$ $(x, y \in \mathcal{M}^{(0)})$ are adjacent if and only if the vertices x, y are adjacent in $\mathcal{M}^{(1)}$. For any two adjacent chambers there is a reflection in W, permuting these chambers and fixing the intersection of their closures pointwise.*

Proof. The lemma is about Coxeter cell, thus it follows from the description of its structure as a Coxeter complex. □

Definition 3.1.9 The base chamber D_0 of $\mathcal{M}$ is the chamber, containing the base vertex x_0 of $\mathcal{M}$. For each s from the generating set S of W, we denote by H_s^- those open halfspace of the wall H_s, which contains the base vertex x_0.

Lemma 3.1.10 $D_0 = \cap\{H_-^s : s \in S\}$.

Proof. Since $D = \cap\{H^s_- : s \in S\}$ contains x_0, it contains also D_0. Let B_J be a block of a maximal cell X_J, containing $x_J = x_0$. Then $B_J \subset D_0$ – indeed it follows from the description of the chambers in the Coxeter complex that B_J is bounded by the hyperplanes $H_s = e_s^\perp, s \in J$. Suppose now that D strictly contains D_0 and let $x \in D - D_0$. Since D is convex, the whole segment $[x, x_0]$ lies in D. Let $T = (x_0, x_1, \dots, x_m)$ be a taut chain from x_0 to $x_m = x$. The first piece $[x_0, x_1]$ lies entirely in some maximal cell of the form X_K and we know that the block $B_K = D_0 \cap X_K$ is the maximal block in X_K and it is bounded by the hyperplanes $H_s^-, s \in K$.
If x_1 is a vertex of X_K, then it is separated by some $H^s, s \in K$ from x_0. If x_1 is not a vertex of X_K, then x_1 is the boundary point of X_K and hence it is contained in the interior of some face F of X_K. If F contains x_0, then all three points x_0, x_1, x_2 lie in some cell contradicting to the choice. Hence F does not contain x_0 and thus the open interval (x_0, x_1) lies entirely in the interior of X_J and hence crosses some wall $H^s, s \in J$ – contradiction. □

3.2 Galleries

Definitions 3.2.1 A gallery is a sequence of chambers $\Gamma = D_1 D_2 \cdots D_k$ such that any two consecutive ones are adjacent.

Recall that the chambers are in one-to-one correspondence with the vertices of $\mathcal{M}$ and chambers are adjacent if and only if the correspondent vertices are adjacent in the 1-skeleton of $\mathcal{M}$. It follows immediately that the following lemma is true.

Lemma 3.2.2 *1) Any two chambers D, D' can be connected by a gallery of length $d(D, D')$. 2) A gallery is geodesic if and only if and only if it does not cross any wall more than once. 3) Given $s_1, \dots, s_d \in S$, there is a gallery of the form $D_0(s_1 D_0)(s_1 s_2 D_0) \cdots (s_1 s_2 \cdots s_d D_0)$. Conversely, any gallery starting at C has this form. 4) The action of W is simply transitive on the set of chambers.*

□

Lemma 3.2.3 *There is a constant $c(\mathcal{M})$ such that for any two distinct chambers D, D' with nonempty intersection $\overline{D} \cap \overline{D'}$, there is a geodesic gallery $\Gamma = D_0 D_1 \cdots D_k$ from $D_0 = D$ to $D_k = D'$ whose length k does not exceed $c(\mathcal{M})$.*

Proof. Let $\mathcal{H}_0$ be the set of walls separating D from D'. In view of Lemma 3.2.2, it is enough to bound the cardinality of $\mathcal{H}_0$. According to Lemma 3.1.7 each $H \in \mathcal{H}_0$ contains $\overline{D} \cap \overline{D'}$. Let $x \in \overline{D} \cap \overline{D'}$. Clearly the number of cells containing x is uniformly bounded and for each such a cell C the number of midplanes in C containing x is also uniformly bounded. Since a wall is uniquely determined by any of its midplanes, this proves the lemma. □

3.3 Approximation property

Definition 3.3.1 Let (X, G) be a pair consisting of a geodesic metric space X and a graph G embedded into X. We say that (X, G) satisfies the approximation property if X-geodesics between the vertices of G can be uniformly approximated by geodesics in G. This means that there is a constant δ such that for any X-geodesic α_X between the vertices of G there is a G-geodesic α_G between the same vertices such that both α_X and α_G lie entirely in the δ-neighborhoods of each other. We will express this by saying that α_X, α_G are δ-close to each other.

Of particular interest is the case when G is the embedded Cayley graph of a group acting on X.

Theorem 3.3.2 *Let (W, S) be a Coxeter group and let $\mathcal{M}$ be its Moussong complex. Embed the Cayley graph $\mathcal{C}_W$ as an orbit Wx_0 for a point x_0 in a base chamber D_0 of $\mathcal{M}$. Then the pair $(\mathcal{M}_W, \mathcal{C}_W)$ satisfies the approximation property.*

Proof. Let $\sigma = [a, b]$ be a nondegenerate segment in $\mathcal{M}$ and $\mathcal{H}_\sigma$ be the set of walls having a nonempty intersection with the interior (a, b). Since the family of all walls is locally finite and the walls are totally geodesic, we have a partition

$$\mathcal{H}_\sigma = \mathcal{H}'_\sigma \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots \cup \mathcal{H}_n,$$

where the walls from $\mathcal{H}'_\sigma$ contain σ and the walls from $\mathcal{H}_i$ cross σ precisely in the point $a_i, i = 1, \cdots, n$, and $a = a_0 < a_1 < \cdots < a_n < a_{n+1} = b$.
Now we define a gallery Γ along the geodesic $\sigma = [a, b]$ as the gallery

$$\Gamma = D_1\Gamma_1 D_2\Gamma_2 D_3 \cdots D_n\Gamma_n D_{n+1}$$

such that
1) $\overline{D_i} \cap [a, b] = [a_{i-1}, a_i]$ $(i = 1, 2 \ldots, n+1)$,
2) Each spherical piece $D_i\Gamma_i D_{i+1}$ is a geodesic gallery and the lengths of spherical pieces are bounded from above by the constant $c(\mathcal{M})$ from Lemma 3.2.3,
3) Each spherical piece $D_i\Gamma_i D_{i+1}, i = 1, \ldots, n$ crosses the walls only from the set $\mathcal{H}'_\sigma \cup \mathcal{H}_i$.

Lemma 3.3.3 *For any geodesic $\sigma = [a, b]$ in $\mathcal{M}$ there is a geodesic gallery along σ.*

Proof of the lemma. By construction of the sequence $\{a_i\}$, for each $i = 1, \ldots, n+1$ there is a chamber D_i such that $\overline{D_i} \cap [a, b] = [a_{i-1}, a_i]$. The corresponding sequence of chambers $D_1, D_2, \ldots D_n, D_1 = D, D_n = D'$ is the first approximation to the required gallery. In general, this sequence is not a gallery, since two consecutive chambers are not necessarily adjacent. For each $1 \leq i \leq n$, the intersection of neighbors $\overline{D_i} \cap \overline{D_{i+1}}$ contains the point a_i.

Application of Lemmas 3.1.7, 3.2.3 enables us to inscribe a spherical geodesic subgallery of bounded length between these neighbors and get a gallery

$$\Gamma = D_1\Gamma_1 D_2\Gamma_2 \cdots D_{n-1}\Gamma_{n-1}D_n$$

such that the spherical pieces $D_i\Gamma_i D_{i+1}$ are geodesic galleries of uniformly bounded length satisfying condition 3) from the definition above. We will show that Γ can be modified to a geodesic gallery along $[a,b]$. If Γ is not geodesic then by Lemma 3.2.2 it crosses some wall H at least twice. Clearly $H \in \mathcal{H}'_\sigma$ i.e., H contains σ. Then there are indices $i+1 < j$ and subgalleries Γ_1, Γ_2 each of length 1 such that
a) Γ_1, Γ_2 belong to i-th and $j-$th spherical piece respectively,
b) Γ_1, Γ_2 cross H and moreover there are no crossing subgalleries in between.
Let $\Gamma_1 = DD', \Gamma_2 = D''D'''$. In particular the chambers D and D''' lie on the same side of H, say H^-, and the subgallery Γ' of Γ, joining D' with D'' lies on the opposite side, say H^+.
Let $w \in W$ be the reflection in the wall H. If we modify Γ by applying w to the portion $\tilde{\Gamma}$, we obtain the gallery from D to D''' that is strictly shorter than $D\Gamma' D'''$. Replacing $D\Gamma' D'''$ by $w(\tilde{\Gamma})$ we get the gallery Γ' that is strictly shorter than Γ but still is the gallery along σ. Repeating the previous process will construct a geodesic gallery along σ. This proves Lemma 3.3.3.
The theorem now follows easily from Lemma 3.3.3. Namely, given two chambers D, D' we take the points d, d' inside them and build a geodesic gallery $\Gamma = D_1 \cdots D_n$ along $[d,d']$. Γ not necessarily joins D to D' but the intersections $\overline{D}\cap\overline{D_1}, \overline{D'}\cap\overline{D_n}$ are nonempty, so we can join D to D_1 and D' to D_2 respectively by the galleries of uniformly bounded length thereby getting the gallery joining D to D' and that is $\delta(\mathcal{M})-$close to σ for some universal constant $\delta(\mathcal{M})$. $\square$

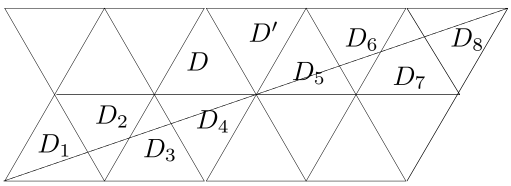


Figure 3: Gallery along geodesic. The spherical piece is $D_4DD'D_5$.

4 Word length on abelian subgroups of a Coxeter group

4.1 Straightness

Definition 4.1.1 Let G be a group with a fixed word metric $x \mapsto \ell(x)$. We say that an element $x \neq 0$ is straight if $\ell(v^n) = n\ell(v)$ for all natural n.

Remark 4.1.2 Straight elements have been studied for Coxeter groups in [Kra94] and for small cancellation groups in [Kap97] (in the last paper they are called periodically geodesic.

Example 4.1.3 (An element that is not straight.) Let W be an affine Coxeter group generated by reflections s_1, s_2, s_3 in the sides of an equilateral triangle C of a Euclidean plane. Let L_1, L_2, L_3 be the corresponding reflecting lines of this triangle. It is easily seen that there is nontrivial translation $u \in W$ with an axis L_1. We assert that nor $s_1 u$ neither any of it conjugates $v = ws_1uw^{-1}$ are straight. Indeed, the length $|ws_1uw^{-1}|$ is the length of a geodesic gallery Γ from C to $ws_1uw^{-1}C$. Any such a gallery intersects the line wL_1. The concatenation $\Gamma(v\Gamma)$ is a gallery from C to v^2C of length $2|ws_1uw^{-1}|$. But $\Gamma(v\Gamma)$ can not be geodesic, since it intersects wL_1 twice. Hence $|v^2| < 2|v|$. $\square$

Definition 4.1.4 Let $\mathcal{M}$ be the Moussong complex of a Coxeter group W. Recall that $\mathcal{M}$ is a proper complete CAT(0) space and W acts properly and cocompactly on $\mathcal{M}$ by isometries. In particular, any element $w \in W$ of infinite order acts as an axial isometry i.e., there is a geodesic axis A_w in $\mathcal{M}$, isometrical to $\mathbb{R}$, on which w acts as a nonzero translation [Bal95]. We say that w is generic if A_w intersects any wall in at most one point. In view of Lemma 2.2.6, this is equivalent to saying that no nondegenerate segment of A_w is contained in a wall.

Theorem 4.1.5 *Let (W, S) be a Coxeter system of finite type. For any generic element w of W of infinite order there is a conjugate v which is straight, that is $\ell(v^n) = n\ell(v)$ for all $n \in \mathbb{N}$, where $\ell(v)$ is a word length in generators S.*

Proof. We make use of the action of W on the Moussong complex $\mathcal{M}$. Since the family of all walls is locally finite, there is a point a on the axis A_w such that a does not belong to any wall of $\mathcal{M}$. Every point $w^i a (i \in \mathbb{Z})$ also does not belong to any wall of $\mathcal{M}$. Let $\mathcal{H}$ be the set of walls crossed by the segment $[a, wa]$ and let $a < a_1 < a_2 < \cdots < a_k < wa$ be the crossing points, so that $\mathcal{H}$ is a disjoint union $\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_k$ of subsets $\mathcal{H}_i$ crossing $[a, wa]$ in $a_i, i = 1, 2, \ldots, k$. There are the chambers $D_1, D_2, \cdots, D_k$ such that $\overline{D_1} \cap [a, wa] = [a, a_1], \overline{D_i} \cap [a, wa] = [a_{i-1}, a_i] (i = 1, 2, \ldots, k)$. Inscribe into the sequence $D_1, D_2, \cdots, D_k(wD_1)$ subgalleries $\Gamma_1, \ldots, \Gamma_k$, so that the concatenation $\Gamma = D_1\Gamma_1 D_2 \cdots \Gamma_{k-1} D_k \Gamma_k (wD_1)$ is a gallery, crossing only the walls from $\mathcal{H}$ and crossing each wall precisely once. In particular this gallery is geodesic. Let $\Gamma_0 = D_1\Gamma_1 D_2 \cdots \Gamma_{k-1} D_k \Gamma_k$. Translating by w and concatenating, we get a gallery $\tilde{\Gamma} = \Gamma_0(w\Gamma_0))(w^2\Gamma_0) \cdots (w^{n-1}\Gamma_0) w^n D_1$. The walls that it crosses are precisely those from the union $\mathcal{H} \cup w\mathcal{H} \cup w^2\mathcal{H} \cup \cdots \cup w^{n-1}\mathcal{H}$, and each wall is crossed precisely once. Hence the gallery $\tilde{\Gamma}$ is geodesic. Now let $D_1 = uD_0, u \in W$, where D_0 is the base chamber. Being a geodesic path in the Cayley graph, the gallery $\tilde{\Gamma}$ joins the vertex u to the vertex $w^n u = u(u^{-1}w^n u)$. Hence its length $n\ell(\Gamma_0)$ equals the word length of the element $u^{-1}w^n u \in W$.

We conclude that for $v = u^{-1}wu$ the equality $|v^n| = n|v|$ holds for all $n \in \mathbb{N}$. $\square$

Theorem 4.1.6 *Let (W, S) be a Coxeter group of finite type. There is a constant $c = c(W)$ such that for any element w of W of infinite order there is a conjugate v of w^c which is straight.*

Proof. Let $w \in W$ be of infinite order and let A_w be an axis of w. Let $\mathcal{H}_u = \mathcal{H}_u(A_u)$ denote the set of walls in the Moussong complex $\mathcal{M}_W$, containing A_u. It is easy to see that the cardinality of $\mathcal{H}_w$ is bounded by a constant depending only on W and we take $c = c(W)$ to be the number

$$2 \times \text{l.c.m.} \times (\text{card}\{\mathcal{H}_w : w \in W \text{ is of infinite order }\}).$$

Clearly A_w is an axis of w^c as well. Furthermore, w^c leaves invariant each wall $H \in \mathcal{H}_w$; moreover, it leaves invariant each of the two components of $\mathcal{M}_W - H, H \in \mathcal{H}_w$. It follows that for any chamber D, a geodesic gallery from D to w^cD does not cross a wall H from $\mathcal{H}_w$. Indeed, otherwise D and w^cD would lie in different components of $\mathcal{M}_W - H$ implying that w^c interchanges these components, contradicting the property above. Take a chamber D such that $\overline{D} \cap A_w$ is a nondegenerate segment and fix a point a in the interior of this segment. Let $\mathcal{H}$ denote the set of walls H that are crossed by the segment $[a, w^ca]$ but do not contain it. Clearly any $H \in \mathcal{H}$ separates D from w^cD. And conversely, if H separates, then the points a, w^ca lie in different components of $\mathcal{M}_W - H$ implying that H crosses the segment $[a, w^ca]$ in precisely one point. Let Γ be a geodesic gallery from D to w^cD then the walls that it crosses are precisely those from $\mathcal{H}$, and each wall $H \in \mathcal{H}$ is crossed by Γ precisely once. Iterating we obtain a gallery $\tilde{\Gamma} = \Gamma(w\Gamma))(w^2\Gamma) \cdots (w^{n-1}\Gamma)w^nD(n \in \mathbb{N})$ of the length $n\ell(\Gamma)$. This gallery crosses the walls only from (disjoint) union $\mathcal{H} \cup w^c\mathcal{H} \cup w^{2c}\mathcal{H} \cup \cdots \cup w^{(n-1)c}\mathcal{H}$, each precisely once. Hence the gallery $\tilde{\Gamma}$ is geodesic. Now let $D = uD_0, u \in W$, where D_0 is the base chamber. Being a geodesic path in the Cayley graph, the gallery $\tilde{\Gamma}$ joins the vertex u to the vertex $w^{nc}u = u(u^{-1}w^{nc}u)$. Hence its length $n\ell(\Gamma)$ equals the word length of the element $u^{-1}w^{nc}u \in W$. We conclude that for $v = u^{-1}w^cu$ the equality $|v^n| = n|v|$ holds for all $n \in \mathbb{N}$. $\square$

For elements which are not necessarily generic we have the following

Lemma 4.1.7 *Let (W, S) be a Coxeter group of finite type and let $w \in W$ be an element of infinite order. Fix an axis A_w of w in the Moussong complex $\mathcal{M}_W$. There is a chamber D such that for all $n \in \mathbb{Z}$*

$$d(D, w^nD) = n\ d(D, wD) - n \text{ card } (w^{\mathbb{Z}} \backslash \mathcal{H}_w) + c_n,$$

where $|c_n|$ is bounded by a constant depending only on W and $\mathcal{H}_w$ is the set of all walls H in $\mathcal{M}_W$, containing A_w and such that H separates w^iD from $w^{i+1}D$ for some $i \in \mathbb{Z}$.

Proof. We follow the proof of Theorem 4.1.5. Take a chamber D, such $\overline{D} \cap A_w$ is a nondegenerate segment. Let $\mathcal{H}$ be the set of walls, separating D from wD and do not containing A_w. By total geodesicity, any $H \in \mathcal{H}$ crosses A_w precisely in one point. Let Γ be a geodesic gallery from D to wD then it crosses all $H \in \mathcal{H}$, each precisely once, and some of the walls from $\mathcal{H}_w$. Iterating we get the gallery $\tilde{\Gamma} = \Gamma(w\Gamma))(w^2\Gamma) \cdots (w^{n-1}\Gamma)w^n D$. This gallery crosses the walls from (disjoint) union $\mathcal{H} \cup w\mathcal{H} \cup w^2\mathcal{H} \cup \cdots \cup w^{n-1}\mathcal{H}$, each precisely once. Also, it crosses some walls from $\mathcal{H}_w$. Note that, whenever $\tilde{\Gamma}$ crosses $H \in \mathcal{H}_w$, it crosses it periodically with a period $r_H =$ card $w^{\mathbb{Z}}H$. Hence, the integer part $[n/r_H]$ is the number of times the gallery $\tilde{\Gamma}$ crosses each $H' \in w^{\mathbb{Z}}H$. Hence it crosses the walls from the orbit $w^{\mathbb{Z}}H$ approximately n times, up to a universal constant. Hence, the number $d(D, w^n D)$ of walls, separating D from $w^n D$, equals $n\, d(D, wD) - n$ card $(w^{\mathbb{Z}} \backslash \mathcal{H}_w) + c_n$, where c_n is uniformly bounded. □

Theorem 4.1.8 *If, under conditions of Lemma 4.1.7, $D = uD_0, u \in W$, where D_0 is the base chamber, then $d(D, wD)$ is the word length of the conjugate $v = u^{-1}wu \in W$ and we get the following formula*

$$\ell(v^n) = n\ell(v) - \text{ card } (w^{\mathbb{Z}} \backslash \mathcal{H}_W) + c_n.$$

From this we get the following formula for a translation length $||w||$ *of* w:

$$||w|| \overset{\text{def}}{=} \lim_{n\to\infty} \frac{\ell(w^n)}{n} = \lim_{n\to\infty} \frac{\ell(v^n)}{n} = \ell(v) - \text{card}(w^{\mathbb{Z}} \backslash \mathcal{H}_w).$$

In particular, translation length of any element of W is rational (even integral).

Remark 4.1.9 The formula for translation length is similar to the one given in [Kra94], where it follows from the classification of roots. It seems unknown whether translation length is rational in an arbitrary "semihyperbolic group".

4.2 Norms and Burago's inequality

Let A be a normed abelian group, so A is equipped with a function $\ell : A \to \mathbb{R}$ satisfying (1) $\ell(a^{-1}) = \ell(a)$, (2) $\ell(ab) \leq \ell(a) + \ell(b)$, and (3) $\ell(a) \geq 0$ with $\ell(a) = 0$ iff $a = 1$, for $a, b \in A$. If (3) is replaced by (3') $\ell(a) \geq 0$ for $a \in A$, we call A a pseudonormed abelian group. Two pseudonorms ℓ and ℓ' on the abelian group A are called Hausdorff equivalent if there is a constant $k > 0$ so that $|\ell(a) - \ell'(a)| \leq k$ for all $a \in A$. The (pseudo)norm ℓ on the abelian group A is called regular if $\ell(a^n) = n\ell(a)$ for all $a \in A$ and all positive natural numbers n. Let ℓ be a norm on the abelian group A. We define the regularization $R\ell$ of ℓ by

$$R\ell(a) = \lim_{n\to\infty} \frac{\ell(a^n)}{n}.$$

By [PS78], p. 23, Exercise 99, this limit always exists, and it is an exercise to see that $R\ell$ is a regular pseudonorm.

Lemma 4.2.1 *The norm ℓ on the abelian group A is regular iff $R\ell = \ell$.*

Proof. If ℓ is regular, then clearly $R\ell = \ell$. Conversely, if $\ell(a^n) < n\ell(a)$ for some positive number n and some $a \in A$, then

$$R\ell(a) = \lim_{m\to\infty} \frac{\ell(a^{mn})}{mn} \le \frac{\ell(a^n)}{n} < \ell(a),$$

thus the lemma. $\square$

In general positivity of $R\ell$ fails, so it is possible that $R\ell(a) = 0$ but $a \neq 0$. Also it may easily happen that $R\ell$ is not Hausdorff equivalent to ℓ. We give a criterion for positivity and Hausdorff equivalence in terms of Burago's inequality [Gro93].

Definition 4.2.2 We say that a norm ℓ on an abelian group A satisfies the Burago's inequality if there exists a constant $c = c(A) > 0$ such that

$$\ell(a^2) \ge 2\ell(a) - c \quad \text{for all} \quad a \in A.$$

The norm is discrete if for all $n \in \mathbb{N}$ the ball $B_n = \{x \in A : \ell(x) \le n\}$ is finite. For example any word metric, corresponding to a finite generating set, is discrete.

Lemma 4.2.3 *If a discrete norm ℓ on a torsionfree abelian group A satisfies Burago's inequality then its regularization $R\ell$ is a norm also and, furthermore, $R\ell$ is Hausdorff equivalent to ℓ.*

Proof. By induction from Burago's inequality we deduce that $\ell(a^{2^n}) \ge 2^n\ell(a) - (2^n - 1)c$, for all $a \in A, n \in \mathbb{N}$. This implies that

$$\ell(a) \ge R\ell(a) = \lim_{m\to\infty} \frac{\ell(a^{2^m})}{2^m} \ge \ell(a) - c$$

for all $a \in A$. Thus the regularization $R\ell$ is Hausdorff equivalent to ℓ. As any regularization, this pseudonorm is regular. It remains to prove that $R\ell$ is a norm on A, i.e., it does not vanish on nonzero $a \in A$. If $a \in A$ is such that $\ell(a) \ge 1 + c$, then

$$R\ell(a) = \lim_{m\to\infty} \frac{\ell(a^{2^m})}{2^m} \ge \ell(a) - c \ge 1.$$

Now suppose $a \in A$ is arbitrary nonzero, then by the discreteness assumption $\ell(a^n) \ge 1+c$ for sufficiently large n, and since $R\ell$ is regular, $R\ell(a) = \frac{1}{n}R\ell(a^n) > 0$. $\square$

4.3 Approximation and Burago's inequality

Lemma 4.3.1 *Let Γ be a finitely generated group of isometries of a proper CAT(0) space X, acting cocompactly and properly on X. Suppose that $x_0 \in X$ has a trivial stabilizer so that the Cayley graph $\mathcal{C}$ of Γ can be considered as embedded into X via the orbit map $g \mapsto gx_0 (g \in \Gamma)$. Suppose that the pair $(X, \Gamma x_0)$ satisfies the approximation property. Then the restriction of the word length ℓ on Γ to any finitely generated free abelian subgroup A satisfies the Burago's inequality.*

Proof. By assumption there is a $\delta > 0$ such that for any $g \in \Gamma$ the X-geodesic α_X from x_0 to gx_0 and some $\mathcal{C}$–geodesic $\alpha_{\mathcal{C}}$ from x_0 to gx_0 are δ–close to each other. By the flat torus theorem [Bow95], [Bri95] there is a Euclidean subspace F in X on which A acts by translation. Fix the point $y_0 \in F$ and let a be an arbitrary nontrivial element in A. We will show that ax_0 is contained in a c-neighborhood of $\alpha_{\mathcal{C}}$ for a suitable $c > 0$. Clearly $d_X(a^2x_0, a^2y_0) = d_X(x_0, y_0)$. Parameterize the segments $[x_0, a^2x_0], [y_0, a^2y_0]$ by the segment $[0, 1]$ proportionally to arc length. It follows from the convexity of X–metric that the corresponding points on the segments are distance at most $d_X(x_0, y_0)$ from each other. Let u be the point on $[x_0, a^2x_0]$ corresponding to the point ay_0. By assumption u is distance at most δ from some point v on $\alpha_{\mathcal{C}}$. Hence we have bounded the X-distance from $ax_0 \in \mathcal{C}$ to $v \in \mathcal{C}$. (This key observation is illustrated in Figure 4). Since the Cayley graph $\mathcal{C}$ is quasiisometric to X this bounds the Cayley graph distance also. Thus, there is a constant $c = c(\Gamma, X) > 0$ such that $d_{\mathcal{C}}(ax_0, v) \leq c$. We have $\ell(a^2) = d_{\mathcal{C}}(x_0, v) + d_{\mathcal{C}}(v, a^2x_0) \geq (d_{\mathcal{C}}(x_0, ax_0) - c) + (d_{\mathcal{C}}(ax_0, a^2x_0) - c) = (\ell(a) - c) + (\ell(a) - c) = 2\ell(a) - 2c$, that is the Burago's inequality. $\square$

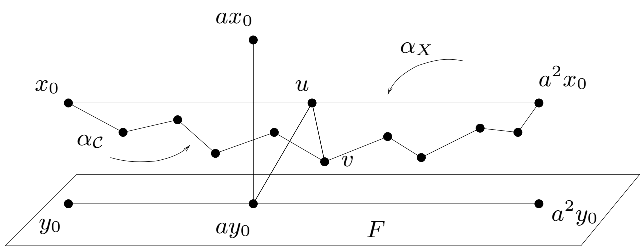


Figure 4: Lemma 4.3.1

Theorem 4.3.2 *Let (W, S) be a Coxeter group and let ℓ be the word length in generators S. Then the restriction of ℓ to any free abelian subgroup A of W is Hausdorff equivalent to a regular norm on A.*

Proof. Consider the pair $(\mathcal{M}_W, \mathcal{C}_W)$ where the Cayley graph $\mathcal{C}_W$ is embedded into the Moussong complex as an orbit Wx_0. By Theorem 3.3.2 $(\mathcal{M}_W, \mathcal{C}_W)$

satisfies the approximation property. Therefore by Lemma 4.3.1 the restriction of the word length ℓ on W to any finitely generated free abelian subgroup A satisfies the Burago's inequality. Finally, by Lemma 4.2.3 ℓ is Hausdorff equivalent to its regularization $R\ell$ and thus $R\ell$ is the required norm on A.

$\square$

I am grateful to Herbert Abels for asking the question that leads to the theorem above.

References

[Bal95] Ballmann, Werner. *Lectures on spaces of nonpositive curvature. With an appendix by Misha Brin: Ergodicity of geodesic flows.* DMV Seminar. Bd. 25. Basel: Birkhaeuser Verlag, 1995.

[BB97] Ballmann, Werner. *Rank rigidity of Euclidean polyhedra.* American Journal of Mathematics, 122(5):873–885, 2000.

[BH99] Martin Bridson and Andre Haefliger. *Metric Spaces of Non-Positive Curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften.* Springer, Berlin, Heidelberg, first edition, 1999.

[Bow95] B.H Bowditch. *Minkowskian subspaces of non-positively curved metric spaces. Bull. London Math. Soc.*, 27(6):575–584, 1995.

[Bri95] M. Bridson. *On the existence of flat planes in spaces of non-positive curvature.* Proc. Am. Math. Soc., 123(1):223–235, 1995.

[Bro96] Kenneth S. Brown. *Buildings.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, Heidelberg, second edition, 1996.

[Gro87] M. Gromov. *Hyperbolic groups.* In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[Gro93] M. Gromov. *Asymptotic invariants of infinite groups.* In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[Hum90] James E. Humphreys. *Reflection groups and Coxeter groups.* Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, 1990.

[Kap97] Ilya Kapovich. Small cancellation groups and translation numbers. *Trans. Amer. Math. Soc.*, 349(5):1851–1875, 1997.

[Kra94] Krammer Daan. *The conjugacy problem for Coxeter groups.* PhD thesis, Utrecht University, 1994.

[Mou88] Moussong Gabor. *Hyperbolic Coxeter groups.* PhD thesis, Ohio State University, 1988.

[NR98] G. A. Niblo and L. D. Reeves. *The geometry of cube complexes and the complexity of their fundamental groups.* Topology, 37(3):621–633, 1998.

[PS78] George Polia and Gabor Szege. *Problems and theorems in analysis. I*, volume 193 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, 1978. Series, integral calculus, theory of functions, Translated from the German by D. Aeppli, Corrected printing of the revised translation of the fourth German edition.

G. A. Noskov
Sobolev Institute of
Mathematics,
Omsk 644099, Pevtsova 13,
Russia
and
Fakultät für Mathematik ,
Universität Bielefeld,
Postfach 100131, D-33501
Bielefeld, Germany

Moduli of Framed Sheaves on Projective Surfaces

Ugo Bruzzo and Dimitri Markushevish[1]

Received: October 19, 2009
Revised: November 11, 2010

Communicated by Thomas Peternell

Abstract. We show that there exists a fine moduli space for torsion-free sheaves on a projective surface which have a "good framing" on a big and nef divisor. This moduli space is a quasi-projective scheme. This is accomplished by showing that such framed sheaves may be considered as stable pairs in the sense of Huybrechts and Lehn. We characterize the obstruction to the smoothness of the moduli space and discuss some examples on rational surfaces.

2010 Mathematics Subject Classification: 14D20; 14D21;14J60

Keywords and Phrases: Framed sheaves, moduli spaces, stable pairs, instantons

1. Introduction

There has been recently some interest in the moduli spaces of framed sheaves. One reason is that they are often smooth and provide desingularizations of the moduli spaces of ideal instantons, which in turn are singular [17, 19, 18]. For this reason, their equivariant cohomology under suitable toric actions is relevant to the computation of partition functions, and more generally expectation values of quantum observables in topological quantum field theory [20, 2, 19, 6, 3]. On the other hand, these moduli spaces can be regarded as higher-rank generalizations of Hilbert schemes of points, and as such they have interesting connections with integrable systems [12, 1], representation theory [26], etc.

While it is widely assumed that such moduli spaces exist and are well behaved, an explicit analysis, showing that they are quasi-projective schemes and are fine moduli spaces, is missing in the literature. In the present paper we provide such a construction for the case of framed sheaves on smooth projective surfaces under some mild conditions. We show that if D is a big and nef curve in a smooth projective surface X, there is a fine quasi-projective moduli space for

[1]This research was partly supported by prin "Geometria delle varietà algebriche" and by the European Science Foundation Programme Misgam.

sheaves that have a "good framing" on D (Theorem 3.1). The point here is that the sheaves under consideration are not assumed a priori to be semistable, and the basic idea is to show that there exists a stability condition making all of them stable, so that our moduli space is an open subscheme of the moduli space of stable pairs in the sense of Huybrechts and Lehn [8, 9].

In the papers [21, 22] T. Nevins constructed a scheme structure for these moduli spaces, however we obtain a stronger result, showing that these schemes are quasi-projective, and in particular are separated and of finite type. Moreover we compute the obstruction to the smoothness of these moduli spaces (Theorem 4.3). In fact, the tangent space is well known, but we provide a more precise description of the obstruction space than the one given by Lehn [14]. We show that it lies in the kernel of the trace map, thus extending a previous result of Lübke [15] to the non-locally free case.

In some cases there is another way to give the moduli spaces $\mathfrak{M}(r, c, n)$ a structure of algebraic variety, namely, by using ADHM data. This was done for vector bundles on $\mathbb{P}^2$ by Donaldson [5], while (always in the locally free case) the case of the blowup of $\mathbb{P}^2$ at a point is studied in A. King's thesis [13], and $\mathbb{P}^2$ blown-up at an arbitrary number of points was analyzed by Buchdahl [4]. The general case (i.e., including torsion-free sheaves) is studied by C. Rava for Hirzebruch surfaces [24] and A.A. Henni for multiple blowups of $\mathbb{P}^2$ at distinct points [7]. The equivalence between the two approaches follows from the fact that in both cases one has *fine* moduli spaces. On the ADHM side, this is shown by constructing a universal monad on the moduli space [23, 7, 25].

In the final section we discuss some examples, i.e. framed bundles on Hirzebruch surfaces with "minimal invariants", and rank 2 framed bundles on the blowup of $\mathbb{P}^2$ at one point.

In the present article, all the schemes we consider are separated and are of finite type over $\mathbb{C}$, and "a variety" is a reduced irreducible scheme of finite type over $\mathbb{C}$. A "sheaf" is always coherent, the term "(semi)stable" always means "μ-(semi)stable", and the prefix μ- will be omitted. Framed sheaves are always assumed to be torsion-free.

2. Framed sheaves

Let us characterize the objects that we shall study.

Definition 2.1. *Let X be a scheme over $\mathbb{C}$, $D \subset X$ an effective Weil divisor, and $\mathcal{E}_D$ a sheaf on D. We say that a sheaf $\mathcal{E}$ on X is $(D, \mathcal{E}_D)$-framable if $\mathcal{E}$ is torsion-free and there is an epimorphism $\mathcal{E} \to \mathcal{E}_D$ of $\mathcal{O}_X$-modules inducing an isomorphism $\mathcal{E}_{|D} \xrightarrow{\sim} \mathcal{E}_D$. An isomorphism $\phi\colon \mathcal{E}_{|D} \xrightarrow{\sim} \mathcal{E}_D$ will be called a $(D, \mathcal{E}_D)$-framing of $\mathcal{E}$. A framed sheaf is a pair $(\mathcal{E}, \phi)$ consisting of a $(D, \mathcal{E}_D)$-framable sheaf $\mathcal{E}$ and a framing ϕ. Two framed sheaves $(\mathcal{E}, \phi)$ and $(\mathcal{E}', \phi')$ are isomorphic if there is an isomorphism $f\colon \mathcal{E} \to \mathcal{E}'$ and a nonzero constant $\lambda \in \mathbb{C}$ such that $\phi' \circ f_{|D} = \lambda\phi$.*

Let us remark that our notion of framing is the same as the one used in [14, 22, 21], but is more restrictive than that of [8], where a framing is any homomorphism $\alpha : \mathcal{E} \to \mathcal{E}_D$ of $\mathcal{O}_X$-modules, not necessarily factoring through an *isomorphism* $\mathcal{E}_{|D} \xrightarrow{\sim} \mathcal{E}_D$. To distinguish between the two definitions, we will call such a pair $(\mathcal{E}, \alpha)$ a *framed pair*, whilst the term *framed sheaf* will refer to the notion introduced in Definition 2.1

Our strategy to show that framed sheaves on a projective variety make up "good" moduli spaces will consist in proving that, under some conditions, the framed sheaves $(\mathcal{E}, \phi)$ are stable according to a notion of stability introduced by Huybrechts and Lehn [8, 9]. The definition of stability for framed pairs depends on the choice of a polarization H on X and a positive real number δ (in our notation, δ is the leading coefficient of the polynomial δ in the definition of (semi)stability in [8]).

Definition 2.2 ([8, 9]). *A framed pair $(\mathcal{E}, \alpha)$ on an n-dimensional projective variety X, consisting of a torsion-free sheaf $\mathcal{E}$ and its framing $\alpha : \mathcal{E} \to \mathcal{E}_D$, is said to be (H, δ)-stable, if for any subsheaf $\mathcal{G} \subset \mathcal{E}$ with $0 < \operatorname{rk} \mathcal{G} \leq \operatorname{rk} \mathcal{E}$, the following inequalities hold:*

(1) $\dfrac{c_1(\mathcal{G}) \cdot H^{n-1}}{\operatorname{rk}(\mathcal{G})} < \dfrac{c_1(\mathcal{E}) \cdot H^{n-1} - \delta}{\operatorname{rk}(\mathcal{E})}$ *when $\mathcal{G}$ is contained in $\ker \alpha$;*

(2) $\dfrac{c_1(\mathcal{G}) \cdot H^{n-1} - \delta}{\operatorname{rk}(\mathcal{G})} < \dfrac{c_1(\mathcal{E}) \cdot H^{n-1} - \delta}{\operatorname{rk}(\mathcal{E})}$ *otherwise.*

Remark, that according to this definition, any rank-1 framed sheaf is (H, δ)-stable for any ample H and any $0 \leq \delta < \deg D$.

For any sheaf $\mathcal{F}$ on X, $P^H_{\mathcal{F}}$ denotes the Hilbert polynomial $P^H_{\mathcal{F}}(k) = \chi(\mathcal{F} \otimes \mathcal{O}_X(kH))$. For a non-torsion sheaf $\mathcal{F}$ on X, μ^H denotes the slope of $\mathcal{F}$: $\mu^H(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot H^{n-1}}{\operatorname{rk} \mathcal{F}}$.

Theorem 2.3 ([8, 9]). *Let X be a smooth projective variety, H an ample divisor on X and δ a positive real number. Let $D \subset X$ be an effective divisor, and $\mathcal{E}_D$ a sheaf on D. Then there exists a fine moduli space $\mathfrak{M} = \mathfrak{M}^H_X(P)$ of (H, δ)-stable $(D, \mathcal{E}_D)$-framed sheaves $(\mathcal{E}, \phi)$ with fixed Hilbert polynomial $P = P^H_{\mathcal{E}}$, and this moduli space is a quasi-projective scheme.*

Since we are using slope stability, and a more restrictive definition of framing with respect to that of [8, 9], our moduli space $\mathfrak{M}^H_X(P)$ is actually an open subscheme of the moduli space constructed by Huybrechts and Lehn.

Another general result on framed sheaves we shall need is a boundedness theorem due to M. Lehn. Given $X, D, \mathcal{E}_D$ as above, a set $\mathcal{M}$ of $(D, \mathcal{E}_D)$-framed pairs $(\mathcal{E}, \phi)$ is bounded if there exists a scheme of finite type S over $\mathbb{C}$ together with a family $(\boldsymbol{\mathcal{G}}, \boldsymbol{\phi})$ of $(D, \mathcal{E}_D)$-framed pairs over S such that for any $(\mathcal{E}, \phi) \in \mathcal{M}$, there exist $s \in S$ and an isomorphism $(\boldsymbol{\mathcal{G}}_s, \boldsymbol{\phi}_{|D \times s}) \simeq (\mathcal{E}, \phi)$.

Definition 2.4. *Let X be a smooth projective variety. An effective divisor D on X is called a good framing divisor if we can write $D = \sum n_i D_i$, where*

D_i are prime divisors and $n_i > 0$, and there exists a nef and big divisor of the form $\sum a_i D_i$ with $a_i \geq 0$. For a sheaf $\mathcal{E}_D$ on D, we shall say that $\mathcal{E}_D$ is a good framing sheaf, if it is locally free and there exists a real number A_0, $0 \leq A_0 < \frac{1}{r} D^2 \cdot H^{n-2}$, such that for any locally free subsheaf $\mathcal{F} \subset \mathcal{E}_D$ of constant positive rank, $\frac{1}{\operatorname{rk}\mathcal{F}} \deg c_1(\mathcal{F}) \leq \frac{1}{\operatorname{rk}\mathcal{E}_D} \deg c_1(\mathcal{E}_D) + A_0$.

Theorem 2.5. *Let X be a smooth projective variety of dimension $n \geq 2$, H an ample divisor on X, $D \subset X$ an effective divisor, and $\mathcal{E}_D$ a vector bundle on D. Assume that D is a good framing divisor. Then for every polynomial P with coefficients in $\mathbb{Q}$, the set of torsion-free sheaves $\mathcal{E}$ on X that satisfy the conditions $P^H_{\mathcal{E}} = P$ and $\mathcal{E}_{|D} \simeq \mathcal{E}_D$ is bounded.*

This is proved in [14], Theorem 3.2.4, for locally free sheaves, but the proof goes through also in the torsion-free case, provided that $\mathcal{E}_D$ is locally free, as we are assuming.

3. Quasi-projective moduli spaces

Using the notions introduced in the previous section, we now can state the main existence result for quasi-projective moduli spaces:

Theorem 3.1. *Let X be a smooth projective surface, $D \subset X$ a big and nef curve, and $\mathcal{E}_D$ a good framing sheaf on D. Then for any $c \in H^*(X, \mathbb{Q})$, there exists an ample divisor H on X and a real number $\delta > 0$ such that all the $(D, \mathcal{E}_D)$-framed sheaves $\mathcal{E}$ on X with Chern character* $\operatorname{ch}(\mathcal{E}) = c$ *are (H, δ)-stable, so that there exists a quasi-projective scheme $\mathfrak{M}_X(c)$ which is a fine moduli space for these framed sheaves.*

Proof. Let us fix an ample divisor C on X. Set $\mathcal{O}_X(k) = \mathcal{O}_X(kC)$ and $\mathcal{E}(k) = \mathcal{E} \otimes \mathcal{O}_X(k)$ for any sheaf $\mathcal{E}$ on X and for any $k \in \mathbb{Z}$. Recall that the Castelnuovo-Mumford regularity $\rho(\mathcal{E})$ of a sheaf $\mathcal{E}$ on X is the minimal integer m such that $h^i(X, \mathcal{E}(m-i)) = 0$ for all $i > 0$. According to Lehn's Theorem (Theorem 2.5), the family $\mathcal{M}$ of all the sheaves $\mathcal{E}$ on X with $\operatorname{ch}(\mathcal{E}) = c$ and $\mathcal{E}_{|D} \simeq \mathcal{E}_D$ is bounded. Hence $\rho(\mathcal{E})$ is uniformly bounded over all $\mathcal{E} \in \mathcal{M}$. By Grothendieck's Lemma (Lemma 1.7.9 in [10]), there exists $A_1 \geq 0$, depending only on $\mathcal{E}_D$, c and C, such that $\mu^C(\mathcal{F}) \leq \mu^C(\mathcal{E}) + A_1$ for all $\mathcal{E} \in \mathcal{M}$ and for all nonzero subsheaves $\mathcal{F} \subset \mathcal{E}$.

For $n > 0$, denote by H_n the ample divisor $C + nD$. We shall verify that there exists a positive integer n such that the range of positive real numbers δ, for which all the framed sheaves $\mathcal{E}$ from $\mathcal{M}$ are (H_n, δ)-stable, is nonempty.

Let $\mathcal{F} \subset \mathcal{E}$, $0 < r' = \operatorname{rk}\mathcal{F} \leq r = \operatorname{rk}\mathcal{E}$. Assume first that $\mathcal{F} \not\subset \ker\left(\mathcal{E} \to \mathcal{E}_{|D}\right)$. Then we may only consider the case $r' < r$, and the (H_n, δ)-stability condition for $\mathcal{E}$ reads:

$$\mu^{H_n}(\mathcal{F}) < \mu^{H_n}(\mathcal{E}) + \left(\frac{1}{r'} - \frac{1}{r}\right)\delta. \tag{1}$$

Saturating $\mathcal{F}$, we make $\mu^{H_n}(\mathcal{F})$ bigger, so we may assume that $\mathcal{F}$ is a saturated subsheaf of $\mathcal{E}$, and hence that it is locally free. Then $\mathcal{F}_{|D} \subset \mathcal{E}_{|D}$ and we have:

$$\mu^{H_n}(\mathcal{F}) = \frac{n}{r'} \deg c_1(\mathcal{F}_{|D}) + \mu^C(\mathcal{F}) \leq \mu^{H_n}(\mathcal{E}) + nA_0 + A_1. \tag{2}$$

Thus we see that (2) implies (1) whenever

$$\frac{rr'}{r-r'}(nA_0 + A_1) < \delta. \tag{3}$$

Assume now that $\mathcal{F}$ is a saturated, and hence a locally free subsheaf of $\ker\big(\mathcal{E} \to \mathcal{E}_{|D}\big) \simeq \mathcal{E}(-D)$. Then the (H_n, δ)-stability condition for $\mathcal{E}$ is

$$\mu^{H_n}(\mathcal{F}) < \mu^{H_n}(\mathcal{E}) - \frac{1}{r}\delta, \tag{4}$$

and the inclusion $\mathcal{F}(D) \subset \mathcal{E}$ yields:

$$\mu^{H_n}(\mathcal{F}) < \mu^{H_n}(\mathcal{E}) - H_n D + nA_0 + A_1 = \mu^{H_n}(\mathcal{E}) - (D^2 - A_0)n + A_1 - DC. \tag{5}$$

We see that (5) implies (4) whenever

$$\delta < r[(D^2 - A_0)n - A_1 + DC]. \tag{6}$$

The inequalities (3), (6) for all $r' = 1, \dots, r-1$ have a nonempty interval of common solutions δ if

$$n > \max\left\{\frac{rA_1 - CD}{D^2 - rA_0}, 0\right\}.$$

□

Remark 3.2. Grothedieck's Lemma is stated in [10] in terms of the so called $\hat{\mu}$ slope. However, for torsion-free sheaves, the $\hat{\mu}$ slope and the usual slope differ by constants depending only on $(X, \mathcal{O}_X(1))$, see Definition 1.6.8 in [10] and the following remark. △

Note that up to isomorphism, the quasi-projective structure making $\mathfrak{M}_X(c)$ a fine moduli space is unique, which follows from the existence of a universal family of framed sheaves over it.

If D is a smooth and irreducible curve and $D^2 > 0$, then our definition of a good framing sheaf with $A_0 = 0$ is just the definition of semistability. The following is thus an immediate consequence of the theorem:

Corollary 3.3. *Let X be a smooth projective surface, $D \subset X$ a smooth, irreducible, big and nef curve, and $\mathcal{E}_D$ a semistable vector bundle on D. Then for any $c \in H^*(X, \mathbb{Q})$, there exists a quasi-projective scheme $\mathfrak{M}_X(c)$ which is a fine moduli space of $(D, \mathcal{E}_D)$-framed sheaves on X with Chern character c.*

4. Infinitesimal study

Let X be a smooth projective variety, D an effective divisor on X, $\mathcal{E}_D$ a vector bundle on D. We shall consider sheaves $\mathcal{E}$ on X framed to $\mathcal{E}_D$ on D. We recall the notion of a simplifying framing bundle introduced by Lehn.

Definition 4.1. *$\mathcal{E}_D$ is simplifying if for any two vector bundles $\mathcal{E}$, $\mathcal{E}'$ on X such that $\mathcal{E}_{|D} \simeq \mathcal{E}'_{|D} \simeq \mathcal{E}_D$, the group $H^0(X, \mathcal{H}om(\mathcal{E}, \mathcal{E}')(-D))$ vanishes.*

An easy sufficient condition for $\mathcal{E}_D$ to be simplifying is $H^0(D, \mathcal{E}nd(\mathcal{E}_D) \otimes \mathcal{O}_X(-kD)_{|D}) = 0$ for all $k > 0$.

Lehn [14] proved that if D is good and $\mathcal{E}_D$ is simplifying, there exists a fine moduli space $\mathfrak{M}$ of $(D, \mathcal{E}_D)$-framed vector bundles on X in the category of separated algebraic spaces. Lübke [15] proved a similar result: if X is a compact complex manifold, D a smooth hypersurface (not necessarily "good") and if $\mathcal{E}_D$ is simplifying, then the moduli space $\mathfrak{M}$ of $(D, \mathcal{E}_D)$-framed vector bundles exists as a Hausdorff complex space. In both cases the tangent space $T_{[\mathcal{E}]}\mathfrak{M}$ at a point representing the isomorphism class of a framed bundle $\mathcal{E}$ is naturally identified with $H^1(X, \mathcal{E}nd(\mathcal{E})(-D))$, and the moduli space is smooth at $[\mathcal{E}]$ if $H^2(X, \mathcal{E}nd(\mathcal{E})(-D)) = 0$. Lübke gives a more precise statement about smoothness: $[\mathcal{E}]$ is a smooth point of $\mathfrak{M}$ if $H^2(X, \mathcal{E}nd_0(\mathcal{E})(-D)) = 0$, where $\mathcal{E}nd_0$ denotes the traceless endomorphisms. Huybrechts and Lehn in [9] define the tangent space and give a smoothness criterion for the moduli space of stable pairs that are more general objects than our framed sheaves. In this section, we adapt Lübke's criterion to our moduli space $\mathfrak{M}_X(c)$, parametrizing not only vector bundles, but also some non-locally-free sheaves. When we work with stable framed sheaves, we do not need the assumption that $\mathcal{E}_D$ is simplifying.

We shall use the notions of the trace map and traceless exts, see Definition 10.1.4 from [10]. Assuming X is a smooth algebraic variety, $\mathcal{F}$ any (coherent) sheaf on it, and $\mathcal{N}$ a locally free sheaf (of finite rank), the trace map is defined

$$\operatorname{tr} : \operatorname{Ext}^i(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}) \to H^i(X, \mathcal{N}), \quad i \in \mathbb{Z}, \tag{7}$$

and the traceless part of the ext-group, denoted by $\operatorname{Ext}^i(\mathcal{F}, \mathcal{F} \otimes \mathcal{N})_0$, is the kernel of this map.

We shall need the following property of the trace:

Lemma 4.2. *Let $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{E} \to 0$ be an exact triple of sheaves and $\mathcal{N}$ a locally free sheaf. Then there are two long exact sequences of ext-functors giving rise to the natural maps*

$$\mu_i : \operatorname{Ext}^i(\mathcal{F}, \mathcal{E} \otimes \mathcal{N}) \to \operatorname{Ext}^{i+1}(\mathcal{E}, \mathcal{E} \otimes \mathcal{N}) \ ,$$
$$\tau_i : \operatorname{Ext}^i(\mathcal{F}, \mathcal{E} \otimes \mathcal{N}) \to \operatorname{Ext}^{i+1}(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}) \ ,$$

and we have $\operatorname{tr} \circ \mu_i = (-1)^i \operatorname{tr} \circ \tau_i$ *as maps* $\operatorname{Ext}^i(\mathcal{F}, \mathcal{E} \otimes \mathcal{N}) \to H^{i+1}(X, \mathcal{N})$.

Proof. This is a particular case of the graded commutativity of the trace with respect to cup-products on Homs in the the derived category (see Section

V.3.8 in [11]): if $\xi \in \mathrm{Hom}(\mathcal{F}, \mathcal{E} \otimes \mathcal{N}[i])$, $\eta \in \mathrm{Hom}(\mathcal{E}, \mathcal{F}[j])$, then $\mathrm{tr}\,(\xi \circ \eta) = (-1)^{ij}\mathrm{tr}\,((\eta \otimes \mathrm{id}_{\mathcal{N}}) \circ \xi)$. This should be applied to $\xi \in \mathrm{Hom}(\mathcal{F}, \mathcal{E} \otimes \mathcal{N}[i])$ and $\eta = \partial \in \mathrm{Hom}(\mathcal{E}, \mathcal{F}[1])$, where ∂ is the connecting homomorphism in the distinguished triangle associated to the given exact triple:

$$\mathcal{E}[-1] \xrightarrow{-\partial} \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{E} \xrightarrow{\partial} \mathcal{F}[1].$$

□

Theorem 4.3. *Let X be a smooth projective surface, $D \subset X$ an effective divisor, $\mathcal{E}_D$ a locally free sheaf on D, and $c \in H^*(X, \mathbb{Q})$ the Chern character of a $(D, \mathcal{E}_D)$-framed sheaf $\mathcal{E}$ on X. Assume that there exists an ample divisor H on X and a positive real number δ such that $\mathcal{E}$ is (H, δ)-stable, and denote by $\mathfrak{M}_X(c)$ the moduli space of $(D, \mathcal{E}_D)$-framed sheaves on X with Chern character c which are (H, δ)-stable. Then the tangent space to $\mathfrak{M}_X(c)$ is given by*

$$T_{[\mathcal{E}]}\mathfrak{M}_X(c) = \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_X(-D)),$$

and $\mathfrak{M}_X(c)$ is smooth at $[\mathcal{E}]$ if the traceless ext-group

$$\mathrm{Ext}^2(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_X(-D))_0 = \ker\left[\ \mathrm{tr}\ : \mathrm{Ext}^2(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_X(-D)) \to H^2(X, \mathcal{O}(-D))\ \right]$$

vanishes.

Proof. We prove this result by a combination of arguments of Huybrechts-Lehn and Mukai, so we just give a sketch, referring to [9, 16] for details. As in Section 4.iv) of [9], the smoothness of $\mathfrak{M} = \mathfrak{M}_X(c)$ follows from the T^1-lifting property for the complex $\mathcal{E} \to \mathcal{E}_D$.

Let $A_n = k[t]/(t^{n+1})$, $X_n = X \times \mathrm{Spec}A_n$, $D_n = D \times \mathrm{Spec}A_n$, $\mathcal{E}_{D_n} = \mathcal{E}_D \boxtimes A_n$, and let $\mathcal{E}_n \xrightarrow{\alpha_n} \mathcal{E}_{D_n}$ be an A_n-flat lifting of $\mathcal{E} \to \mathcal{E}_D$ to X_n. Then the infinitesimal deformations of α_n over $k[\epsilon]/(\epsilon^2)$ are classified by the hyper-ext $\mathbb{E}\mathrm{xt}^1(\mathcal{E}_n, \mathcal{E}_n \xrightarrow{\alpha_n} \mathcal{E}_{D_n})$, and one says that the T^1-lifting property is verified for $\mathcal{E} \to \mathcal{E}_D$ if all the natural maps

$$T^1_n : \mathbb{E}\mathrm{xt}^1(\mathcal{E}_n, \mathcal{E}_n \xrightarrow{\alpha_n} \mathcal{E}_{D_n}) \to \mathbb{E}\mathrm{xt}^1(\mathcal{E}_{n-1}, \mathcal{E}_{n-1} \xrightarrow{\alpha_{n-1}} \mathcal{E}_{D_{n-1}})$$

are surjective whenever $(\mathcal{E}_n, \alpha_n) \equiv (\mathcal{E}_{n-1}, \alpha_{n-1}) \mod (t^n)$. In loc. cit., the authors remark that there is an obstruction map ob on the target of T^1_n which embeds the cokernel of T^1_n into $\mathbb{E}\mathrm{xt}^2(\mathcal{E}, \mathcal{E} \to \mathcal{E}_D)$, so that if the latter vanishes, the T^1-lifting property holds.

In our case, $\mathcal{E}$ is locally free along D, so the complex $\mathcal{E} \to \mathcal{E}_D$ is quasi-isomorphic to $\mathcal{E}(-D)$ and $\mathbb{E}\mathrm{xt}^i(\mathcal{E}, \mathcal{E} \to \mathcal{E}_D) = \mathrm{Ext}^i(\mathcal{E}, \mathcal{E}(-D))$. It remains to prove that the image of ob is contained in the traceless part of $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}(-D))$. This is done by a modification of Mukai's proof in the non-framed case.

First we assume that $\mathcal{E}$ is locally free. Then the elements of $\mathrm{Ext}^1(\mathcal{E}_{n-1}, \mathcal{E}_{n-1}(-D_{n-1}))$ can be given by Čech 1-cocycles with values in $\mathcal{E}nd(\mathcal{E}_{n-1})(-D_{n-1})$ for some open covering of X, and the image of such a 1-cocycle (a_{ij}) under the obstruction map $\mathrm{Ext}^1(\mathcal{E}_{n-1}, \mathcal{E}_{n-1}(-D_{n-1})) \to$

$\mathrm{Ext}^2(\mathcal{E},\mathcal{E}(-D))$ is a Čech 2-cocycle (c_{ijk}) with values in $\mathcal{E}nd(\mathcal{E})(-D)$. A direct calculation shows that $(\mathrm{tr}\, c_{ijk})$ is a Čech 2-cocycle with values in $\mathcal{O}_X(-D)$ which is the obstruction to the lifting of the infinitesimal deformation of the framed line bundle $\det \mathcal{E}_{n-1}$ from A_{n-1} to A_n. As we know that the moduli space of line bundles, whether framed or not, is smooth, this obstruction vanishes, so the cocycle $(\mathrm{tr}\, c_{ijk})$ is cohomologous to 0.

Now consider the case when $\mathcal{E}$ is not locally free. Replacing $\mathcal{E}, \mathcal{E}_D$ by their twists $\mathcal{E}(n)$, $\mathcal{E}_D(n)$ for some $n > 0$, we may assume that $H^i(X,\mathcal{E}) = H^i(X,\mathcal{E}(-D)) = 0$ for $i = 1, 2$ and that $\mathcal{E}$ is generated by global sections. Then we get the exact triple of framed sheaves

$$0 \to (\mathcal{G},\gamma) \to (H^0(X,\mathcal{E}) \otimes \mathcal{O}_X, \beta) \to (\mathcal{E},\alpha) \to 0,$$

where $\mathcal{G}$ is locally free (at this point it is essential that $\dim X = 2$ and X is smooth). Then we verify the T^1-lifting property for the exact triples

$$0 \to (\mathcal{G}_n,\gamma_n) \to (\mathcal{O}_{X_n}^N, \beta_n) \to (\mathcal{E}_n,\alpha_n) \to 0.$$

The infinitesimal deformations of such exact triples are classified by $\mathrm{Hom}(\mathcal{G}_n,\mathcal{E}_n(-D_n))$, and the obstructions lie in $\mathrm{Ext}^1(\mathcal{G},\mathcal{E}(-D))$. We have two connecting homomorphisms $\mu_1 : \mathrm{Ext}^1(\mathcal{G},\mathcal{E}(-D)) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E}(-D))$ and $\tau_1 : \mathrm{Ext}^1(\mathcal{G},\mathcal{E}(-D)) \to \mathrm{Ext}^2(\mathcal{G},\mathcal{G}(-D))$. Our hypotheses on $\mathcal{E}$ imply that: 1) every infinitesimal deformation of $(\mathcal{E}_n,\alpha_n)$ lifts to that of the triple, and 2) μ_1 is an isomorphism, that is, the infinitesimal deformation of $\mathcal{E}_n$ is unobstructed if and only if that of the triple is. By Lemma 4.2, $\mathrm{tr}\,(\mu_1(\xi)) = -\mathrm{tr}\,(\tau_1(\xi))$ in $H^2(X,\mathcal{O}_X(-D))$. As in 1.10 of [16], $\tau_1(\xi)$ is the obstruction $\mathrm{ob}(G_{n-1},\gamma_{n-1})$ to lifting (G_{n-1},γ_{n-1}) from A_{n-1} to A_n. As G_{n-1} is locally free, we can use the Čech cocycles as above and see that $\mathrm{tr}\,(\tau_1(\xi)) \in H^2(X,\mathcal{O}_X(-D))$ is the obstruction to lifting $(\det G_{n-1}, \det\gamma_{n-1})$, hence it is zero and we are done. □

The following Corollary describes a situation where the moduli space $\mathfrak{M}_X(c)$ is smooth (hence, every connected component is a smooth quasi-projective variety).

Corollary 4.4. *In addition to the hypothesis of Theorem 4.3, let us assume that D is irreducible, that $(K_X + D)\cdot D < 0$, and choose the framing bundle to be trivial. Then the moduli space $\mathfrak{M}_X(c)$ is smooth.*

This happens for instance when X is a Hirzebruch surface, or the blowup of $\mathbb{P}^2$ at a number of distinct points, taking for D the inverse image of a generic line in $\mathbb{P}^2$ via the birational morphism $X \to \mathbb{P}^2$. In this case one can also compute the dimension of the moduli space, obtaining $\dim \mathfrak{M}_X(c) = 2rn$, with $r = \mathrm{rk}(\mathcal{E})$ and

$$c_2(\mathcal{E}) - \frac{r-1}{2r} c_1(\mathcal{E})^2 = n\,\varpi,$$

where ϖ is the fundamental class of X. When X is the p-th Hirzebruch surface $\mathbb{F}_p$ we shall denote this moduli space by $\mathfrak{M}^p(r,k,n)$ if $c_1(\mathcal{E}) = kC$, where C is the unique curve in $\mathbb{F}_p$ having negative self-intersection.

The next example shows that the moduli space may be nonsingular even if the group $\operatorname{Ext}^2(\mathcal{E},\mathcal{E}\otimes\mathcal{O}_X(-D))$ does not vanish.

Example 4.5. For $r=1$ the moduli space $\mathfrak{M}(1,0,n)$ is isomorphic to the Hilbert scheme $X_0^{[n]}$ parametrizing length n 0-cycles in $X_0 = X\backslash D$. Of course this space is a smooth quasi-projective variety of dimension $2n$. Indeed in this case the trace morphism $\operatorname{Ext}^2(\mathcal{E},\mathcal{E}\otimes\mathcal{O}_X(-D)) \to H^2(X,\mathcal{O}(-D))$ is an isomorphism.

5. Examples

5.1. Bundles with small invariants on Hirzebruch surfaces. Let X be the p-th Hirzebruch surface $\mathbb{F}_p$, and normalize the Chern character by twisting by powers of the line bundle $\mathcal{O}_{\mathbb{F}_p}(C)$ so that $0 \le k \le r-1$. It has been shown in [3] that the moduli space $\mathfrak{M}^p(r,k,n)$ is nonempty if and only if the bound

$$n \ge N = \frac{pk}{2r}(r-k)$$

is satisfied. The moduli spaces $\mathfrak{M}^p(r,k,N)$ can be explicitly characterized: $\mathfrak{M}^p(r,k,N)$ is a rank $k(r-k)(p-1)$ vector bundle on the Grassmannian $G(k,r)$ of k-planes in $\mathbb{C}^r$ [25]; in particular, $\mathfrak{M}^1(r,k,N) \simeq G(k,r)$, and $\mathfrak{M}^2(r,k,N)$ is isomorphic to the tangent bundle of $G(k,r)$. This is consistent with instanton counting, which shows that the spaces $\mathfrak{M}^p(r,k,N)$ have the same Betti numbers as $G(k,r)$ [3].

5.2. Rank 2 vector bundles on $\mathbb{F}_1$. We study in some detail the moduli spaces $\mathfrak{M}^1(2,k,n)$. As [27] and [28] show, the non-locally free case turns out to be very complicated as soon as the value of n exceeds the rank. So we consider only locally free sheaves. To simplify notation we call this moduli space $\hat{M}(k,n)$, where n denotes now the second Chern class. We normalize k so that it will assume only the values 0 and -1. Moreover we shall denote by $M(n)$ the moduli space of rank 2 bundles on $\mathbb{P}^2$, with second Chern class n, that are framed on the "line at infinity" $\ell_\infty \subset \mathbb{P}^2$ (which we identify with the image of D via the blow-down morphism $\pi\colon \mathbb{F}_1 \to \mathbb{P}^2$).

Let us start with the case $k=-1$. We introduce a stratification on $\hat{M}(-1,n)$ according to the splitting type of the bundles it parametrizes on the exceptional line $E \subset \mathbb{F}_1$

$$\hat{M}(-1,n) = Z_0(-1,n) \supset Z_1(-1,n) \supset Z_2(-1,n) \supset \dots$$

defined as follows: if $Z_k^0(-1,n) = Z_k(-1,n) \setminus Z_{k+1}(-1,n)$ then

$$Z_k^0(-1,n) = \{\mathcal{E} \in \hat{M}(-1,n) \,|\, \mathcal{E}_{|E} \simeq \mathcal{O}_E(-k)\oplus\mathcal{O}_E(k+1)\}\,.$$

Proposition 5.1. *There is a map*

$$F_1\colon \hat{M}(-1,n) \to \coprod_{k=0}^{n} M(n-k)$$

which restricted to the subset $Z_k^0(-1,n)$ *yields a morphism*

$$Z_k^0(-1,n) \to M(n-k)$$

whose fibre is an open set in $\operatorname{Hom}(\sigma^*\mathcal{E}_{|E}, \mathcal{O}_E(k))/\mathbb{C}^* \simeq \mathbb{P}^{2k+1}$*, made by* k*-linear forms that have no common zeroes on the exceptional line.*

Proof. We start by considering $Z_0^0(-1,n)$. The morphism $Z_0^0(-1,n) \to M(n)$ is given by $\mathcal{E}_1 \mapsto \mathcal{E} = (\pi_*\mathcal{E})^{**}$. The fibre of this morphism includes a $\mathbb{P}^1$. To show that this is indeed a $\mathbb{P}^1$-fibration we need to check that $\mathcal{E}_1$ has no other deformations than those coming from the choice of a point in $M(n)$ and a point in this $\mathbb{P}^1$. This follows from the equalities

$$\dim \operatorname{Ext}^1(\mathcal{E}_1, \mathcal{E}_1(-E)) = \dim \operatorname{Ext}^1(\mathcal{E}, \mathcal{E}(-\ell_\infty) + 1$$

$$\operatorname{Ext}^2(\mathcal{E}_1, \mathcal{E}_1(-E)) = 0$$

Note that this result is compatible with the isomorphism $\mathfrak{M}^1(r,k,N) \simeq G(k,r)$ mentioned in Section 5.1.

In general, if $\mathcal{E}_1 \in Z_k^0(-1,n)$ with $k \geq 1$, so that $\mathcal{E}_{1|E} \simeq \mathcal{O}_E(k+1) \oplus \mathcal{O}_E(-k)$, the direct image $\pi_*(\mathcal{E}_1(kE))$ is locally free. This defines the morphism $Z_k^0(-1,n) \to M(n-k)$. $\square$

We consider now the case $k=0$. One has $Z_0^0(0,n) \simeq M(n)$. We study the other strata by reducing to the odd case. If $\mathcal{E}_1 \in Z_k^0(0,n)$, there is a unique surjection $\alpha\colon \mathcal{E}_1 \to \mathcal{O}_E(-k)$; let $\mathcal{F}$ be the kernel. Restricting $0 \to \mathcal{F} \to \mathcal{E}_1 \to \mathcal{O}_E(-k) \to 0$ we get an exact sequence

$$0 \to \mathcal{O}_E(1-k) \to \mathcal{F}_{|E} \to \mathcal{O}_E(k) \to 0$$

so that

$$\mathcal{F}_{|E} \simeq \mathcal{O}_E(a+1) \oplus \mathcal{O}_E(-a) \qquad \text{with} \qquad -k \leq a \leq k-1.$$

A detailed analysis shows that $a = k-1$. As a result we have:

Proposition 5.2. *For all* $k \geq 1$ *there is a morphism*

$$Z_k^0(0,n) \to M(n-2k+1)$$

whose fibres have dimension $2k-1$.

References

[1] D. Ben-Zvi and T. Nevins, *Flows of Calogero-Moser systems*, Int. Math. Res. Not. IMRN (2007), no. 23, Art. ID rnm105, 38 pp.

[2] U. Bruzzo, F. Fucito, J. F. Morales, and A. Tanzini, *Multi-instanton calculus and equivariant cohomology*, J. High Energy Phys. (2003), no. 5, 054, 24 pp. (electronic).

[3] U. Bruzzo, R. Poghossian, and A. Tanzini, *Poincaré polynomial of moduli spaces of framed sheaves on (stacky) Hirzebruch surfaces.* `arXiv:0909.1458 [math.AG]`. Comm. Math. Phys. (2011), to appear.

[4] N. P. Buchdahl, *Blowups and gauge fields*, Pacific J. Math. 196 (2000), pp. 69–111.

[5] S. K. Donaldson, *Instantons and geometric invariant theory*, Comm. Math. Phys. 93 (1984), pp. 453–460.

[6] E. Gasparim and C.-C. M. Liu, *The Nekrasov conjecture for toric surfaces,* Comm. Math. Phys. 293 (2010), pp. 661–700.

[7] A. A. Henni, *Monads for torsion-free sheaves on multi-blowups of the projective plane*. `arXiv:0903.3190 [math.AG]`.

[8] D. Huybrechts and M. Lehn, *Stable pairs on curves and surfaces*, J. Alg. Geom. 4 (1995), pp. 67–104.

[9] ——, *Framed modules and their moduli*, Internat. J. Math. 6 (1995), pp. 297–324.

[10] ——, *The geometry of moduli spaces of sheaves*, 2nd edition, Cambridge University Press, Cambridge, 2010.

[11] L. Illusie, *Complexe cotangent et déformations. I*, vol. 239 of Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1971.

[12] A. Kapustin, A. Kuznetsov, and D. Orlov, *Noncommutative instantons and twistor transform*, Comm. Math. Phys. 221 (2001), pp. 385–432.

[13] A. King, *Instantons and holomorphic bundles on the blown-up plane*. Thesis, Oxford University, 1989.

[14] M. Lehn, *Modulräume gerahmter Vektorbündel*. Ph.D. thesis, Bonn, 1992.

[15] M. Lübke, *The analytic moduli space of framed vector bundles*, J. Reine Angew. Math. 441 (1993), pp. 45–59.

[16] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface*, Invent. Math. 77 (1984), pp. 101–116.

[17] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, vol. 18 of University Lecture Series, American Mathematical Society, Providence, RI, 1999.

[18] H. Nakajima and K. Yoshioka, *Lectures on instanton counting*, in Algebraic structures and moduli spaces, vol. 38 of CRM Proc. Lecture Notes, Amer. Math. Soc., Providence, RI, 2004, pp. 31–101.

[19] ——, *Instanton counting on blowup. I. 4-dimensional pure gauge theory*, Invent. Math. 162 (2005), pp. 313–355.

[20] N. A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. 7 (2003), pp. 831–864.

[21] T. A. Nevins, *Moduli spaces of framed sheaves on certain ruled surfaces over elliptic curves*, Int. J. Math. 13 (2002), pp. 1117–1151.

[22] ——, *Representability for some moduli stacks of framed sheaves*, Manuscr. Math. 109 (2002), pp. 85–91.

[23] C. Okonek, M. Schneider, and H. Spindler, *Vector bundles on complex projective spaces*, vol. 3 of Progress in Mathematics, Birkhäuser Boston, Mass., 1980.

[24] C. L. S. Rava. Ph.D. thesis, SISSA (Trieste), 2011.

[25] ——, *ADHM data for framed sheaves on Hirzebruch surfaces*. In preparation, 2011.

[26] F. Sala and P. Tortella, *Representations of the Heisenberg algebra and moduli spaces of framed sheaves*. `arXiv:1004.2814 [math.AG]`.
[27] M. E. Sorokina, *Birational properties of moduli spaces of rank 2 semistable sheaves on the projective plane*. Ph.D. thesis (in Russian), Yaroslavl, 2006.
[28] A. S. Tikhomirov, *On birational transformations of Hilbert schemes of an algebraic surface*, Mat. Zametki 73 (2003), pp. 281–294.

Ugo Bruzzo
Scuola Internazionale Superiore
di Studi Avanzati
Via Bonomea 265
34136 Trieste
and
Istituto Nazionale
di Fisica Nucleare
Sezione di Trieste
Italia
bruzzo@sissa.it

Dimitri Markushevich
Mathématiques — Bât. M2
Université Lille 1
F-59655 Villeneuve d'Ascq Cedex
France
markushe@math.univ.lille1.fr

Modules Homotopiques

F. Déglise[(1)]

Received: January 22, 2010
Revised: November 30, 2010

Communicated by Alexander Merkurjev

Abstract. Based on previous works, we compare over a perfect field k the category of homotopy invariant sheaves with transfers introduced by V. Voevodsky and the category of cycle modules introduced by M. Rost: the former is a full subcategory of the latter. Using the recent construction by D.C. Cisinski and the author of a non effective version $DM(k)$ of the category of motivic complexes, we show that cycle modules form the heart of a natural t-structure on $DM(k)$, generalizing the homotopy t-structure on motivic complexes.

2010 Mathematics Subject Classification: 14F42, (14C15, 14C35)
Keywords and Phrases: motifs mixtes, complexes motiviques, modules de cycles, filtration par coniveau

Table des matières

(1) L'auteur est partiellement financé par l'ANR, projet no. ANR-07-BLAN-0142 "Méthodes à la Voevodsky, motifs mixtes et Géométrie d'Arakelov".

Introduction. —

Théorie de Voevodsky. — Dans sa théorie des complexes motiviques sur un corps parfait k, V. Voevodsky introduit le concept central de faisceau Nisnevich invariant par homotopie avec transferts, que nous appellerons simplement faisceau homotopique. Rappelons qu'un faisceau homotopique F est un préfaisceau de groupes abéliens sur la catégorie des k-schémas algébriques lisses, fonctoriel par rapport aux correspondances finies à homotopie près, qui est un faisceau pour la topologie de Nisnevich. Un exemple central d'un tel faisceau est donné par le préfaisceau $\mathbb{G}_m$ qui à un schéma lisse X associe le groupe des sections globales inversibles sur X. La catégorie des faisceaux homotopiques, notée ici $HI(k)$, a de bonnes propriétés que l'on peut résumer essentiellement en disant que c'est une catégorie abélienne de Grothendieck, monoïdale symétrique fermée.

Un des points centraux de la théorie est la démonstration par Voevodsky que tout faisceau homotopique F admet une *résolution de Gersten*[2]. Un cas

[2]Les complexes du type $(\mathcal{G})$, ci-dessous, ont été introduits par Grothendieck sous le nom *résolution de Cousin*, remplaçant la théorie des faisceaux homotopiques par celle des faisceaux cohérents. Grâce à la suite spectrale associée à la filtration par coniveau d'après Grothendieck, ils ont été réintroduits un peu plus tard dans le contexte des théories cohomologiques, par Brown et Gersten en K-théorie et finallement par Bloch et Ogus dans une version axiomatique. Notons que ces derniers auteurs parlent plutôt de "*arithmetic resolution*" et il semble que le terme de *résolution de Gersten* se soit imposé par la suite. La fonctorialité de la résolution de Gersten par rapport à un morphisme de schémas lisses a été

particulier de ce résultat est le fait que pour tout schéma lisse X, le groupe abélien $F(X)$ admet une résolution par un complexe de la forme:

$$(\mathcal{G}) \qquad C^*(X, \hat{F}_*) : \bigoplus_{x \in X^{(0)}} \hat{F}(\kappa(x)) \to \dots \bigoplus_{x' \in X^{(n)}} \hat{F}_{-n}(\kappa(x')) \to \dots$$

Suivant Voevodsky, $F_{-n} = \underline{\mathrm{Hom}}_{HI(k)}(\mathbb{G}_m^{\otimes n}, F)$. On a noté $X^{(n)}$ l'ensemble des points de codimension n de X. Pour un entier $r \geq 0$ et un point x de X, $\hat{F}_r(\kappa(x))$ désigne la fibre du faisceau homotopique F_r au point Nisnevich qui correspond au corps résiduel $\kappa(x)$, vu comme un corps de fonctions.
Un corollaire de cette résolution de Gersten est que les faisceaux homotopiques sont essentiellement déterminés par leurs fibres en un corps de fonctions. La question centrale de cet article est de savoir jusqu'à quel point ils le sont.

Théorie de Rost. — Pour définir un complexe de Gersten, du type $(\mathcal{G})$, on remarque qu'il faut essentiellement se donner un groupe abélien pour chaque corps résiduel d'un point de X. M. Rost axiomatise cette situation en introduisant les modules de cycles. Un module de cycles est un foncteur ϕ de la catégorie des corps de fonctions au-dessus de k vers les groupes abéliens gradués, muni d'une fonctorialité étendue qui permet de définir un complexe $C^*(X, \phi)$ du type $(\mathcal{G})$. Pour avoir une idée de cette fonctorialité, le lecteur peut se référer aux propriétés de la K-théorie de Milnor – mais aussi à la théorie des modules galoisiens. Rost note l'analogie entre ce complexe et le groupe des cycles de X – comme l'avaient fait Bloch et Quillen avant lui – et utilise le traitement de la théorie de l'intersection par Fulton pour montrer que la *cohomologie* du complexe, notée $A^*(X, \phi)$, est naturelle en X par rapport aux morphismes de schémas lisses.

Une comparaison. — Répondant à la question finale du premier paragraphe, nous comparons la théorie de Rost et celle de Voevodsky. D'une manière vague, notre résultat principal affirme que l'association $F \mapsto \hat{F}_*$ définit un foncteur pleinement fidèle des faisceaux homotopiques dans les modules de cycles, avec pour quasi-inverse à gauche le foncteur $\phi \mapsto A^0(., \phi)$.
Pour être plus précis dans la formulation de ce résultat, on est conduit à élargir la catégorie des faisceaux homotopiques. On définit un module homotopique F_* comme un faisceau homotopique $\mathbb{Z}$-gradué muni d'isomorphismes $\epsilon_n : F_n \to (F_{n+1})_{-1}$. La catégorie obtenue, notée $HI_*(k)$, est encore abélienne de Grothendieck, symétrique monoïdale fermée. De plus, elle contient comme sous-catégorie pleine la catégorie $HI(k)$ – si F est un faisceau homotopique, le module homotopique associé a pour valeur $\mathbb{G}_m^{\otimes n} \otimes F$ (resp. F_{-n}) en degré $n \geq 0$ (resp. $n < 0$).
Dès lors, on peut montrer que le système $\hat{F}_*$ des fibres d'un module homotopique F_* en un corps de fonctions définit un module de cycles. De plus, pour

traitée dans le cas de la K-théorie par H. Gillet (voir [Gil85]) puis étendue dans le cas des modules de cycles par M. Rost.

tout module de cycles ϕ, le groupe $A^0(X,\phi)$, dépendant fonctoriellement d'un schéma lisse X, définit un module homotopique.

Théorème (*cf.* 3.7). — *Les deux associations décrites ci-dessus définissent des fonteurs quasi-inverses l'un de l'autre.*

La résolution de Gersten obtenue par Voevodsky est maintenant équivalente au résultat suivant:

Corollaire (*cf.* 3.12). — *Si F_* est un module homotopique et X un schéma lisse, $H^n(X,F_*) = A^n(X,\hat{F}_*)$.*[3]

Notons que ce corollaire est étendu au cas singulier à la fin de l'article (Proposition 6.10). Cette extension nécessite d'interpréter le théorème 3.7 en termes motiviques.

L'interprétation motivique. — Rappelons qu'un complexe motivique suivant Voevodsky est un complexe[4] de faisceaux Nisnevich avec transferts dont les faisceaux de cohomologie sont des faisceaux homotopiques. La catégorie des complexes motiviques $DM^{eff}(k)$ porte ainsi naturellement une t-structure au sens de Beilinson, Bernstein et Deligne dont le coeur est la catégorie $HI(k)$.
La catégorie $DM^{eff}(k)$ est triangulée monoïdale symétrique fermée. Elle contient comme sous catégorie pleine la catégorie des motifs purs modulo équivalence rationnelle définie par Grothendieck. C'est ainsi une catégorie "effective ", dans le sens où le motif de Tate $\mathbb{1}(1)$ n'a pas de $\otimes$-inverse. Suivant l'approche initiale de Grothendieck, on est conduit à introduire une version non effective des complexes motiviques ; c'est ce qui est fait par D.C. Cisinski et l'auteur dans [CD09b]. Il est naturel dans le contexte des complexes motiviques de remplacer la construction habituelle pour inverser $\mathbb{1}(1)$ par l'approche des topologues pour définir la catégorie homotopique stable. La catégorie $DM(k)$, dont les objets seront appelés les spectres motiviques, est ainsi construite à partir du formalisme des spectres et des catégories de modèles. C'est la catégorie monoïdale homotopique[5] *universelle* munie d'un foncteur dérivé monoïdal

$$\Sigma^\infty : DM^{eff}(k) \to DM(k)$$

admettant un adjoint à droite Ω^∞ et telle que l'objet $\Sigma^\infty\,\mathbb{1}(1)$ est $\otimes$-inversible. Notons que dans le cadre des complexes motiviques, le foncteur Σ^∞ est pleinement fidèle d'après le théorème de simplification de Voevodsky [Voe02].
Dans cet article, nous montrons que l'on peut étendre la définition de la t-structure homotopique à la catégorie $DM(k)$, de telle manière que le foncteur Ω^∞ est t-exact. Le coeur de la t-structure homotopique sur $DM(k)$ est

(3) L'identification obtenue ici est naturelle, non seulement par rapport au pullback (lemme 3.3), mais aussi par rapport aux correspondances finies (proposition 3.10) et par rapport au pushout par un morphisme projectif (proposition 3.16).
(4) Originellement, ces complexes sont supposés bornés supérieurement. Nous abandonnons cette hypothèse dans tout l'article suivant [CD09b].
(5) c'est-à-dire la catégorie homotopique associée à une catégorie de modèles monoïdale.

la catégorie $HI_*(k)$ des modules homotopiques, qui est donc canoniquement identifiée à la catégorie des modules de cycles d'après le théorème 3.7 déjà cité. Ceci nous permet de donner une interprétation frappante du module de cycles $\hat{F}_*$ associé à un module homotopique F_*, à travers la notion de motifs génériques de [Dég08b].[(6)] Le motif générique associé à un corps de fonctions E est le pro-motif défini par tous les modèles lisses de E. On considère la catégorie $DM^{(0)}_{gm}(k)$ formée par tous les twists de motifs génériques par $\mathbb{1}(n)[n] = \mathbb{1}\{n\}$ pour $n \in \mathbb{Z}$. Alors, $\hat{F}_*$ est simplement la restriction du foncteur représenté par F_* dans $DM(k)$ à la catégorie $DM^{(0)}_{gm}(k)$. La catégorie $DM^{(0)}_{gm}(k)$ est une catégorie de "points " pour les spectres motiviques, et la fonctorialité des modules de cycles est interprétée en termes de *morphismes de spécialisations* entre ces points. De ce point de vue, les modules homotopiques correspondent à des systèmes locaux où le groupoïde fondamental est remplacé par la catégorie $DM^{(0)}_{gm}(k)$.
L'interprétation motivique nous sert finalement à introduire une condition de finitude (définition 6.6) sur les modules de cycles qui implique que leur graduation naturelle est bornée inférieurement (Corollaire 6.8) – comme c'est le cas de la plupart des modules de cycles définis par des moyens géométriques.

Plan du travail. — L'article est divisé en deux parties, l'une consacée au théorème principal 3.7 et l'autre à sa signification en termes de la théorie motivique de Voevodsky.
La première partie est faite de trois sections. Dans la section 1, on rappelle les propriétés principales des faisceaux homotopiques, et on introduit la catégorie des module homotopiques. Dans la section 2, on rappelle brièvement la théorie des modules de cycle de M. Rost et on établit quelques résultats supplémentaires utiles dans cet article. La section 3 et consacrée à la preuve du théorème central 3.7 cité précédemment. De plus, on établit plusieurs propriétés concernant la fonctorialité de l'identification 3.12 citée ci-dessus.
La deuxième partie est aussi constituée de trois sections. La section 4 contient des rappels concernant la théorie des complexes motiviques de Voevodsky ainsi que la version stable qu'on a introduite avec Cisinski dans [CD09b]. La section 5 est consacrée à la définition de la t-structure homotopique et à l'identification de son coeur avec les modules homotopiques. La section 6 est consacrée aux applications du point de vue motivique: construction de modules de cycles (section 6.1), borne inférieure (section 6.2) et extension du corollaire 3.12 au cas singulier (section 6.3).

Mise en perspective. — Ce travail a été utilisé récemment par B. Kahn dans [Kah10] pour étendre un théorème de Merkurjev. Kahn démontre par exemple que le théorème de Merkurjev est conséquence de notre théorème 3.7 (voir remarque 6.2).
Nous avons aussi utilisé les résultats de cet article dans deux travaux indépendants:

(6) Cette notion a aussi été introduite par A.Beilinson dans [Bei02].

– F. Morel a introduit une t-structure homotopique sur la catégorie homotopique stable des schémas, analogue à celle qu'on définit sur $DM(k)$. Il a conjecturé une relation très précise entre le coeur de sa t-structure, noté $\Pi_*(k)$, et les modules homotopiques (avec transferts) considérés ici: la catégorie $HI_*(k)$ est une sous-catégorie pleine de $\Pi_*(k)$, formée des objets sur lesquels l'*application de Hopf* agit trivialement. On démontre cette conjecture à partir des résultats de cet article dans [Dég10].
– On approfondit aussi la relation entre modules homotopiques et résolution de Gersten en montrant que la suite spectrale du coniveau associée à la cohomologie représentée par un spectre motivique E s'identifie canoniquement à la suite spectrale d'hyper-cohomologie à coefficients dans E associée à la t-structure homotopique (voir [Dég09, sec. 6]). Ce théorème prolonge un résultat de Bloch-Ogus (*cf.* [BO74, 6.4]).

Remerciements. — Mes remerciements vont en premier lieu à F. Morel qui a dirigé ma thèse, dans laquelle le résultat central de cet article a été établi. L'influence de ses idées est partout dans ce texte. Je remercie aussi A. Suslin et A. Merkurjev qui ont été les rapporteurs de cette thèse et dont les rapports m'ont beaucoup aidés dans la rédaction présente, ainsi que D.C. Cisinski pour sa relecture et son intérêt pour mon mémoire de thèse. Enfin, je remercie J. Ayoub, A. Beilinson, J.B. Bost, B. Kahn, J. Riou, C. Soulé et J. Wildeshaus pour leur intérêt et des discussions autour du sujet de cet article.

Notations. — On fixe un corps parfait k. Tous les schémas considérés sont des k-schémas séparés. Nous dirons qu'un schéma X est lisse si il est lisse de type fini sur k. La catégorie des schémas lisses est notée $\mathscr{L}_k$.
Nous disons qu'un schéma X est *essentiellement de type fini* s'il est localement isomorphe au spectre d'une k-algèbre qui est une localisation d'une k-algèbre de type fini.
On appelle *corps de fonctions* toute extension de corps E/k de degré de transcendance fini. Un *corps de fonctions valué* est un couple (E, v) où E est un corps de fonctions et v est une valuation sur E dont l'anneau des entiers est essentiellement de type fini sur k.
Un *modèle* de E/k est un k-schéma lisse connexe X muni d'un k-isomorphisme entre son corps des fonctions et E. On définit le pro-schéma des modèles de E :

$$(E) = \text{"}\varprojlim_{A\subset E}\text{"}\ \mathrm{Spec}(A)$$

où A parcourt l'ensemble ordonné filtrant des sous-k-algèbres de type fini de E dont le corps des fractions est E.
Voici une liste des catégories principales utilisées dans ce texte:
– $DM^{eff}_{gm}(k)$ (resp. $DM_{gm}(k)$) désigne la catégorie des motifs géométriques effectifs (resp. non nécessairement effectifs).
– $DM^{eff}(k)$ désigne la catégorie des complexes motiviques (que l'on ne suppose pas nécessairement bornés inférieurement).

– $DM(k)$ désigne la catégorie des spectres motiviques, version non effective de $DM^{eff}(k)$.
– $HI(k)$ (resp. $HI_*(k)$) désigne la catégorie des faisceaux (resp. modules) homotopiques. C'est le coeur de la t-structure homotopique sur $DM^{eff}(k)$ (resp. $DM(k)$).
– $\mathscr{M}Cycl(k)$ désigne la catégorie des modules de cycles.

PARTIE I
MODULES HOMOTOPIQUES ET MODULES DE CYCLES

1. Modules homotopiques

1.1. Rappels sur les faisceaux avec transferts. — Dans cette partie préliminaire, on rappelle la théorie de Voevodsky des faisceaux avec transferts et des faisceaux homotopiques. Nous nous référons à [Dég07] pour les détails.[7]

1.1. — Soient X et Y des schémas lisses. Rappelons qu'une *correspondance finie* de X vers Y est un cycle de $X \times Y$ dont le support est fini équidimensionel sur X. La formule habituelle permet de définir un produit de composition pour les correspondances finies qui donne lieu à une catégorie additive $\mathscr{L}_k^{cor}$ (*cf.* [Dég07, 4.1.19]). On obtient un foncteur $\gamma : \mathscr{L}_k \to \mathscr{L}_k^{cor}$, égal à l'identité sur les objets, en associant à tout morphisme le cycle associé à son graphe. La catégorie $\mathscr{L}_k^{cor}$ est enfin monoïdale symétrique. Le produit tensoriel sur les objets est donné par le produit cartésien des schémas lisses; sur les morphismes, il est induit par le produit extérieur des cycles (*cf.* [Dég07, 4.1.23]).

1.2. — Un *faisceau avec transferts* est un foncteur $F : (\mathscr{L}_k^{cor})^{op} \to \mathscr{A}b$ additif contravariant tel que $F \circ \gamma$ est un faisceau Nisnevich. On note $Sh^{tr}(k)$ la catégorie des faisceaux avec transferts munis des transformations naturelles. Cette catégorie est abélienne de Grothendieck (*cf.* [Dég07, 4.2.8]). Une famille génératrice est donnée par les faisceaux représentables par un schéma lisse X :

$$\mathbb{Z}^{tr}(X) : Y \mapsto c(Y, X).$$

Il existe un unique produit tensoriel symétrique $\otimes^{\mathrm{tr}}$ sur $Sh^{tr}(k)$ telle que le foncteur $\mathbb{Z}^{tr}$ est monoïdal symétrique. La catégorie $Sh^{tr}(k)$ est de plus monoïdale symétrique fermée (*cf.* [Dég07, 4.2.14]).

Définition 1.3. — Un *faisceau homotopique* est un faisceau avec transferts F invariant par homotopie : pour tout schéma lisse X, le morphisme induit par la projection canonique $F(X) \to F(\mathbb{A}^1_X)$ est un isomorphisme.

[7] Cette référence contient une relecture des preuves originales de Voevodsky ainsi que quelques compléments qui nous seront utiles.

On note $HI(k)$ la sous-catégorie pleine de $Sh^{tr}(k)$ formée des faisceaux homotopiques. Le foncteur d'oubli évident $\mathcal{O} : HI(k) \to Sh^{tr}(k)$ admet un adjoint à gauche $h_0 : Sh^{tr}(k) \to HI(k)$, $h_0(F)$ étant défini comme le faisceau associé au préfaisceau

$$X \mapsto \mathrm{coKer}\left(F(\mathbb{A}^1_X) \xrightarrow{s_0^*-s_1^*} F(X)\right) \tag{1.3.a}$$

avec s_0 (resp. s_1) la section nulle (resp. unité) de $\mathbb{A}^1_X/X$ (*cf.* [Dég07, 4.4.4, 4.4.15]). D'après *loc. cit.*, le foncteur $\mathcal{O}$ est exact. La catégorie $HI(k)$ est donc une sous-catégorie épaisse de $Sh^{tr}(k)$. En particulier, c'est une catégorie abélienne de Grothendieck dont une famille génératrice est donnée par les faisceaux de la forme $h_0(X) := h_0(\mathbb{Z}^{tr}(X))$. On vérifie aisément que le foncteur $\mathcal{O}$ commute de plus à toutes les limites projectives ce qui implique que $HI(k)$ admet des limites projectives.

1.4. — Pour un corps de fonctions E, on définit la fibre de F en E comme la limite inductive de l'application de F au pro-schéma (E) :

$$\hat{F}(E) = \varinjlim_{A \subset E} F(\mathrm{Spec}(A))$$

Les foncteurs $F \mapsto \hat{F}(E)$ forment une famille conservative de foncteurs fibres[8] de $HI(k)$ (*cf.* [Dég07, 4.4.7]).

Remarque 1.5. — Ce dernier résultat repose sur la propriété très intéressante des faisceaux homotopiques suivante:

Proposition 1.6. — *Pour toute immersion ouverte dense $j : U \to X$ dans un schéma lisse, le morphisme induit*

$$j_* : h_0(U) \to h_0(X)$$

est un épimorphisme dans $HI(k)$.

Cette proposition est une conséquence du corollaire 4.3.22 de [Dég07]: il existe un recouvrement ouvert $W \xrightarrow{\pi} X$ et une correspondance finie $\alpha : W \to U$ telle que le diagramme suivant est commutatif à homotopie près

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[d, "\pi"] \\ U \arrow[r, "j"] & X. \end{tikzcd}$$

On peut la reformuler en disant que pour tout faisceau homotopique F, le morphisme $F(X) \to F(U)$ induit par j est un monomorphisme. On déduit de ce dernier résultat que pour tout schéma lisse connexe X de corps des fontions E, le morphisme canonique $F(X) \to \hat{F}(E)$ est un monomorphisme.

1.7. — Dans une catégorie abélienne de Grothendieck $\mathscr{A}$, une classe de flèches $\mathcal{W}$ est dite localisante si :

(i) $\mathcal{W}$ est stable par limite inductive.

[8] *i.e.* exacts commutant aux limites inductives.

(ii) Soit f et g des flèches composables de $\mathscr{A}$. Si deux des constituants de (f, g, gf) appartiennent à $\mathcal{W}$, le troisième appartient à $\mathcal{W}$.

Si $\mathcal{S}$ est un classe de flèches essentiellement petite, on peut parler de la classe de flèches localisante engendrée par $\mathcal{S}$.

Lemme 1.8. — Il existe un unique produit tensoriel symétrique $\otimes^{\mathrm{Htr}}$ *sur* $HI(k)$ *tel que le fonteur* h_0 *est monoïdal symétrique.*

Démonstration. — D'après ce qui précède, $HI(k)$ s'identifie à la localisation de la catégorie $Sh^{tr}(k)$ par rapport à la classe de flèches localisante engendrée par les morphismes $\mathbb{Z}^{tr}(\mathbb{A}^1_X) \to \mathbb{Z}^{tr}(X)$ pour un schéma lisse X arbitraire. Ainsi, pour tout schéma lisse X, $\mathcal{W} \otimes^{\mathrm{tr}} \mathbb{Z}^{tr}(X) \subset \mathcal{W}$. Donc le produit tensoriel $\otimes^{\mathrm{tr}}$ satisfait la propriété de localisation par rapport à $\mathcal{W}$ ce qui démontre le lemme. $\square$

La catégorie $HI(k)$ munie du produit tensoriel $\otimes^{\mathrm{Htr}}$ obtenu dans le lemme précédent est monoïdale symétrique fermée. Ce produit tensoriel est caractérisé par la relation $h_0(X) \otimes^{\mathrm{Htr}} h_0(Y) = h_0(X \times Y)$ déduite du lemme précédent.

Définition 1.9. — Soit $s : \{1\} \to \mathbb{G}_m$ l'immersion du point unité. On appelle *sphère de Tate* le conoyau de $h_0(s)$ dans la catégorie $HI(k)$. On la note S^1_t. Pour tout entier $n \geq 0$, on note S^n_t la puissance tensorielle n-ième de S^1_t dans la catégorie monoïdale $HI(k)$.

D'après l'invariance par homotopie, on obtient encore une suite exacte courte scindée dans $HI(k)$:

$$0 \to S^1_t \to h_0(\mathbb{G}_m) \xrightarrow{j_*} h_0(\mathbb{A}^1_k) \to 0.$$

où j est l'immersion ouverte évidente.

1.10. — Soit $n \geq 0$ un entier et E/k un corps de fonction. Pour un groupe abélien M, on note $T_n(M)$ la puissance tensorielle n-ième de M pour $\otimes_{\mathbb{Z}}$.
En utilisant le morphisme canonique $\mathbb{G}_m \to h_0(\mathbb{G}_m)$ et la définition du produit tensoriel $\otimes^{\mathrm{Htr}}$, on obtient un morphisme canonique:

$$\lambda^n_E : T_n(E^\times) \to \widehat{S^n_t}(E).$$

Notons encore

$$\pi_n : T_n(E^\times) \to K^M_n(E)$$

l'épimorphisme canonique à valeur dans le n-ème groupe de K-théorie de Milnor de E. On utilisera de manière centrale le résultat suivant dû à Suslin et Voevodsky (voir [SV00, th. 3.4]):

Théorème 1.11 (Suslin-Voevodsky). — *Avec les notations qui précèdent, le morphisme* λ^n_E *se factorise de manière unique par* π_n *et induit un isomorphisme:*

$$K^M_n(E) \to \widehat{S^n_t}(E).$$

On déduit de ce théorème le lemme suivant:

Lemme 1.12. — *L'automorphisme ϵ de permutation des facteurs sur $S_t^2 = S_t^1 \otimes^{\mathrm{Htr}} S_t^1$ est égal à -1.*

Démonstration. — Compte tenu de la proposition 1.6, il suffit de montrer que pour tout corps de fonctions E/k, ϵ agit par -1 sur la fibre $\widehat{S_t^2}(E)$.
D'après le théorème précédent, la flèche canonique:

$$\lambda_E^2 : E^\times \otimes_{\mathbb{Z}} E^\times \to \widehat{S_t^2}(E)$$

est un épimorphisme. De plus, pour tout couple (a, b) d'unités de E, la relation suivante $\lambda_E^2(b, a) = -\lambda_E^2(a, b)$ est vérifiée, d'après la relation analogue bien connue dans $K_2^M(E)$. On conclut du fait que $\epsilon.\lambda_E^2(a, b) = \lambda_E^2(b, a)$. □

1.13. — Pour un entier $n \geq 0$ et un faisceau homotopique F, on pose $F_{-n} = \underline{\mathrm{Hom}}_{HI(k)}(S_t^n, F)$. Par définition, pour tout schéma lisse X,

$$F_{-1}(X) = F(\mathbb{G}_m \times X)/F(X).$$

Le foncteur $?_{-n}$ est le n-ième itéré du fonteur $?_{-1}$. Ainsi la proposition 3.4.3 de [Dég08b] entraine :

Lemme 1.14. — *L'endofoncteur $HI(k) \to HI(k), F \mapsto F_{-n}$ est exact.*

Le résultat suivant est un corollaire du théorème de simplification de Voevodsky [Voe02].

Proposition 1.15. — *L'endofoncteur $HI(k) \to HI(k), F \mapsto S_t^n \otimes^{\mathrm{Htr}} F$ est pleinement fidèle.*

Démonstration. — Il suffit de considérer le cas $n = 1$. La preuve anticipe la suite de l'exposé puisqu'elle utilise la catégorie $DM_-^{eff}(k)$ des complexes motiviques de Voevodsky définie dans [Voe02]. Le théorème central de *loc. cit.* affirme que le twist de Tate est pleinement fidèle dans $DM_-^{eff}(k)$. Il en résulte que le morphisme canonique $F \to \underline{\mathrm{Hom}}_{DM_-^{eff}(k)}(\mathbb{Z}^{tr}(1)[1], F(1)[1])$ est un isomorphisme. D'après [Dég08b, 3.4.4], le membre de droite est égal à $\underline{\mathrm{H}}^0(F(1)[1])_{-1}$. Or par définition, $\underline{\mathrm{H}}^0(F(1)[1]) = S_t^1 \otimes^{\mathrm{Htr}} F$ et la transformation naturelle correspondante $F \to (S_t^1 \otimes^{\mathrm{Htr}} F)_{-1}$ est l'application d'adjonction. □

1.2. Définition. —

1.16. — On note $\mathbb{Z}$–$HI(k)$ la catégorie des faisceaux homotopiques $\mathbb{Z}$-gradués. Pour un tel faisceau F_* et un entier $n \in \mathbb{Z}$, on note $F_*\{n\}$ le faisceau gradué dont la composante en degré i est F_{i+n}. Si F est un faisceau homotopique, on note encore $F\{n\}$ le faisceau gradué concentré en degré $-n$ égal à F. La catégorie $\mathbb{Z} - HI(k)$ est abélienne de Grothendieck avec pour générateurs la famille $(h_0(X)\{i\})$ indexée par les schémas lisses X et les entiers $i \in \mathbb{Z}$.
Cette catégorie est monoïdale symétrique :

$$\left(F_* \hat{\otimes}^{\mathrm{Htr}} G_*\right)_n = \oplus_{p+q=n} F_p \otimes^{\mathrm{Htr}} G_q.$$

Pour la symétrie, on adopte la convention donnée par la règle de Koszul :

$$\oplus_{p+q=n} F_p \otimes^{\mathrm{Htr}} G_q \xrightarrow{\sum(-1)^{pq}.\epsilon_{pq}} \oplus_{p+q=n} G_q \otimes^{\mathrm{Htr}} F_p$$

où ϵ_{pq} désigne l'isomorphisme de symétrie pour la structure monoïdale des faisceaux homotopiques.
On note S_t^* le monoïde libre dans $\mathbb{Z}-HI(k)$ engendré par le faisceau S_t^1 placé en degré 1. Il est égal en degré n à S_t^n. Compte tenu de la règle de Koszul ci-dessus et du lemme 1.12, c'est un monoïde commutatif dans $\mathbb{Z}-HI(k)$. On note $S_t^*-\mathrm{mod}$ la catégorie des modules sur S_t^*. C'est une catégorie abélienne monoïdale de Grothendieck avec pour générateurs $(S_t^* \otimes^{\mathrm{Htr}} h_0(X)\{i\})$ pour X un schéma lisse et $i \in \mathbb{Z}$. Comme S_t^* est un monoïde libre, se donner un S_t^*-module

$$\tau : S_t^* \hat{\otimes}^{\mathrm{Htr}} F_* \to F_*$$

revient à se donner une suite de morphismes

$$S_t^1 \otimes^{\mathrm{Htr}} F_n \xrightarrow{\tau_n} F_{n+1}$$

appelés *morphismes de suspension.*

Définition 1.17. — Un *module homotopique* est un S_t^*-module (F_*, τ) tel que le morphisme adjoint à τ_n

$$\epsilon_n : F_n \to \underline{\mathrm{Hom}}_{HI(k)}(S_t^1, F_{n+1}) = (F_{n+1})_{-1}$$

est un isomorphisme. On note $HI_*(k)$ la sous-catégorie de $S_t^*-\mathrm{mod}$ formée des modules homotopiques.

Il revient au même de se donner la suite de morphismes $(\tau_n)_{n\in\mathbb{N}}$ ou la suite de d'isomorphismes $(\epsilon_n)_{n\in\mathbb{N}}$ pour définir une structure de module homotopique sur un faisceau homotopique gradué F_*. Par la suite, la notation (F_*, ϵ_*) pour un module homotopique fera toujours référence aux isomorphismes ϵ_n.

1.18. — Compte tenu du lemme 1.14, le foncteur d'oubli $HI_*(k) \to S_t^*-\mathrm{mod}$ est exact et conservatif. Il admet de plus un adjoint à gauche L définit pour tout faisceau homotopique F et tout entier $i \in \mathbb{Z}$ par la formule

$$L\big(S_t^* \otimes^{\mathrm{Htr}} F\{i\}\big)_n = \begin{cases} S_t^{n+i} \otimes^{\mathrm{Htr}} F & \text{si } n+i \geq 0 \\ F_{n+i} & \text{si } n+i \leq 0 \end{cases}$$

en adoptant la notation de 1.13. Le fait que L prend ses valeurs dans les faisceaux homotopiques résulte de 1.15. On pose plus simplement $\sigma^\infty F\{i\} = L\big(S_t^* \otimes^{\mathrm{Htr}} F\{i\}\big)$. La catégorie $HI_*(k)$ est donc une sous-catégorie abélienne de $S_t^*-\mathrm{mod}$, avec pour générateurs la famille

$$h_{0,*}(X) = \sigma^\infty h_0(X)\{i\} \qquad (1.18.a)$$

pour un schéma lisse X et un entier $i \in \mathbb{Z}$ – le symbole $*$ correspond à la graduation naturelle de module homotopique.

Si (F_*, ϵ_*) est un module homotopique, on pose $\omega^\infty F_* = F_0$. On obtient ainsi un couple de foncteurs adjoints

$$\sigma^\infty : HI(k) \leftrightarrows HI_*(k) : \omega^\infty \tag{1.18.b}$$

tels que σ^∞ est pleinement fidèle (prop. 1.15) et ω^∞ est exact (lemme 1.14). Ainsi, pour tout schéma lisse X, tout module homotopique F_* et tout $(n, i) \in \mathbb{Z}^2$,

$$\mathrm{Hom}_{HI_*(k)}(h_{0,*}(X), F_*\{i\}[n]) = H^n_{\mathrm{Nis}}(X; F_i). \tag{1.18.c}$$

Lemme 1.19. — Il existe sur $HI_*(k)$ une unique structure monoïdale symétrique telle que le foncteur L est monoïdal symétrique.

Démonstration. — Compte tenu de ce qui précède, le foncteur L est un foncteur de localisation: pour tout schéma lisse X et tout entier $n \in \mathbb{Z}$, on obtient par définition $(S^*_t \otimes^{\mathrm{Htr}} h_0(X)\{n\})_{-n+1} = S^1_t \otimes^{\mathrm{Htr}} h_0(X)$. Par adjonction, l'identité de $S^1_t \otimes^{\mathrm{Htr}} h_0(X)$ induit donc un morphisme de S^*_t-modules

$$S^*_t \otimes^{\mathrm{Htr}} (S^1_t \otimes^{\mathrm{Htr}} h_0(X)\{n-1\}) \to S^*_t \otimes^{\mathrm{Htr}} h_0(X)\{n\}.$$

Utilisant à nouveau le jeu des adjonctions introduites ci-dessus, $HI_*(k)$ est la localisation de S^*_t–mod par rapport à la classe de flèches localisante $\mathcal{W}$ (*cf.* 1.7) engendrée par les morphismes précédents. Pour tout couple (Y, m), Y schéma lisse, $m \in \mathbb{Z}$, il est évident que $\mathcal{W} \hat{\otimes}^{\mathrm{Htr}} (S^*_t \otimes^{\mathrm{Htr}} h_0(Y)\{m\}) \subset \mathcal{W}$. Ainsi, $\hat{\otimes}^{\mathrm{Htr}}$ vérifie la propriété de localisation par rapport à $\mathcal{W}$ ce qui conclut. $\square$

La catégorie $HI_*(k)$ est donc monoïdale symétrique fermée avec pour neutre le module homotopique S^*_t. Le foncteur σ^∞ est de plus monoïdal symétrique. Enfin, l'objet $\sigma^\infty S^1_t$ est inversible pour le produit tensoriel avec pour inverse $\sigma^\infty \mathbb{Z}^{tr}\{-1\}$.

Remarque 1.20. — La catégorie $HI_*(k)$ est la catégorie monoïdale abélienne de Grothendieck *universelle* pour les propriétés qui viennent d'être énoncées. La construction donnée ici est parfaitement analogue à la construction de la catégorie des spectres en topologie algébrique, comme le suggère nos notations – en particulier pour le faisceau S^1_t qui joue le rôle de la sphère topologique. La construction ici est facilitée parce que nous sommes dans un cadre abélien et que la sphère S^1_t est anti-commutative. Le théorème de simplification 1.15 rend la construction du foncteur L plus facile mais n'est pas indispensable.

1.3. Réalisation des motifs géométriques. — Rappelons que la catégorie des motifs géométriques effectifs $DM^{eff}_{gm}(k)$ définie par Voevodsky est l'enveloppe pseudo-abélienne de la localisation de la catégorie $K^b(\mathscr{L}^{cor}_k)$ des complexes de $\mathscr{L}^{cor}_k$ à équivalence d'homotopie près par la sous-catégorie triangulée épaisse engendrée par les complexes suivants :

1. $\ldots 0 \to U \cap V \to U \oplus V \to X \to 0 \ldots$
   pour un recouvrement ouvert $U \cup V$ d'un schéma lisse X.
2. $\ldots 0 \to \mathbb{A}^1_X \to X \to 0 \ldots$
   induit par la projection canonique pour un schéma lisse X.

Rappelons que cette catégorie est triangulée monoïdale symétrique. Pour un schéma lisse X, on note simplement $M(X)$ le complexe concentré en degré 0 égal à X vu dans $DM_{gm}^{eff}(k)$.

Pour tout complexe borné C de $\mathscr{L}_k^{cor}$, on note $\mathbb{Z}^{tr}(C)$ le complexe de faisceau avec transferts évident. Pour un faisceau homotopique F, posons $\varphi_F(C) = \mathrm{Hom}_{D(Sh^{tr}(k))}(\mathbb{Z}^{tr}(C), F)$. Rappelons que pour un schéma lisse X, $\mathrm{Hom}_{D(Sh^{tr}(k))}(\mathbb{Z}^{tr}(X)[-n], F) = H^n_{\mathrm{Nis}}(X; F)$ (*cf.* [Voe00b, 3.1.9]); la cohomologie Nisnevich de F est de plus invariante par homotopie (*cf.* [Voe00a, 5.6]). On en déduit que le foncteur φ_F ainsi défini se factorise et induit un foncteur cohomologique encore noté $\varphi_F : DM_{gm}^{eff}(k)^{op} \to \mathscr{A}b$.

On définit le motif de Tate *suspendu*[(9)] $\mathbb{Z}\{1\}$ comme le complexe

$$\ldots \to \mathrm{Spec}(k) \to \mathbb{G}_m \to 0 \ldots$$

où $\mathbb{G}_m$ est placé en degré 0, vu dans $DM_{gm}^{eff}(k)$. Avec une convention légèrement différente de celle de Voevodsky, adpatée à nos besoins, on définit la catégorie des motifs géométriques $DM_{gm}(k)$ comme la catégorie monoïdale symétrique universelle obtenue en inversant $\mathbb{Z}\{1\}$ pour le produit tensoriel. Un objet de $DM_{gm}(k)$ est un couple (C, n) où C est un complexe de $\mathscr{L}_k^{cor}$ et n un entier, noté suggestivement $C\{n\}$. Les morphismes sont définis par la formule

$$\mathrm{Hom}_{DM_{gm}(k)}(C\{n\}, D\{m\}) = \varinjlim_{r \geq -n, -m} \mathrm{Hom}\, DM_{gm}^{eff}(k)(C\{r+n\}, D\{r+n\}).$$

Cette catégorie est de manière évidente équivalente à la catégorie définie dans [Voe00b] obtenue en inversant le motif de Tate $\mathbb{Z}(1) = \mathbb{Z}\{1\}[-1]$. Elle est donc triangulée monoïdale symétrique.

Considérons maintenant un module homotopique (F_*, ϵ_*). Pour tout motif géométrique $C\{n\}$, on pose

$$\varphi(C\{n\}) = \varinjlim_{r \geq -n} \mathrm{Hom}_{D(Sh^{tr}(k))}(\mathbb{Z}^{tr}(C)\{r+n\}, F_r)$$

où les morphismes de transitions sont

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}^{tr}(C)\{r+n\}, F_r) \xrightarrow{\epsilon_{r*}} &\mathrm{Hom}(\mathbb{Z}^{tr}(C)\{r+n\}, (F_{r+1})_{-1}) \\ &= \mathrm{Hom}(\mathbb{Z}^{tr}(C)\{r+n+1\}, F_{r+1}),\end{aligned}$$

les morphismes étant considérés dans la catégorie $D(Sh^{tr}(k))$. Comme dans le cas des motifs effectifs, ceci induit un foncteur de *réalisation cohomologique* associé à (F_*, ϵ_*) :

$$\varphi : DM_{gm}(k)^{op} \to \mathscr{A}b$$

Notons que ce foncteur est naturellement gradué $\varphi_n(\mathbb{Z}^{tr}(C)\{r\}) = \varphi(\mathbb{Z}^{tr}(C)\{r-n\})$ de sorte que, d'après le théorème de simplification 1.15, pour tout schéma lisse X, $\varphi_n(\mathbb{Z}^{tr}(X)) = F_n(X)$

Remarque 1.21. — On déduit du théorème de simplification 1.15 la relation suivante: $\varphi(M(X)\{n\}) = F_{-n}(X)$.

(9) En effet, $\mathbb{Z}\{1\} = \mathbb{Z}(1)[1]$.

2. Modules de cycles

Dans cette section, on rappelle la théorie de Rost des modules de cycles exposée dans [Ros96] ainsi que les compléments que nous lui avons apportés dans [Dég08a]. L'étude de la fonctorialité de la suite exacte longue de localisation à l'aide d'un morphisme de Gysin raffiné est nouvelle (*cf.* proposition 2.6).

2.1. Rappels. — Un *pré-module* de cycles ϕ (*cf.* [Ros96, (1.1)]) est la donnée pour tout corps de fonctions E d'un groupe abélien $\mathbb{Z}$-gradué $\phi(E)$ satisfaisant à la fonctorialité suivante :

(D1) Pour toute extension de corps $f : E \to L$, on se donne un morphisme appelé *restriction* $f_* : \phi(E) \to \phi(L)$ de degré 0.

(D2) Pour toute extension finie de corps $f : E \to L$, on se donne un morphisme appelé *norme* $f^* : \phi(L) \to \phi(E)$ de degré 0.

(D3) Pour tout élément $\sigma \in K_r^M(E)$ du r-ième groupe de K-théorie de Milnor de E, on se donne un morphisme $\gamma_\sigma : \phi(E) \to \phi(E)$ de degré r.

(D4) Pour tout corps de fonctions valué (E, v), on se donne un morphisme appelé *résidu* $\partial_v : \phi(E) \to \phi(\kappa(v))$ de degré -1.

Considérant ces données, on introduit fréquemment un cinquième type de morphisme, associé à un corps de fonctions valué (E, v) et à une uniformisante π de v, de degré 0, $s_v^\pi = \partial_v \circ \gamma_\pi$, appelé *spécialisation*.

Ces données sont soumises à un ensemble de relations (*cf.* [Ros96, (1.1)]). On peut se faire une idée de ces relations en considérant le foncteur de K-théorie de Milnor qui est l'exemple le plus simple de pré-module de cycles.

Considérons un schéma X essentiellement de type fini sur k. Soit x, y deux points de X. Soit Z l'adhérence réduite de x dans X, $\tilde{Z}$ sa normalisation et $f : \tilde{Z} \to Z$ le morphisme canonique. Supposons que y est un point de codimension 1 dans Z et notons $\tilde{Z}_y^{(0)}$ l'ensemble des points génériques de $f^{-1}(y)$. Tout point $z \in \tilde{Z}_y^{(0)}$ correspond alors à une valuation v_z sur $\kappa(x)$ de corps résiduel $\kappa(z)$. On note encore $\varphi_z : \kappa(y) \to \kappa(z)$ le morphisme induit par f. On définit un morphisme $\partial_y^x : \phi(\kappa(x)) \to \phi(\kappa(y))$ par la formule suivante :

$$\partial_y^x = \begin{cases} \sum_{z \in \tilde{Z}_y^{(0)}} \varphi_z^* \circ \partial_{v_z} & \text{si } y \in Z^{(1)}, \\ 0 & \text{sinon.} \end{cases}$$

Considérons ensuite le groupe abélien :

$$C^p(X; \phi) = \bigoplus_{x \in X^{(p)}} \phi(\kappa(x)).$$

On dit que le pré-module de cycles ϕ est un *module de cycles* (*cf.* [Ros96, (2.1)]) si pour tout schéma essentiellement de type fini X,

(FD) Le morphisme

$$d_{X,\phi}^p : \sum_{x \in X^{(p)}, y \in X^{(p+1)}} \partial_y^x : C^p(X; \phi) \to C^{p+1}(X; \phi)$$

est bien définit.

(C) La suite

$$\ldots \to C^p(X;\phi) \xrightarrow{d^p_{X,\phi}} C^{p+1}(X;\phi) \to \ldots$$

est un complexe.

Les modules de cycles forment de manière évidente une catégorie que l'on note $\mathscr{M}Cycl(k)$.

On introduit une graduation sur le complexe de la propriété (C) :

$$C^p(X;\phi)_n = \bigoplus_{x \in X^{(p)}} M_{n-p}(\kappa(x)).$$

On note $A^p(X;\phi)_n$ le p-ième groupe de cohomologie de ce complexe, appelé parfois *groupe de Chow à coefficients dans* ϕ.

Pour un schéma lisse X de corps des fonctions E, le groupe $A^0(X;\phi)_n$ est donc le noyau de l'application bien définie

$$\phi_n(E) \xrightarrow{\sum_{x \in X^{(1)}} \partial_x} \phi_{n-1}(\kappa(x))$$

où ∂_x désigne le morphisme résidu associé à la valuation sur E correspondant au point x.

2.2. Fonctorialité. —

2.1. — Le complexe gradué $C^*(X;\phi)_*$ est contravariant en X par rapport aux morphismes plats (*cf.* [Ros96, (3.4)]). Il est covariant par rapport aux morphismes propres équidimensionnels (*cf.* [Ros96, (3.5)]).

2.2. — Dans [Dég06, 3.18], nous avons prolongé le travail original de Rost et nous avons associé à tout morphisme $f : Y \to X$ localement d'intersection complète ([Dég06, 3.12]) tel que Y est lissifiable ([Dég06, 3.13]) un *morphisme de Gysin*

$$f^* : C^*(X;\phi) \to C^*(Y;\phi)$$

qui est un composé d'un morphisme de complexes et de l'inverse formel d'un morphisme de complexe qui est un quasi-isomorphisme (plus précisément, il s'agit de l'inverse formel d'un morphisme p^* pour p la projection d'un fibré vectoriel). Pour désigner une telle flèche formelle, on utilise la notation abrégée $f^* : X \bullet\!\!\longrightarrow Y$.[10]

Ce morphisme de Gysin f^* satisfait les propriétés suivantes :

1. Lorsque f est de plus plat, f^* coincide avec le pullback plat évoqué plus haut.
2. Si $g : Z \to Y$ est un morphisme localement d'intersection complète avec Z lissifiable, $(fg)^* = g^* f^*$.

[10] Les flèches de ce type sont bien définies dans la catégorie dérivée des groupes abéliens et induisent en particulier un morphisme sur les groupes de cohomologie.

Dans le cas où f est une immersion fermée régulière, l'hypothèse que Y est lissifiable est inutile ; le morphisme f^* est défini en utilisant la déformation au cône normal, suivant l'idée originale de Rost (*cf.* [Dég06, 3.3]). On utilisera par ailleurs le résultat suivant dû à Rost ([Ros96, (12.4)]) qui décrit partiellement ce morphisme de Gysin :

Proposition 2.3. — Soit X un schéma intègre de corps des fonctions E, et $i : Z \to X$ l'immersion fermée d'un diviseur principal régulier irréductible paramétré par $\pi \in \mathcal{O}_X(X)$. Soit v la valuation de E correspondant au diviseur Z. Alors, le morphisme $i^* : A^0(X;\phi) \to A^0(Z;\phi)$ est la restriction de $s_v^\pi : \phi(E) \to \phi(\kappa(v))$.

2.4. — A tout carré cartésien

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[i]{} & X \end{array}$$

tel que i est une immersion fermée régulière, on associe un *morphisme de Gysin raffiné* $\Delta^* : X' \bullet\!\!\longrightarrow Y'$. Ce morphisme Δ^* vérifie les propriétés suivantes :

1. Si j est régulière et le morphisme des cônes normaux $N_{Y'}(X') \to g^{-1}N_Y(X)$ est un isomorphisme, $\Delta^* = j^*$.
2. Si f est propre, $i^*f_* = g_*\Delta^*$.

De plus, si l'immersion canonique $C_{Y'}(X') \to g^{-1}N_Y(X)$ du cône de j dans le fibré normal de i est de codimension pure égale à e, le morphisme Δ^* est de degré cohomologique e.

2.5. — Pour tout couple de schémas lisses (X, Y) et pour toute correspondance finie $\alpha \in c(X, Y)$, on définit un morphisme $\alpha^* : Y \bullet\!\!\longrightarrow X$ (*cf.* [Dég06, 6.9]). On peut décrire ce dernier comme suit. Supposons que α est la classe d'un sous-schéma fermé irréductible Z de $X \times Y$. Considérons les morphismes:

$$X \xleftarrow{p} Z \xrightarrow{i} Z \times X \times Y \xrightarrow{q} Y$$

où p et q désignent les projections canoniques et i le graphe de l'immersion fermée $Z \to X \times Y$. Alors,

$$\alpha^* = p_* i^* q^* \tag{2.5.a}$$

où i^* désigne le morphisme de Gysin de l'immersion fermée régulière i, q^* le pullback plat et p_* le pushout fini.

La propriété $(\beta\alpha)^* = \alpha^*\beta^*$ est démontrée dans [Dég06, 6.5].

2.3. Suite exacte de localisation. — La suite exacte de localisation n'est pas étudiée (ni rappelée) dans [Dég06]. Nous la rappelons maintenant suivant [Ros96] et démontrons un résultat supplémentaire concernant sa fonctorialité. Pour une immersion fermée $i : Z \to X$ purement de codimension c, d'immersion ouverte complémentaire $j : U \to X$, on obtient en utilisant la fonctorialité rappelée ci-dessus une suite exacte courte scindée de complexes

$$0 \to C^{p-c}(Z;\phi)_{n-c} \xrightarrow{i_*} C^p(X;\phi)_n \xrightarrow{j^*} C^p(U;\phi)_n \to 0. \tag{2.5.b}$$

On en déduit une suite exacte longue de localisation
(2.5.c)

$$\ldots \to A^{p-c}(Z;\phi)_{n-c} \xrightarrow{i_*} A^p(X;\phi)_n \xrightarrow{j^*} A^p(U;\phi)_n \xrightarrow{\partial_Z^U} A^{p-c+1}(Z;\phi)_{n-c} \to \ldots$$

où le morphisme ∂_Z^U est définit au niveau des complexes par la formule $\sum_{x \in U^{(p)}, z \in Z^{(p-c+1)}} \partial_z^x$.
Cette suite est naturelle par rapport au pushout propre et au pullback plat. La proposition suivante est nouvelle :

Proposition 2.6. — Considérons un carré cartésien

$$\begin{tikzcd} T \arrow[r, "\iota'"] \arrow[d, "k"'] \arrow[dr, phantom, "\Delta"] & Z \arrow[d, "i"] \\ Y \arrow[r, "\iota"'] & X \end{tikzcd}$$

tel que ι est une immersion fermée régulière. Supposons que i (resp. k) est une immersion fermée d'immersion ouverte complémentaire $j : U \to X$ (resp. $l : V \to X$). Notons $h : V \to U$ le morphisme induit par ι. Supposons enfin que i (resp. k) est de codimension pure égale à c (resp. d). Alors, le diagramme suivant est commutatif :

$$\begin{tikzcd} \cdots \arrow[r] & A^{p-c}(Z;\phi)_{n-c} \arrow[r, "i_*"] \arrow[d, "\Delta^*"] & A^p(X;\phi)_n \arrow[r, "j^*"] \arrow[d, "\iota^*"] & A^p(U;\phi)_n \arrow[r, "\partial_Z^U"] \arrow[d, "h^*"] & A^{p-c+1}(Z;\phi)_{n-c} \arrow[r] \arrow[d, "\Delta^*"] & \cdots \\ \cdots \arrow[r] & A^{p-d}(T;\phi)_{n-d} \arrow[r, "k_*"] & A^p(Y;\phi)_n \arrow[r, "l^*"] & A^p(V;\phi)_n \arrow[r, "\partial_T^V"] & A^{p-d+1}(T;\phi)_{n-d} \arrow[r] & \cdots \end{tikzcd}$$

Remarque 2.7. —
1. On peut généraliser la proposition précédente au cas des morphismes de Gysin raffinés comme dans la proposition 4.5 de [Dég06]. Nous laissons au lecteur le soin de formuler cette généralisation.
2. Alors que l'hypothèse sur la codimension pure de i est naturelle, celle sur k ne l'est pas, en particulier dans un cas non transverse. Elle ne nous sert qu'à exprimer les degrés cohomologiques de tous les morphismes et peut aisément être supprimée si on accepte des morphismes non homogènes par rapport au degré cohomologique.

Démonstration. — Il suffit de reprendre la preuve de la proposition 4.5 de *loc. cit.* dans le cas du diagramme commutatif :

$$\begin{tikzcd} T \arrow[rr] \arrow[dr, "k"] \arrow[dd] & & Z \arrow[dr, "i"] \arrow[dd] & \\ & Y \arrow[rr] \arrow[dl, equal] & & X \arrow[dl, equal] \\ Y \arrow[rr, "\iota"'] & & X. & \end{tikzcd}$$

On obtient ainsi un diagramme commutatif[11], avec les notations analogues de *loc. cit.*

$$\begin{tikzcd}
Z \arrow[r, "\sigma'"] \arrow[d, "i_*"'] \arrow[dr, phantom, "(1)"] & C_T Z \arrow[r, "\nu'_*"] \arrow[d, "k''_*"] \arrow[dr, phantom, "(2)"] & k^* N_Y X \arrow[d, "k'_*"'] \arrow[dr, phantom, "(3)"] & T \arrow[l, "p_T^*"'] \arrow[d, "k_*"] \\
X \arrow[r, "\sigma"'] \arrow[d, "j^*"'] \arrow[dr, phantom, "(1')"] & N_Y X \arrow[r, equal] \arrow[d, "l'^*"] \arrow[dr, phantom, "(2')"] & N_Y X \arrow[d, "l'^*"'] \arrow[dr, phantom, "(3')"] & Y \arrow[l, "p^*"'] \arrow[d, "l^*"] \\
U \arrow[r, "\sigma_U"'] & N_U V \arrow[r, equal] & N_U V & U \arrow[l, "p_U^*"]
\end{tikzcd}$$

Les carrés (1), (2), (3) sont commutatifs d'après *loc. cit.* et les carrés (1'), (2'), (3') le sont pour des raisons triviales. Les flèches $\bullet\!\!\longrightarrow$ qui apparaissent dans ce diagramme sont bien des morphismes de complexes et induisent donc des morphismes de suite exacte longue de localisation. Il suffit alors d'appliquer le fait que les morphismes p^*, p_T^* et p_U^* sont des quasi-isomorphismes pour conclure. □

Corollaire 2.8. — Considérons un carré cartésien

$$\begin{tikzcd}
T \arrow[r, "g"] \arrow[d, "k"'] \arrow[dr, phantom, "\Delta"] & Z \arrow[d, "i"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}$$

de schémas lisses tels que i (resp. k) est une immersion fermée de codimension pure égale à c, d'immersion ouverte complémentaire $j : U \to X$ (resp. $l : V \to X$). Notons $h : V \to U$ le morphisme induit par f. Alors, le diagramme suivant est commutatif :

$$\begin{tikzcd}
\cdots \arrow[r] & A^{p-c}(Z;\phi)_{n-c} \arrow[r, "i_*"] \arrow[d, "g^*"] & A^p(X;\phi)_n \arrow[r, "j^*"] \arrow[d, "f^*"] & A^p(U;\phi)_n \arrow[r, "\partial_Z^U"] \arrow[d, "h^*"] & A^{p-c+1}(Z;\phi)_{n-c} \arrow[r] \arrow[d, "g^*"] & \cdots \\
\cdots \arrow[r] & A^{p-c}(T;\phi)_{n-c} \arrow[r, "k_*"] & A^p(Y;\phi)_n \arrow[r, "l^*"] & A^p(V;\phi)_n \arrow[r, "\partial_T^V"] & A^{p-c+1}(T;\phi)_{n-c} \arrow[r] & \cdots
\end{tikzcd}$$

Remarque 2.9. — Dans l'article [Dég08b], une *paire fermée* est un couple (X, Z) tel que X est un schéma lisse et Z un sous-schéma fermé. On dit que (X, Z) est lisse (resp. de codimension n) si Z est lisse (resp. purement de codimension n dans X).

Si $i : Z \to X$ est l'immersion fermée associée, un *morphisme de paires fermées* (f, g) est un carré commutatif

$$\begin{tikzcd}
T \arrow[r, "g"] \arrow[d, "k"'] & Z \arrow[d, "i"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}$$

qui est topologiquement cartésien. On dit que (f, g) est *cartésien* (resp. *transverse*) quand le carré est cartésien (resp. et le morphisme induit sur les cônes

[11] Il y a une faute de frappe dans le diagramme commutatif de *loc. cit.* Il faut lire $t^* N_Z X$ au lieu de $N_Y X$.

normaux $C_T Y \to g^{-1} C_Z X$ est un isomorphisme). [12]
Le corollaire précédent montre que la suite de localisation associée à un module de cycles ϕ et une paire fermée (X, Z) est naturelle par rapport aux morphismes transverses.

2.4. Module homotopique associé. —

2.10. — Considérons un module de cycles ϕ. D'après 2.5, $A^0(.;\phi)_*$ définit un préfaisceau gradué avec transferts. D'après [Dég06, 6.9], c'est un faisceau homotopique gradué. On le note F_*^ϕ et on lui définit une structure de module homotopique comme suit:
Soit X un schéma lisse. On considère le début de la suite exacte longue de localisation (2.5.c) associée à la section nulle $X \to \mathbb{A}^1_X$:

$$0 \to F_n^\phi(\mathbb{A}^1_X) \xrightarrow{j_X^*} F_n^\phi(\mathbb{G}_m \times X) \xrightarrow{\partial_0^X} F_{n-1}^\phi(X) \to \dots$$

On peut décrire le morphisme ∂_0^X si X est connexe de corps des fonctions E comme étant induit par le morphisme

$$\partial_0^E : \phi_n(E(t)) \to \phi_{n-1}(E)$$

associé à la valuation standard de $E(t)$.
Soit $s_1 : X \to \mathbb{G}_m \times X$ la section unité. Rappelons que $(F_n^\phi)_{-1}(X) = \mathrm{Ker}(s_1^*)$. Or par invariance par homotopie de F_n^ϕ, le morphisme canonique $\mathrm{Ker}(s_1^*) \to \mathrm{coKer}(j^*)$ est un isomorphisme. Ainsi, le morphisme ∂_0^X induit un morphisme

$$\epsilon_{n,X} : (F_n^\phi)_{-1}(X) \to F_{n-1}^\phi(X).$$

On vérifie que la suite de localisation précédente est compatible aux transferts en X, comme cela résulte de la description des transferts rappelée en 2.5 et du corollaire 2.8. Ainsi, ϵ_n définit un morphisme de faisceaux homotopiques. Pour tout corps de fonctions E, $A^1(\mathbb{A}^1_E;\phi) = 0$ (*cf.* [Ros96, (2.2)(H)]). Donc la fibre de ϵ_n en E est un isomorphisme ce qui implique que c'est un isomorphisme de faisceaux homotopiques d'après 1.4.
Ainsi, $(F_*^\phi, \epsilon_*^{-1})$ définit un module homotopique qui dépend fonctoriellement de ϕ.

3. Equivalence de catégories

3.1. Transformée générique. — Considérons un couple (E, n) formé d'un corps de fonctions E et d'un entier relatif n. Rappelons que l'on a associé dans [Dég08b, 3.3.1] au couple (E, n) un *motif générique*

$$M(E)\{n\} = \text{"}\varprojlim_{A \subset E}\text{"}\ M(\mathrm{Spec}(A))\{n\}$$

[12] Lorsque (X, Z) est lisse de codimension n le fait que le morphisme (f, g) est transverse entraîne que (Y, T) est lisse de codimension n (k est régulier).

dans la catégorie des pro-objets de $DM_{gm}(k)$. On note $DM_{gm}^{(0)}(k)$ la catégorie des motifs génériques.

3.1. — Considérons un module homotopique (F_*, ϵ_*) ainsi que le foncteur de réalisation $\varphi : DM_{gm}(k)^{op} \to \mathscr{A}b$ qui lui est associé dans la section 1.3. On note $\hat{\varphi}$ le prolongement évident de φ à la catégorie des pro-objets. Il résulte de [Dég08b, 6.2.1] que la restriction de $\hat{\varphi}$ à la catégorie $DM_{gm}^{(0)}(k)$ est un module de cycles, que l'on note $\hat{F}_*$ et que l'on appelle la *transformée générique* de F_*. Rappelons brièvement certaines parties de la construction de [Dég08b]. Notons d'abord que pour tout motif générique $M(E)\{n\}$, $\hat{\varphi}(M(E)\{n\}) = \hat{F}_{-n}(E)$ n'est autre que la fibre de F_{-n} en E (*cf.* 1.4). La transformée $\hat{F}_*$ s'interprète donc comme le *système des fibres* de F_*. Ce sont les *morphismes de spécialisation* entre ces fibres qui donnent la structure de pré-module de cycles :

(D1) Fonctorialité évidente de F_*.

(D2) ([Dég08b, 5.2]) Pour une extension finie L/E, on trouve des modèles respectifs X et Y de E et L ainsi qu'un morphisme fini surjectif $f : Y \to X$ dont l'extension induite des corps de fonctions est isomorphe à L/E. Le graphe de f vu comme cycle de $X \times Y$ définit une correspondance finie de X vers Y notée ${}^{t}f$ – la *transposée* de f. On en déduit un morphisme $({}^{t}f)^* : F_*(X) \to F_*(Y)$. On montre que ce morphisme est compatible à la restriction à un ouvert de X et il induit donc la fonctorialité attendue.

(D3) ([Dég08b, 5.3]) Soit E un corps de fonctions et $x \in E^\times$ une unité. Considérons un modèle X de E munit d'une section inversible $X \to \mathbb{G}_m$ qui correspond à x. Considérons l'immersion fermée $s_x : X \to \mathbb{G}_m \times X$ induite par cette section. On en déduit un morphisme

$$\gamma_x : F_{n-1}(X) \xrightarrow{\epsilon_{n-1}} (F_n)_{-1}(X) \xrightarrow{\nu} F_n(\mathbb{G}_m \times X) \xrightarrow{s_x^*} F_n(X)$$

où ν est l'inclusion canonique. Ce morphisme est compatible à la restriction suivant un ouvert de X et induit la donnée D3 pour $\hat{F}_*$.

(D4) ([Dég08b, 5.4]) Soit (E, v) un corps de fonctions valué. On peut trouver un schéma lisse X munit d'un point x de codimension 1 tel que l'adhérence réduite Z de x dans X est lisse et l'anneau local $\mathcal{O}_{X,x}$ est isomorphe à l'anneau des entiers de v. On pose $U = X - Z$, $j : U \to X$ l'immersion ouverte évidente. Rappelons que le motif $M_Z(X)$ de la paire (X, Z) est définie comme l'objet de $DM_{gm}^{eff}(k)$ représenté par le complexe concentré en degré 0 et -1 avec pour seule différentielle non nulle le morphisme j. Ce motif s'inscrit naturellement dans le triangle distingué

$$M_Z(X)[-1] \xrightarrow{\partial'_{X,Z}} M(U) \xrightarrow{j_*} M(X) \xrightarrow{+1}$$

On a définit dans [Dég08b, sec. 2.2.5] un *isomorphisme de pureté*

$$\mathfrak{p}_{X,Z} : M_Z(X) \to M(Z)(1)[2].$$

On en déduit un morphisme

$$\partial_{X,Z} : F_n(U) = \varphi_n(M(U)) \xrightarrow{\varphi_n(\partial'_{X,Z})} \varphi_n(M_Z(X)[-1])$$
$$\xrightarrow{(\varphi_n(\mathfrak{p}_{X,Z}^{-1})} \varphi_n(M(Z)\{1\}) = (F_n)_{-1}(Z) \xrightarrow{\epsilon_n^{-1}} F_{n-1}(Z),$$

ayant posé $\varphi_n(\mathcal{M}) = \varphi(\mathcal{M}\{-n\})$ pour un motif $\mathcal{M}$. Le morphisme résidu du module de cycles $\hat{F}_*$ est donné par la limite inductive des morphismes $\partial_{U,Z\cap U}$ suivant les voisinages ouverts U de x dans X.

3.2. Résolution de Gersten: fonctorialité I. —

3.2. — Considérons un module de cycles ϕ et F^ϕ le module homotopique qui lui est associé dans le paragraphe 2.10 – jusqu'au paragraphe 3.5, on n'indique pas la graduation pour alléger les notations. D'après [Ros96, 6.5], on dispose pour tout schéma lisse X et tout entier $p \in \mathbb{Z}$ d'un isomorphisme canonique $A^p(X;\phi) = H^p_{\mathrm{Zar}}(X;F^\phi)$.
On rappelle la construction de cet isomorphisme tout en le généralisant au cas de la topologie Nisnevich. Soit X un schéma lisse et X_{Nis} le petit site Nisnevich associé. Les morphismes de X_{Nis} étant étales, on obtient, en utilisant la fonctorialité rappelée dans 2.1, un préfaisceau de complexes de groupes abéliens sur X_{Nis}:

$$C^*_X(\phi) : V/X \mapsto C^*(V;\phi).$$

On vérifie que c'est un faisceau Nisnevich (voir [Dég08b], preuve de 6.10). On pose de plus:

$$F^\phi_X = \underline{\mathrm{H}}^0(C^*_X(\phi)).$$

Ainsi, F^ϕ_X est la restriction du faisceau F^ϕ, défini sur le site Nisnevich $\mathscr{L}_k$, au petit site X_{Nis}. D'après [Ros96, 6.1], le morphisme évident

$$F^\phi_X \to C^*_X(\phi) \tag{3.2.a}$$

est un quasi-isomorphisme.[13] Il induit donc un isomorphisme

$$H^p_{\mathrm{Nis}}(X;F^\phi_X) \to H^p_{\mathrm{Nis}}(X;C^*_X(\phi)).$$

Notons par ailleurs que le complexe $C^*_X(\phi)$ vérifie la propriété de Brown-Gersten au sens de [CD09a, 1.1.9] (voir à nouveau [Dég08b], preuve de 6.10). D'après la démonstration de [CD09a, 1.1.10], on en déduit que le morphisme canonique

$$H^p(C^*(X;\phi)) \to H^p_{\mathrm{Nis}}(X;C^*_X(\phi))$$

est un isomorphisme. Ces deux isomorphismes définissent comme annoncé :

$$\rho_X : A^p(X;\phi) = H^p(C^*(X;\phi)) \xrightarrow{\sim} H^p_{\mathrm{Nis}}(X;F^\phi_X) \simeq H^p_{\mathrm{Nis}}(X;F^\phi). \tag{3.2.b}$$

Lemme 3.3. — L'isomorphisme ρ_X construit ci-dessus est naturel en X par rapport aux morphismes de schémas.

---

(13) Le complexe de faisceaux $C^*_X(\phi)$ est la *résolution de Gersten* du faisceau F^ϕ_X. C'est en fait la version Nisnevich de la résolution de Cousin au sens de [Har66].

Démonstration. — Notons que, du fait que F^{ϕ}_X est la restriction d'un faisceau F^{ϕ} sur $\mathscr{L}_k$, pour tout morphisme $f : Y \to X$ de schémas lisses, on obtient une transformation naturelle canonique:

$$F^{\phi}_X \to f_* F^{\phi}_Y$$

qui induit dans la catégorie dérivée:

$$\tau_f : F^{\phi}_X \to \mathrm{R}f_* F^{\phi}_Y.$$

La preuve consiste à relever cette transformation naturelle au niveau de la résolution $C^*_X(\phi)$.

On considère d'abord le cas où f est plat. Suivant le paragraphe 2.1, on dispose d'un morphisme de complexes

$$f^* : C^*(X;\phi) \to C^*(Y;\phi)$$

qui est naturel en X par rapport aux morphismes étales. La transformation naturelle sur X_{Nis} correspondante définit un morphisme dans la catégorie dérivée des faisceaux abéliens sur X_{Nis}:

$$\eta_f : C^*_X(\phi) \to f_* C^*_Y(\phi)) = \mathrm{R}f_* C^*_Y(\phi). \tag{3.3.a}$$

(La dernière identification résulte du fait que $C^*_Y(\phi)$ vérifie la propriété de Brown-Gersten.) Par définition de la structure de faisceau sur F^{ϕ}, le diagramme suivant est commutatif:

$$\begin{tikzcd} F^{\phi}_X \arrow[r, "\tau_f"] \arrow[d] & \mathrm{R}f_* F^{\phi}_Y \arrow[d] \\ C^*_X(\phi) \arrow[r, "\eta_f"] & \mathrm{R}f_* C^*_Y(\phi). \end{tikzcd}$$

On en déduit la naturalité de ρ par rapport aux morphismes plats. Remarquons au passage que si f est la projection d'un fibré vectoriel, η_f est un quasi-isomorphisme.

Il reste à considérer le cas d'une immersion fermée $f = i : Z \to X$ entre schémas lisses. Notons N le fibré normal associé à i. La spécialisation au fibré normal définie par Rost (*cf.* [Ros96, sec. 11]) est un morphisme de complexes

$$\sigma_Z X : C^*(X;\phi) \to C^*(N;\phi)$$

qui est de plus naturel en X par rapport aux morphismes étales (*cf.* [Dég06, 2.2]). Notons ν le morphisme composé

$$N \xrightarrow{p} Z \xrightarrow{i} X.$$

On en déduit dans la catégorie dérivée un morphisme canonique

$$\sigma_i : C^*_X(\phi) \to \mathrm{R}\nu_* C^*_N(\phi).$$

Puisque le morphisme η_p est un quasi-isomorphisme, on obtient alors un morphisme canonique dans la catégorie dérivée

$$\eta_i : C^*_X(\phi) \to \mathrm{R}i_* C^*_Z(\phi). \tag{3.3.b}$$

Rappelons enfin que, par définition du pullback sur F^ϕ, le diagramme suivant est commutatif:

$$\begin{tikzcd} F^\phi(X) \arrow[rr, "i^*"] \arrow[d] & & F^\phi(Z) \arrow[d] \\ C^*(X;\phi) \arrow[r, "\sigma_Z(X)"] & C^*(N;\phi) & C^*(Z;\phi). \arrow[l, "p^*"'] \end{tikzcd}$$

On en déduit que le diagramme suivant est commutatif:

$$\begin{tikzcd} F_X^\phi \arrow[r, "\tau_i"] \arrow[d] & \mathrm{R}i_*F_Z^\phi \arrow[d] \\ C_X^*(\phi) \arrow[r, "\eta_i"] & \mathrm{R}i_*C_Z^*(\phi) \end{tikzcd}$$

ce qui conclut. $\square$

Remarque 3.4. — 1. On généralisera ce lemme au cas des correspondances finies dans la proposition 3.10.

2. Les constructions (3.3.a) et (3.3.b) de la preuve précédente permettent d'associer à tout morphisme de schémas $f : Y \to X$ un diagramme commutatif dans la catégorie dérivée des faisceaux sur X_{Nis}:

$$\begin{tikzcd} F_X^\phi \arrow[r, "\tau_f"] \arrow[d] & \mathrm{R}f_*F_Y^\phi \arrow[d] \\ C_X^*(\phi) \arrow[r, "\eta_f"] & \mathrm{R}f_*C_Y^*(\phi), \end{tikzcd}$$

en considérant la factorisation de f par son morphisme graphe qui est une immersion régulière. On peut montrer par ailleurs que η_f est compatible à la composition des morphismes.

3.5. — On reprend les notations du paragraphe 3.2 Considérons par ailleurs le foncteur de réalisation

$$\varphi : DM_{gm}(k)^{op} \to \mathscr{A}b$$

associé au module homotopique F^ϕ suivant la section 1.3. L'isomorphisme ρ_X correspond par définition à un isomorphisme:

$$A^p(X,\phi)_n \to \varphi_n(M(X)[-p]).$$

Considérons de plus une immersion fermée $i : Z \to X$ entre schémas lisses et $j : U \to X$ l'immersion ouverte du complémentaire. Supposons que i est de codimension pure égale à c. On déduit de la suite exacte de localisation (2.5.b) une unique flèche pointillée qui fait commuter le diagramme de complexes suivant (on utilise à nouveau le fait que $C_X^*(\phi)$ vérifie la propriété de Brown-Gersten):

$$\begin{tikzcd} 0 \arrow[r] & C^*(Z,\phi)_{n-c}[-c] \arrow[r, "i_*"] \arrow[d, dashed, "(1)"'] & C^*(X,\phi)_n \arrow[r, "j^*"] \arrow[d] & C^*(U,\phi)_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{R}\Gamma_Z(X, C_X^*(\phi))_n \arrow[r] & \mathrm{R}\Gamma(X, C_X^*(\phi))_n \arrow[r, "j^*"] & \mathrm{R}\Gamma(U, C_X^*(\phi))_n \arrow[r] & 0. \end{tikzcd}$$

La flèche (1) est un quasi-isomorphisme, puisqu'il en est de même des deux autres flèches verticales. Considérons le motif $M_Z(X)$ associé à la paire fermée (X, Z) – *cf.* 3.1, (D4). En utilisant l'isomorphisme (1) et l'identification canonique $H^p_Z(X; F^\phi)_n = \varphi_n(M_Z(X)[-p])$, on obtient un diagramme commutatif:

$$\begin{tikzcd}
A^{p-1}(U,\phi)_n \arrow[r, "\partial^U_Z"] \arrow[d, "\rho_U"'] & A^{p-c}(Z,\phi)_{n-c} \arrow[r, "i_*"] \arrow[d, "\rho'_{X,Z}"] & A^p(X,\phi)_n \arrow[d, "\rho_X"] \\
\varphi_n(M(U)[-p]) \arrow[r] & \varphi_n(M_Z(X)[-p]) \arrow[r] & \varphi_n(M(X)[-p])
\end{tikzcd}$$

dans lequel les flèches verticales sont des isomorphismes. Le morphisme $\rho'_{X,Z}$ est de plus naturel en (X, Z) par rapport aux morphismes transverses (définis en 2.9). Cela résulte en effet du corollaire 2.8, ou plus précisément du diagramme commutatif apparaissant dans la démonstration de 2.6, en utilisant d'une part l'unicité de la flèche pointillée (1) et d'autre part la description de la fonctorialité dérivée de $C^*_X(\phi)$ établie ci-dessus – *i.e.* les transformations naturelles τ_f et τ_i.
Comme conséquence de cette construction, on obtient le lemme clé suivant:

Lemme 3.6. — Reprenons les notations qui précèdent. Considérons le triangle de Gysin (cf. [Voe00b, 3.5.4]*) associé à* (X, Z)*:*

$$M(U) \to M(X) \xrightarrow{i^*} M(Z)(c)[2c] \xrightarrow{\partial_{X,Z}} M(U)[1].$$

Alors, le diagramme suivant est commutatif:

$$\begin{tikzcd}
A^{p-1}(U,\phi)_n \arrow[r, "\partial^U_Z"] \arrow[dd, "\rho_U"'] & A^{p-c}(Z,\phi)_{n-c} \arrow[r, "i_*"] \arrow[d, "\rho_Z"] & A^p(X,\phi)_n \arrow[dd, "\rho_X"] \\
 & \varphi_{n-c}(M(Z)[c-p]) \arrow[d, equal] & \\
\varphi_n(M(U)[-p]) \arrow[r, "\varphi_n(\partial_{X,Z})"] & \varphi_n(M(Z)(c)[2c-p]) \arrow[r, "\varphi_n(i^*)"] & \varphi_n(M(X)[-p]).
\end{tikzcd}$$

Démonstration. — On utilise la construction du triangle de Gysin effectuée dans [Dég08b]. Considérons l'isomorphisme de pureté définit dans [Dég08b, sec. 2.2.5]

$$\mathfrak{p}_{X,Z} : M_Z(X) \to M(Z)(c)[2c].$$

D'après ce qui précède, l'isomorphisme composé

$$\rho_{X,Z} : A^{p-c}(Z,\phi)_{n-c} \xrightarrow{\rho'_{X,Z}} \varphi_n(M_Z(X)[-p])$$
$$\xrightarrow{\varphi(\mathfrak{p}_{X,Z})} \varphi_n(M(Z)(c)[2c-p]) = \varphi_{n-c}(M(Z)[c-p])$$

s'inscrit dans le diagramme commutatif:

$$\begin{tikzcd}
A^{p-1}(U,\phi)_n \arrow[r, "\partial_Z^U"] \arrow[dd, "\rho_U"'] & A^{p-c}(Z,\phi)_{n-c} \arrow[r, "i_*"] \arrow[d, "\rho_{X,Z}"] & A^p(X,\phi)_n \arrow[dd, "\rho_X"] \\
& \varphi_{n-c}(M(Z)[c-p]) \arrow[d, equal] & \\
\varphi_n(M(U)[-p]) \arrow[r, "\varphi_n(\partial_{X,Z})"] & \varphi_n(M(Z)(c)[2c-p]) \arrow[r, "\varphi_n(i^*)"] & \varphi_n(M(X)[-p]).
\end{tikzcd}$$

Il s'agit de voir que $\rho_{X,Z} = \rho_Z$. Notons que d'après ce qui précède, le morphisme $\rho_{X,Z} - \rho_Z$ est naturel en (X,Z) par rapport aux morphisme transverses (définis en 2.9). Soit $P_Z X$ la complétion projective du fibré normal de Z dans X. Considérons l'éclatement $B_Z(\mathbb{A}^1_X)$ de $Z \times \{0\}$ dans X, ainsi que le diagramme de déformation classique qui lui est associé

$$(X,Z) \xrightarrow{(d,i_1)} (B_Z(\mathbb{A}^1_X), \mathbb{A}^1_Z) \xleftarrow{(d',i_0)} (P_Z X, Z).$$

Les carrés correspondants à (d,i_1) et (d',i_0) sont transverses. On est donc réduit au cas où $(X,Z) = (P_Z X, Z)$. Dans ce cas, l'immersion fermée i admet une rétraction et le morphisme $\rho_{X,Z}$ (resp. ρ_Z) est déterminé de manière unique par ρ_X. □

3.3. Théorème et démonstration. —

Théorème 3.7. — Les foncteurs

$$\begin{array}{rcl} HI_*(k) & \leftrightarrows & \mathscr{M}Cycl(k) \\ F_* & \mapsto & \hat{F}_* \\ F_*^\phi & \leftarrow\!\shortmid & \phi \end{array}$$

sont des équivalences de catégories quasi-inverses l'une de l'autre.

Démonstration. — Il s'agit de construire les deux isomorphismes naturels qui réalisent l'équivalence.

Premier isomorphisme : Considérons un module de cycles ϕ, F_*^ϕ le module homotopique associé. Par définition, pour tout corps de fonctions E, il existe une flèche canonique

$$a_E : \hat{F}_n^\phi(E) = \varinjlim_{A \subset E} A^0(\mathrm{Spec}(A);\phi)_n \to \phi_n(E).$$

C'est trivialement un isomorphisme et il reste à montrer que a définit un morphisme de modules de cycles. La compatibilité à (D1) est évidente. La compatibilité à (D2) résulte du fait que pour un morphisme fini surjectif $f : Y \to X$, le morphisme $A^0({}^{\mathrm{t}}f;\phi)$ est le pushout f_* propre (*cf.* [Dég08b, 6.6]).

Compatibilité à (D3) : On reprend les notations du point (D3) de 3.1 pour le module homotopique F_*^ϕ et pour une unité $x \in E^\times$. On considère la flèche canonique

$$a'_E : \hat{F}_n^\phi(\mathbb{G}_m \times (E)) = \varinjlim_{A \subset E} A^0\big(\mathrm{Spec}(A[t,t^{-1}]);\phi\big)_n \longrightarrow \phi_n(E(t)).$$

Pour tout E-point y de $\mathrm{Spec}(E[t])$, on note v_y la valuation de $E(t)$ correspondante, d'uniformisante $t-y$. D'après la proposition 2.3, le diagramme suivant est commutatif :

$$\begin{array}{ccc} \hat{F}_n^{\phi}(\mathbb{G}_m \times (E)) & \xrightarrow{s_x^*} & \hat{F}_n^{\phi}(E) \\ {\scriptstyle a'_E}\downarrow & & \downarrow{\scriptstyle a_E} \\ \phi_n(E(t)) & \xrightarrow{s_{v_x}^{t-x}} & \phi_n(E). \end{array}$$

Par définition du morphisme structural ϵ_* de F_*^{ϕ} (*cf.* 2.10), le morphisme $\nu' : \hat{F}_{n-1}^{\phi}(E) \xrightarrow{\epsilon_{n-1}} (\hat{F}_n^{\phi})_{-1}(E) \xrightarrow{\nu} \bar{F}_n^{\phi}(\mathbb{G}_m \times (E))$ est la section de la suite exacte courte

$$0 \to \hat{F}_n^{\phi}(E) \xrightarrow{p^*} \hat{F}_n^{\phi}(\mathbb{G}_m \times (E)) \xrightarrow{\partial} \hat{F}_{n-1}^{\phi}(E) \to 0$$

qui correspond à la rétraction s_1^* de p^*, pour $s_1 : (E) \to \mathbb{G}_m \times (E)$ la section unité de la projection $p : \mathbb{G}_m \times (E) \to (E)$. En particulier, ν' est caractérisé par les propriétés $\partial\nu' = 1$ et $s_1^*\nu' = 0$.
Notons $\varphi : E \to E(t)$ l'inclusion canonique. On peut vérifier en utilisant les relations des pré-modules de cycles les formules suivantes :

(1) $\forall \rho \in \phi_n(E), \partial_{v_0}(\{t\}.\varphi_*(\rho)) = \rho.$

(2) $\forall y \in E^{\times}, \forall \rho \in \phi_n(E), \partial_{v_y}(\{t\}.\varphi_*(\rho)) = 0.$

(3) $\forall y \in E^{\times}, \forall \rho \in \phi_n(E), s_{v_y}^{t-y}(\{t-y\}.\varphi_*(\rho)) = \{y\}.\rho.$

D'après (2), l'application $\phi_n(E) \to \phi_n(E(t)), \rho \mapsto \{t\}.\varphi_*(\rho)$ induit une unique flèche pointillée rendant le diagramme suivant commutatif :

$$\begin{array}{ccc} \hat{F}_n^{\phi}(E) & \dashrightarrow & \hat{F}_n^{\phi}(\mathbb{G}_m \times (E)) \\ {\scriptstyle a_E}\downarrow & & \downarrow{\scriptstyle a'_E} \\ \phi_n(E(t)) & \xrightarrow{\{t\}.\varphi_*} & \phi_n(E). \end{array}$$

D'après la relation (1) et la relation (3) avec $y = 1$, cette flèche pointillée satisfait les deux propriétés caractérisant ν'. On déduit donc de la relation (3) avec $y = x$ que $\nu' \circ s_x^*(\rho) = \{x\}.\rho$ ce qui prouve la relation attendue.
Compatibilité à (D4) : Considérons les notations du point (D4) dans 3.1. La compatibilité au résidu est alors une conséquence directe du lemme 3.6 appliqué, pour tout voisinage ouvert U de x dans X, à l'immersion fermée $i : Z \cap U \to U$ dans le cas $c = 1$, $p = 1$.
Deuxième isomorphisme : Considérons un module homotopique (F_*, ϵ_*). Pour tout schéma lisse X, en considérant la limite inductive des morphismes de restriction $F(X) \to F(U)$ pour les ouverts U de X, on obtient une flèche $F_*(X) \to C^0(X; \hat{F}_*)$ qui induit par définition des différentielles un morphisme $b_X : F_*(X) \to A^0(X; \hat{F}_*)$ homogène de degré 0.
Le point clé est de montrer que cette flèche est naturelle par rapport aux correspondances finies. Soit $\alpha \in c(X, Y)$ une correspondance finie entre schémas lisses, que l'on peut supposer connexes. Rappelons que pour tout ouvert dense $j : U \to X$, le morphisme $j^* : A^0(X; \hat{F}_*) \to A^0(U; \hat{F}_*)$ est injectif d'après la

suite exacte de localisation (2.5.c). Ainsi, on peut remplacer α par $\alpha \circ j$ et X par U. Par additivité, on se ramène encore au cas où α est la classe d'un sous-schéma fermé intègre Z de $X \times Y$, fini et dominant sur X. Dès lors, $\alpha \circ j = [Z \times_X U]$. Donc puisque k est parfait, quitte à réduire X, on peut supposer que Z est lisse sur k. Rappelons que d'après 2.5, $\alpha^* = p_* i^* q^*$ pour les morphismes évidents suivants

$$X \xleftarrow{p} Z \xrightarrow{i} Z \times X \times Y \xrightarrow{q} Y.$$

On est donc ramené à vérifier la naturalité dans les trois cas suivants :
Premier cas : Si $\alpha = q$ est un morphisme plat, la compatibilité résulte alors de la définition du pullback plat sur $A^0(.;\hat{F}_*)$ est de la définition de D1.
Deuxième cas : Si $\alpha = {}^t p$, $p : Z \to X$ morphisme fini surjectif entre schémas lisses. Ce cas résulte de la définition du pushout propre sur A^0 et de la définition de D2.
Troisième cas : Supposons $\alpha = i$, pour $i : Z \to X$ immersion fermée régulière entre schémas lisses. Comme on l'a déjà vu, l'assertion est locale en X. On se réduit donc en factorisant i au cas de codimension 1. On peut aussi supposer que Z est un diviseur principal paramétré par $\pi \in \mathcal{O}_X(U)$, pour $U = X - Z$. D'après la proposition 2.3, on est ramené à montrer que le diagramme suivant est commutatif :

$$\begin{tikzcd} F_*(X) \arrow[r, "i^*"] \arrow[d] & F_*(Z) \arrow[d] \\ \hat{F}_*(\kappa(X)) \arrow[r, "s_v^\pi"] & \hat{F}_*(\kappa(Z)). \end{tikzcd}$$

Tenant compte de la naturalité du morphisme structural ϵ_* du module homotopique F_*, on se ramène à la commutativité du diagramme :

$$\begin{tikzcd} \varphi(M(X)\{1\}) \arrow[rrr, "i^*"] \arrow[d, "j^*"'] & & & \varphi(M(Z)\{1\}) \arrow[d, equal] \\ \varphi(M(U)\{1\}) \arrow[r, "\nu"] & \varphi(M(\mathbb{G}_m \times U)) \arrow[r, "\gamma_\pi^*"] & \varphi(M(U)) \arrow[r, "\partial_{X,Z}"] & \varphi(M(Z)\{1\}) \end{tikzcd}$$

où ν est l'inclusion canonique, γ_π est induit par $\pi : U \to \mathbb{G}_m$ et $\partial_{X,Z} = \partial'_{X,Z} \circ \mathfrak{p}_{X,Z}^{-1}$ avec les notations de 3.1(D4) est le morphisme résidu au niveau des motifs. Or la commutativité de ce diagramme résulte exactement de [Dég08b, 2.6.5].
Le morphisme $b : F_* \to A^0(.;\hat{F}_*)$ est donc un morphisme de faisceaux avec transferts. Or, il est évident que le morphisme induit sur les fibres en un corps de fontions quelconque est un isomorphisme. Il en résulte (*cf.* 1.4) que b est un isomorphisme. Enfin, on établit facilement la compatibilité de b avec les morphismes structuraux des modules homotopiques F_* et $A^0(.;\hat{F}_*)$ compte tenu de la construction 2.10 – on utilise simplement la fonctorialité de b par rapport à $j_X : \mathbb{G}_m \times X \to \mathbb{A}^1_X$ et $s_1 : X \to \mathbb{G}_m \times X$. $\square$

3.8. — Le théorème précédent montre que la catégorie des modules de cycles est monoïdale symétrique avec pour élément neutre le foncteur de K-théorie de

Milnor. Le produit tensoriel est de plus compatible au foncteur de décalage de la graduation des modules de cycles – *i.e.* le foncteur noté $\{\pm 1\}$ dans $HI_*(k)$. A tout schéma lisse X, on associe un module de cycles

$$\hat{h}_{0,*}(X) = (h_{0,*}(X))^\wedge.$$

D'après le théorème précédent, la famille de modules de cycles $(\hat{h}_{0,*}(X)\{n\})$ pour un schéma lisse X et un entier $n \in \mathbb{Z}$ est génératrice dans la catégorie abélienne $\mathscr{M}Cycl(k)$.
Notons que ces générateurs caractérisent le produit tensoriel des modules de cycles:

$$\hat{h}_{0,*}(X)\{n\} \otimes \hat{h}_{0,*}(Y)\{m\} = \hat{h}_{0,*}(X \times Y)\{n+m\}.$$

On peut enfin donner une formule explicite pour calculer ces modules de cycles. Considérons pour tous schémas lisses X et Y le groupe

$$\pi(Y, X) = \mathrm{coKer}\left(c(\mathbb{A}^1_Y, X) \xrightarrow{s_0^* - s_1^*} c(Y, X)\right).$$

Notons que ce groupe s'étend de manière évidente aux schémas réguliers essentiellement de type fini sur k et que l'on dipose d'un théorème de commutation aux limites projectives de schémas pour ces groupes étendus (*cf.* [Dég07, 4.1.24]). Par ailleurs, si E est un corps de fonctions, et X un schéma projectif lisse, $\pi(\mathrm{Spec}(E), X) = CH_0(X_E)$, groupe de Chow des 0-cycles de X étendu à E.
On déduit de tout cela les calculs suivants: pour tout corps de fonctions E et tout schéma projectif lisse X,

$$\hat{h}_{0,0}(X).E = CH_0(X_E).$$

De plus, pour tout entier $n > 0$,

$$\hat{h}_{0,n}(X).E = \mathrm{coKer}\left(\oplus_{i=0}^n CH_0(\mathbb{G}_m^{n-1} \times X_E) \to CH_0(\mathbb{G}_m^n \times X_E)\right)$$
$$\hat{h}_{0,-n}(X).E = \mathrm{Ker}\left(\pi(\mathbb{G}_{m,E}^n, X) \to \oplus_{i=0}^n \pi(\mathbb{G}_{m,E}^{n-1}, X)\right)$$

où les flèches sont induites par les injections évidentes $\mathbb{G}_m^i \times \{1\} \times \mathbb{G}_m^{n-1-i} \to \mathbb{G}_m^n$.

3.4. Résolution de Gersten: fonctorialité II. —

3.9. — Dans ce paragraphe, on complète les résultats du paragraphe 3.2. On fixe un module de cycles ϕ et on note F^ϕ le module homotopique qui lui est associé. On peut étendre la construction de *loc. cit.* au cas d'un k-schéma de type fini X: on associe à ce schéma un complexe de faisceaux sur X_{Nis}:

$$C_X^*(\phi) : V/X \mapsto C^*(V; \phi)$$

et un faisceau $F_X^\phi = \underline{\mathrm{H}}^0(C_X^\phi)$. Le complexe $C_X^*(\phi)$ vérifie encore la propriété de Brown-Gersten mais par contre, le morphisme canonique:

$$F_X^\phi \to C_X^*(\phi)$$

n'est plus nécessairement un isomorphisme. Cette construction nous sert à montrer le résultat suivant:

*Proposition 3.10. — Considérons les notations ci-dessus. Alors, l'isomorphisme $\rho_X : A^p(X,\phi) \to H^p(X;F^\phi)$ pour un schéma lisse X (*cf. (3.2.b)*) est naturel par rapport aux correspondances finies.*

Démonstration. — La preuve reprend le principe de la preuve du lemme 3.3. Soit $f : Y \to X$ un morphisme plat ou une immersion fermée régulière entre schémas de type fini. Il est clair que les constructions de *loc. cit.* se généralisent et permettent de définir un morphisme canonique η_f qui s'insère dans un diagramme commutatif de la catégorie dérivée des faisceaux abéliens sur X_{Nis}:

$$\begin{tikzcd} F_X^\phi \arrow[r, "\tau_f"] \arrow[d] & \mathrm{R}f_* F_Y^\phi \arrow[d] \\ C_X^*(\phi) \arrow[r, "\eta_f"] & \mathrm{R}f_* C_Y^*(\phi). \end{tikzcd}$$

Par ailleurs, si $p : Z \to X$ est un morphisme fini, il induit (*cf.* 2.1) un morphisme de complexes

$$p_* : C^*(Z;\phi) \to C^*(X;\phi)$$

qui est naturel en X par rapport aux morphismes étales (*cf.* [Ros96, (4.1)]). On en déduit un morphisme canonique $tr_p : p_* C_Z^*(\phi) \to C_X^*(\phi)$ qui induit un diagramme commutatif dans la catégorie dérivée (p_* est exact):

$$\begin{tikzcd} \mathrm{R}p_*(F_Z^\phi) \arrow[r, "tr_p^0"] \arrow[d] & F_X^\phi \arrow[d] \\ \mathrm{R}p_* C_Z^*(\phi) \arrow[r, "tr_p"] & C_X^*(\phi). \end{tikzcd}$$

Revenons à la preuve de la proposition. Il suffit de montrer la naturalité de ρ_X pour une correspondance finie $\alpha \in c(X,Y)$ telle que α est la classe d'un sous-schéma fermé intègre Z de $X \times Y$. Suivant le paragraphe 2.4, on considère les morphismes:

$$X \xleftarrow{p} Z \xrightarrow{i} ZXY \xrightarrow{q} Y.$$

D'après *loc. cit.*, $\alpha^* = p_* i^* q^*$. Appliquant les constructions qui précèdent, on obtient un diagramme commutatif dans la catégorie dérivée des groupes abéliens:

$$\begin{tikzcd} H^p(Y;F_Y^\phi) \arrow[r, "(\tau_q)_*"] \arrow[d] & H^p(ZXY;F_Y^\phi) \arrow[r, "(\tau_i)_*"] \arrow[d] & H^p(Z;F_Z^\phi) \arrow[r, "(tr_p^0)_*"] \arrow[d] & H^p(X;F_X^\phi) \arrow[d] \\ H^p(Y;C_Y^*(\phi)) \arrow[r, "(\eta_q)_*"] & H^p(ZXY;C_Z^*(\phi)) \arrow[r, "(\eta_i)_*"] & H^p(Z;C_Z^*(\phi)) \arrow[r, "(tr_p)_*"] & H^p(X;C_X^*(\phi)) \end{tikzcd}$$

On vérifie que la composée des flèches de la première ligne coincide avec α^* et cela permet de conclure. □

3.11. — Soit F_* un module homotopique, $\phi = \hat{F}_*$ sa transformée générique. Considérons l'isomorphisme $b : F_* \to F_*^\phi$ qui lui est associé d'après le théorème 3.7. Compte tenu de l'isomorphisme (3.2.b), on en déduit un isomorphisme

$$\epsilon_X : H^n_{\mathrm{Nis}}(X; F_*) \xrightarrow{b_*} H^n_{\mathrm{Nis}}(X; F_*^\phi) \xrightarrow{\rho_X^{-1}} A^n(X; \hat{F}_*). \tag{3.11.a}$$

La proposition précédente a comme corollaire immédiat:

Corollaire 3.12. — Avec les notations ci-dessus, ϵ_X est un isomorphisme naturel en X par rapport aux correspondances finies.

3.13. — Considérons le module homotopique S_t^*. D'après le théorème de Suslin-Voevodsky rappelé en 1.11, pour tout corps de fonctions E, $\widehat{S_t^*}(E) \simeq K_*^M(E)$. Cet isomorphisme est de plus compatible aux structures de module de cycles. Pour la norme, cela résulte de [SV00, 3.4.1]. Pour le résidu associé à un corps de fontions valué (E, v), on se réduit à montrer que $\partial_v(\pi) = 1$ pour le module de cycle $\hat{S}_t^*$, ce qui résulte de [Dég07, 2.6.5].
On en déduit l'isomorphisme de Bloch[14] pour tout schéma lisse X :

$$\epsilon_X^{\mathrm{B}} : H^n_{\mathrm{Nis}}(X; S_t^n) \to A^n(X; K_*^M)_n = CH^n(X).$$

On a obtenu ci-dessus que cet isomorphisme est compatible aux transferts.
Rappelons que pour tout module de cycles ϕ, il existe un accouplement de modules de cycles $K_*^M \times \phi \to \phi$ au sens de [Ros96, 1.2]. Il induit d'après [Ros96, par. 14] un accouplement

$$CH^n(X) \otimes A^m(X; \phi)_r \to A^{m+n}(X; \phi)_{r+n}.$$

Considérant un module homotopique F_*, on dispose d'un (iso)morphisme de modules homotopiques $S_t^* \otimes F_* \to F_*$. Pour un schéma lisse X, de diagonale $\delta : X \to X \times X$, on en déduit un accouplement

$$H^n(X; S_t^*)_n \otimes H^m(X; F_*)_r \to H^{m+n}(X; F_*)_{r+n}$$

définit en associant à deux morphismes $a : h_{0,*}(X) \to S_t^*\{n\}[n]$ et $b : h_{0,*}(X) \to F_*\{r\}[m]$ la composée

$$h_{0,*}(X) \xrightarrow{\delta_*} h_{0,*}(X) \otimes h_{0,*}(X) \xrightarrow{a \otimes b} S_t^* \otimes F_*\{n+r\}[n+m] \xrightarrow{\sim} F_*\{n+r\}[n+m].$$

Nous laissons au lecteur le soin de vérifier la compatibilité suivante :

Lemme 3.14. — Avec les notations introduites ci-dessus, le diagramme suivant est commutatif :

$$\begin{tikzcd}
H^n(X; S_t^*)_n \otimes H^m(X; F_*)_r \arrow[r] \arrow[d, "\epsilon_X^{\mathrm{B}} \otimes \epsilon_X"'] & H^{n+m}(X; F_*)_{n+r} \arrow[d, "\epsilon_X"] \\
CH^n(X) \otimes A^m(X; \hat{F}_*)_r \arrow[r] & A^{m+n}(X; \hat{F}_*)_{n+r}
\end{tikzcd}$$

[14] En effet, d'après l'isomorphisme que l'on vient d'expliciter, le faisceau gradué S_t^* est le faisceau de K-théorie de Milnor non ramifié.

Ainsi, l'isomorphisme ϵ_X^{B} est compatible au produit, et l'isomorphisme ϵ_X est compatible aux structures de module décrites ci-dessus.

3.15. — Notons $\varphi : DM_{gm}(k)^{op} \to \mathscr{A}b$ le foncteur de réalisation associé à F_* (*cf.* section 1.3). D'après la proposition précédente, le foncteur φ prolonge le foncteur $A^*(.;\hat{F}_*)$. Ainsi, on a étendu canoniquement la cohomologie à coefficients dans un module de cycles quelconque en un foncteur de réalisation triangulé de $DM_{gm}(k)$. Nous notons encore

$$\epsilon_X : \varphi(M(X)\{-r\}[-n]) \to A^n(X;\hat{F}_*)_r$$

l'isomorphisme qui se déduit par construction de l'isomorphisme (3.11.a).
Soit $f : Y \to X$ un morphisme projectif entre schémas lisses, de dimension relative constante d. Dans [Dég08a, 2.7], on a construit $f^* : M(X)(d)[2d] \to M(Y)$, morphisme de Gysin associé à f dans $DM_{gm}(k)$.

Proposition 3.16. — Considérons les notations introduites ci-dessus. Alors, le carré suivant est commutatif :

$$\begin{tikzcd}
\varphi\big(M(X)\{d-r\}[d-n]\big) \arrow[r, "\varphi(f^*)"] \arrow[d, "\epsilon_X"'] & \varphi\big(M(Y)\{-r\}[-n]\big) \arrow[d, "\epsilon_Y"] \\
A^{n-d}(X;\hat{F}_*)_{r-d} \arrow[r, "f_*"] & A^n(Y;\hat{F}_*)_r
\end{tikzcd}$$

Démonstration. — Dans cette preuve, on utilisera particulièrement le lemme suivant :

Lemme 3.17. — Soit X un schéma lisse et E/X un fibré vectoriel de rang n. Soit $p : P \to X$ le fibré projectif associé, et λ le fibré inversible canonique sur P tel que $\lambda \subset p^{-1}(E)$. On note $c = c_1(\lambda) \in CH^1(X)$ la première classe de Chern de λ.
Alors, le morphisme suivant est un isomorphisme :

$$\begin{array}{rcl} \bigoplus_{i=0}^n A^*(X;\hat{F}_*) & \to & A^*(P;\hat{F}_*) \\ x_i & \mapsto & p^*(x_i).c^i. \end{array}$$

en utilisant la structure de $CH^*(X)$-module (ici notée à droite) de $A^*(X;\hat{F}_*)$ rappelée en 3.13.

Pour obtenir ce lemme, il suffit d'appliquer le théorème du fibré projectif dans $DM_{gm}(k)$ (*cf.* [Voe00b, 2.5.1]) et de regarder son image par φ compte tenu du lemme 3.14.
Soit E un fibré vectoriel sur X et P sa complétion projective. On déduit de ce lemme le cas où $f = p$. En effet, d'après la formule de projection

$$p_*(p^*(x_i).c^i) = x_i.p_*(c^i)$$

pour les groupes de Chow à coefficients (*cf.* [Dég06, 5.9]), on déduit que p_* est la projection évidente à travers le théorème du fibré projectif. L'analogue de ce calcul pour $\varphi(p^*)$ résulte des définitions de [Dég08a].

Compte tenu de la définition du morphisme de Gysin et du cas que l'on vient de traiter, nous sommes ramenés au cas où $f = i : Z \to X$ est une immersion fermée, que l'on peut supposer de codimension pure égale à c. Ce cas est alors une conséquence directe du lemme 3.6. ☐

PARTIE II
MOTIFS MIXTES TRIANGULÉS

4. Rappels

Dans cette section, on rappelle la théorie de Voevodsky des complexes motiviques et l'extension qu'on lui a donnée avec D.C. Cisinski suivant les lignes de Morel et Voevodsky.

4.1. Catégorie effective. —

4.1. — La catégorie $Sh^{tr}(k)$ (*cf.* paragraphe 1.2) est abélienne de Grothendieck. On note $\mathscr{T}_{\mathbb{A}^1}$ la sous-catégorie triangulée localisante de la catégorie dérivée $\mathrm{D}(Sh^{tr}(k))$ engendrée par les complexes de la forme:

$$\cdots \to \mathbb{Z}^{tr}(\mathbb{A}^1_X) \to \mathbb{Z}^{tr}(X) \to 0 \cdots$$

Définition 4.2 (Voevodsky). — On définit la catégorie des motifs effectifs comme le quotient de Verdier:

$$DM^{eff}(k) = \mathrm{D}(Sh^{tr}(k))/\mathscr{T}_{\mathbb{A}^1}.$$

Suivant les idées de la théorie de l'homotopie des schémas de Morel et Voevodsky, on peut décrire cette catégorie grâce à la notion de localisation de Bousfield. Le concept central dans cette théorie est le suivant:

Définition 4.3. —
1. Soit C un complexe de faisceaux avec transferts.
   On dit que C est $\mathbb{A}^1$-*local* si pour tout schéma lisse X et tout entier $n \in \mathbb{Z}$, le morphisme suivant, induit par la projection canonique, est un isomorphisme:
   $$H^n_{\mathrm{Nis}}(X, C) \to H^n_{\mathrm{Nis}}(\mathbb{A}^1_X, C).$$
   On dit que C est *Nis-fibrant* si pour tout schéma lisse X et tout entier $n \in \mathbb{Z}$, le morphisme canonique suivant est un isomorphisme:
   $$H^n(C(X)) \to H^n_{\mathrm{Nis}}(X, C).$$
   On dit que C est $\mathbb{A}^1$-*fibrant* si il est $\mathbb{A}^1$-local et Nis-fibrant.
2. Soit $f : C \to D$ un morphisme de $\mathrm{C}(Sh^{tr}(k))$.
   On dit que f est une $\mathbb{A}^1$-*équivalence* si pour tout complexe $\mathbb{A}^1$-local L, le morphisme induit
   $$\mathrm{Hom}_{\mathrm{D}(Sh^{tr}(k))}(D, L) \to \mathrm{Hom}_{\mathrm{D}(Sh^{tr}(k))}(C, L)$$

est un isomorphisme.

On dit que f est une $\mathbb{A}^1$-*fibration* si c'est un épimorphisme dans $\mathrm{C}(Sh^{tr}(k))$ et son noyau est $\mathbb{A}^1$-fibrant.

La proposition suivante (voir [CD09B, ex. 3.3, 4.12]) donne une bonne structure homotopique à la catégorie $DM^{eff}(k)$.

Proposition 4.4. — 1. *La catégorie* $\mathrm{C}(Sh^{tr}(k))$ *avec pour équivalences faibles les quasi-isomorphismes et pour fibrations les épimorphismes dont le noyau est Nis-fibrant est une catégorie de modèles symétrique monoïdale dont la catégorie homotopique associée est* $\mathrm{D}(Sh^{tr}(k))$.

2. *La localisation de Bousfield de la catégorie de modèles précédente par rapport à la classe d'objets* $\mathscr{T}_{\mathbb{A}^1}$ *est encore une catégorie de modèles symétrique monoïdale avec pour équivalences faibles les* $\mathbb{A}^1$*-équivalences et pour fibrations les* $\mathbb{A}^1$*-fibrations.*

On déduit de cette proposition que le morphisme de projection canonique

$$L_{\mathbb{A}^1} : \mathrm{D}(Sh^{tr}(k)) \to DM^{eff}(k)$$

est monoïdal et admet un adjoint à droite $\mathcal{O} : DM^{eff}(k) \to \mathrm{D}(Sh^{tr}(k))$ pleinement fidèle. L'image essentielle de ce dernier est formée des complexes $\mathbb{A}^1$-locaux. Par la suite, on identifie $DM^{eff}(k)$ à cette image essentielle. En particulier, pour tout complexe C, $L_{\mathbb{A}^1}(C)$ est $\mathbb{A}^1$-local.

On notera simplement $\otimes$ le produit tensoriel sur $DM^{eff}(k)$ et $\mathbb{1}$ l'unité pour $\otimes$.

Remarque 4.5. — Cette description des objets de $DM^{eff}(k)$ comme complexes $\mathbb{A}^1$-locaux n'a rien d'original. Elle est essentiellement due à Voevodsky suivant son article fondateur [VOE00A] (voir aussi le théorème 5.1 ci-dessous).

Rappelons au passage que d'après [VOE00A, 3.2.6], le foncteur canonique

$$DM^{eff}_{gm}(k) \to DM^{eff}(k) \tag{4.5.a}$$

est pleinement fidèle. Notons de plus que son image essentielle est égale à la sous-catégorie pleine formée des objets compacts de $DM^{eff}(k)$ (*cf.* [CD09B, ex. 5.5]).

4.2. CATÉGORIE NON EFFECTIVE. —

4.6. — Soit T le conoyau du morphisme $\mathbb{Z}^{tr}(k) \xrightarrow{s_*} \mathbb{Z}^{tr}(\mathbb{G}_m)$ induit par la section unité. Utilisant la notation de la section 1.3, on définit le *complexe motivique de Tate suspendu* dans $DM^{eff}(k)$ par la formule: $\mathbb{1}\{1\} := L_{\mathbb{A}^1}(T)$.[15]

Dans l'énoncé qui suit, nous dirons qu'une catégorie est *monoïdale homotopique* si c'est un quotient de Verdier d'une catégorie dérivée munie d'un produit tensoriel dérivé. La proposition qui suit est bien connue (*cf.* [HOV01] pour

[15] Rappelons à nouveau qu'avec la notation habituelle $\mathbb{1}\{1\} = \mathbb{1}(1)[1]$. Le twist $\mathbb{1}\{1\}$ apparaît plus naturel pour notre propos c'est pourquoi on adopte cette notation ici.

l'aspect catégorie de modèles pure et [CD09b, 7.15] pour l'aspect catégorie dérivée):

Proposition 4.7. — Il existe une unique catégorie monoïdale homotopique $DM(k)$ munie d'une adjonction de catégories homotopiques

$$\Sigma^\infty : DM^{eff}(k) \leftrightarrows DM(k) : \Omega^\infty \tag{4.7.a}$$

vérifiant

1. Σ^∞ *est monoïdal,*
2. $\Sigma^\infty L_{\mathbb{A}^1}(T)$ *est inversible pour le produit tensoriel sur $DM(k)$,*

et qui soit universelle (initiale) pour ces propriétés.

On note encore $\otimes$ le produit tensoriel sur $DM(k)$ et $\mathbb{1}$ l'objet unité. Pour tout entier $n \in \mathbb{Z}$, on note $\mathbb{1}\{n\}$ la puissance tensorielle n-ième de l'objet inversible $\mathbb{1}\{1\}$ dans $DM(k)$; pour tout spectre motivique K, on pose $K\{n\} := K \otimes \mathbb{1}\{n\}$.

4.8. — Rappelons que la construction de $DM(k)$ reprend celle de la catégorie d'homotopie stable de la topologie algébrique. On utilise ici la version classique (*i.e.* non symétrique) des spectres qui sera plus commode pour notre propos.(16)
On définit un *spectre motivique* comme une famille $(E_n, \sigma_n)_{n\in\mathbb{N}}$ telle que E_n est un complexe de faisceaux avec transferts et $\sigma_n : T \otimes^{\mathrm{tr}} E_n \to E_{n+1}$ un morphisme de faisceaux avec transferts dit *de suspension*. On notera simplement E pour le spectre $(E_n, \sigma_n)_{n\in\mathbb{N}}$.
On dit que E est un Ω-*spectre* si pour tout $n \in \mathbb{N}$, le complexe de faisceaux avec transferts E_n est $\mathbb{A}^1$-fibrant et le morphisme adjoint à σ_n:

$$\tau_n : E_n \to \underline{\mathrm{Hom}}(T, E_{n+1}) \tag{4.8.a}$$

est un quasi-isomorphisme (voir [Hov01, 3.1]).
Un morphisme f de spectres motiviques est un morphisme de complexes gradués compatible avec les morphismes de suspensions. On dit que f est une *équivalence stable* (resp. *quasi-isomorphisme*) si pour tout Ω-spectre E, $\mathrm{Hom}(f, E)$ est un isomorphisme (resp. f est un quasi-isomorphisme degré par degré). La catégorie $DM(k)$ est la localisation de la catégorie des spectres motiviques par rapport aux équivalences stables (voir [Hov01, 3.4]).

Exemple 4.9. — Soit X un schéma lisse. On peut donner la description suivante du spectre associé au complexe $\mathbb{A}^1$-local $L_{\mathbb{A}^1}\mathbb{Z}^{tr}(X)$ par le foncteur Σ^∞:

$$\big(\Sigma^\infty L_{\mathbb{A}^1}(\mathbb{Z}^{tr}(X))\big)_n := L_{\mathbb{A}^1}\big(T^{\otimes^{\mathrm{tr}},n} \otimes^{\mathrm{tr}} \mathbb{Z}^{tr}(X)\big),$$

les morphismes de suspensions étant donnés par les applications évidentes. On déduit de plus du théorème de simplification de Voevodsky [Voe00a], que $\Sigma^\infty L_{\mathbb{A}^1}(\mathbb{Z}^{tr}(X))$ est un Ω-spectre. Par la suite, on le notera simplement $M(X)$.

(16) Dans [CD09b], on utilise les spectres symétriques pour définir $DM(k)$ et sa structure monoïdale symétrique. L'équivalence de cette définition avec celle présentée ici résulte de [Hov01, 9.1, 9.3].

Remarque 4.10. — Nous utiliserons par la suite les points techniques suivants concernant les spectres motiviques(17):

1. Si E et E' sont des Ω-spectres, un morphisme $f : E \to E'$ est une équivalence faible si et seulement si c'est un quasi-isomorphisme. De plus, la catégorie $DM(k)$ s'identifie à la localisation de la catégorie des Ω-spectres par rapport aux quasi-isomorphismes.
2. Un triangle entre Ω-spectres
$$E' \to E \to E'' \to E[1]$$
est distingué dans $DM(k)$ si et seulement si pour tout entier $n \geq 0$, le triangle correspondant
$$E'_n \to E_n \to E''_n \to E_n[1]$$
est distingué dans $DM^{eff}(k)$. Ce dernier triangle est en particulier distingué dans $\mathrm{D}(Sh^{tr}(k))$.
3. Si E est un Ω-spectre et $n \in \mathbb{Z}$ un entier, $\Omega^\infty\big(E\{n\}\big) = E_n$.

Remarque 4.11. — D'après la propriété universelle de $DM(k)$, le foncteur (4.5.a) s'étend de manière unique en un foncteur:

$$DM_{gm}(k) \to DM(k). \tag{4.11.a}$$

On démontre dans [CD07] – à la suite de la définition 10.1.4 – que ce foncteur est pleinement fidèle avec pour image essentielle la sous-catégorie formée des objets compacts.

5. t-structure homotopique

Notre référence pour les t-structures est [BBD82, sec. 1.3].

5.1. Cas effectif.

— Le théorème suivant est une reformulation du résultat central de la théorie des complexes motiviques (*cf.* [Voe00b, 3.1.12]):

Théorème 5.1 (Voevodsky). — *Soit C un complexe de faisceaux avec transferts. Les conditions suivantes sont équivalentes :*

(i) *C est $\mathbb{A}^1$-local.*
(ii) *Pour tout entier $n \in \mathbb{Z}$, $\underline{\mathrm{H}}^n(C)$ est $\mathbb{A}^1$-local.*
(iii) *Pour tout entier $n \in \mathbb{Z}$, $\underline{\mathrm{H}}^n(C)$ est invariant par homotopie.*

Démonstration. — L'équivalence de (i) et (ii) résulte de la suite spectrale d'hypercohomologie Nisnevich. L'implication $(ii) \Rightarrow (iii)$ est évidente et sa réciproque résulte du théorème de Voevodsky *loc. cit.* □

(17) Ces assertions résultent de [Hov01, 3.4] ; plus précisément du fait que les Ω-spectres sont les objets fibrants pour la *structure de catégorie de modèles stable* sur la catégorie des spectres motiviques.

Rappelons que la catégorie $\mathrm{D}(Sh^{tr}(k))$ porte naturellement une t-structure (*cf.* [BBD82, 1.3.2]): un complexe C est positif (resp. négatif) si pour tout $i < 0$ (resp. $i > 0$) le faisceau de cohomologie $\underline{\mathrm{H}}^i(C)$ est nul.

Corollaire 5.2. — *Il existe une unique t-structure sur $DM^{eff}(k)$ telle que le foncteur $\mathcal{O} : DM^{eff}(k) \to \mathrm{D}(Sh^{tr}(k))$ est t-exact*[18].
Le foncteur canonique:

$$DM^{eff}(k) \to HI(k), C \mapsto \underline{\mathrm{H}}^0(C)$$

induit une équivalence de catégories entre $HI(k)$ et le coeur de $DM^{eff}(k)$ pour cette t-structure.

Suivant Voevodsky, on appelle cette t-structure sur $DM^{eff}(k)$ la *t-structure homotopique.*
Considérons les notations qui suivent la définition 1.3. Si F est un faisceau avec transferts, on obtient l'identification: $\underline{\mathrm{H}}^0(L_{\mathbb{A}^1}F) = h_0(F)$.[19] En particulier, pour tout schéma lisse X:

$$\underline{\mathrm{H}}^0\left(L_{\mathbb{A}^1}\mathbb{Z}^{tr}(X)\right) = h_0(X). \tag{5.2.a}$$

Rappelons au passage le calcul du motif de Tate (voir [SV00, 3.2]):

$$\mathbb{1}\{1\} := L_{\mathbb{A}^1}(T) = S^1_t. \tag{5.2.b}$$

5.2. Cas non effectif. — Notons que le Hom interne des complexes de faisceaux avec transferts se dérive à droite pour la structure de catégorie de modèles du point 2 de la proposition 4.4. On le note $\mathrm{R}_{\mathbb{A}^1}\underline{\mathrm{Hom}}$. Le théorème suivant nous sera essentiel. C'est un corollaire de la théorie de Voevodsky: il résulte de [Voe00a, 4.34].

Théorème 5.3. — *L'endofoncteur de $DM^{eff}(k)$ défini par la formule*

$$K \mapsto \mathrm{R}_{\mathbb{A}^1}\underline{\mathrm{Hom}}(S^1_t, K)$$

est t-exact et sa restriction au coeur de $DM^{eff}(k)$, identifié avec $HI(k)$, coincide avec le foncteur $F \mapsto F_{-1}$ défini au paragraphe 1.13.

5.4. — Considérons un Ω-spectre $E = (E_n, \sigma_n)$ au sens du paragraphe 4.8. Fixons un entier $n \geq 0$. On associe à E un faisceau homotopique par la formule:

$$\underline{\mathrm{H}}^0_n(E) = \underline{\mathrm{H}}^0(E_n).$$

Considérons l'isomorphisme (4.8.a). Notons que, puisque T est cofibrant et E_{n+1} est $\mathbb{A}^1$-fibrant pour la structure de catégorie de modèles du point (2) de 4.4, on obtient:

$$\underline{\mathrm{Hom}}(T, E_{n+1}) = \mathrm{R}_{\mathbb{A}^1}\underline{\mathrm{Hom}}(T, E_{n+1}) = \mathrm{R}_{\mathbb{A}^1}\underline{\mathrm{Hom}}(S^1_t, E_{n+1})$$

(18) *i.e.* il respecte les objets positifs ainsi que les objets négatifs.
(19) En effet, $h_0(F)$ est $\mathbb{A}^1$-local d'après le théorème précédent et le morphisme canonique $F \to h_0(F)$ est une $\mathbb{A}^1$-équivalence faible (voir aussi [Voe00b, 3.2.3]).

D'après le théorème précédent, le morphisme $\underline{\mathrm{H}}^0(\tau_n)$ induit donc un isomorphisme:

$$\epsilon_n : \underline{\mathrm{H}}^0_n(E) \to \left(\underline{\mathrm{H}}^0_{n+1}(E)\right)_{-1}$$

Si l'on pose $\underline{\mathrm{H}}^0_{-n}(E) := \left(\underline{\mathrm{H}}^0(E_0)\right)_{-n}$, on a définit ainsi un module homotopique (*cf.* définition 1.17) que l'on note $\underline{\mathrm{H}}^0_*(E)$.
On a construit ainsi un foncteur $\underline{\mathrm{H}}^0_*$ sur les Ω-spectres. Puisqu'il respecte manifestement les quasi-isomorphismes, il induit, d'après le premier point de la remarque 4.10, un unique foncteur:

$$\underline{\mathrm{H}}^0_* : DM(k) \to HI_*(k).$$

Pour tout entier $m \in \mathbb{Z}$, on pose: $\underline{\mathrm{H}}^m_*(E) = \underline{\mathrm{H}}^0_*(E[m])$.

Lemme 5.5. — Considérons les notations introduites ci-dessus.

1. *Le foncteur $\underline{\mathrm{H}}^0_* : DM(k) \to HI_*(k)$ est un foncteur cohomologique qui commute aux sommes quelconques.*
2. *La famille de foncteurs $(\underline{\mathrm{H}}^m_*)_{m\in\mathbb{Z}}$ est conservative.*

Démonstration. — Le point 1 résulte des propriétés analogues du foncteur $\underline{\mathrm{H}}^0 : DM^{eff}(k) \to HI(k)$, du deuxième point de la remarque 4.10, et du lemme 1.14. Concernant le point 2, d'après la remarque 4.10(1), on se ramène à montrer que pour un morphisme $f : \mathbb{E} \to \mathbb{E}'$ entre Ω-spectres, les conditions suivantes sont équivalentes:

(i) f est un quasi-isomorphisme (au sens du paragraphe 4.8).
(ii) pour tout $m \in \mathbb{Z}$, $\underline{\mathrm{H}}^m_*(f)$ est un isomorphisme de modules homotopiques.

Par définition du foncteur $\underline{\mathrm{H}}^m_0$, cette équivalence résulte du corollaire 5.2 et du lemme 1.14. □

On dit qu'un spectre motivique est *positif* (resp. *négatif*) si pour tout $n < 0$ (resp. $n > 0$), $\underline{\mathrm{H}}^n_*(E) = 0$. Soit $\tau_{\leq 0}$ le foncteur de troncation négative pour la t-structure homotopique sur $DM^{eff}(k)$. On vérifie en utilisant à nouveau le théorème 5.3 que l'application de $\tau_{\leq 0}$ degré par degré à un Ω-spectre E définit un $|Omega$-spectre négatif $\tau_{\leq 0}E$ et un morphisme canonique:

$$\tau_{\leq 0}E \to E.$$

Proposition 5.6. — La catégorie DM(k), munie de la notion d'objets négatifs et positifs introduite ci-dessus, est une t-structure dont le foncteur de troncation négatif est le foncteur $\tau_{\leq 0}$ introduit ci-dessus et dont le foncteur cohomologique associé est le foncteur $\underline{\mathrm{H}}^0_*$.

On appelle cette t-structure la *t-structure homotopique* sur $DM(k)$.

5.7. — Notons les propriétés caractéristiques suivantes de cette t-structure:

1. Le diagramme suivant est commutatif:

$$\begin{tikzcd} DM(k) \arrow[r, "\underline{\mathrm{H}}^0_*"] \arrow[d, "\Omega^\infty"'] & HI_*(k) \arrow[d, "\omega^\infty"] \\ DM^{eff}(k) \arrow[r, "\underline{\mathrm{H}}^0"] & HI(k) \end{tikzcd}$$

   avec la notation de (1.18.b) pour ω^∞. En particulier, Ω^∞ est t-exact.
2. Pour tout objet E de $DM(k)$ et tout entier $n \in \mathbb{Z}$,

$$\underline{\mathrm{H}}^0_*\big(E\{n\}\big) = \underline{\mathrm{H}}^0_*(E)\{n\}$$

   en utilisant la notation du paragraphe 1.16 pour le membre de droite. Ainsi, le produit tensoriel par $\mathbb{1}\{1\}$ est t-exact.
3. Pour tout schéma lisse X,

$$\underline{\mathrm{H}}^0_*\big(M(X)\big) = h_{0,*}(X)$$

   avec la notation de l'exemple 4.9 (resp. (1.18.a)) à gauche (resp. à droite).

La première assertion résulte du troisième point de la remarque 4.10. La deuxième assertion se déduit du cas $n = -1$ qui résulte lui-même de la définition. La troisième résulte de la remarque 4.9 et de (5.2.a).

Notons finalement qu'un objet E de $DM(k)$ est positif si et seulement si pour tout schéma lisse X et tout couple d'entiers $(n, i) \in \mathbb{Z} \times \mathbb{N}^*$,

$$\mathrm{Hom}_{DM(k)}\big(M(X)\{n\}[i], E\big) = 0.$$

On en déduit aisément la proposition suivante:

*Proposition 5.8. — Le produit tensoriel sur $DM(k)$ est t-exact à droite (*i.e.* préserve les objets négatifs).*

5.3. Coeur homotopique. — Notons $\underline{\mathrm{H}}^0_*(DM(k))$ le coeur de $DM(k)$ pour la t-structure homotopique de la proposition 5.6. Notons le corollaire suivant de la proposition 5.8:

Corollaire 5.9. — Le produit tensoriel sur $DM(k)$ induit une structure monoïdale symétrique sur $\underline{\mathrm{H}}^0_*(DM(k))$ dont l'objet unité est $\underline{\mathrm{H}}^0_*(\mathbb{1})$ et le produit tensoriel est défini par la formule:

$$E \bar{\otimes} E' := \underline{\mathrm{H}}^0_*(E \otimes E'). \tag{5.9.a}$$

Ce corollaire résulte plus précisément du lemme suivant laissé au lecteur[20]:

Lemme 5.10. — Soit $\mathscr{T}$ une catégorie triangulée monoïdale munie d'une t-structure telle que le produit tensoriel est t-exact à droite.
Alors, pour tous objets K et L négatifs de $\mathscr{T}$,

$$\underline{\mathrm{H}}^0(K \otimes L) \simeq \underline{\mathrm{H}}^0\big(\underline{\mathrm{H}}^0(K) \otimes L\big).$$

[20] On fera attention toutefois que ce lemme est faux sans bornes sur K et L.

L'identification du coeur de la t-structure homotopique sur $DM(k)$ est maintenant aisée:

Théorème 5.11. — *Le foncteur* $\underline{\mathrm{H}}^0_*$ *induit une équivalence de catégories abéliennes monoïdales:*

$$\underline{\mathrm{H}}^0_*(DM(k)) \to HI_*(k) \simeq \mathscr{M}Cycl(k),$$

l'équivalence de droite étant celle du théorème 3.7.

Démonstration. — Tout module homotopique définit évidemment un Ω-spectre au sens du paragraphe 4.8. On vérifie facilement que cela défini un quasi-inverse au foncteur de l'énoncé.
Concernant les structures monoïdales, on note tout d'abord que $\underline{\mathrm{H}}^0_*(\mathbb{1}) = K^M_*$ compte tenu du théorème 1.11 et du point 2 du paragraphe 5.7. D'après la formule (5.9.a), et les points 2 et 3 du paragraphe *loc. cit.*, pour tous schémas lisses X, Y et tout couple d'entiers $(n, m) \in \mathbb{Z}^2$, on obtient:

$$\begin{aligned} h_{0,*}(X)\{n\}\bar{\otimes}h_{0,*}(Y)\{m\} &= \underline{\mathrm{H}}^0_*(\Sigma^\infty M(X)\{n\})\bar{\otimes}\underline{\mathrm{H}}^0_*(\Sigma^\infty M(X)\{n\}) \\ &= \underline{\mathrm{H}}^0_*(\Sigma^\infty M(X \times Y)\{n+m\}) = h_{0,*}(X \times Y)\{n+m\}. \end{aligned}$$

Cela conclut d'après le lemme 1.19. □

Remarque 5.12. — Reprenons les notations de la section 3.1. Utilisant le foncteur pleinement fidèle (4.11.a), on définit un foncteur:

$$DM^{(0)}_{gm}(k) \to \text{pro–}DM(k) \xrightarrow{\underline{\mathrm{H}}^0_*} \text{pro–}HI_*(k).$$

On vérifie que ce foncteur est pleinement fidèle.[21]
Il en résulte que la catégorie des motifs génériques est la sous-catégorie pleine de la catégorie abélienne pro–$HI_*(k)$, formée des pro-objets de la forme $\underline{\mathrm{H}}^0_*(M(L)\{n\})$ pour un corps de fonctions L/k et un entier $n \in \mathbb{Z}$. Ces pro-objets correspondent à des foncteurs fibres de la catégorie $HI_*(k)$ (*i.e.* exacts, commutant aux sommes infinies). Cette interprétation des motifs génériques est donc très proche de la notion de points d'un topos. La transformée générique d'un module homotopique F_* est finalement donnée par la *restriction de* F_* *à cette catégorie de points.*[22]

6. Applications et compléments

6.1. Construction de modules de cycles. —

[21] Par le théorème de simplification de Voevodsky [Voe02], on se ramène au cas effectif qui est démontré dans [Dég08b, 3.4.7].
[22] Dans un topos arbitraire, il est très rare que la restriction d'un faisceau à une famille conservative de points donnée définisse une équivalence de catégories.

6.1. — Pour un objet M de $DM(k)$, on note $\underline{\hat{\mathrm{H}}}^0_*(M)$ la transformée générique (par. 3.1) du module homotopique $\underline{\mathrm{H}}^0_*(M)$ (par. 5.4). Cette construction nous permet de construire des modules de cycles intéressants.

Ainsi, pour tout schéma algébrique X, on peut définir suivant [Voe00b] un complexe motivique $\underline{\mathrm{C}}^*_{\mathrm{sing}}\, \mathbb{Z}^{tr}(X)$ – qui coincide avec le complexe motivique $M(X)$ lorsque X est lisse. Pour tout entier $i \geq 0$, on pose donc:

$$\hat{h}_{i,*}(X) := \underline{\mathrm{H}}^{-i}_* \left(\Sigma^\infty\, \underline{\mathrm{C}}^*_{\mathrm{sing}}\, \mathbb{Z}^{tr}(X)\right). \tag{6.1.a}$$

Pour tout corps de fonctions L, le gradué de degré 0 de ce module de cycles est donné par l'homologie de Suslin de X:

$$\hat{h}_{i,0}(X).L = H^{sing}_i(X_L/L)$$

avec les notations de [SV96].(23)

Si X est projectif lisse de dimension pure d, le motif $M(X) = \Sigma^\infty\, \underline{\mathrm{C}}^*_{\mathrm{sing}}\, \mathbb{Z}^{tr}(X)$ dans $DM(k)$ est fortement dualisable avec pour dual fort $M(X)(-d)[-2d]$.(24) Il en résulte que pour tout corps de fonctions L,

$$\hat{h}_{i,n}(X).L = H^{2d+i+n,d+n}_{\mathcal{M}}(X_L), \tag{6.1.b}$$

où $H^{s,t}_{\mathcal{M}}(X_L)$ désigne la cohomologie motivique de X étendue à L en degré s et twist t.

Remarque 6.2. — Supposons k de caractéristique 0. Comme remarqué par B. Kahn dans [Kah10], on obtient un théorème de Merkurjev (*cf.* [Mer08, Th. 2.10]) comme corollaire du théorème 3.7. En effet, pour X propre et lisse, on peut identifier le module de cycles $\hat{h}_{i,*}(X)$ introduit ici avec le module de cycles K^X construit par Merkurjev. On renvoie le lecteur à [Kah10] pour d'autres utilisations des modules de cycles $\hat{h}_{0,*}(X)$.

6.2. Modules de cycles constructibles. — On introduit l'hypothèse suivante sur le corps k:

($\mathcal{M}_k$) Pour tout corps de fonctions E/k, il existe un k-schéma projectif lisse dont le corps des fonctions est k-isomorphe à E.

Cette hypothèse est évidemment une conséquence de la résolution des singularités au sens classique pour k.

(23)Si X est lisse ou si l'on admet la résolution des singularités pour k, on peut montrer que pour tout couple $(i,n) \in \mathbb{Z} \times \mathbb{N}$, le groupe $\hat{h}_{i,n}(X).L$ est le conoyau de la flèche

$$\bigoplus_{i=1}^{n} H^{sing}_i(\mathbb{G}_m^{n-1} \times X_L/L) \to H^{sing}_i(\mathbb{G}_m^n \times X_L/L)$$

induite par la somme des inclusions $\mathbb{G}_m^{n-1} \to \mathbb{G}_m^n$ de la forme $Id \times s_1 \times Id$ où s_1 est la section unité.

(24)C'est une conséquence de [Voe00b, chap. 5, 2.1.4] et de la dualité dans les motifs de Chow. On obtient une démonstration plus directe à l'aide du morphisme de Gysin suivant [Dég08a, 2.16].

Le résultat suivant est bien connu[25]:

Proposition 6.3. — Soit d un entier et $\mathscr{P}_{\leq d}$ la sous-catégorie triangulée de $DM(k)$ engendrée par les motifs de schémas projectifs lisses de dimension inférieure à d.
Soit X un schéma lisse de dimension inférieure à d.

(i) *Si $(\mathcal{M}_k)$ est vérifiée, $M(X)$ appartient à $\mathscr{P}_{\leq d}$.*
(ii) *Le motif rationel $M(X) \otimes \mathbb{Q}$ appartient à $\mathscr{P}_{\leq d} \otimes \mathbb{Q}$.*

On en déduit le résultat suivant:

Proposition 6.4. — Soit X un schéma lisse de dimension d et $(n, i) \in \mathbb{Z}^2$ un couple d'entiers.

(i) *Si X est projectif lisse, $\hat{h}_{i,-n}(X) = 0$ si $n > d$.*
(ii) *Si $(\mathcal{M}_k)$ est vérifiée, $\hat{h}_{i,-n}(X) = 0$ si $n > d$.*
(iii) *Dans tous les cas, $\hat{h}_{i,-n}(X) \otimes \mathbb{Q} = 0$ si $n > d$.*

Démonstration. — Le point (i) est un corollaire de la formule (6.1.b) et du théorème de simplification de Voevodsky car ce dernier affirme qu'il n'y a pas de cohomologie motivique en poids strictement négatif.
Soit $\mathscr{C}_{\leq d}$ la sous-catégorie pleine de $DM(k)$ formée des motifs $\mathcal{M}$ tel que pour tout corps de fonctions E et tout couple $(n, i) \in \mathbb{Z}^2$, $n > d$,

$$\mathrm{Hom}_{DM(k)}(M(E), \mathcal{M}\{-n\}[-i]) = 0.$$

Cette catégorie est une sous-catégorie triangulée. D'après (i), elle contient les motifs $M(P)$ pour P projectif lisse de dimension inférieure à d. La proposition précédente permet donc de conclure. □

Remarque 6.5. — Considérons un schéma algébrique X de dimension d. Sous l'hypothèse de résolution des singularités, on peut trouver un hyper-recouvrement $p : \mathcal{X} \to X$ pour la topologie cdh tel que pour tout entier $n \geq 0$, $\mathcal{X}_n$ est projectif lisse de dimension inférieure à d. Utilisant les techniques de [Voe00a], on peut montrer que le morphisme induit $\mathbb{Z}^{tr}(\mathcal{X}) \to \mathbb{Z}^{tr}(X)$ est un isomorphisme dans $DM^{eff}(k)$. Le point (ii) de la proposition ci-dessus est dès lors valable sans hypothèse de lissité sur X.
Notons que d'après le théorème de De Jong, on peut toujours trouver un hyper-recouvrement p comme ci-dessus pour la h-topologie. D'après [CD07, 10.4.4, 15.1.2], le morphisme $p_* : M(\mathcal{X}) \to M(X)$ est un isomorphisme dans $DM(k) \otimes \mathbb{Q}$. Le point (iii) est donc valable sans hypothèse de lissité.

Définition 6.6. — Nous dirons qu'un module homotopique (resp. module de cycles) est *constructible* s'il appartient à la sous-catégorie épaisse[26] de $HI_*(k)$ (resp. $\mathscr{M}Cycl(k)$) engendrée par les objets $\sigma^\infty h_i(X)\{n\}$ (resp. $\hat{h}_i(X)\{n\}$) pour un schéma lisse X et un couple d'entiers $(n, i) \in \mathbb{Z}^2$.

---

(25) On obtient une preuve très élégante en utilisant un argument dû à J. Riou facilement adapté de la preuve de [Rio05, th. 1.4].
(26) *i.e.* stable par noyau, conoyau, extension, sous-objet et quotient.

Remarque 6.7. — 1. Grâce à la t-structure homotopique, on peut considérer une autre condition de finitude sur les modules homotopiques. Un module homotopique F_* est dit *fortement constructible* s'il est de la forme $\underline{\mathrm{H}}^0_*(\mathbb{E})$ pour un motif géométrique $\mathbb{E}$.[27] Dans ce cas, F_* est constructible dans le sens précédent mais la réciproque n'est pas claire.

2. Les modules homotopiques constructibles ne jouissent pas des propriétés de finitude de leur analogue l-adique. Ainsi, il y a lieu de considérer parallèlement la notion plus forte de module homotopique *de type fini*[28]: F_* est de type fini s'il existe un épimorphisme $\sigma^\infty h_0(X) \to F_*$. Ces subtilités interviennent car le foncteur $\underline{\mathrm{H}}^0$ ne préserve pas la propriété d'être géométrique (*i.e.* compact) – contrairement à son analogue l-adique, le foncteur cohomologique associé à la t-structure canonique, qui préserve la constructibilité.
3. Dans le prolongement de la remarque précédente, notons qu'il est probable que la plupart des modules homotopiques constructibles ne soient pas fortement dualisables. La seule exception que l'on connaisse à cette règle est le cas d'un k-schéma étale X et du module homotopique $\sigma^\infty h_0(X)$. Ce dernier est fortement dualisable dans $HI_*(k)$ (ou même dans $HI(k)$) et il est son propre dual fort.

Corollaire 6.8. — La graduation d'un module de cycles constructible M est bornée inférieurement dès que l'une des deux propriétés suivantes est réalisée:

– *La propriété $(\mathcal{M}_k)$ est satisfaite.*
– *M est sans torsion.*

6.3. Homologie de Borel-Moore. —

6.9. — Pour la proposition qui suit, on suppose l'existence pour tout schéma algébrique X d'un *motif à support compact* $M^c(X)$ dans $DM(k)$ satisfaisant les propriétés suivantes:

(C1) $M^c(X)$ est covariant par rapport aux immersion fermées et contravariant par rapport aux immersions ouvertes.

(C2) Si $i : Z \to X$ est une immersion fermée et $j : U \to X$ l'immersion ouverte complémentaire, il existe un triangle distingué canonique:

$$M^c(Z) \xrightarrow{i_*} M^c(X) \xrightarrow{j^*} M^c(U) \xrightarrow{+1}$$

(C3) Si X est lisse de dimension pure d, $M(X)$ est fortement dualisable avec pour dual fort $M^c(X)\{-d\}[-d]$. De plus, la contravariance de $M^c(X)$ par rapport aux immersions ouvertes correspond par dualité à la covariance naturelle de $M(X)$.

[27] De même, un module de cycles est fortement constructible si le module homotopique associé l'est.

[28] Cette notion, introduite dans la thèse de l'auteur [DÉG02], a été étudiée indépendamment par J. Ayoub dans l'appendice de [HK06].

Si k vérifie la résolution des singularités, Voevodsky a obtenu cette construction dans [Voe00b, par. 4].
Soit ϕ un module de cycles. Rappelons la numérotation homologique du complexe des cycles à coefficients dans ϕ (*cf.* [Ros96, §5]): pour $(n,s) \in \mathbb{Z}^2$, et un schéma algébrique X,

$$C_n(X;\phi)_s = \bigoplus_{x \in X_{(n)}} \phi_{s+n}(\kappa(x))$$

où $X_{(n)}$ désigne l'ensemble des points de dimension n de X.

Proposition 6.10. — Soit ϕ un module de cycles et F_* le module homotopique associé (théorème 3.7).
Alors, utilisant l'hypothèse et les notations qui précèdent, pour tout schéma lissifiable[29] *X et tout couple $(n,s) \in \mathbb{Z}^2$,*

$$A_i(X,\phi)_s \simeq \mathrm{Hom}_{DM(k)}(\mathbb{1}[i], M^c(X) \otimes F_*\{s\}).$$

Démonstration. — La catégorie $DM(k)$ est naturellement munie d'un Hom enrichi en complexes (en tant que localisation d'une catégorie dérivée). On introduit les complexes suivants, gradués par rapport à $s \in \mathbb{Z}$:

$$\begin{aligned} C_*(X)_s &= C_*(X;\phi)_s, \\ D_*(X)_s &= \mathrm{R}\,\mathrm{Hom}(\mathbb{1}, M^c(X) \otimes F_*\{s\}). \end{aligned}$$

Supposons tout d'abord que X est lisse de dimension pure d. On obtient alors un quasi-isomorphisme canonique:

$$\begin{aligned} \epsilon_X : D_*(X)_s &= \mathrm{R}\,\mathrm{Hom}(\mathbb{1}, M^c(X) \otimes F_*\{s\}) \\ &\overset{(1)}{\simeq} \mathrm{R}\,\mathrm{Hom}(M(X), F_*\{s+d\}[d]) \simeq \mathrm{R}\Gamma(X, F_{s+d})[d] \\ &\overset{(2)}{\simeq} C^*(X,\phi)_{s+d}[d] = C_*(X,\phi)_s. \end{aligned}$$

L'isomorphisme (1) résulte de la propriété (C3) ci-dessus et l'isomorphisme (2) est induit par (3.2.a). Cet isomorphisme est naturel en X par rapport aux immersions ouvertes.
Dans le cas général, on peut supposer que X est connexe. Puisque il est lissifiable, il existe un schéma lisse Y, qu'on peut supposer connexe, et une immersion fermée $i : X \to Y$. Soit $j : U \to Y$ l'immersion ouverte complémentaire. Utilisant la propriété (C2) et la suite (2.5.c), on obtient une flèche pointillée dans $D(\mathscr{A}b)$ formant un morphisme de triangles distingués:

(6.10.a)
$$\begin{tikzcd} D_*(X) \arrow[r] \arrow[d, dashed, "\epsilon_X"'] & D_*(Y) \arrow[r] \arrow[d, "\epsilon_Y"] & D_*(U) \arrow[r, "+1"] \arrow[d, "\epsilon_U"] & {} \\ C_*(X) \arrow[r] & C_*(Y) \arrow[r] & C_*(U) \arrow[r, "+1"] & {} \end{tikzcd}$$

[29] *i.e.* il existe une immersion fermée de X dans un schéma lisse

Alors, ϵ_X est un quasi-isomorphisme compatible à la graduation. Il induit l'isomorphisme attendu. □

Remarque 6.11. — Rappelons que suivant [Voe00b, par. 4], $M^c(X)$ est covariant par rapport aux morphismes propres et contravariant par rapport aux morphismes plats équidimensionnels. On peut montrer que l'isomorphisme de la proposition précédente est canonique, contravariant par rapport aux morphismes plats équidimensionnels et covariant par rapport aux morphismes propres, en utilisant les techniques des sections 3.2 et 3.4.
En caratéristique 0, on pourrait aussi utiliser la méthode de *descente par hyper-enveloppes* de [GS96] pour obtenir la proposition précédente, remplaçant le choix d'une lissification par celui d'un hyper-recouvrement cdh – on exploite la fonctorialité *covariante* de $C_*(X)$ et $D_*(X)$. Ceci permet de se débarrasser de l'hypothèse: X lissifiable.

Références

[BBD82] A. A. Beĭlinson, J. Bernstein & P. Deligne – "Faisceaux pervers", Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, p. 5–171.

[Bei02] A. Beilinson – "Remarks on n-motives and correspondences at the generic point", Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), Int. Press Lect. Ser., vol. 3, Int. Press, Somerville, MA, 2002, p. 35–46.

[BO74] S. Bloch & A. Ogus – "Gersten's conjecture and the homology of schemes", *Ann. Sci. École Norm. Sup. (4)* 7 (1974), p. 181–201 (1975).

[CD07] D.-C. Cisinski & F. Déglise – "Triangulated categories of motives", 2007.

[CD09a] ———, "Mixed weil cohomologies", arXiv:0712.3291v2, 2007, revised in 2009.

[CD09b] ———, "Local and stable homological algebra in Grothendieck abelian categories", *HHA* 11 (2009), no. 1, p. 219–260.

[Dég02] F. Déglise – "Modules homotopiques avec transferts et motifs génériques", Thèse, Univ. Paris VII, 2002.

[Dég06] ———, "Transferts sur les groupes de chow à coefficients", *Mathematische Zeitschrift* 252 (2006), no. 2, p. 315–343.

[Dég07] ———, "Correspondences and transfers", *Algebraic cycles and motives* (J. Nagel & P. Chris, éds.), Cambridge university press, April 2007.

[Dég08a] ———, "Around the gysin triangle I", arXiv:0804.2415, avril 2008.

[Dég08b] ———, "Motifs génériques", *Rend. Semin. Mat. Univ. Padova* 119 (2008), p. 173–244.

[Dég09] ———, "Modules homotopiques", arXiv:0904.4747v1, 2009.

[Dég10] ———, "Orientable homotopy modules", arXiv:1005.4187, 2010.

[Gil85] H. Gillet – "Some new Gysin homomorphisms for the Chow homology of varieties", *Proc. London Math. Soc. (3)* 50 (1985), no. 1, p. 57–68.

[GS96] H. Gillet & C. Soulé – "Descent, motives and K-theory", *J. Reine Angew. Math.* 478 (1996), p. 127–176.

[Har66] R. Hartshorne – *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966.

[HK06] A. Huber & B. Kahn – "The slice filtration and mixed Tate motives", *Compos. Math.* 142 (2006), no. 4, p. 907–936.

[Hov01] M. Hovey – "Spectra and symmetric spectra in general model categories", *J. Pure Appl. Alg.* 165 (2001), p. 63–127.

[Kah10] B. Kahn – "Relatively unramified elements in cycle modules", `http://www.math.jussieu.fr/~kahn/preprints/modcyclnr.pdf`, 2010.

[Mer08] A. Merkurjev – "Unramified elements in cycle modules", *J. Lond. Math. Soc. (2)* 78 (2008), no. 1, p. 51–64.

[Rio05] J. Riou – "Dualité de Spanier-Whitehead en géométrie algébrique", *C. R. Math. Acad. Sci. Paris* 340 (2005), no. 6, p. 431–436.

[Ros96] M. Rost – "Chow groups with coefficients", *Doc. Math. J.* (1996), p. 319–393.

[SV96] A. Suslin & V. Voevodsky – "Singular homology of abstract algebraic varieties", *Invent. Math.* 123 (1996), no. 1, p. 61–94.

[SV00] ———, "Bloch-Kato conjecture and motivic cohomology with finite coefficients", The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, p. 117–189.

[Voe00a] V. Voevodsky – "Cohomological theory of presheaves with transfers", Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, p. 87–137.

[Voe00b] ———, "Triangulated categories of motives over a field", Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, p. 188–238.

[Voe02] ———, "Cancellation theorem", preprint, 2002.

F. Déglise
CNRS, LAGA (UMR 7539)
Institut Galilée
Université Paris 13
99 avenue Jean-Baptiste Clément
93430 Villetaneuse
France

The Additivity Theorem in Algebraic K-Theory

Daniel R. Grayson

Received: April 20, 2011

Communicated by Max Karoubi

Abstract. The additivity theorem in algebraic K-theory, due to Quillen and Waldhausen, is a basic tool. In this paper we present a new proof, which proceeds by constructing an explicit homotopy combinatorially.

2010 Mathematics Subject Classification: 19D99
Keywords and Phrases: additivity, algebraic K-theory

Introduction

In this paper[1], we present a new proof of the additivity theorem of Quillen [7, §3, Theorem 2] and Waldhausen [8, 1.3.2(4)]. See also [6] and [5]. Previous proofs used Theorem A or Theorem B of Quillen [7], but this one proceeds by constructing an explicit combinatorial homotopy, which is made possible by suitably subdividing one of the spaces involved.

This research was partially supported by the National Science Foundation under grant NSF DMS 08-10948. I thank Mona Merling for valuable remarks that helped me improve the exposition substantially.

1. The additivity theorem

Let Ord denote the category of finite nonempty ordered sets. We regard a simplicial object in a category $\mathcal{C}$ as a functor $Ord^{\mathrm{op}} \to \mathcal{C}$. For $A \in Ord$ let Δ^A denote the simplicial set it represents. For each $n \in \mathbb{N}$ let $[n]$ denote the ordered set $\{0 < 1 < \cdots < n\} \in Ord$, and let Δ^n denote the simplicial set it represents. Let Δ^A_{top} denote the corresponding topological simplex, consisting of the functions $p : A \to [0, 1]$ that sum to 1; for $A = [n]$ we may also write $p = (p_0, \ldots, p_n)$.

If X is a simplicial set, we let $[A, x, p]$ denote the point of the geometric realization $|X|$ corresponding to $A \in Ord$, $x \in X(A)$, and $p \in \Delta^A_{top}$.

For objects A and B in Ord, let $A * B \in Ord$ denote their concatenation; it is the disjoint union, with the ordering extended so the elements of A are smaller

[1] Permanent ID of this document: 822f09c4645830c51423e431c01cd592; Date: 2011.04.18.

than the elements of B. We make that precise by setting $A * B := (\{0\} \times A) \cup (\{1\} \times B)$, so $(0, a)$ and $(1, b)$ denote typical elements, and the ordering is lexicographic. We do the analogous thing with multiple concatenation, e.g., $A * B * C := (\{0\} \times A) \cup (\{1\} \times B) \cup (\{2\} \times C)$. Given functions $p : A \to \mathbb{R}$ and $q : B \to \mathbb{R}$, we let $p * q : A * B \to \mathbb{R}$ be the function defined by $(0, a) \mapsto p(a)$ and $(1, b) \mapsto q(b)$. An embedding $\Delta^A_{top} \times \Delta^B_{top} \to \Delta^{A*B}_{top}$ is defined by $(p, q) \mapsto (p/2) * (q/2)$.

The reason for using *Ord* in this paper, instead of its full subcategory whose objects are the ordered sets $[n]$, is that it is closed under the concatenation operation $(A, B) \mapsto A * B$ and under various other constructions used later in the paper. Since the two categories are equivalent, nothing essential is changed. Since *Ord* is not a small category, to make the definition of geometric realization of a simplicial set work, one should either replace *Ord* by a small subcategory containing each $[n]$ and closed under the constructions used in this paper, or one should interpret the point $[A, x, p]$ introduced above as the point $[[n], \theta^* x, p\theta]$ where $\theta : [n] \to A$ is the unique isomorphism of its form.

For a simplicial object X, its *two-fold edge-wise subdivision* $sub_2 X$ (see [3, §4], [2], and [1]) is the simplicial object defined by $A \mapsto X(A * A)$. For a simplicial set X, there is a natural homeomorphism $\Psi : |sub_2 X| \xrightarrow{\cong} |X|$ (defined in [3, §4]). It can be defined on each simplex as the affine map that sends each vertex of $|sub_2 X|$ to the midpoint of the corresponding (possibly degenerate) edge of $|X|$. More precisely, it sends a point $[A, x, p] \in |sub_2 X|$ to $[A * A, x, (p/2) * (p/2)] \in |X|$.

The edges of $|sub_2 X|$ that map onto the two parts of each edge of $|X|$ are oriented in the same direction. There is another *edge-wise subdivision* where the edges are oriented in opposite directions, defined by $A \mapsto X(A * A^{\mathrm{op}})$. Subdivision into more parts can be accomplished by adding additional factors of A or A^{op}. Our use of $sub_2 X$ in this paper, rather than one of the other available subdivisions, was based on rough sketches in low dimension of the homotopy Θ produced in Lemma 7 below.

Let $\mathcal{C}$ be a category. Let $Ar\, \mathcal{C}$ denote the category of arrows in $\mathcal{C}$. If f is an arrow of $\mathcal{C}$, let $[f]$ denote the corresponding object of $Ar\, \mathcal{C}$.

As defined in [8, 1.1 and 1.2] a *category with cofibrations and weak equivalences* consists of a category $\mathcal{N}$ equipped with a subcategory $co\mathcal{N}$ of *cofibrations* and a subcategory $w\mathcal{N}$ of *weak equivalences* satisfying five axioms, not repeated here. Its K-theory space is denoted by $K\mathcal{N}$ or $Kw\mathcal{N}$, and is defined as the loop space $\Omega|wS.\mathcal{N}|$, where $wS.\mathcal{N}$ is defined in [8, (1.3)] as follows. Given $A \in Ord$, we regard it as a category in the usual way, and we let $Exact(Ar\, A, \mathcal{N})$ denote the category of functors $N : Ar\, A \to \mathcal{N}$ that are *exact* in the sense that (1) $N[a \to a] = *$ for all $a \in A$, and (2) the sequence

$$N[a \to b] \rightarrowtail N[a \to c] \twoheadrightarrow N[b \to c]$$

is a *cofibration sequence*, for all $a \le b \le c$ in A. (In the presence of condition (1), condition (2) is equivalent to

$$\begin{tikzcd} N[a \rightarrowtail b] \arrow[r] \arrow[d] & N[b \rightarrowtail b] \arrow[d] \\ N[a \to c] \arrow[r] & N[b \to c] \end{tikzcd}$$

being a pushout square.) Then $S.\mathcal{N}$ is the simplicial category that is defined on objects by sending $A \in Ord$ to $Exact(Ar\,A, \mathcal{N})$, and is defined on arrows in the natural way. Since $\mathcal{N}$ is equipped with a category of weak equivalences $w\mathcal{N}$, so is the exact category $Exact(Ar\,A, \mathcal{N})$, as Waldhausen proves, yielding a simplicial category denoted $wS.\mathcal{N}$.

Now suppose F and G are exact functors $\mathcal{M} \to \mathcal{N}$ between categories with cofibrations and weak equivalences. Choose a coproduct operation on $\mathcal{N}$ satisfying the identities $N \vee * = N$ and $* \vee N = N$. We define a map $\Phi = \Phi_{F,G} : sub_2 S.\mathcal{M} \to S.\mathcal{N}$ by $(\Phi M)[a \to b] := FM[(0,a) \to (0,b)] \vee GM[(1,a) \to (1,b)]$; here we have $A \in Ord$, an exact functor $M : Ar(A * A) \to \mathcal{M}$ regarded as an element of $(sub_2 S.\mathcal{M})(A)$, and an arrow $a \to b$ in A. One extends the definition of ΦM from objects to arrows by naturality and checks that it is exact (using the identity $(\Phi M)[a \to a] = * \vee * = *$ and exactness of the coproduct of two cofibration sequences), so Φ is well defined. The idea is that each edge of $S.\mathcal{M}$ gets subdivided into two parts, and we apply F to the first part and G to the second. (The same thing works for two homomorphisms between abelian groups, with $S.$ replaced by the nerve of the group.) Let $sub_2 wS.\mathcal{M}$ denote the simplicial category obtained by applying edge-wise subdivision in the simplicial direction. The functor Φ preserves weak equivalences, because F, G, and sum do, yielding a map $\Phi : sub_2 wS.\mathcal{M} \to wS.\mathcal{N}$ of simplicial categories.

The following definition comes from the text above [8, Proposition 1.3.2].

Definition 1. A sequence $F \rightarrowtail G \twoheadrightarrow H : \mathcal{M} \to \mathcal{N}$ of exact functors between categories with cofibrations and weak equivalences is a *cofibration sequence* if: (1) for all $M \in \mathcal{M}$ the sequence $F(M) \rightarrowtail G(M) \twoheadrightarrow H(M)$ is a cofibration sequence of $\mathcal{M}$; and (2) for any cofibration $M' \rightarrowtail M$ in $\mathcal{M}$ the map $G(M') \cup_{F(M')} F(M) \rightarrowtail G(M)$ is a cofibration in $\mathcal{N}$.

Given a cofibration sequence $F \rightarrowtail G \twoheadrightarrow H$ as in the definition above, the additivity theorem (Theorem 8 below) states that $F \vee H$ and G yield homotopic maps $wS.\mathcal{M} \to wS.\mathcal{N}$. We will prove it by showing first that G and $\Phi_{H,F}$ yield homotopic maps, and then composing two such homotopies. To construct this homotopy we need a new triangulation of the cylinder $[0,1] \times |wS.\mathcal{M}|$ that agrees with that of $|wS.\mathcal{M}|$ at one end and with that of $|sub_2 wS.\mathcal{M}|$ at the other end. Geometrically, it's sort of clear that such a thing should exist, for another description of the triangulation on $|sub_2 X|$ for a simplicial set X, or rather of its bisimplicial variant, is that it comes by intersecting the simplices of $|\Delta^1 \times X| \cong |\Delta^1| \times |X|$ with $\{p\} \times |X|$, where p denotes the midpoint of $|\Delta^1|$. The new triangulation (called IX in Definition 4 below), or rather a bisimplicial variant of it, arises by intersecting the simplices of $|\Delta^2 \times X| \cong |\Delta^2| \times |X|$ with

$\ell \times |X|$, where ℓ is the line segment in $|\Delta^2|$ joining the first vertex with the midpoint of the opposite edge. However, we ignore that interpretation and give a direct construction, as follows.

Definition 2. Given objects A and B of Ord, define $A \ltimes B \in Ord$ to be $A \times B$ equipped with the lexicographic ordering, where $(a,b) \leq (a',b')$ if and only if (1) $a < a'$, or (2) $a = a'$ and $b \leq b'$. (The notation is chosen to suggest that the projection $A \ltimes B \to A$ is an order preserving map, but the projection $A \ltimes B \to B$ is, in general, not.)

Definition 3. Given maps $A \xrightarrow{\sigma} C \xleftarrow{\varphi} B$ in Ord, define $\varphi^{-1}(\sigma) \in Ord$ to be the ordered subset $\{(a,b) \mid \sigma a = \varphi b\} \subseteq A \ltimes B$. (The notation is chosen as a reminder that when σ is injective, then projection to the second factor gives an isomorphism $\varphi^{-1}(\sigma) \xrightarrow{\cong} \varphi^{-1}(\sigma(A)) \subseteq B$. On the other hand, if σ is the map $[n] \to [0]$, then $\varphi^{-1}(\sigma) = B * \ldots * B$, the concatenation of $n+1$ copies of B.)

Definition 4. Let $s : [2] \to [1]$ be the map in Ord defined by $s(0) = 0$, $s(1) = 1$, and $s(2) = 1$. For a simplicial set X we define a simplicial set IX on objects by setting $IX(A) := \{(\varphi, x) \mid \varphi : A \to [1], x \in X(\varphi^{-1}(s))\}$ for $A \in Ord$; its definition on arrows arises from naturality. We point out that $\varphi^{-1}(s) = \varphi^{-1}\{0\} * \varphi^{-1}\{1\} * \varphi^{-1}\{1\}$, so $\varphi^{-1}(s) \cong A$ if $\varphi = 0$, and $\varphi^{-1}(s) \cong A * A$ if $\varphi = 1$. Consequently, the simplicial subset of IX defined by the equation $\varphi = 0$ is isomorphic to X, and the simplicial subset of IX defined by the equation $\varphi = 1$ is isomorphic to $sub_2 X$. We regard those isomorphisms as identifications.

Definition 5. We define a map $\Psi : |IX| \to |\Delta^1| \times |X|$ as follows. The first component $|IX| \to |\Delta^1|$ arises from the simplicial map $IX \to \Delta^1$ defined by $(\varphi, x) \mapsto \varphi$, and thus it sends a point $[A, (\varphi, x), p]$ to the point $[A, \varphi, p]$. The second component $|IX| \to |X|$ is the unique map, affine on each simplex, whose behavior on vertices (each of which has either $\varphi = 0$ or $\varphi = 1$) is that it sends those with $\varphi = 0$ to the corresponding vertex of $|X|$ and those with $\varphi = 1$ to the midpoint of the corresponding (possibly degenerate) edge of $|X|$. More precisely, the map sends a point $[A, (\varphi, x), p] \in |IX|$ to $[\varphi^{-1}(s), x, \varphi \diamond p] \in |X|$, where $\varphi \diamond p \in \Delta_{top}^{\varphi^{-1}(s)}$ is defined by $(0,a) \mapsto p(a)$ for $a \in \varphi^{-1}(0)$, and by $(1,a) \mapsto p(a)/2$ and $(2,a) \mapsto p(a)/2$ for $a \in \varphi^{-1}(1)$. (Writing p' for the restriction of p to $\varphi^{-1}(0)$ and p'' for the restriction of p to $\varphi^{-1}(1)$, we see that $\varphi \diamond p = p' * (p''/2) * (p''/2)$.)

Lemma 6. *For a simplicial set X, the map $\Psi : |IX| \to |\Delta^1| \times |X|$ is a homeomorphism.*

Proof. By commutativity with colimits, we may assume $X = \Delta^n$. The simplicial set IX has only a finite number of nondegenerate simplices, so the source and target of Ψ are compact Hausdorff spaces, and thus it is enough to show that Ψ is a bijection.

To show surjectivity, consider a point $([[1], \beta, q], [[t], x, r])$ in $|\Delta^1| \times |X|$, with r in the interior of Δ_{top}^t. Let $k = q(0)$. We may assume that the partial sums $s_j :=$

$\sum_{i=0}^{j-1} r_i$, for $j = 0, \ldots, t+1$, include k, for if not, then picking j so that $s_j < k < s_{j+1}$, we may construct $r' = (r_0, \ldots, r_{j-1}, k - s_j, s_{j+1} - k, r_{j+1}, \ldots, r_t) \in \Delta^{t+1}_{top}$; its partial sums are those of r, together with k, and there is a surjective map $f : [t+1] \to [t]$ that collapses r' to r. Letting $x' = f^*(x) = x \circ f$ be the corresponding degeneracy of x, we have $[[t], x, r] = [[t+1], x', r']$. Similarly, we may assume that each number w with $k \leq w \leq k + (1-k)/2$ is a partial sum of r if and only if $w + (1-k)/2$ is. Pick b with $s_b = k$ and c with $s_{b+c} = k + (1-k)/2$. Then, due to the symmetry of the partial sums, $r_{b+i} = r_{b+c+i}$ if $0 \leq i < c$, and $b + 2c = t + 1$. In more detail, one deduces the equality as follows: one has $r_{b+i} = s_{b+i+1} - s_{b+i}$, in which s_{b+i+1} and s_{b+i} are adjacent partial sums between k and $k + (1-k)/2$, so by symmetry of the partial sums, $s_{b+i+1} + (1-k)/2 = s_{b+c+i+1}$ and $s_{b+i} + (1-k)/2 = s_{b+c+i}$, hence $r_{b+c+i} = s_{b+c+i+1} - s_{b+c+i} = s_{b+i+1} - s_{b+i} = r_{b+i}$. Now let $p \in \Delta^{b+c-1}_{top}$ be defined by $p = (r_0, \ldots, r_{b-1}, 2r_b, \ldots, 2r_{b+c-1})$, and let $\varphi : [b+c-1] \to [1]$ be defined by $\varphi(i) = 0$ for $0 \leq i < b$ and $\varphi(i) = 1$ for $b \leq i < b + c$. Then $([[1], \beta, q], [[t], x, r]) = \Psi([[b+c-1], (\varphi, x'), p])$, where $x' \in X(\varphi^{-1}(s))$ corresponds to $x \in X([t])$ via the unique isomorphism $\varphi^{-1}(s) \cong [t]$.
To show injectivity, consider a point $[A, (\varphi, x), p] \in |IX|$ where (φ, x) is nondegenerate and p is an interior point of Δ^A_{top}. Observe that x is a function $\varphi^{-1}(s) \to [n]$, and that $\varphi \diamond p$ is an interior point of its simplex. The deterministic procedure described in the previous paragraph recovers A, φ, x, and p, up to isomorphism, from the unique nondegenerate interior representatives of the two components of $\Psi([A, (\varphi, x), p])$, showing injectivity. $\square$

Lemma 7. *Let $F \rightarrowtail G \twoheadrightarrow H : \mathcal{M} \to \mathcal{N}$ be a cofibration sequence of exact functors between categories with cofibrations and weak equivalences. There is a map $\Theta : IwS.\mathcal{M} \to wS.\mathcal{N}$ such that Θ agrees with G on the simplicial subset of $IwS.\mathcal{M}$ where $\varphi = 0$ and with $\Phi_{H,F}$ on the simplicial subset of $IwS.\mathcal{M}$ where $\varphi = 1$.*

Proof. The construction will be natural in the direction of the nerve of the weak equivalences, so we don't explicitly mention the weak equivalences in the rest of the proof. For each object $[M' \xrightarrow{f} M]$ of $Ar\,\mathcal{M}$ we choose a value in $\mathcal{N}$ for

$$P[f] := colim \left(\begin{array}{ccc} F(M') & \xrightarrow{F(f)} & F(M) \\ \big\downarrow & & \\ G(M') & & \end{array} \right).$$

The colimit exists because the vertical map in the diagram is a cofibration, and, in the case where f is a cofibration, is the same as the pushout referred to in part (2) of definition 1. We may ensure $P[f] = *$ if $M' = M = *$. Having made those choices, one defines P on maps in $Ar\,\mathcal{M}$ to get a functor $P : Ar\,\mathcal{M} \to \mathcal{N}$. Recall from [8, Lemma 1.1.1] that the full subcategory $\mathcal{F}_1\mathcal{N}$ of $Ar\,\mathcal{N}$, consisting of the arrows of $\mathcal{N}$ that are cofibrations, is a category with cofibrations, where a *cofibration* $[A \rightarrowtail B] \rightarrowtail [A' \rightarrowtail B']$ is an arrow having the property that both $A \rightarrowtail A'$ and $A \cup_{A'} B \rightarrowtail B'$ are cofibrations; the latter part of the condition ensures that cofibrations are stable under pushout. It follows that P sends each (horizontal) *cofibration sequence*

$$\begin{tikzcd} L' \arrow[r, tail] \arrow[d, "f"] & M' \arrow[r, two heads] \arrow[d, "g"] & N' \arrow[d, "h"] \\ L \arrow[r, tail] & M \arrow[r, two heads] & N \end{tikzcd}$$

of (vertical) maps (in which the rows are cofibration sequences of $\mathcal{M}$) to a cofibration sequence $P[f] \rightarrowtail P[g] \twoheadrightarrow P[h]$ of $\mathcal{N}$. The point is that, according to definition 1, the left vertical map in the pushout diagram

$$\begin{tikzcd} {[FL' \rightarrowtail FM']} \arrow[r] \arrow[d, tail] & {[FL \rightarrowtail FM]} \arrow[d, tail] \\ {[GL' \rightarrowtail GM']} \arrow[r] & {[P[f] \rightarrowtail P[g]]} \end{tikzcd}$$

is a cofibration in $\mathcal{F}_1\mathcal{N}$, that the upper horizontal map is an arrow in $\mathcal{F}_1\mathcal{N}$, and thus that the pushout $[P[f] \rightarrowtail P[g]]$ lies in $\mathcal{F}_1\mathcal{N}$ and is therefore a cofibration. One also sees, using the gluing lemma [8, 1.2: Weq 2], that P sends each (horizontal) *weak equivalence*

$$\begin{tikzcd} L' \arrow[r, "\sim"] \arrow[d, "f"] & M' \arrow[d, "g"] \\ L \arrow[r, "\sim"] & M \end{tikzcd}$$

of (vertical) maps (in which the horizontal maps are weak equivalences of $\mathcal{M}$) to a weak equivalence $P[f] \xrightarrow{\sim} P[g]$ in $w\mathcal{N}$.

We say that P is an *exact functor*, in the sense that it preserves cofibration sequences and weak equivalences, as proved above.

We point out two special cases.

(A) if $f = 1$ is an identity map (or an isomorphism), then there is a natural isomorphism $P[f] \cong G(M')$

(B) if $f = 0$ is a map that factors through $*$, then there is a natural isomorphism $P[f] \cong F(M) \vee H(M')$

Thus, in a precise sense, P includes G and $F \vee H$ as special cases, allowing it to play the lead role in the construction of Θ, which somehow deforms $f = 1$ to $f = 0$ continuously. (This basic idea was also used in [4, (10.3) and (10.4)] to prove a different sort of additivity theorem.)

We define $\Theta : IwS.\mathcal{M} \to wS.\mathcal{N}$ as follows. Given $A \in Ord$ and $(\varphi, M) \in (IwS.\mathcal{M})(A)$, we define $\Theta(\varphi, M) \in (wS.\mathcal{N})(A)$ as follows. Recall from definition 4 that φ is a map $A \to [1]$, that s is a certain map $s : [2] \to [1]$, and that $M \in (wS.\mathcal{M})(\varphi^{-1}(s))$. Introduce maps $d \leq e : [1] \to [2]$ defined by $d(0) = e(0) = 0$, $d(1) = 1$, and $e(1) = 2$; they are the sections of s, and thus, for any $a \in A$, we have $(d\varphi a, a) \in \varphi^{-1}(s)$ and $(e\varphi a, a) \in \varphi^{-1}(s)$. Our task is to define an exact functor $\Theta(\varphi, M) : Ar\, A \to \mathcal{N}$, so given an object $[a \to b]$ in $Ar\, A$, we define an object of $\mathcal{N}$ as follows, introducing the label f for future reference.

$$(\Theta(\varphi, M))[a \to b] := P[M[(d\varphi a, a) \to (d\varphi b, b)] \xrightarrow{f} M[(e\varphi a, a) \to (e\varphi b, b)]]$$

We extend the definition of $\Theta(\varphi, M)$ to arrows by naturality and by pointing out that the construction preserves weak equivalences. Exactness of $\Theta(\varphi, M)$ follows from exactness of M and of P, completing the definition of Θ.
The rest of the statement follows from the following two special cases, which result from the previous ones.

(A) if $\varphi a = \varphi b = 0$ then $f = 1$ is an identity map, and thus there is a natural isomorphism
$$(\Theta(\varphi, M))[a \to b] \cong GM[(0, a) \to (0, b)]$$

(B) if $\varphi a = \varphi b = 1$, then $(d\varphi b, b) = (1, b) < (2, a) = (e\varphi a, a)$, which implies that $f = 0$ (because it factors through the object $M[(1, b) \to (1, b)] = *$), and thus that there is a natural isomorphism
$$(\Theta(\varphi, M))[a \to b] \cong HM[(1, a) \to (1, b)] \vee FM[(2, a) \to (2, b)]$$

□

THEOREM 8 (Additivity, [8, 1.3.2(4)]). *Let $F \rightarrowtail G \twoheadrightarrow H$ be a cofibration sequence of exact functors $\mathcal{M} \to \mathcal{N}$ between categories with cofibrations and weak equivalences. Then $F \vee H$ and G induce homotopic maps $K\mathcal{M} \to K\mathcal{N}$.*

PROOF. Combining lemma 7 and lemma 6 we see that G and $\Phi_{H,F}$ induce homotopic maps $|wS.\mathcal{M}| \to |wS.\mathcal{N}|$. There is a cofibration sequence $F \rightarrowtail F \vee H \twoheadrightarrow H$, so $F \vee H$ and $\Phi_{H,F}$ also induce homotopic maps. Composing the two homotopies (after reversing one of them) yields the result. □

REMARK 9. Waldhausen's Additivity Theorem provides four equivalent formulations of the result, so it is sufficient to prove only the fourth of them, as we do here. Quillen's version [7, §3, Theorem 2] of the additivity theorem was stated for the Q-construction as a homotopy equivalence $(s, q) : Q\mathcal{E} \to Q\mathcal{M} \times Q\mathcal{M}$, where $\mathcal{M}$ is an exact category, and $\mathcal{E}$ is the exact category of short exact sequences $E = (0 \to sE \to tE \to qE \to 0)$ in $\mathcal{M}$. Here $s, q : \mathcal{E} \to \mathcal{M}$ are the exact functors that extract sE and qE from the exact sequence E. Quillen's formulation is analogous to Waldhausen's first formulation [8, 1.3.2(1)] and is implied by it.

References

[1] Herbert Edelsbrunner and Daniel R. Grayson, *Edgewise subdivision of a simplex*, Proceedings of the Fifteenth Annual Symposium on Computational Geometry (Miami Beach, FL, 1999) (New York), ACM, 1999, pp. 24–30 (electronic), links: cv art doi pdf.

[2] ______, *Edgewise subdivision of a simplex*, Discrete Comput. Geom. 24 (2000), no. 4, 707–719, ACM Symposium on Computational Geometry (Miami, FL, 1999), links: cv doi pdf.

[3] Daniel R. Grayson, *Exterior power operations on higher K-theory*, K-Theory 3 (1989), no. 3, 247–260, links: cv art doi pdf.

[4] ______, *Weight filtrations via commuting automorphisms*, K-Theory 9 (1995), no. 2, 139–172, links: cv art doi pre pdf.

[5] Kevin Charles Jones, Youngsoo Kim, Andrea H. Mhoon, Rekha Santhanam, Barry J. Walker, and Daniel R. Grayson, *The additivity theorem in K-theory*, K-Theory 32 (2004), no. 2, 181–191, links: cv art doi pre pdf.

[6] Randy McCarthy, *On fundamental theorems of algebraic K-theory*, Topology 32 (1993), no. 2, 325–328.

[7] Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.

[8] Friedhelm Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318–419, links: homepage pdf djvu pdf-amended djvu-amended.

Daniel R. Grayson
2409 S. Vine St.
Urbana, IL 61801
USA
dan@math.uiuc.edu

Witt Groups of Complex Cellular Varieties

Marcus Zibrowius

Received: December 30, 2010
Revised: May 10, 2011

Communicated by Max Karoubi

Abstract. We show that the Grothendieck-Witt and Witt groups of smooth complex cellular varieties can be identified with their topological KO-groups. As an application, we deduce the values of the Witt groups of all irreducible hermitian symmetric spaces, including smooth complex quadrics, spinor varieties and symplectic Grassmannians.

2010 Mathematics Subject Classification: Primary 11E81; Secondary 19G99, 19L99, 32M15
Keywords and Phrases: Witt groups, KO-theory, projective homogeneous varieties, hermitian symmetric spaces

Contents

Introduction

The purpose of this paper is to demonstrate that the Grothendieck-Witt and Witt groups of complex projective homogeneous varieties can be computed in a purely topological way. That is, we show in Theorem 2.5 how to identify them with the topological KO-groups of these varieties, and we illustrate this with a series of known and new examples.

Our theorem holds more generally for any smooth complex cellular variety. By this we mean a smooth complex variety X with a filtration by closed subvarieties $\emptyset = Z_0 \subset Z_1 \subset Z_2 \cdots \subset Z_N = X$ such that the complement of Z_k in Z_{k+1} is an open "cell" isomorphic to $\mathbb{A}^{n_k}$ for some n_k. Let us put our result into perspective. It is well-known that for such cellular X we have an isomorphism

$$\mathrm{K}_0(X) \xrightarrow{\cong} \mathrm{K}^0(X(\mathbb{C}))$$

between the algebraic K-group of X and the complex K-group of the underlying topological space $X(\mathbb{C})$. In fact, both sides are easy to compute: they decompose as direct sums of the K-groups of the cells, each of which is isomorphic to $\mathbb{Z}$. Such decompositions are characteristic of oriented cohomology theories. Witt groups, however, are strictly non-oriented, and this makes computations much harder. It is true that the Witt groups of complex varieties decompose into copies of $\mathbb{Z}/2$, the Witt group of $\mathbb{C}$, but even in the cellular case there is no general understanding of how many copies to expect.

Nonetheless, we can prove our theorem by an induction over the number of cells of X. The main issue is to define the map from Witt groups to the relevant KO-groups in such a way that it respects various exact sequences. The basic idea is clear: the Witt group $\mathrm{W}^0(X)$ classifies vector bundles equipped with non-degenerate symmetric forms, and in topology symmetric complex vector bundles are in one-to-one correspondence with real vector bundles, classified by $\mathrm{KO}^0(X)$. More precisely, we have two natural maps:

$$\mathrm{GW}^0(X) \to \mathrm{KO}^0(X(\mathbb{C}))$$
$$\mathrm{W}^0(X) \to \frac{\mathrm{KO}^0(X(\mathbb{C}))}{\mathrm{K}^0(X(\mathbb{C}))}$$

Here, $\mathrm{GW}^0(X)$ is the Grothendieck-Witt group of X, and in the second line $\mathrm{K}^0(X)$ is mapped to $\mathrm{KO}^0(X)$ by sending a complex vector bundle to the underlying real bundle. It is possible to extend these maps to shifted groups and groups with support in a concrete and "elementary" way, as was done in [Zib09]. The method advocated here is to rely instead on a result in $\mathbb{A}^1$-homotopy theory: the representability of hermitian K-theory by a spectrum whose complex realization is the usual topological KO-spectrum. Currently, our only reference is a draft paper of Morel [Mor06], but the result is well-known to the experts and a full published account will undoubtedly become available in due course. In the unstable homotopy category at least, the statement is immediate from Schlichting and Tripathi's recent description of a geometric representing space for hermitian K-theory (see Section 1.5).

The structure of the paper is as follows: In the first section we assemble the basic definitions, reviewing some representability results along the way before finally stating in 1.9 the results in $\mathbb{A}^1$-homotopy theory that we ultimately take as our starting point. Our main result, Theorem 2.5, is stated and proved in the second section. Section 3 reviews mostly well-known facts about the Atiyah-Hirzebruch spectral sequence, on which the computations of examples in the final section rely.

1 Preliminaries

1.1 Witt groups and hermitian K-theory

From a modern point of view, the theory of Witt groups represents a K-theoretic approach to the study of quadratic forms. We briefly run through some of the basic definitions.

Recall that the algebraic K-group $\mathrm{K}_0(X)$ of a scheme X can be defined as the free abelian group on isomorphism classes of vector bundles over X modulo the following relation: for any short exact sequence of vector bundles

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

over X we have $[\mathcal{F}] = [\mathcal{E}] + [\mathcal{G}]$ in $\mathrm{K}_0(X)$. In particular, as far as $\mathrm{K}_0(X)$ is concerned, we may pretend that all exact sequences of vector bundles over X split.

Now let $(\mathcal{E}, \epsilon)$ be a symmetric vector bundle, by which we mean a vector bundle $\mathcal{E}$ equipped with a non-degenerate symmetric bilinear form ϵ. We may view ϵ as an isomorphism from $\mathcal{E}$ to its dual bundle $\mathcal{E}^\vee$, in which case its symmetry may be expressed by saying that ϵ and $\epsilon^\vee$ agree under the canonical identification of the double-dual $(\mathcal{E}^\vee)^\vee$ with $\mathcal{E}$. Two symmetric vector bundles $(\mathcal{E}, \epsilon)$ and $(\mathcal{F}, \phi)$ are isometric if there is an isomorphism of vector bundles $i\colon \mathcal{E} \to \mathcal{F}$ compatible with the symmetries, i.e. such that $i^\vee \phi i = \epsilon$. The orthogonal sum of two symmetric bundles has the obvious definition $(\mathcal{E}, \epsilon) \perp (\mathcal{F}, \phi) := (\mathcal{E} \oplus \mathcal{F}, \epsilon \oplus \phi)$. Any vector bundle $\mathcal{E}$ gives rise to a symmetric bundle $H(\mathcal{E}) := (\mathcal{E} \oplus \mathcal{E}^\vee, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, the hyperbolic bundle associated with $\mathcal{E}$. These hyperbolic bundles are the simplest members of a wider class of so-called metabolic bundles: symmetric bundles $(\mathcal{M}, \mu)$ which contain a subbundle $j\colon \mathcal{L} \to \mathcal{M}$ of half their own rank on which μ vanishes. In other words, $(\mathcal{M}, \mu)$ is metabolic if it fits into a short exact sequence of the form

$$0 \to \mathcal{L} \xrightarrow{j} \mathcal{M} \xrightarrow{j^\vee \mu} \mathcal{L}^\vee \to 0$$

The subbundle $\mathcal{L}$ is then called a Lagrangian of $\mathcal{M}$. If the sequence splits, $(\mathcal{M}, \mu)$ is isometric to $H(\mathcal{L})$, at least in any characteristic other than two. This motivates the definition of the Grothendieck-Witt group.

Definition 1.1 ([Wal03a, Sch10a]). The Grothendieck-Witt group $\mathrm{GW}^0(X)$ of a scheme X is the free abelian group on isometry classes of symmetric vector bundles over X modulo the following two relations:

- $[(\mathcal{E}, \epsilon) \perp (\mathcal{G}, \gamma)] = [(\mathcal{E}, \epsilon)] + [(\mathcal{G}, \gamma)]$
- $[(M, \mu)] = [H(\mathcal{L})]$ for any metabolic bundle (M, μ) with Lagrangian $\mathcal{L}$

The Witt group $\mathrm{W}^0(X)$ is defined similarly, except that the second relation reads $[(M, \mu)] = 0$. Equivalently, we may define $\mathrm{W}^0(X)$ by the exact sequence

$$\mathrm{K}_0(X) \xrightarrow{H} \mathrm{GW}^0(X) \longrightarrow \mathrm{W}^0(X) \to 0$$

Shifted Witt groups. The groups above can be defined more generally in the context of exact or triangulated categories with dualities. The previous definitions are then recovered by considering the category of vector bundles over X or its bounded derived category. However, the abstract point of view allows for greater flexibility. In particular, a number of useful variants of Witt groups can be introduced by passing to related categories or dualities. For example, if we take a line bundle $\mathcal{L}$ over X and replace the usual duality $\mathcal{E}^\vee := \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ on vector bundles by $\mathcal{H}om(-, \mathcal{L})$ we obtain "twisted" Witt groups $\mathrm{W}^0(X; \mathcal{L})$. On the bounded derived category, we can consider dualities that involve shifting complexes, leading to the definition of "shifted" Witt groups $\mathrm{W}^i(X)$. This approach, pioneered by Paul Balmer in [Bal00, Bal01a], elevates the theory of Witt groups into the realm of cohomology theories. We illustrate the meaning and significance of these remarks with a few of the key properties of the theory, concentrating on the case when X is a smooth scheme over a field of characteristic not equal to two. The interested but unacquainted reader may prefer to consult [Bal01b] or [Bal05].

- For any line bundle $\mathcal{L}$ over X and any integer i, we have a Witt group

  $$\mathrm{W}^i(X; \mathcal{L})$$

  This is the i^{th} Witt group of X "with coefficients in $\mathcal{L}$", or "twisted by $\mathcal{L}$". When $\mathcal{L}$ is trivial it is frequently dropped from the notation.

- The Witt groups are four-periodic in i and "two-periodic in $\mathcal{L}$" in the sense that, for any i and any line bundles $\mathcal{L}$ and $\mathcal{M}$ over X, we have canonical isomorphisms

$$\mathrm{W}^i(X;\mathcal{L}) \cong \mathrm{W}^{i+4}(X;\mathcal{L})$$
$$\mathrm{W}^i(X;\mathcal{L}) \cong \mathrm{W}^i(X;\mathcal{L}\otimes\mathcal{M}^{\otimes 2})$$

- More generally, for any closed subset Z of X we have Witt groups "with support on Z", written $\mathrm{W}^i_Z(X;\mathcal{L})$. For $Z = X$ these agree with $\mathrm{W}^i(X;\mathcal{L})$.
- We have long exact "localization sequences" relating the Witt groups of X and $X - Z$, which can be arranged as 12-term exact loops by periodicity.

Balmer's approach already works on the level of Grothendieck-Witt groups, as shown in [Wal03a]. In this context, the localization sequences take the form

$$\begin{aligned}&\mathrm{GW}^i_Z(X) \to \mathrm{GW}^i(X) \to \mathrm{GW}^i(X-Z) \to \\ &\quad \mathrm{W}^{i+1}_Z(X) \to \mathrm{W}^{i+1}(X) \to \mathrm{W}^{i+1}(X-Z) \to \mathrm{W}^{i+2}_Z(X) \to \cdots\end{aligned} \tag{1}$$

continuing to the right with shifted Witt groups of X, and similarly for arbitrary twists $\mathcal{L}$ [Wal03a, Theorem 2.4]. However, if one wishes to continue the sequences to the left, one has to revert to the methods of higher algebraic K-theory.

Hermitian K-theory. Recall that the higher algebraic K-groups of a scheme X can be defined as the homotopy groups of a topological space $\mathrm{K}(X)$ associated with X. If one replaces $\mathrm{K}(X)$ by an appropriate spectrum one can similarly define groups $\mathrm{K}_n(X)$ in all degrees $n \in \mathbb{Z}$. On a smooth scheme X, however, the groups in negative degrees vanish.
An analogous construction of hermitian K-theory is developed in [Sch10b]. Given a scheme X and a line bundle $\mathcal{L}$ over X, Schlichting constructs a family of spectra $\mathbb{G}\mathrm{W}^i(X;\mathcal{L})$ from which hermitian K-groups can be defined as

$$\mathrm{GW}^i_n(X;\mathcal{L}) := \pi_n(\mathbb{G}\mathrm{W}^i(X;\mathcal{L}))$$

In degree $n = 0$, one recovers Balmer and Walter's Grothendieck-Witt groups, and the Witt groups appear as hermitian K-groups in negative degrees. To be precise, for any smooth scheme X over a field of characteristic not equal to two one has the following natural identifications:

$$\mathrm{GW}^i_0(X;\mathcal{L}) \cong \mathrm{GW}^i(X;\mathcal{L}) \tag{2}$$
$$\mathrm{GW}^i_n(X;\mathcal{L}) \cong \mathrm{W}^{i-n}(X;\mathcal{L}) \text{ for } n < 0 \tag{3}$$

For affine varieties, the identifications of Witt groups may be found in [Hor05]: see Proposition A.4 and Corollary A.5. For a general smooth scheme X, we can pass to a vector bundle torsor T over X such that T is affine [Jou73, Lemma 1.5;

Hor05, Lemma 2.1].[1] Both Balmer's Witt groups and Schlichting's hermitian K-groups are homotopy invariant in the sense that the groups of T may naturally be identified with those of X. This is proved for Witt groups in [Gil03, Corollary 4.2] and may be deduced for hermitian K-theory from the Mayer-Vietoris sequences established in [Sch10b, Theorem 1]. The identifications also hold more generally for hermitian K-groups with support $\mathrm{GW}^i_{n,Z}(X)$ [Sch]. They will be used implicitly throughout.
For completeness, we mention how the 4-periodic notation used here translates into the traditional notation in terms of KO- and U-theory, as used for example in [Hor05]. Namely, we have

$$\mathrm{GW}^i_n(X) = \begin{cases} \mathrm{KO}_n(X) & \text{for } i \equiv 0 \mod 4 \\ \mathrm{U}_n(X) & \text{for } i \equiv -1 \\ {}_{-}\mathrm{KO}_n(X) & \text{for } i \equiv -2 \\ {}_{-}\mathrm{U}_n(X) & \text{for } i \equiv -3 \end{cases}$$

(This notation will not be used elsewhere in this paper.)

1.2 KO-theory

We now turn to the corresponding theories in topology. To ensure that the definitions given here are consistent with the literature, we restrict our attention to finite-dimensional CW complexes.[2] Since we are ultimately only interested in topological spaces that arise as complex varieties, this is not a problem. The definitions of K_0 and GW^0 given above applied to complex vector bundles over a finite-dimensional CW complex X yield its complex and real topological K-groups $\mathrm{K}^0(X)$ and $\mathrm{KO}^0(X)$. Since short exact sequences of vector bundles over X always split, the definitions may even be simplified:

Definition 1.2. For a finite-dimensional CW complex X, the complex K-group $\mathrm{K}^0(X)$ is the free abelian group on isomorphism classes of complex vector bundles over X modulo the relation $[\mathcal{E}\oplus\mathcal{G}] = [\mathcal{E}]+[\mathcal{G}]$. Likewise, the KO-group $\mathrm{KO}^0(X)$ is the free abelian group on isometry classes of symmetric complex vector bundles over X modulo the relation $[(\mathcal{E},\epsilon) \perp (\mathcal{G},\gamma)] = [(\mathcal{E},\epsilon)] + [(\mathcal{G},\gamma)]$.

There is a more common description of $\mathrm{KO}^0(X)$ as the K-group of real vector bundles. The equivalence with the definition given here can be traced back to the fact that the orthogonal group $\mathrm{O}(n)$ is a maximal compact subgroup of both $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{O}_n(\mathbb{C})$, but also seen very concretely along the following lines. We say that a complex bilinear form ϵ on a real vector bundle $\mathcal{F}$ is real if $\epsilon\colon \mathcal{F}\otimes\mathcal{F}\to\mathbb{C}$ factors through $\mathbb{R}$.

[1] This step is known as Jouanolou's trick.
[2] The key property we need is that any vector bundle over a finite-dimensional CW complex has a stable inverse. See the proof of Theorem 1.5.

Lemma 1.3. *Let $(\mathcal{E},\epsilon)$ be a symmetric complex vector bundle. There exists a unique real subbundle $\Re(\mathcal{E},\epsilon)\subset\mathcal{E}$ such that $\Re(\mathcal{E},\epsilon)\otimes_{\mathbb{R}}\mathbb{C}=\mathcal{E}$ and such that the restriction of ϵ to $\Re(\mathcal{E},\epsilon)$ is real and positive definite. Concretely, a fibre of $\Re(\mathcal{E},\epsilon)$ is given by the real span of any orthonormal basis of the corresponding fibre of $\mathcal{E}$.*

Corollary 1.4. *For any CW complex X, the monoid of isomorphism classes of real vector bundles over X is isomorphic to the monoid of isometry classes of symmetric complex vector bundles over X (with respect to the operations $\oplus$ and $\perp$, respectively).*

Proof of Lemma 1.3. In the case of a vector bundle over a point we may assume without loss of generality that

$$(\mathcal{E},\epsilon)=\left(\mathbb{C}^r,\begin{pmatrix}1&&0\\&\ddots&\\0&&1\end{pmatrix}\right)$$

Clearly, the subspace $\mathbb{R}^r\subset\mathbb{C}^r$ has the required properties. Uniqueness follows from elementary linear algebra. If $(\mathcal{E},\epsilon)$ is an arbitrary symmetric complex vector bundle over a space X, then any point of X has some neighbourhood over which $(\mathcal{E},\epsilon)$ can be trivialized in the form above. We know how to define $\Re(\mathcal{E},\epsilon)$ over each such neighbourhood, and by uniqueness these local bundles can be glued together. □

Proof of Corollary 1.4. A map in one direction is given by sending a symmetric complex vector bundle $(\mathcal{E},\epsilon)$ to $\Re(\mathcal{E},\epsilon)$. Conversely, with any real vector bundle $\mathcal{E}$ over X we may associate a symmetric complex vector bundle $(\mathcal{E}\otimes_{\mathbb{R}}\mathbb{C},\sigma_{\mathbb{C}})$, where $\sigma_{\mathbb{C}}$ is the $\mathbb{C}$-linear extension of some inner product σ on $\mathcal{E}$. Since σ is defined uniquely up to isometry, so is $(\mathcal{E}\otimes_{\mathbb{R}}\mathbb{C},\sigma_{\mathbb{C}})$. See [MH73, Chapter V, § 2] for a proof that avoids the uniqueness part of the preceding lemma. □

Representing topological K-groups. A standard construction of the cohomology theories associated with K^0 and KO^0 is based on the fact that these functors are representable in the homotopy category $\mathcal{H}$ of topological spaces. The starting point is the homotopy classification of vector bundles: Let us write $\mathrm{Gr}_{r,n}$ for the Grassmannian $\mathrm{Gr}(r,\mathbb{C}^{r+n})$ of complex r-bundles in $\mathbb{C}^{r+n}$, and let Gr_r be the union of $\mathrm{Gr}_{r,n}\subset\mathrm{Gr}_{r,n+1}\subset\cdots$ under the obvious inclusions. Denote by $\mathcal{U}_{r,n}$ and $\mathcal{U}_r$ the universal r-bundles over these spaces. For any connected paracompact Hausdorff space X we have a one-to-one correspondence between the set $\mathrm{Vect}_r(X)$ of isomorphism classes of rank r complex vector bundles over X and homotopy classes of maps from X to Gr_r: a homotopy class $[f]$ in $\mathcal{H}(X,\mathrm{Gr}_r)$ corresponds to the pullback of $\mathcal{U}_r$ along f [Hus94, Chapter 3, Theorem 7.2].

To describe $\mathrm{K}^0(X)$, we need to pass to Gr, the union of the Gr_r under the embeddings $\mathrm{Gr}_r\hookrightarrow\mathrm{Gr}_{r+1}$ that send a complex r-plane W to $\mathbb{C}\oplus W$.

Theorem 1.5. *For finite-dimensional CW complexes X we have natural isomorphisms*

$$\mathrm{K}^0(X) \cong \mathcal{H}(X, \mathbb{Z} \times \mathrm{Gr}) \tag{4}$$

such that, for $X = \mathrm{Gr}_{r,n}$, the class $[\mathcal{U}_{r,n}] + (d-r)[\mathbb{C}]$ in $\mathrm{K}^0(\mathrm{Gr}_{r,n})$ corresponds to the inclusion $\mathrm{Gr}_{r,n} \hookrightarrow \{d\} \times \mathrm{Gr}_{r,n} \hookrightarrow \mathbb{Z} \times \mathrm{Gr}$.

Proof. The theorem is of course well-known, see for example [Ada95, page 204]. To deduce it from the homotopy classification of vector bundles, we note first that any CW complex is paracompact and Hausdorff [Hat09, Proposition 1.20]. Moreover, we may assume that X is connected. The product $\mathbb{Z} \times \mathrm{Gr}$ can be viewed as the colimit of the inductive system

$$\coprod_{d\geq 0} \{d\} \times \mathrm{Gr}_d \hookrightarrow \coprod_{d\geq -1} \{d\} \times \mathrm{Gr}_{d+1} \hookrightarrow \coprod_{d\geq -2} \{d\} \times \mathrm{Gr}_{d+2} \hookrightarrow \cdots \subset \mathbb{Z} \times \mathrm{Gr}$$

Any continuous map from X to $\mathbb{Z} \times \mathrm{Gr}$ factors though one of the components $\mathrm{colim}_n(\{d\} \times \mathrm{Gr}_n)$. By cellular approximation, it is in fact homotopic to a map that factors through $\{d\} \times \mathrm{Gr}_n$ for some n. Thus,

$$\mathcal{H}(X, \mathbb{Z} \times \mathrm{Gr}) \cong \coprod_{d\in\mathbb{Z}} \mathrm{colim}_n \, \mathrm{Vect}_n(X)$$

where the colimit is taken over the maps $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$ sending a vector bundle $\mathcal{E}$ to $\mathbb{C} \oplus \mathcal{E}$. We define a map from the coproduct to $\mathrm{K}^0(X)$ by sending a vector bundle $\mathcal{E}$ in the d^{th} component to the class $[\mathcal{E}] + (d - \mathrm{rank}\,\mathcal{E})[\mathbb{C}]$ in $\mathrm{K}^0(X)$. To see that this is a bijection, we use the fact that every vector bundle $\mathcal{E}$ over a finite-dimensional CW complex has a stable inverse: a vector bundle $\mathcal{E}^\perp$ over X such that $\mathcal{E} \oplus \mathcal{E}^\perp$ is a trivial bundle [Hus94, Chapter 3, Proposition 5.8]. □

If we replace the complex Grassmannians by real Grassmannians $\mathbb{R}\mathrm{Gr}_{r,n}$, we obtain the analogous statement that KO^0 can be represented by $\mathbb{Z} \times \mathbb{R}\mathrm{Gr}$. Equivalently, but more in the spirit of Definition 1.2, we could work with the following spaces:

Definition 1.6. Let (V, ν) be a symmetric complex vector space, and let $\mathrm{Gr}(r, V)$ be the Grassmannian of complex k-planes in V. The "non-degenerate Grassmannian"

$$\mathrm{Gr}^{\mathrm{nd}}(r, (V, \nu))$$

is the open subspace of $\mathrm{Gr}(r, V)$ given by r-planes T for which the restriction $\nu|_T$ is non-degenerate.

Complexification induces an inclusion of $\mathbb{R}\mathrm{Gr}(k, \Re(V, \nu))$ into $\mathrm{Gr}^{\mathrm{nd}}(r, (V, \nu))$, which, by Lemma 1.7 below, is a homotopy equivalence. So let $\mathrm{Gr}^{\mathrm{nd}}_{r,n}$ abbreviate $\mathrm{Gr}^{\mathrm{nd}}(r, \mathbb{H}^{r+n})$, where $\mathbb{H}$ is the hyperbolic plane $(\mathbb{C}^2, (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))$, and let $\mathcal{U}^{\mathrm{nd}}_{r,n}$ denote the restriction of the universal bundle over $\mathrm{Gr}(r, \mathbb{C}^{2r+2n})$ to $\mathrm{Gr}^{\mathrm{nd}}_{r,n}$. Then colimits $\mathrm{Gr}_r^{\mathrm{nd}}$ and $\mathrm{Gr}^{\mathrm{nd}}$ can be defined in the same way as for the usual Grassmannians, and, for finite-dimensional CW complexes X, we obtain natural isomorphisms

$$\mathrm{KO}^0(X) \cong \mathcal{H}(X, \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}) \tag{5}$$

Here, for even $(d-r)$, the inclusion $\mathrm{Gr}_{r,n}^{\mathrm{nd}} \hookrightarrow \{d\}\times\mathrm{Gr}_{r,n}^{\mathrm{nd}} \hookrightarrow \mathbb{Z}\times\mathrm{Gr}^{\mathrm{nd}}$ corresponds to the class of $[\mathcal{U}_{r,n}^{\mathrm{nd}}] + \frac{d-r}{2}[\mathbb{H}]$ in $\mathrm{GW}^0(\mathrm{Gr}_{r,n}^{\mathrm{nd}})$.

Lemma 1.7. *For any symmetric complex vector space (V,ν), the following inclusion is a homotopy equivalence:*

$$\begin{aligned} \mathbb{R}\mathrm{Gr}(k, \Re(V,\nu)) &\overset{j}{\hookrightarrow} \mathrm{Gr}^{nd}(k,(V,\nu)) \\ U &\mapsto U \otimes_{\mathbb{R}} \mathbb{C} \end{aligned}$$

Proof. Consider the projection $\pi\colon V = \Re(V,\nu) \oplus i\Re(V,\nu) \twoheadrightarrow \Re(V,\nu)$. We define a retract r of j by sending a complex k-plane $T \in \mathrm{Gr}^{\mathrm{nd}}(k,(V,\nu))$ to $\pi(\Re(T,\nu|_T)) \subset \Re(V,\nu)$. This is indeed a linear subspace of real dimension k: since ν is positive definite on $\Re(T,\nu|_T)$ but negative definite on $i\Re(V,\nu)$, the intersection $\Re(T,\nu|_T) \cap i\Re(V,\nu)$ is trivial.
More generally, we can define a family of endomorphisms of V parametrized by $t \in [0,1]$ by

$$\begin{aligned} \pi_t\colon \Re(V,\nu) \oplus i\Re(V,\nu) &\twoheadrightarrow \Re(V,\nu) \oplus i\Re(V,\nu) \\ (x,y) &\mapsto (x,ty) \end{aligned}$$

This family interpolates between the identity π_1 and the projection π_0, which we can identify with π. We claim that

$$\pi_t(\Re(T,\nu|_T)) \subset V$$

is a real linear subspace of dimension k on which ν is real and positive definite. The claim concerning the dimension has already been verified in the case $t=0$ and follows for non-zero t from the fact that π_t is an isomorphism. Now take a non-zero vector $v \in \pi_t(\Re(T,\nu|_T))$ and write it as $v = x + tiy$, where $x, y \in \Re(V,\nu)$ and $x + iy \in \Re(T,\nu|_T)$. Since $\nu(x,x)$, $\nu(y,y)$ and $\nu(x+iy,x+iy)$ are all real we deduce that $\nu(x,y)=0$; it follows that $\nu(v,v)$ is real as well. Moreover, since $\nu(x+iy,x+iy)$ is positive we have $\nu(x,x) > \nu(y,y)$, so that $\nu(v,v) > (1-t^2)\nu(y,y)$. In particular, $\nu(v,v) > 0$ for all $t \in [0,1]$, as claimed. It follows that $T \mapsto \pi_t(\Re(T,\nu|_T)) \otimes_{\mathbb{R}} \mathbb{C}$ defines a homotopy from $j \circ r$ to the identity on $\mathrm{Gr}^{\mathrm{nd}}(k,(V,\nu))$. □

K-spectra and cohomology theories. The infinite Grassmannian Gr can be identified with the classifying space BU of the infinite unitary group. Consequently, K^0 can be represented by $\mathbb{Z} \times \mathrm{BU}$, which by Bott periodicity is equivalent to its own two-fold loopspace $\Omega^2(\mathbb{Z} \times \mathrm{BU})$. This can be used to construct a 2-periodic Ω-spectrum $\mathbb{K}^{\mathrm{top}}$ in the stable homotopy category $\mathcal{SH}$ whose even terms are all given by $\mathbb{Z} \times \mathrm{BU}$. Similarly, $\mathbb{R}\mathrm{Gr}$ is equivalent to the classifying space BO of the infinite orthogonal group, and Bott periodicity in this case says that $\mathbb{Z} \times \mathrm{BO}$ is equivalent to $\Omega^8(\mathbb{Z} \times \mathrm{BO})$. Thus, one obtains a spectrum $\mathbb{K}\mathbf{O}^{\mathrm{top}}$ in $\mathcal{SH}$ which is 8-periodic. The associated cohomology theories are given by

$$\mathrm{K}^i(X) := \mathcal{SH}(\Sigma^\infty(X_+), S^i \wedge \mathbb{K}^{\mathrm{top}})$$
$$\mathrm{KO}^i(X) := \mathcal{SH}(\Sigma^\infty(X_+), S^i \wedge \mathbb{K}\mathbf{O}^{\mathrm{top}})$$

where X_+ denotes the union of X and a disjoint base point, and Σ^∞ is the functor assigning to a pointed space its suspension spectrum. We refer the reader to [Ada95, III.2] for background and details.
For convenience and later reference, we include here the values of the theories on a point. Since we are in fact dealing with multiplicative theories, these can be summarized in the form of coefficient rings:

$$\mathrm{K}^*(\mathrm{point}) = \mathbb{Z}\big[g, g^{-1}\big] \tag{6}$$
$$\mathrm{KO}^*(\mathrm{point}) = \mathbb{Z}\big[\eta, \alpha, \lambda, \lambda^{-1}\big]\Big/(2\eta,\ \eta^3,\ \eta\alpha,\ \alpha^2 - 4\lambda) \tag{7}$$

where g is of degree -2 and η, α and λ have degrees -1, -4 and -8, respectively [Bot69, pages 66–74[3]].

1.3 Comparison

Now suppose X is a smooth complex variety. We write $X(\mathbb{C})$ for the set of complex points of X equipped with the analytic topology. If $\mathcal{E}$ is a vector bundle over X then $\mathcal{E}(\mathbb{C})$ has the structure of a complex vector bundle over $X(\mathbb{C})$, so that we obtain natural maps

$$\mathrm{K}_0(X) \to \mathrm{K}^0(X(\mathbb{C})) \tag{8}$$
$$\mathrm{GW}^0(X) \to \mathrm{KO}^0(X(\mathbb{C})) \tag{9}$$

and an induced map

$$\mathrm{W}^0(X) \to \frac{\mathrm{KO}^0(X(\mathbb{C}))}{\mathrm{K}^0(X(\mathbb{C}))} \tag{10}$$

We now wish to extend these maps to be defined on $\mathrm{GW}^i(X)$ and $\mathrm{W}^i(X)$ for arbitrary i, and also on groups with support and twisted groups. Let us comment on some "elementary" constructions that are possible before outlining the approach that we will ultimately follow here.

[3] The multiplicative relations among the generators are given on page 74, but unfortunately the relation $\eta\alpha = 0$ is missing. This omission seems to have pervaded much of the literature, and I am indebted to Ian Grojnowski for pointing out the same mistake in an earlier version of this paper. Of course, the relation follows from the fact that $\mathrm{KO}^{-5}(\mathrm{point}) = 0$.

Firstly, one way to extend the maps to the groups $\mathrm{GW}^i(X)$ and $\mathrm{W}^i(X)$ is to use the multiplicative structure of the theories together with Walter's results on projective bundles [Wal03b]. Namely, for any variety X one has isomorphisms

$$\mathrm{GW}^i(X \times \mathbb{P}^1) \cong \mathrm{GW}^i(X) \oplus \mathrm{GW}^{i-1}(X)$$
$$\mathrm{KO}^{2i}(X(\mathbb{C}) \times S^2) \cong \mathrm{KO}^{2i}(X(\mathbb{C})) \oplus \mathrm{KO}^{2i-2}(X(\mathbb{C}))$$

This allows an inductive definition of comparison maps, at least for all negative i. Basic properties of these maps, for example compatibility with the periodicities of Grothendieck-Witt and KO-groups, can be checked by direct calculations.

It is less clear how to obtain maps on Witt groups with restricted supports. One possibility, pursued in [Zib09], is to work on the level of complexes of vector bundles and adapt a construction of classes in relative K-groups described in [Seg68] to the case of KO-theory. However, it remains unclear to the author how to see in this approach that the resulting maps are compatible with the boundary morphisms in localization sequences.

Theorem 2.5 below could in fact be proved without knowing that the comparison maps respect the boundary morphisms in localization sequences in general. However, $\mathbb{A}^1$-homotopy theory provides an alternative construction of a comparison map for which this property immediately follows from the construction, and which in any case is so compellingly elegant that it would be difficult to argue in favour of any other approach.

1.4 $\mathbb{A}^1$-homotopy theory

Theorem 1.5 describing K^0 in terms of homotopy classes of maps to Grassmannians has an analogue in algebraic geometry, in the context of $\mathbb{A}^1$-homotopy theory. Developed mainly by Morel and Voevodsky, the theory provides a general framework for a homotopy theory of schemes emulating the situation for topological spaces. The authoritative reference is [MV99]; closely related texts by the same authors are [Voe98], [Mor99] and [Mor04]. See [DLØ$^+$07] for a textbook introduction and [Dug01] for an enlightening perspective on one of the main ideas.

We summarize the main points relevant for us in just a few sentences. The category Sm_k of smooth schemes over a field k can be embedded into some larger category of "spaces" Spc_k which is closed under small limits and colimits, and which can be equipped with a model structure. The $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ over k is the homotopy category associated with this model category.

In fact, there are several possible choices for Spc_k and many possible model structures yielding the same homotopy category $\mathcal{H}(k)$. One possibility is to consider the category of simplicial presheaves over Sm_k, or the category of simplicial sheaves with respect to the Nisnevich topology. Both categories contain Sm_k as full subcategories via the Yoneda embedding, and they also contain simplicial sets viewed as constant (pre)sheaves. One may thus apply

a general recipe for equipping the category of simplicial (pre)sheaves over a site with a model structure (see [Jar87]). In a crucial last step, one forces the affine line $\mathbb{A}^1$ to become contractible by localizing with respect to the set of all projections $\mathbb{A}^1 \times X \twoheadrightarrow X$.
As in topology, we also have a pointed version $\mathcal{H}_\bullet(k)$ of $\mathcal{H}(k)$. Remarkably, these categories contain several distinct "circles": the simplicial circle S^1, the "Tate circle" $\mathbb{G}_m = \mathbb{A}^1 - 0$ (pointed at 1) and the projective line $\mathbb{P}^1$ (pointed at ∞). These are related by the intriguing formula $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$. A common notational convention which we will follow is to define

$$S^{p,q} := (S^1)^{\wedge(p-q)} \wedge \mathbb{G}_m^{\wedge q}$$

for any $p \geq q$. In particular, we then have $S^1 = S^{1,0}$, $\mathbb{G}_m = S^{1,1}$ and $\mathbb{P}^1 = S^{2,1}$.
One can take the theory one step further by passing to the stable homotopy category $\mathcal{SH}(k)$, a triangulated category in which the suspension functors $S^{p,q} \wedge -$ become invertible. This category is usually constructed using $\mathbb{P}^1$-spectra. The triangulated shift functor is given by suspension with the simplicial sphere $S^{1,0}$.
Finally and crucially, the analogy with topology can be made precise: when we take our ground field k to be the complex numbers, or more general any subfield of $\mathbb{C}$, we have a complex realization functor

$$\mathcal{H}(k) \to \mathcal{H} \tag{11}$$

that sends a smooth scheme X to its set of complex points $X(\mathbb{C})$ equipped with the analytic topology. There is also a pointed realization functor and, moreover, a triangulated functor of the stable homotopy categories

$$\mathcal{SH}(k) \to \mathcal{SH} \tag{12}$$

which takes $\Sigma^\infty(X_+)$ to $\Sigma^\infty(X(\mathbb{C})_+)$ for any smooth scheme X [Rio06, Théorème I.123; Rio07a, Théorème 5.26].

1.5 Representing algebraic and hermitian K-theory

Grassmannians of r-planes in k^{n+r} can be constructed as smooth projective varieties over any field k. Viewing them as objects in Spc_k, we can form their colimits Gr_r and Gr in the same way as in topology. The following analogue of Theorem 1.5 is established in [MV99, § 4]; see Théorème III.3 and Assertion III.4 in [Rio06].

Theorem 1.8. *For smooth schemes X over k we have natural isomorphisms*

$$\mathrm{K}_0(X) \cong \mathcal{H}(k)(X, \mathbb{Z} \times \mathrm{Gr}) \tag{13}$$

such that the inclusion of $\mathrm{Gr}_{r,n} \hookrightarrow \{d\} \times \mathrm{Gr}_{r,n} \hookrightarrow \mathbb{Z} \times \mathrm{Gr}$ corresponds to the class $[\mathcal{U}_{r,n}] + (d-r)[\mathcal{O}]$ in $\mathrm{K}_0(\mathrm{Gr}_{r,n})$.

An analogous result for hermitian K-theory has recently been obtained by Schlichting and Tripathi[4]: Let $\mathrm{Gr}^{\mathrm{nd}}_{r,n}$ denote the "non-degenerate Grassmannians" defined as open subvarieties of $\mathrm{Gr}_{r,r+2n}$ as above, and let $\mathrm{Gr}^{\mathrm{nd}}_{r}$ and $\mathrm{Gr}^{\mathrm{nd}}$ be the respective colimits. Then for smooth schemes over k we have natural isomorphisms

$$\mathrm{GW}^0(X) \cong \mathcal{H}(k)(X, \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}) \tag{14}$$

It follows from the construction that, when $(d-r)$ is even, the inclusion of $\mathrm{Gr}^{\mathrm{nd}}_{r,n} \hookrightarrow \{d\} \times \mathrm{Gr}^{\mathrm{nd}}_{r,n} \hookrightarrow \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}$ corresponds to the class of $[\mathcal{U}^{\mathrm{nd}}_{r,n}] + \frac{d-r}{2}[\mathbb{H}]$ in $\mathrm{GW}^0(\mathrm{Gr}^{\mathrm{nd}}_{r,n})$, where $\mathcal{U}^{\mathrm{nd}}_{r,n}$ is the universal symmetric bundle over $\mathrm{Gr}^{\mathrm{nd}}_{r,n}$.
The fact that hermitian K-theory is representable in $\mathcal{H}(k)$ has been known for longer, see [Hor05]. One of the advantages of having a geometric description of a representing space, however, is that one can easily see what its complex realization is. In particular, this gives us an alternative way to define the comparison maps. For any smooth complex scheme X we have the following commutative squares, in which the left vertical arrows are the comparison maps (8) and (9), the right vertical arrows are induced by the complex realization functor (11).

$$\begin{array}{ccc} \mathrm{K}_0(X) & \cong & \mathcal{H}(\mathbb{C})(X, \mathbb{Z}\times \mathrm{Gr}) \\ \downarrow & & \downarrow \\ \mathrm{K}^0(X(\mathbb{C})) & \cong & \mathcal{H}(X(\mathbb{C}), \mathbb{Z}\times \mathrm{Gr}) \end{array} \qquad \begin{array}{ccc} \mathrm{GW}^0(X) & \cong & \mathcal{H}(\mathbb{C})(X, \mathbb{Z}\times \mathrm{Gr}^{\mathrm{nd}}) \\ \downarrow & & \downarrow \\ \mathrm{KO}^0(X(\mathbb{C})) & \cong & \mathcal{H}(X(\mathbb{C}), \mathbb{Z}\times \mathrm{Gr}^{\mathrm{nd}}) \end{array}$$

Some of the results quoted here are in fact known in a much greater generality. Firstly, higher algebraic and hermitian K-groups of X are obtained by passing to suspensions of X in (13) and (14). Even better, algebraic and hermitian K-theory are representable in the stable $\mathbb{A}^1$-homotopy category $\mathcal{SH}(k)$. Let us make the statement a little more precise by fixing some notation. Given a spectrum $\mathbb{E}$ in $\mathcal{SH}(k)$, we obtain a bigraded reduced cohomology theory $\widetilde{E}^{*,*}$ on $\mathcal{H}_\bullet(k)$ and a corresponding unreduced theory $E^{*,*}$ on $\mathcal{H}(k)$ by setting

$$\begin{aligned} \widetilde{E}^{p,q}(\mathcal{X}) &:= \mathcal{SH}(k)(\Sigma^\infty \mathcal{X}, S^{p,q} \wedge \mathbb{E}) && \text{for } \mathcal{X} \in \mathcal{H}_\bullet(k) \\ E^{p,q}(X) &:= \widetilde{E}^{p,q}(X_+) && \text{for } X \in \mathcal{H}(k) \end{aligned}$$

A spectrum $\mathbb{K}$ representing algebraic K-theory was first constructed in [Voe98, § 6.2]; see [Rio06] or [Rio07b] for some further discussion. It is (2,1)-periodic, meaning that in $\mathcal{SH}(k)$ we have an isomorphism

$$S^{2,1} \wedge \mathbb{K} \xrightarrow{\cong} \mathbb{K}$$

[4]Talk "Geometric representation of hermitian K-theory in $\mathbb{A}^1$-homotopy theory" at the Workshop "Geometric Aspects of Motivic Homotopy Theory", 6.–10. September 2010 at the Hausdorff Center for Mathematics, Bonn

Thus, the bigrading of the corresponding cohomology theory $\mathrm{K}^{p,q}$ is slightly artificial. The identification with the usual notation for algebraic K-theory is given by

$$\mathrm{K}^{p,q}(X) = \mathrm{K}_{2q-p}(X) \tag{15}$$

For hermitian K-theory we have an (8,4)-periodic spectrum $\mathbb{K}\mathbf{O}$, and the corresponding cohomology groups $\mathrm{KO}^{p,q}$ are honestly bigraded. The translation into the notation used for hermitian K-groups in Section 1.1 is given by

$$\mathrm{KO}^{p,q}(X) = \mathrm{GW}^{q}_{2q-p}(X) \tag{16}$$

We will refer to the number $2q-p$ as the degree of the group $\mathrm{KO}^{p,q}(X)$. The relation with Balmer's Witt groups obtained by combining (16) and (2) is illustrated by the following table:

$\mathbf{KO^{p,q}}$	$p=0$	1	2	3	4	5	6	7
$q=0$	$\mathbf{GW^0}$	$\mathbf{W^1}$	$\mathbf{W^2}$	$\mathbf{W^3}$	$\mathbf{W^0}$	$\mathbf{W^1}$	$\mathbf{W^2}$	$\mathbf{W^3}$
$q=1$	GW^1_2	GW^1_1	$\mathbf{GW^1}$	$\mathbf{W^2}$	$\mathbf{W^3}$	$\mathbf{W^0}$	$\mathbf{W^1}$	$\mathbf{W^2}$
$q=2$	GW^2_4	GW^2_3	GW^2_2	GW^2_1	$\mathbf{GW^2}$	$\mathbf{W^3}$	$\mathbf{W^0}$	$\mathbf{W^1}$
$q=3$	GW^3_6	GW^3_5	GW^3_4	GW^3_3	GW^3_2	GW^3_1	$\mathbf{GW^3}$	$\mathbf{W^0}$

As for the representing spaces in the unstable homotopy category, it is known that the complex realizations of $\mathbb{K}\mathbf{O}$ and $\mathbb{K}$ represent real and complex topological K-theory. This is well-documented in the latter case, see for example [Rio06, Proposition VI.12]. For $\mathbb{K}\mathbf{O}$, our references are slightly thin. Since the emphasis in this article is on showing how such a result in $\mathbb{A}^1$-homotopy theory can be used for some concrete computations, we will at this point succumb to an "axiomatic" approach — the key statements we will be using are as follows:

Standing assumptions 1.9. *There exist spectra $\mathbb{K}$ and $\mathbb{K}\mathbf{O}$ in $\mathcal{SH}(\mathbb{C})$ representing algebraic K-theory and hermitian K-theory in the sense described above, such that:*

(a) The complex realization functor (12) *takes $\mathbb{K}$ to $\mathbb{K}^{top}$ and $\mathbb{K}\mathbf{O}$ to $\mathbb{K}\mathbf{O}^{top}$.*

(b) We have an exact triangle in $\mathcal{SH}(\mathbb{C})$ of the form

$$\mathbb{K}\mathbf{O} \wedge S^{1,1} \xrightarrow{\eta} \mathbb{K}\mathbf{O} \to \mathbb{K} \to S^{1,0} \wedge \dots \tag{17}$$

which corresponds to the usual triangle in $\mathcal{SH}$.

These results are announced in [Mor06]. Independent constructions of spectra representing hermitian K-theory can be found in [Hor05] and in a recent preprint of Panin and Walter [PW10].

2 The comparison maps

It follows immediately from 1.9 that complex realization induces comparison maps

$$\begin{aligned} \widetilde{k}^{p,q}\colon &\qquad \widetilde{\mathrm{K}}^{p,q}(\mathcal{X}) \to \widetilde{\mathrm{K}}^{p}(\mathcal{X}(\mathbb{C})) \\ \widetilde{k}_h^{p,q}\colon &\qquad \widetilde{\mathrm{KO}}^{p,q}(\mathcal{X}) \to \widetilde{\mathrm{KO}}^{p}(\mathcal{X}(\mathbb{C})) \end{aligned}$$

and hence comparison maps $k^{p,q}$ and $k_h^{p,q}$ on K- and hermitian K-groups. In particular, in degrees 0 and -1 we have maps

$$\begin{aligned} k^{0,0}\colon &\qquad \mathrm{K}_0(X) \to \mathrm{K}^0(X(\mathbb{C})) \\ \mathrm{gw}^q := k_h^{2q,q}\colon &\qquad \mathrm{GW}^q(X) \to \mathrm{KO}^{2q}(X(\mathbb{C})) \\ \mathrm{w}^q := k_h^{2q-1,q-1}\colon &\qquad \mathrm{W}^q(X) \to \mathrm{KO}^{2q-1}(X(\mathbb{C})) \end{aligned}$$

for any smooth complex scheme X. Some good properties of these maps follow directly from the construction:

- They commute with pullbacks along morphisms of smooth schemes.
- They are compatible with suspension isomorphisms.
- They are compatible with the periodicity isomorphisms, so we can identify $k_h^{p,q}$ with $k_h^{p+8,q+4}$ (and hence w^q with w^{q+4} and gw^q with gw^{q+4}).

It is also clear that they are compatible with long exact sequences arising from exact triangles in $\mathcal{SH}(\mathbb{C})$. This will be particularly useful in the following two cases.

Localization sequences. Given a smooth closed subscheme Z of a smooth scheme X, we have an exact triangle

$$\Sigma^\infty(X-Z)_+ \to \Sigma^\infty X_+ \to \Sigma^\infty\left(\tfrac{X}{X-Z}\right) \to S^{1,0}\wedge\ldots$$

in $\mathcal{SH}(\mathbb{C})$. It induces long exact "localization sequences" for cohomology theories. For example, for hermitian K-theory we obtain sequences of the form

$$\cdots \to \widetilde{\mathrm{KO}}^{p,q}\left(\tfrac{X}{X-Z}\right) \to \mathrm{KO}^{p,q}(X) \to \mathrm{KO}^{p,q}(X-Z) \to \widetilde{\mathrm{KO}}^{p+1,q}\left(\tfrac{X}{X-Z}\right) \to \ldots \tag{18}$$

The comparison maps commute with all maps appearing in this sequence and the corresponding sequence of topological KO-groups.

The space $X/(X-Z)$ depends only on the normal bundle $\mathcal{N}$ of Z in X. To make this precise, we introduce the Thom space of a vector bundle $\mathcal{E}$ over an arbitrary smooth scheme Z, defined as the homotopy quotient of $\mathcal{E}$ by the complement of the zero section:

$$\mathrm{Thom}_Z(\mathcal{E}) := \mathcal{E}\big/(\mathcal{E}-Z)$$

Using a geometric construction known as deformation to the normal bundle, Morel and Voevodsky show in Theorem 2.23 of [MV99, Chapter 3] that $X/(X-Z)$ is canonically isomorphic to $\mathrm{Thom}_Z(\mathcal{N})$ in the unstable pointed $\mathbb{A}^1$-homotopy category. Thus, sequence (18) can be rewritten in the following form:

$$\begin{aligned}\dots \to \widetilde{\mathrm{KO}}^{p,q}(\mathrm{Thom}_Z\,\mathcal{N}) \to \mathrm{KO}^{p,q}(X) \to \mathrm{KO}^{p,q}(X-Z) \to \\ \widetilde{\mathrm{KO}}^{p+1,q}(\mathrm{Thom}_Z\,\mathcal{N}) \to \mathrm{KO}^{p+1,q}(X) \to \mathrm{KO}^{p+1,q}(X-Z) \to \dots\end{aligned}$$

Karoubi/Bott sequences. The KO- and K-groups of a topological space X fit into a long exact sequence known as the Bott sequence [Bot69, pages 75 and 112[5]; BG10, 4.I.B]. It has the form

$$\begin{aligned}\dots \to \mathrm{KO}^{2i-1}X \to \mathrm{KO}^{2i-2}X \to \mathrm{K}^0X \to \mathrm{KO}^{2i}X \to \mathrm{KO}^{2i-1}X \to \mathrm{K}^1X \\ \to \mathrm{KO}^{2i+1}X \to \mathrm{KO}^{2i}X \to \mathrm{K}^0X \to \mathrm{KO}^{2i+2}X \to \mathrm{KO}^{2i+1}X \to \dots\end{aligned} \tag{19}$$

The maps from KO- to K-groups are essentially given by complexification (or, depending on our choice of description of KO-groups, by forgetting the symmetric structure of a complex symmetric bundle), and the maps from K- to KO-groups are given by sending a complex vector bundle to its underlying real bundle (or to the associated hyperbolic bundle). The maps between KO-groups are given by multiplication with the generator η of $\mathrm{KO}^{-1}(\text{point})$ (see (7)).
This long exact sequence is induced by the triangle described in 1.9. The sequence arising from the corresponding triangle (17) in the stable $\mathbb{A}^1$-homotopy category is known as the Karoubi sequence. The comparison maps induce a commutative ladder diagram that allows us to compare the two. Near degree zero, this takes the following form:

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{KO}^{2i-1,i}X \arrow[r] \arrow[d] & \mathrm{GW}^{i-1}X \arrow[r] \arrow[d] & \mathrm{K}_0X \arrow[r] \arrow[d] & \mathrm{GW}^iX \arrow[r] \arrow[d] & \mathrm{W}^iX \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & \mathrm{KO}^{2i-1}X \arrow[r] & \mathrm{KO}^{2i-2}X \arrow[r] & \mathrm{K}^0X \arrow[r] & \mathrm{KO}^{2i}X \arrow[r] & \mathrm{KO}^{2i-1}X \arrow[r] & \mathrm{K}^1X \arrow[r] & \dots \end{tikzcd} \tag{20}$$

As a consequence, the comparison maps w^i factor as

$$\mathrm{W}^i(X) \to \frac{\mathrm{KO}^{2i}(X)}{\mathrm{K}^0(X)} \to \mathrm{KO}^{2i-1}(X)$$

For cellular varieties, or more generally for spaces for which the odd topological K-groups vanish, the second map in this factorization is an isomorphism.

[5]Unfortunately, there are misprints on both pages. In particular, the central group in the diagram on page 112 should be K^0.

Groups with restricted support. Comparing the localization sequences (1) and (18), we see that the groups $\widetilde{\mathrm{KO}}^{p,q}(\frac{X}{X-Z})$ play the role of hermitian K-groups of X supported on Z. This should be viewed as part of any representability statement, see for example [PW10, Theorem 6.5]. Alternatively, a formal identification of the groups in degrees zero and below using only the minimal assumptions we have stated could be achieved as follows:

Lemma 2.1. *Let Z be a smooth closed subvariety of a smooth quasi-projective variety X. We have the following isomorphisms:*

$$\begin{aligned}\widetilde{\mathrm{KO}}^{2q,q}(\tfrac{X}{X-Z}) &\cong \mathrm{GW}^q_Z(X)\\ \widetilde{\mathrm{KO}}^{p,q}(\tfrac{X}{X-Z}) &\cong \mathrm{W}^{p-q}_Z(X) \quad \textit{for} \quad 2q-p<0\end{aligned}$$

Proof. Consider $Z = Z\times\{0\}$ as a subvariety of $X\times\mathbb{A}^1$. Its open complement $(X\times\mathbb{A}^1)-Z$ contains $X = X\times\{1\}$ as a retract. Thus, the projection from $X\times\mathbb{A}^1$ onto X induces a splitting of the localization sequences associated with $(X\times\mathbb{A}^1-Z)\hookrightarrow X\times\mathbb{A}^1$, and we have

$$\begin{aligned}\mathrm{GW}^{i+1}_Z(X\times\mathbb{A}^1) &\cong \operatorname{coker}\Big(\mathrm{GW}^{i+1}_1(X\times\mathbb{A}^1)\hookrightarrow \mathrm{GW}^{i+1}_1(X\times\mathbb{A}^1-Z)\Big)\\ \widetilde{\mathrm{KO}}^{2i+2,i+1}(\tfrac{X\times\mathbb{A}^1}{X\times\mathbb{A}^1-Z}) &\cong \operatorname{coker}\Big(\mathrm{KO}^{2i+1,i+1}(X\times\mathbb{A}^1)\hookrightarrow \mathrm{KO}^{2i+1,i+1}(X\times\mathbb{A}^1-Z)\Big)\end{aligned}$$

By (16), we can identify the groups appearing on the right, so we obtain an induced isomorphism of the cokernels. The quotient $X\times\mathbb{A}^1/(X\times\mathbb{A}^1-Z)$ can be identified with the suspension of $X/(X-Z)$ by $S^{2,1}$, so we have an isomorphism

$$\widetilde{\mathrm{KO}}^{2i+2,i+1}(\tfrac{X\times\mathbb{A}^1}{X\times\mathbb{A}^1-Z}) \cong \widetilde{\mathrm{KO}}^{2i,i}(\tfrac{X}{X-Z})$$

On the other hand, we have analogous isomorphisms

$$\mathrm{GW}^{i+1}_Z(X\times\mathbb{A}^1)\cong \mathrm{GW}^i_Z(X)$$

for Grothendieck-Witt and Witt groups. For Witt groups, this is a special case of Theorem 2.5 in [Nen07], the case when $Z = X$ being Theorem 8.2 in [BG05]. The corresponding isomorphisms of Grothendieck-Witt groups may be deduced via Karoubi induction. The proof in lower degrees is analogous. □

2.1 Twisting by line bundles

As described in Section 1.1, there is a natural notion of Witt groups twisted by line bundles. In the homotopy theoretic approach, such a twist can be encoded by passing to the Thom space of the bundle.

Definition 2.2. For a vector bundle $\mathcal{E}$ of constant rank r over a smooth scheme X, we define the hermitian K-groups of X with coefficients in $\mathcal{E}$ by

$$\mathrm{KO}^{p,q}(X;\mathcal{E}) := \widetilde{\mathrm{KO}}^{p+2r,q+r}(\operatorname{Thom}\mathcal{E})$$

Likewise, for any complex vector bundle of rank r over a topological space X, we define

$$\mathrm{KO}^p(X;\mathcal{E}) := \widetilde{\mathrm{KO}}^{p+2r}(\mathrm{Thom}\,\mathcal{E})$$

When $\mathcal{E}$ is a trivial bundle, its Thom space is just a suspension of X, so that $\mathrm{KO}^{p,q}(X;\mathcal{E})$ and $\mathrm{KO}^{p,q}(X)$ agree.

Lemma 2.3. *For any vector bundle $\mathcal{E}$ over a smooth quasi-projective variety X, we have isomorphisms*

$$\begin{aligned} \mathrm{KO}^{2q,q}(X;\mathcal{E}) &\cong \mathrm{GW}^q(X;\det\mathcal{E}) \\ \mathrm{KO}^{p,q}(X;\mathcal{E}) &\cong \mathrm{W}^{p-q}(X;\det\mathcal{E}) \quad \textit{for} \quad 2q-p<0 \end{aligned}$$

Proof. This follows from Lemma 2.1 and Nenashev's Thom isomorphisms for Witt groups: for any vector bundle $\mathcal{E}$ of rank r there is a canonical Thom class in $\mathrm{W}^r_X(\mathcal{E})$ which induces an isomorphism $\mathrm{W}^i(X;\det\mathcal{E}) \cong \mathrm{W}^{i+r}_X(\mathcal{E})$ by multiplication [Nen07, Theorem 2.5]. This Thom class actually comes from a class in $\mathrm{GW}^r_X(\mathcal{E})$, and, as in the proof of Lemma 2.1, we can deduce that it induces an analogous isomorphism on Grothendieck-Witt groups via Karoubi induction. □

Remark. The isomorphisms of Lemmas 2.1 and 2.3 are constructed here in a rather ad hoc fashion, and we have taken little care in recording their precise form. Whenever we give an argument concerning the comparison maps on "twisted groups" in the following, we do all constructions on the level of representable groups of Thom spaces. The identifications with the usual twisted groups are only needed to identify the final output of concrete calculations as in Section 4.

It follows similarly from Thom isomorphisms in topology that the groups $\mathrm{KO}(X;\mathcal{E})$ only depend on the determinant line bundle of $\mathcal{E}$:

Lemma 2.4. *For complex vector bundles $\mathcal{E}$ and $\mathcal{F}$ on a topological space X with identical first Chern class modulo 2, we have*

$$\mathrm{KO}^p(X;\mathcal{E}) \cong \mathrm{KO}^p(X;\mathcal{F})$$

Proof. A complex vector bundle $\mathcal{E}$ whose first Chern class vanishes modulo 2 has a spin structure and is therefore oriented with respect to KO-theory [ABS64, § 12]. That is, we have a Thom isomorphism

$$\mathrm{KO}^p\,X \xrightarrow{\cong} \widetilde{\mathrm{KO}}^{p+2r}(\mathrm{Thom}\,\mathcal{E})$$

Now suppose $c_1(\mathcal{E}) \equiv c_1(\mathcal{F}) \bmod 2$. We may view $\mathcal{E}\oplus\mathcal{E}\oplus\mathcal{F}$ both as a vector bundle over $\mathcal{E}$ and as a vector bundle over $\mathcal{F}$, and by assumption it is oriented with respect to KO-theory in both cases. Thus, both groups in the lemma can be identified with $\mathrm{KO}^p(X;\mathcal{E}\oplus\mathcal{E}\oplus\mathcal{F})$. □

Remark 2.1. In general the identifications of Lemma 2.4 are non-canonical. Given a spin structure on a real vector bundle, the constructions in [ABS64] do yield a canonical Thom class, but there may be several different spin structures on the same bundle. Still, canonical identifications exist in many cases. For example, there is a canonical spin structure on the square of any complex line bundle, yielding canonical identifications

$$\mathrm{KO}^p(X;\mathcal{L}) \cong \mathrm{KO}^p(X;\mathcal{L}\otimes\mathcal{M}^{\otimes 2})$$

for any two complex line bundles $\mathcal{L}$ and $\mathcal{M}$ over X. Moreover, as different spin structures on a spin bundle over X are classified by the singular cohomology group $H^1(X;\mathbb{Z}/2)$, all spin structures arising in the context of complex cellular varieties below will be unique.

2.2 The comparison for cellular varieties

Theorem 2.5. *For a smooth cellular complex variety X, the following comparison maps are isomorphisms:*

$$\begin{aligned} \mathrm{K}_0(X) &\xrightarrow{\cong} \mathrm{K}^0(X(\mathbb{C})) \\ \mathrm{gw}^q\colon \mathrm{GW}^q(X) &\xrightarrow{\cong} \mathrm{KO}^{2q}(X(\mathbb{C})) \\ \mathrm{w}^q\colon \mathrm{W}^q(X) &\xrightarrow{\cong} \mathrm{KO}^{2q-1}(X(\mathbb{C})) \end{aligned}$$

This remains true for twisted groups (see Section 2.1).

As indicated in the introduction, the first isomorphism is well-known and almost self-evident, given that both $\mathrm{K}_0(X)$ and $\mathrm{K}^0(X(\mathbb{C}))$ are free abelian of rank equal to the number of cells of X. In particular, both the algebraic group $\mathrm{K}_0(\mathbb{C})$ and the topological K-group $\mathrm{K}^0(\text{point})$ are isomorphic to the integers, generated by the trivial line bundle, and the comparison map is the obvious isomorphism.

Let us begin the proof of the theorem by also considering the other two maps first in the case when X is just a point $\mathrm{Spec}(\mathbb{C})$. We can easily see that the corresponding groups are isomorphic by direct comparison:

$\mathbf{KO}^{p,q}(\mathbb{C})$	$p=0$	1	2	3	4	5	6	7
$q=0$	$\mathbf{\mathbb{Z}}$	$\mathbf{0}$	0	0	$\mathbb{Z}/2$	0	0	0
$q=1$	...	...	$\mathbf{0}$	$\mathbf{0}$	0	$\mathbb{Z}/2$	0	0
$q=2$	...	...	...	...	$\mathbf{\mathbb{Z}}$	$\mathbf{0}$	$\mathbb{Z}/2$	0
$q=3$	...	...	...	...	...	...	$\mathbf{\mathbb{Z}/2}$	$\mathbf{\mathbb{Z}/2}$
$\mathbf{KO}^{p}(\text{point})$	$\mathbf{\mathbb{Z}}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{\mathbb{Z}}$	$\mathbf{0}$	$\mathbf{\mathbb{Z}/2}$	$\mathbf{\mathbb{Z}/2}$

Table 1: (Grothendieck-)Witt and KO-groups of a point.

To see that the isomorphisms are given by our comparison maps, we can use the comparison of the Karoubi and Bott sequences. First, setting $i = 0$ in Diagram (20), we see that gw^0 and w^0 are isomorphisms on a point. As $\mathrm{W}^0(\mathbb{C})$ is the only non-trivial Witt group of a point, it follows that w^q is an isomorphism on a point in general, so that we have

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{GW}^{q-1} \arrow[r] \arrow[d, "\mathrm{gw}^{q-1}"] & \mathrm{K}_0 \arrow[r] \arrow[d, "\cong"] & \mathrm{GW}^q \arrow[r] \arrow[d, "\mathrm{gw}^q"] & \mathrm{W}^q \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \mathrm{KO}^{2q-2} \arrow[r] & \mathrm{K}^0 \arrow[r] & \mathrm{KO}^{2q} \arrow[r] & \mathrm{KO}^{2q-1} \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Given the periodicity of the Grothendieck-Witt groups, repeated applications of the Five Lemma now show that gw^q is an isomorphism on a point for all values of q. (This strategy of proof is known as “Karoubi induction”.)

We now treat the hermitian case in general. The case of algebraic/complex K-theory could be dealt with similarly, or deduced from the hermitian case using triangle (17). It will be helpful to consider not only the maps $\mathrm{gw}^q = k_h^{2q,q}$ and $\mathrm{w}^{q+1} = k_h^{2q+1,q}$ in degrees 0 and -1, respectively, but also the maps $k_h^{2q-1,q}$ in degree 1 and the maps $k_h^{2q+2,q}$ in degree -2. We will prove the following extended statement:

Theorem 2.6. *For a smooth cellular variety X, the hermitian comparison maps in degrees* 1*,* 0*,* -1 *and* -2 *have the properties indicated:*

$$\begin{aligned}
\mathrm{KO}^{2q-1,q}(X) &\twoheadrightarrow \mathrm{KO}^{2q-1}(X(\mathbb{C})) \\
\mathrm{KO}^{2q,q}(X) &\xrightarrow{\cong} \mathrm{KO}^{2q}(X(\mathbb{C})) \\
\mathrm{KO}^{2q+1,q}(X) &\xrightarrow{\cong} \mathrm{KO}^{2q+1}(X(\mathbb{C})) \\
\mathrm{KO}^{2q+2,q}(X) &\rightarrowtail \mathrm{KO}^{2q+2}(X(\mathbb{C}))
\end{aligned}$$

The analogous statements for twisted groups are also true.

Remark 2.2. The map in degree 1 is not an isomorphism even when X is a point. For example, it is known that $\mathrm{KO}^{-1,0}(\mathbb{C}) = \mathbb{Z}/2$ (see [Kar05, Example 18]), from which we may deduce via the Karoubi sequence that $\mathrm{KO}^{1,1}(\mathbb{C}) \cong \mathbb{C}^*$. In particular, $\mathrm{KO}^{1,1}(\mathbb{C})$ cannot be isomorphic to $\mathrm{KO}^1(\text{point}) = 0$.
The map in degree -2 can be identified with the inclusion of the 2-torsion subgroup of $\mathrm{KO}^{2q+2}(X(\mathbb{C}))$ into $\mathrm{KO}^{2q+2}(X(\mathbb{C}))$ for any cellular variety X. This follows from the theorem and the description of the KO-groups of cellular varieties given in Lemma 3.1.
In degrees less than -2, the comparison map is necessarily zero. The problem is that while $\eta\colon \mathrm{W}^{p-q}(X) \to \mathrm{W}^{p-q}(X)$ is an isomorphism in all negative degrees, the topological η is nilpotent ($\eta^3 = 0$).

The proof of Theorem 2.6 will proceed by induction over the number of cells of X and occupy the remainder of this section. To begin the induction, we need to consider the case of only one cell, which immediately reduces to the case of a point by homotopy invariance. In this case, degrees 0 and -1 have already been dealt with above. In degrees 1 and -2, on the other hand, most of the statements are trivial, and we only need to look at a few particular cases, which we postpone to the end of the proof.

Spheres. Assuming the theorem to be true for a point, the compatibility of the comparison maps with suspensions immediately shows that the theorem is also true for the reduced cohomology of the spheres $(\mathbb{P}^1)^{\wedge d} = S^{2d,d}$. To be precise, the following maps in degrees 1, 0, -1 and -2 have the properties indicated:

$$\begin{aligned}
\widetilde{\mathrm{KO}}^{2q-1,q}(S^{2d,d}) &\twoheadrightarrow \widetilde{\mathrm{KO}}^{2q-1}(S^{2d}) \\
\widetilde{\mathrm{KO}}^{2q,q}(S^{2d,d}) &\xrightarrow{\cong} \widetilde{\mathrm{KO}}^{2q}(S^{2d}) \\
\widetilde{\mathrm{KO}}^{2q+1,q}(S^{2d,d}) &\xrightarrow{\cong} \widetilde{\mathrm{KO}}^{2q+1}(S^{2d}) \\
\widetilde{\mathrm{KO}}^{2q+2,q}(S^{2d,d}) &\rightarrowtail \widetilde{\mathrm{KO}}^{2q+2}(S^{2d})
\end{aligned}$$

Cellular varieties. Now let X be a smooth cellular variety. By definition, X has a filtration by closed subvarieties $\emptyset = Z_0 \subset Z_1 \subset Z_2 \cdots \subset Z_N = X$ such that the open complement of Z_k in Z_{k+1} is isomorphic to $\mathbb{A}^{n_k}$ for some n_k. In general, the subvarieties Z_k will not be smooth. Their complements $U_k := X - Z_k$ in X, however, are always smooth as they are open in X. So we obtain an alternative filtration $X = U_0 \supset U_1 \supset U_2 \cdots \supset U_N = \emptyset$ of X by smooth open subvarieties U_k. Each U_k contains a closed cell $C_k \cong \mathbb{A}^{n_k}$ with open complement U_{k+1}.

Our inductive hypothesis is that we have already proved the theorem for U_{k+1}, and we now want to prove it for U_k. We can use the following exact triangle in $\mathcal{SH}(\mathbb{C})$:

$$\Sigma^\infty(U_{k+1})_+ \to \Sigma^\infty(U_k)_+ \to \Sigma^\infty \mathrm{Thom}(\mathcal{N}_{C_k\setminus U_k}) \to S^{1,0} \wedge \ldots$$

As C_k is a cell, the Quillen-Suslin theorem tells us that the normal bundle $\mathcal{N}_{C_k\setminus U_k}$ of C_k in U_k has to be trivial. Thus, $\mathrm{Thom}(\mathcal{N}_{C_k\setminus U_k})$ is $\mathbb{A}^1$-weakly equivalent to $S^{2d,d}$, where d is the codimension of C_k in U_k. Figure 1 displays the comparison between the long exact cohomology sequences induced by this triangle. The inductive step is completed by applying the Five Lemma to each dotted map in the diagram.

The twisted case. To obtain the theorem in the case of coefficients in a vector bundle $\mathcal{E}$ over X, we replace the exact triangle above by the triangle

$$\Sigma^\infty \mathrm{Thom}(\mathcal{E}_{|U_{k+1}}) \to \Sigma^\infty \mathrm{Thom}(\mathcal{E}_{|U_k}) \to \Sigma^\infty \mathrm{Thom}(\mathcal{E}_{|C_k} \oplus \mathcal{N}_{C_k\setminus U_k}) \to S^{1,0} \wedge \ldots$$

The existence of this exact triangle is shown in the next lemma. The Thom space on the right is again just a sphere, so we can proceed as in the untwisted case.

$$
\begin{tikzcd}
{} \arrow[d] \arrow[r, phantom, "\cdots"] & {} \arrow[d] \\
\widetilde{\mathrm{KO}}^{2q-1,q}(S^{2d,d}) \arrow[r, two heads] \arrow[d] & \widetilde{\mathrm{KO}}^{2q-1}(S^{2d}) \arrow[d] \\
\mathrm{KO}^{2q-1,q}(U_k) \arrow[r, dashed, "k_h^{2q-1,q}"] \arrow[d] & \mathrm{KO}^{2q-1}(U_k) \arrow[d] \\
\mathrm{KO}^{2q-1,q}(U_{k+1}) \arrow[r, two heads] \arrow[d] & \mathrm{KO}^{2q-1}(U_{k+1}) \arrow[d] \\
\widetilde{\mathrm{KO}}^{2q,q}(S^{2d,d}) \arrow[r, "\cong"] \arrow[d] & \widetilde{\mathrm{KO}}^{2q}(S^{2d}) \arrow[d] \\
\mathrm{KO}^{2q,q}(U_k) \arrow[r, dashed, "k_h^{2q,q}"] \arrow[d] & \mathrm{KO}^{2q}(U_k) \arrow[d] \\
\mathrm{KO}^{2q,q}(U_{k+1}) \arrow[r, "\cong"] \arrow[d] & \mathrm{KO}^{2q}(U_{k+1}) \arrow[d] \\
\widetilde{\mathrm{KO}}^{2q+1,q}(S^{2d,d}) \arrow[r, "\cong"] \arrow[d] & \widetilde{\mathrm{KO}}^{2q+1}(S^{2d}) \arrow[d] \\
\mathrm{KO}^{2q+1,q}(U_k) \arrow[r, dashed, "k_h^{2q+1,q}"] \arrow[d] & \mathrm{KO}^{2q+1}(U_k) \arrow[d] \\
\mathrm{KO}^{2q+1,q}(U_{k+1}) \arrow[r, "\cong"] \arrow[d] & \mathrm{KO}^{2q+1}(U_{k+1}) \arrow[d] \\
\widetilde{\mathrm{KO}}^{2q+2,q}(S^{2d,d}) \arrow[r, tail] \arrow[d] & \widetilde{\mathrm{KO}}^{2q+2}(S^{2d}) \arrow[d] \\
\mathrm{KO}^{2q+2,q}(U_k) \arrow[r, dashed, "k_h^{2q+2,q}"] \arrow[d] & \mathrm{KO}^{2q+2}(U_k) \arrow[d] \\
\mathrm{KO}^{2q+2,q}(U_{k+1}) \arrow[r, tail] \arrow[d] & \mathrm{KO}^{2q+2}(U_{k+1}) \arrow[d] \\
{} \arrow[r, phantom, "\cdots"] & {}
\end{tikzcd}
$$

Figure 1: The inductive step.

Lemma 2.7. *Given a smooth subvariety Z of a smooth variety X with complement U, and given any vector bundle $\mathcal{E}$ over X, we have an exact triangle*

$$\Sigma^\infty \operatorname{Thom}(\mathcal{E}_{|U}) \to \Sigma^\infty \operatorname{Thom}\mathcal{E} \to \Sigma^\infty \operatorname{Thom}(\mathcal{E}_{|Z} \oplus \mathcal{N}_{Z\setminus X}) \to S^{1,0} \wedge \ldots$$

Proof. From the Thom isomorphism theorem we know that the Thom space of a vector bundle over a smooth base is $\mathbb{A}^1$-weakly equivalent to the quotient of the vector bundle by the complement of the zero section. Consider the closed embeddings $U \hookrightarrow (\mathcal{E}-Z)$, $X \hookrightarrow \mathcal{E}$ and $Z \hookrightarrow \mathcal{E}$. Computing the normal bundles, we obtain

$$\begin{aligned}
(\mathcal{E} - Z)/(\mathcal{E} - X) &\cong \operatorname{Thom}_U(\mathcal{E}_{|U}) \\
\mathcal{E}/(\mathcal{E} - X) &\cong \operatorname{Thom}_X \mathcal{E} \\
\mathcal{E}/(\mathcal{E} - Z) &\cong \operatorname{Thom}_Z(\mathcal{E}_{|Z} \oplus \mathcal{N}_{Z\setminus X})
\end{aligned}$$

The claim follows by passing to the stable homotopy category and applying the octahedral axiom to the composition of the embeddings $(\mathcal{E} - X) \subseteq (\mathcal{E} - Z) \subseteq \mathcal{E}$. $\square$

Remaining details concerning a point. To finish the proof of Theorem 2.6, we now return to the maps of degrees 1 and -2 in the case of a point, which we skipped above. First, let us deal with degree 1. The odd KO-groups of a point are all trivial except for KO^{-1}, so $k_h^{2q-1,q}$ is trivially a surjection unless $q \equiv 0 \mod 4$. In that case, surjectivity of $k_h^{-1,0}$ is clear from the following diagram:

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{KO}^{-1,0} \arrow[r] \arrow[d, "k_h^{-1,0}"] & \mathrm{GW}^{-1} \arrow[r] \arrow[d, "\cong"] & \mathrm{K}_0 \arrow[r] \arrow[d, "\cong"] & \ldots \\
\ldots \arrow[r] & \mathrm{KO}^{-1} \arrow[r] & \mathrm{KO}^{-2} \arrow[r] & \mathrm{K}^0 \arrow[r] & \ldots
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{KO}^{-1,0} \arrow[r, two heads] \arrow[d, "k_h^{-1,0}"] & \mathbb{Z}/2 \arrow[r, "0"] \arrow[d, "\cong"] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \ldots \\
\ldots \arrow[r] & \mathbb{Z}/2 \arrow[r, "\cong"] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z} \arrow[r] & \ldots
\end{tikzcd}$$

Lastly, we consider what happens in degree -2. Again, three out of four cases are trivial as $\mathrm{KO}^{2q+2,q} = \mathrm{W}^{q+2}$ is zero unless $q \equiv 2 \mod 4$. For the non-trivial case, consider the map η appearing in triangle (17). As the negative algebraic K-groups of $\mathbb{C}$ are zero, η yields automorphisms of W^{p-q} in negative degrees. In topology, the corresponding maps are given by multiplication with a generator

η of KO^{-1}, and η^2 generates KO^{-2}. So the commutative square

$$\begin{tikzcd} \mathrm{W}^0 \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{W}^0 \arrow[d, "k_h^{0,-2}"] \\ \mathrm{KO}^{-1} \arrow[r, "\eta", "\cong"'] & \mathrm{KO}^{-2} \end{tikzcd}$$

shows that $k_h^{0,-2}$ is an injection (in fact, an isomorphism), as claimed. This completes the proof of Theorem 2.6.

Remark 2.3. We indicate briefly how Theorem 2.5 can alternatively be obtained by working only with the maps in degrees 0 and -1 that can be defined by more elementary means. The basic strategy — comparing the localization sequences arising from the inclusion of a closed cell C_k into the union of "higher" cells U_k — still works. But we cannot deduce that the comparison maps are isomorphisms on U_k from the fact that they are isomorphisms on U_{k+1} because the parts of the sequences that we can actually compare are now too short. We can, however, still deduce that the maps in degree 0 with domains the Grothendieck-Witt groups of U_k are surjective, and that the maps in degree -1 with domains the Witt groups of U_k are injective. The inductive step can then be completed with the help of the Bott/Karoubi sequences. This argument works even without assuming that the comparison maps are compatible with the boundary maps in localization sequences in general: in the relevant cases the cohomology groups involved are so simple that this property can be checked by hand.

3 The Atiyah-Hirzebruch spectral sequence

We now aim to prepare the ground for the discussion of the KO-theory of some examples in the next section. The main computational tool will be the Atiyah-Hirzebruch spectral sequence, which in topology exists for any generalized cohomology theory and any finite-dimensional CW complex X [Ada95, III.7; Koc96, Theorem 4.2.7]. For KO-theory, it has the form

$$E_2^{p,q} = H^p(X; \mathrm{KO}^q(\text{point})) \Rightarrow \mathrm{KO}^{p+q}(X)$$

with differential d_r of bidegree $(r, -r+1)$. The E_2-page is thus concentrated in the half-plane $p \geq 0$ and 8-periodic in q: we have the integral cohomology of X in rows $q \equiv 0$ and $q \equiv -4 \bmod 8$, its cohomology with $\mathbb{Z}/2$-coefficients in rows $q \equiv -1$ and $q \equiv -2$, and all other rows are zero. The differential d_2 is given by $\mathrm{Sq}^2 \circ \pi_2$ and Sq^2 on rows $q \equiv 0$ and $q \equiv -1$, respectively, where

$$\mathrm{Sq}^2 \colon H^*(X; \mathbb{Z}/2) \to H^{*+2}(X; \mathbb{Z}/2)$$

is the second Steenrod square and π_2 is mod-2 reduction [Fuj67, 1.3].

The spectral sequence is multiplicative [Koc96, Proposition 4.2.9]. That is, the multiplication on the E_2-page induced by the cup product on singular cohomology and the ring structure of $\mathrm{KO}^*(\text{point})$ (see (7)) descends to a multiplication on all subsequent pages, such that the multiplication on the E_∞-page is compatible with the multiplication on $\mathrm{KO}^*(X)$. In particular, each page is a module over $\mathrm{KO}^*(\text{point})$. The differentials of the spectral sequence are derivations, i. e. they satisfy a Leibniz rule.

3.1 The Atiyah-Hirzebruch spectral sequence for cellular varieties

For cellular varieties, or more generally for CW complexes with only even-dimensional cells, the spectral sequence becomes simple enough to make some general deductions. We summarize some lemmas of Hoggar and Kono and Hara.

Lemma 3.1. [Hog69, 2.1 and 2.2] *Let X be a CW complex with only even-dimensional cells. Then:*

- *The ranks of the free parts of $\mathrm{KO}^0 X$ and $\mathrm{KO}^4 X$ are equal to the number t_0 of cells of X of dimension a multiple of 4.*
- *The ranks of the free parts of $\mathrm{KO}^2 X$ and $\mathrm{KO}^6 X$ are equal to the number t_1 of cells of X of dimension 2 modulo 4.*
- *The groups of odd degrees are two-torsion, i. e. $\mathrm{KO}^{2i-1} X = (\mathbb{Z}/2)^{s_i}$ for some s_i.*
- *$\mathrm{KO}^{2i} X$ is isomorphic to the direct sum of its free part and $\mathrm{KO}^{2i+1} X$.*

Table 3 in Section 4.1 summarizes these statements.

Proof. The cohomology of X is free on generators given by the cells and concentrated in even degrees. The first two statements thus follow easily from the Atiyah-Hirzebruch spectral sequence for KO-theory (e. g. after tensoring with $\mathbb{Q}$). On the other hand, we see from the Atiyah-Hirzebruch spectral sequence for complex K-theory that $\mathrm{K}^0(X)$ is a free abelian group on the cells while $\mathrm{K}^1(X)$ is zero. The last two statements thus become consequences of the Bott sequence (19). □

The free part of KO^* is thus very simple. In good cases, the spectral sequence also provides a nice description of the 2-torsion. To see this, note that $\mathrm{Sq}^2\,\mathrm{Sq}^2 = \mathrm{Sq}^3\,\mathrm{Sq}^1$ must vanish when the cohomology of X with $\mathbb{Z}/2$-coefficients is concentrated in even degrees. So we can view $(H^*(X;\mathbb{Z}/2), \mathrm{Sq}^2)$ as a differential graded algebra over $\mathbb{Z}/2$. To lighten the notation, we will write

$$H^*(X, \mathrm{Sq}^2) := H^*(H^*(X;\mathbb{Z}/2), \mathrm{Sq}^2)$$

for the cohomology of this algebra. We keep the same grading as before, so that it is concentrated in even degrees. The row $q \equiv -1$ on the E_3-page is given by $H^*(X, \mathrm{Sq}^2) \cdot \eta$, where η is the generator of $\mathrm{KO}^{-1}(\text{point})$. Since it is the only row that contributes to KO^* in odd degrees, we arrive at the following lemma, which will be central to our computations.

LEMMA 3.2. *Let X be as above. If the Atiyah-Hirzebruch spectral sequence of $\mathrm{KO}^*(X)$ degenerates on the E_3-page, then*

$$\mathrm{KO}^{2i-1}(X) \cong \bigoplus_k H^{2i+8k}(X, \mathrm{Sq}^2)$$

In all the examples we consider below, the spectral sequence does indeed degenerate at this stage. However, showing that it does can be tricky. One step in the right direction is the following observation of Kono and Hara [KH91, Proposition 1].

LEMMA 3.3. *Let X be as above. If the differentials d_3, d_4, ..., d_{r-1} are trivial and d_r is non-trivial, then $r \equiv 2 \mod 8$. In other words, the first non-trivial differential after d_2 can only appear on a page E_r with page number $r \equiv 2 \mod 8$.*
Such a differential is non-zero only on rows $q \equiv 0$ and $q \equiv -1 \mod 8$. If it is non-zero on some x in row $q \equiv 0$, then it is also non-zero on ηx in row $q \equiv -1$. Conversely, if it is non-zero on some y in row $q \equiv -1$, there exists some x in row $q \equiv 0$ such that $y = x\eta$ and d_r is non-zero on x.

Proof. We see from the spectral sequence of a point that $d_r\eta = 0$ for all differentials. Thus, multiplication by η gives a map of bidegree $(0, -1)$ on the spectral sequence that commutes with the differentials. On the E_2-page this map is mod-2 reduction from row $q \equiv 0$ to row $q \equiv -1$ and the identity between rows $q \equiv -1$ and $q \equiv -2$. It follows that on the E_3-page multiplication by η induces a surjection from row $q \equiv 0$ to row $q \equiv -1$ and an injection of row $q \equiv -1$ into row $q \equiv -2$. This implies all statements above. $\square$

We derive a corollary that we will use to deduce that the spectral sequence collapses for certain Thom spaces:

COROLLARY 3.4. *Suppose we have a continuous map $p\colon X \to T$ of CW complexes with only even-dimensional cells. Suppose further that the Atiyah-Hirzebruch spectral sequence for $\mathrm{KO}^*(X)$ collapses on the E_3-page, and that p^* induces an injection in row $q \equiv -1$:*

$$p^*\colon H^*(T, \mathrm{Sq}^2) \hookrightarrow H^*(X, \mathrm{Sq}^2)$$

Then the spectral sequence for $\mathrm{KO}^*(T)$ also collapses at this stage.

Proof. Write d_r for the first non-trivial higher differential, so $r \equiv 2 \mod 8$. Then, for any element x in row $q \equiv 0$, we have $p^*(d_r x) = d_r p^*(x) = 0$ since the spectral sequence for X collapses. From our assumption on p^* we can deduce that $d_r x = 0$. By the preceding lemma, this is all we need to show. $\square$

3.2 The Atiyah-Hirzebruch spectral sequence for Thom spaces

In order to compute twisted KO-groups, we need to apply the Atiyah-Hirzebruch spectral sequence of KO-theory to Thom spaces. So let X be a finite-dimensional CW complex, and let $\pi\colon \mathcal{E} \to X$ be a vector bundle of constant rank over X. Though we will be mainly interested in the case when $\mathcal{E}$ is complex, we may more generally assume here that $\mathcal{E}$ is any real vector bundle which is oriented. Then the Thom isomorphism for singular cohomology tells us that the reduced cohomology of the Thom space $\operatorname{Thom}\mathcal{E}$ is additively isomorphic to the cohomology of X itself, apart from a shift in degrees by $r := \operatorname{rank}_{\mathbb{R}} \mathcal{E}$. The isomorphism is given by multiplication with a Thom class θ in $\widetilde{H}^r(\operatorname{Thom}\mathcal{E};\mathbb{Z})$:

$$\begin{array}{ccc} H^*(X;\mathbb{Z}) & \xrightarrow{\cong} & \widetilde{H}^{*+r}(\operatorname{Thom}\mathcal{E};\mathbb{Z}) \\ x & \mapsto & \pi^*(x)\cdot\theta \end{array}$$

Similarly, the reduction of θ modulo two induces an isomorphism of the respective singular cohomology groups with $\mathbb{Z}/2$-coefficients. Thus, apart from a shift of columns, the entries on the E_2-page of the spectral sequence for $\widetilde{\mathrm{KO}}^*(\operatorname{Thom}\mathcal{E})$ are identical to those on the E_2-page for $\mathrm{KO}^*(X)$. However, the differentials may differ.

Lemma 3.5. *Let $\mathcal{E} \xrightarrow{\pi} X$ be a complex vector bundle of constant rank over a topological space X, with Thom class θ as above. The second Steenrod square on $\widetilde{H}^*(\operatorname{Thom}\mathcal{E};\mathbb{Z}/2)$ is given by "$\mathrm{Sq}^2 + c_1(\mathcal{E})$", where $c_1(\mathcal{E})$ is the first Chern class of $\mathcal{E}$ modulo two. That is,*

$$\mathrm{Sq}^2(\pi^*x\cdot\theta) = \pi^*\left(\mathrm{Sq}^2(x) + c_1(\mathcal{E})x\right)\cdot\theta$$

for any $x \in H^*(X;\mathbb{Z}/2)$. More generally, if $\mathcal{E}$ is a real oriented vector bundle, the second Steenrod square on the cohomology of its Thom space is given by "$\mathrm{Sq}^2 + w_2(\mathcal{E})$", where w_2 is the second Stiefel-Whitney class of $\mathcal{E}$.

Proof. This is a special case of an identity of Thom, which he in fact used to *define* Stiefel-Whitney classes:

$$\mathrm{Sq}^i(\pi^*x\cdot\theta) = \pi^*\left(\mathrm{Sq}^i(x) + w_i(\mathcal{E})x\right)\cdot\theta$$

See [MS74, page 91]. □

When X is a CW complex with cells only in even dimensions, the operation $\mathrm{Sq}^2 + c_1$ can be viewed as a differential on $H^*(X;\mathbb{Z}/2)$ for any c_1 in $H^2(X;\mathbb{Z}/2)$. Extending our previous notation, we denote the cohomology with respect to this differential by

$$H^*(X, \mathrm{Sq}^2 + c_1) := H^*(H^*(X;\mathbb{Z}/2), \mathrm{Sq}^2 + c_1) \tag{21}$$

Corollary 3.6 (of Lemmas 3.2 and 3.5). *If the Atiyah-Hirzebruch spectral sequence of* $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E})$ *degenerates on the* E_3*-page, then*

$$\mathrm{KO}^{2i-1}(X;\mathcal{E}) \cong \bigoplus_k H^{2i+8k}(X, \mathrm{Sq}^2 + c_1\mathcal{E})$$

It is true more generally that the differentials in the spectral sequence for $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E})$ depend only on the second Stiefel-Whitney class of $\mathcal{E}$. This follows from the observation that the Atiyah-Hirzebruch spectral sequence is compatible with Thom isomorphisms, as is made more precise by the next lemma:

Fix a vector bundle $\mathcal{E}$ of constant rank r over a finite-dimensional CW complex X. Suppose $\mathcal{E}$ is oriented with respect to ordinary cohomology and let $\theta \in \widetilde{H}^*(\mathrm{Thom}\,\mathcal{E};\mathbb{Z})$ be a Thom class.

Lemma 3.7. *If* $\mathcal{E}$ *is oriented with respect to* KO^**, then* θ *survives to the* E_∞*-page of the Atiyah-Hirzebruch spectral sequence computing* $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E})$*, and the Thom isomorphism for* H^* *extends to an isomorphism of spectral sequences. That is, for each page right multiplication with the class of* θ *in* $\widetilde{E}_s^{r,0}(\mathrm{Thom}\,\mathcal{E})$ *gives an isomorphism of* $E_s^{*,*}(X)$*-modules*

$$E_s^{*,*}(X) \xrightarrow[\cong]{\cdot\theta} \widetilde{E}_s^{*+r,*}(\mathrm{Thom}\,\mathcal{E})$$

Moreover, any lift of $\theta \in \widetilde{E}_\infty^{r,0}(\mathrm{Thom}\,\mathcal{E})$ *to* $\widetilde{\mathrm{KO}}^r(\mathrm{Thom}\,\mathcal{E})$ *defines a Thom class of* $\mathcal{E}$ *with respect to* KO^**. The isomorphism of the* E_∞*-pages of the spectral sequences is induced by the Thom isomorphism given by multiplication with any such class.*

Proof. We may assume without loss of generality that X is connected. Fix a point x on X. The inclusion of the fibre over x into $\mathcal{E}$ induces a map $i_x\colon S^r \hookrightarrow \mathrm{Thom}\,\mathcal{E}$. By assumption, the pullback i_x^* on ordinary cohomology maps θ to a generator of $\widetilde{H}^r(S^r)$, and the pullback on $\widetilde{\mathrm{KO}}^*$ gives a surjection

$$\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E}) \overset{i_x^*}{\twoheadrightarrow} \widetilde{\mathrm{KO}}^r(S^r)$$

Consider the pullback along i_x on the E_∞-pages of the spectral sequences for S^r and $\mathrm{Thom}\,\mathcal{E}$. Since we can identify $\widetilde{E}_\infty^{r,0}(\mathrm{Thom}\,\mathcal{E})$ with a quotient of $\widetilde{\mathrm{KO}}^r(\mathrm{Thom}\,\mathcal{E})$ and $\widetilde{E}_\infty^{r,0}(S^r)$ with $\widetilde{\mathrm{KO}}^r(S^r)$, we must have a surjection

$$i_x^*\colon \widetilde{E}_\infty^{r,0}(\mathrm{Thom}\,\mathcal{E}) \twoheadrightarrow \widetilde{E}_\infty^{r,0}(S^r)$$

On the other hand, the behaviour of i_x^* on $\widetilde{E}_\infty^{r,0}$ is determined by its behaviour on $\widetilde{H}^r$, whence we can only have such a surjection if θ survives to the $\widetilde{E}_\infty$-page of $\mathrm{Thom}\,\mathcal{E}$. Thus, all differentials vanish on θ, and if multiplication by θ induces an isomorphism from $E_s^{*,*}(X)$ to $\widetilde{E}_s^{*+r,*}$ on page s, it also induces an isomorphism on the next page. Lastly, consider any lift of θ to an element Θ of $\widetilde{\mathrm{KO}}^r(\operatorname{Thom}\mathcal{E})$. It is clear by construction that right multiplication with Θ gives an isomorphism from $E_\infty(X)$ to $\widetilde{E}_\infty(\operatorname{Thom}E)$, and thus it also gives an isomorphism from $\mathrm{KO}^*(X)$ to $\widetilde{\mathrm{KO}}^*(\operatorname{Thom}\mathcal{E})$. Thus, Θ is a Thom class for $\mathcal{E}$ with respect to KO^*. □

Lemma 3.7 allows the following strengthening of Lemma 2.4:

Corollary 3.8. *For complex vector bundles $\mathcal{E}$ and $\mathcal{F}$ over X with identical first Chern class modulo 2, the spectral sequences computing $\widetilde{\mathrm{KO}}^*(\operatorname{Thom}\mathcal{E})$ and $\widetilde{\mathrm{KO}}^*(\operatorname{Thom}\mathcal{F})$ can be identified up to a possible shift of columns when $\mathcal{E}$ and $\mathcal{F}$ have different ranks.*

4 Examples

We now turn to the study of projective homogeneous varieties, that is, varieties of the form G/P for some complex simple linear algebraic group G with a parabolic subgroup P. Any such variety has a cell decomposition [BGG73, Proposition 5.1], so that our comparison theorem applies. As far as we are only interested in the topology of G/P, we may alternatively view it as a homogeneous space for the compact real Lie group G^c corresponding to G:

Proposition 4.1. *Let P be a parabolic subgroup of a simple complex algebraic group G. Then we have a diffeomorphism*

$$G\big/P \cong G^c\big/K$$

where K is a compact subgroup of maximal rank in a maximal compact subgroup G^c of G. More precisely, K is a maximal compact subgroup of a Levi subgroup of P.

Proof. The Iwasawa decomposition for G viewed as a real Lie group implies that we have a diffeomorphism $G \cong G^c \cdot P$ [GOV94, Ch. 6, Prop. 1.7], inducing a diffeomorphism of quotients as claimed for $K = G^c \cap P$. Since $G^c \hookrightarrow G$ is a homotopy equivalence, so is the inclusion $G^c \cap P \hookrightarrow P$. On the other hand, if L is a Levi subgroup of P then $P = U \rtimes L$, where U is unipotent and hence contractible. So the inclusion $L \hookrightarrow P$ is also a homotopy equivalence. It follows that any maximal compact subgroup L^c of L is also maximal compact in P, and conversely that any maximal compact subgroup of P will be contained as a maximal compact subgroup in some Levi subgroup of P. We may therefore assume that $K \subset L^c \subset L \subset P$ and conclude that $K \hookrightarrow L^c$ is a homotopy equivalence. Since both groups are compact, it follows that in fact $K \cong L^c$. □

The KO-theory of homogeneous varieties has been studied intensively. In particular, the papers [KH91] and [KH92] of Kono and Hara provide complete computations of the (untwisted) KO-theory of all compact irreducible hermitian symmetric spaces, which we list in Table 2. For the convenience of the

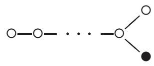
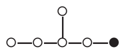
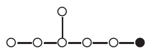
	G/P	G	Diagram of P	G^c/K
Grassmannians (AIII)	$\mathrm{Gr}_{m,n}$	SL_{m+n}	1 n $n+m-1$	$\dfrac{\mathrm{U}(m+n)}{\mathrm{U}(m)\times\mathrm{U}(n)}$
Maximal symplectic Grassmannians (CI)	X_n	Sp_{2n}		$\mathrm{Sp}(n)/\mathrm{U}(n)$
Projective quadrics of dimension $n\geq 3$ (BDI)	Q^n	SO_{n+2}	(n odd)  (n even)	$\dfrac{\mathrm{SO}(n+2)}{\mathrm{SO}(n)\times\mathrm{SO}(2)}$
Spinor varieties (DIII)	S_n	SO_{2n}		$\mathrm{SO}(2n)/\mathrm{U}(n)$
Exceptional hermitian symmetric spaces:	EIII	E_6		$\dfrac{E_6^c}{\mathrm{Spin}(10)\cdot S^1}$ $(\mathrm{Spin}(10)\cap S^1=\mathbb{Z}/4)$
	EVII	E_7		$\dfrac{E_7^c}{E_6^c\cdot S^1}$ $(E_6^c\cap S^1=\mathbb{Z}/3)$

Table 2: List of irreducible compact hermitian symmetric spaces. The symbols AIII, CI, ... refer to E. Cartan's classification. In the description of $\mathrm{Gr}_{m,n}$ we use $\mathrm{U}(m+n)$ instead of $G^c=\mathrm{SU}(m+n)$.

reader, we indicate how each of these arises as a quotient of a simple complex algebraic group G by a parabolic subgroup P, describing the latter in terms of marked nodes on the Dynkin diagram of G as in [FH91, § 23.3]. The last column gives an alternative description of each space as a quotient of a compact real Lie group.

On the following pages, we will run through this list of examples and, in each case, extend Kono and Hara's computations to include KO-groups twisted by a line bundle. Since each of these spaces is a "Grassmannian" in the sense that the parabolic subgroup P in G is maximal, its Picard group is free abelian on a single generator. Thus, there is exactly one non-trivial twist that we need to consider. In most cases, we — reassuringly — recover results for Witt groups that are already known. In a few other cases, we consider our results new.

The untwisted KO-theory of complete flag varieties is also known in all three classical cases thanks to Kishimoto, Kono and Ohsita. We do not reproduce their result here but instead refer the reader directly to [KKO04]. By a recent result of Calmès and Fasel, all Witt groups with non-trivial twists vanish for these varieties [CF11].

4.1 Notation

Topologically, a cellular variety is a CW complex with cells only in even (real) dimensions. For such a CW complex X the KO-groups can be written in the form displayed in Table 3 below. This was shown in Section 3.1 in the case when the twist $\mathcal{L}$ is trivial, and the general case follows: if X is a CW complex with only even-dimensional cells, so is the Thom space of any complex vector bundle over X [MS74, Lemma 18.1].

In the following examples, results on KO^* will be displayed by listing the values of the t_i and s_i. Since the t_i are just given by counting cells, and since the numbers of odd- and even-dimensional cells of a Thom space $\mathrm{Thom}_X\mathcal{E}$ only depend on X and the rank of $\mathcal{E}$, the t_i are in fact independent of $\mathcal{L}$. The s_i, on the other hand, certainly will depend on the twist, and we will sometimes acknowledge this by writing $s_i(\mathcal{L})$.

$$
\begin{aligned}
\mathrm{KO}^6(X;\mathcal{L}) &= \mathbb{Z}^{t_1} \oplus (\mathbb{Z}/2)^{s_0} = \mathrm{GW}^3(X;\mathcal{L})\\
\mathrm{KO}^7(X;\mathcal{L}) &= \phantom{\mathbb{Z}^{t_1} \oplus{}} (\mathbb{Z}/2)^{s_0} = \mathrm{W}^0(X;\mathcal{L})\\
\mathrm{KO}^0(X;\mathcal{L}) &= \mathbb{Z}^{t_0} \oplus (\mathbb{Z}/2)^{s_1} = \mathrm{GW}^0(X;\mathcal{L})\\
\mathrm{KO}^1(X;\mathcal{L}) &= \phantom{\mathbb{Z}^{t_1} \oplus{}} (\mathbb{Z}/2)^{s_1} = \mathrm{W}^1(X;\mathcal{L})\\
\mathrm{KO}^2(X;\mathcal{L}) &= \mathbb{Z}^{t_1} \oplus (\mathbb{Z}/2)^{s_2} = \mathrm{GW}^1(X;\mathcal{L})\\
\mathrm{KO}^3(X;\mathcal{L}) &= \phantom{\mathbb{Z}^{t_1} \oplus{}} (\mathbb{Z}/2)^{s_2} = \mathrm{W}^2(X;\mathcal{L})\\
\mathrm{KO}^4(X;\mathcal{L}) &= \mathbb{Z}^{t_0} \oplus (\mathbb{Z}/2)^{s_3} = \mathrm{GW}^2(X;\mathcal{L})\\
\mathrm{KO}^5(X;\mathcal{L}) &= \phantom{\mathbb{Z}^{t_1} \oplus{}} (\mathbb{Z}/2)^{s_3} = \mathrm{W}^3(X;\mathcal{L})
\end{aligned}
$$

Table 3: Notational conventions in the examples. Only the s_i depend on $\mathcal{L}$.

4.2 Projective spaces

Complex projective spaces are perhaps the simplest examples for which Theorem 2.5 asserts something non-trivial, so we describe the results here separately before turning to complex Grassmannians in general. The computations of the Witt groups of projective spaces were certainly landmark events in the history of the theory. In 1980, Arason was able to show that the Witt group $\mathrm{W}^0(\mathbb{P}^n)$ of $\mathbb{P}^n$ over a field k agrees with the Witt group of k [Ara80]. The shifted Witt groups of projective spaces, and more generally of arbitrary projective bundles, were first computed by Walter in [Wal03b]. Quite recently, Nenashev deduced the same results via different methods [Nen09].

In the topological world, complete computations of $\mathrm{KO}^i(\mathbb{CP}^n)$ were first published in a 1967 paper by Fujii [Fuj67]. It is not difficult to deduce the values of the twisted groups $\mathrm{KO}^i(\mathbb{CP}^n;\mathcal{O}(1))$ from these: the Thom space $\mathrm{Thom}(\mathcal{O}_{\mathbb{CP}^n}(1))$ can be identified with $\mathbb{CP}^{n+1}$, so

$$\begin{aligned}\mathrm{KO}^i(\mathbb{CP}^n;\mathcal{O}(1)) &= \widetilde{\mathrm{KO}}^{i+2}(\mathrm{Thom}(\mathcal{O}(1)))\\ &= \widetilde{\mathrm{KO}}^{i+2}(\mathbb{CP}^{n+1})\end{aligned}$$

Alternatively, we could do all required computations directly following the methods outlined in Section 3. The result, in any case, is displayed in Table 4, coinciding with the known results for the (Grothendieck-)Witt groups.

$\mathrm{KO}^*(\mathbb{CP}^n;\mathcal{L})$			$\mathcal{L}\equiv\mathcal{O}$				$\mathcal{L}\equiv\mathcal{O}(1)$			
	t_0	t_1	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3
$n\equiv 0 \mod 4$	$(n/2)+1$	$n/2$	1	0	0	0	1	0	0	0
$n\equiv 1$	$(n+1)/2$	$(n+1)/2$	1	1	0	0	0	0	0	0
$n\equiv 2$	$(n/2)+1$	$n/2$	1	0	0	0	0	0	1	0
$n\equiv 3$	$(n+1)/2$	$(n+1)/2$	1	0	0	1	0	0	0	0

Table 4: KO-groups of projective spaces

4.3 Grassmannians

We now consider the Grassmannians $\mathrm{Gr}_{m,n}$ of complex m-planes in $\mathbb{C}^{m+n}$. Again both the Witt groups and the untwisted KO-groups are already known: the latter by Kono and Hara [KH91], the former by the work of Balmer and Calmès [BC08]. A detailed comparison of the two sets of results in the untwisted case has been carried out by Yagita [Yag09]. We provide here an alternative topological computation of the twisted groups.

Balmer and Calmès state their result by describing an additive basis of the total Witt group of $\mathrm{Gr}_{m,n}$ in terms of certain "even Young diagrams". This is probably the most elegant approach, but needs some space to explain. We will stick instead to the tabular exposition used in the other examples. Let $\mathcal{O}(1)$ be a generator of $\mathrm{Pic}(\mathrm{Gr}_{m,n})$, say the dual of the determinant line bundle of the universal m-bundle over $\mathrm{Gr}_{m,n}$. The result is displayed in Table 5.

Our computation is based on the following geometric observation. Let $\mathcal{U}_{m,n}$ and $\mathcal{U}^{\perp}_{m,n}$ be the universal m-bundle and the orthogonal n-bundle on $\mathrm{Gr}_{m,n}$, so that $\mathcal{U}\oplus\mathcal{U}^{\perp}=\mathcal{O}^{\oplus(m+n)}$. We have various natural inclusions between the Grassmannians of different dimensions, of which we fix two:

$\mathrm{Gr}_{m,n-1}\hookrightarrow\mathrm{Gr}_{m,n}$ via the inclusion of the first $m+n-1$ coordinates into $\mathbb{C}^{m+n}$

$\mathrm{Gr}_{m-1,n}\hookrightarrow\mathrm{Gr}_{m,n}$ by sending an $(m-1)$-plane Λ to the m-plane $\Lambda\oplus\langle e_{m+n}\rangle$, where $e_1,e_2,\ldots,e_{m+n}$ are the canonical basis vectors of $\mathbb{C}^{m+n}$

$\mathrm{KO}^*(\mathrm{Gr}_{m,n};\mathcal{L})$			$\mathcal{L}\equiv\mathcal{O}$				$\mathcal{L}\equiv\mathcal{O}(1)$			
	t_0	t_1	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3
m and n odd s.t. $m\equiv n$	$\frac{a}{2}$	$\frac{a}{2}$	b	b	0	0	0	0	0	0
m and n odd s.t. $m\not\equiv n$	$\frac{a}{2}$	$\frac{a}{2}$	b	0	0	b	0	0	0	0
$\begin{cases} m\equiv n\equiv 0 \\ m\equiv 0 \text{ and } n \text{ odd} \\ n\equiv 0 \text{ and } m \text{ odd}\end{cases}$	$\frac{a+b}{2}$	$\frac{a-b}{2}$	b	0	0	0	b	0	0	0
$\begin{cases} m\equiv n\equiv 2 \\ m\equiv 2 \text{ and } n \text{ odd} \\ n\equiv 2 \text{ and } m \text{ odd}\end{cases}$	$\frac{a+b}{2}$	$\frac{a-b}{2}$	b	0	0	0	0	0	b	0
$m\equiv 0$ and $n\equiv 2$	$\frac{a+b}{2}$	$\frac{a-b}{2}$	b	0	0	0	b_1	0	b_2	0
$m\equiv 2$ and $n\equiv 0$	$\frac{a+b}{2}$	$\frac{a-b}{2}$	b	0	0	0	b_2	0	b_1	0

All equivalences ($\equiv$) are modulo 4. For the values of a and $b=b_1+b_2$, put $k:=\lfloor m/2\rfloor$ and $l:=\lfloor n/2\rfloor$. Then

$$a:=\binom{m+n}{m}\qquad b:=\binom{k+l}{k}\qquad b_1:=\binom{k+l-1}{k}\qquad b_2:=\binom{k+l-1}{k-1}$$

Table 5: KO-groups of Grassmannians

Lemma 4.2. *The normal bundle of $\mathrm{Gr}_{m,n-1}$ in $\mathrm{Gr}_{m,n}$ is the dual $\mathcal{U}^\vee_{m,n-1}$ of the universal m-bundle. Similarly, the normal bundle of $\mathrm{Gr}_{m-1,n}$ in $\mathrm{Gr}_{m,n}$ is given by $\mathcal{U}^\perp_{m-1,n}$. In both cases, the embeddings of the subspaces extend to embeddings of their normal bundles, such that one subspace is the closed complement of the normal bundle of the other.*

This gives us two cofibration sequences of pointed spaces:

$$\mathrm{Gr}_{m-1,n_+}\overset{i}{\hookrightarrow}\mathrm{Gr}_{m,n_+}\overset{p}{\twoheadrightarrow}\mathrm{Thom}(\mathcal{U}^\vee_{m,n-1})\tag{22}$$

$$\mathrm{Gr}_{m,n-1_+}\overset{i}{\hookrightarrow}\mathrm{Gr}_{m,n_+}\overset{p}{\twoheadrightarrow}\mathrm{Thom}(\mathcal{U}^\perp_{m-1,n})\tag{23}$$

These sequences are the key to relating the untwisted KO-groups to the twisted ones. Following the notation in [KH91], we write $A_{m,n}$ for the cohomology of $\mathrm{Gr}_{m,n}$ with $\mathbb{Z}/2$-coefficients, denoting by a_i and b_i the Chern classes of $\mathcal{U}$ and $\mathcal{U}^\perp$, respectively, and by a and b the total Chern classes $1+a_1+\cdots+a_m$ and

$1 + b_1 + \cdots + b_n$:

$$A_{m,n} = \frac{\mathbb{Z}/2\,[a_1, a_2, \ldots, a_m, b_1, b_2, \ldots b_n]}{a \cdot b = 1}$$

We write d for the differential given by the second Steenrod square Sq^2, and d' for $\mathrm{Sq}^2 + a_1$. To describe the cohomology of $A_{m,n}$ with respect to these differentials, it is convenient to introduce the algebra

$$B_{k,l} = \frac{\mathbb{Z}/2\,\left[a_2^2, a_4^2, \ldots, a_{2k}^2, b_2^2, b_4^2, \ldots, b_{2l}^2\right]}{(1 + a_2^2 + \cdots + a_{2k}^2)(1 + b_2^2 + \cdots + b_{2l}^2) = 1}$$

Note that this subquotient of $A_{2k,2l}$ is isomorphic to $A_{k,l}$ up to a "dilatation" in grading. Proposition 2 in [KH91] tells us that

$$H^*(A_{m,n}, d) = \begin{cases} B_{k,l} & \text{if } (m,n) = (2k, 2l),\ (2k+1, 2l) \text{ or} \\ & \phantom{\text{if } (m,n) = } (2k, 2l+1) \\ B_{k,l} \oplus B_{k,l} \cdot a_m b_{n-1} & \text{if } (m,n) = (2k+1, 2l+1) \end{cases}$$

Here, the algebra structure in the case where both m and n are odd is determined by $(a_m b_{n-1})^2 = 0$.

Lemma 4.3. *The cohomology of $A_{m,n}$ with respect to the twisted differential d' is as follows:*

$$H^*(A_{m,n}, d') = \begin{cases} B_{k,l-1} \cdot a_m \oplus B_{k-1,l} \cdot b_n & \text{if } (m,n) = (2k, 2l) \\ B_{k,l} \cdot a_m & \text{if } (m,n) = (2k, 2l+1) \\ B_{k,l} \cdot b_n & \text{if } (m,n) = (2k+1, 2l) \\ 0 & \text{if } (m,n) = (2k+1, 2l+1) \end{cases}$$

Proof. Let us shift the dimensions in the cofibration sequences (22) and (23) in such a way that we have the Thom spaces of $\mathcal{U}^\vee_{m,n}$ and $\mathcal{U}^\perp_{m,n}$ on the right. Since the cohomologies of the spaces involved are concentrated in even degrees, the associated long exact sequence of cohomology groups falls apart into short exact sequences. Reassembling these, we obtain two short exact sequences of differential $(A_{m,n+1}, d)$- and $(A_{m+1,n}, d)$-modules, respectively:

$$0 \to (A_{m,n}, d') \cdot \theta^\vee \xrightarrow{p^*} (A_{m,n+1}, d) \xrightarrow{i^*} (A_{m-1,n+1}, d) \to 0 \tag{24}$$

$$0 \to (A_{m,n}, d') \cdot \theta^\perp \xrightarrow{p^*} (A_{m+1,n}, d) \xrightarrow{i^*} (A_{m+1,n-1}, d) \to 0 \tag{25}$$

Here, $\theta^\vee$ and $\theta^\perp$ are the respective Thom classes of $\mathcal{U}^\vee_{m,n}$ and $\mathcal{U}^\perp_{m,n}$. The map i^* in the first row is the obvious quotient map annihilating a_m. Its kernel, the image of $A_{m,n}$ under multiplication by a_m, is generated as an $A_{m,n+1}$-module by its unique element in degree $2m$, and thus we must have $p^*(\theta^\vee) = a_m$. Likewise, in the second row we have $p^*(\theta^\perp) = b_n$.

The lemma can be deduced from here case by case. For example, when both m and n are even, i^* maps $H^*(A_{m,n+1}, d) = B_{k,l}$ to the first summand of $H^*(A_{m-1,n+1}, d) = B_{k-1,l} \oplus B_{k-1,l} \cdot a_{m-1}b_n$ by annihilating a_m^2. We know by comparison with the short exact sequences for the $A_{m,n}$ that the kernel of this map is $B_{k,l-1}$ mapping to $B_{k,l}$ under multiplication by a_m^2. Thus, we obtain a short exact sequence

$$0 \to B_{k-1,l} \cdot a_{m-1}b_n \xrightarrow{\partial} H^*(A_{m,n}, d') \cdot \theta^\vee \xrightarrow{p^*} B_{k,l-1} \cdot a_m^2 \to 0 \qquad (26)$$

For the Steenrod square Sq^2 of the top Chern class a_m of $\mathcal{U}$, we have $\mathrm{Sq}^2(a_m) = a_1a_m$. This can be checked, for example, by expressing a_m as the product of the Chern roots of $\mathcal{U}$. Consequently, $d'(a_m) = 0$. Together with the fact that $H^*(A_{m,n}, d')$ is a module over $H^*(A_{m,n+1}, d)$, this shows that we can define a splitting of p^* by sending a_m^2 to $a_m\theta^\vee$. Thus, $H^*(A_{m,n}, d')$ contains $B_{k,l-1} \cdot a_m$ as a direct summand. If instead of working with sequence (24) we work with sequence (25), we see that $H^*(A_{m,n}, d')$ also contains a direct summand $B_{k-1,l} \cdot b_n$. These two summands intersect trivially, and a dimension count shows that together they encompass all of $H^*(A_{m,n}, d')$. Alternatively, one may check explicitly that the boundary map ∂ above sends $a_{m-1}b_n$ to $b_n\theta$. The other cases are simpler. □

Lemma 4.4. *The Atiyah-Hirzebruch spectral sequence for* $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{U}_{m,n}^\vee)$ *collapses at the* E_3*-page.*

Proof. By Proposition 4 of [KH91] we know that the spectral sequence for $\mathrm{KO}^*(\mathrm{Gr}_{m,n})$ collapses as this stage, for any m and n. Now, if both m and n are even, we have

$$(B_{k,l-1} \cdot a_m \oplus B_{k-1,l} \cdot b_n) \cdot \theta$$

in the $(-1)^{\text{st}}$ row of the E_3-pages of the spectral sequences for $\mathrm{Thom}\,\mathcal{U}^\vee$ and $\mathrm{Thom}\,\mathcal{U}^\perp$, where $\theta = \theta^\vee$ or $\theta^\perp$, respectively. In the case of $\mathcal{U}^\vee$ we see from (26) that p^* maps the second summand injectively to the E_3-page of the spectral sequence for $\mathrm{KO}^*(\mathrm{Gr}_{m,n+1})$. Similarly, in the case of $\mathcal{U}^\perp$, the first summand is mapped injectively to the E_3-page of $\mathrm{KO}^*(\mathrm{Gr}_{m+1,n})$. Since the spectral sequences for $\mathrm{Thom}\,\mathcal{U}^\vee$ and $\mathrm{Thom}\,\mathcal{U}^\perp$ can be identified via Corollary 3.8, we can argue as in Corollary 3.4 to see that they must collapse at this stage. Again, the cases when at least one of m, n is odd are similar but simpler. □

We may now apply Corollary 3.6. The entries of Table 5 that do not appear in [KH91], i.e. those of the last four columns, follow from Lemma 4.3 by noting that $B_{k,l}$ is concentrated in degrees $8i$ and of dimension $\dim B_{k,l} = \dim A_{k,l} = \binom{k+l}{k}$.

4.4 Maximal symplectic Grassmannians

The Grassmannian of isotropic n-planes in $\mathbb{C}^{2n}$ with respect to a non-degenerate skew-symmetric bilinear form is given by $X_n = \mathrm{Sp}(n)/U(n)$. The universal bundle $\mathcal{U}$ on the usual Grassmannian $\mathrm{Gr}(n,2n)$ restricts to the universal bundle on X_n, and so does the orthogonal complement bundle $\mathcal{U}^\perp$. We will continue to denote these restrictions by the same letters. Thus, $\mathcal{U}\oplus\mathcal{U}^\perp\cong\mathbb{C}^{2n}$ on X_n, and the fibres of $\mathcal{U}$ are orthogonal to those of $\mathcal{U}^\perp$ with respect to the standard hermitian metric on $\mathbb{C}^{2n}$. The determinant line bundles of $\mathcal{U}$ and $\mathcal{U}^\perp$ give dual generators $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ of the Picard group of X_n.

Theorem 4.5. *The additive structure of* $\mathrm{KO}^*(X_n;\mathcal{L})$ *is as follows:*

	t_0	t_1	$s_i(\mathcal{O})$	$s_i(\mathcal{O}(1))$
n *even*	2^{n-1}	2^{n-1}	$\rho(\frac{n}{2},i)$	$\rho(\frac{n}{2},i-n)$
n *odd*	2^{n-1}	2^{n-1}	$\rho(\frac{n+1}{2},i)$	0

Here, for any $i\in\mathbb{Z}/4$ we write $\rho(n,i)$ for the dimension of the i-graded piece of a $\mathbb{Z}/4$-graded exterior algebra $\Lambda_{\mathbb{Z}/2}(g_1,g_2,\dots,g_n)$ on n homogeneous generators g_1, g_2, $\dots$, g_n of degree 1, i. e.

$$\rho(n,i)=\sum_{\substack{d\equiv i\\ \bmod 4}}\binom{n}{d}$$

A table of the values of $\rho(n,i)$ can be found in [KH92, Proposition 4.1].

It turns out to be convenient to work with the vector bundle $\mathcal{U}^\perp\oplus\mathcal{O}$ for the computation of the twisted groups $\mathrm{KO}^*(X_n;\mathcal{O}(1))$. Namely, we have the following analogue of Lemma 4.2.

Lemma 4.6. *There is an open embedding of the bundle $\mathcal{U}^\perp\oplus\mathcal{O}$ over the symplectic Grassmannian X_n into the symplectic Grassmannian X_{n+1} whose closed complement is again isomorphic to X_n.*

Proof. To fix notation, let e_1,e_2 be the first two canonical basis vectors of $\mathbb{C}^{2n+2}$, and embed $\mathbb{C}^{2n}$ into $\mathbb{C}^{2n+2}$ via the remaining coordinates. Assuming X_n is defined in terms of a skew-symmetric form Q_{2n}, define X_{n+1} with respect to the form

$$Q_{2n+2}:=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&Q_{2n}\end{pmatrix}$$

Then we have embeddings i_1 and i_2 of X_n into X_{n+1} sending an n-plane $\Lambda\subset\mathbb{C}^{2n}$ to $e_1\oplus\Lambda$ or $e_2\oplus\Lambda$ in $\mathbb{C}^{2n+2}$, respectively.
We extend i_1 to an embedding of $\mathcal{U}^\perp\oplus\mathcal{O}$ by sending an n-plane $\Lambda\in X_n$ together with a vector v in $\Lambda^\perp\subset\mathbb{C}^{2n}$ and a complex scalar z to the graph $\Gamma_{\Lambda,v,z}\subset\mathbb{C}^{2n+2}$ of the linear map

$$\begin{pmatrix}z&Q_{2n}(-,v)\\v&0\end{pmatrix}:\langle e_1\rangle\oplus\Lambda\to\langle e_2\rangle\oplus\Lambda^\perp$$

To avoid confusion, we emphasize that v is orthogonal to Λ with respect to a *hermitian* metric on $\mathbb{C}^{2n}$. The value of $Q_{2n}(-,v)$, on the other hand, may well be non-zero on Λ. Consider the above embedding of $\mathcal{U}^{\perp} \oplus \mathcal{O}$ together with the embedding i_2:

$$\begin{array}{ccccc} \mathcal{U}^{\perp} \oplus \mathcal{O} & \hookrightarrow & X_{n+1} & \xleftarrow{i_2} & X_n \\ (\Lambda, v, z) & \mapsto & \Gamma_{\Lambda,v,z} & & \\ & & \langle e_2 \rangle \oplus \Lambda & \leftarrow & \Lambda \end{array}$$

To see that the two embeddings are complementary, take an arbitrary $(n+1)$-plane W in X_{n+1}. If $e_2 \in W$ then we can consider a basis

$$e_2, \begin{pmatrix} a_1 \\ 0 \\ v_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ 0 \\ v_n \end{pmatrix}$$

of W, and the fact that Q_{2n+2} vanishes on W implies that all a_i are zero. Thus W can be identified with $i_2(\langle v_1, \dots, v_n \rangle)$.
If, on the other hand, e_2 is not contained in W then we must have a vector of the form ${}^t(1, z', v')$ in W, for some $z' \in \mathbb{C}$ and $v' \in \mathbb{C}^{2n}$. Extend this vector to a basis of W of the form

$$\begin{pmatrix} 1 \\ z' \\ v' \end{pmatrix}, \begin{pmatrix} 0 \\ b_1 \\ v_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ b_n \\ v_n \end{pmatrix}$$

and let $\Lambda := \langle v_1, \dots, v_n \rangle$. The condition that Q_{2n+2} vanishes on W implies that Q vanishes on Λ and that $b_i = Q_{2n}(v_i, v')$ for each i. In particular, Λ is n-dimensional. Moreover, we can replace the first vector of our basis by a vector ${}^t(1, z, v)$ with $v \in \Lambda^{\perp}$, by subtracting appropriate multiples of the remaining basis vectors. Since Q vanishes on Λ we have $Q_{2n}(v_i, v') = Q_{2n}(v_i, v)$ and our new basis has the form

$$\begin{pmatrix} 1 \\ z \\ v \end{pmatrix}, \begin{pmatrix} 0 \\ Q(v_1, v) \\ v_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ Q(v_n, v) \\ v_n \end{pmatrix}$$

This shows that $W = \Gamma_{\Lambda,v,z}$. □

Corollary 4.7. *We have a cofibration sequence*

$$X_{n+} \overset{i}{\hookrightarrow} X_{n+1+} \overset{p}{\twoheadrightarrow} \mathrm{Thom}_{X_n}(\mathcal{U}^{\perp} \oplus \mathcal{O})$$

The associated long exact cohomology sequence splits into a short exact sequence of $H^*(X_{n+1})$-modules since all cohomology here is concentrated in even degrees:

$$0 \to \widetilde{H}^*(\mathrm{Thom}_{X_n}(\mathcal{U}^{\perp} \oplus \mathcal{O})) \xrightarrow{p^*} H^*(X_{n+1}) \xrightarrow{i^*} H^*(X_n) \to 0 \tag{27}$$

Lemma 4.8. *Let c_i denote the i^{th} Chern classes of $\mathcal{U}$ over X_n. We have*

$$H^*(X_n, \mathrm{Sq}^2) = \begin{cases} \Lambda(a_1, a_5, a_9, \dots, a_{4m-3}) & \text{if } n = 2m \\ \Lambda(a_1, a_5, a_9, \dots, a_{4m-3}, a_{4m+1}) & \text{if } n = 2m+1 \end{cases}$$

$$H^*(X_n, \mathrm{Sq}^2 + c_1) = \begin{cases} \Lambda(a_1, a_5, \dots, a_{4m-3}) \cdot c_{2m} & \text{if } n = 2m \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

for certain generators a_i of degree $2i$.

Proof. Consider the short exact sequence (27). The mod-2 cohomology of X_n is an exterior algebra on the Chern classes c_i of $\mathcal{U}$,

$$H^*(X_n;\mathbb{Z}/2) = \Lambda(c_1, c_2, \dots, c_n)$$

and i^* is given by sending c_{n+1} to zero. Thus, p^* is the unique morphism of $H^*(X_{n+1};\mathbb{Z}/2)$-modules that sends the Thom class θ of $\mathcal{U}^\perp \oplus \mathcal{O}$ to c_{n+1}.
This short exact sequence induces a long exact sequence of cohomology groups with respect to the Steenrod square Sq^2. The algebra $H^*(X_n, \mathrm{Sq}^2)$ was computed in [KH92, 2–2], with the result displayed above, so we already know two thirds of this sequence. Explicitly, we have $a_{4i+1} = c_{2i}c_{2i+1}$,[6] so i^* is the obvious surjection sending a_i to a_i (or to zero). Thus, the long exact sequence once again splits.
If $n = 2m$ we obtain a short exact sequence

$$0 \to H^*(X_{2m}, \mathrm{Sq}^2 + c_1)\cdot\theta \xrightarrow{p^*} \Lambda(a_1, \dots, a_{4m-3}, a_{4m+1}) \xrightarrow{i^*} \Lambda(a_1, \dots, a_{4m-3}) \to 0$$

We see that $H^*(X_{2m}, \mathrm{Sq}^2 + c_1)\cdot\theta$ is isomorphic to $\Lambda(a_1, \dots, a_{4m-3})\cdot a_{4m+1}$ as a $\Lambda(a_1, \dots, a_{4m+1})$-module. It is thus generated by a single element, which is the unique element of degree $8m+2$. Since $p^*(c_{2m}\theta) = a_{4m+1}$, the class of $c_{2m}\theta$ is the element we are looking for, and the result displayed above follows. If, on the other hand, n is odd, then i^* is an isomorphism and $H^*(X_n, \mathrm{Sq}^2 + c_1)$ must be trivial. ☐

We see from the proof that p^* induces an injection of $H^*(X_n, \mathrm{Sq}^2 + c_1)\cdot\theta$ into $H^*(X_n, \mathrm{Sq}^2)$. Since we already know from [KH92, Theorem 2.1] that the Atiyah-Hirzebruch spectral sequence for $\mathrm{KO}^*(X_n)$ collapses, we can apply Corollary 3.4 to deduce that the spectral sequence for $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}_{X_n}(\mathcal{U}^\perp \oplus \mathcal{O}))$ collapses at the E_3-page as well. This completes the proof of Theorem 4.5.

4.5 Quadrics

We next consider smooth complex quadrics Q^n in $\mathbb{P}^{n+1}$. As far as we are aware, the first complete results on (shifted) Witt groups of split quadrics were due to Walter: they are mentioned together with the results for projective bundles in [Wal03a] as the main applications of that paper. Unfortunately, they seem to have remained unpublished. Partial results are also included in Yagita's preprint [Yag04], see Corollary 8.3. More recently, Nenashev obtained almost complete results by considering the localization sequences arising from the inclusion of a linear subspace of maximal dimension [Nen09]. Calmès informs me that the geometric description of the boundary map given in [BC09] can be used to show that these localization sequences split in general, yielding a complete computation. The calculation described here is completely independent of these results.

[6] In [KH92] the generators are written as $c_{2i}c'_{2i+1}$ with $c'_{2i+1} = c_{2i+1} + c_1c_{2i}$.

$\mathrm{KO}^*(Q^n;\mathcal{L})$			$\mathcal{L}\equiv\mathcal{O}$				$\mathcal{L}\equiv\mathcal{O}(1)$			
	t_0	t_1	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3
$n\equiv 0 \mod 8$	$(n/2)+2$	$n/2$	2	0	0	0	2	0	0	0
$n\equiv 1$	$(n+1)/2$	$(n+1)/2$	1	1	0	0	1	1	0	0
$n\equiv 2$	$(n/2)+1$	$(n/2)+1$	1	2	1	0	0	0	0	0
$n\equiv 3$	$(n+1)/2$	$(n+1)/2$	1	1	0	0	0	0	1	1
$n\equiv 4$	$(n/2)+2$	$n/2$	2	0	0	0	0	0	2	0
$n\equiv 5$	$(n+1)/2$	$(n+1)/2$	1	0	0	1	0	1	1	0
$n\equiv 6$	$(n/2)+1$	$(n/2)+1$	1	0	1	2	0	0	0	0
$n\equiv 7$	$(n+1)/2$	$(n+1)/2$	1	0	0	1	1	0	0	1

Table 6: KO-groups of projective quadrics ($n\geq 3$)

For $n\geq 3$ the Picard group of Q^n is free abelian on a single generator given by the restriction of the universal line bundle $\mathcal{O}(1)$ over $\mathbb{P}^{n+1}$. We will use the same notation $\mathcal{O}(1)$ for this restriction.

Theorem 4.9. *The KO-theory of a smooth complex quadric Q^n of dimension $n\geq 3$ is as described in Table 6.*

Untwisted KO-groups. Before turning to $\mathrm{KO}^*(Q^n;\mathcal{O}(1))$ we review the initial steps in the computation of the untwisted KO-groups. The integral cohomology of Q^n is well-known:
If n is even, write $n=2m$. We have a class x in $H^2(Q^n)$ given by a hyperplane section, and two classes a and b in $H^n(Q^n)$ represented by linear subspaces of Q of maximal dimension. These three classes generate the cohomology multiplicatively, modulo the relations

$$x^m = a+b \qquad\qquad x^{m+1}=2ax$$
$$ab=\begin{cases}0 & \text{if } n\equiv 0\\ ax^m & \text{if } n\equiv 2\end{cases}\qquad a^2=b^2=\begin{cases}ax^m & \text{if } n\equiv 0 \mod 4\\ 0 & \text{if } n\equiv 2 \mod 4\end{cases}$$

Additive generators can thus be given as follows:

d	0	2	4	$\ldots$	$n-2$	n	$n+2$	$n+4$	$\ldots$	$2n$
$H^d(Q^n)$	1	x	x^2	$\ldots$	x^{m-1}	a,b	ax	ax^2	$\ldots$	ax^m

If n is odd, write $n=2m+1$. Then similarly multiplicative generators are given by the class of a hyperplane section x in $H^2(Q^n)$ and the class of a linear subspace a in $H^{n+1}(Q^n)$ modulo the relations $x^{m+1}=2a$ and $a^2=0$.

d	0	2	4	$\ldots$	$n-1$	$n+1$	$n+3$	$n+5$	$\ldots$	$2n$
$H^d(Q^n)$	1	x	x^2	$\ldots$	x^m	a	ax	ax^2	$\ldots$	ax^m

The action of the Steenrod square on $H^*(Q^n;\mathbb{Z}/2)$ is also well-known; see for example [Ish92, Theorem 1.4 and Corollary 1.5] or [EKM08, § 78]:

$$\mathrm{Sq}^2(x) = x^2$$
$$\mathrm{Sq}^2(a) = \begin{cases} ax & \text{if } n \equiv 0 \text{ or } 3 \mod 4 \\ 0 & \text{if } n \equiv 1 \text{ or } 2 \end{cases}$$
$$\mathrm{Sq}^2(b) = \mathrm{Sq}^2(a) \quad \text{(for even } n\text{)}$$

As before, we write $H^*(Q^n, \mathrm{Sq}^2)$ for the cohomology of $H^*(Q^n;\mathbb{Z}/2)$ with respect to the differential Sq^2.

Lemma 4.10. *Write $n = 2m$ or $n = 2m+1$ as above. The following table gives a complete list of the additive generators of $H^*(Q^n, \mathrm{Sq}^2)$.*

d	0	$\dots$	$n-1$	n	$n+1$	$\dots$	$2n$	
$H^d(Q^n, \mathrm{Sq}^2)$	1						ax^m	*if* $n \equiv 0 \mod 4$
	1				a			*if* $n \equiv 1$
	1			a, b			ab	*if* $n \equiv 2$
	1		x^m					*if* $n \equiv 3$

The results of Kono and Hara on $\mathrm{KO}^*(Q)$ follow from here provided there are no non-trivial higher differentials in the Atiyah-Hirzebruch spectral sequence. This is fairly clear in all cases except for the case $n \equiv 2 \mod 4$. In that case, the class $a + b = x^m$ can be pulled back from Q^{n+1}, and therefore all higher differentials must vanish on $a + b$. But one has to work harder to see that all higher differentials vanish on a (or b). Kono and Hara proceed by relating the KO-theory of Q^n to that of the spinor variety $S_{\frac{n}{2}+1}$ discussed in Section 4.6.

Twisted KO-groups. We now compute $\mathrm{KO}^*(Q^n;\mathcal{O}(1))$.
Let $\theta \in H^2(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ be the Thom class of $\mathcal{O}(1)$, so that multiplication by θ maps the cohomology of Q^n isomorphically to the reduced cohomology of $\mathrm{Thom}_{Q^n}\mathcal{O}(1)$. The Steenrod square on $\widetilde{H}^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1);\mathbb{Z}/2)$ is determined by Lemma 3.5: for any $y \in H^*(Q^n;\mathbb{Z}/2)$ we have $\mathrm{Sq}^2(y\cdot\theta) = (\mathrm{Sq}^2 y + xy)\cdot\theta$. We thus arrive at

Lemma 4.11. *The following table gives a complete list of the additive generators of $\widetilde{H}^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$.*

d	$\dots$	$n+1$	$n+2$	$n+3$	$\dots$	$2n+2$	
$\widetilde{H}^d(\dots)$			$a\theta, b\theta$				*if* $n \equiv 0 \mod 4$
		$x^m\theta$				$ax^m\theta$	*if* $n \equiv 1$
							if $n \equiv 2$
				$a\theta$		$ax^m\theta$	*if* $n \equiv 3$

We claim that all higher differentials in the Atiyah-Hirzebruch spectral sequence for $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ vanish. For even n this is clear. But for $n = 8k+1$ the differential d_{8k+2} might a priori take $x^m\theta$ to $ax^m\theta$, and for $n = 8k+3$ the differential d_{8k+2} might take $a\theta$ to $ax^m\theta$.

We therefore need some geometric considerations. Namely, the Thom space $\mathrm{Thom}_{Q^n}\mathcal{O}(1)$ can be identified with the projective cone over Q^n embedded in $\mathbb{P}^{n+2}$. This projective cone can be realized as the intersection of a smooth quadric $Q^{n+2} \subset \mathbb{P}^{n+3}$ with its projective tangent space at the vertex of the cone [Har92, p. 283]. Thus, we can consider the following inclusions:

$$Q^n \overset{j}{\hookrightarrow} \mathrm{Thom}_{Q^n}\mathcal{O}(1) \overset{i}{\hookrightarrow} Q^{n+2}$$

The composition is the inclusion of the intersection of Q^{n+2} with two transversal hyperplanes.

Lemma 4.12. *All higher differentials (d_k with $k > 2$) in the Atiyah-Hirzebruch spectral sequence for* $\mathrm{KO}^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ *vanish.*

Proof. We need only consider the cases when n is odd. Write $n = 2m+1$.

When $n \equiv 1 \bmod 4$ we claim that i^* maps x^{m+1} in $H^{n+1}(Q^{n+2}, \mathrm{Sq}^2)$ to $x^m\theta$ in $H^{n+1}(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$. Indeed, j^*i^* maps the class of the hyperplane section x in $H^2(Q^{n+2})$ to the class of the hyperplane section x in $H^2(Q^n)$. So i^*x in $H^2(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ must be non-zero, hence equal to θ modulo 2. It follows that $i^*(x^{m+1}) = \theta^{m+1}$. Since $\theta^2 = \mathrm{Sq}^2(\theta) = x\theta$, we have $\theta^{m+1} = x^m\theta$, proving the claim. As we already know that all higher differentials vanish on $H^*(Q^{n+2}, \mathrm{Sq}^2)$, we may now deduce that they also vanish on $H^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$.

When $n \equiv 3 \bmod 4$ we claim that i^* maps a in $H^{n+3}(Q^{n+2}, \mathrm{Sq}^2)$ to $a\theta$ in $H^{n+3}(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$. Indeed, a represents a linear subspace of codimension $m+2$ in Q^{n+2} and is thus mapped to the class of a linear subspace of the same codimension in Q^n: $j^*i^*(a) = ax$ in $H^{n+3}(Q^n)$. Thus, $i^*(a)$ is non-zero in $H^{n+3}(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$, equal to $a\theta$ modulo 2. Again, this implies that all higher differentials vanish on $H^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$ since they vanish on $H^*(Q^{n+2}, \mathrm{Sq}^2)$. ☐

The additive structure of $\mathrm{KO}^*(Q^n; \mathcal{O}(1))$ thus follows directly from the result for $H^d(Q^n, \mathrm{Sq}^2 + x) = \widetilde{H}^{d+2}(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ displayed in Lemma 4.11 via Corollary 3.6.

4.6 Spinor varieties

Let $\mathrm{Gr}_{\mathrm{SO}}(n, N)$ be the Grassmannian of n-planes in $\mathbb{C}^N$ isotropic with respect to a fixed non-degenerate symmetric bilinear form, or, equivalently, the Fano variety of projective $(n-1)$-planes contained in the quadric Q^{N-2}. For each $N > 2n$, this is an irreducible homogeneous variety. In particular, for $N = 2n+1$ we obtain the spinor variety $S_{n+1} = \mathrm{Gr}_{\mathrm{SO}}(n, 2n+1)$. The variety $\mathrm{Gr}_{\mathrm{SO}}(n, 2n)$

falls apart into two connected components, both of which are isomorphic to S_n. This is reflected by the fact that we can equivalently identify S_n with $\mathrm{SO}(2n-1)/U(n-1)$ or $\mathrm{SO}(2n)/U(n)$.
As for all Grassmannians, the Picard group of S_n is isomorphic to $\mathbb{Z}$; we fix a line bundle $\mathcal{S}$ which generates it. The KO-theory twisted by $\mathcal{S}$ vanishes:

Theorem 4.13. *For all $n \geq 2$ the additive structure of $\mathrm{KO}^*(S_n;\mathcal{L})$ is as follows:*

	t_0	t_1	$s_i(\mathcal{O})$	$s_i(\mathcal{S})$
$n \equiv 2 \mod 4$	2^{n-2}	2^{n-2}	$\rho(\frac{n}{2}, 1-i)$	0
otherwise	2^{n-2}	2^{n-2}	$\rho(\lfloor\frac{n}{2}\rfloor, -i)$	0

Here, the values $\rho(n,i)$ are defined as in Theorem 4.5.

Proof. The cohomology of S_n with $\mathbb{Z}/2$-coefficients has simple generators e_2, e_4, …, e_{2n-2}, i. e. it is additively generated by products of distinct elements of this list. Its multiplicative structure is determined by the rule $e_{2i}^2 = e_{4i}$, and the second Steenrod square is given by $\mathrm{Sq}^2(e_{2i}) = ie_{2i+2}$ [Ish92, Proposition 1.1]. In both formulae it is of course understood that $e_{2j} = 0$ for $j \geq n$. What we need to show is that for all $n \geq 2$ we have

$$H^*(S_n, \mathrm{Sq}^2 + e_2) = 0$$

Let us abbreviate $H^*(S_n, \mathrm{Sq}^2 + e_2)$ to (H_n, d'). We claim that we have the following short exact sequence of differential $\mathbb{Z}/2$-modules:

$$0 \to (H_n, d') \overset{\cdot e_{2n}}{\to} (H_{n+1}, d') \to (H_n, d') \to 0 \tag{28}$$

This can be checked by a direct calculation. Alternatively, it can be deduced from the geometric considerations below. Namely, it follows from the cofibration sequence of Corollary 4.15 that we have such an exact sequence of $\mathbb{Z}/2$-modules with maps respecting the differentials given by Sq^2 on all three modules. Since they also commute with multiplication by e_2, they likewise respect the differential $d' = \mathrm{Sq}^2 + e_2$.
The long exact cohomology sequence associated with (28) allows us to argue by induction: if $H^*(H_n, d') = 0$ then also $H^*(H_{n+1}, d') = 0$. Since we can see by hand that $H^*(H_2, d') = 0$, this completes the proof. $\square$

We close with a geometric interpretation of the exact sequence (28), via an analogue of Lemmas 4.2 and 4.6. Let us write $\mathcal{U}$ for the universal bundle over S_n, i. e. for the restriction of the universal bundle over $\mathrm{Gr}(n-1, 2n-1)$ to S_n, and $\mathcal{U}^\perp$ for the restriction of the orthogonal complement bundle, so that $\mathcal{U} \oplus \mathcal{U}^\perp$ is the trivial $(2n-1)$-bundle over S_n. As in Section 4.4, we emphasize that under these conventions the fibres of $\mathcal{U}$ and $\mathcal{U}^\perp$ are perpendicular with respect to a hermitian metric on $\mathbb{C}^{2n-1}$ — they are not orthogonal with respect to the chosen symmetric form.

Lemma 4.14. *The spinor variety S_n embeds into the spinor variety S_{n+1} with normal bundle $\mathcal{U}^\perp$ such that the embedding extends to an embedding of this bundle. The closed complement of $\mathcal{U}^\perp$ in S_{n+1} is again isomorphic to S_n.*

Corollary 4.15. *We have a cofibration sequence*

$$S_{n+} \xhookrightarrow{i} S_{n+1_+} \xrightarrow{p} \mathrm{Thom}_{S_n}\,\mathcal{U}^\perp$$

Note however that, unlike in the symplectic case, the first Chern classes of $\mathcal{U}$ and $\mathcal{U}^\perp$ pull back to *twice* a generator of the Picard group of S_n. For example, the embedding of S_2 into $\mathrm{Gr}(1,3)$ can be identified with the embedding of the one-dimensional smooth quadric into the projective plane, of degree 2, and the higher dimensional cases can be reduced to this example. Thus, $c_1(\mathcal{U})$ and $c_1(\mathcal{U}^\perp)$ are trivial in $\mathrm{Pic}(S_n)/2$.

proof of Lemma 4.14. The proof is similar to the proof of Lemma 4.6. Let e_1, e_2 be the first two canonical basis vectors of $\mathbb{C}^{2n+1}$, and let $\mathbb{C}^{2n-1}$ be embedded into $\mathbb{C}^{2n+1}$ via the remaining coordinates. Let S_n be defined in terms of a symmetric form Q on $\mathbb{C}^{2n-1}$, and define S_{n+1} in terms of

$$Q_{2n+1} := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & Q \end{pmatrix}$$

Let i_1 and i_2 be the embeddings of S_n into S_{n+1} sending an $(n-1)$-plane $\Lambda \subset \mathbb{C}^{2n-1}$ to $e_1 \oplus \Lambda$ or $e_2 \oplus \Lambda$ in $\mathbb{C}^{2n+1}$, respectively. Given an $(n-1)$-plane $\Lambda \in S_n$ together with a vector v in $\Lambda^\perp \subset \mathbb{C}^{2n-1}$, consider the linear map

$$\begin{pmatrix} -\frac{1}{2}Q(v,v) & -Q(-,v) \\ v & 0 \end{pmatrix} : \langle e_1 \rangle \oplus \Lambda \to \langle e_2 \rangle \oplus \Lambda^\perp$$

Sending (Λ, v) to the graph of this function defines an open embedding of $\mathcal{U}^\perp$ whose closed complement is the image of i_2. □

4.7 Exceptional hermitian symmetric spaces

Lastly, we turn to the exceptional hermitian symmetric spaces EIII and EVII. We write $\mathcal{O}(1)$ for a generator of the Picard group in both cases.

Theorem 4.16. *The KO-groups of the exceptional hermitian symmetric spaces* EIII *and* EVII *are as follows:*

			$\mathcal{L} \equiv \mathcal{O}$				$\mathcal{L} \equiv \mathcal{O}(1)$			
	t_0	t_1	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3
$\mathrm{KO}^*(\mathrm{EIII};\mathcal{L})$	15	12	3	0	0	0	3	0	0	0
$\mathrm{KO}^*(\mathrm{EVII};\mathcal{L})$	28	28	1	3	3	1	0	0	0	0

Proof. The untwisted KO-groups have been computed in [KH92], the main difficulty as always being to prove that the Atiyah-Hirzebruch spectral sequence collapses. For the twisted groups, however, there are no problems. We quote from § 3 of said paper that the cohomologies of the spaces in question can be written as

$$H^*(\mathrm{EIII};\mathbb{Z}/2) = \mathbb{Z}/2\big[t,u\big]\Big/\big(u^2t,\ u^3+t^{12}\big)$$
$$H^*(\mathrm{EVII};\mathbb{Z}/2) = \mathbb{Z}/2\big[t,v,w\big]\Big/\big(t^{14},\ v^2,\ w^2\big)$$

with t of degree 2 in both cases, and u, v and w of degrees $8, 10$ and 18, respectively. The Steenrod squares are determined by $\mathrm{Sq}^2\,u = ut$ and $\mathrm{Sq}^2\,v = \mathrm{Sq}^2\,w = 0$. Thus, we find

$$H^*(\mathrm{EIII},\mathrm{Sq}^2+t) = \mathbb{Z}/2\cdot u \oplus \mathbb{Z}/2\cdot u^2 \oplus \mathbb{Z}/2\cdot u^3$$
$$H^*(\mathrm{EVII},\mathrm{Sq}^2+t) = 0$$

By Lemma 3.3 the Atiyah-Hirzebruch spectral sequence for EIII must collapse. This gives the result displayed above. □

Acknowledgements

The present paper is based on research carried out in the context of the author's PhD studies, which have been generously funded by the Engineering and Physical Sciences Research Council (EPSRC). I wish to thank my supervisor Burt Totaro for his commitment, time and patience, for directing me towards the problems and ideas discussed here, and for his continuing support. Special thanks are moreover due to Baptiste Calmès and Marco Schlichting for highly helpful conversations and explanations of their respective work. Finally, I am grateful to Jens Hornbostel for constructive comments on an earlier version of this paper, and to the anonymous referee for careful proofreading.

References

[Ada95] Frank Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1995. Reprint of the 1974 original.

[Ara80] Jón Kristinn Arason, *Der Wittring projektiver Räume*, Math. Ann. 253 (1980), no. 3, 205–212.

[ABS64] Michael Atiyah, Raoul Bott, and Arnold Shapiro, *Clifford modules*, Topology 3 (1964), no. suppl. 1, 3–38.

[Bal00] Paul Balmer, *Triangular Witt groups. I. The 12-term localization exact sequence*, K-Theory 19 (2000), no. 4, 311–363.

[Bal01a] ———, *Triangular Witt groups. II. From usual to derived*, Math. Z. 236 (2001), no. 2, 351–382.

[Bal01b] ———, *Witt cohomology, Mayer-Vietoris, homotopy invariance and the Gersten conjecture*, K-Theory 23 (2001), no. 1, 15–30.

[Bal05] ———, *Witt groups*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 539–576.

[BC09] Paul Balmer and Baptiste Calmès, *Geometric description of the connecting homomorphism for Witt groups*, Doc. Math. 14 (2009), 525–550.

[BC08] ———, *Witt groups of Grassmann varieties* (2008), available at www.math.uiuc.edu/K-theory/0903/. Preprint.

[BG05] Paul Balmer and Stefan Gille, *Koszul complexes and symmetric forms over the punctured affine space*, Proc. London Math. Soc. (3) 91 (2005), no. 2, 273–299.

[BGG73] Joseph Bernstein, Israel Gelfand, and Sergei Gelfand, *Schubert cells, and the cohomology of the spaces G/P*, Uspehi Mat. Nauk 28 (1973), no. 3(171), 3–26 (Russian); English transl., Russian Math. Surveys 28 (1973), no. 3, 1–26.

[Bot69] Raoul Bott, *Lectures on $K(X)$*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[BG10] Robert Bruner and John Greenlees, *Connective real K-theory of finite groups*, Mathematical Surveys and Monographs, vol. 169, American Mathematical Society, Providence, RI, 2010.

[CF11] Baptiste Calmès and Jean Fasel, *Trivial Witt groups of flag varieties* (2011), available at arxiv.org/abs/1103.4412. Preprint.

[Dug01] Daniel Dugger, *Universal homotopy theories*, Adv. Math. 164 (2001), no. 1, 144–176.

[DLØ+07] Bjørn Ian Dundas, Marc Levine, Paul Arne Østvær, Oliver Röndigs, and Vladimir Voevodsky, *Motivic homotopy theory*, Universitext, Springer-Verlag, Berlin, 2007. Lectures from the Summer School held in Nordfjordeid, August 2002.

[EKM08] Richard Elman, Nikita Karpenko, and Alexander Merkurjev, *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications, vol. 56, American Mathematical Society, Providence, RI, 2008.

[Fuj67] Michikazu Fujii, *K_0-groups of projective spaces*, Osaka J. Math. 4 (1967), 141–149.

[FH91] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991.

[Gil03] Stefan Gille, *Homotopy invariance of coherent Witt groups*, Math. Z. 244 (2003), no. 2, 211–233.

[Har92] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1992. A first course.

[Hat09] Allen Hatcher, *Vector bundles and K-theory*, 2009. Version 2.1, available at www.math.cornell.edu/~hatcher/VBKT/VBpage.html.

[Hog69] Stuart Hoggar, *On* KO *theory of Grassmannians*, Quart. J. Math. Oxford Ser. (2) 20 (1969), 447–463.

[Hor05] Jens Hornbostel, *$\mathbb{A}^1$-representability of Hermitian K-theory and Witt groups*, Topology 44 (2005), no. 3, 661–687.

[Hus94] Dale Husemoller, *Fibre bundles*, 3rd ed., Graduate Texts in Mathematics, vol. 20, Springer-Verlag, New York, 1994.

[Ish92] Kiminao Ishitoya, *Squaring operations in the Hermitian symmetric spaces*, J. Math. Kyoto Univ. 32 (1992), no. 1, 235–244.

[Jar87] John Frederick Jardine, *Simplicial presheaves*, J. Pure Appl. Algebra 47 (1987), no. 1, 35–87.

[Jou73] Jean-Pierre Jouanolou, *Une suite exacte de Mayer-Vietoris en K-théorie algébrique*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 293–316. Lecture Notes in Math., Vol. 341.

[Kar05] Max Karoubi, *Bott periodicity in topological, algebraic and Hermitian K-theory*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 111–137.

[KKO04] Daisuke Kishimoto, Akira Kono, and Akihiro Ohsita, *KO-theory of flag manifolds*, J. Math. Kyoto Univ. 44 (2004), no. 1, 217–227.

[Koc96] Stanley Kochman, *Bordism, stable homotopy and Adams spectral sequences*, Fields Institute Monographs, vol. 7, American Mathematical Society, Providence, RI, 1996.

[KH91] Akira Kono and Shin-ichiro Hara, *KO-theory of complex Grassmannians*, J. Math. Kyoto Univ. 31 (1991), no. 3, 827–833.

[KH92] ______, *KO-theory of Hermitian symmetric spaces*, Hokkaido Math. J. 21 (1992), no. 1, 103–116.

[MH73] John Milnor and Dale Husemoller, *Symmetric bilinear forms*, Springer-Verlag, New York, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.

[MS74] John Milnor and James Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.

[Mor99] Fabien Morel, *Théorie homotopique des schémas*, Astérisque 256 (1999).

[MV99] Fabien Morel and Vladimir Voevodsky, *$\mathbb{A}^1$-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. 90 (1999), 45–143 (2001).

[Mor04] Fabien Morel, *An introduction to $\mathbb{A}^1$-homotopy theory*, Contemporary developments in algebraic K-theory, ICTP Lect. Notes, XV, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, pp. 357–441.

[Mor06] ______, *Rational stable splitting of Grassmanians and rational motivic sphere spectrum* (2006), available at `www.mathematik.uni-muenchen.de/~morel/preprint.html`. Draft.

[Nen07] Alexander Nenashev, *Gysin maps in Balmer-Witt theory*, J. Pure Appl. Algebra 211 (2007), no. 1, 203–221.

[Nen09] ______, *On the Witt groups of projective bundles and split quadrics: geometric reasoning*, J. K-Theory 3 (2009), no. 3, 533–546.

[GOV94] Vladimir Gorbatsevich, Arkadiĭ Onishchik, and Èrnest Vinberg, *Lie Groups and Lie Algebras III: Structure of Lie Groups and Lie Algebras*, Encyclopaedia of Mathematical Sciences, vol. 41, Springer-Verlag, Berlin, 1994.

[PW10] Ivan Panin and Charles Walter, *On the motivic commutative ring spectrum BO* (2010), available at `www.math.uiuc.edu/K-theory/0978/`. Preprint.

[Rio06] Joël Riou, *Opérations sur la K-théorie algébrique et régulateurs via la théorie homotopique des schémas*, Ph. D. Thesis, l'Université Paris 7 – Denis Diderot, 2006, `www.math.u-psud.fr/~riou/these/these.pdf`.

[Rio07a] ______, *Catégorie homotopique stable d'un site suspendu avec intervalle*, Bull. Soc. Math. France 135 (2007), no. 4, 495–547.

[Rio07b] ______, *Opérations sur la K-théorie algébrique et régulateurs via la théorie homotopique des schémas*, C. R. Math. Acad. Sci. Paris 344 (2007), no. 1, 27–32.

[Sch10a] Marco Schlichting, *Hermitian K-theory of exact categories*, J. K-Theory 5 (2010), no. 1, 105–165.

[Sch10b] ______, *The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes*, Invent. Math. 179 (2010), no. 2, 349–433.

[Sch] ______, *Hermitian K-theory, derived equivalences and Karoubi's fundamental theorem*. In preparation.

[Seg68] Graeme Segal, *Equivariant K-theory*, Inst. Hautes Études Sci. Publ. Math. 34 (1968), 129–151.

[Voe98] Vladimir Voevodsky, $\mathbb{A}^1$-*homotopy theory*, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998), 1998, pp. 579–604.

[Wal03a] Charles Walter, *Grothendieck-Witt groups of triangulated categories* (2003), available at `www.math.uiuc.edu/K-theory/0643/`. Preprint.

[Wal03b] ———, *Grothendieck-Witt groups of projective bundles* (2003), available at `www.math.uiuc.edu/K-theory/0644/`. Preprint.

[Yag04] Nobuaki Yagita, *Coniveau spectral sequence and motivic cohomology of quadrics and classifying spaces* (2004), available at `www.math.uiuc.edu/K-theory/0709/`. Preprint.

[Yag09] ———, *Note on Witt group and KO-theory of complex Grassmannians* (2009), available at `www.math.uni-bielefeld.de/LAG/man/325.html`. Preprint.

[Zib09] Marcus Zibrowius, *Comparing Grothendieck-Witt groups of a complex variety to its real topological K-groups* (2009), available at `www.dspace.cam.ac.uk/handle/1810/236624`. Smith-Knight/Rayleigh-Knight Essay.

Marcus Zibrowius
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge
CB3 0WB
United Kingdom

The Hilbert-Chow Morphism and the Incidence Divisor

Joseph Ross

Received: October 18, 2010
Revised: June 20, 2011

Communicated by Gavril Farkas

Abstract. For a smooth projective variety P of dimension n, we construct a Cartier divisor supported on the incidence locus in the product of Chow varieties $\mathscr{C}_a(P) \times \mathscr{C}_{n-a-1}(P)$. There is a natural definition of the corresponding line bundle on a product of Hilbert schemes, and we show this bundle descends to the Chow varieties. This answers a question posed by Barry Mazur.

2010 Mathematics Subject Classification: Primary 14C05
Keywords and Phrases: Chow variety, Hilbert scheme

Section 1. Introduction

Let $(P, \mathcal{O}_P(1))$ be a smooth projective variety of dimension n over an algebraically closed field k. The *Chow variety* $\mathscr{C}_{d,d'}(P)$ parameterizes algebraic cycles on P. In particular, there is a bijection between $\mathscr{C}_{d,d'}(P)(k)$ and the set of effective algebraic cycles on P of dimension d and degree d'. (We suppress the degree since it plays no role.) Let a and b be nonnegative integers such that $a + b + 1 = n$. The product $\mathscr{C}_a(P) \times \mathscr{C}_b(P)$ parameterizes pairs (A, B) of an a-dimensional cycle A and a b-dimensional cycle B. Since $a + b$ is less than n, one expects a generic A and B to be disjoint. But one expects that the *incidence locus* $\mathscr{I}$ parameterizing incident pairs is a codimension 1 closed subvariety in $\mathscr{C}_a(P) \times \mathscr{C}_b(P)$.

For k the field of complex numbers, Mazur [14] constructed a Weil divisor supported on the incidence locus as follows. Consider the diagram of schemes:

$$\begin{tikzcd} P \times \mathscr{C}_a(P) \times \mathscr{C}_b(P) \arrow[r, "\Delta"] \arrow[d, "pr_{23}"] & P \times P \times \mathscr{C}_a(P) \times \mathscr{C}_b(P) \\ \mathscr{C}_a(P) \times \mathscr{C}_b(P) & \end{tikzcd}$$

Let U_a, U_b denote the universal cycles on $P \times \mathscr{C}_a(P)$, $P \times \mathscr{C}_b(P)$ respectively (these exist in characteristic zero). Since Δ is a local complete intersection morphism, there is a refined Gysin homomorphism $\Delta^!$, as constructed in [7, 6.2]. Using standard operations in intersection theory, one has $pr_{23*}\Delta^!(U_a \boxtimes U_b)$, a cycle of codimension 1 on $\mathscr{C}_a(P) \times \mathscr{C}_b(P)$. The main question of [14] is whether $pr_{23*}\Delta^!(U_a \boxtimes U_b)$ is a Cartier divisor. The main result of this paper is a positive answer to Mazur's question.

Theorem 1.1. *Let $U \subset \mathscr{C}_a(P) \times \mathscr{C}_b(P)$ denote the locus of disjoint cycles, i.e., the complement of the incidence locus $\mathscr{I}$. Let $U' \subset \overline{U}$ denote the union of products $C_a \times C_b$ of irreducible components $C_a \subset \mathscr{C}_a(P), C_b \subset \mathscr{C}_b(P)$ over which the universal cycles intersect properly.*

- *There is a Cartier divisor D on $\overline{U}$ which is supported on $\overline{U} - U$.*
- *The restriction of D to U' is an effective Cartier divisor.*
- *Let T be the spectrum of a discrete valuation ring $R \supset k$, with generic point η. Let $g : T \to \overline{U} \subset \mathscr{C}_a(P) \times \mathscr{C}_b(P)$ be a morphism corresponding to cycles Z, W on $P \times T$, and such that $g(\eta) \in U$. Let s_D denote the canonical section of the line bundle $\mathcal{O}_{\overline{U}}(D)$. Then we have*

$$\operatorname{ord} g^*(s_D) = \deg(Z \cdot W) \in \mathbb{Z},$$

where $Z \cdot W \in A_0(P \times T)$ is the class constructed in [7, 20.2].

Remark 1.2. Our methods will suggest there is a line bundle on the whole of the product $\mathscr{C}_a(P) \times \mathscr{C}_b(P)$, but it does not seem reasonable to expect a Cartier divisor beyond the locus $\overline{U}$. On the locus U', the operation $\Delta^!$ is defined on the cycle level, and all of the coefficients appearing in $pr_{23*}\Delta^!(U_a \boxtimes U_b)$ are positive. On the locus $\overline{U} - U'$, negative coefficients may appear.

Techniques. Our approach to Mazur's question, initiated in [20], is to define the incidence line bundle $\mathcal{L}$ on a product of Hilbert schemes mapping to the corresponding Chow varieties, and then show $\mathcal{L}$ is the pullback of a line bundle $\mathcal{M}$ on the Chow varieties. Our $\mathcal{L}$ will be equipped with a canonical nonvanishing rational section on the locus of disjoint subschemes, and we will show this section is induced by a trivialization of $\mathcal{M}$ on U. Briefly, $\mathcal{L}$ is the determinant of a perfect complex formed from the universal flat families. Then we form a proper hypercovering of $\overline{U}$ along the Hilbert-Chow morphism, and a descent datum for $\mathcal{L}$ on this hypercovering. This amounts to an identification ϕ between two pullbacks of $\mathcal{L}$, satisfying a cocycle condition.

At first we define the descent datum ϕ over a normal base provided the incidence has the expected dimension (5.18); this boils down to the Serre Tor-formula for intersection multiplicities and basic properties of the determinant functor (additivity on short exact sequences). To extend the descent datum over families with more complicated incidence structure, we establish some moving lemmas to produce local trivializations (5.13, 5.14), then apply Grothendieck-Riemann-Roch to show these local sections glue (5.17, 5.32). A useful tool is the following result, which characterizes functions on a seminormal scheme (4.3; see also Definition 4.2): a Noetherian ring A is seminormal if and only if every pointwise function on $\operatorname{Spec} A$ which varies algebraically along (complete) DVRs is induced by an element of A.

As for the effectiveness of $(\mathcal{L}, \phi)$, i.e., that $\mathcal{L}$ is induced by a line bundle on the Chow varieties, an outgrowth of 4.3 is a criterion for effective descent (4.6) applicable to our Hilbert-Chow hypercovering: the bundle $\mathcal{L}$ descends to $\mathcal{M} \in \operatorname{Pic}(\overline{U})$ if it can be trivialized locally on $\overline{U}$, compatibly with the descent datum ϕ. The compatible local trivializations are built into the definition of the descent datum.

Motivation. In the classical construction, the Chow variety $\mathscr{C}_{d,d'}(\mathbb{P}^n)$ is realized as a closed subvariety of the scheme of Cartier divisors of the Grassmannian $\mathscr{G}$ of $(n-d-1)$-planes: to a d-dimensional cycle Z on $\mathbb{P}^n$ we associate the codimension one set of $(n-d-1)$-planes in $\mathbb{P}^n$ which intersect Z. Thus the natural ample line bundle on $\mathscr{G} \times \operatorname{CDiv}(\mathscr{G})$ simultaneously shows the projectivity of the Chow variety, and endows the incidence locus (in $\mathscr{C}_{d,d'}(\mathbb{P}^n) \times \mathscr{G}$, a special case of the $\mathscr{I}$ considered above) with the structure of an effective Cartier divisor. This generalizes to the case $\mathscr{C}_d(\mathbb{P}^n) \times \mathscr{C}_{n-d-1}(\mathbb{P}^n)$ using the ruled join; see [17].

This direct geometric construction does not extend to general smooth projective P. However, the Hilbert scheme (the moduli space for closed subschemes of P) and the Hilbert-Chow morphism $\mathscr{H} \to \mathscr{C}$ suggest another approach. The pullback of the line bundle associated to the incidence divisor via $\mathscr{H}(\mathbb{P}^n) \times \mathscr{G} \to \mathscr{C}(\mathbb{P}^n) \times \mathscr{G}$ is the determinant of a perfect complex formed from the universal flat families (see the end of Section 3), and the determinant construction can be defined for any smooth projective P. Thus one is naturally led to wonder, for a general pair of Hilbert schemes parameterizing subschemes of dimension a, b as above, whether the determinant line bundle descends to the corresponding product of Chow varieties. The direct geometric construction for $P = \mathbb{P}^n$ and the determinant formula are in fact compatible; see the end of Section 3.

Further motivation comes from the case of zero-cycles and divisors, where the Hilbert-Chow morphism admits a reasonably explicit description. The equality of the families of zero-cycles associated to two families of zero-dimensional subschemes has a natural expression in terms of determinants; and similarly two families of codimension one subschemes determine the same family of cycles if the determinants of their structure sheaves agree. For a detailed study of the determinant bundle in the case of zero-cycles and divisors, in particular the descent to the Chow varieties, see [20].

Contents. In Section 2 we recall background material on determinant functors and K-theory. In Section 3 we discuss the relevant properties of the Chow variety and the Hilbert-Chow morphism, and define the incidence line bundle and the Hilbert-Chow hypercovering along which the incidence bundle descends. In Section 4 we explain the role of seminormality both in defining the descent datum and demonstrating its effectiveness. In Section 5 we construct the descent datum and show it is effective.

OTHER WORK. In [24] Wang proved that a certain multiple of the incidence divisor, namely $(n-1)!(pr_{23*}\Delta^!(U_a \boxtimes U_b))$, is Cartier by using the Archimedean height pairing on algebraic cycles (over $\mathbb{C}$). (See the references in [24] for history of the height pairing.) Given disjoint cycles A, B on P as above, one has the pairing $\langle A, B\rangle := \int_A [G_B]$ defined by integrating a normalized Green's current for B over A. Wang views $\langle A, B\rangle$ as a function on the open set U in $\mathscr{C}_a(P) \times \mathscr{C}_b(P)$ consisting of disjoint cycles, and by studying the behavior of the function as the cycles collide, obtains [24, Thm. 1.1.2] a metrized line bundle L on $\overline{U}$ and a rational section s that is regular and nowhere zero on U, such that $\log ||s(A, B)||^2 = (\dim(P) - 1)!\langle A, B\rangle$. Using relative fundamental classes in Deligne cohomology (and again over $\mathbb{C}$), Barlet and Kaddar [4] constructed an incidence Cartier divisor in the analytic setting under the assumption that the incidence is generically finite over the parameter space. It would be interesting to "go back" from the Chow varieties to the height pairing.

CONVENTIONS. We use the definition of the Chow variety from [13]. In characteristic 0, there is a functor of effective algebraic cycles (of dimension d and degree d') defined on the category of seminormal k-schemes; and this functor is represented by a seminormal, projective k-scheme $\mathscr{C}_{d,d'}(P)$ [13, I.3.21]. In characteristic $p > 0$, there are several plausible notions of a family of effective algebraic cycles, stemming from the ambiguity of the field of definition of a cycle [13, I.4.11]. In this case we work with the seminormal, projective k-scheme $\mathscr{C}_{d,d'}(P)$ constructed in [13, I.4.13]. This coarsely represents at least two reasonable functors of effective algebraic cycles. Since we rely on the method of "seminormal descent," our methods do not apply to other definitions of Chow varieties (e.g., those of Barlet [3], Angéniol [1], and Rydh [21]) when those constructions yield Chow varieties/schemes which are not seminormal.
All schemes considered in this paper are locally Noetherian. A variety over a field k is an integral separated scheme of finite type over k.

ACKNOWLEDGMENTS. This paper is derived from and improves upon the author's PhD thesis. The author thanks his thesis advisor Aise Johan de Jong. This work was completed while the author was a wissenschaftlicher Mitarbeiter at the Universität Duisburg-Essen.

SECTION 2. DETERMINANT FUNCTORS AND K-GROUPS

A determinant functor assigns an invertible sheaf to each perfect complex. We discuss this notion following [12]. Then we quickly review some background material on K-groups.

NOTATION 2.1. Let X be a scheme. Let $D(X)$ denote the derived category of the abelian category $\mathrm{Mod}(X)$ of $\mathcal{O}_X$-modules. Denote by $D^+(X)$, respectively $D^-(X)$, the full subcategory of $D(X)$ whose objects are complexes of $\mathcal{O}_X$-modules which are bounded below, respectively bounded above. Denote by $D^b(X)$ the full subcategory whose objects are complexes which are both bounded below and bounded above.

We denote by $D_{(q)coh}(X)$ the full triangulated subcategory of $D(X)$ consisting of pseudo-(quasi)coherent complexes, and by $D^*_{(q)coh}(X)$ the corresponding bounded category for $* = +, -, b$ [10, 1.4 p.38, 2.1 p.85]. We denote by $\mathrm{Parf}(X) \subset D^b(X)$ the full triangulated subcategory consisting of perfect complexes [9, Exp. I Def. 4.7 p.44].
Let Parf-is(X) denote the category whose objects are perfect complexes on X, with morphisms isomorphisms in $D(X)$. Let $\mathrm{Pic}(X)$ denote the category whose objects are invertible sheaves on X, and whose morphisms are isomorphisms. This is a Picard category in the sense of [2, Exp. XVIII Def. 1.4.2].

Determinants. The main result of [12, Thm. 2] is that there exists up to canonical isomorphism a unique *determinant functor* $\det_X : \text{Parf-is}(X) \to \mathrm{Pic}(X)$ extending the usual determinant (top exterior power) of a locally free sheaf. Indeed the idea is to locally replace a perfect complex by a bounded complex of locally free sheaves, and take the signed tensor product of the usual determinants of the locally free terms. Then one shows this patches to give a global invertible sheaf. For every true triangle of complexes $0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3 \to 0$ in Parf-is(X), we require an isomorphism $i_X(\alpha, \beta) : \det(\mathcal{F}_1) \otimes \det(\mathcal{F}_3) \xrightarrow{\sim} \det(\mathcal{F}_2)$, and the isomorphisms i (extending the obvious i for short exact sequences of locally free sheaves) are required to be compatible with isomorphisms of triangles, and more generally triangles of triangles. Associated to morphisms of schemes we have base change isomorphisms interchanging the determinant with pullback, and these are required to be compatible with composition of morphisms of schemes.

Remark 2.2. When X is reduced, i extends to the class of distinguished triangles, is functorial over isomorphisms (in $D(X)$) of distinguished triangles, and is compatible with distinguished triangles of distinguished triangles [12, Prop. 7].

Associated Cartier divisors. If $\mathcal{F} \in \mathrm{Parf}(X)$ is acyclic at every $x \in X$ of depth 0, then [12, Ch.II] constructs a Cartier divisor $\mathrm{Div}(\mathcal{F})$ on X and a canonical isomorphism $\mathcal{O}_X(\mathrm{Div}(\mathcal{F})) \cong \det_X(\mathcal{F})$ extending the trivialization $\mathcal{O}_{X,x} \cong \det_x \mathcal{F}$ at $x \in X$ of depth 0. The formation of this divisor and isomorphism is additive on short exact sequences [12, Thm.3(ii)], and is compatible with base change $f : X' \to X$ such that $\mathrm{L}f^*(\mathcal{F})$ is acyclic at every $x' \in X'$ of depth 0 [12, Thm.3(v)]. Furthermore, in case X is normal and $x \in X$ is a point of depth 1, the coefficient of $\overline{\{x\}}$ in the Weil divisor associated to $\mathrm{Div}(\mathcal{F})$ is the alternating sum of the lengths (at x) of the cohomology sheaves, i.e., $\sum_i (-1)^i \ell_x(\mathcal{H}^i(\mathcal{F}))$. This construction is also studied in [6, Sect. 3] and [15, Ch. 5 Sect. 3].
We mention two further properties implicit in [12, Thm.3]. If the complex $\mathcal{F}$ is acyclic, then $\mathrm{Div}(\mathcal{F}) = 0$ and the canonical isomorphism $\mathcal{O}_X(\mathrm{Div}(\mathcal{F})) = \mathcal{O}_X \cong \det_X(\mathcal{F})$ is the trivialization of the determinant of an acyclic complex [12, Lemma 2]. Finally, the construction is determined by the quasi-isomorphism class of the perfect complex $\mathcal{F}$: since all filtration levels and subquotients

appearing in the canonical filtration ("good truncation") of $\mathcal{F}$ are generically acyclic so long as $\mathcal{F}$ is, the additivity implies $\operatorname{Div}(\mathcal{F}) = \sum_i (-1)^i \operatorname{Div}(\mathcal{H}^i(\mathcal{F}))$.

K-GROUPS. Let X be a variety. Then $K_0(X)$ is the Grothendieck group of X, generated by coherent sheaves on X with relations for short exact sequences of sheaves; $K^0(X)$ is the Grothendieck group of vector bundles. When X is regular, we have an isomorphism $K_0(X) \cong K^0(X)$. We have also the Chern character $\mathrm{ch} : K_0(X) \to A_*(X)$, where $A_*(X)$ is the Chow group of cycles on X, graded by dimension. We note $\mathcal{F} \in \operatorname{Parf}(X)$ determines a class in $K_0(X)$ since for any abelian category $\mathscr{A}$ (e.g., $\operatorname{Coh}(X)$), $K_0(\mathscr{A}) \cong K_0(D^b(\mathscr{A}))$; the latter group is generated by objects of the triangulated category $D^b(\mathscr{A})$ with relations for distinguished triangles.

The group $K_0(X)$ has a topological filtration: the subgroup $F_k(K_0(X))$ is generated by those $\mathcal{F} \in \operatorname{Coh}(X)$ such that $\dim(\operatorname{Supp}(\mathcal{F})) \leq k$. For a proper morphism of schemes $f : X \to T$ we obtain a homomorphism $f_* : K_0(X) \to K_0(T)$ sending (the class of) a coherent sheaf to the alternating sum of (the classes of) its higher direct image sheaves. This preserves the topological filtration. If T is a point and $\mathcal{F} \in \operatorname{Coh}(X)$, then $\chi(\mathcal{F}) = \mathrm{ch}(f_*(\mathcal{F}))$.

SECTION 3. THE HILBERT-CHOW MORPHISM AND THE INCIDENCE DIVISOR

In this section we define the Chow variety, the Hilbert-Chow morphism, and construct our proper hypercovering. Then we define the incidence line bundle on the product of Hilbert schemes.

We recall an application of the characterization of seminormal schemes [19, 5.1], where it is shown that properties (1)-(5) below characterize the Chow variety. For properties (6) and (7) we refer to [13].

DEFINITION-THEOREM 3.1 (Existence of the Chow variety). *Let P be a smooth projective variety over a field k. The Chow variety $\mathscr{C}_{d,d'}$ of P is a k-scheme with the following properties:*

(1) It is projective over k.
(2) It is seminormal.
(3) For every point $w \in \mathscr{C}_{d,d'}$ there exist purely inseparable field extensions $\kappa(w) \subset L_i$ and cycles Z_i on P_{L_i} such that:
  (a) Z_i and Z_j are essentially equivalent [13, I.3.8]*: they agree as cycles over the perfection $\kappa(w)^{perf}$ of $\kappa(w)$;*
  (b) the intersection of the fields L_i is $\kappa(w)$, which is the Chow field (field of definition of the Chow form in any projective embedding of P) of any of the Z_i [13, I.3.24.1]*; and*
  (c) for any cycle Z on P_M defined over a subfield $k \subset M \subset \kappa(w)^{perf}$ which agrees with the Z_i over $\kappa(w)^{perf}$ (equivalently, agrees with one Z_i), we have $\kappa(w) \subset M$ (the Chow field is the intersection of all fields of definition of the cycle).
(4) Points w of $\mathscr{C}_{d,d'}$ are in bijective correspondence with systems $(k \subset \kappa(w), \{\kappa(w) \subset L_i, Z_i\}_{i \in I})$ up to an obvious equivalence relation.

(5) *For any DVR $R \supset k$ and any cycle Z on P_R of relative dimension d and degree d' in the generic fiber, we obtain a morphism $g : \operatorname{Spec} R \to \mathscr{C}_{d,d'}$ such that the generic fiber Z_η and the special fiber Z_s agree with the systems of cycles of the previous property at $g(\eta)$ and $g(s)$.*

(6) *For any numerical polynomial q of degree d and with leading coefficient $d'/(d!)$, we obtain a morphism (the Hilbert-Chow morphism)*

$$FC : (\mathscr{H}^q)^{sn}_{red} \to \mathscr{C}_{d,d'}$$

by taking the fundamental cycle of the components of maximal relative dimension $(= d)$[13, I.6.3.1]. *A finite number of $(\mathscr{H}^q)^{sn}_{red}$'s surject onto $\mathscr{C}_{d,d'}$.*

(7) *Let $\eta \in \mathscr{C}_{d,d'}$ be a generic point. Then either $\dim \overline{\{\eta\}} = 0$ or there exists a cycle Z_η on P_η defined over $\kappa(\eta)$. In particular, if k is perfect then there exists a Z_η for every generic point η of $\mathscr{C}_{d,d'}$* [13, I.4.14].

CONSTRUCTION 3.2. Let P be a smooth projective variety, and $r \in \mathbb{Z}_{\geq 0}$. Let $\mathscr{H}'_r$ denote the Hilbert scheme of r-dimensional subschemes of P. Let $\mathscr{H}_r$ denote the seminormalization of the (closed) subscheme of $\mathscr{H}'_r$ consisting of subschemes Z such that Z has pure r-dimensional support (this is different from the notion of a pure sheaf: Z may have embedded components of smaller dimension so long as they are set-theoretically contained in the top-dimensional components). We have the product of the Hilbert-Chow morphisms (3.1) $\pi : Y_0 = \mathscr{H}_a \times \mathscr{H}_b \to \mathscr{C}_a \times \mathscr{C}_b =: C$. Because seminormalization is a functor, we may form a proper hypercovering $\pi_\bullet : Y_\bullet \to C$ augmented towards C whose i-th term Y_i is the seminormalization of $Y_0 \times_C \ldots \times_C Y_0$ ($i+1$ factors), with the (seminormalizations of the) canonical morphisms.

Remarks 3.3. (3.3.1) We explain property (7) in more detail. For any positive-dimensional component of $\mathscr{C}_{d,d'}$, its generic point corresponds to a cycle all of whose coefficients are 1, i.e., a subscheme [13, 1.4.14]. Hence we can find a (component of some) $(\mathscr{H}^q)^{sn}_{red}$ admitting a birational morphism to that component.

(3.3.2) Over a field of characteristic zero, the seminormality of the Chow variety and [19, 4.1] imply $\mathcal{O}_{\mathscr{C}} = \pi_{\bullet *}(\mathcal{O}_{X_\bullet})$ for a proper hypercovering $X_\bullet$ augmented towards $\mathscr{C}$. In characteristic $p > 0$, we have the characterization of the residue fields on the Chow variety as the intersection of all fields of definition [13, I.4.5]. So by [19, 4.1 Corrigendum], we have $\mathcal{O}_{\mathscr{C}} = \pi_{\bullet *}(\mathcal{O}_{X_\bullet})$ for a proper hypercovering such that $X_0 = \mathscr{H}^{sn}_{red}$.

In more detail and in the language of [19, 4.1], we explain how to construct (locally) a pointwise function on $\mathscr{C}$ from a pointwise function on $\mathscr{H}^{sn}_{red}$ which belongs to $\pi_{\bullet *}(\mathcal{O}_{X_\bullet})$. So suppose $z \in \mathscr{C}(P_k)$ corresponds via a morphism $\operatorname{Spec} \kappa(z) \to \mathscr{C}(P_k)$ to the cycle Z on $P_{\kappa(z)}$. Consider an algebraically closed field $\overline{K}$ containing $\kappa(z)$ and the cycle associated to the base change $Z_{\overline{K}}$. Then by [13, I.4.5], the residue field $\kappa(z)$ is characterized as the intersection in $\overline{K}$ of all fields of definition of Z, i.e., the intersection of all E_i such that $k \subset E_i \subset \overline{K}$ and there exists a subscheme $Y_i \subset P_{E_i}$ whose associated cycle agrees with $Z_{\overline{K}}$ upon base change. Consider fields E_0, E_1 satisfying these conditions. Then we have morphisms $\operatorname{Spec} E_i \to \mathscr{H}$ with the property that the compositions $\operatorname{Spec} \overline{K} \to \operatorname{Spec} E_i \to \mathscr{H} \to \mathscr{C}$ are the same. Thus we have a commutative diagram:

$$\begin{tikzcd} \operatorname{Spec} \overline{K} \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & (\mathscr{H} \times_{\mathscr{C}} \mathscr{H})^{sn}_{red} \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\ \operatorname{Spec} E_0, \operatorname{Spec} E_1 \arrow[r] & \mathscr{H} \end{tikzcd}$$

(The morphism from $\operatorname{Spec} \overline{K}$ factors through the seminormalization.) Considering $a \in \pi_{\bullet *}(\mathcal{O}_{X_\bullet})$ as a pointwise function, we obtain elements $a_i \in E_i$. The preceding diagram shows $a_0 = a_1$ in $\overline{K}$, therefore $a_0 \in E_0 \cap E_1$. By the same argument we find $a_0 \in E_0 \cap E_i$ for all i, therefore $a_0 \in \kappa(z)$. Thus we made an element in the residue field $\kappa(z)$. It varies algebraically along DVRs by [19, 4.1].

(3.3.3) If $X = Y \cup Z$ is a reducible scheme with irreducible components Y, Z, then the field of definition of Y is contained in the field of definition of X. Also, a scheme and its seminormalization have the same residue fields. Hence to cut out the residue fields on $\mathscr{C}$, it is enough to consider the subscheme $\mathscr{H}_r \hookrightarrow \mathscr{H}'_r$ defined in 3.2. So we have $\mathcal{O}_{\mathscr{C}} = \pi_{\bullet *}(\mathcal{O}_{X_\bullet})$ for a proper hypercovering such that $X_0 = \mathscr{H}_r$.

We record the (presumably known) fact that the Hilbert-Chow morphism is compatible with products.

LEMMA 3.4. *If P, P' are smooth projective varieties over a field k, then the following diagram commutes. (We suppose p has leading coefficient $d'/(d!)$ and $\deg(p) = d$; and q has leading coefficient $e'/(e!)$ and $\deg(q) = e$.)*

$$\begin{tikzcd} (\mathscr{H}^p(P))^{sn}_{red} \times (\mathscr{H}^q(P'))^{sn}_{red} \arrow[r] \arrow[d, "FC \times FC"] & (\mathscr{H}^{pq}(P \times P'))^{sn}_{red} \arrow[d, "FC"] \\ \mathscr{C}_{d,d'}(P) \times \mathscr{C}_{e,e'}(P') \arrow[r] & \mathscr{C}_{d+e,d'e'}(P \times P') \end{tikzcd}$$

Proof. We describe the map in the top row: if $Z \hookrightarrow P \times T, Z' \hookrightarrow P' \times T$ constitute a T-point of $(\mathscr{H}^p(P))^{sn}_{red} \times (\mathscr{H}^q(P'))^{sn}_{red}$, then the scheme theoretic intersection $pr_{13}^* Z \cap pr_{23}^* Z'$ in $P \times P' \times T$ is a T-point of $(\mathscr{H}^{pq}(P \times P'))^{sn}_{red}$. A top-dimensional component in the product scheme is induced by a pair of top-dimensional components; and length multiplies, so the coefficients in the product cycle are the products of the coefficients of the factors. $\square$

The main goal of this paper is to construct a Cartier divisor supported on the incidence locus. Now we define an invertible sheaf (the "incidence bundle") on a product of Hilbert schemes, and show the incidence bundle is pulled back from the product of Chow varieties in the case $P = \mathbb{P}^n$.

Construction 3.5. Let B be a base scheme, and P a smooth projective B-scheme. Let $\mathscr{H}^1, \mathscr{H}^2$ denote the Hilbert schemes corresponding to numerical polynomials q_1, q_2, and set $\mathscr{H}^{1,2} := \mathscr{H}^1 \times_B \mathscr{H}^2$.
Over each $\mathscr{H}^i$ we have a universal flat family, a closed subscheme of $P \times_B \mathscr{H}^i$. Denote by $\mathscr{U}_i$ its pullback to $P \times_B \mathscr{H}^{1,2}$. Standard facts about the behavior of perfect complexes under certain operations (stability under tensor product; pullback; and pushfoward via a proper morphism of finite Tor-dimension; and the perfectness of a coherent sheaf on the source of a smooth morphism which is flat over the target) imply $\mathrm{R}pr_{23*}(\mathcal{O}_{\mathscr{U}_1} \otimes^{\mathrm{L}} \mathcal{O}_{\mathscr{U}_2})$ is a perfect complex on $\mathscr{H}^{1,2}$. For details on the necessary facts about perfect complexes, see section 2 of [20]. The incidence bundle $\mathcal{L}$ is defined to be its determinant:

$$\mathcal{L} := \det_{\mathscr{H}^{1,2}} \mathrm{R}pr_{23*}(\mathcal{O}_{\mathscr{U}_1} \otimes^{\mathrm{L}} \mathcal{O}_{\mathscr{U}_2}).$$

In fact we will be interested in this construction only on the locus Y_0 defined earlier in this section. Furthermore, since the complex $\mathrm{R}pr_{23*}(\mathcal{O}_{\mathscr{U}_1} \otimes^{\mathrm{L}} \mathcal{O}_{\mathscr{U}_2})$ is acyclic over the locus U_0 of disjoint pairs of subschemes, this construction determines a Cartier divisor on the closure of the locus of disjoint pairs of subschemes (the Hilbert scheme analogue of the locus $\overline{U}$ defined in 1.1). See [20, 2.5].
As motivation for pursuing the determinant formula (mentioned in the Introduction), we make contact with the classical construction of the Chow variety of $P = \mathbb{P}^n$. As explained in the Introduction, the construction of the Chow variety endows the incidence locus $\mathscr{I} \hookrightarrow \mathscr{C}_d(\mathbb{P}^n) \times \mathscr{G}$ with the structure of a Cartier divisor. Let $FC_{\mathbb{P}^n} : \mathscr{H}(\mathbb{P}^n) \to \mathscr{C}(\mathbb{P}^n)$ denote the Hilbert-Chow morphism (and its product with $\mathscr{G}$). In the special case of Construction 3.5 with $P = \mathbb{P}^n$ and $\mathscr{H}^2 = \mathscr{G}$, it follows from [5, Thms. 1.2, 1.4] that there is a canonical isomorphism $\mathcal{L} \cong {FC_{\mathbb{P}^n}}^* \mathcal{O}(\mathscr{I})$ of invertible sheaves on $\mathscr{H} \times \mathscr{G}$. This isomorphism is canonical in the following sense. Over the locus U_0 of disjoint subschemes, there is a canonical trivialization $\mathcal{L}|_{U_0} \cong \mathcal{O}_{U_0}$. This rational section is the pullback via $FC_{\mathbb{P}^n}$ of the canonical trivialization of $\mathcal{O}(\mathscr{I})$ on the complement of $\mathscr{I}$.

Section 4. Seminormal schemes and descent criteria

In this section we explain the role of seminormality both in defining the descent datum and demonstrating its effectiveness.

Definition 4.1 ([8]). A ring A is a *Mori ring* if it is reduced and its integral closure A^ν (in its total quotient ring Q) is finite over it; if A is a Mori ring, A^{sn} denotes its seminormalization, the largest subring $A \subset A^{sn} \subset A^\nu$ such that $\operatorname{Spec} A^{sn} \to \operatorname{Spec} A$ is bijective and all maps on residue fields are isomorphisms. The seminormalization is described elementwise in [23, 1.1]. We say A is seminormal if $A = A^{sn}$ (so we only define seminormality for Mori rings). A locally Noetherian scheme X is Mori if and only if it has an affine cover by Noetherian Mori rings [8, Def. 3.1].

DEFINITION 4.2. Let A be a ring, and let $S = \{f_y \in \kappa(y) | y \in \operatorname{Spec} A\}$ be a collection of elements, one in each residue field. Then we say S *is a pointwise function on* $\operatorname{Spec} A$. We say the pointwise function S *varies algebraically along (complete) DVRs* if it has the following property: for every specialization $\mathfrak{p}_1 \subset \mathfrak{p}_2$ in A and every (complete) discrete valuation ring R covering that specialization via a ring homomorphism $g : A \to R$, there exists a (necessarily) unique $f_R \in R$ such that $\overline{g_{\mathfrak{p}_1}}(f_{\mathfrak{p}_1}) = f_R$ (in K) and $\overline{g_{\mathfrak{p}_2}}(f_{\mathfrak{p}_2}) = \overline{f_R}$ (in k_0).

The main result of [19, 2.2, 2.6] is the following.

THEOREM 4.3. *Let A be a seminormal (in particular, Mori) ring which is Noetherian. Let $\{f_y \in \kappa(y) | y \in \operatorname{Spec} A\}$ be a pointwise function on $\operatorname{Spec} A$ which varies algebraically along (complete) DVRs. Then there exists a unique $f \in A$ whose image in $\kappa(y)$ is f_y for all $y \in \operatorname{Spec} A$.*

This simplifies greatly the problem of defining a descent datum for a line bundle on a seminormal scheme, as seen in the following corollary.

COROLLARY 4.4. *Let X be a seminormal locally Noetherian (in particular, Mori) scheme, and let $L, M \in \operatorname{Pic}(X)$. Then an isomorphism $L \cong M$ is equivalent to an "identification of fibers varying algebraically along DVRs," that is:*

> *for any field or (complete) DVR R, any $\operatorname{Spec} R \xrightarrow{f} X$, an identification $\beta_f : f^*L \cong f^*M$ compatible with restriction to the closed and generic points: if $s \xrightarrow{i} \operatorname{Spec} R, \eta \xrightarrow{j} \operatorname{Spec} R$ denote the inclusions, then $\beta_{fi} = i^*\beta_f$ and $\beta_{fj} = j^*\beta_f$.*

Proof. Fix an open cover $X = \cup_i \operatorname{Spec} S_i$ with S_i a seminormal (Noetherian and Mori) ring which trivializes both L and M, and fix trivializations $\varphi_i : L_i := L|_{\operatorname{Spec} S_i} \cong \mathcal{O}_{\operatorname{Spec} S_i}$, $\psi_i : M_i := M|_{\operatorname{Spec} S_i} \cong \mathcal{O}_{\operatorname{Spec} S_i}$. Then defining $L \cong M$ is equivalent to identifying $\Gamma(\operatorname{Spec} S_i, L_i) \cong \Gamma(\operatorname{Spec} S_i, M_i)$ as S_i-modules (for all i), compatibly with restrictions. Then considering the diagram:

$$\begin{tikzcd} \Gamma(L_i) \arrow[r, "\cong"] \arrow[d, "\varphi_i"] & \Gamma(M_i) \arrow[d, "\psi_i"] \\ S_i \arrow[r] & S_i \end{tikzcd}$$

and its pullbacks to spectra of fields and DVRs, we see that relative to the fixed φ_i, ψ_i, a family β_f as in the statement is equivalent to an invertible pointwise function on each S_i varying algebraically along DVRs. By Theorem 4.3 this is equivalent to a family of elements $f_i \in {S_i}^\times = \operatorname{Isom}_{S_i}(S_i, S_i)$. The f_i thus obtained agree on overlaps by the uniqueness statement in Theorem 4.3. Then using the above diagram again we see that relative to the fixed trivializations, the family f_i is equivalent to a family of isomorphisms $\Gamma(L_i) \cong \Gamma(M_i)$ compatible with restrictions. $\square$

We will need the following general fact later.

LEMMA 4.5. *Let X be a seminormal k-scheme and $Y \hookrightarrow X$ a closed subscheme. Suppose every $y \in Y$ admits a generization to a point in $X - Y$, i.e., that Y does not contain any generic points of X. Let S be a pointwise function on X which varies algebraically along DVRs covering specializations within $X - Y$, and along those from $X - Y$ to Y. Then S varies algebraically along DVRs.*

Proof. This follows readily from the techniques used in [19]. We may assume X is affine. Let $\nu : X^\nu \to X$ denote the normalization. Then our pointwise function S determines a pointwise function on X^ν which is constant along the fibers of ν. The normalization is birational, so identifies generic points of X^ν with those of X. Hence as a pointwise function on X^ν, S varies algebraically along specializations of the form $\eta \rightsquigarrow x$, with η generic. This is enough to conclude S is induced by an element of $\Gamma(X^\nu, \mathcal{O}_{X^\nu})$ (see [19, 2.4]). But then because S is constant along the fibers of ν, this element comes from $\Gamma(X, \mathcal{O}_X)$ and *a fortiori* varies algebraically along all DVRs. $\square$

As for the effectiveness of a descent datum, we recall the following result from [20]. Let $\pi_\bullet : X_\bullet \to X$ be a proper hypercovering augmented towards a scheme X. We denote by $(\mathcal{L}, \phi)$ an element of $\mathrm{Pic}(X_\bullet)$, i.e., $\mathcal{L}$ is an element of $\mathrm{Pic}(X_0)$ and $\phi : {p_0}^*\mathcal{L} \xrightarrow{\sim} {p_1}^*\mathcal{L}$ is an isomorphism on X_1 satisfying the cocycle condition on X_2. As in [20, 3.3], we say $(\mathcal{L}, \phi) \in \mathrm{Pic}(X_\bullet)$ is *Zariski locally effective* if for every $x \in X$, there exists an open $U \subset X$ containing x and a trivialization $T_x : \mathcal{L}|_{{\pi_0}^{-1}(U)} \xrightarrow{\sim} \mathcal{O}_{{\pi_0}^{-1}(U)}$ compatible with ϕ in the sense that the diagram

$$\begin{tikzcd} {p_0}^*(\mathcal{L}|_{{\pi_0}^{-1}(U)}) \arrow[r, "{p_0}^*T"] \arrow[d, "\phi"] & \mathcal{O}_{(p_0)^{-1}({\pi_0}^{-1}(U))} \arrow[d, "="] \\ {p_1}^*(\mathcal{L}|_{{\pi_0}^{-1}(U)}) \arrow[r, "{p_1}^*T"] & \mathcal{O}_{(p_1)^{-1}({\pi_0}^{-1}(U))} \end{tikzcd}$$

commutes.

PROPOSITION 4.6. [20, 3.4] *Let X be a scheme, and let $\pi_\bullet : X_\bullet \to X$ be a proper hypercovering augmented towards X which satisfies $\mathcal{O}_X = \pi_{\bullet *}(\mathcal{O}_{X_\bullet})$. Then:*

- ${\pi_\bullet}^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X_\bullet)$ *is injective; and*
- *the image of ${\pi_\bullet}^*$ consists of those $(\mathcal{L}, \phi)$ that are Zariski locally effective.*

Remark 4.7. The Proposition applies when X is seminormal and $X_\bullet$ satisfies any of the conditions in [19, 4.1 Corrigendum], for example the proper hypercovering $\pi_\bullet : Y_\bullet \to C$ defined in 3.2.

SECTION 5. DEFINITION OF THE DESCENT DATUM

In this section we prove the main result, in the following form. Having established this result, we will consider the refinements and further properties stated in 1.1.

Theorem 5.1 (5.36, 5.37). *With the notation as in 3.2, let* $\mathscr{U}_r \hookrightarrow P \times \mathscr{H}_r$ *denote the (pullback of the) universal flat family. Using 3.5 we may form the determinant line bundle* $\mathcal{L}$ *on* Y_0*. Now base change everything to* $\overline{U} \subset C$*, the closure of the locus of disjoint cycles. Then the following hold.*

- *The sheaf* $\mathcal{L}$ *lifts to an invertible sheaf on* $Y_\bullet$*, i.e., there is an isomorphism* $\phi : {p_0}^*\mathcal{L} \cong {p_1}^*\mathcal{L}$ *on* Y_1 *satisfying the cocycle condition on* Y_2*.*
- *The descent datum* ϕ *is effective: there is a unique* $\mathcal{M} \in \mathrm{Pic}(\overline{U})$ *such that* $(\pi^*\mathcal{M}, can) \cong (\mathcal{L}, \phi)$*.*

Subsection 5.1. Notation and preliminary reductions.

Definition 5.2. Let P be a smooth projective k-variety of dimension n. A *Hilbert datum for* P *over* T consists of the following:

(1) a seminormal k-scheme T;
(2) $Z \hookrightarrow P_T := P \times_k T$ a T-flat closed subscheme of relative dimension a, such that the support of Z has pure dimension a in every fiber; and
(3) $W \hookrightarrow P_T$ a T-flat closed subscheme of relative dimension b, such that the support of W has pure dimension b in every fiber;

such that $a + b + 1 \leq n$; and every point $t \in T$ admits a generization to the locus of disjoint subschemes. Thus a Hilbert datum (Z, W) is simply a morphism $T \to \mathscr{H}_a \times \mathscr{H}_b$ such that the image of every generic point of T lies in a component of $\mathscr{H}_a \times \mathscr{H}_b$ with at least one pair of disjoint subschemes.
Typically we will make some construction from (Z, W) and then show the construction only depends on $[Z], [W]$, the cycles underlying Z and W. Therefore we make the following definition. A *Hilbert-Chow datum for* P *over* T is a pair of Hilbert data $(Z, W), (Z', W')$ for P over T such that $[Z] = [Z']$ and $[W] = [W']$. Since the supports of Z, W are assumed pure-dimensional, we have also $\mathrm{Supp}(Z) = \mathrm{Supp}(Z')$ and $\mathrm{Supp}(W) = \mathrm{Supp}(W')$. Thus a Hilbert-Chow datum (Z, Z', W, W') for P over T is nothing more than a morphism $T \to (\mathscr{H}_a \times \mathscr{H}_b \times_{\mathscr{C}_a \times \mathscr{C}_b} \mathscr{H}_a \times \mathscr{H}_b)^{sn}$ such that (after projecting to either $\mathscr{H}_a \times \mathscr{H}_b$ factor) every generic point of T lands in a pair of irreducible components with at least one pair of disjoint subschemes.
Because we work on the subscheme $\mathscr{H}_r$ of the Hilbert scheme, disjointness of subschemes on $\mathscr{H}_a \times \mathscr{H}_b$ corresponds exactly to disjointness of their associated cycles on $\mathscr{C}_a \times \mathscr{C}_b$. In general, two subschemes could have disjoint associated cycles but lower-dimensional components which coincide. So in the notation above we have $Z \cap W = \emptyset$ if and only if $Z' \cap W = \emptyset$.
Note that given a morphism $S \to T$ of seminormal k-schemes and a Hilbert-Chow datum for P over T, by pullback we obtain a Hilbert-Chow datum for P over S.

Notation 5.3. The structure morphism $P_T \to T$ will be called π.
For $\mathcal{F}, \mathcal{G} \in \mathrm{Parf}(P_T)$, we set $f_T(\mathcal{F}, \mathcal{G}) := \det_T \mathrm{R}\pi_*(\mathcal{F} \otimes^{\mathrm{L}} \mathcal{G}) \in \mathrm{Pic}(T)$.
If α is a b-dimensional cycle on P_T with $\alpha = \sum a_i W_i$, we put $f_T(\mathcal{O}_Z, \alpha) := \otimes_i (f_T(\mathcal{O}_Z, \mathcal{O}_{W_i})^{\otimes a_i})$. In general we use the notation $[-]$ to denote the cycle

associated to a subscheme or coherent sheaf: this means the top-dimensional components and their geometric multiplicities, even if, for example, $b < n - a - 1$. (In fact we used this in the preceding definition.)

If T is affine and equal to $\operatorname{Spec} R$, we may write f_R for f_T.

We will use the subscripts $(-)_0$ and $(-)_\eta$ to denote the base change of some object to closed and generic fibers, respectively.

By the incidence $Z \cap W$, we mean the underlying reduced algebraic subset $\operatorname{Supp}(Z) \cap \operatorname{Supp}(W)$. Stated properties of $Z \cap W$ will depend only on the underlying supports $\operatorname{Supp}(Z), \operatorname{Supp}(W)$.

Goal. For every Hilbert-Chow datum, we aim to construct an isomorphism $\phi_T^{Z,Z',W,W'} : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_{W'})$ varying functorially in T, in such a way that the resulting descent datum $\{\phi_T\}$ is Zariski locally effective. The essential case is $b = n - a - 1$.

Proposition 5.4 (reduction to fields and DVRs). *To define an isomorphism $\phi_T : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_{W'})$ for each Hilbert-Chow datum, for all smooth projective P, so that for each P, the collection $\{\phi_T\}$*

(1) is compatible with base change $S \to T$; and

(2) satisfies the cocycle condition;

it is sufficient to define an isomorphism ϕ_T for each Hilbert-Chow datum with T the spectrum of a field or complete DVR, compatible with base change among fields and complete DVRs, and satisfying the cocycle condition on fields.

Proof. This is a consequence of 4.4. □

Proposition 5.5 (reduction to the diagonal). *To define an isomorphism $\phi_T : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_{W'})$ for each Hilbert-Chow datum, for all smooth projective P, so that for each P, the collection $\{\phi_T\}$:*

(1) is compatible with base change $S \to T$;

(2) satisfies the cocycle condition; and

(3) is Zariski locally effective;

it is sufficient to define an isomorphism ϕ_T for each Hilbert-Chow datum with $W = W'$, for all P, so that for each P, the collection $\{\phi_T\}$ has the stated properties.

Proof. On $P \times P \times T$, let $\mathcal{O}_\Delta$ denote the structure sheaf of the diagonal $(\times T)$, i.e., the image of the closed immersion $P \times T \xrightarrow{\Delta \times 1_T} P \times P \times T$. Given T-flat closed subschemes $Z, W \hookrightarrow P \times T$, we let $Z \times W \hookrightarrow P \times P \times T$ denote the scheme-theoretic intersection $pr_{13}^* Z \cap pr_{23}^* W$. Then there is a canonical isomorphism of line bundles on T: $f_T(\mathcal{O}_Z, \mathcal{O}_W; P) \cong f_T(\mathcal{O}_{Z \times W}, \mathcal{O}_\Delta; P \times P)$. Then the proposition follows from the fact that the Hilbert-Chow morphism is compatible with products (3.4). □

Remark 5.6. We may even assume W is constant, i.e., there is a k-subscheme W_k such that $W = W_k \times_k T$; and we may assume W_k is integral (even smooth).

Notation 5.7. When W and W' are omitted from the notations, this means $W = W'$.

We start with some easy cases of our goal.

Lemma 5.8. *Let (Z, W) be a Hilbert datum over any base T such that $Z \cap W = \emptyset$. Then there is a canonical trivialization $\varphi_T^Z : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong \mathcal{O}_T$ which is compatible with base change.*

Proof. The hypothesis implies $\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W$, hence also $\mathrm{R}\pi_*(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W)$, is acyclic. The pullback via $S \to T$ is also acyclic, and the trivialization of the determinant of an acyclic complex is compatible with base change. □

Remark 5.9. The canonical isomorphism φ of the lemma has an additivity property in each variable. For example, if $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to^{+1}$ is a distinguished triangle in $\mathrm{Parf}(P_T)$ such that $\mathrm{Supp}(\mathcal{F}_i) \cap W = \emptyset$ for all i, then the isomorphism $f_T(\mathcal{F}_1, \mathcal{O}_W) \otimes f_T(\mathcal{F}_3, \mathcal{O}_W) \cong f_T(\mathcal{F}_2, \mathcal{O}_W)$ induced by the triangle corresponds, via the identifications $\varphi_T^{\mathcal{F}_i}$, to multiplication $\mathcal{O}_T \otimes \mathcal{O}_T \to \mathcal{O}_T$.

Corollary 5.10. *Among Hilbert-Chow data satisfying $Z \cap W = Z' \cap W = \emptyset$, there exists a collection of isomorphisms $\phi_T^{Z,Z'} : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_W)$ which is compatible with base change and satisfies the cocycle condition.*

Proof. We define $\phi_T^{Z,Z';W} := (\varphi_T^{Z'})^{-1} \circ \varphi_T^Z : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_W)$ to be the composition of the canonical trivializations. This is compatible with base change because each φ_T^Z is. We check the cocycle condition:

$$\phi_T^{Z',Z'';W} \circ \phi_T^{Z,Z';W} = ((\varphi_T^{Z''})^{-1} \circ \varphi_T^{Z'}) \circ ((\varphi_T^{Z'})^{-1} \circ \varphi_T^{Z}) = (\varphi_T^{Z''})^{-1} \circ \varphi_T^{Z} = \phi_T^{Z,Z'';W}.$$

□

From now on we keep the collection $\{\phi_T\}$ whose existence is asserted in 5.10. The idea is to gradually extend it to a collection over Hilbert-Chow data with increasingly complicated incidence structure, until we have covered the whole moduli space. Note that an isomorphism of line bundles on a reduced (e.g., seminormal) scheme is determined by its restriction to generic points (i.e., points of depth 0). Since our base T is always reduced, when we have defined an isomorphism ϕ for a more general class of Hilbert-Chow data, to check agreement with previously defined isomorphisms it suffices to check agreement on generic points.

Lemma 5.11. *Let (Z, W) be a Hilbert datum over T, and suppose that $(Z \cap W)_\eta = \emptyset$ for all generic points $\eta \in T$. (This holds, for example, whenever Z and W intersect properly on P_T.) Then there exists a Cartier divisor $D_{Z,W}$ on T and a canonical isomorphism $\varphi_T^Z : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong \mathcal{O}_T(D_{Z,W})$ characterized by agreeing with the trivialization $\varphi_\eta^{Z_\eta}$ for every generic point $\eta \in T$. When $Z \cap W = \emptyset$, $D_{Z,W} = 0$ and φ_T^Z is the canonical trivialization. The formation of the divisor $D_{Z,W}$ and the isomorphism φ_T^Z are compatible with base change $S \to T$ preserving the generic disjointness.*

Furthermore, the formation of $D_{Z,W}$ is additive in each variable: if $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to^{+1}$ is a distinguished triangle in $\mathrm{Parf}(P_T)$ *such that* $(\mathrm{Supp}(\mathcal{F}_i) \cap W)_\eta = \emptyset$ *for all generic points $\eta \in T$, all i; then $D_{\mathcal{F}_1,W} + D_{\mathcal{F}_3,W} = D_{\mathcal{F}_2,W}$; and the triangle induces, upon application of $\varphi_T^{\mathcal{F}_i}$, the canonical isomorphism $\mathcal{O}_T(D_{\mathcal{F}_1,W}) \otimes \mathcal{O}_T(D_{\mathcal{F}_3,W}) \cong \mathcal{O}_T(D_{\mathcal{F}_2,W})$. Similarly we have an additivity property in the variable W.*

Proof. To see the generic disjointness is satisfied when Z and W intersect properly, note that Z (resp. W) has codimension $\geq b+1$ (resp. $\geq a+1$) in P_T, hence $Z \cap W$ has codimension $\geq a+b+2$ in P_T. Therefore $\dim(Z \cap W) < \dim(T)$, so the support of $\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W$ cannot dominate any component of T. The hypothesis on the incidence means the construction of [12, Ch.II] applies. The compatibility with base change is a consequence of [12, Thm.3(v)]; and the additivity is inherited from [12, Thm.3(ii)]. □

Remark 5.12. Our essential task is to show that given a Hilbert-Chow datum (Z, Z', W), we have $D_{Z,W} = D_{Z',W}$.

SUBSECTION 5.2. MOVING LEMMAS.

PROPOSITION 5.13. *Let (Z, W) be a Hilbert datum over T the spectrum of a local ring, with $W = W_k \times_k T$ for a b-dimensional k-scheme W_k, and suppose $(Z \cap W)_\eta = \emptyset$ for every generic point $\eta \in T$. Then there exist subvarieties $B_1, \ldots, B_n \subset P$ of dimension $b+1$, $M_i \in \mathrm{Pic}(B_i)$, and short exact sequences:*

$$0 \to M_i \xrightarrow{s_0^i} \mathcal{O}_{B_i} \to Q_0^i \to 0$$

$$0 \to M_i \xrightarrow{s_\infty^i} \mathcal{O}_{B_i} \to Q_\infty^i \to 0$$

such that:

(1) $(Z \cap \mathrm{Supp}(Q_*^i))_\eta = \emptyset$ *for all i, all generic η; and*
(2) *the b-dimensional cycle $[W] + \sum_i([Q_0^i] - [Q_\infty^i])$ is disjoint from Z.*

Proof. We let Z_0 denote the cycle over the closed fiber of T. By Chow's moving lemma [16, Thm.], we can find a cycle α rationally equivalent to $[W_k]$ and satisfying $\alpha \cap Z_0 = \emptyset$; hence also $\alpha \cap Z = \emptyset$ on P_T. This shows we can achieve the second property; the issue is to show we can move W in such a way that the first property is satisfied.

Suppose we have a closed immersion $P \hookrightarrow \mathbb{P}^{2n+1}$. Then every step in moving a cycle involves essentially two choices: a linear space $L \cong \mathbb{P}^n \hookrightarrow \mathbb{P}^{2n+1}$, disjoint from P, from which projection induces a finite morphism $\pi_L : P \to \mathbb{P}^n$; and an element $g \in PGL(n+1)$. The excess intersection $e(Z_0, \pi_L^* \pi_{L*}[W] - [W])$ is smaller than $e(Z_0, [W])$ for generic L; and $\pi_L^*(g \cdot \pi_{L*} W)$ is disjoint from Z_0 for generic g.

If $(Z \cap W)_\eta = \emptyset$ for all generic points $\eta \in T$, then for generic choices of L, g, we have $(Z \cap \pi_L^*(g \cdot \pi_{L*}[W]))_\eta = (Z \cap \pi_L^*(\pi_{L*}[W]))_\eta = \emptyset$. The Q_*s are supported in subsets of the form $\pi_L^*(g \cdot \pi_{L*} W)$ and $\pi_L^*(\pi_{L*} W)$, hence the result. □

We need a slight variation for subvarieties W as in 5.13 of dimension strictly smaller than $n-a-1$; eventually we need that such subvarieties do not contribute to $D_{Z,W}$.

PROPOSITION 5.14. *Let (Z,W) be a Hilbert datum over T a base of dimension ≤ 1, with $W = W_k \times_k T$ for a k-scheme W_k, and suppose $(Z \cap W)_\eta = \emptyset$ for every generic point $\eta \in T$.*
Suppose further that $\dim(W_k) = b \leq n-a-2$. Then there exist subvarieties $B_1, \ldots, B_n \subset P$ of dimension $b+1$, $M_i \in \mathrm{Pic}(B_i)$, and short exact sequences:

$$0 \to M_i \xrightarrow{s_0^i} \mathcal{O}_{B_i} \to Q_0^i \to 0$$

$$0 \to M_i \xrightarrow{s_\infty^i} \mathcal{O}_{B_i} \to Q_\infty^i \to 0$$

such that:

(1) $(Z \cap B_i)_\eta = \emptyset$ for all i, all generic η; and
(2) the b-dimensional cycle $[W] + \sum_i([Q_0^i] - [Q_\infty^i])$ is disjoint from Z.

Remark 5.15. The first condition in the conclusion implies $(Z \cap \mathrm{Supp}(Q_*^i))_\eta = \emptyset$ for all i, all generic η.

Proof. Again we are intersecting a finite number of open conditions. Without loss of generality we may assume W_k is an integral subscheme of dimension $n-a-2$. Let $pr_1(Z) \hookrightarrow P$ denote the "sweep" of the family Z (with the reduced structure); this is a subscheme of dimension $\leq a+1$.
Now $pr_1(Z)$ and W_k are not expected to meet, and we have an open dense $U \subset T$ such that $pr_1(Z_U) \cap W_k = \emptyset$. For a generic finite morphism $\pi : P \to \mathbb{P}^n$ (as in the proof of 5.13) we have, possibly after shrinking U, that $\pi(pr_1(Z_U)) \cap \pi(W_k) = \emptyset$; and that the pair $(Z, \pi^*\pi_*(W_k) - W_k)$ has smaller excess intersection than does (Z, W_k). Now we move $\pi_*(W_k)$ along a general smooth (affine) rational curve $C \hookrightarrow PGL(n+1)$. Let $\mathscr{Y} \hookrightarrow \mathbb{P}^n \times C$ note the total space of the resulting family,
Write $\mathscr{Y} = \sum m_i Y_i$, and let ${Y_i}^{fl} \hookrightarrow \mathbb{P}^n \times \mathbb{P}^1$ denote the flat limit of the family $Y_i \hookrightarrow \mathbb{P}^n \times C$. Then $\mathscr{Y}^{fl} := \sum m_i {Y_i}^{fl}$ is the unique way to complete $\mathscr{Y}$ to a family of cycles over $\mathbb{P}^1$. Let $pr_1(\mathscr{Y}^{fl}) \hookrightarrow \mathbb{P}^n$ be the sweep; this is a subscheme of dimension $n-a-1$. Choose some $t \in T$ such that $Z_t \cap W_k = \emptyset$. For a general choice of C, since $\dim(Z_t) + \dim(\mathscr{Y}^{fl}) = a + (n-a-1) < n$, we will have $\pi(pr_1(Z_t)) \cap \mathscr{Y}^{fl} = \emptyset$. Hence the disjointness holds on an open dense of T. This process can be iterated until we have a cycle $\alpha \sim W_k$ such that $pr_1(Z) \cap \alpha = \emptyset$.
The subvarieties $B_i \hookrightarrow P$ lie in subsets of the form $\pi^{-1}(\mathscr{Y}^{fl})$. (This follows from the proof that flat pullback preserves rational equivalence [7, 1.7].) Since a general $\mathscr{Y}^{fl}$ used in one step of the moving process is disjoint from a general member of the family Z, this holds after pullback by π as well. □

SUBSECTION 5.3. GROTHENDIECK-RIEMANN-ROCH.

LEMMA 5.16. *Let T be the spectrum of a field, and suppose $\mathcal{F}, \mathcal{G} \in \mathrm{Parf}(P_T)$ satisfy $\dim(\mathrm{Supp}(\mathcal{F})) + \dim(\mathrm{Supp}(\mathcal{G})) < n = \dim(P)$. Then $\chi(\mathcal{F} \otimes^{\mathrm{L}} \mathcal{G}) = 0$.*

Proof. Since $F_a(K_0(P))$ is generated by $[\mathcal{O}_V]$, $V \subset P$ a subvariety of dimension $\leq a$ [7, Ex. 15.1.5], we may assume $\mathcal{F}, \mathcal{G}$ are structure sheaves of subvarieties of dimensions a, b respectively, with $a + b < n$.

Now since P is smooth, any coherent sheaf has a finite length resolution by finite rank locally free sheaves, so we may apply [7, 18.3.1 (c)] to the closed immersion $i : V \to P$ with $\beta = \mathcal{O}_V$. This gives, in the Chow group $A_*(P)$:

$$i_*(\mathrm{ch}(\mathcal{O}_V) \cap \mathrm{Td}(V)) = \mathrm{ch}(i_*\mathcal{O}_V) \cap \mathrm{Td}(P).$$

As $\mathrm{ch}(\mathcal{O}_V) = 1$ and $\mathrm{Td}(V) = [V] + r_V$ with $r_V \in A_{<a}(V)$, the left hand side lies in $A_{\leq a}(P)$.

Since P is smooth, $A^p(P) \cap A_q(P) \subset A_{q-p}(P)$ by [7, 8.3 (b)]. As $\mathrm{Td}(P) = [P] + r_P$ with $r_P \in A_{<n}(P)$, by equating terms in each degree, we find $\mathrm{ch}(i_*\mathcal{O}_V) \in A^{\geq n-a}(P)$.

By Grothendieck-Riemann-Roch (for the smooth P, as in [7, 15.2.1]) and the action of ch on $\otimes$, $\chi(\mathcal{F} \otimes^{\mathrm{L}} \mathcal{G}) = \int_P \mathrm{ch}(\mathcal{F}) \cdot \mathrm{ch}(\mathcal{G}) \cdot \mathrm{Td}_P$. Here $\cdot$ means intersection product of cycle classes. The first possible nonzero term in $\mathrm{ch}(\mathcal{F}) \cdot \mathrm{ch}(\mathcal{G})$ would come from $\mathrm{ch}_{n-a}(\mathcal{F}) \cdot \mathrm{ch}_{n-b}(\mathcal{G})$, but this term is zero for degree reasons. □

Proposition 5.17. *Let T be the spectrum of a field, and $Z \hookrightarrow P_T$ an a-dimensional subscheme. Let $M, N \in \mathrm{Coh}(P_T)$ be invertible sheaves on some subvariety of P_T of dimension $\leq n - a$, and suppose we have exact sequences:*

$$0 \to M \xrightarrow{s} N \to Q_s \to 0$$

$$0 \to M \xrightarrow{t} N \to Q_t \to 0$$

such that $Z \cap \mathrm{Supp}(Q_s) = Z \cap \mathrm{Supp}(Q_t) = \emptyset$.
Then the unique $a_Z \in \Gamma(T, \mathcal{O}_T^*)$ making the following diagram commute:

$$\begin{tikzcd}
f_T(\mathcal{O}_Z, Q_s) \arrow[r, "\varphi_T^Z"] \arrow[d, "via\ s"'] & \mathcal{O}_T \arrow[dd, "a_Z"] \\
(f_T(\mathcal{O}_Z, M))^{-1} \otimes f_T(\mathcal{O}_Z, N) & \\
f_T(\mathcal{O}_Z, Q_t) \arrow[r, "\varphi_T^Z"] \arrow[u, "via\ t"] & \mathcal{O}_T
\end{tikzcd}$$

depends only on $[Z]$, i.e., $a_Z = a_{Z'}$ if $[Z] = [Z']$.

Proof. To prove the claim it is equivalent to show that the difference between $f(1 \otimes s), f(1 \otimes t) : f_T(\mathcal{O}_Z, M) \cong f_T(\mathcal{O}_Z, N)$ depends only on $[Z]$.

Step 1. By taking a filtration of $\mathcal{O}_Z$ such that the graded pieces are isomorphic to (twists of) structure sheaves of subvarieties, and using the additivity of the determinant on filtrations, we are reduced to showing that if $\mathcal{F} \in \mathrm{Coh}(P_T)$ with $\mathrm{Supp}(\mathcal{F}) \subset \mathrm{Supp}(Z)$ and $\dim(\mathrm{Supp}(\mathcal{F})) \leq a - 1$, the induced isomorphisms $f(1 \otimes s), f(1 \otimes t) : f(\mathcal{F}, M) \cong f(\mathcal{F}, N)$ are equal. (Such a filtration exists by

[11, I.7.4].) The subquotients of the filtration of $\mathcal{O}_Z$ depend on the filtration chosen, but the top-dimensional components always appear with their correct multiplicities, i.e., the cycle $[Z]$ can be extracted from the filtration.
Step 2. Let Q_U denote the cokernel of the universal $\mathcal{O}_P$-homomorphism $M \to N$; note Q_U is flat over $\mathrm{Hom}_{\mathcal{O}_P}(M,N) \setminus 0$ since a morphism between invertible sheaves is either injective or zero, hence the Euler characteristic of every cokernel is $\chi(M) - \chi(N)$. We consider the line bundle $\det \mathrm{R}\pi_*(p_1^*(\mathcal{F}) \otimes^{\mathrm{L}} Q_U)$ on $\mathrm{Hom}_{\mathcal{O}_P}(M,N) \setminus 0$. Its fiber over $s \in \mathrm{Hom}_{\mathcal{O}_P}(M,N)$ is precisely $f(\mathcal{F}, Q_s)$. Since $Q_s = Q_{\lambda s}$ for $\lambda \in \Gamma(T, \mathcal{O}_T^*)$, we consider $\det \mathrm{R}\pi_*(p_1^*(\mathcal{F}) \otimes^{\mathrm{L}} Q_U)$ as a line bundle on the projective space $\mathbb{P}(\mathrm{Hom}_{\mathcal{O}_P}(M,N) \setminus 0)$.
We claim this line bundle is trivial. To prove this it suffices to show it is trivial along a line $\mathbb{P}^1 \cong L \hookrightarrow \mathbb{P}(\mathrm{Hom}_{\mathcal{O}_P}(M,N) \setminus 0)$. For this purpose Grothendieck-Riemann-Roch (i.e., ignoring torsion) is adequate. More precisely, we consider the GRR diagram:

$$\begin{tikzcd}
K_0(P \times L) \arrow[r, "\mathrm{ch}(-)\cdot\mathrm{Td}(P)\cdot\mathrm{Td}(L)"] \arrow[d, "\mathrm{R}\pi_*"] & A_*(P \times L)_{\mathbb{Q}} \arrow[d, "\pi_*"] \\
K_0(L) \arrow[r, "\mathrm{ch}(-)\cdot\mathrm{Td}(L)"] \arrow[d, "\det"] & A_*(L)_{\mathbb{Q}} \\
\mathrm{Pic}(L) \arrow[ur, "c_1"] &
\end{tikzcd}$$

We have $\dim(\mathrm{Supp}(p_1^*(\mathcal{F}))) \leq a, \dim(\mathrm{Supp}(Q_U)) \leq n-a$, and $\dim(P \times L) = n+1$. Hence $\mathrm{ch}(p_1^*(\mathcal{F})) \cdot \mathrm{ch}(Q_U) = 0$ in $A_*(P \times L)_{\mathbb{Q}}$ for degree reasons (as in the proof of 5.16), so $p_1^*(\mathcal{F}) \otimes^{\mathrm{L}} Q_U \in K_0(P \times L)$ maps to 0 in the top row. Hence $c_1(\det \mathrm{R}\pi_*(p_1^*(\mathcal{F}) \otimes^{\mathrm{L}} Q_U))$ is a torsion class, and therefore $\det \mathrm{R}\pi_*(p_1^*(\mathcal{F}) \otimes^{\mathrm{L}} Q_U)$ is trivial.
Step 3. For $s \in \mathrm{Hom}_{\mathcal{O}_P}(M,N) \setminus 0$, consider the induced identification $f(s) : f(\mathcal{F}, Q_s) \otimes f(\mathcal{F}, M) \cong f(\mathcal{F}, N)$. Since $f(\mathcal{F}, Q_s)$ is canonically trivial, i.e., the trivialization induced by $\mathrm{Supp}(\mathcal{F}) \cap \mathrm{Supp}(Q_s) = \emptyset$ extends over all $\mathrm{Hom}_{\mathcal{O}_P}(M,N) \setminus 0$, we consider $f(s)$ as an isomorphism $f(\mathcal{F}, M) \cong f(\mathcal{F}, N)$. Since $\chi(\mathcal{F} \otimes^{\mathrm{L}} M) = \chi(\mathcal{F} \otimes^{\mathrm{L}} N) = 0$ by 5.16, we have $f(s) = f(\lambda s)$ for $\lambda \in \Gamma(T, \mathcal{O}_T^*)$. Therefore we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(M,N) \setminus 0 \arrow[r, "s \mapsto f(s)"] \arrow[d] & \mathrm{Isom}(f(\mathcal{F}, M), f(\mathcal{F}, N)) \\
\mathbb{P}(\mathrm{Hom}(M,N) \setminus 0) \arrow[ur] &
\end{tikzcd}$$

But there are no nonconstant functions on $\mathbb{P}(\mathrm{Hom}(M,N) \setminus 0)$, hence $f(s) = f(t) : f(\mathcal{F}, M) \cong f(\mathcal{F}, N)$.

□

Subsection 5.4. Further properties of $D_{Z,W}$. Assuming the base and incidence are optimal, we can already prove the following important property of $D_{Z,W}$.

Proposition 5.18. *Let (Z, Z', W) be a Hilbert-Chow datum over a normal base T, and suppose the incidence $Z \cap W$ satisfies:*

(1) $(Z \cap W)_\eta = \emptyset$ for all generic points $\eta \in T$; and
(2) $Z \cap W$ is finite over all points of depth 1 in T.

Then the Cartier divisors $D_{Z,W}, D_{Z',W}$ are equal.

Proof. Since the smooth locus of T contains all points of depth 1, and the formation of $D_{Z,W}$ is compatible with the inclusion $T^{sm} \subset T$, we may assume T is smooth. For T regular there is a canonical isomorphism [12, Prop. 8]:

$$f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong \otimes_{p,q}(\det{}_T R^q \pi_*(\mathcal{H}^p(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W)))^{(-1)^{p+q}}.$$

To calculate the coefficient of a depth 1 point $t \in T$ in $D_{Z,W}$, we may replace T with the spectrum of the DVR $\mathcal{O}_{T,t}$. Then the support of $\mathcal{H}^p(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W)$ is finite over T (indeed, over t), so in the displayed expression only the terms with $q = 0$ can contribute.

By [12, Thm.3(vi)], the multiplicity of a depth one point is determined by the sum $\sum_i (-1)^i \ell_t(\mathcal{H}^i(\mathrm{R}\pi_*(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W)))$. This last sum is equal to

$$\sum_i (-1)^i \ell_t(\pi_* \mathcal{H}^i(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W)) = (\deg \pi)(\sum_{p,t' \to t} (-1)^p \ell_{t'}(\mathcal{H}^p(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W))),$$

hence it suffices to show $\gamma(\mathcal{O}_Z) := \sum_{p,t' \to t} (-1)^p \ell_{t'}(\mathcal{H}^p(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W))$ depends only the underlying cycle $[Z]$. We remark that if Z and W are integral and $b = n - a - 1$, then the contribution of a point t' (lying over t) to $\gamma(\mathcal{O}_Z)$ is exactly Serre's Tor-formula for the intersection index of Z and W at t' [22, V.C.Thm.1(b)].

Without loss of generality we assume W is integral and $\dim(W) = b = n-a-1$; we will see in the proof all sums are 0 if $b < n-a-1$. Given an exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of coherent sheaves on P_T with support of relative dimension $\leq a$ and satisfying the incidence hypothesis with respect to W, by the long exact cohomology sequence we obtain $\gamma(\mathcal{F}_1) + \gamma(\mathcal{F}_3) = \gamma(\mathcal{F}_2)$. It then follows γ is additive on filtrations.

Write $[Z] = \sum_i a_i Z_i$. Again by [11, I.7.4], the sheaf $\mathcal{O}_Z$ admits a filtration whose subquotients are invertible sheaves L_i on subvarieties contained in Z; and each top-dimensional component Z_i appears exactly a_i times. Since every L_i is some tensor power of a very ample class on P_T, we may assume there is either an injective map $\mathcal{O}_{Z_i} \to L_i$ or an injective map $L_i \to \mathcal{O}_{Z_i}$. Therefore $\gamma(\mathcal{O}_Z) = \sum_i \gamma(L_i) = \sum_i a_i \gamma(\mathcal{O}_{Z_i})$ modulo summands of the form $\gamma(\mathcal{F})$ where $\mathcal{F}$ is a sheaf on P_T whose support over the generic point of T has dimension $\leq a - 1$. So it suffices to show γ vanishes on sheaves of this type. Again we may assume $\mathcal{F}$ is isomorphic to the structure sheaf of a subvariety $Y \hookrightarrow Z$ in P_T. Note that $\dim Y \leq a$, else, being contained in Z, Y would dominate T and

have all fibers of dimension $\geq a$; and then Y would contribute to the cycle $[Z]$ of Z.
There are two cases to consider: if $Y \cap W = \emptyset$, then $\gamma(\mathcal{F}) = 0$ since $\mathcal{F} \otimes^{\mathrm{L}} \mathcal{O}_W$ is acyclic. If $Y \cap W \neq \emptyset$, then the intersection $Y \cap W$ is improper, and the Tor-formula vanishes at components of improper intersection (due to Serre in the equal characteristic case [22, V.C.Thm.1(a)]). □

Remarks 5.19. (5.19.1) The proof shows the relation between the incidence line bundle and Serre's Tor-formula for intersection multiplicities; and also that our construction agrees with Mazur's at least on the normalization of the locus U'. Thus the essential tasks are to extend the divisor through the locus where the expected incidence condition (the hypothesis in 5.18) fails, and to remove the assumption of normality.
(5.19.2) The assumption (C2) of [4] is that the incidence is generically finite over, and nowhere dense in, its image in the base S. These assumptions imply the map $S \to \mathscr{C}_a(P) \times \mathscr{C}_{n-a-1}(P)$ factors through U'.
(5.19.3) Considering the GRR diagram as in the proof of 5.17 with L replaced by a general regular base T, one sees that the first Chern class of the incidence line bundle $f_T(\mathcal{O}_Z, \mathcal{O}_W)$ modulo torsion depends only on the underlying cycles $[Z], [W]$, independent of any assumption of properness of intersection. By contrast in 5.18 we have the result integrally.
(5.19.4) We may write $D_{[Z],W}$ in the case we have a Hilbert datum as in 5.18, e.g., with proper intersection over a regular base T.

Corollary 5.20. *Among Hilbert-Chow data:*

1. *over normal bases T; and*
2. *such that Z, W (and hence Z', W) are generically disjoint and have finite incidence over points of depth 1 in T;*

there exists a collection of isomorphisms $\phi_T^{Z,Z'} : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_W)$ which:

1. *is compatible with base change preserving the incidence condition (2);*
2. *satisfies the cocycle condition; and*
3. *agrees with the collection on disjoint families defined in 5.10.*

Proof. We define $\phi_T^{Z,Z'} := (\varphi_T^{Z'})^{-1} \circ \varphi_T^Z : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong \mathcal{O}_T(D_{[Z],W}) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_W)$. □

Construction-Notation 5.21. We continue with the Cartier divisor $D_{Z,W} \hookrightarrow T$ associated to a Hilbert datum (5.11). For $Z \hookrightarrow P_T$ a T-flat family of a-dimensional subschemes of P and $s \in \mathbb{Z}_{\geq 0}$, let $\mathrm{Coh}_{\leq s; Z_T}(P)$ denote the abelian category of coherent sheaves $\mathcal{G}$ on P such that $\dim(\mathrm{Supp}(\mathcal{G})) \leq s$ and $(Z \cap \mathrm{Supp}(\mathcal{G}))_\eta = \emptyset$ for all generic points $\eta \in T$. For $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-1; Z_T}(P)$, we obtain a Cartier divisor $D_{Z,\mathcal{G}} \hookrightarrow T$ and a canonical isomorphism $f_T(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_T(D_{Z,\mathcal{G}})$.
We denote by $K_0^{s;Z}(P)$ the K_0-group of the abelian category $\mathrm{Coh}_{\leq s; Z_T}(P)$: we take the free abelian group on sheaves in $\mathrm{Coh}_{\leq s; Z_T}(P)$, then impose relations

from short exact sequences whose terms all lie in $\mathrm{Coh}_{\leq s;Z_T}(P)$. Let $\mathrm{CDiv}(T)$ denote the group of Cartier divisors on T.

We summarize some elementary properties of this construction in the following proposition.

Proposition 5.22. *(1) The map $D_{Z,-} : \mathrm{Coh}_{\leq n-a-1;Z_T}(P) \to \mathrm{CDiv}(T)$ defined by $\mathcal{G} \mapsto D_{Z,\mathcal{G}}$ descends to a homomorphism $K_0^{n-a-1;Z}(P) \to \mathrm{CDiv}(T)$ (which we also denote by $D_{Z,-}$).*

(2) If $f : S \to T$ is a morphism and $\mathcal{G} \in \mathrm{Coh}_{\leq s;Z_S}(P)$, then $f^*(D_{Z,\mathcal{G}}) = D_{Z,\mathcal{G}}$ as divisors on S.

(3) If $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-1;Z_T}(P)$ satisfies $Z \cap \mathcal{G} = \emptyset$, then $D_{Z,\mathcal{G}} = 0$.

(4) If $\mathcal{G} \in \mathrm{Coh}_{\leq 0;Z_T}(P)$ and the divisorial part of the family $Z \hookrightarrow P_T$ is trivial (i.e., $a \leq \dim(P) - 2$), then $D_{Z,\mathcal{G}} = 0$.

Proof. The first three properties follow immediately from the additivity, compatibility with base change, and compatibility with the trivialization of an acyclic complex, of the associated divisor construction discussed in Section 2. To prove the last property, by the first property we may assume $\mathcal{G}$ is the structure sheaf of a zero-dimensional subvariety $W \hookrightarrow P$. (If $k = \overline{k}$, this is just a single closed point, but we give an argument here valid for any T-flat family $W \hookrightarrow P_T$ of zero-dimensional subschemes, such that W is integral.)
By [20, 5.3] there is a canonical isomorphism:

$$\det{}_T \mathrm{R}\pi_*(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{G}) \cong (\det{}_T \pi_* \mathcal{G})^{\mathrm{rk}(\mathcal{O}_Z)-1} \otimes \det{}_T(\pi_*(\det{}_{P_T}(\mathcal{O}_Z)|_W)).$$

Since the divisorial part of $\mathcal{O}_Z$ was assumed to be empty, the line bundle $\det_{P_T}(\mathcal{O}_Z)$ is canonically trivial; and $\mathrm{rk}(\mathcal{O}_Z) - 1 = -1$. Therefore the right hand side is canonically trivial, so $D_{Z,\mathcal{G}} = 0$. □

Remark 5.23. In case $a = \dim(P) - 1$, we refer to [20]. The reduction to the diagonal shows we may assume $\dim(P) = 1$ or $a \leq \dim(P) - 2$; but to the extent we rely on (4) in 5.22, we must understand the case of zero-cycles and divisors.

Proposition 5.24. *Suppose T has dimension ≤ 1 and $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-2;Z_T}(P)$. Then $D_{Z,\mathcal{G}} = 0$.*

Remark 5.25. The condition $D_{Z,\mathcal{G}} = 0$ is equivalent to the canonical trivialization (induced by the generic acyclicity of $\mathrm{R}\pi_*(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{G})$) extending over all of T.

Proof. We may also assume $\mathcal{G}$ is the structure sheaf of a subvariety $W \hookrightarrow P$ of dimension $b \leq n - a - 2$, as these sheaves generate the K_0-group.
Since $D_{Z,\mathcal{G}} = 0$ for $\dim(\mathrm{Supp}(\mathcal{G})) \leq 0$ (again by 5.22), it suffices to prove the following claim: if $D_{Z,\mathcal{G}} = 0$ for all $\mathcal{G} \in \mathrm{Coh}_{\leq b-1;Z}(P)$, and $1 \leq b \leq n - a - 2$, then $D_{Z,\mathcal{G}} = 0$ for all $\mathcal{G} \in \mathrm{Coh}_{\leq b;Z}(P)$.
We prove this claim: by 5.14 there are short exact sequences:

$$0 \to M_i \xrightarrow{s_0^i} \mathcal{O}_{B_i} \to Q_0^i \to 0$$

$$0 \to M_i \xrightarrow{s^i_\infty} \mathcal{O}_{B_i} \to Q^i_\infty \to 0$$

in $\mathrm{Coh}_{\leq n-a-1;Z_T}(P)$, such that the b-dimensional cycle $W + \sum_i([Q^i_0] - [Q^i_\infty])$ is disjoint from Z.

If $[Q] = \sum_j a_j W_j$, then we write $D_{Z,[Q]} := \sum_j a_j D_{Z,W_j}$. The short exact sequences give $\sum_i D_{Z,Q^i_0} = \sum_i D_{Z,Q^i_\infty}$ in $K_0^{n-a-1;Z}(P)$. Furthermore the difference $Q^i_* - [Q^i_*]$ lies in $F_{b-1}(K_0(P))$.

But then

$D_{Z,W} = D_{Z,W} + \sum_i(D_{Z,Q^i_0} - D_{Z,Q^i_\infty})$ from the SES

$= D_{Z,W} + \sum_i(D_{Z,[Q^i_0]} - D_{Z,[Q^i_\infty]})$ since we assumed $D_{Z,-}$ vanishes on $F_{b-1}(K_0(P))$

$= D_{Z,W+\sum_i([Q^i_0]-[Q^i_\infty])} = 0$ by the disjointness from Z. □

Corollary 5.26. *Suppose T is normal and $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-2;Z_T}(P)$. Then $D_{Z,\mathcal{G}} = 0$.*

Proof. For $t \in T$ of depth 1, the formation of $D_{Z,\mathcal{G}}$ is compatible with the morphism $g_t : \operatorname{Spec} \mathcal{O}_{T,t} \to T$. By 5.24, we have $g_t^*(D_{Z,\mathcal{G}}) = 0$ for all such t. Since $D_{Z,\mathcal{G}}$ is determined its restriction to points of depth 1, the result follows. □

Proposition 5.27. *Suppose T is seminormal and $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-2;Z_T}(P)$. Then $D_{Z,\mathcal{G}} = 0$.*

Proof. The formation of the divisor $D_{Z,\mathcal{G}}$ is compatible with the (finite, birational) normalization morphism $\nu : T^\nu \to T$. By the previous result we know local equations for $\nu^*(D_{Z,\mathcal{G}})$ are units. We need to show these units are constant along the fibers of ν. Suppose $t \in T$ has branches $b_1, \ldots, b_r$ in T^ν. For each b_i there exists a DVR R_i and a morphism $g_i : \operatorname{Spec} R_i \to T$ such that: R_i has residue field $\kappa(t)$, and g_i covers a generization of t to the locus of subschemes disjoint from $\mathrm{Supp}(\mathcal{G})$. Denote by S the union of the $\operatorname{Spec} R_i$s glued along $\operatorname{Spec} \kappa(t)$. Then we have a morphism $g : S \to T$, and by 5.24 we conclude $g^*(D_{Z,\mathcal{G}}) = 0$. The corresponding trivialization $f_S(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_S$ is our candidate for extending the trivialization through t.

Let $\mathrm{Exc}(\nu) \hookrightarrow T$ denote the locus over which ν is not an isomorphism, and let $I_\mathcal{G} \hookrightarrow T$ denote the locus of subschemes Z such that $Z \cap \mathrm{Supp}(\mathcal{G}) \neq \emptyset$. Set $U := T - (\mathrm{Exc}(\nu) \cap I_\mathcal{G})$. Then we have a trivialization $\varphi_U : f_U(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_U$, and this extends to $\varphi_{T^\nu} : f_{T^\nu}(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_{T^\nu}$. For $t \notin U$, we constructed in the previous paragraph the isomorphism $t_S : f_S(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_S$, which we may restrict to t. Together these define a pointwise trivialization $f_T(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_T$, i.e., a nonzero element in every fiber of the line bundle $f_T(\mathcal{O}_Z, \mathcal{G})$. We form the cartesian diagram:

$$\begin{tikzcd} S^\nu \arrow[r, "g^\nu"] \arrow[d, "\nu|_S"'] & T^\nu \arrow[d, "\nu"] \\ S \arrow[r, "g"] & T \end{tikzcd}$$

Since the formation of $D_{Z,\mathcal{G}}$ is compatible with all of the morphisms appearing in this diagram, we obtain $(\nu|_S)^*(t_S) = (g^\nu)^*(\varphi_{T^\nu})$. It then follows φ_{T^ν} is constant along the fibers of ν, hence it descends to a trivialization $\varphi_T : f_T(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_T$. □

Corollary 5.28. *Suppose T is seminormal and $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-1;Z_T}(P)$. Then $D_{Z,\mathcal{G}} = D_{Z,[\mathcal{G}]}$.*

Remark 5.29. For T smooth and $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-2;Z_T}(P)$, one can deduce the line bundle $f_T(\mathcal{O}_Z, \mathcal{G})$ is trivial as follows. The filtration of the $K_0(X)$-group by dimension of support is compatible with multiplication, if X is a smooth quasi-projective scheme over a field [9, Exp.0 Ch.2 Sect.4 Thm.2.12 Cor.1]. From $\mathcal{O}_Z \in F_{a+\dim T}(K_0(P_T))$ and $\mathcal{G} \in F_{n-a-2+\dim T}(K_0(P_T))$ it follows that $\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{G} \in F_{\dim T-2}(K_0(P_T))$. Therefore $\mathrm{R}\pi_*(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{G}) \in F_{\dim T-2}(K_0(T))$, and hence $f_T(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_T$.
For a general base T, this reasoning is valid rationally, hence we can conclude $f_T(\mathcal{O}_Z, \mathcal{G})$ is a torsion line bundle.
Since the dimension filtration's compatibility with multiplication can be viewed as a consequence of the moving lemma, in some sense we have given this proof. Because we need to keep track of the trivialization and not just the abstract invertible sheaf, we work with Cartier divisors rather than line bundles.

Corollary 5.30. *Let (Z, W) be a Hilbert datum over any base T such that $(Z \cap W)_\eta = \emptyset$ for every generic point $\eta \in T$. Suppose further that $W = W_k \times_k T$ for a k-scheme W_k with $[W_k] = \sum_i a_i W_i$. Then there is a canonical isomorphism $f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong \otimes_i (f_T(\mathcal{O}_Z, \mathcal{O}_{W_i}))^{\otimes a_i}$.*

Subsection 5.5. Application to the incidence line bundle.

Construction 5.31. The facts 5.13 and 5.30 produce a construction; in the description we suppress the base T and we use $f(-) := f(\mathcal{O}_Z, -)$ as before, since the first factor is constant. We use the identification of 5.30: $f(\mathcal{O}_W) \cong \otimes_i (f(\mathcal{O}_{W_i}))^{\otimes a_i} =: f([\mathcal{O}_W])$. (In our situation W is a Cartier divisor on a $(b+1)-$dimensional subvariety $B \hookrightarrow P$ with $[W] = \sum_i a_i W_i$.)
Now given $(Z, W); B_i, M_i, s^i_{0,\infty}$ as in 5.13, let $f(s_*) : f(M) \otimes f(Q_*) \cong f(\mathcal{O}_B)$ denote the isomorphism induced by the short exact sequence. We set

$$\alpha := [W] + \sum_i ([Q^i_0] - [Q^i_\infty])$$

to be the moved b-dimensional cycle. Then we have a canonical isomorphism $f([\mathcal{O}_W]) \otimes (\otimes_i f([Q^i_0])) = f(\alpha) \otimes (\otimes_i f([Q^i_\infty]))$.
Let $\beta^Z_{W,\alpha} : f([\mathcal{O}_W]) \cong f(\alpha)$ be the unique isomorphism making the following diagram commute:

$$\begin{tikzcd}
f([\mathcal{O}_W]) \otimes (\otimes_i (f([Q^i_0]) \otimes f(M_i))) \arrow[r, "="] \arrow[d, "1\otimes(\otimes_i f(s^i_0))"] & f(\alpha) \otimes (\otimes_i (f([Q^i_\infty]) \otimes f(M_i))) \arrow[d, "1\otimes(\otimes_i f(s^i_\infty))"] \\
f([\mathcal{O}_W]) \otimes (\otimes_i f(\mathcal{O}_{B_i})) \arrow[r, "\beta^Z_{W,\alpha}\otimes 1"] & f(\alpha) \otimes (\otimes_i f(\mathcal{O}_{B_i}))
\end{tikzcd}$$

In the top row, we tensored the canonical isomorphism with the identity on the $f(M_i)$ factors.

This construction extends the collection $\{\phi_T\}$ to spectra of local rings T.

PROPOSITION 5.32. *Let (Z, Z', W) be a Hilbert-Chow datum over T the spectrum of a local ring with $W = W_k \times_k T$, and suppose $(Z\cap W)_\eta = (Z'\cap W)_\eta = \emptyset$ for all generic points $\eta \in T$. Suppose subvarieties $B_1, \dots, B_n \subset P$ and short exact sequences as in 5.13 have been chosen. Then the isomorphism:*

$$\left(\varphi^{Z';\alpha} \circ \beta^{Z'}_{W,\alpha}\right)^{-1} \circ \left(\varphi^{Z;\alpha} \circ \beta^{Z}_{W,\alpha}\right) : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong$$
$$\cong f_T(\mathcal{O}_Z, \alpha) \cong \mathcal{O}_T \cong f_T(\mathcal{O}_{Z'}, \alpha) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_W)$$

is independent of the choice of move: given another collection of data $(\hat{B}_i, \hat{M}_i, \hat{s^i}_{0,\infty})$ producing a b-dimensional cycle $\hat{\alpha}$ (also disjoint from Z), we have:

$$\left(\varphi^{Z';\alpha} \circ \beta^{Z'}_{W,\alpha}\right)^{-1} \circ \left(\varphi^{Z;\alpha} \circ \beta^{Z}_{W,\alpha}\right) = \left(\varphi^{Z';\hat{\alpha}} \circ \beta^{Z'}_{W,\hat{\alpha}}\right)^{-1} \circ \left(\varphi^{Z;\hat{\alpha}} \circ \beta^{Z}_{W,\hat{\alpha}}\right).$$

Furthermore $\left(\varphi^{Z';\alpha} \circ \beta^{Z'}_{W,\alpha}\right)^{-1} \circ \left(\varphi^{Z;\alpha} \circ \beta^{Z}_{W,\alpha}\right)$ agrees with the canonical identifications at every generic $\eta \in T$. If in addition T is the spectrum of a regular local ring and the incidence $Z \cap W$ satisfies the hypotheses of 5.20, then $\phi_T^{Z,Z'} = \left(\varphi^{Z';\alpha} \circ \beta^{Z'}_{W,\alpha}\right)^{-1} \circ \left(\varphi^{Z;\alpha} \circ \beta^{Z}_{W,\alpha}\right)$.

Proof. We claim any choice of moving data produces the canonical isomorphism over the generic points of T. Now 5.17 shows the choice of short exact sequences affects the map $\varphi_\eta^{Z;\alpha} \circ \left(\beta^Z_{W,\alpha}\right)_\eta : f_\eta(\mathcal{O}_Z, \mathcal{O}_W) \cong f_\eta(\mathcal{O}_Z, \alpha) \cong \mathcal{O}_\eta$ by a constant depending only on $[Z]$, and this is exactly canceled by the map back to $f_T(\mathcal{O}_{Z'}, \mathcal{O}_W)$. In other words, since $a_Z = a_{Z'}$, the following diagram commutes.

$$\begin{tikzcd}
f_\eta(\mathcal{O}_Z, \mathcal{O}_W) \arrow[r, "\varphi_\eta^{Z;W}"] \arrow[d, "(\beta^Z_{W,\alpha})_\eta"'] \arrow[rrr, bend left, "\phi_\eta^{Z,Z'}"] & \mathcal{O}_{T,\eta} \arrow[r, "="] \arrow[d, "a_Z"'] & \mathcal{O}_{T,\eta} \arrow[d, "a_{Z'}"] & f_\eta(\mathcal{O}_{Z'}, \mathcal{O}_W) \arrow[l, "\varphi_\eta^{Z';W}"'] \arrow[d, "(\beta^{Z'}_{W,\alpha})_\eta"] \\
f_\eta(\mathcal{O}_Z, \alpha) \arrow[r, "\varphi_\eta^{Z;\alpha}"'] & \mathcal{O}_{T,\eta} \arrow[r, "="'] & \mathcal{O}_{T,\eta} & f_\eta(\mathcal{O}_{Z'}, \alpha) \arrow[l, "\varphi_\eta^{Z';\alpha}"]
\end{tikzcd}$$

From this our claims follow: two choices of moving data produce isomorphisms which agree at the generic points of T, hence they agree; and the isomorphism $\phi_T^{Z,Z'}$ for T regular is characterized by agreeing with the composition of the canonical trivializations over the generic points of T. $\square$

COROLLARY 5.33. *Among Hilbert-Chow data over bases T satisfying:*

(1) T is either regular, or local; and
(2) all generic points of T correspond to disjoint subschemes;

there exists a collection of isomorphisms $\{\phi_T\}$ which:

(1) is compatible with base change preserving generic disjointness;
(2) satisfies the cocycle condition; and
(3) extends the collection of 5.20.

Proof. To see our construction commutes with base change preserving generic disjointness, note that moves valid over T (i.e., producing $\beta_{W,\alpha}$) pullback via $S \to T$ to suitable moves on S. The cocycle condition (an equality of two isomorphisms of line bundles on a reduced scheme) holds at the generic points, hence it holds everywhere. ☐

The previous corollary provides us an isomorphism $f_s(\mathcal{O}_Z, \mathcal{O}_W) \cong f_s(\mathcal{O}_{Z'}, \mathcal{O}_W)$ for a Hilbert-Chow datum over s the spectrum of a field corresponding to a point in the incidence locus, namely the restriction of the isomorphism over the local ring, possibly followed by a field extension. In the next proposition we observe this is compatible with specializations from the locus of disjoint subschemes into the incidence locus.

PROPOSITION 5.34. *Let (Z, Z', W) be a Hilbert-Chow datum over s the spectrum of a field $\kappa(s)$ corresponding to a point of incidence, i.e., $Z \cap W, Z' \cap W \neq \emptyset$. Let (Z_T, Z'_T, W) be a Hilbert-Chow datum over T the spectrum of a DVR covering a generization from s to the locus of disjoint subschemes, and with $T_0 = s$. Then the isomorphism*

$$f_s(\mathcal{O}_Z, \mathcal{O}_W) \xrightarrow{can} f_T(\mathcal{O}_{Z_T}, \mathcal{O}_W) \times_T s$$
$$\xrightarrow{(\phi_T^{Z_T, Z'_T}) \times_T s} f_T(\mathcal{O}_{Z'_T}, \mathcal{O}_W) \times_T s \xleftarrow{can} f_s(\mathcal{O}_{Z'}, \mathcal{O}_W)$$

is equal to the isomorphism induced by $\phi_R := \phi_{\operatorname{Spec} R}$, where $(R, \mathfrak{m}, K = R/\mathfrak{m})$ is the (seminormal) local ring of the image of s on $(\mathscr{H}_a \times_{\mathscr{C}_a} \mathscr{H}_a)^{sn}$. In other words, the previously displayed isomorphism is equal to $(-) \otimes_K \kappa(s)$ of the following isomorphism:

$$f_K(\mathcal{O}_Z, \mathcal{O}_W) \xrightarrow{can} f_R(\mathcal{O}_{Z_R}, \mathcal{O}_W) \otimes_R K$$
$$\xrightarrow{(\phi_R^{Z_R, Z'_R}) \otimes_R K} f_R(\mathcal{O}_{Z'_R}, \mathcal{O}_W) \otimes_R K \xleftarrow{can} f_K(\mathcal{O}_{Z'}, \mathcal{O}_W).$$

In particular, if we generize to the locus of disjoint subschemes and then restrict, the resulting isomorphism at s is independent of the choice of generization.

Proof. Since the collection $\{\phi_T\}$ is compatible with base change preserving generic disjointness, we may replace R by R/I for some ideal $I \subset R$ such that R/I is a local domain whose generic point $\operatorname{Spec} L$ is the image of the generic point of T. Therefore we have commutative squares:

$$\begin{tikzcd} L \arrow[r] & \kappa(\eta) \\ R \arrow[u] \arrow[d] \arrow[r] & \Gamma(T, \mathcal{O}_T) \arrow[u] \arrow[d] \\ K \arrow[r] & \kappa(s) \end{tikzcd}$$

To show $\phi_R \times_R T = \phi_T$, it suffices to show they agree at η, i.e., that $(\phi_R \times_R T) \times_T \eta = \phi_T \times_T \eta$. But this is equivalent to $(\phi_L) \times_L \eta = \phi_\eta$, which is a consequence of the compatibility with base change on pairs of disjoint subschemes (5.10). $\square$

COROLLARY 5.35. *Among Hilbert-Chow data satisfying at least one of the following conditions:*

(1) the conditions of 5.33;
(2) the base T is a field;

there exists a collection of isomorphisms $\{\phi_T\}$ which:

(1) satisfies the cocycle condition;
(2) extends the collection of 5.33 (so is compatible with base change preserving generic disjointness); and
(3) is compatible with specialization from the locus of disjoint subschemes to the incidence locus.

Proof. The compatibility with specialization is built into the construction. The new feature to check is the cocycle condition on field points mapping to the incidence locus. But if the cocycle condition holds after generization and the covering isomorphism is compatible with base change, then the collection must also satisfy the cocycle condition at new (field) points. $\square$

We augment 5.35 to include specializations fully within the incidence locus, hence we have the first part of 5.1.

THEOREM 5.36. *Among Hilbert-Chow data satisfying at least one of the following conditions:*

(1) the conditions of 5.33;
(2) the base T is a field;
(3) the base T is a DVR;

there exists a collection of isomorphisms $\{\phi_T\}$ which:

(1) satisfies the cocycle condition;
(2) extends the collection of 5.35 (so is compatible with base change preserving generic disjointness); and
(3) is compatible with arbitrary specialization.

Therefore, in the notation of 5.1, the incidence bundle $\mathcal{L} \in \mathrm{Pic}(Y_0)$ lifts to an element $(\mathcal{L}, \phi) \in \mathrm{Pic}(Y_\bullet)$.

Proof. This follows from the general lemma 4.5, the preceding result 5.35, and the hypothesis that T is seminormal. □

Now we conclude the proof of 5.1.

Theorem 5.37 (Zariski local effectiveness). *The element $(\mathcal{L}, \phi = \{\phi_T\})$ of 5.36 is Zariski locally effective: for any cycle $(z, W) \in \overline{U} \subset \mathscr{C}_a \times \mathscr{C}_b$ there exists an open subscheme $V \subset \overline{U}$ containing (z, W) and an isomorphism $t : \mathcal{L}|_{\pi_0^{-1}(V)} \cong \mathcal{O}_{\pi_0^{-1}(V)}$ which is compatible with ϕ. Therefore, the incidence bundle $\mathcal{L}$ descends to $\overline{U}$.*

Proof. This structure is built into the definition of the descent datum $\phi = \{\phi_T\}$. By 5.5 we may assume W is fixed. Suppose z and W are disjoint. Then W is disjoint from all cycles in a neighborhood V of z, and also from all subschemes in $V_0 := \pi_0^{-1}(V) \subset \mathscr{H}_a$. We let $Z_{V_0} \hookrightarrow P \times V_0$ denote the corresponding family. On V_0 we use the canonical trivialization $\varphi_{V_0}^{Z_{V_0}} : f_{V_0}(\mathcal{O}_{Z_{V_0}}, \mathcal{O}_W) := \det_{V_0}(\mathrm{R}\pi_*(\mathcal{O}_{Z_{V_0}} \otimes^{\mathrm{L}} \mathcal{O}_W)) \cong \mathcal{O}_{V_0}$ induced by the acyclicity of $\mathcal{O}_{Z_{V_0}} \otimes^{\mathrm{L}} \mathcal{O}_W$. Then by our definition of ϕ on disjoint subschemes, the following diagram commutes:

$$\begin{array}{ccc} f_{V_0}(\mathcal{O}_{Z_{V_0}}, \mathcal{O}_W) & \xrightarrow{\varphi_{V_0}^{Z_{V_0}}} & \mathcal{O}_{V_0} \\ \downarrow{\scriptstyle \phi_{V_0}^{Z,Z'}} & & \downarrow{\scriptstyle =} \\ f_{V_0}(\mathcal{O}_{Z'_{V_0}}, \mathcal{O}_W) & \xrightarrow{\varphi_{V_0}^{Z'_{V_0}}} & \mathcal{O}_{V_0} \end{array}$$

For a pair (z, W) in the incidence locus, choose a collection of short exact sequences as in 5.13 moving W to a rationally equivalent α such that $z \cap \alpha = \emptyset$. Then also $z' \cap \alpha = \emptyset$ for z' in a neighborhood $V \ni z$, and α is disjoint from all subschemes parameterized by $V_0 := \pi_0^{-1}(V)$. Then we define t to be the trivialization induced by the move, then the acyclicity of $\mathcal{O}_{Z_{V_0}} \otimes^{\mathrm{L}} \mathcal{O}_{W_i}$ (for all $W_i \in \mathrm{Supp}(\alpha)$):

$$t : f_{V_0}(\mathcal{O}_{Z_{V_0}}, \mathcal{O}_W) \cong f_{V_0}(\mathcal{O}_{Z_{V_0}}, \alpha) \xrightarrow{\varphi_{V_0}^{Z_{V_0}}} \mathcal{O}_{V_0}.$$

This is compatible with ϕ by 5.32. Zariski local effectiveness implies effectiveness by 3.1 and 4.6. □

Remark 5.38. In [18] we proved 1.1 for pairs of 1-dimensional cycles on a threefold P (the case $n = 3, a = b = 1$) by constructing isomorphisms $\varphi_T^{Z,W} : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T([Z],[W])$ for all Hilbert data (Z, W) for P over spectra of fields and complete DVRs T. (We use the evident extension of the "determinant of a cycle" notation from 5.3 when both variables are cycles.) Thus we obtained for all Hilbert-Chow data $(Z, W), (Z', W')$ over spectra of fields and DVRs a system of isomorphisms $\phi_T^{Z,Z';W,W'} := (\varphi_T^{Z',W'})^{-1} \circ \varphi_T^{Z,W} : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_{W'})$.

We explain the strategy used in [18] to construct the φ isomorphisms. Consider first the case where T is the spectrum of a field, and write $[Z] = \sum n_i Z_i$. Then $\mathcal{O}_Z, \mathcal{O}_{Z_i} \in F_1(K_0(P_T))$ and the difference $\mathcal{O}_Z - \sum n_i \mathcal{O}_{Z_i}$ lies in $F_0(K_0(P_T))$. Since the determinant is additive on filtrations, to define $\varphi_T^{Z,W}$ it is enough to trivialize in a sufficiently canonical way the determinant $f_T(\mathcal{O}_Z, \mathcal{O}_W)$ when at least one of Z, W is zero-dimensional. To achieve this we used the explicit form of the isomorphism in [20, 5.3]. For T the spectrum of a DVR, one has also to trivialize the determinant $f_T(\mathcal{O}_Z, \mathcal{O}_W)$ when at least one of Z, W is supported in the closed fiber. For this we used an exact sequence given by a uniformizer. To summarize, as far as the incidence bundle is concerned, [20, 5.3] trivializes canonically the difference between the Hilbert scheme and the Chow variety. Thus what was lacking in [18] was a generalization of [20, 5.3] to higher dimensions.

To obtain the functoriality of the resulting collection $\{\phi_T\}$ in [18], we deduced from the construction of the φ isomorphisms a natural list of properties (essentially: being additive on triangles, and agreeing with prescribed normalizations on structure sheaves of subvarieties) sufficient to characterize them, then checked the properties were stable under base change. This method worked on all of $\mathscr{C} \times \mathscr{C}$, not just the locus $\overline{U}$. By contrast, our approach here is to restrict to the locus $\overline{U}$ and compare with the canonical trivialization on the locus U of disjoint cycles. As for the effectiveness of the descent datum, both in [18] and in the present work the verification of the Zariski local effectiveness of the collection $\{\phi_T\}$ employs Grothendieck-Riemann-Roch to analyze the effect on φ of a choice of "moving collection" of short exact sequences.

As a final point of contrast, in [18] we made use of the product structure of the Hilbert-Chow proper hypercovering and showed, using [20, 3.12], that the incidence bundle could be descended to the product of a Hilbert scheme and a Chow variety. Then we showed the descended bundle inherited a Zariski locally effective descent datum of its own. In other words, we wrote $\pi = (FC \times \mathrm{Id}) \circ (\mathrm{Id} \times FC) : \mathscr{H} \times \mathscr{H} \to \mathscr{C} \times \mathscr{C}$ and descended along $(FC \times \mathrm{Id})$ and $(\mathrm{Id} \times FC)$ separately.

SUBSECTION 5.6. CONCLUSION OF THE PROOF OF 1.1. Finally we verify the properties stated in 1.1.

DESCENT OF RATIONAL SECTION. To see we have actually constructed a Cartier divisor in $\overline{U} \subset \mathscr{C}_a \times \mathscr{C}_b$ supported on the incidence locus, consider the diagram whose vertical arrows are the restriction maps:

$$\begin{tikzcd} \mathrm{Pic}(\overline{U}) \arrow[r] \arrow[d] & \mathrm{Pic}(Y_\bullet) \arrow[d] \\ \mathrm{Pic}(U) \arrow[r] & \mathrm{Pic}((\pi^{-1}(U))_\bullet) \end{tikzcd}$$

The arrow in the bottom row is injective by 4.6. On the locus of disjoint subschemes, the isomorphism constructed in 5.8, $\varphi : \mathcal{L}|_{U_0} \cong \mathcal{O}_{U_0}$, is an isomorphism of pairs $\mathcal{L}|_{(\pi^{-1}(U))_\bullet} = (\mathcal{L}|_{U_0}, \phi|_{U_0 \times_U U_0}) \cong (\mathcal{O}_{U_0}, id)$. Hence by the injectivity of the bottom row, the trivialization φ descends to $U \subset \overline{U}$.

Restriction to U' is effective. To check that the restriction $D|_{U'}$ is effective, we may replace U' with its normalization U'^{ν}. Then we may replace U'^{ν} with the local ring of some depth 1 point t on U'^{ν}. By the assumption that the universal cycles intersect properly, over a given component $C \subset U'$, the incidence has dimension $\dim(C) - 1$. This is preserved by the finite base change $U'^{\nu} \to U'$.

Suppose first the incidence is generically finite onto its image. Then 5.18 applies, and the coefficient of t in $D|_{U'^{\nu}}$ is a sum of intersection multiplicities of properly intersecting components (weighted with positive coefficients). If the incidence dominates t, this coefficient is positive by [22, V.C.Thm.1(b)]; in any case the coefficient is nonnegative.

If the incidence has generic positive dimension over its image, then its image must have dimension $\leq \dim(C)-2$. Hence in this case the associated coefficient is 0.

Intersection multiplicity. On the Chow varieties we have the incidence bundle $\mathcal{M}$ and its rational section over the locus of disjoint cycles, giving the Cartier divisor $D \hookrightarrow \overline{U}$. This pulls back via the Hilbert-Chow morphism $\pi : Y_0 \to \overline{U}$ to the determinant line bundle $\mathcal{L}$ and its rational section over the locus of disjoint subschemes. Our goal is to relate the order of vanishing of a local defining equation of D, to intersection numbers. So let s_D be the canonical (rational) section of the line bundle $\mathcal{O}_{\overline{U}}(D)$.

If $g : T \to \overline{U}$ is a morphism from the spectrum of a discrete valuation ring $R \supset k$ (corresponding to cycles Z, W), there exists a discrete valuation ring R' which is finite over R, and such that the composition $g' : T' := \operatorname{Spec} R' \to T \to \overline{U}$ factors through Y_0. (Note that if we start with a specialization from a generic point of $\overline{U}$, we can find a component of the Hilbert scheme so that no generic extension is necessary.) If $\operatorname{ord} g'^*(s_D) = \deg(Z_{T'} \cdot W_{T'})$, it follows that $\operatorname{ord} g^*(s_D) = \deg(Z \cdot W)$. Thus we may assume our specialization factors through the Hilbert scheme, corresponding to subschemes $\widetilde{Z}, \widetilde{W}$ such that $[\widetilde{Z}] = Z$ and $[\widetilde{W}] = W$. Now we assume disjointness over the generic point $\eta \in T$, i.e., $g(\eta) \in U$. Let $t \in T$ denote the closed point.

First we have:

$$\operatorname{ord}_T(s_D) = \operatorname{ord}_T(s_{\pi^* D}) = \sum_p (-1)^p \ell_t(\mathcal{H}^p(\mathrm{R}\pi_*(\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}})))$$

since each $\mathcal{H}^p(\mathrm{R}\pi_*(\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}}))$ is a torsion T-module, and by [12, Thm.3(v)]. Since the scheme P_T is smooth, the filtration of the K_0-groups by dimension is compatible with multiplication, thus $\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}}$ and $\mathrm{R}\pi_*(\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}})$ are classes of dimension zero. Then $\sum_p (-1)^p \ell_t(\mathcal{H}^p(\mathrm{R}\pi_*(\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}})))$ is equal to the degree of the K_0-classes $\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}}$ and $\mathrm{R}\pi_*(\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}})$. Note also the refined class $Z \cdot W$ is of the expected dimension, i.e., $Z \cdot W \in A_0(P_T)$.

We have $\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}} = \sum_i (-1)^i [\mathrm{Tor}_i^{P_T}(\mathcal{O}_{\widetilde{Z}}, \mathcal{O}_{\widetilde{W}})] \in K_0(P_T)$. The degree of this class is computed by [7, 20.4]: it is simply the degree of the refined class $Z \cdot W \in A_0(P_T)$, since the terms of dimension < 0 necessarily vanish.

References

[1] B. Angéniol. *Familles de cycles algébriques - Schéma de Chow*, volume 896. Springer-Verlag, 1981.

[2] M. Artin, A. Grothendieck, and Verdier J.-L. *Théorie des topos et cohomologie étale des schémas: séminaire de géométrie algébrique du Bois Marie, 1963-64, SGA 4*, volume 305. Springer-Verlag, 1972.

[3] D. Barlet. *Espace analytique réduit des cycles analytiques complexes compacts d'un espace analytique complexe de dimension finie*, volume 482. Springer-Verlag, 1975.

[4] D. Barlet and M. Kaddar. Incidence divisor. *Internat. J. Math.*, 14(4):339–359, 2003.

[5] David Eisenbud, Frank-Olaf Schreyer, and Jerzy Weyman. Resultants and Chow forms via exterior syzygies. *J. Amer. Math. Soc.*, 16(3):537–579 (electronic), 2003.

[6] John Fogarty. Truncated Hilbert functors. *J. Reine Angew. Math.*, 234:65–88, 1969.

[7] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[8] S. Greco and C. Traverso. On seminormal schemes. *Compositio Math.*, 40(3):325–365, 1980.

[9] A. Grothendieck, P. Berthelot, and L. Illusie. SGA 6. *Lecture Notes in Mathematics*, 225.

[10] Robin Hartshorne. *Residues and duality*. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin, 1966.

[11] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[12] Finn Faye Knudsen and David Mumford. The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div". *Math. Scand.*, 39(1):19–55, 1976.

[13] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern*

Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1996.

[14] Barry Mazur. Private letter to Bruno Harris, Mar 10, 1993.

[15] David Mumford and John Fogarty. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin, second edition, 1982.

[16] Joel Roberts. Chow's moving lemma. In *Algebraic geometry, Oslo 1970 (Proc. Fifth Nordic Summer School in Math.)*, pages 89–96. Wolters-Noordhoff, Groningen, 1972. Appendix 2 to: "Motives" (ıt Algebraic geometry, Oslo 1970 (Proc. Fifth Nordic Summer School in Math.), pp. 53–82, Wolters-Noordhoff, Groningen, 1972) by Steven L. Kleiman.

[17] Joseph Ross. The incidence divisor of cycles in projective space. To appear in *Communications in Algebra*.

[18] Joseph Ross. The Hilbert-Chow morphism and the incidence divisor. PhD thesis, Columbia University, 2009.

[19] Joseph Ross. A characterization of seminormal schemes. *J. Algebra*, 323(5):1494–1502, 2010. Corrigendum 323(11):3155-3156, 2010.

[20] Joseph Ross. The Hilbert-Chow morphism and the incidence divisor: Zero-cycles and divisors. *Journal of Pure and Applied Algebra*, 215(11):2586 – 2599, 2011.

[21] D. Rydh. *Families of cycles and the Chow scheme*. PhD thesis, Kungliga Tekniska högskolan, 2008.

[22] Jean-Pierre Serre. *Local algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000. Translated from the French by CheeWhye Chin and revised by the author.

[23] Carlo Traverso. Seminormality and Picard group. *Ann. Scuola Norm. Sup. Pisa (3)*, 24:585–595, 1970.

[24] Bin Wang. Mazur's incidence structure for projective varieties. I. *Compositio Math.*, 115(3):303–327, 1999.

Joseph Ross
Universität Duisburg-Essen
Campus Essen
Fachbereich Mathematik
Germany
joseph.ross@uni-due.de

Presentation of an Iwasawa Algebra: The Case of $\Gamma_1 SL(2,\mathbb{Z}_p)$

Laurent Clozel [1]

Received: July 1, 2010
Revised: February 17, 2011

Communicated by Takeshi Saito

Abstract. We give an explicit presentation of the p-adic Iwasawa algebra of the subgroup of level one of $SL(2,\mathbb{Z}_p)$ for $p \neq 2$.

2010 Mathematics Subject Classification: 11F85, 11S31,22E50
Keywords and Phrases: Non-commutative Iwasawa agebras, p-adic groups

Assume G is a semi–simple Chevalley group, so $G(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ is a maximal compact subgroup. Both the p–adic representation theory of $G(\mathbb{Q}_p)$ and non–commutative Iwasawa theory involve the Iwasawa algebra of $G(\mathbb{Z}_p)$ or suitable congruence subgroups. It seems to have been assumed that explicit descriptions, by generators and relations, of these algebras were inaccessible. However, it is a general principle that natural objects coming from semi–simple (split) groups have explicit presentations. Famous examples are Serre's presentation of the semi–simple algebras and Steinberg's presentation of the Chevalley groups [7, 8]. In this paper we will give a presentation for the Iwasawa algebra of the subgroup of level 1 in $SL(2,\mathbb{Z}_p)$ ($p \neq 2$).

I thank M. Duflo and, once more, J.–P. Serre for useful discussions. Thanks are also due to P. Schneider for reading a first draft and suggesting several corrections, and to the referee for his careful work which eliminated an embarrassing mistake.

[1] Membre de l'Institut Universitaire de France

1

Let $\underline{G} = SL(2)$ and let G be the subgroup of level 1 in $\underline{G}(\mathbb{Z}_p)$:

$$G = \{g \in SL(2, \mathbb{Z}_p) : g \equiv 1[p]\} .$$

We assume $p > 2$, so G has no p–torsion. It has a triangular decomposition

$$G = N^- \ T \ N^+$$

where $N^- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, $N^+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ (entries $*$ in $p\mathbb{Z}_p$) and $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ (entries in $1 + p\mathbb{Z}_p$). We identify N^-, N^+ with $\mathbb{Z}_p$ by $* = px$ ($x \in \mathbb{Z}_p$). Similarly $T \cong \mathbb{Z}_p$ by

$$x \longmapsto \begin{pmatrix} (1+p)^x & \\ & (1+p)^{-x} \end{pmatrix} \qquad (x \in \mathbb{Z}_p) .$$

We consider the Iwasawa algebra Λ_G of $\mathbb{Z}_p$–valued measures (or distributions, in the sense of [9]), on G, which we will denote by $\mathcal{D}(G, \mathbb{Z}_p)$. The triangular decomposition of G, as an analytic manifold, yields a decomposition of $\mathcal{D}(G, \mathbb{Z}_p)$ as a topological $\mathbb{Z}_p$–MODULE :

$$\mathcal{D}(G, \mathbb{Z}_p) = \mathcal{D}(N^-, \mathbb{Z}_p) \widehat{\otimes} \mathcal{D}(T, \mathbb{Z}_p) \widehat{\otimes} \mathcal{D}(N^+, \mathbb{Z}_p) , \tag{1.1}$$

the factors of (1.1) being the spaces of distributions on the factors of G. If f is a function on G and U, V, W distributions on N^-, T, N^+,

$$< U \otimes V \otimes W, f > := < U \otimes V \otimes W, f(uhn) > \tag{1.2}$$

where $u \in N^-$, $h \in T$, $n \in N^+$ and f is therefore seen as a function on $N^- \times T \times N^+$. The natural definition of the completed tensor product is equivalent to the explicit description of $\mathcal{D}(G, \mathbb{Z}_p)$ reviewed below.
The algebra $\Lambda_{\mathbb{Z}_p} = \mathcal{D}(\mathbb{Z}_p, \ \mathbb{Z}_p)$ is identified with the ring of power series $\mathbb{Z}_p[[T]]$ by Iwasawa's theorem. For $\mu \in \Lambda_{\mathbb{Z}_p}$, the associated series is given by the Fourier–Amice transform

$$\widehat{\mu}(t) = \int_{\mathbb{Z}_p} (1+t)^x d\mu(x) \qquad (t \in \mathbb{Z}_p, |t| < 1) .$$

In particular, $\delta(x)$ being the Dirac measure at x :

$$\widehat{\delta(1)} = 1 + T ,$$

$$\text{so} \qquad T = \widehat{\delta(1)} - \widehat{\delta(0)} .$$

In each factor of the decomposition (1.1), we therefore have the Dirac measures :

$$
\begin{aligned}
\mu_- &= \delta(\begin{pmatrix} 1 & \\ p & 1 \end{pmatrix}), & \widehat{\mu}_- &= 1 + Y \in \mathcal{D}(N^-, \mathbb{Z}_p) \cong \\
& & &\cong \mathbb{Z}_p[[Y]] \\
\mu_+ &= \delta(\begin{pmatrix} 1 & p \\ & 1 \end{pmatrix}), & \widehat{\mu}_+ &= 1 + X \in \mathcal{D}(N^+, \mathbb{Z}_p) \cong \\
& & &\cong \mathbb{Z}_p[[X]] \\
\mu_0 &= \delta(\begin{pmatrix} (1+p) & \\ & (1+p)^{-1} \end{pmatrix}), & \widehat{\mu}_0 &= 1 + H \in \mathcal{D}(T, \mathbb{Z}_p) \cong \\
& & &\cong \mathbb{Z}_p[[H]]
\end{aligned}
$$

For each factor, $U = N^-$, T or N^+ of G, $\mathcal{D}(U, \mathbb{Z}_p)$ is naturally sent to $\mathcal{D}(G, \mathbb{Z}_p)$, by integrating a function $f \in C(G, \mathbb{Z}_p)$ against $\mu \in \mathcal{D}(U, \mathbb{Z}_p)$ on the U–factor. This map is compatible with the convolution product. We therefore write, unambiguously, Y^n, X^n, H^n $(n \geq 0)$ in $\mathcal{D}(G, \mathbb{Z}_p)$. A distribution λ in this space can then be written uniquely

$$
\lambda = \sum_n \lambda_n \; Y^{n_1} \; H^{n_2} \; X^{n_3} \qquad (n \in \mathbb{N}^3) \tag{1.3}
$$

with $\lambda_n \in \mathbb{Z}_p$. This is the meaning of the completed tensor product (1.1). The expansion is convergent in $\mathcal{D}(G, \mathbb{Z}_p)$. Of course the product $Y^{n_1} H^{n_2} X^{n_3} := Y^{n_1} \otimes H^{n_2} \otimes X^{n_3}$ is defined as above. This easily follows from Mahler's theorem in several variables (cf. Lazard [4, Théorème 1.2.4]).

It immediately follows from formula (1.2) that the distributions Y, H, $X \in \mathcal{D}(G, \mathbb{Z}_p)$ multiply in the obvious fashion when the variables are taken in the "natural order", i.e.

$$
\begin{aligned}
Y \otimes H &= Y * H \\
Y \otimes X &= Y * X \\
H \otimes X &= H * X \,,
\end{aligned}
$$

the convolution product being taken on G. We will simply write, consistent with previous notation :

$$
YH = Y * H \,, \; YX = Y * X \,, \; HX = H * X \,. \tag{1.4}
$$

To determine the product structure in $\mathcal{D}(G, \mathbb{Z}_p)$ is to understand first the product of monomials in a different order.

Consider first the product HY. It suffices to compute, in G, the product $\mu_0 \mu_- = \delta(h_0)\delta(u_0)$, say. We compute $h_0 u_0 h_0^{-1}$.

Since

$$
\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} = \begin{pmatrix} 1 & \\ t^{-2}x & 1 \end{pmatrix},
$$

we have $h_0u_0h_0^{-1} = u_0^{(1+p)^{-2}}$ if we write the group N^- multiplicatively. The equation

$$\mu_0\mu_- = \delta(h_0u_0h_0^{-1})\delta(h_0)\,,$$

and the fact that $\mathcal{D}(N^-,\mathbb{Z}_p) \cong \mathbb{Z}_p[[Y]]$ is a homomorphism, show that

$$(1+H)(1+Y) = (1+Y)^q(1+H) \tag{1.5}$$

where we have set

$$q = (1+p)^{-2} \equiv 1 \qquad [p]\,. \tag{1.6}$$

Similarly consider XH. Let n_0 be the generator of N^+. Now $\delta(n_0)\delta(h_0)$ reduces to $h_0^{-1}n_0h_0$. Again

$$\begin{pmatrix} t^{-1} & \\ & t \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = \begin{pmatrix} 1 & t^{-2}x \\ & 1 \end{pmatrix}\,,$$

so $h_0^{-1}n_0h_0 = n_0^{(1+p)^{-2}} = n_0^q$, whence

$$(1+X)(1+H) = (1+H)(1+X)^q\,. \tag{1.7}$$

Finally, to express XY we have to decompose

$$n_0u_0 = \begin{pmatrix} 1 & p \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ p & 1 \end{pmatrix} = \begin{pmatrix} 1+p^2 & p \\ p & 1 \end{pmatrix}\,.$$

Since

$$\begin{pmatrix} 1 & \\ a & 1 \end{pmatrix}\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} = \begin{pmatrix} t & tb \\ ta & abt+t^{-1} \end{pmatrix}\,,$$

we see that

$$p = ta = tb$$

with

$$1+p^2 = t = (1+p)^P\,, \qquad P \in \mathbb{Z}_p\,.$$

This yields, since $t_0 = 1+p$ is the parameter of h_0 :

$$(1+X)(1+Y) = (1+Y)^Q(1+H)^P(1+X)^Q \tag{1.8}$$

with

$$Q = (1+p^2)^{-1} \equiv 1[p^2]\,,\ P = \frac{\log(1+p^2)}{\log(1+p)}\,. \tag{1.9}$$

For $p > 2$, we have

$$\begin{aligned}\log(1+p) &= p - \tfrac{p^2}{2} + \tfrac{p^3}{3} \cdots = p(1+O(p))\\ \log(1+p^2) &= p^2(1+O(p^2))\end{aligned}$$

whence

$$P = p(1 + O(p))\,. \tag{1.10}$$

Note that we have simply written HY for $H * Y$, etc... This will cause no confusion if we remember that a product such as HY, for variables not in the natural order, is not given by the ostensible product of monomials in the expression (1.3).
To summarize, we have :

Proposition 1.1. *Set $Q = (1+p^2)^{-1}$, $q = (1+p)^{-2}$, $P = \frac{\log(1+p^2)}{\log(1+p)}$. Then the elements X, Y, H of $\mathcal{D}(G, \mathbb{Z}_p)$ verify the relations*

(a) $(1+H)(1+Y) = (1+Y)^q(1+H)$

(b) $(1+X)(1+H) = (1+H)(1+X)^q$

(c) $(1+X)(1+Y) = (1+Y)^Q(1+H)^P(1+X)^Q$.

Consider now the universal, non–commutative p–adic algebra in the variables Y, H, X : thus

$$\mathcal{A} = \mathbb{Z}_p\{\{Y, H, X\}\}$$

is composed of all the non–commutative series

$$f = \sum_{n \geq 0} \sum_i a_i x^i \tag{1.11}$$

where the coefficients $a_i \in \mathbb{Z}_p$ and, for all $n \geq 0$, i runs over all maps $\{1, 2, \dots n\} \longrightarrow \{1, 2, 3\}$; we set $x_1 = Y$, $x_2 = H$, $x_3 = X$ and $x^i = x_{i(1)} \cdots x_{i(n)}$. The topology on $\mathcal{A}$ is the product topology on $\prod_n \mathbb{Z}_p^{I(n)}$ where $I(n)$ is the set of maps ($\equiv$ of non–commutative monomials of degree n). The algebra $\mathcal{A}$ has a maximal ideal $\mathcal{M}_{\mathcal{A}}$ generated by (p, x_1, x_2, x_3) and a prime ideal $\mathcal{P}_{\mathcal{A}}$ generated by (x_1, x_2, x_3). Its topology is given by the powers of $\mathcal{M}_{\mathcal{A}}$.
The non–commutative polynomial algebra

$$A = \mathbb{Z}_p\{Y, H, X\}$$

is a dense subalgebra of $\mathcal{A}$.
Let $\mathcal{R}$ be the closed two–sided ideal generated in $\mathcal{A}$ by the relations (a, b, c).
Our main result is

Theorem 1.2. *The Iwasawa algebra Λ_G is naturally isomorphic to $\mathcal{A}/\mathcal{R}$.*

The proof will in fact rely on the corresponding result with coefficients in $\mathbb{F}_p$. So let $\Omega_G = \Lambda_G \bigotimes_{\mathbb{Z}_p} \mathbb{F}_p$ be the Iwasawa algebra with finite coefficients, $\overline{\mathcal{A}} = \mathbb{F}_p\{\{Y, H, X\}\}$ the algebra of non–commutative series with coefficients in $\mathbb{F}_p$, with its natural linearly compact topology, given by its maximal ideal $\mathcal{M}_{\overline{\mathcal{A}}}$.
Let $\overline{\mathcal{R}}$ be the image of $\mathcal{R}$ in $\overline{\mathcal{A}}$.

LEMMA 1.3. *$\overline{\mathcal{R}}$ is the closed two–sided ideal generated in $\overline{\mathcal{A}}$ by the image of the relations (a, b, c).*

Proof.– Denote by $\mathcal{I} \subset \mathcal{A}$ the IDEAL generated by the relations ; let $\mathcal{J} \subset \overline{\mathcal{A}}$ be the similar ideal. Then $\mathcal{J}$ is obviously the image of $\mathcal{I}$ in $\overline{\mathcal{A}}$; we denote it by $\overline{\mathcal{I}}$. Let $\overline{\mathcal{R}}$ be the reduction of $\mathcal{R}$, and consider the closure $cl(\overline{\mathcal{I}})$ of $\overline{\mathcal{I}}$ in $\overline{\mathcal{A}}$. If $f \in \mathcal{R}$, we have $f = \lim_n f_n$ $(f_n \in \mathcal{I})$ for the topology given by $(\mathcal{M}_{\mathcal{A}}^N)$. This implies that $\overline{f} = \lim \overline{f}_n$ for the topology given by $\mathcal{M}_{\overline{\mathcal{A}}}^N$ on $\overline{\mathcal{A}}$, thus $\overline{f} \in cl(\overline{\mathcal{I}})$. Conversely assume $\overline{f} \in \overline{\mathcal{A}}$ can be written $\overline{f} = \lim \overline{f}_n$ with $\overline{f}_n \in \overline{\mathcal{I}}$. Then $\overline{f}_n$ is the reduction of a series $f_n \in \mathcal{I} \subset \mathcal{R}$. Since $\mathcal{R}$ is closed and $\mathcal{A}$ compact, we may assume that f_n converges to $g \in \mathcal{R}$. Then, by definition of the topologies, $\overline{f} = \lim \overline{f}_n = \overline{g}$. Thus $cl(\overline{\mathcal{I}}) = \overline{\mathcal{R}}$, which finishes the proof.

THEOREM 1.4. *The Iwasawa algebra mod p, Ω_G, is naturally isomorphic to $\overline{\mathcal{A}}/\overline{\mathcal{R}}$.*

The proof of these results will occupy § 2, 3.

2

We consider the natural map

$$A \longrightarrow \Lambda_G$$

given by the universal property of A. Note that the topology of Λ_G, as a distribution algebra, coincides with its topology when it is seen as the algebra of distributions on the commutative group $\mathbb{Z}_p^3$. In particular a basis of neighbourhoods of 0 is given by the family of $\mathbb{Z}_p$–MODULES $\mathcal{M}_\Lambda^N (\Lambda = \Lambda_G)$, where

$$\mathcal{M}_\Lambda^N = \{\lambda \in \Lambda_G\,,\ \lambda = \sum_n \lambda_n Y^{n_1} H^{n_2} X^{n_3}\,,\ v(\lambda_n) + |n| \geq N\} \tag{2.1}$$

with the usual notation $|n| = n_1 + n_2 + n_3$. For a linear monomial $x = Y$, H or X, we have $w(x) = 1$, w being the function on Λ given by

$$w(\lambda) = \inf_n (v(\lambda_n) + |n|)\,. \tag{2.2}$$

We will use the following deep result of Lazard :

PROPOSITION 2.1 (Lazard). *The valuation w is additive : $w(\lambda * \mu) = w(\lambda) + w(\mu)$ $(\lambda, \mu \in \Lambda_G)$.*

Cf. [4, III 2.3.3]. Lazard proves, in fact, that the associated graded ring is an enveloping algebra, thus an integral domain, and this implies the additivity. I am indebted to the paper of Schneider and Teitelbaum [6] for a lucid exposition of Lazard's results.
In fact, it follows from Lazard's results that $\mathcal{M}_\Lambda^N$ is indeed the N-th power of the maximal ideal $\mathcal{M}_\Lambda$ of Λ_G. Indeed, let J_N be defined by $w(\lambda) \geq N$. It is

easy to check that $J_1 = \mathcal{M}_\Lambda$. The additivity implies that $\mathcal{M}_\Lambda^N$ is contained in J_N. Since every linear monomial belongs to the maximal ideal, the expression (2.1) implies the converse inclusion since $\mathcal{M}_\Lambda^N$ is closed.

Consider now the filtration of $\mathcal{A}$ by the powers of its maximal ideal. It is defined by a valuation $w_\mathcal{A}$ given by a formula similar to (2.2) : if

$$f = \sum_i a_i \; x^i \,,$$

$$w_\mathcal{A}(f) = \inf_i (v(a_i) + |i|)$$

where $|i| = n$ is the degree of i (cf. after (1.11)). We now have the following ("ideal" means two–sided ideal unless otherwise indicated).

Proposition 2.2. *The natural map $\varphi : A \to \Lambda_G$ extends continuously to a surjective homomorphism $\mathcal{A} \to \Lambda_G$. In fact,*

$$\varphi(\mathcal{M}_\mathcal{A}^N) \subset \mathcal{M}_\Lambda^N \qquad (N \geq 0)\,.$$

Proof : The continuity is implied by the stronger property

$$w(\varphi(x^i)) = n = |i| \tag{2.3}$$

where n, as after (1.11), is the degree of the monomial. By induction on n, this follows from Proposition 2.1. If $f \in \mathcal{M}_\mathcal{A}^N$, we have $w_\mathcal{A}(f) \geq N$ and the continuity follows from (2.3) by $\mathbb{Z}_p$–linearity. The surjectivity follows from the fact that φ is already surjective if $\mathcal{A}$ is replaced by the set of linear combinations of well-ordered monomials (i increasing).

Corollary 2.3. *There is a natural, continuous surjection*

$$\mathcal{B} = \mathcal{A}/\mathcal{R} \longrightarrow \Lambda_G\,.$$

Corollary 2.4. *There is a continuous surjection*

$$\overline{\varphi} : \overline{\mathcal{B}} = \overline{\mathcal{A}}/\overline{\mathcal{R}} \longrightarrow \Omega_G\,.$$

This follows from Lemma 1.3.

It follows from Abelian distribution theory that Ω_G is, as a space, isomorphic to

$$\mathbb{F}_p[[Y, H, X]]$$

with the compact topology. An obvious computation shows that

$$\mathcal{M}_\Omega^N = \{\lambda \in \Omega_G : v_\Omega(\lambda) \geq N\}\,,$$

v_Ω being the usual valuation on power series, is the image of $\mathcal{M}_\Lambda^N$. In particular it is an ideal ; for $N = 1$, $\mathcal{M}_\Omega$ is the maximal (two–sided) ideal, and $(\mathcal{M}_\Omega)^N \subset \mathcal{M}_\Omega^N$. (Reduce mod p the corresponding property for Λ.)

Similarly in $\mathcal{A}$, we find that the reduction $\mod p$ (image in $\overline{\mathcal{A}}$) of $\mathcal{M}_{\mathcal{A}}^N$ is the ideal of series

$$\overline{f} = \sum_i \alpha_i \; x^i \qquad (\alpha_i \in \mathbb{F}_p)$$

such that $|i| \geq N$. For $N = 1$ we obtain the maximal ideal in $\overline{\mathcal{A}}$. Furthermore in this case too $(\mathcal{M}_{\overline{\mathcal{A}}})^N = \mathcal{M}_{\overline{\mathcal{A}}}^N$.

3

In this paragraph we will directly study the quotient algebra $\overline{\mathcal{B}} = \overline{\mathcal{A}}/\overline{\mathcal{R}}$, using the properties of the relations (a, b, c).

Consider the natural filtration of $\overline{\mathcal{A}}$ by the powers of $\mathcal{M}_{\overline{\mathcal{A}}}$, which we denote by $F^n\overline{\mathcal{A}}$. We have $F^n\overline{\mathcal{A}}/F^{n+1}\overline{\mathcal{A}} = gr^n\overline{\mathcal{A}} \cong \mathbb{F}_p^{I(n)}$ where $I(n)$ is the set of maps $\{1, \ldots, n\} \to \{1, 2, 3\}$ (§1). The filtration F^n induces a filtration on $\overline{\mathcal{B}} = \overline{\mathcal{A}}/\overline{\mathcal{R}}$:

$$F^n\overline{\mathcal{B}} = F^n\overline{\mathcal{A}} + \overline{\mathcal{R}}$$

whence a graduation

$$\begin{aligned} gr^n\overline{\mathcal{B}} &= F^n\overline{\mathcal{A}} + \overline{\mathcal{R}}/F^{n+1}\overline{\mathcal{A}} + \overline{\mathcal{R}} \\ &= F^n\overline{\mathcal{A}}/F^{n+1}\overline{\mathcal{A}} + (F^n\overline{\mathcal{A}} \cap \overline{\mathcal{R}})\,. \end{aligned}$$

Let $S_n = S_n(X, Y, Z)$ be the space of commutative polynomials over $\mathbb{F}_p$ of degree n ; thus $\dim S_n = \frac{(n+1)(n+2)}{2}$. Let Σ_n be the space of homogeneous non–commutative polynomials of degree n ; thus $\Sigma_n \to F^n\overline{\mathcal{A}}/F^{n+1}\overline{\mathcal{A}}$, and therefore $\Sigma_n \to gr^n\overline{\mathcal{B}}$, is surjective.

Proposition 3.1. $\dim gr^n\overline{\mathcal{B}} \leq \dim S_n$.

In order to prove this we consider the relations defining $\mathcal{R}$ (or rather $\overline{\mathcal{R}}$). Consider first the relation (a) :

$$(1 + H)(1 + Y) = (1 + Y)^q(1 + H)$$

with $q \equiv 1$ $[p]$. Expanding the power series gives

$$1 + H + Y + HY = (1 + qY + \binom{q}{2} Y^2 + \cdots)(1 + H)\,.$$

We note that $\displaystyle\binom{q}{2} = \frac{q(q-1)}{2} \equiv 0$ $[p]$. Thus in $\overline{\mathcal{A}}/\overline{\mathcal{R}}$:

$$1 + H + Y + HY = (1 + qY)(1 + H) + R(Y)(1 + H)\,,$$

the term $R(Y)$ being of degree ≥ 3, so

$$\begin{aligned} HY &= (q-1)Y + qYH + R_1(Y, H) \\ &= YH + R_1(Y, H) \end{aligned}$$

since $q \equiv 1$, $R_1(Y,H)$ of degree ≥ 3. This shows that in $\overline{\mathcal{B}} = \overline{\mathcal{A}}/\overline{\mathcal{R}}$:

$$HY = YH \mod F^3\overline{\mathcal{B}} \text{ i.e.} \tag{3.1}$$

$$HY = YH \text{ in } gr^2\overline{\mathcal{B}}\,.$$

The computation for relation (b) is obviously similar, yielding in $\overline{\mathcal{B}}$

$$XH = HX \qquad \mod F^3\overline{\mathcal{B}}\,. \tag{3.2}$$

Consider now the identity (c) :

$$(1+X)(1+Y) = (1+Y)^Q(1+H)^P(1+X)^Q\,.$$

We have $Q \equiv 1$ $[p^2]$, $P \equiv p$ $[p^2]$. Again the coefficients $\dfrac{Q(Q-1)}{2}$ of Y^2, X^2 in the power series vanish $\mod p$. Modulo $\mathcal{M}^3_{\overline{\mathcal{A}}}$, whose image is in $F^3\overline{\mathcal{B}}$, we then have

$$(1+X)(1+Y) \equiv (1+QY)(1+H)^P(1+QX)\,.$$

Since $P \equiv p$ $[p^2]$ and since 2 is invertible, $(1+H)^P \equiv 1 \mod (p,H^3)$. Thus

$$1+X+Y+XY \equiv 1+QX+QY+Q^2YX \;(\mod F^3\overline{\mathcal{B}})\,,$$

and since $Q \equiv 1$:

$$XY \equiv YX \qquad (\mod F^3\overline{\mathcal{B}})\,. \tag{3.3}$$

Since $gr^2\overline{\mathcal{B}}$ is generated by these three monomials and the squares Y^2, H^2, X^2 , the identities (3.1)–(3.3) show that dim $gr^2\overline{\mathcal{B}} \leq 6$, whence the result for $n=2$. The proposition for general n is deduced from this case. Consider an arbitrary monomial of degree n,

$$x^i = x_{i_1}...x_{i_n}\,.$$

The following lemma is obvious :

Lemma 3.2. *We can change x^i into a well–ordered monomial $x^{i'}$ (i' increasing) by a sequence of transpositions $x_{i_\alpha}x_{i_{\alpha+1}} \mapsto x_{i_{\alpha+1}}x_{i_\alpha}$.*

(Consider the set of inversions $\{\alpha < \beta : i_\alpha > i_\beta\}$. Assume $i_\gamma > i_{\gamma+1}$, and replace in x^i the term $x_{i_\gamma}x_{i_{\gamma+1}}$ by $x_{i_{\gamma+1}}x_{i_\gamma}$. It is easy to check that the set of inversions decreases by one element.)

We now write $x^i = x^j x_{i_\alpha}x_{i_{\alpha+1}}x^\ell$. We will prove by induction

Lemma 3.3. *In $\overline{\mathcal{B}}$, $x^i \equiv x^{i'} \mod F^{n+1}\overline{\mathcal{B}}$, where i' is well–ordered.*

But this is now equally obvious. Let r, s be the degrees of x^j, x^ℓ, so $n = r+s+2$. Then $x^j \equiv x^{j'}[F^{r+1}\overline{\mathcal{B}}]$, $x^\ell \equiv x^{\ell'}[F^{s+1}\overline{\mathcal{B}}]$ and $x_{i_\alpha}x_{i_{\alpha+1}} \equiv x_{i_{\alpha+1}}x_{i_\alpha}[F^3\overline{\mathcal{B}}]$; we are of course assuming $i_\alpha > i_{\alpha+1}$. Factoring the congruences gives

$x^i \equiv x^{j'} x_{i_{\alpha+1}} x_{i_\alpha} x^{\ell'} [F^{n+1}\overline{\mathcal{B}}]$ since the filtration F^n, image of F^n on $\overline{\mathcal{A}}$, verifies $F^n F^m \subset F^{n+m}$. Using induction if necessary, we obtain the Lemma, whence Proposition 3.1.

Proof of Theorem 1.4.– The natural map $\varphi : \mathcal{A} \to \Lambda_G$ sends $\mathcal{M}^n_{\mathcal{A}}$ to $\mathcal{M}^n_{\Lambda}$. Since $F^\bullet$ is on $\overline{\mathcal{B}}$ the filtration inherited from the natural filtration on $\overline{\mathcal{A}}$, we see that $\overline{\varphi}$ sends $F^n\overline{\mathcal{B}}$ to $\mathcal{M}^n_{\Omega}$. We then have a natural map $gr\overline{\varphi} : gr^\bullet\overline{\mathcal{B}} \to gr^\bullet\Omega_G$, surjective since $\overline{\varphi}$ is so. It is an isomorphism since dim $gr^n\overline{\mathcal{B}} \leq$ dim $S_n =$ dim $gr^n\Omega_G$. (The last equality follows from the considerations after Cor.2.4 ; cf also [3, Theorem 7.24]). Therefore $\overline{\varphi}$ is isomorphic since the filtration on $\overline{\mathcal{B}}$ is complete. The last point follows from the fact that $\overline{\mathcal{B}} = \overline{\mathcal{A}}/\overline{\mathcal{R}}$ where $\overline{\mathcal{R}}$ is closed and therefore complete for the filtration induced from that of $\overline{\mathcal{A}}$: see e.g. [5, Thm 4 (5) p. 31].

Proof of Theorem 1.2.– The reduction of $\varphi : \mathcal{A}/\mathcal{R} \to \Lambda_G$ is $\overline{\varphi}$. Recall that $\overline{\mathcal{R}}$ is the image of $\mathcal{R}$ in $\overline{\mathcal{A}}$. Assume $f \in \mathcal{A}$ satisfies $\varphi(f) = 0$. We then have $\overline{f} \in \overline{\mathcal{R}}$ by Theorem 1.3, so $f = r_1 + pf_1$, $r_1 \in \mathcal{R}$, $f_1 \in \mathcal{A}$. Then $\varphi(f_1) = 0$. Inductively, we obtain an expression $f = r_n + p^n f_n$ of the same type. Since $p^n f_n \to 0$ in $\mathcal{A}$ and $\mathcal{R}$ is closed, we see that $f \in \mathcal{R}$, QED.

4

In this section, we show that the description of Λ_G given in § 1 allows one to give different proofs of some results of Ardakov and to understand them in terms of the growth of coefficients in the Iwasawa expansion.

Ardakov's main result in [1] is that the centre of the Iwasawa algebra reduces to the Iwasawa algebra of the centre of G, trivial in our case. We will see that the fact of being central is incompatible with the boundedness of the Iwasawa coefficients.

It will be instructive to compare this behaviour with what happens for the centre of the enveloping algebra. Recall that instead of the Iwasawa distributions, or measures, we can consider the analytic distributions (or hyperfunctions), dual to the locally analytic functions on G (cf. Schneider–Teitelbaum [6]). They admit an expansion (1.3), but with now

$$|\lambda_n| r^{|n|} \longrightarrow 0 \qquad \forall r < 1\,,\ |n| = n_1 + n_2 + n_3\,. \tag{4.1}$$

Among these we have the Casimir operator (seen as a distribution with support at 1)

$$\omega = h^2 + 2(xy + yx) = h^2 - 2h + 4xy$$

(cf. e.g. Borel [2, p. 19]) where h, x, y are the infinitesimal generators of the groups T, N^+, N^-. It suffices to compute ω on a function f given by

$$f(utn) = (1+Y)^{x_1}(1+H)^{x_2}(1+X)^{x_3}$$

where u, t, n have parameters x_1, x_2, $x_3 \in \mathbb{Z}_p$ and Y, H, X belong to the disc $|w| < 1$ in $\mathbb{C}_p$ or even $\mathbb{Q}_p$ (such functions are dense). Now

$$\begin{aligned}
(xyf)(1) &= \frac{d}{dt}\Big|_0 yf(e^{tx}) = \frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 f(e^{sy}e^{tx}) \\
&= \frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 f\left(\begin{pmatrix} 1 & \\ s & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\right) \\
&= \tfrac{\partial^2}{\partial s \partial t}\Big|_0 (1+Y)^{s/p}(1+X)^{t/p} \\
&= \tfrac{1}{p^2}\log(1+Y)\log(1+X)\,,
\end{aligned}$$

$$\begin{aligned}
hf(1) &= \frac{d}{dt}\Big|_0 f\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \\
&= \frac{1}{\log(1+p)}\frac{d}{dt}\Big|_0 f\begin{pmatrix} (1+p)^t & \\ & (1+p)^{-t} \end{pmatrix} \\
&= \frac{1}{\log(1+p)}\log(1+H)\,, \\
h^2 f(1) &= \frac{1}{\log^2(1+p)}\frac{d^2}{dp^2}\Big|_0 f\begin{pmatrix} (1+p)^t & \\ & (1+p)^{-t} \end{pmatrix} \\
&= \frac{1}{\log^2(1+p)}[\log(1+H)]\,,
\end{aligned}$$

Thus the Amice transform of ω is

$$F(Y,H,X) =$$
$$= \frac{1}{\log^2(1+p)}\log^2(1+H) - \frac{2}{\log(1+p)}\log(1+H) + \frac{4}{p^2}\log(1+Y)\log(1+X)\,.$$

This obviously has an expansion (4.1) – and is an element of the ring of convergent series on $D(1)^3$, $D(1) \subset \overline{\mathbb{Q}}_p$ being the open unit disc – but it is not an element of Λ_G.
We will see that the invariance under T suffices to impose such a logarithmic behaviour. This leads to :

Theorem 4.1. *The space of elements on Λ_G invariant by conjugation under T is equal to the Iwasawa algebra $\Lambda_T \subset \Lambda_G$.*

Assume indeed $\lambda \in \Lambda_G$ is T–invariant, with Amice transform

$$F(Y,H,X)\,.$$

We have $Y = u_0 - 1$, with $h_0 u_0 h_0^{-1} = u_0^{(1+p)^{-2}}$; thus the automorphism $Ad(h_0)$ of G sends $1+Y$ to $(1+Y)^{(1+p)^{-2}}$. Similarly, $h_0 n_0 h_0^{-1} = n_0^{(1+p)^2}$, so $1+X$

is sent to $(1+X)^{(1+p)^2}$. Of course H is left invariant. If λ is T–invariant we therefore have

$$F(Y,H,X) = F(Y',H,X') \tag{4.2}$$

where $1+Y' = (1+Y)^{(1+p)^{-2}}$, $1+X' = (1+X)^{(1+p)^2}$. Since $p \neq 2$, $(1+p)^2$ is a topological generator of $1+\mathbb{Z}_p$. Therefore (4.2) remains true if

$$1+Y' = (1+Y)^u\,,\ 1+X' = (1+X)^{u^{-1}}\,,\ u \in 1+p\mathbb{Z}_p\,. \tag{4.3}$$

In the following computations consider F as an element of the Lazard ring in three variables. If we fix a value of H in $\mathbb{C}_p$ such that $|H| < 1$, say H_0, $F(Y,H_0,X) := F_1(Y,X)$ becomes an Iwasawa series in the two variables, still invariant under (4.3). Now set

$$U = \log(1+Y)\,,\ V = \log(1+X)\,, \tag{4.4}$$

two series convergent in $D(1)$. We have

$$F_1(Y,X) = G_1(U,V)$$

where G_1 converges absolutely in the domain of convergence of the exponential, i.e. for $|U|,|V| < r_0 = p^{-\frac{1}{p-1}}$. Moreover G_1 is invariant by $U \mapsto uU$, $V \mapsto u^{-1}V$, $|u-1| < p^{-1}$. This implies that

$$G_1(U,V) = G_2(UV)$$

with $G_2(z)$ convergent for $|z| < r_0^2$.
Let

$$\begin{array}{rrcl} & G_2(z) & = & \displaystyle\sum_0^\infty b_q z^q\,, \\ & F_1(Y,X) & = & \displaystyle\sum_{m,n} a_{mn} Y^m X^n \quad (|a_{mn}| \leq 1)\,. \\ \text{Then} & F_1(Y,X) & = & G_2(\log(1+Y)\log(1+X))\,, \\ & \log(1+Y) & = & Y\displaystyle\sum_0^\infty \frac{(-1)^k}{k+1} Y^k := YL_1(Y) \\ & \log(1+X) & = & X\displaystyle\sum_0^\infty \frac{(-1)^\ell}{\ell+1} Y^\ell := XL_1(X) \end{array}$$

Thus $(\log(1+Y)\log(1+X))^q$ contains only terms the degree of which in Y AND X is at least q. We have of course $b_0 = a_0$, and the previous remark implies that

$$\sum_{n\geq 0} a_{1n}\ Y\ X^n + \sum_{m\geq 0} a_{m_1} Y^m X$$

is identical with the sum of terms of these degrees in

$$b_1\ YX\ L_1(Y)L_1(X)\,,$$

i.e. with

$$b_1\ YX\ (L_1(Y)+L_1(X)-1)\,.$$

Since the a_{mn} are integral, this implies that $b_1=0$ as the denominators in the log–series are not bounded.
By induction assume that $b_1=\cdots b_{N-1}=0$, so

$$G_2=\sum_N^\infty b_q z^q\,.$$

We then find that

$$F_1(Y,X)=b_N Y^N X^N L_1(Y)^N L_1(X)^N \tag{4.4}$$

$+$ terms of degree $>N$ in X and Y.

$$\begin{aligned} \text{Now}\quad L_1(Y) &= 1+YM_1(Y)\,,\quad \text{say}\,,\\ L_1(X) &= 1+XM_1(X)\end{aligned}$$

so (4.4) implies that

$$\begin{aligned} &F_1(Y,X)=b_N Y^N X^N(1+NYM_1(Y)+NXM_1(X))\\ &+\text{ terms of degree }>N\text{ in }X\text{ and }Y\,.\end{aligned}$$

Since M_1 does not have bounded denominators, we deduce that $b_N=0$.
Finally we have proved that $F_1=b_0$, i.e. $F(Y,H,X)\equiv b_0(H)$ for any $H\in\mathbb{C}_p$, $|H|<1$, This implies that $F(Y,H,X)=F(H)$ has no terms involving X or Y, whence the result.

Corollary 4.2. *The centre of Λ_G is composed of the multiples of the Dirac measure at* 1.

For assume that $\lambda\in\Lambda_G$ is central, so invariant by all conjugates of T in G. By Thm. 4.1 its support is contained in the intersection of the tori gTg^{-1} $(g\in G)$. This intersection is reduced to $\{1\}$.
We note that Theorem 4.1 itself follows from Ardakov's results [1, Proposition 2.2]: a simple computation shows that the only finite orbits of T in G are the elements of T (use the triangular decomposition).

5

This section is devoted to conjectural remarks on a formal extension of the main result.
Consider the formulas of Proposition 1, for example

(a) $(1+H)(1+Y) = (1+Y)^{(1+p)^{-2}}(1+H)$

(c) $(1+X)(1+Y) = (1+Y)^{(1+p^2)^{-1}}(1+H)^{\frac{\log(1+p^2)}{\log(1+p)}}(1+X)^{(1+p^2)^{-1}}$.

In the p–adic computation the series for, say, $(1+X)^x$ ($x \in \mathbb{Z}_p$) converges as an Iwasawa expansion because of the integrality of the binomial function $\binom{x}{n}$. However, replace now Λ_G by $k[[Y,H,X]]$ where k is a field of characteristic zero. Set $p = \varepsilon$, another formal variable, which should however be considered as a small parameter. The binomial coefficients, namely

$$\binom{(1+\varepsilon)^{-2}}{n} = \frac{(1+\varepsilon)^{-2}((1+\varepsilon)^{-2}-1)\cdots((1+\varepsilon)^{-2}-n+1)}{n!} \tag{5.1}$$

and similarly

$$\binom{\log(1+\varepsilon^2)/\log(1+\varepsilon)}{n}$$

are well–defined series in $k[[\varepsilon]]$. Formulas (a,b,c) therefore define the products HY, XH and XY in $k[[\varepsilon]][[Y,H,X]]$. The p–adic results do not seem to imply that this extends to an associative product in this ring of power series. Note that if it were so, equations (a,b,c) at $\varepsilon = 0$ would simply yield $HY = YH$, $XH = HX$ and $XY = YX$. Such an extension would therefore define, quite naturally, a formal deformation of the algebra of power series $k[[Y,H,X]]$ associated to the group $SL(2)$. It would be interesting to understand this deformation in group–theoretic terms (or in terms of the Lie algebra) –assuming, of course, it exists. In this respect one should note that formulas (a,b) allow one to define inductively the products H^nY^m and X^nH^m. However I do not see how to define X^nY^m, even granting (c).

References

[1] K. Ardakov, *The centre of completed algebras of pro–p groups*, Documenta Math. 9 (2004), 599–606.

[2] A. Borel, *Automorphic forms on $SL(2,\mathbb{R})$,* Cambridge Univ. Press, 1997.

[3] J.D. Dixon, M.P.F. Du Sautoy, A. Mann, D. Segal, *Analytic pro-p groups*, 2nd edition, Cambridge University Press (1999).

[4] M. Lazard, *Groupes analytiques p–adiques,* Publ. IHES 26 (1965), 389–603.

[5] Li Huishi, F. van Oystaeyen, *Zariskian filtrations,* Kluwer, 1997.

[6] P. Schneider, J. Teitelbaum, *Algebras of p–adic distributions and admissible representations,* Inv. Math. 153 (2003), 145–196.

[7] J.–P. Serre, *Algèbres de Lie semi–simples complexes,* Benjamin, 1966.

[8] R. Steinberg, *Lectures on Chevalley groups,* mimeographed notes, Yale University, 1967.

[9] L. Washington, *Introduction to cyclotomic fields,* Springer, 1982.

Laurent Clozel
Mathématique
Université Paris-Sud 11
Bât. 425
91405 Orsay CEDEX
France

On the Existence of Stationary Solutions for Some Non-Fredholm Integro-Differential Equations

Vitali Vougalter, Vitaly Volpert

Received: February 4, 2011

Communicated by Heinz Siedentop

Abstract. We show the existence of stationary solutions for some reaction-diffusion type equations in the appropriate H^2 spaces using the fixed point technique when the elliptic problem contains second order differential operators with and without Fredholm property.

2010 Mathematics Subject Classification: 35J10, 35P30, 35K57
Keywords and Phrases: solvability conditions, non Fredholm operators, integro-differential equations, stationary solutions

1 Introduction

Let us recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. As a consequence, the equation $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. These properties of Fredholm operators are widely used in many methods of linear and nonlinear analysis.

Elliptic problems in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, proper ellipticity and Lopatinskii conditions are satisfied (see e.g. [1], [9], [10]). This is the main result of the theory of linear elliptic problems. In the case of unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For example, Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ does not satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Linear elliptic problems in unbounded domains satisfy the Fredholm property if and only if, in addition to the conditions cited above, limiting operators are invertible (see [11]). In some simple cases, limiting operators can be explicitly constructed. For example, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have limits at infinity,

$$a_\pm = \lim_{x\to\pm\infty} a(x), \quad b_\pm = \lim_{x\to\pm\infty} b(x), \quad c_\pm = \lim_{x\to\pm\infty} c(x),$$

the limiting operators are:

$$L_\pm u = a_\pm u'' + b_\pm u' + c_\pm u.$$

Since the coefficients are constant, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L-\lambda$ does not satisfy the Fredholm property, can be explicitly found by means of the Fourier transform:

$$\lambda_\pm(\xi) = -a_\pm \xi^2 + b_\pm i\xi + c_\pm, \quad \xi \in \mathbb{R}.$$

Invertibility of limiting operators is equivalent to the condition that the essential spectrum does not contain the origin.
In the case of general elliptic problems, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, these conditions may not be explicitly written.
In the case of non-Fredholm operators the usual solvability conditions may not be applicable and solvability conditions are, in general, not known. There are some classes of operators for which solvability conditions are obtained. Let us illustrate them with the following example. Consider the equation

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, where a is a positive constant. The operator L coincides with its limiting operators. The homogeneous equation has a nonzero bounded solution. Hence the Fredholm property is not satisfied. However, since the operator has constant coefficients, we can apply the Fourier transform and find the solution explicitly. Solvability conditions can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exist a solution of this equation in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [19]). Here and further down S^d_r denotes the sphere in $\mathbb{R}^d$ of radius r centered at the origin. Thus, though the operator does not satisfy the Fredholm property, solvability conditions are formulated in a similar way. However, this similarity is only formal since the range of the operator is not closed.
In the case of the operator with a potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

Fourier transform is not directly applicable. Nevertheless, solvability conditions in $\mathbb{R}^3$ can be obtained by a rather sophisticated application of the theory of self-adjoint operators (see [13]). As before, solvability conditions are formulated in terms of orthogonality to solutions of the homogeneous adjoint equation. There are several other examples of linear elliptic operators without Fredholm property for which solvability conditions can be obtained (see [11]-[19]).
Solvability conditions play an important role in the analysis of nonlinear elliptic problems. In the case of non-Fredholm operators, in spite of some progress in understanding of linear problems, there exist only few examples where nonlinear non-Fredholm operators are analyzed (see [4]-[6]). In the present article we consider another class of nonlinear equations, for which the Fredholm property may not be satisfied:

$$\frac{\partial u}{\partial t} = \Delta u + au + \int_\Omega G(x-y)F(u(y),y)dy = 0, \quad a \geq 0. \tag{1.2}$$

Here Ω is a domain in $\mathbb{R}^d$, $d = 1, 2, 3$, the more physically interesting dimensions. In population dynamics the integro-differential equations describe models with intra-specific competition and nonlocal consumption of resources (see e.g. [2], [3], [7]). The linear part of the corresponding operator is the same as in equation (1.1) above. We will use the explicit form of solvability conditions and will study the existence of stationary solutions of the nonlinear equation.

2 Formulation of the results

The nonlinear part of equation (1.2) will satisfy the following regularity conditions.

Assumption 1. *Function $F(u,x) : \mathbb{R} \times \Omega \to \mathbb{R}$ is such that*

$$|F(u,x)| \leq k|u| + h(x) \quad for \quad u \in \mathbb{R},\ x \in \Omega \tag{2.1}$$

with a constant $k > 0$ and $h(x) : \Omega \to \mathbb{R}^+$, $h(x) \in L^2(\Omega)$. Moreover, it is a Lipschitz continuous function, such that

$$|F(u_1,x) - F(u_2,x)| \leq l|u_1 - u_2| \quad for \quad any \quad u_{1,2} \in \mathbb{R}, \quad x \in \Omega \tag{2.2}$$

with a constant $l > 0$.

Clearly, the stationary solutions of (1.2), if they exist, will satisfy the nonlocal elliptic equation

$$\Delta u + \int_\Omega G(x-y)F(u(y),y)dy + au = 0,\ a \geq 0.$$

Let us introduce the auxiliary problem

$$-\Delta u - au = \int_{\Omega} G(x-y)F(v(y),y)dy. \tag{2.3}$$

We denote $(f_1(x), f_2(x))_{L^2(\Omega)} := \int_{\Omega} f_1(x)\bar{f}_2(x)dx$, with a slight abuse of notations when these functions are not square integrable, like for instance those used in the one dimensional Lemma A1 of the Appendix. In the first part of the article we study the case of $\Omega = \mathbb{R}^d$, such that the appropriate Sobolev space is equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|\Delta u\|^2_{L^2(\mathbb{R}^d)}.$$

The main issue for the problem above is that the operator $-\Delta - a : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d),\ a \geq 0$ does not satisfy the Fredholm property, which is the obstacle to solve equation (2.3). The similar situations but in linear problems, both self- adjoint and non self-adjoint involving non Fredholm second or fourth order differential operators or even systems of equations with non Fredholm operators have been studied extensively in recent years (see [13]-[18]). However, we manage to show that equation (2.3) in this case defines a map $T_a : H^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d),\ a \geq 0$, which is a strict contraction under certain technical conditions.

Theorem 1. *Let $\Omega = \mathbb{R}^d,\ G(x) : \mathbb{R}^d \to \mathbb{R},\ G(x) \in L^1(\mathbb{R}^d)$ and Assumption 1 holds.*
I) When $a > 0$ we assume that $xG(x) \in L^1(\mathbb{R}^d)$, orthogonality relations (6.4) hold if $d = 1$ and (6.9) when $d = 2, 3$ and $\sqrt{2}(2\pi)^{\frac{d}{2}} N_{a,\ d}\, l < 1$. Then the map $T_a v = u$ on $H^2(\mathbb{R}^d)$ defined by equation (2.3) has a unique fixed point v_a, which is the only stationary solution of problem (1.2) in $H^2(\mathbb{R}^d)$.
II) When $a = 0$ we assume that $x^2G(x) \in L^1(\mathbb{R}^d)$, orthogonality relations (6.10) hold, $d = 1, 2, 3$ and $\sqrt{2}(2\pi)^{\frac{d}{2}} N_{0,\ d}\, l < 1$. Then the map $T_0 v = u$ on $H^2(\mathbb{R}^d)$ defined by equation (2.3) admits a unique fixed point v_0, which is the only stationary solution of problem (1.2) with $a = 0$ in $H^2(\mathbb{R}^d)$.
In both cases I) and II) the fixed point $v_a,\ a \geq 0$ is nontrivial provided the intersection of supports of the Fourier transforms of functions $supp\widehat{F(0,x)} \cap supp\widehat{G}$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$.

In the second part of the work we study the analogous problem on the finite interval with periodic boundary conditions, i.e. $\Omega = I := [0,\ 2\pi]$ and the appropriate functional space is

$$H^2(I) = \{u(x) : I \to \mathbb{R} \mid u(x), u''(x) \in L^2(I),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi)\}.$$

Let us introduce the following auxiliary constrained subspaces

$$H^2_0(I) := \{u \in H^2(I) \mid \Big(u(x), \frac{e^{\pm i n_0 x}}{\sqrt{2\pi}}\Big)_{L^2(I)} = 0\},\ n_0 \in \mathbb{N} \tag{2.4}$$

and

$$H^2_{0,\ 0}(I) = \{u \in H^2(I) \mid (u(x), 1)_{L^2(I)} = 0\}, \tag{2.5}$$

which are Hilbert spaces as well (see e.g. Chapter 2.1 of [8]). We prove that equation (2.3) in this situation defines a map $\tau_a,\ a \geq 0$ on the above mentioned spaces which will be a strict contraction under our assumptions.

Theorem 2. *Let $\Omega = I$, $G(x) : I \to \mathbb{R}$, $G(x) \in L^1(I)$, $G(0) = G(2\pi)$, $F(u, 0) = F(u, 2\pi)$ for $u \in \mathbb{R}$ and Assumption 1 holds.*
I) When $a > 0$ and $a \neq n^2$, $n \in \mathbb{Z}$ we assume that $2\sqrt{\pi}\mathcal{N}_a l < 1$. Then the map $\tau_a v = u$ on $H^2(I)$ defined by equation (2.3) has a unique fixed point v_a, the only stationary solution of problem (1.2) in $H^2(I)$.
II) When $a = n_0^2$, $n_0 \in \mathbb{N}$ assume that orthogonality relations (6.17) hold and $2\sqrt{\pi}\mathcal{N}_{n_0^2} l < 1$. Then the map $\tau_{n_0^2} v = u$ on $H^2_0(I)$ defined by equation (2.3) has a unique fixed point $v_{n_0^2}$, the only stationary solution of problem (1.2) in $H^2_0(I)$.
III) When $a = 0$ assume that orthogonality relation (6.18) holds and $2\sqrt{\pi}\mathcal{N}_0 l < 1$. Then the map $\tau_0 v = u$ on $H^2_{0,\ 0}(I)$ defined by equation (2.3) has a unique fixed point v_0, the only stationary solution of problem (1.2) in $H^2_{0,\ 0}(I)$.
In all cases I), II) and III) the fixed point $v_a,\ a \geq 0$ is nontrivial provided the Fourier coefficients $G_n F(0, x)_n \neq 0$ for some $n \in \mathbb{Z}$.

Remark. *We use the constrained subspaces $H^2_0(I)$ and $H^2_{0,\ 0}(I)$ in cases II) and III) respectively, such that the operators $-\dfrac{d^2}{dx^2} - n_0^2 : H^2_0(I) \to L^2(I)$ and $-\dfrac{d^2}{dx^2} : H^2_{0,\ 0}(I) \to L^2(I)$, which possess the Fredholm property, have empty kernels.*

We conclude the article with the studies of our problem on the product of spaces, where one is the finite interval with periodic boundary conditions as before and another is the whole space of dimension not exceeding two, such that in our notations $\Omega = I \times \mathbb{R}^d = [0, 2\pi] \times \mathbb{R}^d$, $d = 1, 2$ and $x = (x_1, x_\perp)$ with $x_1 \in I$ and $x_\perp \in \mathbb{R}^d$. The appropriate Sobolev space for the problem is $H^2(\Omega)$ defined as

$$\{u(x) : \Omega \to \mathbb{R} \mid u(x), \Delta u(x) \in L^2(\Omega),$$
$$u(0, x_\perp) = u(2\pi, x_\perp),\ u_{x_1}(0, x_\perp) = u_{x_1}(2\pi, x_\perp)\},$$

where $x_\perp \in \mathbb{R}^d$ *a.e.* and u_{x_1} stands for the derivative of $u(x)$ with respect to the first variable x_1. As in the whole space case covered in Theorem 1, the operator $-\Delta - a : H^2(\Omega) \to L^2(\Omega)$, $a \geq 0$ does not possess the Fredholm property. Let us show that problem (2.3) in this context defines a map $t_a : H^2(\Omega) \to H^2(\Omega)$, $a \geq 0$, a strict contraction under appropriate technical conditions.

Theorem 3. *Let* $\Omega = I \times \mathbb{R}^d$, $d = 1, 2$, $G(x) : \Omega \to \mathbb{R}$, $G(x) \in L^1(\Omega)$, $G(0, x_\perp) = G(2\pi, x_\perp)$, $F(u, 0, x_\perp) = F(u, 2\pi, x_\perp)$ *for* $x_\perp \in \mathbb{R}^d$ *a.e. and* $u \in \mathbb{R}$ *and Assumption 1 holds.*
I) When $n_0^2 < a < (n_0+1)^2$, $n_0 \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ *let* $x_\perp G(x) \in L^1(\Omega)$, *condition (6.29) holds if dimension* $d = 1$ *and (6.30) if* $d = 2$ *and* $\sqrt{2}(2\pi)^{\frac{d+1}{2}} M_a l < 1$. *Then the map* $t_a v = u$ *on* $H^2(\Omega)$ *defined by equation (2.3) has a unique fixed point* v_a, *the only stationary solution of problem (1.2) in* $H^2(\Omega)$.
II) When $a = n_0^2$, $n_0 \in \mathbb{N}$ *let* $x_\perp^2 G(x) \in L^1(\Omega)$, *conditions (6.25), (6.27) hold if dimension* $d = 1$ *and conditions (6.26), (6.27) hold if* $d = 2$ *and* $\sqrt{2}(2\pi)^{\frac{d+1}{2}} M_{n_0^2} l < 1$. *Then the map* $t_{n_0^2} v = u$ *on* $H^2(\Omega)$ *defined by equation (2.3) has a unique fixed point* $v_{n_0^2}$, *the only stationary solution of problem (1.2) in* $H^2(\Omega)$.
III) When $a = 0$ *let* $x_\perp^2 G(x) \in L^1(\Omega)$, *conditions (6.23) hold and* $\sqrt{2}(2\pi)^{\frac{d+1}{2}} M_0 l < 1$. *Then the map* $t_0 v = u$ *on* $H^2(\Omega)$ *defined by equation (2.3) has a unique fixed point* v_0, *the only stationary solution of problem (1.2) in* $H^2(\Omega)$.
In all cases I), II) and III) the fixed point v_a, $a \geq 0$ *is nontrivial provided that for some* $n \in \mathbb{Z}$ *the intersection of supports of the Fourier images of functions* $supp\widehat{F(0,x)}_n \cap supp\widehat{G}_n$ *is a set of nonzero Lebesgue measure in* $\mathbb{R}^d$.

Remark. *Note that the maps discussed above act on real valued functions due to the assumptions on* $F(u, x)$ *and* $G(x)$ *involved in the nonlocal term of (2.3).*

3 The Whole Space Case

Proof of Theorem 1. We present the proof of the theorem in case I) and when $a = 0$ the argument will be similar. Let us first suppose that in the case of $\Omega = \mathbb{R}^d$ for some $v \in H^2(\mathbb{R}^d)$ there exist two solutions $u_{1,2} \in H^2(\mathbb{R}^d)$ of problem (2.3). Then their difference $w := u_1 - u_2 \in H^2(\mathbb{R}^d)$ will satisfy the homogeneous problem $-\Delta w = aw$. Since the Laplacian operator acting in the whole space does not have any nontrivial square integrable eigenfunctions, $w(x)$ vanishes a.e. in $\mathbb{R}^d$. Let $v(x) \in H^2(\mathbb{R}^d)$ be arbitrary. We apply the standard Fourier transform to both sides of (2.3) and arrive at

$$\widehat{u}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G}(p)\widehat{f}(p)}{p^2 - a} \tag{3.1}$$

with $\widehat{f}(p)$ denoting the Fourier image of $F(v(x), x)$. Clearly, we have the upper bounds

$$|\widehat{u}(p)| \leq (2\pi)^{\frac{d}{2}} N_{a,\ d} |\widehat{f}(p)| \quad and \quad |p^2 \widehat{u}(p)| \leq (2\pi)^{\frac{d}{2}} N_{a,\ d} |\widehat{f}(p)|$$

with $N_{a,\ d} < \infty$ by means of Lemma A1 of the Appendix in one dimension and via Lemma A2 for $d = 2, 3$ under orthogonality relations (6.4) and (6.9) respectively. This enables us to estimate the norm

$$\|u\|^2_{H^2(\mathbb{R}^d)} = \|\widehat{u}(p)\|^2_{L^2(\mathbb{R}^d)} + \|p^2\widehat{u}(p)\|^2_{L^2(\mathbb{R}^d)} \leq 2(2\pi)^d N^2_{a,\ d}\|F(v(x),x)\|^2_{L^2(\mathbb{R}^d)},$$

which is finite by means of (2.1) of Assumption 1. Therefore, for any $v(x) \in H^2(\mathbb{R}^d)$ there is a unique solution $u(x) \in H^2(\mathbb{R}^d)$ of problem (2.3) with its Fourier image given by (3.1) and the map $T_a : H^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)$ is well defined. This enables us to choose arbitrarily $v_{1,2}(x) \in H^2(\mathbb{R}^d)$ such that their images $u_{1,2} = T_a v_{1,2} \in H^2(\mathbb{R}^d)$ and estimate

$$|\widehat{u}_1(p) - \widehat{u}_2(p)| \leq (2\pi)^{\frac{d}{2}} N_{a,\ d}|\widehat{f}_1(p) - \widehat{f}_2(p)|,$$
$$|p^2\widehat{u}_1(p) - p^2\widehat{u}_2(p)| \leq (2\pi)^{\frac{d}{2}} N_{a,\ d}|\widehat{f}_1(p) - \widehat{f}_2(p)|,$$

where $\widehat{f}_{1,2}(p)$ stand for the Fourier images of $F(v_{1,2}(x), x)$. For the appropriate norms of functions this yields

$$\|u_1 - u_2\|^2_{H^2(\mathbb{R}^d)} \leq 2(2\pi)^d N^2_{a,\ d}\|F(v_1(x),x) - F(v_2(x),x)\|^2_{L^2(\mathbb{R}^d)}.$$

Note that $v_{1,2}(x) \in H^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d),\ d \leq 3$ by means of the Sobolev embedding. Using condition (2.2) we easily arrive at

$$\|T_a v_1 - T_a v_2\|_{H^2(\mathbb{R}^d)} \leq \sqrt{2}(2\pi)^{\frac{d}{2}} N_{a,\ d} l \|v_1 - v_2\|_{H^2(\mathbb{R}^d)}$$

with the constant in the right side of this estimate less than one by the assumption of the theorem. Therefore, by means of the Fixed Point Theorem, there exists a unique function $v_a \in H^2(\mathbb{R}^d)$ with the property $T_a v_a = v_a$, which is the only stationary solution of equation (1.2) in $H^2(\mathbb{R}^d)$. Suppose $v_a(x)$ vanishes a.e. in $\mathbb{R}^d$. This will contradict to the assumption that the Fourier images of $G(x)$ and $F(0,x)$ do not vanish on a set of nonzero Lebesgue measure in $\mathbb{R}^d$. ∎

4 The Problem on the Finite Interval

Proof of Theorem 2. Let us demonstrate the proof of the theorem in case I) and when $a = n_0^2,\ n_0 \in \mathbb{N}$ or $a = 0$ the ideas will be similar, using the constrained subspaces (2.4) and (2.5) respectively instead of $H^2(I)$. First we suppose that for $v \in H^2(I)$ there are two solutions $u_{1,2} \in H^2(I)$ of problem (2.3) with $\Omega = I$. Then function $w := u_1 - u_2 \in H^2(I)$ will be a solution to the problem $-w'' = aw$. But $a \neq n^2,\ n \in \mathbb{Z}$ and therefore, it is not an eigenvalue of the operator $-\dfrac{d^2}{dx^2}$ on $L^2(I)$ with periodic boundary conditions. Therefore, $w(x)$ vanishes a.e. in I. Suppose $v(x) \in H^2(I)$ is arbitrary. Let us apply the Fourier transform to problem (2.3) considered on the interval I which yields

$$u_n = \sqrt{2\pi}\frac{G_n f_n}{n^2 - a}, \quad n \in \mathbb{Z} \tag{4.1}$$

with $f_n := F(v(x), x)_n$. Clearly for the transform of the second derivative we have

$$(-u'')_n = \sqrt{2\pi}\frac{n^2 G_n f_n}{n^2 - a}, \quad n \in \mathbb{Z},$$

which enables us to estimate

$$\|u\|^2_{H^2(I)} = \sum_{n=-\infty}^{\infty} |u_n|^2 + \sum_{n=-\infty}^{\infty} |n^2 u_n|^2 \leq 4\pi \mathcal{N}_a^2 \|F(v(x), x)\|^2_{L^2(I)} < \infty$$

due to (2.1) of Assumption 1 and Lemma A3 of the Appendix. Hence, for an arbitrary $v(x) \in H^2(I)$ there is a unique $u(x) \in H^2(I)$ solving equation (2.3) with its Fourier image given by (4.1) and the map $\tau_a : H^2(I) \to H^2(I)$ in case I) is well defined. Let us consider any $v_{1,2} \in H^2(I)$ with their images under the map mentioned above $u_{1,2} = \tau_a v_{1,2} \in H^2(I)$ and arrive easily at the upper bound

$$\|u_1 - u_2\|^2_{H^2(I)} = \sum_{n=-\infty}^{\infty} |u_{1n} - u_{2n}|^2 + \sum_{n=-\infty}^{\infty} |n^2(u_{1n} - u_{2n})|^2 \leq$$

$$\leq 4\pi \mathcal{N}_a^2 \|F(v_1(x), x) - F(v_2(x), x)\|^2_{L^2(I)}.$$

Obviously $v_{1,2}(x) \in H^2(I) \subset L^\infty(I)$ due to the Sobolev embedding. By means of (2.2) we easily obtain

$$\|\tau_a v_1 - \tau_a v_2\|_{H^2(I)} \leq 2\sqrt{\pi}\mathcal{N}_a l \|v_1 - v_2\|_{H^2(I)},$$

such that the constant in the right side of this upper bound is less than one as assumed. Thus, the Fixed Point Theorem implies the existence and uniqueness of a function $v_a \in H^2(I)$ satisfying $\tau_a v_a = v_a$, which is the only stationary solution of problem (1.2) in $H^2(I)$. Suppose $v_a(x) = 0$ a.e. in I. Then we obtain the contradiction to the assumption that $G_n F(0, x)_n \neq 0$ for some $n \in \mathbb{Z}$. Note that in the case of $a \neq n^2,\ n \in \mathbb{Z}$ the argument does not require any orthogonality conditions. ■

5 The Problem on the Product of Spaces

Proof of Theorem 3. We present the proof of the theorem for case II) since when the parameter a vanishes or is located on the open interval between squares of two nonnegative integers the ideas are similar. Suppose there exists $v(x) \in H^2(\Omega)$ which generates $u_{1,2}(x) \in H^2(\Omega)$ solving equation (2.3). Then the difference $w := u_1 - u_2 \in H^2(\Omega)$ will satisfy $-\Delta w = n_0^2 w$ in our domain Ω. By applying the partial Fourier transform to this equation we easily arrive at $-\Delta_\perp w_n(x_\perp) = (n_0^2 - n^2) w_n(x_\perp)$. Clearly $\|w\|^2_{L^2(\Omega)} = \sum_{n=-\infty}^{\infty} \|w_n\|^2_{L^2(\mathbb{R}^d)}$ such that $w_n(x_\perp) \in L^2(\mathbb{R}^d),\ n \in \mathbb{Z}$. Since the transversal Laplacian operator $-\Delta_\perp$ on $L^2(\mathbb{R}^d)$ does not have any nontrivial square integrable eigenfunctions,

$w(x)$ is vanishing a.e. in Ω. Let $v(x) \in H^2(\Omega)$ be arbitrary. We apply the Fourier transform to both sides of problem (2.3) and obtain

$$\widehat{u}_n(p) = (2\pi)^{\frac{d+1}{2}} \frac{\widehat{G}_n(p)\widehat{f}_n(p)}{p^2 + n^2 - n_0^2}, \quad n \in \mathbb{Z}, \quad p \in \mathbb{R}^d, \quad d = 1, 2, \tag{5.1}$$

where $\widehat{f}_n(p)$ stands for the Fourier image of $F(v(x), x)$. Obviously,

$$|\widehat{u}_n(p)| \leq (2\pi)^{\frac{d+1}{2}} M_{n_0^2} |\widehat{f}_n(p)| \quad \textit{and} \quad |(p^2 + n^2)\widehat{u}_n(p)| \leq (2\pi)^{\frac{d+1}{2}} M_{n_0^2} |\widehat{f}_n(p)|,$$

where $M_{n_0^2} < \infty$ by means of Lemma A5 of the Appendix under the appropriate orthogonality conditions stated in it. Thus

$$\|u\|^2_{H^2(\Omega)} = \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |\widehat{u}_n(p)|^2 dp + \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |(p^2 + n^2)\widehat{u}_n(p)|^2 dp \leq$$
$$\leq 2(2\pi)^{d+1} {M_{n_0^2}}^2 \|F(v(x), x)\|^2_{L^2(\Omega)} < \infty$$

by means of (2.1) of Assumption 1, such that for any $v(x) \in H^2(\Omega)$ there exists a unique $u(x) \in H^2(\Omega)$ solving equation (2.3) with its Fourier image given by (5.1) and the map $t_a : H^2(\Omega) \to H^2(\Omega)$ in case II) of the Theorem is well defined. Then we consider arbitrary $v_{1,2} \in H^2(\Omega)$ such that their images under the map are $u_{1,2} = t_{n_0^2} v_{1,2} \in H^2(\Omega)$ and obtain

$$\|u_1 - u_2\|^2_{H^2(\Omega)} =$$
$$= \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |\widehat{u_1}_n(p) - \widehat{u_2}_n(p)|^2 dp + \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |(p^2 + n^2)(\widehat{u_1}_n(p) - \widehat{u_2}_n(p))|^2 dp \leq$$
$$\leq 2(2\pi)^{d+1} {M_{n_0^2}}^2 \|F(v_1(x), x) - F(v_2(x), x)\|^2_{L^2(\Omega)}.$$

Clearly $v_{1,2} \in H^2(\Omega) \subset L^\infty(\Omega)$ via the Sobolev embedding theorem. Using (2.2) we easily arrive at the estimate

$$\|t_{n_0^2} v_1 - t_{n_0^2} v_2\|_{H^2(\Omega)} \leq \sqrt{2}(2\pi)^{\frac{d+1}{2}} M_{n_0^2} l \|v_1 - v_2\|_{H^2(\Omega)}$$

with the constant in the right side of it less than one by assumption. Therefore, the Fixed Point Theorem yields the existence and uniqueness of a function $v_{n_0^2} \in H^2(\Omega)$ which satisfies $t_{n_0^2} v_{n_0^2} = v_{n_0^2}$ and is the only stationary solution of problem (1.2) in $H^2(\Omega)$ in case II) of the theorem. Suppose $v_{n_0^2}(x) = 0$ a.e. in Ω. This yields the contradiction to the assumption that there exists $n \in \mathbb{Z}$ for which $supp \widehat{G}_n \cap supp \widehat{F(0,x)}_n$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$. ■

6 Appendix

Let $G(x)$ be a function, $G(x) : \mathbb{R}^d \to \mathbb{R}, \quad d \leq 3$ for which we denote its standard Fourier transform using the hat symbol as

$$\widehat{G}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} G(x) e^{-ipx} dx, \ p \in \mathbb{R}^d$$

such that

$$\|\widehat{G}(p)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{\frac{d}{2}}}\|G\|_{L^1(\mathbb{R}^d)} \tag{6.1}$$

and $G(x) = \dfrac{1}{(2\pi)^{\frac{d}{2}}}\displaystyle\int_{\mathbb{R}^d}\widehat{G}(q)e^{iqx}dq,\ x\in\mathbb{R}^d$. Let us define the auxiliary quantities

$$N_{a,\ d} := max\Big\{\Big\|\frac{\widehat{G}(p)}{p^2-a}\Big\|_{L^\infty(\mathbb{R}^d)},\quad \Big\|\frac{p^2\widehat{G}(p)}{p^2-a}\Big\|_{L^\infty(\mathbb{R}^d)}\Big\} \tag{6.2}$$

for $a>0$ and

$$N_{0,\ d} := max\Big\{\Big\|\frac{\widehat{G}(p)}{p^2}\Big\|_{L^\infty(\mathbb{R}^d)},\quad \Big\|\widehat{G}(p)\Big\|_{L^\infty(\mathbb{R}^d)}\Big\} \tag{6.3}$$

when $a=0$.

Lemma A1. *Let $G(x)\in L^1(\mathbb{R})$.*
a) If $a>0$ and $xG(x)\in L^1(\mathbb{R})$ then $N_{a,\ 1}<\infty$ if and only if

$$\Big(G(x), \frac{e^{\pm i\sqrt{a}x}}{\sqrt{2\pi}}\Big)_{L^2(\mathbb{R})} = 0. \tag{6.4}$$

b) If $a=0$ and $x^2G(x)\in L^1(\mathbb{R})$ then $N_{0,\ 1}<\infty$ if and only if

$$(G(x),1)_{L^2(\mathbb{R})} = 0 \quad and \quad (G(x),x)_{L^2(\mathbb{R})} = 0. \tag{6.5}$$

Proof. In order to prove part a) of the lemma we write the function

$$\frac{\widehat{G}(p)}{p^2-a} = \frac{\widehat{G}(p)}{p^2-a}\chi_{I_\delta} + \frac{\widehat{G}(p)}{p^2-a}\chi_{{I_\delta}^c}, \tag{6.6}$$

where χ_A here and further down stands for the characteristic function of a set A, A^c for its complement, the set $I_\delta = I_\delta^+\cup I_\delta^-$ with $I_\delta^+ = \{p\in\mathbb{R}\mid \sqrt{a}-\delta < p < \sqrt{a}+\delta\}$, $I_\delta^- = \{p\in\mathbb{R}\mid -\sqrt{a}-\delta < p < -\sqrt{a}+\delta\}$ and $0<\delta<\sqrt{a}$. The second term in the right side of (6.6) can be easily estimated in absolute value from above using (6.1) as $\dfrac{1}{\sqrt{2\pi}\delta^2}\|G\|_{L^1(\mathbb{R})}<\infty$ and the remaining term in the right side of (6.6) can be written as

$$\frac{\widehat{G}(p)}{p^2-a}\chi_{I_\delta^+} + \frac{\widehat{G}(p)}{p^2-a}\chi_{I_\delta^-}.$$

We will use the expansions near the points of singularity given by

$$\widehat{G}(p) = \widehat{G}(\sqrt{a}) + \int_{\sqrt{a}}^p \frac{d\widehat{G}(s)}{ds}ds,\quad \widehat{G}(p) = \widehat{G}(-\sqrt{a}) + \int_{-\sqrt{a}}^p \frac{d\widehat{G}(s)}{ds}ds$$

with $\left\|\dfrac{d\widehat{G}(p)}{dp}\right\|_{L^\infty(\mathbb{R})} \leq \dfrac{1}{\sqrt{2\pi}}\|xG\|_{L^1(\mathbb{R})} < \infty$ by the assumption of the lemma. This enables us to obtain the bound

$$\left|\frac{\int_{\sqrt{a}}^p \frac{d\widehat{G}(s)}{ds}ds}{p^2-a}\chi_{I_\delta^+}\right| \leq \frac{C}{2\sqrt{a}-\delta} < \infty, \quad \left|\frac{\int_{-\sqrt{a}}^p \frac{d\widehat{G}(s)}{ds}ds}{p^2-a}\chi_{I_\delta^-}\right| \leq \frac{C}{2\sqrt{a}-\delta} < \infty.$$

Therefore it remains to estimate

$$\frac{\widehat{G}(\sqrt{a})}{p^2-a}\chi_{I_\delta^+} + \frac{\widehat{G}(-\sqrt{a})}{p^2-a}\chi_{I_\delta^-},$$

which belongs to $L^\infty(\mathbb{R})$ if and only if $\widehat{G}(\pm\sqrt{a}) = 0$, which is equivalent to the orthogonality relations (6.4). To estimate the second term in the right side of (6.2) under these orthogonality relations we consider the two situations. The first one is when $|p| \leq \sqrt{a}+\delta$ and we have the bound

$$\left|\frac{p^2\widehat{G}(p)}{p^2-a}\right| \leq (\sqrt{a}+\delta)^2\left\|\frac{\widehat{G}(p)}{p^2-a}\right\|_{L^\infty(\mathbb{R})} < \infty.$$

In the second one $|p| > \sqrt{a}+\delta$ which yields $\dfrac{p^2}{p^2-a} \in L^\infty(\mathbb{R})$ and $\widehat{G}(p)$ is bounded via (6.1), which completes the proof of part a) of the lemma. Then we turn our attention to the situation of $a = 0$, such that

$$\frac{\widehat{G}(p)}{p^2} = \frac{\widehat{G}(p)}{p^2}\chi_{\{|p|\leq 1\}} + \frac{\widehat{G}(p)}{p^2}\chi_{\{|p|>1\}}. \tag{6.7}$$

The second term in the right side of the identity above can be easily estimated as

$$\left|\frac{\widehat{G}(p)}{p^2}\chi_{\{|p|>1\}}\right| \leq \|\widehat{G}(p)\|_{L^\infty(\mathbb{R})} < \infty \tag{6.8}$$

due to (6.1). We will make use of the representation

$$\widehat{G}(p) = \widehat{G}(0) + p\frac{d\widehat{G}}{dp}(0) + \int_0^p \Big(\int_0^s \frac{d^2\widehat{G}(q)}{dq^2}dq\Big)ds.$$

Obviously $\left|\dfrac{d^2\widehat{G}(p)}{dp^2}\right| \leq \dfrac{1}{\sqrt{2\pi}}\|x^2G(x)\|_{L^1(\mathbb{R})} < \infty$ by the assumption of the lemma. Hence

$$\left|\frac{\int_0^p \Big(\int_0^s \frac{d^2\widehat{G}(q)}{dq^2}dq\Big)ds}{p^2}\chi_{\{|p|\leq 1\}}\right| \leq \frac{C}{2} < \infty$$

and the only expression which remains to estimate is given by $\left[\dfrac{\widehat{G}(0)}{p^2} + \dfrac{\frac{d\widehat{G}}{dp}(0)}{p}\right]\chi_{\{|p|\leq 1\}}$, which is contained in $L^\infty(\mathbb{R})$ if and only if $\widehat{G}(0)$ and

$\frac{d\widehat{G}}{dp}(0)$ vanish. This is equivalent to the orthogonality relations (6.5). Note that $\|\widehat{G}(p)\|_{L^\infty(\mathbb{R})} < \infty$ by means of (6.1). ■

The proposition above can be generalized to higher dimensions in the following statement.

LEMMA A2. *Let $G(x) \in L^1(\mathbb{R}^d),\ d = 2,3.$*
a) If $a > 0$ and $xG(x) \in L^1(\mathbb{R}^d)$ then $N_{a,\ d} < \infty$ if and only if

$$\Big(G(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\Big)_{L^2(\mathbb{R}^d)} = 0 \quad for \quad p \in S^d_{\sqrt{a}} \quad a.e. \tag{6.9}$$

b) If $a = 0$ and $x^2G(x) \in L^1(\mathbb{R}^d)$ then $N_{0,\ d} < \infty$ if and only if

$$(G(x), 1)_{L^2(\mathbb{R}^d)} = 0 \quad and \quad (G(x), x_k)_{L^2(\mathbb{R}^d)} = 0, \quad 1 \le k \le d. \tag{6.10}$$

Proof. To prove part a) of the lemma we introduce the auxiliary spherical layer in the space of $d = 2,3$ dimensions

$$A_\delta := \{p \in \mathbb{R}^d \mid \sqrt{a} - \delta < |p| < \sqrt{a} + \delta\}, \quad 0 < \delta < \sqrt{a}$$

and write

$$\frac{\widehat{G}(p)}{p^2 - a} = \frac{\widehat{G}(p)}{p^2 - a}\chi_{A_\delta} + \frac{\widehat{G}(p)}{p^2 - a}\chi_{A^c_\delta}. \tag{6.11}$$

For the second term in the right side of (6.11) we have the upper bound in the absolute value as $\frac{\|\widehat{G}(p)\|_{L^\infty(\mathbb{R}^d)}}{\sqrt{a}\delta} < \infty$ due to (6.1). Let us expand

$$\widehat{G}(p) = \int_{\sqrt{a}}^{|p|} \frac{\partial \widehat{G}(|s|, \sigma)}{\partial |s|} d|s| + \widehat{G}(\sqrt{a}, \sigma),$$

where σ stands for the angle variables on the sphere. Using the elementary inequality $\left|\frac{\partial \widehat{G}(p)}{\partial |p|}\right| \le \frac{1}{(2\pi)^{\frac{d}{2}}}\|xG(x)\|_{L^1(\mathbb{R}^d)}$ with its right side finite by the assumption of the lemma we estimate

$$\left|\frac{\int_{\sqrt{a}}^{|p|} \frac{\partial \widehat{G}(|s|,\sigma)}{\partial |s|} d|s|}{p^2 - a}\chi_{A_\sigma}\right| \le \frac{C}{\sqrt{a}} < \infty.$$

The only remaining term $\frac{\widehat{G}(\sqrt{a}, \sigma)}{p^2 - a}\chi_{A_\delta} \in L^\infty(\mathbb{R}^d), \quad d = 2,3$ if and only if $\widehat{G}(\sqrt{a}, \sigma)$ vanishes a.e. on the sphere $S^d_{\sqrt{a}}$, which is equivalent to orthogonality relations (6.9). The proof of the fact that the second norm in the right side of (6.2) under conditions (6.9) is finite is analogous to the one presented in Lemma A1 in one dimension. For the proof of part b) of the lemma we apply the two and three dimensional analog of formula (6.7), such that for the second term in its right side there is a bound analogous to (6.8). Let us use the representation formula

$$\widehat{G}(p) = \widehat{G}(0) + |p|\frac{\partial \widehat{G}}{\partial |p|}(0,\sigma) + \int_0^{|p|}\left(\int_0^s \frac{\partial^2 \widehat{G}(|q|,\sigma)}{\partial |q|^2} d|q|\right) ds.$$

Apparently

$$\frac{\partial \widehat{G}}{\partial |p|}(0,\sigma) = -\frac{i}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} G(x)|x| cos\theta dx, \tag{6.12}$$

where θ is the angle between vectors p and x in $\mathbb{R}^d$ and for the second derivative

$$\left|\frac{\partial^2 \widehat{G}(p)}{\partial |p|^2}\right| \leq \frac{1}{(2\pi)^{\frac{d}{2}}}\|x^2 G(x)\|_{L^1(\mathbb{R}^d)} < \infty$$

by the assumption of the lemma. This yields

$$\left|\frac{\int_0^{|p|}\left(\int_0^s \frac{\partial^2 \widehat{G}(|q|,\sigma)}{\partial |q|^2} d|q|\right) ds}{p^2}\chi_{\{|p|\leq 1\}}\right| \leq \frac{C}{2} < \infty,$$

such that the only expression remaining to estimate is given by

$$\left[\frac{\widehat{G}(0)}{p^2} + \frac{\frac{\partial \widehat{G}}{\partial |p|}(0,\sigma)}{|p|}\right]\chi_{\{|p|\leq 1\}} \tag{6.13}$$

with the first derivative (6.12) containing the angular dependence. We consider first the case of $d=2$ such that $p = (|p|,\theta_p),\ x = (|x|,\theta_x) \in \mathbb{R}^2$ and the angle between them $\theta = \theta_p - \theta_x$. A straightforward computation yields that the right side of (6.12) is given by $-\frac{i}{2\pi}\sqrt{Q_1^2+Q_2^2}cos(\theta_p - \alpha)$ with

$$Q_1 := \int_{\mathbb{R}^2} G(x)x_1 dx, \quad Q_2 := \int_{\mathbb{R}^2} G(x)x_2 dx, \quad \tan\alpha := \frac{Q_2}{Q_1} \tag{6.14}$$

and $x = (x_1,x_2) \in \mathbb{R}^2$ such that (6.13) is equal to

$$\left[\frac{\widehat{G}(0)}{p^2} - \frac{i}{2\pi}\sqrt{Q_1^2+Q_2^2}\frac{cos(\theta_p-\alpha)}{|p|}\right]\chi_{\{|p|\leq 1\}}.$$

Note that the situation of $Q_1 = 0$ and $Q_2 \neq 0$ corresponds to the cases of α equal to $\frac{\pi}{2}$ or $-\frac{\pi}{2}$. Obviously, the expression above is contained in $L^\infty(\mathbb{R}^2)$

if and only if the quantities $\widehat{G}(0)$, Q_1 and Q_2 vanish, which is equivalent to orthogonality relations (6.10) in two dimensions. In the case of $d = 3$ the argument is quite similar. The coordinates of vectors

$$x = (x_1, x_2, x_3) = (|x| sin\theta_x cos\varphi_x, |x| sin\theta_x sin\varphi_x, |x| cos\theta_x) \in \mathbb{R}^3$$

and

$$p = (|p| sin\theta_p cos\varphi_p, |p| sin\theta_p sin\varphi_p, |p| cos\theta_p) \in \mathbb{R}^3$$

are being used to compute $cos\theta = \dfrac{(p,x)_{\mathbb{R}^3}}{|p||x|}$ involved in the right side of (6.12). Here $(p,x)_{\mathbb{R}^3}$ stands for the scalar product of the vectors in three dimensions. An easy calculation shows that when $d = 3$ the right side of (6.12) can be written as

$$-\frac{i}{(2\pi)^{\frac{3}{2}}}\{\sqrt{Q_1^2 + Q_2^2} sin\theta_p cos(\varphi_p - \alpha) + Q_3 cos\theta_p\}$$

with α given by (6.14) and here $Q_k = \int_{\mathbb{R}^3} G(x) x_k dx,\ k = 1, 2, 3$, which are the three dimensional generalizations of the correspondent expressions given by (6.14) and term (6.13) will be equal to

$$\left[\frac{\widehat{G}(0)}{p^2} - \frac{i}{(2\pi)^{\frac{3}{2}}|p|}\{\sqrt{Q_1^2 + Q_2^2} sin\theta_p cos(\varphi_p - \alpha) + Q_3 cos\theta_p\}\right]\chi_{\{|p|\le 1\}}$$

and will belong to $L^\infty(\mathbb{R}^3)$ if and only if $\widehat{G}(0)$ along with $Q_k,\ k = 1, 2, 3$ vanish, which is equivalent to orthogonality conditions (6.10) in three dimensions. The second norm in the right side of (6.3) is finite under relations (6.1). ∎

Let the function $G(x) : I \to \mathbb{R}$, $\quad G(0) = G(2\pi)$ and its Fourier transform on the finite interval is given by

$$G_n := \int_0^{2\pi} G(x)\frac{e^{-inx}}{\sqrt{2\pi}}dx, \quad n \in \mathbb{Z}$$

and $G(x) = \sum_{n=-\infty}^{\infty} G_n \dfrac{e^{inx}}{\sqrt{2\pi}}$. Similarly to the whole space case we define

$$\mathcal{N}_a := max\left\{\left\|\frac{G_n}{n^2 - a}\right\|_{l^\infty}, \quad \left\|\frac{n^2 G_n}{n^2 - a}\right\|_{l^\infty}\right\} \tag{6.15}$$

for $a > 0$. In the situation of $a = 0$

$$\mathcal{N}_0 := max\left\{\left\|\frac{G_n}{n^2}\right\|_{l^\infty}, \quad \left\|G_n\right\|_{l^\infty}\right\}. \tag{6.16}$$

We have the following elementary statement.

Lemma A3. *Let $G(x) \in L^1(I)$ and $G(0) = G(2\pi)$.*
a) If $a > 0$ and $a \neq n^2,\ n \in \mathbb{Z}$ then $\mathcal{N}_a < \infty$.
b) If $a = n_0^2,\ n_0 \in \mathbb{N}$ then $\mathcal{N}_a < \infty$ if and only if

$$\left(G(x), \frac{e^{\pm i n_0 x}}{\sqrt{2\pi}} \right)_{L^2(I)} = 0. \tag{6.17}$$

c) If $a = 0$ then $\mathcal{N}_0 < \infty$ if and only if

$$(G(x), 1)_{L^2(I)} = 0. \tag{6.18}$$

Proof. Clearly we have the bound

$$\|G_n\|_{l^\infty} \leq \frac{1}{\sqrt{2\pi}} \|G\|_{L^1(I)} < \infty. \tag{6.19}$$

Thus in case a) when $a \neq n^2,\ n \in \mathbb{Z}$ the expressions under the norms in the right side of (6.15) do not contain any singularities and the result of the lemma is obvious. When $a = n_0^2$ for some $n_0 \in \mathbb{N}$ or $a = 0$ conditions (6.17) and (6.18) respectively are necessary and sufficient for eliminating the existing singularities by making the corresponding Fourier coefficients equal to zero: $G_{\pm n_0}$ in case b) and G_0 in case c). ■

Let $G(x)$ be a function on the product of spaces studied in Theorem 3, $G(x) : \Omega = I \times \mathbb{R}^d \to \mathbb{R},\ d = 1, 2,\ G(0, x_\perp) = G(2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e. and its Fourier transform on the product of spaces equals to

$$\widehat{G}_n(p) := \frac{1}{(2\pi)^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} dx_\perp e^{-ipx_\perp} \int_0^{2\pi} G(x_1, x_\perp) e^{-inx_1} dx_1, \quad p \in \mathbb{R}^d,\ n \in \mathbb{Z}$$

such that

$$\|\widehat{G}_n(p)\|_{L^\infty_{n,p}} := \sup_{\{p \in \mathbb{R}^d,\ n \in \mathbb{Z}\}} |\widehat{G}_n(p)| \leq \frac{1}{(2\pi)^{\frac{d+1}{2}}} \|G\|_{L^1(\Omega)} \tag{6.20}$$

and $G(x) = \frac{1}{(2\pi)^{\frac{d+1}{2}}} \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} \widehat{G}_n(p) e^{ipx_\perp} e^{inx_1} dp$. It is also useful to consider the Fourier transform only in the first variable, such that

$$G_n(x_\perp) := \int_0^{2\pi} G(x_1, x_\perp) \frac{e^{-inx_1}}{\sqrt{2\pi}} dx_1, \quad n \in \mathbb{Z}.$$

Let us introduce $\xi_n^a(p) := \dfrac{\widehat{G}_n(p)}{p^2 + n^2 - a}$ and define

$$M_a := max\{\|\xi_n^a(p)\|_{L^\infty_{n,p}}, \quad \|(p^2 + n^2)\xi_n^a(p)\|_{L^\infty_{n,p}}\} \tag{6.21}$$

for $a > 0$ and

$$M_0 := max\left\{\left\|\frac{\widehat{G}_n(p)}{p^2+n^2}\right\|_{L^\infty_{n,p}}, \quad \|\widehat{G}_n(p)\|_{L^\infty_{n,p}}\right\} \tag{6.22}$$

when $a = 0$. Here the momentum vector $p \in \mathbb{R}^d$.

Lemma A4. *Let $G(x) \in L^1(\Omega)$, $x_\perp^2 G(x) \in L^1(\Omega)$ and $G(0, x_\perp) = G(2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e., $d = 1, 2$.* Then $M_0 < \infty$ if and only if

$$(G(x), 1)_{L^2(\Omega)} = 0, \ (G(x), x_{\perp,\ k})_{L^2(\Omega)} = 0, \ 1 \le k \le d, \ d = 1, 2. \tag{6.23}$$

Proof. Let us expand

$$\frac{\widehat{G}_n(p)}{p^2+n^2} = \frac{\widehat{G}_0(p)}{p^2}\chi_{\{p\in\mathbb{R}^d,\ n=0\}} + \frac{\widehat{G}_n(p)}{p^2+n^2}\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ n\neq 0\}}.$$

The second term in the right side of this identity can be estimated above in the absolute value by means of (6.20) by $\frac{1}{(2\pi)^{\frac{d+1}{2}}}\|G\|_{L^1(\Omega)} < \infty$. Clearly we have the bound on the norm

$$\|x_\perp^2 G_n(x_\perp)\|_{L^1(\mathbb{R}^d)} \le \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} dx_1 \int_{\mathbb{R}^d} dx_\perp x_\perp^2 |G(x)| < \infty, \quad n \in \mathbb{Z} \tag{6.24}$$

by the assumption of the lemma. Thus the term $\frac{\widehat{G}_0(p)}{p^2} \in L^\infty(\mathbb{R}^d)$ if and only if the orthogonality conditions (6.23) hold, which is guaranteed for $d = 1$ by Lemma A1 and when dimension $d = 2$ by Lemma A2. Note that the last term in the right side of (6.22) is bounded via (6.20). ■

Next we turn our attention to the situation when the parameter a is nontrivial.

Lemma A5. *Let $G(x) \in L^1(\Omega)$, $x_\perp^2 G(x) \in L^1(\Omega)$ and $G(0, x_\perp) = G(2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e., $d = 1, 2$ and $a = n_0^2$, $n_0 \in \mathbb{N}$. Then $M_a < \infty$ if and only if*

$$\left(G(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{\pm i\sqrt{n_0^2-n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad |n| \le n_0 - 1, \quad d = 1, \tag{6.25}$$

$$\left(G(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0, \quad p \in S^2_{\sqrt{n_0^2-n^2}} \quad a.e., \quad |n| \le n_0 - 1, \ d = 2, \tag{6.26}$$

$$\left(G(x_1, x_\perp), \frac{e^{\pm in_0x_1}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \ \left(G(x_1, x_\perp), \frac{e^{\pm in_0x_1}}{\sqrt{2\pi}}x_{\perp,\ k}\right)_{L^2(\Omega)} = 0, \ 1 \le k \le d. \tag{6.27}$$

Proof. We will use the representation of the function $\xi_n^a(p),\ n\in\mathbb{Z},\ p\in\mathbb{R}^d$ as the sum

$$\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ |n|>n_0\}}+\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ |n|<n_0\}}+ \\ +\xi_{n_0}^a(p)\chi_{\{p\in\mathbb{R}^d,\ n=n_0\}}+\xi_{-n_0}^a(p)\chi_{\{p\in\mathbb{R}^d,\ n=-n_0\}}. \tag{6.28}$$

Obviously $|\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ |n|>n_0\}}|\le\|\widehat{G}_n(p)\|_{L^\infty_{n,p}}<\infty$ by means of (6.20). We have trivial estimates on the norms for $n\in\mathbb{Z}$

$$\|G_n(x_\perp)\|_{L^1(\mathbb{R}^d)}\le\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}dx_1\int_{\mathbb{R}^d}dx_\perp|G(x_1,x_\perp)|<\infty$$

and

$$\|x_\perp G_n(x_\perp)\|_{L^1(\mathbb{R}^d)}\le\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}dx_1\int_{\mathbb{R}^d}dx_\perp|x_\perp||G(x_1,x_\perp)|<\infty.$$

Note that $G(x)\in L^1(\Omega)$ and $x_\perp^2G(x)\in L^1(\Omega)$ by the assumptions of the lemma, which yields $x_\perp G(x)\in L^1(\Omega)$. Thus when dimension $d=1$ by means of Lemma A1 $\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ |n|<n_0\}}\in L^\infty_{n,p}$ if and only if orthogonality relations (6.25) hold. For $d=2$ the necessary and sufficient conditions for the boundedness of the second term in (6.28) via Lemma A2 are given by (6.26). Lemmas A1 and A2 yield that the third term in (6.28) belongs to $L^\infty_{n,p}$ if and only if conditions (6.27) with the positive sign under the exponents are satisfied. Clearly $x_\perp^2G_n(x_\perp)\in L^1(\mathbb{R}^d)$ due to the assumption of the lemma and estimate (6.24). Similarly, we obtain that the necessary and sufficient conditions for the the last term in (6.28) to be contained in $L^\infty_{n,p}$ are given by (6.27) with the negative sign under the exponents. Then we represent $(p^2+n^2)\xi_n^a(p)$ as the sum

$$(p^2+n^2)\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ p^2+n^2\le n_0^2+1\}}+ \\ (p^2+n^2)\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ p^2+n^2>n_0^2+1\}}$$

in which the absolute value of the first term has the upper bound $(n_0^2+1)\|\xi_n^a(p)\|_{L^\infty_{n,p}}<\infty$ under the orthogonality conditions of the lemma and of the second one $(1+n_0^2)\|\widehat{G}_n(p)\|_{L^\infty_{n,p}}<\infty$ via (6.20). ∎

Finally, we study the case when the parameter a is located on an open interval between the squares of two consecutive nonnegative integers.

Lemma A6. *Let $G(x)\in L^1(\Omega)$, $\quad x_\perp G(x)\in L^1(\Omega)$ and $G(0,x_\perp)=G(2\pi,x_\perp)$ for $x_\perp\in\mathbb{R}^d$ a.e., $d=1,2$ and $n_0^2<a<(n_0+1)^2$, $n_0\in\mathbb{Z}^+=\mathbb{N}\cup\{0\}$. Then $M_a<\infty$ if and only if*

$$\left(G(x_1,x_\perp),\frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{\pm i\sqrt{a-n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)}=0,\quad |n|\le n_0,\quad d=1, \tag{6.29}$$

$$\left(G(x_1,x_\perp),\frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)}=0,\quad p\in S^2_{\sqrt{a-n^2}}\ a.e.,\quad |n|\le n_0,\quad d=2. \tag{6.30}$$

Proof. Let us expand $\xi_n^a(p)$ as the sum of two terms

$$\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ |n|\ge n_0+1\}}+\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ |n|\le n_0\}},$$

such that the absolute value of the first one is bounded above by $\dfrac{\|\widehat{G}_n(p)\|_{L^\infty_{n,p}}}{(n_0+1)^2-a}<\infty$ and the second one belongs to $L^\infty_{n,p}$ if and only if orthogonality relations (6.29) are satisfied in one dimension by means of Lemma A1 and if and only if conditions (6.30) are fulfilled in two dimensions via Lemma A2. We write $(p^2+n^2)\xi_n^a(p)$ as the sum

$$(p^2+n^2)\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ p^2+n^2\ge(n_0+1)^2\}}+ \\ +(p^2+n^2)\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ p^2+n^2<(n_0+1)^2\}}$$

in which the first and the second terms can be easily bounded above in their absolute values by the quantities finite under the conditions of the lemma, namely

$$\left(1+\frac{a}{(n_0+1)^2-a}\right)\|\widehat{G}_n(p)\|_{L^\infty_{n,p}}\quad \textit{and}\quad (n_0+1)^2\|\xi_n^a(p)\|_{L^\infty_{n,p}}$$

respectively. ■

Acknowledgement. *The first author thanks James Colliander for stimulating discussions.*

References

[1] M.S. Agranovich. *Elliptic boundary problems,* Encyclopaedia Math. Sci., vol. 79, Partial Differential Equations, IX, Springer, Berlin, 1997, pp. 1–144.

[2] N. Apreutesei, N. Bessonov, V. Volpert, V. Vougalter: *Spatial Structures and Generalized Travelling Waves for an Integro- Differential Equation,* Discrete Contin. Dyn. Syst. Ser. B 13, 3 (2010), 537–557

[3] H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik: *The non-local Fisher-KPP equation: traveling waves and steady states,* Nonlinearity 22, 12 (2009), 2813–2844

[4] A. Ducrot, M. Marion and V. Volpert: *Systemes de réaction-diffusion sans propriété de Fredholm,* CRAS, 340, 9 (2005), 659–664

[5] A. Ducrot, M. Marion and V. Volpert: *Reaction-diffusion problems with non Fredholm operators,* Advances Diff. Equations , 13, 11-12 (2008), 1151–1192

[6] A. Ducrot, M. Marion and V. Volpert: Reaction-diffusion waves (with the Lewis number different from 1). Publibook, Paris, 2009.

[7] S. Genieys, V. Volpert, P. Auger: *Pattern and waves for a model in population dynamics with nonlocal consumption of resources,* Math. Model. Nat. Phenom. 1, 1 (2006), 63–80

[8] P.D. Hislop, I.M. Sigal: *Introduction to spectral theory with applications to Schrödinger operators.* Springer, 1996

[9] J.L. Lions, E. Magenes. *Problèmes aux limites non homogènes et applications.* Volume 1. Dunod, Paris, 1968.

[10] L.R. Volevich. *Solubility of boundary problems for general elliptic systems,* Mat. Sbor., 68 (1965), 373–416; English translation: Amer. Math. Soc. Transl., 67 (1968), Ser. 2, 182–225.

[11] V. Volpert. *Elliptic partial differential equations.* Volume I. Fredholm theory of elliptic problems in unbounded domains. Birkhäuser, 2011.

[12] V. Volpert, B. Kazmierczak, M. Massot, Z.Peradzynski: *Solvability conditions for elliptic problems with non-Fredholm operators,* Appl. Math., 29, 2 (2002), 219–238

[13] V. Vougalter, V. Volpert: *Solvability conditions for some non Fredholm operators,* Proc. Edinb. Math. Soc. (2), 54, 1 (2011), 249–271

[14] V. Vougalter, V. Volpert: *On the solvability conditions for some non Fredholm operators,* Int. J. Pure Appl. Math., 60, 2 (2010), 169–191

[15] V. Vougalter, V. Volpert: *On the solvability conditions for the diffusion equation with convection terms.* To appear in Commun. Pure Appl. Anal.

[16] V. Vougalter, V. Volpert: *Solvability relations for some non Fredholm operators,* Int. Electron. J. Pure Appl.Math., 2, 1 (2010), 75–83

[17] V. Volpert, V. Vougalter: *On the solvability conditions for a linearized Cahn-Hilliard equation.* To appear in Rend. Istit. Mat. Univ. Trieste.

[18] V. Vougalter, V. Volpert: *Solvability conditions for some systems with non Fredholm operators,* Int. Electron. J. Pure Appl.Math., 2, 3 (2010), 183–187

[19] V. Volpert, V. Vougalter: *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems.* Preprint 2011.

Vitali Vougalter
Department of Mathematics
and Applied Mathematics,
University of Cape Town
Private Bag
Rondebosch 7701
South Africa
Vitali.Vougalter@uct.ac.za

Vitaly Volpert
Institute Camille Jordan
UMR 5208 CNRS
University Lyon 1
Villeurbanne 69622
France
volpert@math.univ-lyon1.fr

Fundamental Group of Schurian Categories and the Hurewicz Isomorphism

Claude Cibils, María Julia Redondo and Andrea Solotar[1]

Received: October 28, 2010
Revised: February 24, 2011

Communicated by Max Karoubi

Abstract. Let k be a field. We attach a CW-complex to any Schurian k-category and we prove that the fundamental group of this CW-complex is isomorphic to the intrinsic fundamental group of the k-category. This extends previous results by J.C. Bustamante in [8]. We also prove that the Hurewicz morphism from the vector space of abelian characters of the fundamental group to the first Hochschild-Mitchell cohomology vector space of the category is an isomorphism.

2010 Mathematics Subject Classification: 55Q05 18D20 16E40 16W25 16W50
Keywords and Phrases: Fundamental group, linear category, Schurian category, Hurewicz, Hochschild

1 Introduction

In this paper we consider Schurian categories, that is, small linear categories over a field k such that each vector space of morphisms is either of dimension one or zero.
Recall that there is no homotopy theory available for a k-algebra or, more generally, for a k-linear category. More precisely there is neither homotopy equivalence nor a definition of loops as in algebraic topology taking into account the k-linear structure. As an alternative we consider an intrinsic fundamental

[1]This work has been supported by the projects UBACYTX212, PIP-CONICET 112-200801-00487, PICT-2007-02182 and MATHAMSUD-NOCOMALRET. The second and third authors are research members of CONICET (Argentina). The third author is a Regular Associate of ICTP Associate Scheme.

group *à la* Grothendieck, that we have defined in [13] and [14] using connected gradings. In [14] we have computed this group for matrix algebras $M_p(k)$ where p is a prime and k is an algebraically closed field of characteristic zero, obtaining that $\pi_1(M_p(k)) = F_{p-1} \times C_p$ where F_{p-1} is the free group with $p-1$ generators and C_p is the cyclic group of order p, using classifications of gradings provided in [2, 4, 23].

The intrinsic fundamental group is defined in terms of Galois coverings provided by connected gradings. It is the automorphism group of the fibre functor over a fixed object. In case a universal covering $\mathcal{U}$ exists for a k-linear category $\mathcal{C}$, its fundamental group $\pi_1(\mathcal{C})$ is isomorphic to the automorphism group of the covering $\mathcal{U}$.

It is intrinsic in the sense that it does not depend on the presentation of the k-category by generators and relations. If a universal covering exists, then we obtain that the fundamental groups constructed by R. Martínez-Villa and J.A. de la Peña (see [27], and [1, 6, 20]) depending on the choice of a presentation of the category by a quiver and relations are in fact quotients of the intrinsic π_1 that we introduce. Note that those groups can vary according to different presentations of the same k-category (see for instance [1, 10, 25]).

The definition of $\pi_1(\mathcal{C})$ is inspired in the topological case considered for instance in R. Douady and A. Douady's book [16]. They are closely related to the way in which the fundamental group is viewed in algebraic geometry after A. Grothendieck and C. Chevalley.

Note that the existence of a universal covering for a k-linear category is equivalent to the existence of a universal grading, namely a connected grading such that any other connected grading is a quotient of it.

In this paper we will prove that a Schurian category $\mathcal{C}$ admits a universal covering. In fact a universal grading is obtained through the topological fundamental group of a CW-complex $CW(\mathcal{C})$ that we attach to $\mathcal{C}$. We infer that $\pi_1(\mathcal{C}) = \pi_1(CW(\mathcal{C}))$. The CW-complex we define is very close to a simplicial complex. It is a simplicial complex when $\mathcal{C}$ is such that if ${}_y\mathcal{C}_x \neq 0$ then ${}_x\mathcal{C}_y = 0$ for $x \neq y$ (where ${}_y\mathcal{C}_x$ is the vector space of morphisms from x to y).

J.C. Bustamante considers in [8] k-categories with a finite number of objects subject to the above conditions which he calls "Schurian almost triangular" in order to prove a similar result through the fundamental group of a presentation as defined in [6, 21, 27]. He uses the simplicial complex from [5, 7, 28] whose 2-skeleton coincides with ours in the Schurian almost triangular context. We do not require that the category has a finite number of objects, neither that it admits an admissible presentation. Moreover we provide an example of a Schurian category which has no admissible presentation and we compute its fundamental group. Note also that we generalize the result by M. Bardzell and E. Marcos in [3], where they prove that the fundamental group of a Schurian basic algebra does not depend on the admissible presentation.

We thank G. Minian for interesting discussions, and in particular for pointing out that cellular approximation enables to provide homotopy arguments from algebraic topology using the 1-skeleton. In [8], J.C. Bustamante uses the edge

path group instead, which requires a finite number of objects. In [9] a CW-complex attached to a presentation of a finite number of objects category is considered in order to compute the fundamental group of the presentation.
In case of a complete Schurian category $\mathcal{C}$, that is all the vector spaces of morphisms are one dimensional and composition of non-zero morphisms is non-zero, the CW-complex attached to $\mathcal{C}$ enables to retrieve that $\pi_1(\mathcal{C}) = 1$, see [14, Corollary 4.6].
Finally we consider the Hurewicz morphism (see [1, 12, 31]) for a Schurian category $\mathcal{C}$. We show that this morphism from the vector space of abelian characters of $\pi_1(\mathcal{C})$ to the first Hochschild-Mitchell cohomology vector space of $\mathcal{C}$ is an isomorphism.
Even though the best understood class of coverings is that of Galois coverings, general non-Galois coverings have also been considered. For instance, in [17, 30], almost Galois coverings and balanced coverings, respectively, have been defined. The approach is different since the focus is to get results in the representation theory of algebras, but they also use gradings, and some notions and results may have a connection with some parts of this paper.
The authors want to thank the referee for the comments and suggestions he/she made, which contributed to improvements in the presentation of the results.

2 Fundamental group

Recall that, given a field k, a k-category is a small category $\mathcal{C}$ such that each morphism set ${}_y\mathcal{C}_x$ from an object $x \in \mathcal{C}_0$ to an object $y \in \mathcal{C}_0$ is a k-vector space, the composition of morphisms is k-bilinear and the k-multiples of the identity at each object are central in its endomorphism ring.
A grading X of a k-category $\mathcal{C}$ by a group Γ_X is a direct sum decomposition

$$ {}_y\mathcal{C}_x = \bigoplus_{s \in \Gamma_X} X^s{}_y\mathcal{C}_x $$

for each $x, y \in \mathcal{C}_0$, where $X^s{}_y\mathcal{C}_x$ is called the homogeneous component of degree s from x to y, such that for $s, t \in \Gamma_X$

$$ X^t{}_z\mathcal{C}_y \ X^s{}_y\mathcal{C}_x \subset X^{ts}{}_z\mathcal{C}_x. $$

In case $f \in X^s{}_y\mathcal{C}_x$ and $f \neq 0$ we write $\deg_X f = s$ and we say that f is homogeneous of degree s.
In order to define a connected grading, we consider virtual morphisms. More precisely, each non-zero morphism f from its source $s(f) = x$ to its target $t(f) = y$ gives rise to a virtual morphism $(f, -1)$ from y to x, and we put $s(f, -1) = y$ and $t(f, -1) = x$. We consider neither zero virtual morphisms nor composition of virtual morphisms. A non-zero usual morphism f is identified with the virtual morphism $(f, 1)$ with the same source and target objects as f.
A walk w in $\mathcal{C}$ is a sequence of virtual morphisms

$$ (f_n, \epsilon_n), \dots, (f_1, \epsilon_1) $$

where $\epsilon_i \in \{+1, -1\}$, such that the target of (f_i, ϵ_i) is the source of $(f_{i+1}, \epsilon_{i+1})$. We put $s(w) = s(f_1, \epsilon_1)$ and $t(w) = t(f_n, \epsilon_n)$.
The category $\mathcal{C}$ is CONNECTED if for any pair of objects (x, y) there exists a walk w from x to y.
A HOMOGENEOUS VIRTUAL MORPHISM is a virtual morphism (f, ϵ) with f homogeneous. We put $\deg_X(f, 1) = \deg_X(f)$ and $\deg_X(f, -1) = \deg_X(f)^{-1}$.
A HOMOGENEOUS WALK w is a walk made of homogeneous virtual morphisms, and its degree is the ordered product of the degrees of the virtual morphisms.
By definition the grading X is CONNECTED if for any pair of objects (x, y) and any group element $s \in \Gamma_X$ there exists a homogeneous walk w from x to y such that $\deg_X w = s$. Hence if a connected grading exists the category is necessarily connected. In case the category $\mathcal{C}$ is already connected, a grading is connected if for a fixed pair of objects (x_0, y_0) there exists a homogeneous walk from x_0 to y_0 of degree s for any $s \in \Gamma_X$, see [13].
In general a linear category does not admit a universal covering. However, in case a universal covering $\mathcal{U}$ exists, according to the theory developed in [13, 14], we have that the intrinsic fundamental group $\pi_1(\mathcal{C})$ is isomorphic to the automorphism group of the universal covering. In this paper we will not provide the general definition of the fundamental group since we will only consider k-categories with a universal covering.

3 CW-complex

Let $\mathcal{C}$ be a connected SCHURIAN k-category, that is a small linear category over a field k such that each vector space of morphisms is either of dimension one or zero. We choose a non-zero morphism ${}_ye_x$ in each one-dimensional space of morphisms ${}_y\mathcal{C}_x$, where ${}_xe_x = {}_x1_x$ is the unit element of the endomorphism algebra of x.
Observe that ${}_xe_y\,{}_ye_x \neq 0$ is equivalent to ${}_ye_x\,{}_xe_y \neq 0$, since if ${}_xe_y\,{}_ye_x = \lambda\,({}_x1_x)$ with $\lambda \in k^*$, then ${}_ye_x\,{}_xe_y \neq 0$ since otherwise ${}_ye_x\,{}_xe_y\,{}_ye_x$ is simultaneously zero and a non-zero multiple of ${}_ye_x$.

DEFINITION 3.1 *The* ASSOCIATED CW-COMPLEX $CW(\mathcal{C})$ *is defined as follows*

- *The* 0*-cells are given by the set of objects* $\mathcal{C}_0$.
- *Each morphism* ${}_ye_x$ *with* $x \neq y$ *gives rise to a* 1*-cell still denoted* ${}_ye_x$ *attached to* x *and* y.
- *If* x, y *and* z *are pairwise distinct objects such that* ${}_y\mathcal{C}_x$, ${}_z\mathcal{C}_y$ *and* ${}_z\mathcal{C}_x$ *are non-zero, and* ${}_ze_y\,{}_ye_x \neq 0$, *we add a* 2*-cell attached to the* 1*-cells* ${}_ye_x$, ${}_ze_y$ *and* ${}_ze_x$.
- *If* x *and* y *are distinct objects such that* ${}_y\mathcal{C}_x$ *and* ${}_x\mathcal{C}_y$ *are non-zero, and* ${}_xe_y\,{}_ye_x \neq 0$ *(equivalently* ${}_ye_x\,{}_xe_y \neq 0$*, as mentioned above), we add exactly one* 2*-cell attached to the* 1*-cells* ${}_ye_x$ *and* ${}_xe_y$.

Remark 3.2 *Note that in case x and y are distinct objects such that ${}_y\mathcal{C}_x \neq 0 \neq {}_x\mathcal{C}_y$, two 1-cells are attached to x and y.*
Observe that in case x, y and z are distinct objects such that ${}_z e_y\ {}_y e_x = 0$, there is no 2-cell attached, even in case ${}_z\mathcal{C}_x \neq 0$.

The associated CW-complex we have just defined has no n-cells for $n \geq 3$, it coincides with its 2-skeleton. We do not need to go further since the fundamental group of any CW-complex coincides with the fundamental group of its 2-skeleton, see for instance [22, Chapter 2].

Example 3.3 *(see [14, Corollary 4.6]) Let $\mathcal{D}^n$ be a* complete Schurian category *with n-objects $1, \ldots, n$: for each pair of objects (x, y), the morphism space $\mathcal{D}^n_x$ is one dimensional with a basis element ${}_y e_x$, where ${}_x e_x = {}_x 1_x$. Composition is defined by ${}_z e_y\ {}_y e_x = {}_z e_x$ for any triple of objects. Note that the direct sum algebra of morphisms for $\mathcal{D}^n$ is the matrix algebra $M_n(k)$.*
We assert that $CW(\mathcal{D}^n)$ is contractible, that is, it has the homotopy type of a point. Note that $CW(\mathcal{D}^2)$ is a disk. For $n \geq 3$ consider the CW-subcomplex $\mathcal{L}^n$ consisting of all 0-cells of $\mathcal{D}^n$ and a chosen 1-cell attached to i and $i+1$ for $i = 1, \ldots, n-1$ (there are no 2-cells in $\mathcal{L}^n$). This CW-subcomplex is closed and contractible. Consequently the quotient $CW(\mathcal{D}^n)/\mathcal{L}^n$ has the same homotopy type than $CW(\mathcal{D}^n)$, see for instance [22, p.11]. Moreover $CW(\mathcal{D}^n)/\mathcal{L}^n$ has only one 0-cell. We assert that each 1-cell not in $\mathcal{L}^n$ is the border of at least one disk in $CW(\mathcal{D}^n)/\mathcal{L}^n$. Indeed, in case of the 1-cell not in $\mathcal{L}^n$ between j and $j+1$, for $j = 1, \ldots, n-1$, the 2-cell attached to the two 1-cells between j and $j+1$ becomes the required disk in the quotient. In case the 1-cell is between j and $j+k$, for $j = 1, \ldots, n-2$ with $k = 2, \ldots, n-j$, the 2-cells given by the triples $(j, j+1, j+2), (j, j+2, j+3), \ldots, (j, j+k-1, j+k)$ provide a disk in the quotient having the original 1-cell as border. Finally there are two 1-cells attached to n and 1, both are not in $\mathcal{L}^n$ and can be identified since a 2-cell is attached to them; they are the border of the disk obtained with the 2-cells $(1, 2, 3), (1, 3, 4), \ldots, (1, n-1, n)$.

Let $w = (f_n, \epsilon_n), \cdots, (f_1, \epsilon_1)$ be a walk in $\mathcal{C}$ from x to y. The inverse walk w^{-1} is the walk $(f_1, -\epsilon_1), \cdots, (f_n, -\epsilon_n)$ from y to x. Note that in case w is a homogeneous walk for a grading X, then

$$\deg_X w^{-1} = (\deg_X w)^{-1}.$$

Let $\mathcal{C}$ be a connected k-category and let X be a grading of $\mathcal{C}$. Let c_0 be an object of $\mathcal{C}$. A set of connector walks is a set of walks $u = \{{}_x u_{c_0}\}_{x \in \mathcal{C}_0}$ where ${}_x u_{c_0}$ goes from c_0 to x, such that $\deg_X {}_x u_{c_0} = 1$ and ${}_{c_0} u_{c_0} = {}_{c_0} 1_{c_0}$. If the grading is connected a set of connector walks exist.
Let $\mathcal{C}$ be a k-category, x an object in $\mathcal{C}$ and let $w = (f_n, \epsilon_n), \cdots, (f_1, \epsilon_1)$ be a closed walk in $\mathcal{C}$ from x to x. In $CW(\mathcal{C})$ there is a loop counterpart to w that we still denote w and that we call the loop described by w which is defined as follows. This loop is obtained as the continuous map from $[0, 1]$

subdivided in n intervals $I_i = [\frac{i-1}{n}, \frac{i}{n}]$, where I_i corresponds to the 1-cell defined by the non-zero morphism f_i corresponding to the virtual one (f_i, ϵ_i) and where $w(\frac{i-1}{n}) = s(f_i, \epsilon_i)$ and $w(\frac{i}{n}) = t(f_i, \epsilon_i)$.

Proposition 3.4 *Let $\mathcal{C}$ be a connected Schurian k-category, let X be a connected grading of $\mathcal{C}$ and let u be a set of connector walks for X for an object c_0. There exists a connected grading $Z_{X,u}$ of $\mathcal{C}$ by the group $\pi_1(CW(\mathcal{C}), c_0)$, where c_0 is considered as a base point of the CW-complex.*

Proof. Let u be a set of connector walks for X and let ${}_ye_x$ be a non-zero morphism of ${}_y\mathcal{C}_x$. We define its $Z_{X,u}$-degree as the homotopy class of the loop described by the walk ${}_yu_{c_0}^{-1}, {}_ye_x, {}_xu_{c_0}$ in $CW(\mathcal{C})$, that is,

$$\deg_{Z_{X,u}} {}_ye_x = [{}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}].$$

In order to prove that this defines a grading, let x, y, z be objects in $\mathcal{C}$. In case ${}_ze_y\ {}_ye_x = 0$ there is nothing to prove. In case ${}_ze_y\ {}_ye_x \neq 0$ we have that

$${}_ze_y\ {}_ye_x = {}_z\lambda_x^y\ {}_ze_x$$

with ${}_z\lambda_x^y$ a non-zero element in k. We have to show that the following equality holds:

$$(\deg_{Z_{X,u}} {}_ze_y)(\deg_{Z_{X,u}} {}_ye_x) = \deg_{Z_{X,u}} {}_ze_x.$$

The left hand side is the following homotopy class

$$\begin{aligned}
[{}_zu_{c_0}^{-1}\ {}_ze_y\ {}_yu_{c_0}][{}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}] &= [{}_zu_{c_0}^{-1}\ {}_ze_y\ {}_yu_{c_0}{}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}] \\
&= [{}_zu_{c_0}^{-1}\ {}_ze_y\ {}_ye_x\ {}_xu_{c_0}].
\end{aligned}$$

Observe that since ${}_ze_{y\,y}e_x$ is a non-zero morphism in $\mathcal{C}$, the CW-complex has a 2-cell attached, which means that the path described by the walk ${}_ze_y, {}_ye_x$ is homotopic to ${}_ze_x$. This observation provides the required result. ◇

Lemma 3.5 *Let $\mathcal{C}$ be a connected Schurian category with a given base object c_0, let X be a connected grading of $\mathcal{C}$ and let $Z_{X,u}$ be the grading considered above by the group $\pi_1(CW(\mathcal{C}), c_0)$. Let w be a closed walk at c_0 in $\mathcal{C}$. Then*

$$\deg_{Z_{X,u}} w = [w] \in \pi_1(CW(\mathcal{C}), c_0)$$

where $[w]$ is the homotopy class of the loop described by w in $CW(\mathcal{C})$.

Proof. Observe first that the degree of a pure virtual morphism $({}_ye_x, -1)$ is the homotopy class $[{}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}]^{-1} = [{}_xu_{c_0}^{-1}\ {}_ye_x^{-1}\ {}_yu_{c_0}]$. Hence the connector walks ${}_xu_{c_0}$ annihilate succesively in $\pi_1(CW(\mathcal{C}), c_0)$, enabling us to obtain the result (recall that ${}_{c_0}u_{c_0} = {}_{c_0}1_{c_0}$). ◇

Proposition 3.6 *Let $\mathcal{C}$ be a connected Schurian k-category and let X be a connected grading. Then the grading $Z_{X,u}$ obtained in Proposition 3.4 is connected.*

Proof. Since $\mathcal{C}$ is connected, it is enough to prove that for any element $[l] \in \pi_1(CW(\mathcal{C}), c_0)$ there exists a closed walk w at c_0 in $\mathcal{C}$ such that $\deg_{Z_{X,u}} w = [l]$. Recall that $[l]$ is a homotopy class, more precisely l is a continuous map

$$[0,1] \to CW(\mathcal{C})$$

such that $l(0) = l(1) = c_0$. We use cellular approximation (see for instance [22, Theorem 4.8]) in order to obtain a homotopic cellular loop l' such that the image of l' is contained in the 1-skeleton. Its image is compact. A compact set in a CW-complex meets only finitely many cells (see for instance [22, Proposition A.1, page 520]). We infer that l is homotopic to a loop l' such that its image is a closed walk w at c_0 in $\mathcal{C}$. The previous Lemma asserts that the $Z_{X,u}$-degree of w is precisely $[l'] = [l]$. $\diamond$

Definition 3.7 *Let X and Z be gradings of a k-category $\mathcal{C}$. We say that X is a* quotient *of Z if there exists a surjective group map*

$$\varphi : \Gamma_Z \to \Gamma_X$$

such that for any pair of objects (x, y) we have that

$$X^s{}_y\mathcal{C}_x = \bigoplus_{\varphi(r)=s} Z^r{}_y\mathcal{C}_x.$$

Theorem 3.8 *Let $\mathcal{C}$ be a connected Schurian k-category and let X be a connected grading of $\mathcal{C}$. Let $Z_{X,u}$ be the connected grading of $\mathcal{C}$ by $\pi_1(CW(\mathcal{C})), c_0)$ defined in the Proof of Proposition 3.4. Then X is a quotient of $Z_{X,u}$ through a unique group map φ.*

Proof. Let $[l]$ be a homotopy class in $\pi_1(CW(\mathcal{C})), c_0)$. As in the previous proof, using cellular approximation we can assume that the image of l is a closed walk w at c_0 in $\mathcal{C}$. In order to define a group morphism

$$\varphi : \pi_1(CW(\mathcal{C})), c_0) \to \Gamma_X$$

we put $\varphi([l]) = \deg_X w$.

In order to check that φ is well defined, we have to prove that $\deg_X w = \deg_X w'$ whenever w and w' are closed walks at c_0 providing homotopic loops in $CW(\mathcal{C})$. Assume first that w and w' only differ by a 2-cell, that is, ${}_ze_y, {}_ye_x$ is part of w, ${}_ze_y\ {}_ye_x \neq 0$ and w' coincide with w except that ${}_ze_y, {}_ye_x$ is replaced by ${}_ze_x$ through the corresponding 2-cell in $CW(\mathcal{C})$. Since $\mathcal{C}$ is Schurian we have that ${}_ze_y\ {}_ye_x$ is a non-zero multiple of ${}_ze_x$. Now since X is a grading

$$\deg_X({}_ze_y\ {}_ye_x) = \deg_X {}_ze_x$$

and $\deg_X w = \deg_X w'$.
For the general case, let h be a homotopy from w to w'. Using again the result in [22, Proposition A.1, page 520], we can assume that the compact image of h meets a finite number of 2-cells. Consequently w and w' only differ by a finite number of 2-cells. By induction we obtain $\deg_X w = \deg_X w'$.
The map is clearly a group morphism. In order to prove that φ is surjective, let $s \in \Gamma_X$. Since X is connected, there exists a closed homogeneous walk w at c_0 of X-degree s. Clearly there is a loop l with image w, hence $\varphi([l]) = s$.
It remains to prove that the homogeneous component of a given X-degree s is the direct sum of the corresponding $Z_{X,u}$-homogeneous components. Observe that since $\mathcal{C}$ is Schurian, the direct sum decomposition is reduced to only one component. Let ${}_ye_x$ be a morphism which has X-degree s. By definition, its $Z_{X,u}$-degree is $[{}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}]$ and we have to prove that $\varphi[{}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}] = s$, that is, $\deg_X({}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}) = s$. The result follows since the connectors ${}_xu_{c_0}$ have trivial X-degree.
Concerning uniqueness, let $\varphi' : \pi_1(CW(\mathcal{C})), c_0) \to \Gamma_X$ be a surjective group map such that for each morphism ${}_ye_x$ we have $\varphi'(\deg_{Z_{X,u}}\ {}_ye_x) = \deg_X\ {}_ye_x$, that is,

$$\varphi'\left(\left[{}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}\right]\right) = \varphi\left(\left[{}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}\right]\right).$$

This shows that φ and φ' coincide on loops of this form. Let now l be an arbitrary loop. In order to prove that $\varphi'([l]) = \varphi([l])$, we first replace l by a cellular approximation in such a way that l describes a walk in $\mathcal{C}$. Clearly any loop at c_0 in $CW(\mathcal{C})$ is homotopic to a product of loops as above and their inverses. We infer that φ and φ' are equal on any loop. $\diamond$

We will prove next that $Z_{X,u}$ depends neither on the choice of the set u nor on the connected grading X. We will consider a slightly more general situation in order to prove these facts.
First recall that a set of connector walks depends on a given grading. In case there is no grading, a set of connector walks means a set of connector walks for the trivial grading by the trivial group. In other words a set of connector walks for a linear category without a given grading is just a choice of a set of walks from a given object c_0 to each object x, where the walk from c_0 to itself is ${}_{c_0}1_{c_0}$.
Let $\mathcal{C}$ be a connected Schurian k-category with a base object c_0 and let u be a set of connector walks. By definition the grading Z_u of $\mathcal{C}$ with group $\pi_1(CW(\mathcal{C}), c_0)$ is given by $\deg_{Z_u}\ {}_ye_x = [{}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}]$. Next we will prove that given two sets of connector walks u, v, the corresponding gradings Z_u and Z_v differ in a simple way that we will call conjugation.

Definition 3.9 *Let X be a grading of a connected k-category $\mathcal{C}$. Let $a = (a_x)_{x\in\mathcal{C}_0}$ be a set of group elements of Γ_X. The* conjugated grading ${}^aX$ *has the same homogeneous components than X but the degree is changed as follows:*

$$\left({}^aX\right)^s\ {}_y\mathcal{C}_x = X^{a_y s a_x^{-1}}\ {}_y\mathcal{C}_x$$

In order to consider morphisms between gradings, they must be understood in the setting of Galois coverings, see [14]. More precisely any grading gives rise to a Galois covering through a smash product construction, see [11]. The Galois coverings obtained by smash products form a full subcategory of the category of Galois coverings. Moreover, both categories are equivalent. Consequently morphisms between gradings are morphisms between the corresponding smash product Galois coverings.
Now, to each grading X of a k-category $\mathcal{C}$ we associate a new k-category $\mathcal{C}\#X$ and a functor $F_X : \mathcal{C}\#X \to \mathcal{C}$ as follows.

$$\begin{array}{rcl}
(\mathcal{C}\#X)_0 & = & \mathcal{C}_0 \times \Gamma_X \\
{}_{(y,t)}(\mathcal{C}\#X)_{(x,s)} & = & X^{t^{-1}s}\,{}_y\mathcal{C}_x \\
F_X(x,s) & = & x \\
F_X & : & {}_{(y,t)}(\mathcal{C}\#X)_{(x,s)} \hookrightarrow {}_y\mathcal{C}_x
\end{array}$$

In particular F_X is a Galois covering and any Galois covering is isomorphic to one of this type. Note that $\mathcal{C}\#X$ is a connected category if and only if the grading X is connected.

Proposition 3.10 *Let $\mathcal{C}$ be a connected k-category and X be a connected grading of $\mathcal{C}$. Let $a = (a_x)_{x\in\mathcal{C}_0}$ be a set of group elements of Γ_X and ${}^aX$ be the conjugated grading. The Galois coverings $\mathcal{C}\#X$ and $\mathcal{C}\#{}^aX$ are isomorphic, more precisely there exists a functor $H : \mathcal{C}\#{}^aX \to \mathcal{C}\#X$ such that $F_XH = F_{{}^aX}$.*

Proof. Recall that $({}^aX)^s\,{}_y\mathcal{C}_x = X^{a_ysa_x^{-1}}\,{}_y\mathcal{C}_x$. Consequently

$${}_{(y,t)}(\mathcal{C}\#{}^aX)_{(x,s)} = ({}^aX)^{t^{-1}s}\,{}_y\mathcal{C}_x = X^{a_yt^{-1}sa_x^{-1}}\,{}_y\mathcal{C}_x = {}_{(y,ta_y^{-1})}(\mathcal{C}\#{}^aX)_{(x,sa_x^{-1})}.$$

This computation shows that defining H on objects by $H(x,s) = (x, sa_x^{-1})$ and by the identity on morphisms provides the required isomorphism. $\diamond$

Proposition 3.11 *Let $\mathcal{C}$ be a connected Schurian k-category, c_0 a base object and X, Y two connected gradings of $\mathcal{C}$. Let $Z_{X,u}$ and $Z_{Y,v}$ be the connected gradings by the group $\pi_1(CW(\mathcal{C}), c_0)$, associated to the sets u and v of homogeneous connector walks for X and Y respectively, given by the choices ${}_xu_{c_0}$ and ${}_xv_{c_0}$ for any $x \in \mathcal{C}_0$. Then $Z_{Y,v}$ and $Z_{X,u}$ are conjugated through the set of group elements $a_x = {}_xu_{c_0}^{-1}\,{}_xv_{c_0}$.*

Proof. Recall that $\deg_{Z_{X,u}}\,{}_ye_x = [{}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}]$, then by definition

$$\begin{array}{rcl}
\deg_{{}^aZ_{X,u}}\,{}_ye_x & = & a_y^{-1}(\deg_{Z_{X,u}}\,{}_ye_x)\ a_x \\
& = & [{}_yv_{c_0}^{-1}\ {}_yu_{c_0}\ {}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}\ {}_xu_{c_0}^{-1}\ {}_xv_{c_0}] \\
& = & \deg_{Z_{Y,v}}\,{}_ye_x.
\end{array}$$

$\diamond$

Remark 3.12 *Since all the gradings $Z_{X,u}$ are isomorphic, we can choose the trivial grading by the trivial group. However we still need to choose connector walks. Moreover we have shown that each connected grading is a unique quotient of the grading by the group $\pi_1(CW(\mathcal{C}), c_0)$.*

Corollary 3.13 *Let $\mathcal{C}$ be a connected Schurian k-category, and let c_0 be a base object. Then*

$$\pi_1(\mathcal{C}, c_0) = \pi_1(CW(\mathcal{C}), c_0).$$

Proof. From [13], we know that in case a universal covering exists, the fundamental group of the category is its group of automorphisms. The results we have proven show that the grading by the fundamental group of $CW(\mathcal{C})$ is a universal grading, consequently the smash product Galois covering is a universal Galois covering with automorphism group $\pi_1(CW(\mathcal{C}), c_0)$. ⋄

Next we compute the intrinsic fundamental group of a k-category with an infinite number of objects and without admissible presentation.

Example 3.14 *Let $\mathcal{C}$ be the k-category given by the quiver:*

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \\
a_1 \arrow[r, "\alpha_1"] \arrow[d, "\beta_1"'] & b_1 \arrow[u] \\
a_0 \arrow[r, "\alpha_0"] \arrow[d, "\beta_0"'] & b_0 \arrow[u, "\gamma_0"'] \\
a_{-1} \arrow[r, "\alpha_{-1}"] \arrow[d] & b_{-1} \arrow[u, "\gamma_{-1}"'] \\
\vdots & \vdots \arrow[u]
\end{tikzcd}$$

with the relations $\gamma_i\alpha_i\beta_{i+1} = \alpha_{i+1}$ for all $i \neq 0$ and $\gamma_0\alpha_0\beta_1 = 0$.
In $CW(\mathcal{C})$ there is a 2-cell attached to each square except the 0-one. Consequently $\pi_1(\mathcal{C}) = \mathbb{Z}$.

4 Hurewicz isomorphism

Let $\mathcal{C}$ be a k-category. A k-derivation d with coefficients in $\mathcal{C}$ is a set of linear morphisms ${}_yd_x : {}_y\mathcal{C}_x \to {}_y\mathcal{C}_x$ for each pair (x, y) of objects, verifying

$${}_zd_x(gf) = {}_zd_y(g)f + g_yd_x(f)$$

for any $f \in {}_y\mathcal{C}_x$ and $g \in {}_z\mathcal{C}_y$.
Let $a = (a_x)_{x\in\mathcal{C}_0}$ be a family of endomorphisms of each object $x \in \mathcal{C}_0$, namely $a_x \in {}_x\mathcal{C}_x$. The INNER DERIVATION d_a associated to a is defined by

$$ {}_y(d_a)_x(f) = a_y f - f a_x. $$

The FIRST HOCHSCHILD-MITCHELL COHOMOLOGY $\mathrm{HH}^1(\mathcal{C})$ is the quotient of the vector space of derivations by the subspace of inner ones (see [29] for the general definition).

REMARK 4.1 *In fact* $\mathrm{HH}^1(\mathcal{C})$ *has a Lie algebra structure, where the bracket of derivations is given by*

$$ {}_y[d,d']_x = {}_yd_x\ {}_yd'_x - {}_yd'_x\ {}_yd_x. $$

DEFINITION 4.2 *Let X be a grading of a k-category $\mathcal{C}$. The* HUREWICZ MORPHISM

$$ h : Hom(\Gamma_X, k^+) \to \mathrm{HH}^1(\mathcal{C}) $$

is defined as follows. Let $\chi : \Gamma_X \to k^+$ be an abelian character and let f be a homogeneous morphism in ${}_y\mathcal{C}_x$. Then

$$ {}_yh(\chi)_x(f) = \chi(\deg_X f)f. $$

An arbitrary morphism is decomposed as a sum of its homogeneous components in order to extend linearly the definition of ${}_yh(\chi)_x$.

REMARK 4.3 *The set $h(\chi)$ is a derivation. This can be verified in a simple way relying on the fact that X is a grading. Derivations of this type are called "Eulerian derivations", see for instance [18, 19].*

The following result is immediate.

LEMMA 4.4 *The image of the Hurewicz morphism is an abelian Lie subalgebra of* $\mathrm{HH}^1(\mathcal{C})$.

We recall that, under some assumptions, the Hurewicz morphism is injective.

PROPOSITION 4.5 *Let $\mathcal{C}$ be a k-category and assume the endomorphism algebra ${}_x\mathcal{C}_x$ of each object x in $\mathcal{C}_0$ is equal to k. Let X be a connected grading of $\mathcal{C}$. Then the Hurewicz morphism is injective.*

PROOF. If $h(\chi)$ is an inner derivation,

$$ {}_{t(f)}h(\chi)_{s(f)}(f) = \chi(\deg_X f)f = a_{t(f)}f - f a_{s(f)} $$

for any homogeneous non-zero morphism f, where $(a_x)_{x\in\mathcal{C}_0}$ is a set of endomorphisms which are elements of k by hypothesis. Then $\chi(\deg_X f) = a_{t(f)} - a_{s(f)}$.

Now we assert that the same equality holds for any homogeneous walk w, that is,

$$\chi(\deg_X w) = a_{t(w)} - a_{s(w)}.$$

For instance let $w = (g,-1),(f,1)$ be a homogeneous walk where $f \in {}_y\mathcal{C}_x$ and $g \in {}_y\mathcal{C}_z$. Then

$$\begin{aligned}\chi(\deg_X w) &= \chi((\deg_X g)^{-1}(\deg_X f)) = -\chi(\deg_X g) + \chi(\deg_X f)\\ &= a_{s(g)} - a_{t(g)} + a_{t(f)} - a_{s(f)}\\ &= a_z - a_y + a_y - a_x = a_z - a_x = a_{t(w)} - a_{s(w)}.\end{aligned}$$

Let c_0 be any fixed object of $\mathcal{C}$. Since X is a connected grading, for any group element $s \in \Gamma_X$ there exists a homogeneous walk w, closed at c_0, such that $\deg_X w = s$. Consequently

$$\chi(s)w = (a_{c_0} - a_{c_0})w = 0$$

hence $\chi(s) = 0$ for any $s \in \Gamma_X$. $\diamond$

Theorem 4.6 *Let $\mathcal{C}$ be a connected Schurian k-category and let U be its universal grading by the fundamental group $\pi_1(CW(\mathcal{C}), c_0)$. The corresponding Hurewicz morphism is an isomorphism.*

Proof. The previous result insures that h is injective. In order to prove that h is surjective, let d be a derivation. We choose a non-zero morphism ${}_ye_x$ in each 1-dimensional space of morphisms ${}_y\mathcal{C}_x$, with ${}_xe_x = {}_x1_x$. Let c_0 be a fixed object in $\mathcal{C}$. To describe the universal grading, recall that we choose a set of connector walks, hence

$$\deg_U {}_ye_x = [{}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}] \in \pi_1(CW(\mathcal{C}), c_0).$$

Since ${}_y\mathcal{C}_x$ is one dimensional, $d({}_ye_x) = {}_y\lambda_x\ {}_ye_x$ with ${}_y\lambda_x \in k$. In order to define an abelian character χ such that $h(\chi) = d$, let l be a loop at c_0 in $CW(\mathcal{C})$. By cellular approximation we can assume that the image of l is a closed walk w in $\mathcal{C}$. In case w is of the form ${}_yu_{c_0}^{-1}\ {}_ye_x\ {}_xu_{c_0}$ we define $\chi[l] = {}_y\lambda_x$. Otherwise the cellular loop w is homotopic to a product of loops of the previous type or of their inverses, and we define $\chi[l]$ to be the corresponding sum of scalars. We have to verify that χ is well defined. First observe that if ${}_ze_y\ {}_ye_x \neq 0$, the scalars of the derivation d verify

$${}_z\lambda_x = {}_z\lambda_y + {}_y\lambda_x.$$

Indeed, ${}_ze_y\ {}_ye_x = \mu\ {}_ze_x$, with $\mu \neq 0$, hence

$$\begin{aligned}d({}_ze_y\ {}_ye_x) &= {}_ze_y\ d({}_ye_x) + d({}_ze_y)\ {}_ye_x\\ &= \mu\ ({}_z\lambda_y + {}_y\lambda_x)\ {}_ze_x.\end{aligned}$$

We deduce the result since $d(\mu\ {}_z e_x) = \mu\ {}_z\lambda_x\ {}_z e_x$. Consider now two cellular loops l and l' which are homotopic by a 2-cell, meaning that a walk ${}_z e_y, {}_y e_x$ is replaced by ${}_z e_x$. The previous computation shows that $\chi[l] = \chi[l']$. We have already verified that any homotopy of cellular loops decomposes as a finite number of homotopies of the previous type, hence we deduce that χ is a well defined map. By construction $\chi : \pi_1(CW(\mathcal{C}), c_0) \to k^+$ is an abelian character and clearly $h(\chi) = d$. $\diamond$

References

[1] Assem, I.; de la Peña, J. A. The fundamental groups of a triangular algebra. Comm. Algebra 24 (1996), 187–208.

[2] Bahturin, Y.; Zaicev, M. Group gradings on matrix algebras, Canad. Math. Bull. 45 (2002), no. 4, 499–508.

[3] Bardzell, M. J.; Marcos, E. N. H^1 and presentations of finite dimensional algebras. Representations of algebras (São Paulo, 1999), 31–38, Lecture Notes in Pure and Appl. Math., 224, Dekker, New York, 2002.

[4] Boboc, C. ; Dăscălescu, S. Group gradings on $M_3(k)$, Comm. Algebra 35 (2007), 2654ï¿½ï¿½-2670.

[5] Bongartz, K.; A criterion for finite representation type. Math. Ann. 269 (1984), no. 1, 1–12.

[6] Bongartz, K.; Gabriel, P. Covering spaces in representation-theory, Invent. Math. 65 (1981/82) 331–378.

[7] Bretscher, O.; Gabriel, P. The standard form of a representation-finite algebra. Bull. Soc. Math. France 111 (1983), no. 1, 21–40.

[8] Bustamante, J.C. On the fundamental group of a Schurian algebra. Comm. Algebra 30 (2002), no. 11, 5307–5329.

[9] Bustamante, J.C. The classifying space of a bound quiver. J. Algebra 277 (2004), no. 2, 431–455.

[10] Bustamante, J.C.; Castonguay, D. Fundamental groups and presentations of algebras. J. Algebra Appl. 5 (2006), 549–562.

[11] Cibils, C.; Marcos, E. Skew category, Galois covering and smash product of a category over a ring. Proc. Amer. Math. Soc. 134, (2006), no. 1, 39–50.

[12] Cibils, C.; Redondo M.J. Cartan-Leray spectral sequence for Galois coverings of categories. J. Algebra 284 (2005), 310–325.

[13] Cibils, C.; Redondo M.J.; Solotar, A. The intrinsic fundamental group of a linear category. To appear in Algebras and Representation Theory. DOI: 10.1007/s10468-010-9263-1.

[14] Cibils, C.; Redondo M.J.; Solotar, A. Connected gradings and fundamental group. Algebra Number Theory 4 (2010), no. 5, 625–648.

[15] Cibils,C.; Solotar, A. Galois coverings, Morita equivalence and smash extensions of categories over a field. Doc. Math. 11 (2006), 143–159.

[16] Douady, R.; Douady, A. Algèbre et théories galoisiennes. Paris: Cassini. 448 p. (2005).

[17] Dowbor, P; Hajduk,A. On some nice class of non-Galois coverings for algebras, Bol. Soc. Mat. Mex. 39 (in press).

[18] Farkas, D.; Green, E.L.; Marcos, E.N. Diagonalizable derivations of finite-dimensional algebras. II. Pacific J. Math. 196 (2000), no. 2, 341–351.

[19] Farkas, D.R.; Geiss, Ch.; Green, E.L.; Marcos, E.N. Diagonalizable derivations of finite-dimensional algebras. I. Israel J. Math. 117 (2000), 157–181.

[20] Gabriel, P. The universal cover of a representation-finite algebra. Representations of algebras (Puebla, 1980), 68–105, Lecture Notes in Math. 903, Springer, Berlin-New York, 1981.

[21] Green, Edward L. Graphs with relations, coverings and group-graded algebras. Trans. Amer. Math. Soc. 279 (1983), no. 1, 297–310.

[22] Hatcher, A. Algebraic topology. Cambridge University Press, Cambridge, 2002. xii+544 pp.

[23] Khazal, R.; Boboc, C.; Dăscălescu, S. Group gradings of $M_2(K)$, Bull. Austral. Math. Soc. 68 (2003), no. 2, 285–293.

[24] Le Meur, P. The universal cover of an algebra without double bypass, J. Algebra 312 (2007), no. 1, 330–353.

[25] Le Meur, P. The fundamental group of a triangular algebra without double bypasses. C. R. Math. Acad. Sci. Paris 341 (2005), 211–216.

[26] Le Meur, P. Revêtements galoisiens et groupe fondamental d'algèbres de dimension finie. Ph.D. thesis, Université Montpellier 2 (2006). http://tel.archives-ouvertes.fr/tel-00011753

[27] Martínez-Villa, R.; de la Peña, J. A. The universal cover of a quiver with relations. J. Pure Appl. Algebra 30 (1983), 277–292.

[28] Martins, Ma. I. R.; de la Peña, J. A. Comparing the simplicial and the Hochschild cohomologies of a finite-dimensional algebra. J. Pure Appl. Algebra 138 (1999), no. 1, 45–58.

[29] Mitchell, B. Rings with several objects. Adv. Math. 8 (1972), 1–161.

[30] de la Peña, J.A.; Redondo, M.J. Coverings without groups. J. Algebra 321 (2009), 3816–3826.

[31] de la Peña, J.A.; Saorín, M. On the first Hochschild cohomology group of an algebra. Manuscripta Math. 104 (2001), 431–442.

[32] Reynaud, E. Algebraic fundamental group and simplicial complexes. J. Pure Appl. Algebra 177 (2003), no. 2, 203–214.

Claude Cibils
Institut de mathématiques et de
modélisation de Montpellier I3M
UMR 5149
Université Montpellier 2
F-34095 Montpellier cedex 5
France
Claude.Cibils@math.univ-montp2.fr

María Julia Redondo
Departamento de Matemática
Universidad Nacional del Sur
Av. Alem 1253
8000 Bahía Blanca
Argentina
mredondo@criba.edu.ar

Andrea Solotar
Departamento de Matemática
Facultad de Ciencias Exactas
y Naturales
Universidad de Buenos Aires
Ciudad Universitaria, Pabellón 1
1428, Buenos Aires
Argentina
asolotar@dm.uba.ar

Characterising Weak-Operator Continuous Linear Functionals on $\mathcal{B}(H)$ constructively

Douglas S. Bridges

Received: February 7, 2011

Communicated by Anil Nerode

Abstract. Let $\mathcal{B}(H)$ be the space of bounded operators on a not-necessarily-separable Hilbert space H. Working within Bishop-style constructive analysis, we prove that certain weak-operator continuous linear functionals on $\mathcal{B}(H)$ are finite sums of functionals of the form $T \rightsquigarrow \langle Tx, y\rangle$. We also prove that the identification of weak- and strong-operator continuous linear functionals on $\mathcal{B}(H)$ cannot be established constructively.

2010 Mathematics Subject Classification: 03F60, 47L50, 46S30
Keywords and Phrases: Constructive, operators, (ultra)weak operator topology, continuous functionals

1 Introduction

Let H be a complex Hilbert space that is nontrivial (that is, contains a unit vector), $\mathcal{B}(H)$ the space of all bounded operators on H, and $\mathcal{B}_1(H)$ the unit ball of $\mathcal{B}(H)$. In this paper we carry out, within Bishop-style constructive mathematics (BISH),[1] an investigation of weak-operator continuous linear functionals on $\mathcal{B}(H)$.

Depending on the context, we use, for example, $\mathbf{x}$ to represent either the element $(x_1, \ldots, x_N)$ of the finite direct sum $H_N \equiv \bigoplus_{n=1}^{N} H$ of N copies of H or else the element $(x_n)_{n\geq 1}$ of the direct sum $H_\infty \equiv \bigoplus_{n\geq 1} H$ of a sequence of copies of H. We use I to denote the identity projection on H.

The following are the topologies of interest to us here.

[1] That is, mathematics that uses only intuitionistic logic and is based on a suitable set- or type-theoretic foundation [1, 2, 12]. For more on BISH see [3, 4, 8].

- ▷ The weak operator topology: the weakest topology on $\mathcal{B}(H)$ with respect to which the mapping $T \rightsquigarrow \langle Tx, y\rangle$ is continuous for all $x, y \in H$; sets of the form

$$\{T \in \mathcal{B}(H) : |\langle Tx, y\rangle| < \varepsilon\},$$

with $x, y \in H$ and $\varepsilon > 0$, form a sub-base of weak-operator neighbourhoods of 0 in $\mathcal{B}(H)$.

- ▷ The ultraweak operator topology: the weakest topology on $\mathcal{B}(H)$ with respect to which the mapping $T \rightsquigarrow \sum_{n=1}^{\infty} \langle Tx_n, y_n\rangle$ is continuous for all $\mathbf{x}, \mathbf{y} \in H_\infty$; sets of the form

$$\left\{T \in \mathcal{B}(H) : \left|\sum_{n=1}^{\infty} \langle Tx_n, y_n\rangle\right| < \varepsilon\right\},$$

with $\mathbf{x}, \mathbf{y} \in H_\infty$ and $\varepsilon > 0$, form a sub-base of ultraweak-operator neighbourhoods of 0 in $\mathcal{B}(H)$.

These topologies are induced, respectively, by the seminorms of the form $T \rightsquigarrow |\langle Tx, y\rangle|$ with $x, y \in H$, and those of the form $T \rightsquigarrow |\sum_{n=1}^{\infty} \langle Tx_n, y_n\rangle|$ with $\mathbf{x}, \mathbf{y} \in H_\infty$.
An important theorem in classical operator algebra theory states that the weak-operator continuous linear functionals on (any linear subspace of) $\mathcal{B}(H)$ all have the form $T \rightsquigarrow \sum_{n=1}^{N} \langle Tx_n, y_n\rangle$ with $\mathbf{x}, \mathbf{y} \in H_N$ for some N; and the ultraweak-operator continuous linear functionals on $\mathcal{B}(H)$ have the form $T \rightsquigarrow \sum_{n=1}^{\infty} \langle Tx_n, y_n\rangle$, where $\mathbf{x}, \mathbf{y} \in H_\infty$. However, the classical proofs, such as those found in [10, 11, 14], depend on applications of nonconstructive versions of the Hahn-Banach theorem, the Riesz representation theorem, and polar decomposition.
The foregoing characterisation of ultraweak-operator continuous functionals was derived constructively, when H is separable, in [9].[2] A variant of it was derived in [8] (Proposition 5.4.16) without the requirement of separability, and using not the standard ultraweak operator topology but one that is classically, though not constructively, equivalent to it. Our aim in the present work is to provide a constructive proof of the standard classical characterisation of weak-operator continuous linear functionals (Theorem 10) on $\mathcal{B}(H)$, without the requirement of separability but with one hypothesis in addition to the classical ones. In presenting this work, we emphasise that, in contrast to their classical counterparts, our proofs contain extractable, implementable algorithms for the desired representation of weak-operator continuous linear functionals; moreover, the constructive proofs themselves verify that those algorithms meet their specifications.

[2]The characterisation was derived by Spitters in the case where H is separable and the subspace is $\mathcal{B}(H)$ itself ([15],Theorem 5); but his proof uses Brouwer's continuity principle and so is intuitionistic, rather than in the style of Bishop.

2 Preliminary Lemmas

The proof of our main theorem depends on a sequence of (at-times-complicated) lemmas. For the first one, we remind the reader of two elementary definitions in constructive analysis: we say that an inhabited set S—that is, one in which we can construct an element—is FINITELY ENUMERABLE if there exist a positive integer N and a mapping of $\{1,\ldots,N\}$ onto S; if that mapping is one-one, then S is called FINITE.

Lemma 1 *If u is a weak-operator continuous linear functional on $\mathcal{B}(H)$, then there exist a finitely enumerable subset F of $H\times H$ and a positive number C such that $|u(T)|\leq C\sum_{(x,y)\in F}|\langle Tx,y\rangle|$ for all $T\in\mathcal{B}(H)$.*

Proof. This is an immediate consequence of Proposition 5.4.1 in [8]. ■

We shall need some information about locally convex spaces. Let $(p_j)_{j\in J}$ be a family of seminorms defining the topology on a locally convex linear space V, and let A be a subset of V. A subset S of A is said to be LOCATED (in A) if

$$\inf\left\{\sum_{j\in F}p_j(x-s):s\in S\right\}$$

exists for each $x\in A$ and each finitely enumerable subset F of J. We say that A is TOTALLY BOUNDED if for each finitely enumerable subset F of J and each $\varepsilon>0$, there exists a finitely enumerable subset S of A—called an ε-APPROXIMATION TO S RELATIVE TO $(p_j)_{j\in F}$—such that for each $x\in A$ there exists $s\in S$ with $\sum_{j\in F}p_j(x-s)<\varepsilon$.

The unit ball $\mathcal{B}_1(H)$ is weak-operator totally bounded ([8], Proposition 5.4.15); but, in contrast to the classical situation, it cannot be proved constructively that $\mathcal{B}_1(H)$ is weak-operator complete [5].

A mapping f between locally convex spaces $\left(X,(p_j)_{j\in J}\right)$ and $\left(Y,(q_k)_{k\in K}\right)$ is UNIFORMLY CONTINUOUS on a subset S of X if for each $\varepsilon>0$ and each finitely enumerable subset G of K, there exist $\delta>0$ and a finitely enumerable subset F of J such that if $x,x'\in S$ and $\sum_{j\in F}p_j(x-x')<\delta$, then $\sum_{k\in G}q_k\left(f(x)-f(x')\right)<\varepsilon$.

We recall four facts about total boundedness, locatedness, and uniform continuity in a locally convex space V. The proofs are found on pages 129–130 of [8].

- ▷ If f is a uniformly continuous mapping of a totally bounded subset A of V into a locally convex space, then $f(A)$ is totally bounded.
- ▷ If f is a uniformly continuous, real-valued mapping on a totally bounded subset A of V, then $\sup_{x\in A}f(x)$ and $\inf_{x\in A}f(x)$ exist.
- ▷ A totally bounded subset of V is located in V.

▷ If $A \subset V$ is totally bounded and $S \subset A$ is located in A, then S is totally bounded.

We remind the reader that a bounded linear mapping $T : X \to Y$ between normed linear spaces is NORMED if its NORM,

$$\|T\| \equiv \sup \{\|Tx\| : x \in X, \|x\| \leq 1\},$$

exists. If X is finite-dimensional, then $\|T\|$ exists; but the statement 'Every bounded linear functional on an infinite-dimensional Hilbert space is normed'[3] is essentially nonconstructive.

LEMMA 2 *Every weak-operator continuous linear functional on $\mathcal{B}(H)$ is normed.*

PROOF. This follows from observations made above, since, in view of Lemma 1, the linear functional is weak-operator uniformly continuous on the weak-operator totally bounded set $\mathcal{B}_1(H)$. ∎

We note the following stronger form of Lemma 1.

LEMMA 3 *Let u be a weak-operator continuous linear functional on $\mathcal{B}(H)$. Then there exist $\delta > 0$, and finitely many nonzero*[4] *elements $\xi_1, \ldots, \xi_N$ and $\zeta_1, \ldots, \zeta_N$ of H with $\sum_{n=1}^{N} \|\xi_n\|^2 = \sum_{n=1}^{N} \|\zeta_n\|^2 = 1$, such that $|u(T)| \leq \delta \sum_{n=1}^{N} |\langle T\xi_n, \zeta_n\rangle|$ for each $T \in \mathcal{B}(H)$.*

PROOF. By Lemma 1, there exist a positive integer ν,$C > 0$, and vectors $\mathbf{x}, \mathbf{y} \in H_\nu$ such that $|u(T)| \leq C \sum_{n=1}^{\nu} |\langle Tx_n, y_n\rangle|$ for all $T \in \mathcal{B}(H)$.[5] For each $n \leq \nu$, construct nonzero vectors x_n', y_n' such that $x_n' \neq x_n$ and $y_n' \neq y_n$. The desired result follows from the inequality

$$\begin{aligned}\sum_{n=1}^{\nu} |\langle Tx_n, y_n\rangle| \leq \sum_{n=1}^{\nu} |\langle T(x_n - x_n'), y_n - y_n'\rangle| + \sum_{n=1}^{\nu} |\langle Tx_n', y_n - y_n'\rangle| \\ + \sum_{n=1}^{\nu} |\langle T(x_n - x_n'), y_n'\rangle| + \sum_{n=1}^{\nu} |\langle Tx_n', y_n'\rangle|\end{aligned}$$

the fact that each of the vectors $x_n', x_n - x_n', y_n'$, and $y_n - y_n'$ is nonzero, and scaling to get the desired norm sums equal to 1 and then the positive δ. ∎

The next lemma will be used in an application of the separation theorem in the proof of Lemma 6.

[3]In fact, a nonzero linear functional on a normed space is normed if and only its kernel is located ([8], Proposition 2.3.6).

[4]A vector in a locally convex space is NONZERO if it is mapped to a positive number by at least one seminorm.

[5]At this stage, it is trivial to prove Lemma 3 classically by simply deleting terms $\langle Tx_n, y_n\rangle$ when either x_n or y_n is 0. With intuitionistic logic we need to work a little harder, because we cannot generally decide whether a given vector in H is, or is not, equal to 0.

Lemma 4 *Let $\zeta_1, \ldots, \zeta_N$ be elements of H with $\sum_{n=1}^{N} \|\zeta_n\|^2 = 1$. Let K be a finite-dimensional subspace of H_N, and let $\| \, \|^*$ be the standard norm on the dual space K^* of K:*

$$\|f\|^* = \sup\{|f(\mathbf{x})| : \mathbf{x} \in K, \|\mathbf{x}\| \leq 1\} \quad (f \in K^*).$$

Define a mapping F of $\mathcal{B}(H)$ into $\left(K^*, \| \, \|^*\right)$ by

$$F(T)(\mathbf{x}) \equiv \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle \quad (\mathbf{x} \in K).$$

Then F is weak-operator uniformly continuous on $\mathcal{B}_1(H)$.

Proof. Given $\varepsilon > 0$, let $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be an ε-approximation to the (compact) unit ball of K. Writing $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,N})$, consider $S, T \in \mathcal{B}_1(H)$ with

$$\sum_{i=1}^{m} \sum_{n=1}^{N} |\langle (S - T)\, x_{i,n}, \zeta_n \rangle| < \varepsilon.$$

For each $\mathbf{x}$ in the unit ball of K, there exists i such that $\|\mathbf{x} - \mathbf{x}_i\| < \varepsilon$. We compute

$$\begin{aligned}
|F(S)(\mathbf{x}) - F(T)(\mathbf{x})| &\leq |F(S)(\mathbf{x}) - F(S)(\mathbf{x}_i)| + |F(S)(\mathbf{x}_i) - F(T)(\mathbf{x}_i)| \\
&\quad + |F(T)(\mathbf{x}) - F(T)(\mathbf{x}_i)| \\
&\leq \sum_{n=1}^{N} |\langle S\,(x_n - x_{i,n})\,, \zeta_n \rangle| + \sum_{n=1}^{N} |\langle (S - T)\, x_{i,n}, \zeta_n \rangle| \\
&\quad + \sum_{n=1}^{N} |\langle T\,(x_n - x_{i,n})\,, \zeta_n \rangle| \\
&\leq 2 \sum_{n=1}^{N} \|x_n - x_{i,n}\| \, \|\zeta_n\| + \varepsilon \\
&\leq 2 \, \|\mathbf{x} - \mathbf{x}_i\| \, \|\zeta\| + \varepsilon < 3\varepsilon.
\end{aligned}$$

Hence $\|F(S) - F(T)\|^* \leq 3\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that F is uniformly continuous on $\mathcal{B}_1(H)$. ■

In order to ensure that the unit kernel $\mathcal{B}_1(H) \cap \ker u$ of a weak-operator continuous linear functional u on $\mathcal{B}(H)$ is weak-operator totally bounded, and hence weak-operator located, we derive a generalisation of Lemma 5.4.9 of [8].

Lemma 5 *Let $\left(V, (p_j)_{j \in J}\right)$ be a locally convex space. Let V_1 be a balanced, convex, and totally bounded subset of V. Let u be a linear functional on V that, on V_1, is both uniformly continuous and nonzero. Then $V_1 \cap \ker u$ is totally bounded.*

PROOF. Since u is nonzero and uniformly continuous on the totally bounded set V_1,

$$C = \sup\{|u(y)| : y \in V_1\}$$

exists and is positive. Choose y_1 in V_1 such that $u(y_1) > C/2$. Then

$$y_0 \equiv \frac{C}{2u(y_1)} y_1$$

belongs to the balanced set V_1, and $u(y_0) = C/2$. Let $\varepsilon > 0$, and let F be a finitely enumerable subset of J. Since each p_j is uniformly continuous on V, it maps the totally bounded set V_1 onto a totally bounded subset of $\mathbf{R}$.[6] Hence there exists $b > 0$ such that $p_j(x) \leq b$ for each $j \in F$ and each $x \in V_1$. Using Theorem 5.4.6 of [8], compute t with

$$0 < t < \frac{C\varepsilon}{C + 4b}$$

such that

$$S_t = \{y \in V_1 : |u(y)| \leq t\}$$

is totally bounded. Pick a t-approximation $\{s_1, \ldots, s_n\}$ of S_t relative to $(p_j)_{j\in F}$, and set

$$y_k = \frac{C}{C+2t} s_k - \frac{2}{C+2t} u(s_k) y_0 \quad (1 \leq k \leq n).$$

Then $y_k \in \ker(u)$. Since $|u(s_k)| \leq t$ and V_1 is balanced,

$$\frac{-u(s_k)}{t} y_0 \in V_1.$$

Thus

$$y_k = \frac{C}{C+2t} s_k + \left(1 - \frac{C}{C+2t}\right)\left(\frac{-u(s_k)}{t} y_0\right) \in V_1.$$

[6] We use $\mathbf{R}$ and $\mathbf{C}$ for the sets of real and complex numbers, respectively.

Now consider any element y of $V_1 \cap \ker(u)$. Since $y \in S_t$, there exists k such that $\sum_{j \in F} p_j(y - s_k) < t$ and therefore

$$\begin{aligned}
\sum_{j \in F} p_j(y - y_k) &\leq \sum_{j \in F} p_j(y - s_k) + \sum_{j \in F} p_j(s_k - y_k) \\
&< t + \frac{2}{C + 2t} \sum_{j \in F} p_j(ts_k + u(s_k)y_0) \\
&\leq t + \frac{2}{C + 2t} \sum_{j \in F} (tp_j(s_k) + u(s_k)p_j(y_0)) \\
&\leq t + \frac{2t}{C} \sum_{j \in F} (p_j(s_k) + p_j(y_0)) \\
&\leq t \left(1 + \frac{4b}{C}\right) < \varepsilon.
\end{aligned}$$

Thus $\{y_1, \ldots, y_n\}$ is a finitely enumerable ε-approximation to $V_1 \cap \ker(u)$ relative to the family $(p_j)_{j \in F}$ of seminorms. ■

The next lemma, the most complicated in the paper, extracts much of the sting from the proof of our main theorem by showing how to find finitely many mappings of the form $T \rightsquigarrow \langle Tx, \zeta \rangle$ whose sum is small on the unit kernel of u.

Lemma 6 *Let u be a nonzero weak-operator continuous linear functional on $\mathcal{B}(H)$. Let δ be a positive number, and $\xi_1, \ldots, \xi_N$ and $\zeta_1, \ldots, \zeta_N$ nonzero elements of H, such that*[7]

$$\sum_{n=1}^{N} \|\xi_n\|^2 = \sum_{n=1}^{N} \|\zeta_n\|^2 = 1,$$

and

$$|u(T)| \leq \delta \sum_{n=1}^{N} |\langle T\xi_n, \zeta_n \rangle| \quad (T \in \mathcal{B}(H)). \tag{1}$$

Then for each $\varepsilon > 0$, there exists a unit vector $\mathbf{x}$ in the subspace

$$K \equiv \mathbf{C}\xi_1 \times \mathbf{C}\xi_2 \times \cdots \times \mathbf{C}\xi_N$$

of H_N, such that $x_n \neq 0$ for $1 \leq n \leq N$ and $\left|\sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle\right| < \varepsilon$ for all $T \in \mathcal{B}_1(H) \cap \ker u$.

Proof. First note that since each ξ_n is nonzero, K is an N-dimensional subspace of H_N. Now, an application of Lemma 5 tells us that the unit kernel

[7] Such ξ_k, ζ_k, and δ exist, by Lemma 3.

$\mathcal{B}_1(H) \cap \ker u$ of u is weak-operator totally bounded. For each $\mathbf{x} \in H_N$, since the mapping $T \rightsquigarrow \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle$ is weak-operator uniformly continuous on the unit kernel, we see that

$$\|\mathbf{x}\|_0 = \sup \left\{ \left| \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle \right| : T \in \mathcal{B}_1(H) \cap \ker u \right\}$$

exists. The mapping $\mathbf{x} \rightsquigarrow \|\mathbf{x}\|_0$ is a seminorm on H_N satisfying $\|\mathbf{x}\|_0 \leq \|\zeta\| \|\mathbf{x}\| = \|\mathbf{x}\|$; whence the identity mapping from $(H_N, \|\ \|)$ to $(H_N, \|\ \|_0)$ is uniformly continuous. Since the subset

$$\{\mathbf{x} \in K : \|\mathbf{x}\| = 1\}$$

of the finite-dimensional Banach space $(K, \|\ \|)$ is totally bounded, it follows that

$$\beta \equiv \inf \{\|\mathbf{x}\|_0 : \mathbf{x} \in K, \|\mathbf{x}\| = 1\},$$

exists. It will suffice to prove that $\beta = 0$. For then, given ε with $0 < \varepsilon < 1$, we can construct a unit vector $\mathbf{x}' \in K$ such that $\left| \sum_{n=1}^{N} \langle Tx'_n, \zeta_n \rangle \right| < \varepsilon/2$ for all $T \in \mathcal{B}_1(H) \cap \ker u$. Picking nonzero vectors $y_n \in \mathbf{C}\xi_n$ such that $\left(\sum_{n=1}^{N} \|x'_n - y_n\|^2 \right)^{1/2} < \varepsilon/8$, we have

$$\left| 1 - \left(\sum_{n=1}^{N} \|y_n\|^2 \right)^{1/2} \right| < \frac{\varepsilon}{8},$$

so

$$\mathbf{x} \equiv \left(\sum_{n=1}^{N} \|y_n\|^2 \right)^{-1/2} \mathbf{y}$$

is a unit vector in $\mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$ with each $x_n \neq 0$. Moreover,

$$\begin{aligned}
\|\mathbf{x} - \mathbf{y}\|^2 &= \sum_{k=1}^{N} \left| \left(\sum_{n=1}^{N} \|y_n\|^2 \right)^{-1/2} - 1 \right|^2 \|y_k\|^2 \\
&\leq \left(\frac{\varepsilon}{8} \right)^2 \sum_{k=1}^{N} \|y_k\|^2 \\
&\leq \frac{\varepsilon^2}{64} \left(1 + \frac{\varepsilon}{8} \right)^2 < \frac{\varepsilon^2}{16},
\end{aligned}$$

so for each $T \in \mathcal{B}_1(H) \cap \ker u$,

$$\begin{aligned}\left|\sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle\right| &\leq \left|\sum_{n=1}^{N} \langle Tx'_n, \zeta_n \rangle\right| + \sum_{n=1}^{N} \left|\langle T\left(x_n - x'_n\right), \zeta_n \rangle\right| \\ &\leq \frac{\varepsilon}{2} + \sum_{n=1}^{N} \|x_n - x'_n\| \, \|\zeta_n\| \\ &\leq \frac{\varepsilon}{2} + \|\mathbf{x} - \mathbf{x}'\| \, \|\zeta\| \\ &\leq \frac{\varepsilon}{2} + \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{x}'\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} < \varepsilon.\end{aligned}$$

To prove that $\beta = 0$, we suppose that $\beta > 0$. Then $\| \, \|_0$ is a norm equivalent to the original norm on K, so $(K, \| \, \|_0)$ is an N-dimensional Banach space. Define norms $\| \, \|^*$ and $\| \, \|_0^*$ on the dual K^* of K by

$$\begin{aligned}\|f\|^* &\equiv \sup\left\{|f(\mathbf{x})| : \mathbf{x} \in K, \|\mathbf{x}\| \leq 1\right\}, \\ \|f\|_0^* &\equiv \sup\left\{|f(\mathbf{x})| : \mathbf{x} \in K, \|\mathbf{x}\|_0 \leq 1\right\}.\end{aligned}$$

For each $T \in \mathcal{B}(H)$ and each $\mathbf{x} \in K$ let

$$F(T)(\mathbf{x}) \equiv \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle .$$

Then, by Lemma 4, F is weak-operator uniformly continuous as a mapping of $\mathcal{B}_1(H)$ into $\left(K^*, \| \, \|^*\right)$; since the norms $\| \, \|^*$ and $\| \, \|_0^*$ are equivalent on the finite-dimensional dual space K^*, F is therefore weak-operator uniformly continuous as a mapping of $\mathcal{B}_1(H)$ into $\left(K^*, \ \| \, \|_0^*\right)$. Hence

$$D = F(\mathcal{B}_1(H) \cap \ker u)$$

is a totally bounded, and therefore located, subset of $\left(K^*, \| \, \|_0^*\right)$. Moreover, for each $T \in \mathcal{B}_1(H) \cap \ker u$ and each $\mathbf{x} \in K$, $|F(T)(\mathbf{x})| \leq \|\mathbf{x}\|_0$; so D is a subset of the unit ball S_0^* of $\left(K^*, \ \| \, \|_0^*\right)$. We shall use the separation theorem from functional analysis to prove that D is $\| \, \|_0^*$-dense in S_0^*. Consider any ϕ in S_0^*, and suppose that

$$0 < d = \inf\left\{\|\phi - F(T)\|_0^* : T \in \mathcal{B}_1(H) \cap \ker u\right\}.$$

Now, D is bounded, convex, balanced, and located; so, by Corollary 5.2.10 of [8], there exists a linear functional v on $\left(K^*, \| \, \|_0^*\right)$ with norm 1 such that

$$v(\phi) > |v(F(T))| + \frac{d}{2} \qquad (T \in \mathcal{B}_1(H) \cap \ker u).$$

It is a simple exercise[8] to show that since $\left(K^*, \| \, \|_0^*\right)$ is N-dimensional, there exists $\mathbf{y} \in K$ such that $\|\mathbf{y}\|_0 = 1$ and $v(f) = f(\mathbf{y})$ for each $f \in K^*$. Hence

$$\begin{aligned}\phi(\mathbf{y}) &\geq \sup\left\{|F(T)(\mathbf{y})| : T \in \mathcal{B}_1(H) \cap \ker u\right\} + \frac{d}{2} \\ &> \sup\left\{\left|\sum_{n=1}^{N} \langle T y_n, \zeta_n\rangle\right| : T \in \mathcal{B}_1(H) \cap \ker u\right\} = \|\mathbf{y}\|_0\,,\end{aligned}$$

which contradicts the fact that $\phi \in S_0^*$. We conclude that $d = 0$ and therefore that D is $\| \, \|_0^*$-dense in S_0^*.

Continuing our proof that $\beta = 0$, pick $T_0 \in \mathcal{B}_1(H)$ with $u(T_0) > 0$. Replacing u by $u(T_0)^{-1}u$ if necessary, we may assume that $u(T_0) = 1$. Define a linear functional Ψ on $(K, \| \, \|_0)$ by setting

$$\Psi(\mathbf{x}) = \beta \sum_{n=1}^{N} \langle T_0 x_n, \zeta_n\rangle \quad (\mathbf{x} \in K)\ .$$

Note that for $\mathbf{x} \in K$ we have

$$|\Psi(\mathbf{x})| \leq \beta \sum_{n=1}^{N} \|x_n\| \, \|\zeta_n\| \leq \beta \, \|\mathbf{x}\| \, \|\zeta\| \leq \|\mathbf{x}\|_0.$$

Hence $\Psi \in S_0^*$. By the work of the previous paragraph, we can find $T \in \mathcal{B}_1(H) \cap \ker u$ such that $\|\Psi - F(T)\|_0^* < \beta/2\delta$. In particular, since $\|\xi\|_0 \leq \|\xi\| = 1$,

$$\left|\sum_{n=1}^{N} \langle (\beta T_0 - T)\, \xi_n, \zeta_n\rangle\right| < \frac{\beta}{2\delta}. \tag{2}$$

In order to apply the defining property of δ and thereby obtain a contradiction, we need to estimate not the sum on the left hand side of (2), but $\sum_{n=1}^{N} |\langle (\beta T_0 - T)\, \xi_n, \zeta_n\rangle|$. To do so, we write

$$\{n : 1 \leq n \leq N\} = P \cup Q,$$

where P, Q are disjoint sets,

$$n \in P \Rightarrow \langle (\beta T_0 - T)\, \xi_n, \zeta_n\rangle \neq 0, \text{ and}$$

$$n \in Q \Rightarrow |\langle (\beta T_0 - T)\, \xi_n, \zeta_n\rangle| < \frac{\beta}{2\delta N}.$$

If $n \in P$, we set

$$\lambda_n = \frac{1}{\langle (\beta T_0 - T)\, \xi_n, \zeta_n\rangle} \, |\langle (\beta T_0 - T)\, \xi_n, \zeta_n\rangle|\,,$$

[8] Alternatively, we can refer to [3] (page 287, Theorem 10) or [8] (Theorem 5.4.14).

and if $n \in Q$, we set $\lambda_n = 0$; in each case, we define $\gamma_n \equiv \lambda_n \xi_n$. Then $\gamma \equiv (\gamma_1, \dots, \gamma_N) \in K$ and

$$\|\gamma\|_0^2 \le \|\gamma\|^2 = \sum_{n=1}^{N} |\lambda_n|^2 \|\xi_n\|^2 \le \|\xi\|^2 = 1.$$

Hence

$$\left| \sum_{n=1}^{N} \langle (\beta T_0 - T)\, \gamma_n, \zeta_n \rangle \right| \le \|\Psi - F(T)\|_0^* < \frac{\beta}{2\delta}.$$

Moreover,

$$\begin{aligned} \left| \sum_{n=1}^{N} \langle (\beta T_0 - T)\, \gamma_n, \zeta_n \rangle \right| &= \left| \sum_{n \in P} \langle (\beta T_0 - T)\, \lambda_n \xi_n, \zeta_n \rangle \right| \\ &= \sum_{n \in P} |\langle (\beta T_0 - T)\, \xi_n, \zeta_n \rangle| , \end{aligned}$$

so

$$\begin{aligned} &\sum_{n=1}^{N} |\langle (\beta T_0 - T)\, \xi_n, \zeta_n \rangle| \\ &= \sum_{n \in P} |\langle (\beta T_0 - T)\, \xi_n, \zeta_n \rangle| + \sum_{n \in Q} |\langle (\beta T_0 - T)\, \xi_n, \zeta_n \rangle| \\ &\le \left| \sum_{n=1}^{N} \langle (\beta T_0 - T)\, \gamma_n, \zeta_n \rangle \right| + N \left(\frac{\beta}{2\delta N} \right) < \frac{\beta}{\delta} \end{aligned}$$

and therefore $u(\beta T_0 - T) < \beta$. But $u(\beta T_0 - T) = \beta u(T_0) - u(T) = \beta$, a contradiction which ensures that β actually equals 0. ■

We shall apply Lemma 6 shortly; but its application requires another construction.

Lemma 7 *Let N be a positive integer, let $\xi_1, \dots, \xi_N$ be linearly independent vectors in H, and let $\zeta_1, \dots, \zeta_N$ be nonzero elements of H, such that $\sum_{n=1}^{N} \|\xi_n\|^2 = \sum_{n=1}^{N} \|\zeta_n\|^2 = 1$. Then there exists a positive number c with the following property: for each unit vector $\mathbf{z}$ in the subspace*

$$K \equiv \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N,$$

there exists $T \in \mathcal{B}_1(H)$ such that $\sum_{n=1}^{N} \langle T z_n, \zeta_n \rangle > c$.

Proof. Let

$$m \equiv \inf \left\{ \|\zeta_n\|^2 : 1 \le n \le N \right\} > 0.$$

Define a norm on the N-dimensional span V of $\{\xi_1, \ldots, \xi_N\}$ by

$$\left\| \sum_{n=1}^{N} \alpha_n \xi_n \right\|_1 \equiv \max_{1 \leq n \leq N} |\alpha_n| .$$

Since V is finite-dimensional, there exists $b > 0$ such that $\|\mathbf{x}\|_1 \leq b \|\mathbf{x}\|$ for each $\mathbf{x} \in V$. Let $\mathbf{z} \equiv (\lambda_1 \xi_1, \ldots, \lambda_N \xi_N)$ in H_N satisfy $\|\mathbf{z}\| = 1$. If $|\lambda_n| < 1/\sqrt{N}$ for each n, then

$$1 = \sum_{n=1}^{N} \|\lambda_n \xi_n\|^2 = \sum_{n=1}^{N} |\lambda_n|^2 \|\xi_n\|^2 < \sum_{n=1}^{N} \left(\frac{1}{\sqrt{N}} \right)^2 = 1,$$

which is absurd. Hence we can pick ν such that $|\lambda_\nu| > 1/\sqrt{2N}$. Define a linear mapping T on H such that

$$T\xi_\nu = \frac{\lambda_\nu^*}{b |\lambda_\nu|} \zeta_\nu, \quad T\xi_n = 0 \ \ (n \neq \nu),$$

and $Tx = 0$ whenever x is orthogonal to V. Then

$$\left\| T \left(\sum_{n=1}^{N} \alpha_n \xi_n \right) \right\| = \frac{|\lambda_\nu^*|}{b |\lambda_\nu|} |\alpha_\nu| \leq \frac{1}{b} \left\| \sum_{n=1}^{N} \alpha_n \xi_n \right\|_1 \leq \left\| \sum_{n=1}^{N} \alpha_n \xi_n \right\|,$$

so $T \in \mathcal{B}_1(H)$. Moreover,

$$\langle T z_n, \zeta_n \rangle = \begin{cases} 0 & \text{if } n \neq \nu \\ \\ \frac{1}{b} |\lambda_\nu| \|\zeta_\nu\|^2 & \text{if } n = \nu, \end{cases}$$

so

$$\sum_{n=1}^{N} \langle T z_n, \zeta_n \rangle = \frac{1}{b} |\lambda_\nu| \|\zeta_\nu\|^2 > \frac{m}{b\sqrt{2N}}.$$

It remains to take $c \equiv m/b\sqrt{2N}$. ∎

The next lemma takes the information arising from the preceding two, and shows that when the vectors ξ_n in (1) are linearly independent, we can approximate u by a finite sum of mappings of the form $T \rightsquigarrow \langle Tx, y \rangle$, not just on its unit kernel but on the entire unit ball of $\mathcal{B}(H)$. At the same time, we produce a priori bounds on the sums of squares of the norms of the components of the vectors x, y that appear in the terms $\langle Tx, y \rangle$ whose sum approximates $u(T)$. Those bounds will be needed in the proof of our characterisation theorem.

Lemma 8 *Let H be a Hilbert space, and u a nonzero weak-operator continuous linear functional on $\mathcal{B}(H)$. Let δ be a positive number, $\xi_1, \ldots, \xi_N$ linearly independent vectors in H, and $\zeta_1, \cdots, \zeta_N$ nonzero vectors in H, such that*

$$\sum_{n=1}^{N} \|\xi_n\|^2 = \sum_{n=1}^{N} \|\zeta_n\|^2 = 1$$

and (1) *holds. Let* $c > 0$ *be as in* Lemma 7.*Then for each* $\varepsilon > 0$*, there exists* $\mathbf{x} \in \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$ *such that* $x_n \neq 0$ *for each* n*,*

$$\|\mathbf{x}\| < \frac{2\|u\|}{c},$$

and

$$\left| u(T) - \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle \right| < \varepsilon$$

for all $T \in \mathcal{B}_1(H)$*.*

Proof. Pick $T_0 \in \mathcal{B}_1(H)$ with $u(T_0) > 0$. To begin with, take the case where $u(T_0) = 1$ and therefore $\|u\| \geq 1$. Given $\varepsilon > 0$, set

$$\alpha \equiv \frac{\min\{\varepsilon, 1\}}{2\|u\|(1 + \|u\|)}.$$

Applying Lemma 6, we obtain nonzero vectors $z_n \in \mathbf{C}\xi_n$ $(1 \leq n \leq N)$ such that $\sum_{n=1}^{N} \|z_n\|^2 = 1$ and

$$\left| \sum_{n=1}^{N} \langle Tz_n, \zeta_n \rangle \right| < c\alpha \quad (T \in \mathcal{B}_1(H) \cap \ker u).$$

For each $T \in \mathcal{B}_1(H)$, since

$$(1 + \|u\|)^{-1} (T - u(T)T_0) \in \mathcal{B}_1(H) \cap \ker u,$$

we have

$$\left| \sum_{n=1}^{N} \langle (T - u(T)T_0) z_n, \zeta_n \rangle \right| < (1 + \|u\|) c\alpha.$$

By Lemma 7, there exists $T_1 \in \mathcal{B}_1(H)$ such that $\sum_{n=1}^{N} \langle T_1 z_n, \zeta_n \rangle > c$. We compute

$$\begin{aligned} c &< \sum_{n=1}^{N} \langle T_1 z_n, \zeta_n \rangle \\ &\leq \left| \sum_{n=1}^{N} \langle (T_1 - u(T_1)T_0) z_n, \zeta_n \rangle \right| \quad + |u(T_1)| \left| \sum_{n=1}^{N} \langle T_0 z_n, \zeta_n \rangle \right| \\ &\leq (1 + \|u\|)c\alpha + \|u\| \left| \sum_{n=1}^{N} \langle T_0 z_n, \zeta_n \rangle \right|. \end{aligned}$$

Hence

$$\begin{aligned} \left| \sum_{n=1}^{N} \langle T_0 z_n, \zeta_n \rangle \right| &> \frac{c}{\|u\|} (1 - (1 + \|u\|) \alpha) \\ &\geq \frac{c}{\|u\|} \left(1 - \frac{1}{2\|u\|} \right) > \frac{c}{2\|u\|}, \end{aligned}$$

since $\|u\| \geq 1$. Setting

$$\mathbf{x} \equiv \left(\sum_{n=1}^{N} \langle T_0 z_n, \zeta_n \rangle \right)^{-1} \mathbf{z},$$

we have $0 \neq x_n \in \mathbf{C}\xi_n$ for each n, and

$$\|\mathbf{x}\| = \left| \sum_{n=1}^{N} \langle T_0 z_n, \zeta_n \rangle \right|^{-1} \|\mathbf{z}\| < \frac{2\|u\|}{c}.$$

Moreover, for each $T \in \mathcal{B}_1(H)$,

$$\begin{aligned} \left| u(T) - \sum_{n=1}^{N} \langle T x_n, \zeta_n \rangle \right| &= \left| \sum_{n=1}^{N} \langle T_0 z_n, \zeta_n \rangle \right|^{-1} \left| \sum_{n=1}^{N} \langle (u(T)T_0 - T)\, z_n, \zeta_n \rangle \right| \\ &< \frac{2\|u\|}{c}(1 + \|u\|)c\alpha \leq \varepsilon. \end{aligned}$$

We now remove the restriction that $u(T_0) = 1$. Applying the first part of the theorem to $v \equiv u(T_0)^{-1}u$, we construct $\mathbf{y} \in K$ such that each component $y_n \neq 0$, $\|\mathbf{y}\| \leq 2\|v\|/c$, and

$$\left| v(T) - \sum_{n=1}^{N} \langle T y_n, \zeta_n \rangle \right| < u(T_0)^{-1}\varepsilon,$$

and we obtain the desired conclusion by taking $\mathbf{x} \equiv u(T_0)\mathbf{y}$. ■

Lemma 9 *Under the hypotheses of* Lemma 8, but without the assumption that u is nonzero, *for all* $\varepsilon, \varepsilon' > 0$, *there exists* $\mathbf{x} \in \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$ *such that* $x_n \neq 0$ *for each* n*,*

$$\|\mathbf{x}\| < \frac{2(\|u\| + \varepsilon')}{c},$$

and

$$\left| u(T) - \sum_{n=1}^{N} \langle T x_n, \zeta_n \rangle \right| < \varepsilon$$

for all $T \in \mathcal{B}_1(H)$*.*

Proof. Either $\|u\| > 0$ and we can apply Lemma 8, or else $\|u\| < \varepsilon/2$. In the latter event, pick $\mathbf{x}$ in $\mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$ such that $x_n \neq 0$ for each n and

$$\|\mathbf{x}\| < \min\left\{ \frac{\varepsilon}{2}, \frac{2(\|u\| + \varepsilon')}{c} \right\}.$$

Then for each $T \in \mathcal{B}_1(H)$ we have

$$\left| \sum_{n=1}^{N} \langle T x_n, \zeta_n \rangle \right| \leq \sum_{n=1}^{N} \|x_n\| \|\zeta_n\| \leq \|\mathbf{x}\| \|\zeta\| < \frac{\varepsilon}{2}$$

and therefore

$$\left| u(T) - \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle \right| \leq \|u\| + \left| \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle \right| < \varepsilon.$$

■

3 The Characterisation Theorem

We are finally able to prove our main result, by inductively applying Lemma 9.

Theorem 10 *Let H be a nontrivial Hilbert space, and u a nonzero weak-operator continuous linear functional on $\mathcal{B}(H)$. Let δ be a positive number, $\xi_1, \dots, \xi_N$ linearly independent vectors in H,*[9] *and $\zeta_1, \cdots, \zeta_N$ nonzero vectors in H, such that $|u(T)| \leq \delta \sum_{n=1}^{N} |\langle T\xi_n, \zeta_n \rangle|$ for all $T \in \mathcal{B}(H)$. Then there exists $\mathbf{x} \in \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$ such that*

$$u(T) = \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle \tag{3}$$

for all $T \in \mathcal{B}(H)$.

Proof. Re-scaling if necessary, we may assume that $\|u\| < 2^{-3}$. In the notation of, and using, Lemma 9, compute $\mathbf{x}^{(1)}$ in $K \equiv \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$ such that[10]

$$\left\| \mathbf{x}^{(1)} \right\| \leq \frac{2}{c} \left(\|u\| + 2^{-3} \right) < \frac{1}{2c}$$

and

$$\left| u(T) - \sum_{n=1}^{N} \left\langle Tx_n^{(1)}, \zeta_n \right\rangle \right| < 2^{-4} \quad (T \in \mathcal{B}_1(H)).$$

Suppose that for some positive integer k we have constructed vectors $\mathbf{x}^{(i)} \in K$ $(1 \leq i \leq k)$ such that

$$\left\| \mathbf{x}^{(k)} \right\| < \frac{1}{2^k c}, \tag{4}$$

[9] The requirement that the vectors ξ_n be linearly independent is the one place where we have a stronger hypothesis than is needed in the classical theorem. It is worth noting here that *if* $u(T)$ has the desired form $\sum_{n=1}^{N} \langle T\xi_n, \zeta_n \rangle$, then *classically* we can find a set F of indices $n \leq N$ such that (i) the set S of those ξ_n with $n \in F$ is linearly independent and (ii) if $\xi_k \notin S$, then ξ_k is linearly dependent on S. We can then write

$$u(T) = \sum_{n \in F} \langle T\xi_n, \lambda_n \zeta_n \rangle,$$

with each $\lambda_n \in \mathbf{C}$. Constructively, this is not possible, since we cannot necessarily determine whether or not ξ_n is linearly dependent on the vectors $\xi_1, \dots, \xi_{n-1}$.

[10] In this proof we do not need the fact that, according to Lemma 9, we can arrange for the components of the vector $\mathbf{x}^{(1)}$, and of the subsequently constructed vectors $\mathbf{x}^{(k)}$, to be nonzero.

and

$$\left|u(T)-\sum_{n=1}^{N}\left\langle T\left(x_n^{(1)}+\cdots+x_n^{(k)}\right),\zeta_n\right\rangle\right|<2^{-k-3}\quad (T\in\mathcal{B}_1(H)). \tag{5}$$

Consider the weak-operator continuous linear functional

$$v:T\rightsquigarrow u(T)-\sum_{n=1}^{N}\left\langle T\left(x_n^{(1)}+\cdots+x_n^{(k)}\right),\zeta_n\right\rangle$$

on $\mathcal{B}(H)$. Writing

$$x_n^{(1)}+\cdots+x_n^{(k)}=\lambda_n\xi_n$$

and

$$\gamma\equiv\max\{|\lambda_1|,\ldots,|\lambda_n|\},$$

for each $T\in\mathcal{B}(H)$ we have

$$\begin{aligned}|v(T)|&\leq|u(T)|+\sum_{n=1}^{N}\left|\left\langle T\left(x_n^{(1)}+\cdots+x_n^{(k)}\right),\zeta_n\right\rangle\right|\\&\leq\delta\sum_{n=1}^{N}|\langle T\xi_n,\zeta_n\rangle|+\sum_{n=1}^{N}|\lambda_n|\,|\langle T\xi_n,\zeta_n\rangle|\\&\leq(\delta+\gamma)\sum_{n=1}^{N}|\langle T\xi_n,\zeta_n\rangle|.\end{aligned}$$

We can now apply Lemma 9, to obtain

$$\mathbf{x}^{(k+1)}=\left(x_1^{(k+1)},\ldots,x_N^{(k+1)}\right)\in K$$

such that

$$\left\|\mathbf{x}^{(k+1)}\right\|<\frac{2}{c}\left(\|\nu\|+2^{-k-3}\right)<\frac{1}{2^{k+1}c}$$

and

$$\begin{aligned}&\left|u(T)-\sum_{n=1}^{N}\left\langle T\left(x_n^{(1)}+\cdots+x_n^{(k)}+x_n^{(k+1)}\right),\zeta_n\right\rangle\right|\\&=\left|v(T)-\sum_{n=1}^{N}\left\langle Tx_n^{(k+1)},\zeta_n\right\rangle\right|<2^{-k-4}\end{aligned}$$

for all $T\in\mathcal{B}_1(H)$. This completes the inductive construction of a sequence $\left(\mathbf{x}^{(k)}\right)_{k\geq1}$ in K such that (4) and (5) hold for each k. The series $\sum_{k=1}^{\infty}\mathbf{x}^{(k)}$ converges to a sum $\mathbf{x}$ in the finite-dimensional Banach space K, by comparison with $\sum_{k=1}^{\infty}2^{-k}c^{-1}$. Letting $k\to\infty$ in (5), we obtain (3) for all $T\in\mathcal{B}_1(H)$ and hence for all $T\in\mathcal{B}(H)$. ∎

For nonzero u, the proof of our theorem can be simplified at each stage of the induction, since we can use Lemma 8 directly. If H has dimension $> N$, we can then construct the classical representation of u in the general case as follows. Either $\|u\| > 0$ and there is nothing to prove, or else $\|u\| < \delta$ (the same δ as in the statement of the theorem). In the latter case, we construct a unit vector ξ_{N+1} orthogonal to each of the vectors ξ_n $(1 \leq n \leq N)$, set $\zeta_{N+1} = \xi_{N+1}$, and consider the weak-operator continuous linear functional

$$v : T \rightsquigarrow u(T) + \delta \left\langle T\xi_{N+1}, \zeta_{N+1} \right\rangle .$$

We have

$$|v(T)| \leq |u(T)| + \delta \left|\left\langle T\xi_{N+1}, \zeta_{N+1} \right\rangle\right| \leq \delta \sum_{n=1}^{N+1} \left|\left\langle T\xi_n, \zeta_n \right\rangle\right| .$$

Moreover,

$$|v(I)| \geq \delta \left\|\xi_{N+1}\right\|^2 - |u(I)| \geq \delta - \|u\| > 0,$$

where I is the identity operator on H; so v is nonzero. We can therefore apply the nonzero case to v, to produce a vector $\mathbf{y} \in \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_{N+1}$ such that

$$v(T) = \sum_{n=1}^{N+1} \left\langle Ty_n, \zeta_n \right\rangle \qquad (T \in \mathcal{B}(H)) .$$

Setting $x_n = y_n$ $(1 \leq n \leq N)$ and $x_{n+1} = y_{N+1} - \delta\xi_{N+1}$, we obtain

$$u(T) = \sum_{n=1}^{N+1} \left\langle Tx_n, \zeta_n \right\rangle$$

for each $T \in \mathcal{B}(H)$. Note, however, that this proof gives $\mathbf{x}$ in $\mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N \times \mathbf{C}\xi_{N+1}$, not, as in Theorem 10, in $\mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$.
As an immediate consequence of Theorem 10, the functional u therein is a linear combination of the functionals $T \rightsquigarrow \left\langle T\xi_n, \zeta_n \right\rangle$ associated with the seminorms that describe the boundedness of u:

Corollary 11 *Under the hypotheses of* Theorem 10*, there exist complex numbers $\alpha_1, \ldots, \alpha_N$ such that*

$$u(T) = \sum_{n=1}^{N} \alpha_n \left\langle T\xi_n, \zeta_n \right\rangle$$

for each $T \in \mathcal{B}(H)$.

4 Strong-operator Continuous Functionals

Next we turn briefly to the STRONG OPERATOR TOPOLOGY on $\mathcal{B}(H)$: the locally convex topology generated by the seminorms $T \rightsquigarrow \|Tx\|$ with $x \in H$. (That is, the weakest topology with respect to which the mapping $T \rightsquigarrow Tx$ is continuous for each $x \in H$.) Clearly, a weak-operator continuous linear functional on $\mathcal{B}(H)$ is strong-operator continuous. The converse holds classically, but, as we now show by a Brouwerian example, is essentially nonconstructive.

Let $(e_n)_{n\geq1}$ be an orthonormal basis of unit vectors in an infinite-dimensional Hilbert space, and let $(a_n)_{n\geq1}$ be a binary sequence with at most one term equal to 1. Then for $k \geq j$ we have

$$\begin{aligned}\sum_{n=j}^{k} |a_n \langle Te_1, e_n\rangle| &\leq \left(\sum_{n=j}^{k} a_n^2\right)^{1/2} \left(\sum_{n=j}^{k} |\langle Te_1, e_n\rangle|^2\right)^{1/2} \\ &\leq \left(\sum_{n=j}^{k} |\langle Te_1, e_n\rangle|^2\right)^{1/2}.\end{aligned}$$

Since $\sum_{n=1}^{\infty} |\langle Te_1, e_n\rangle|^2$ converges to $\|Te_1\|$, we see that $\sum_{n=j}^{k} |a_n \langle Te_1, e_n\rangle| \to 0$ as $j, k \to \infty$. Hence

$$u(T) \equiv \sum_{n=1}^{\infty} a_n \langle Te_1, e_n\rangle$$

defines a linear functional u on $\mathcal{B}(H)$; moreover, $|u(T)| \leq \|Te_1\|$ for each T, so (by Proposition 5.4.1 of [8]) u is strong-operator continuous. Suppose it is also weak-operator continuous. Then, by Lemma 2, it is normed. Either $\|u\| < 1$ or $\|u\| > 0$. In the first case, if there exists (a unique) ν with $a_\nu = 1$, then $u(T) = \langle Te_1, e_\nu\rangle$ for each $T \in \mathcal{B}(H)$. Defining T such that $Te_1 = e_\nu$ and $Te_n = 0$ for all $n \neq \nu$, we see that $T \in \mathcal{B}_1(H)$ and $u(T) = 1$; whence $\|u\| = 1$, a contradiction. Thus in this case, $a_n = 0$ for all n. On the other hand, in the case $\|u\| > 0$ we can find T such that $u(T) > 0$; whence there exists n such that $a_n = 1$. It now follows that the statement

> If H is an infinite-dimensional Hilbert space, then every strong-operator continuous linear functional on $\mathcal{B}(H)$ is weak-operator continuous

implies the essentially nonconstructive principle

> LPO: For each binary sequence $(a_n)_{n\geq1}$, either $a_n = 0$ for all n or else there exists n such that $a_n = 1$

and so is itself essentially nonconstructive.

5 Concluding Observations

The ideal constructive form of Theorem 10 would have two improvements over the current one. First, the requirement that the vectors ξ_n be linearly independent would be relaxed to have them only nonzero in Lemma 8, Lemma 9, and Theorem 10. Second, $\mathcal{B}(H)$ would be replaced by a suitable linear subspace $\mathcal{R}$ of itself, and our theorem would apply to linear functionals that are weak-operator continuous on $\mathcal{R}$, where "suitable" probably means "having weak-operator totally bounded unit ball $\mathcal{R}_1 \equiv \mathcal{R} \cap \mathcal{B}_1(H)$". With that notion of suitability and with minor adaptations, Lemma 6 holds and the proof of Lemma 8 goes through as far as the construction of the vector $\mathbf{z} \in K$. In fact, Theorem 10 goes through with $\mathcal{B}(H)$ replaced by any linear subspace $\mathcal{R}$ of $\mathcal{B}(H)$ that has weak-operator totally bounded unit ball and satisfies the following condition (cf. Lemma 7):

> (*) Let N be a positive integer, let $\xi_1, \ldots, \xi_N$ be linearly independent vectors in H, and let $\zeta_1, \ldots, \zeta_N$ be nonzero elements of H, such that $\sum_{n=1}^{N} \|\xi_n\|^2 = \sum_{n=1}^{N} \|\zeta_n\|^2 = 1$. Then there exists a positive number c with the following property: for each unit vector $\mathbf{z}$ in the subspace
> $$K \equiv \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$$
> there exists $T \in \mathcal{R}_1$ such that $\sum_{n=1}^{N} \langle T z_n, \zeta_n \rangle > c$.

This condition holds in the special case where $N = 1$, in which case, if also $\mathcal{R}_1$ is weak-operator totally bounded, we obtain Theorem 1 of [6].[11] However, there seems to be no means of establishing (*) for $N > 1$ and a general $\mathcal{R}$. So the ideal form of our theorem remains an ideal and a challenge.

Acknowledgements. This work was supported by a Marie Curie IRSES award from the European Union for the project *Construmath*, with counterpart funding from the New Zealand Ministry of Research, Science & Technology. It was written while the author enjoyed the hospitality of (i) Professor Helmut Schwichtenberg and his colleagues in the Logic Group at the Mathematisches Institut der Ludwig-Maximilians-Universität München, and (ii) Professor Erik Palmgren in the Matematiska Institution, University of Uppsala. The author also thanks the referee for providing helpful comments that improved the presentation of the paper.

[11] But the proof of the theorem in [6] is simpler and more direct than the case $N = 1$ of the proof of our Theorem 10 above.

References

[1] P. Aczel and M. Rathjen, *Notes on Constructive Set Theory*, Report No. 40, Institut Mittag-Leffler, Royal Swedish Academy of Sciences, 2001.

[2] P. Aczel and M. Rathjen, *Constructive Set Theory*, monograph, forthcoming.

[3] E.A. Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967.

[4] E.A. Bishop and D.S. Bridges, *Constructive Analysis*, Grundlehren der Math. Wiss. 279, Springer Verlag, Heidelberg, 1985.

[5] D.S. Bridges, 'On the weak operator compactness of the unit ball of $\mathcal{L}(H)$', Zeit. math. Logik Grundlagen Math. 24, 493–494, 1978.

[6] D.S. Bridges, 'Characterising dominated weak-operator continuous functionals constructively', Ann. Pure Appl. Logic, to appear.

[7] D.S. Bridges and L.S. Dediu (Vîţă), 'Weak operator continuity and the existence of adjoints', Math. Logic Quarterly 45, 203–206,1999.

[8] D.S. Bridges and L.S. Vîţă, *Techniques of Constructive Analysis*, Universitext, Springer Verlag, Heidelberg, 2006.

[9] D.S. Bridges and N.F. Dudley Ward, 'Constructing ultraweakly continuous linear functionals on $\mathcal{B}(H)$', Proc. American Math. Soc. 126(11), 3347–3353, 1998.

[10] J. Dixmier, *Les Algèbres d'Opérateurs dans l'Espace Hilbertien: Algèbres de von Neumann*, Gauthier-Villars, Paris, 1957.

[11] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras* (Vols. 1,2), Academic Press, New York, 1988.

[12] P. Martin-Löf, *Intuitionistic Type Theory,* Bibliopolis, Naples, 1984.

[13] R.Mines, F. Richman, and W. Ruitenburg, *A Course in Constructive Algebra*, Universitext, Springer Verlag, New York, 1988.

[14] S. Sakai, *C* Algebras and W* Algebras*, Springer Verlag, Heidelberg, 1971.

[15] B. Spitters, 'Constructive results on operator algebras', J. Univ.Comp. Sci. 11(12), 2096–2113, 2005.

[16] M. Takesaki, *Theory of Operator Algebra 1*, Encyclopaedia of Math. Sciences, Springer Verlag, Berlin, 2002.

Douglas S. Bridges
Department of Mathematics
& Statistics
University of Canterbury
Private Bag 4800
Christchurch 8140
New Zealand
d.bridges@math.canterbury.ac.nz

The Classification of Real Purely Infinite Simple C*-Algebras

Jeffrey L. Boersema, Efren Ruiz, P. J. Stacey

Received: February 26, 2011

Communicated by Joachim Cuntz

Abstract. We classify real Kirchberg algebras using united K-theory. Precisely, let A and B be real simple separable nuclear purely infinite C*-algebras that satisfy the universal coefficient theorem such that $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ are also simple. In the stable case, A and B are isomorphic if and only if $K^{CRT}(A) \cong K^{CRT}(B)$. In the unital case, A and B are isomorphic if and only if $(K^{CRT}(A), [1_A]) \cong (K^{CRT}(B), [1_B])$. We also prove that the complexification of such a real C*-algebra is purely infinite, resolving a question left open from [43]. Thus the real C*-algebras classified here are exactly those real C*-algebras whose complexification falls under the classification result of Kirchberg [26] and Phillips [35]. As an application, we find all real forms of the complex Cuntz algebras $\mathcal{O}_n$ for $2 \leq n \leq \infty$.

2010 Mathematics Subject Classification: 46L35, 46L80, 19K99
Keywords and Phrases: Real C*-algebras, K-theory, classification

1. Introduction

One of the highlights of the classification theory of simple amenable C*-algebras is the classification of purely infinite nuclear simple C*-algebras, obtained by Kirchberg and Phillips in [26] and [35]. This classification theorem relies in an essential way on the Universal Coefficient Theorem established by Rosenberg and Schochet in [40], where it was observed that "For reasons pointed out already by Atiyah, there can be no good Künneth Theorem or Universal Coefficient Theorem for the KKO groups of real C*-algebras; this explains why we deal only with complex C*-algebras". Thus at the time of the Kirchberg and Phillips classification, the lack of a universal coefficient theorem was the primary barrier to extending the classification result to real C*-algebras.

However, in [8], a new invariant called united K-theory was introduced for real C*-algebras and in [9] a universal coefficient theorem was proven for real

C*-algebras using united K-theory. In the present paper, we take advantage of these developments to provide a complete classification of a class of real simple purely infinite C*-algebras in terms of united K-theory. The real C*-algebras that are classified are exactly those real C*-algebras for which the complexification is covered by the Kirchberg and Phillips theory. As an application of our classification we determine all the real forms of the complex Cuntz algebras $\mathcal{O}_n$ for $1 \leq n \leq \infty$: there are two such forms when n is odd and one when n is even or infinite.

The overall framework of the proof will be the same as that in the paper [35] and the underlying theory on which that paper was built. Furthermore, many of the proofs in the development leading to the main theorems of [35] carry over to the real case without significant change. In those cases, we will simply refer to the established proofs in the literature without reproducing them here. However there are many situations where the arguments in the real case require modification and we will then provide full proofs or full discussion of the necessary modifications.

In Section 2, we describe the invariant of united K-theory and summarize its key properties. In Section 3 we then establish real analogues of some of the fundamental properties of purely infinite algebras, in the course of which we resolve a problem left hanging in [43] and [13] by showing that the complexification of a purely infinite simple real C*-algebra is also purely infinite (using the original definition for simple algebras). Following the complex case, as developed in [38], we then establish (in Theorem 5.2) criteria for two unital homomorphisms from the real Cuntz algebra $\mathcal{O}_n^{\mathbb{R}}$ (n even) to be approximately unitarily equivalent. Modifications of the complex arguments are required to establish some of the preliminary results: in Section 4 we modify the required results about exponential rank, noting that the close link between self-adjoint and skew-adjoint elements is absent in a real C*-algebra, and in Section 5 we modify the result from [15] establishing the Rokhlin property of the Bernoulli shift on the CAR-algebra.

Our next step is to establish real analogues of Kirchberg's tensor product theorems and his embedding theorem. This is achieved in Section 6 by using the relevant complex results and the embedding of $\mathbb{C}$ into $M_2(\mathbb{R})$. In Sections 7, 8, 9 and 10, we closely follow [35] indicating how the results achieved for the complex case can be obtained in the real case. In particular, Section 7 contains a key result about uniqueness of homomorphisms from $\mathcal{O}_\infty^{\mathbb{R}}$ to a real purely infinite C*-algebras. Section 8 contains the theory of asymptotic morphisms in the context of real C*-algebras and Section 9 culminates in a theorem identifying KK-theory to a group of asymptotic unitary equivalence classes of asymptotic morphisms as in Section 4 of [35]. To accomplish this, we make use of the axiomatic characterization of KK-theory for real C*-algebras established in [12]. This development culminates in Section 10, which contains the statements and proofs of our classification theorems, and in Section 11, which uses these results to describe the real forms of Cuntz algebras. The notation we use in these sections closely follows that in [35].

We will use the notation $\mathcal{H}^{\mathbb{R}}$ for a real Hilbert space; and $\mathcal{B}(\mathcal{H}^{\mathbb{R}})$ and $\mathcal{K}^{\mathbb{R}}$ for the real C*-algebras of bounded and compact operators $\mathcal{H}^{\mathbb{R}}$. For the complex versions of these objects we will use $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$, and $\mathcal{K}$. For a C*-algebra A, we will write $M_n(A)$ for the matrix algebra over A; and M_n will stand for $M_n(\mathbb{R})$. Following standard convention, we will use $\mathcal{O}_n$ for the complex Cuntz algebras and $\mathcal{O}_n^{\mathbb{R}}$ for the real versions. The complexification of a real C*-algebra A will be denoted by $A_{\mathbb{C}}$. We will use Φ throughout to denote the conjugate linear automorphism of $A_{\mathbb{C}}$ defined by $a + ib \mapsto a - ib$ (for $a, b \in A$). Note that A can be recovered from Φ as the fixed point set. Finally, a tensor product written as $A \otimes B$ will in most cases be the C*-algebra tensor product over $\mathbb{R}$, but should be understood to be a tensor product over $\mathbb{C}$ if both A and B are known to be complex C*-algebras. Recall that if A and B are real C*-algebras, then $(A \otimes B)_{\mathbb{C}} \cong A_{\mathbb{C}} \otimes B_{\mathbb{C}}$.

2. Preliminaries on United K-Theory

United K-theory was developed in the commutative context in [14] and subsequently extended to the context of real C*-algebras in [8]. United K-theory consists of the three separate K-theory modules as well as several natural transformations among them. In this section, we give the definition of united K-theory and summarize the features needed in this paper. Details are in [8], [9], [10].

Definition 2.1. Let A be a real C*-algebra. The *united K-theory* of A is given by

$$K^{\scriptscriptstyle CRT}(A) = \{KO_*(A), KU_*(A), KT_*(A), r, c, \varepsilon, \zeta, \psi_{\scriptscriptstyle U}, \psi_{\scriptscriptstyle T}, \gamma, \tau\} .$$

In this definition, $KO_*(A) = K_*(A)$ is the standard K-theory of a real C*-algebra, considered as a graded module over the ring $K_*(\mathbb{R})$. This means in particular that there are operations

$$\begin{aligned} \eta_O \colon KO_n(A) &\to KO_{n+1}(A) \\ \xi \colon KO_n(A) &\to KO_{n+4}(A) \\ \beta_O \colon KO_n(A) &\to KO_{n+8}(A) \end{aligned}$$

corresponding to multiplication by the elements of the same name in $KO_*(\mathbb{R})$. The operation β_O is the periodicity isomorphism of real K-theory.

The second item $KU_*(A) = K_*(A_{\mathbb{C}})$ is the K-theory of the complexification of A, having period 2. It is a module over $K_*(\mathbb{C})$, which is to say that that there is an isomorphism of period 2 and the two remaining groups are independent with no operations between them.

Finally, $KT_*(A)$ is the period 4 self-conjugate K-theory originally defined in the topological setting in [1]. In the non-commutative setting, it is more easily defined as $KT_*(A) = K_*(T \otimes A)$ in terms of the algebra $T = \{f \in C([0,1], \mathbb{C}) \mid$

$f(0) = \overline{f(1)}\}$ (see [8]). Self-conjugate K-theory is a module over the ring $K_*(T)$, giving operations

$$\begin{aligned} \eta_O &\colon KT_n(A) \to KT_{n+1}(A) \\ \omega &\colon KT_n(A) \to KT_{n+3}(A) \\ \beta_T &\colon KT_n(A) \to KT_{n+4}(A) \ . \end{aligned}$$

The rest of the information in united K-theory consists of operations

$$\begin{aligned} c_n &\colon KO_n(A) \longrightarrow KU_n(A) & r_n &\colon KU_n(A) \longrightarrow KO_n(A) \\ \varepsilon_n &\colon KO_n(A) \longrightarrow KT_n(A) & \zeta_n &\colon KT_n(A) \longrightarrow KU_n(A) \\ (\psi_U)_n &\colon KU_n(A) \longrightarrow KU_n(A) & (\psi_T)_n &\colon KT_n(A) \longrightarrow KT_n(A) \\ \gamma_n &\colon KU_n(A) \longrightarrow KT_{n-1}(A) & \tau_n &\colon KT_n(A) \longrightarrow KO_{n+1}(A) \end{aligned}$$

among the three K-theory modules.

For example, c is induced by the natural inclusion $A \to A_{\mathbb{C}}$; r by the inclusion $A_{\mathbb{C}} \to M_2(A)$; and ψ_U by the involution Φ on $A_{\mathbb{C}}$. These operations are known to satisfy the following relations (see Proposition 1.7 of [8]):

$$\begin{aligned} rc &= 2 & \psi_U\beta_U &= -\beta_U\psi_U & \xi &= r\beta_U^2 c \\ cr &= 1+\psi_U & \psi_T\beta_T &= \beta_T\psi_T & \omega &= \beta_T\gamma\zeta \\ r &= \tau\gamma & \varepsilon\beta_O &= \beta_T^2\varepsilon & \beta_T\varepsilon\tau &= \varepsilon\tau\beta_T + \eta_T\beta_T \\ c &= \zeta\varepsilon & \zeta\beta_T &= \beta_U^2\zeta & \varepsilon r\zeta &= 1+\psi_T \\ (\psi_U)^2 &= 1 & \gamma\beta_U^2 &= \beta_T\gamma & \gamma c\tau &= 1-\psi_T \\ (\psi_T)^2 &= 1 & \tau\beta_T^2 &= \beta_O\tau & \tau &= -\tau\psi_T \\ \psi_T\varepsilon &= \varepsilon & \gamma &= \gamma\psi_U & \tau\beta_T\varepsilon &= 0 \\ \zeta\gamma &= 0 & \eta_O &= \tau\varepsilon & \varepsilon\xi &= 2\beta_T\varepsilon \\ \zeta &= \psi_U\zeta & \eta_T &= \gamma\beta_U\zeta & \xi\tau &= 2\tau\beta_T \ . \end{aligned}$$

United K-theory takes values in the algebraic category of *CRT*-modules. A *CRT*-module consists of a triple (M^O, M^U, M^T) of graded modules, one over each of the rings $K_*(\mathbb{R})$, $K_*(\mathbb{C})$, and $K_*(T)$; as well as natural transformations $c, r, \varepsilon, \zeta, \psi_U, \psi_T, \gamma, \tau$ that satisfy the above relations.

For any real C*-algebra A, the *CRT*-module $K^{CRT}(A)$ is *acyclic*, which means that the sequences

$$\cdots \longrightarrow KO_n(A) \xrightarrow{\eta_O} KO_{n+1}(A) \xrightarrow{c} KU_{n+1}(A) \xrightarrow{r\beta_U^{-1}} KO_{n-1}(A) \longrightarrow \cdots$$

$$\cdots \longrightarrow KO_n(A) \xrightarrow{\eta_O^2} KO_{n+2}(A) \xrightarrow{\varepsilon} KT_{n+2}(A) \xrightarrow{\tau\beta_T^{-1}} KO_{n-1}(A) \longrightarrow \cdots$$

$$\cdots \longrightarrow KU_{n+1}(A) \xrightarrow{\gamma} KT_n(A) \xrightarrow{\zeta} KU_n(A) \xrightarrow{1-\psi_U} KU_n(A) \longrightarrow \cdots$$

are exact.

The important advantage of the full united K-theory over ordinary K-theory for a real C*-algebra A is that it yields both a Künneth formula (Theorem 4.2 of

[8]) and a univeral coefficient theorem (Theorem 1.1 of [9]). For later reference, we now state two results that follow from those fundamental theorems.

PROPOSITION 2.2. *For any real C*-algebra A,*

(1) $K^{CRT}(\mathcal{O}_2^{\mathbb{R}} \otimes A) = 0$
(2) $K^{CRT}(\mathcal{O}_\infty^{\mathbb{R}} \otimes A) \cong K^{CRT}(A)$.

Proof. By Table IV of [8], we have $K^{CRT}(\mathcal{O}_2^{\mathbb{R}}) = 0$. Then (1) follows by the Künneth formula.
The unital inclusion $\mathbb{R} \to \mathcal{O}_\infty^{\mathbb{R}}$ induces an isomorphism on united K-theory. This follows from Theorem 4 of [10] and the fact that the unital inclusion $\mathbb{C} \to \mathcal{O}_\infty$ induces an isomorphism on (complex) K-theory. Thus, Theorem 3.5 of [8] gives $K^{CRT}(\mathcal{O}_\infty^{\mathbb{R}}) \otimes_{CRT} K^{CRT}(A) \cong K^{CRT}(A)$ and $\operatorname{Tor}(K^{CRT}(\mathcal{O}_\infty^{\mathbb{R}}), K^{CRT}(A)) = 0$. Then the isomorphism of (2) follows by the Main Theorem of [8]. □

Recall from [41] that the bootstrap class $\mathcal{N}$ is the smallest subcategory of complex, separable, nuclear C*-algebras that contains the separable type I C*-algebras; that is closed under the operations of taking inductive limits, stable isomorphisms, and crossed products by $\mathbb{Z}$ and $\mathbb{R}$; and that satisfies the two out of three rule for short exact sequences (i.e. if $0 \to A \to B \to C \to 0$ is exact and two of A, B, C are in $\mathcal{N}$, then the third is also in $\mathcal{N}$).

PROPOSITION 2.3 (Corollary 4.11 of [9]). *Let A and B be real separable C*-algebras such that $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ are in $\mathcal{N}$. Then A and B are KK-equivalent if and only if $K^{CRT}(A) \cong K^{CRT}(B)$.*

This last result is the essential preliminary result for our classification of real purely infinite simple C*-algebras. We will also make use of Theorem 1 of [10], which states that every countable acyclic *CRT*-module can be realized as the united K-theory a real separable C*-algebra, indeed the C*-algebra can even be taken to be simple and purely infinite.
We now describe a simpler variation of united K-theory that, by results from [23], contains as much information as the full version of united K-theory.

DEFINITION 2.4. Let A be a real C*-algebra. Then

$$K^{CR}(A) = \{KO_*(A), KU_*(A), r, c, \psi_U\}$$

For any real C*-algebra, $K^{CR}(A)$ is an acyclic *CR*-module, which means that the relations

$$\begin{array}{ccc} rc = 2 & \psi_U \beta_U = -\beta_U \psi_U & \xi = r\beta_U^2 c \\ cr = 1 + \psi_U & \psi_U^2 = 1 & \psi_U c = c \end{array}$$

are satisfied and that the sequence

$$\cdots \longrightarrow KO_n(A) \xrightarrow{\eta_O} KO_{n+1}(A) \xrightarrow{c} KU_{n+1}(A) \xrightarrow{r\beta_U^{-1}} KO_{n-1}(A) \longrightarrow \cdots$$

is exact.

Let Γ be the forgetful functor from the category *CRT*-modules to the category of *CR*-modules. It is immediate from Theorem 4.2.1 of [23] that Γ is injective (but not surjective) on the class of acyclic *CRT*-modules. Hence we have the following result.

Proposition 2.5. *Let A and B be real C*-algebras. Then $K^{CRT}(A) \cong K^{CRT}(B)$ if and only if $K^{CR}(A) \cong K^{CR}(B)$.*

Note, however, that the results of [10] do not extend to *CR*-modules. Not every countable acyclic *CR*-module can be realized as $K^{CR}(A)$ for a real C*-algebra A.

3. Preliminaries on Real Simple Purely Infinite C*-Algebras

In this section, we provide some preliminaries on simple and purely infinite C*-algebras, including a theorem characterizing simple purely infinite real C*-algebras in terms of their complexification. One direction of this characterization was achieved in [43] and [13].

Let A be a real unital C*-algebra, let $\mathcal{U}(A)$ denote the group of unitary elements in A, and let $\mathcal{U}_0(A)$ denote the connected component of the identity in $\mathcal{U}(A)$. Note that if u is a unitary in a real C*-algebra, then the spectrum $\sigma(u) \subseteq \mathbb{T}$ satisfies $\overline{\sigma(u)} = \sigma(u)$ and the real C*-algebra generated by u is isomorphic to the algebra of complex-valued continuous functions f on $\sigma(u)$ that satisfy $\overline{f(z)} = f(\overline{z})$. (If a is an element of A, then by definition the spectrum $\sigma(a)$ is found by passing to $A_{\mathbb{C}}$.)

We begin by making an explicit mention of a fairly well-known result about real simple C*-algebras.

Definition 3.1. A real C*-algebra A is *c-simple* if $A_{\mathbb{C}}$ is simple.

Proposition 3.2. *A simple real C*-algebra A is either c-simple or is isomorphic to a simple complex C*-algebra.*

Proof. Let I be a proper ideal in $A_{\mathbb{C}}$. Then $J = A \cap I \cap \Phi(I) = 0$ and so $I \cap \Phi(I) = 0$. Furthermore, $I + \Phi(I) = A_{\mathbb{C}}$. It then follows that the map $x \mapsto x + \Phi(x)$ is an isomorphism from the complex C*-algebra I onto A. $\square$

As the structure of simple complex C*-algebras is comparatively well-understood, our primary interest lies in c-simple C*-algebras.

As in the complex case, we will use the tilde $\sim$ to denote the relation of Murray-von Neumann equivalence of projections. A projection is said to be infinite if it is Murray-von Neumann equivalent to a proper subprojection of itself. The following definition of purely infinite is from [43]. Bearing in mind subsequent developments, such as [27] and [28], a different definition should be made in the non-simple case. However the focus in this paper is on simple algebras, for which the definition below is appropriate.

Definition 3.3. Let A be a real simple C*-algebra.

(1) A subalgebra B is a *regular hereditary* subalgebra of A if there is an element $x \in A_+$ such that $B = \overline{xAx}$.
(2) A is *purely infinite* if every regular hereditary subalgebra of A contains an infinite projection.

PROPOSITION 3.4. *Let A be a separable simple purely infinite real C*-algebra. Then either A is unital or there is a real unital simple purely infinite C*-algebra A_0 such that $A \cong \mathcal{K}^{\mathbb{R}} \otimes A_0$.*

Proof. As in Section 27.5 of [2]. □

PROPOSITION 3.5. *Let A be a simple purely infinite C*-algebra and let p be a projection in A. Then pAp and A are stably isomorphic.*

Proof. In the complex case, this result follows from Corollary 2.6 of [16]. The proof of that result and the proofs of the preliminary lemmas of Section 2 of [16] work the same in the real case. □

For the rest of this section, f_ε will denote the real-valued function such that $f_\varepsilon(t) = 0$ for $t \leq \varepsilon/2$, $f_\varepsilon(t) = 1$ for $t \geq \varepsilon$, and $f_\varepsilon(t)$ is linear on $[\varepsilon/2, \varepsilon]$.

LEMMA 3.6. *For any real C*-algebra A, the following are equivalent.*
(1) *For any non-zero $a, b \in A$ there exist $x, y \in A$ with $a = xby$.*
(2) *For any non-zero positive $a, b \in A$ there exists $x \in A$ with $a = xbx^*$.*

Proof. (2) $\Rightarrow$ (1). Let $0 \neq a, b \in A$. As in the complex case, described in 1.4.5 of [33], there exists $u \in A$ with $a = u(a^*a)^{1/4}$. Let $x \in A$ with $(a^*a)^{1/4} = xbb^*x^*$ and observe that $a = (ux)b(b^*x^*)$.
(1) $\Rightarrow$ (2). This uses the argument for the complex case, from Lemma 1.7 and Proposition 1.10 of [18]. If $a, b \in A$ are positive and non-zero and ε is chosen so that $f_\varepsilon(b) \neq 0$ then $a = (zz^*zk)b(zz^*zk)^*$, where x, y are chosen so that $a^{1/6} = xf_\varepsilon(b)y$, $k \geq 0$ is chosen so that $f_{\varepsilon/2}(b) = kbk$ and $z = x(f_\varepsilon(b)yy^*f_\varepsilon(b))^{1/2}$. □

LEMMA 3.7. *Let A be a real C*-algebra such that for all non-zero elements a, b there exist x, y with $a = xby$. Suppose that A contains a non-zero projection and let c be a non-zero positive element such that $\overline{cAc} \neq A$. Then $\overline{cAc}$ contains an infinite projection.*

Proof. The argument from $(vii) \Rightarrow (i)$ of Theorem 2.2 of [31] applies to the real case to show that for any non-trivial projection p and positive element x there is a Murray-von Neumann equivalence between p and a subprojection of x. We will repeatedly use this fact.
In the unital case, this shows that the unit 1 is Murray-van Neumann equivalent to a projection of $\overline{cAc}$, which is necessarily infinite.
Now suppose that A has no unit but has a non-zero projection p. Applying the fact above to a non-zero positive element d in $(1-p)A(1-p)$ gives a projection q such that $p \sim q$ and $p \perp q$. Now apply the fact again using the projection $p + q$ and the positive element p to show that $p + q$ is infinite. Finally, apply the same fact using the projection $p + q$ and the positive element c to show that $p + q$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$. □

Lemma 3.8. *Let A be a real simple C^*-algebra. Then the following are equivalent:*

(1) *A is purely infinite,*
(2) *A is not isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and for each pair of non-zero elements $a, b \in A$ there exist $x, y \in A$ such that $a = xby$,*
(3) *A is not isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and for each pair of non-zero positive elements $a, b \in A$ there exists $x \in A$ such that $a = xbx^*$.*

Furthermore, if these conditions are satisfied, then for all $\varepsilon > 0$ the element x in (3) can be chosen to satisfy $\|x\| \leq (\|a\|/\|b\|)^{1/2} + \varepsilon$.

Proof. As the result is well-known in the complex case, we may assume by Theorem 3.1 that A is c-simple. By Lemma 3.6, (2) and (3) are equivalent.
For (1) $\Rightarrow$ (2), let a, b be non-zero elements of A, identified with $e_{11}(\mathcal{K}^{\mathbb{R}} \otimes A)e_{11}$. We are assuming $A_{\mathbb{C}}$ is simple, so Theorem 2.4 of [17] applied to the unital algebra pAp implies that $\mathcal{K} \otimes pA_{\mathbb{C}}p$ is algebraically simple. Then by Proposition 3.5, $\mathcal{K} \otimes A_{\mathbb{C}}$ is algebraically simple, whence $\mathcal{K}^{\mathbb{R}} \otimes A$ is. The argument from $(ii) \Rightarrow (xi)$ of Theorem 2.2 of [31], then produces $x, y \in \mathcal{K}^{\mathbb{R}} \otimes A$ with $a = xby$, so $a = (e_{11}xe_{11})b(e_{11}ye_{11})$.
For (2) $\Rightarrow$ (1), we use a simplified argument based on the proof of Theorem 1.2 of [31]. Note first that if a nonzero projection can be found in A then Lemma 3.7 gives the result. (In particular, this takes care of the unital case.) Let a, d be non-zero positive elements of A with $da = ad = a$ (for a positive element x with norm 1 take $a = f_{1/2}(x)$ and $d = f_{1/4}(x)$). Then let $s, t \in A$ with $d = sat$ and let $y = (as^*sa)^{1/2}t$. An easy argument shows that $|y||y^*| = |y^*|$ hence $f_{1/2}(|y|)f_{1/8}(|y^*|) = f_{1/8}(|y^*|)$. Unless $f_{1/4}(|y|)$ is a projection, Lemma 4.2 of [7] gives a scaling element $t \in A$. In this case, $p_n = f_n + f_n^{1/2}tf_n^{1/2} + f_n^{1/2}t^*f_n^{1/2}$ (where $f_n = t^n(t^*)^n - t^{n+1}(t^*)^{n+1}$ for $n \geq 2$) is a projection by Theorem 3.1 of [7], .
The final condition holds as in Lemma 2.4 of [28]. □

Theorem 3.9. *A real c-simple C^*-algebra A is purely infinite if and only if $A_{\mathbb{C}}$ is purely infinite.*

Proof. From Theorem 3.3 of [43] we know that A is purely infinite if $A_{\mathbb{C}}$ is.
For the converse, suppose A is purely infinite, let ω be a free ultrafilter on $\mathbb{N}$ and let A_ω be the corresponding ultrapower algebra, defined in Definition 6.2.2 of [39]. Note that the proofs of Proposition 6.2.6 of [39] and the preliminary Lemma 6.2.3 carry over directly to the real case (using Lemma 3.8). Therefore A_ω is simple and purely infinite. Suppose that D is a dimension function, as defined in Definition I.1.2 of [5], on the complexification $(A_\omega)_{\mathbb{C}} \cong (A_{\mathbb{C}})_\omega$. For each positive non-zero a, b in A_ω there exist $x, y \in A_\omega$ with $b = xax^*$ and $a = yby^*$ so $D(a) = D(b)$. For any infinite projection $p \in A_\omega$, there exists a projection $q < p$ with $D(p) = D(q) + D(p - q) = D(p) + D(p)$, so $D(a) = D(p) = 0$ for each positive $a \in A_\omega$. Then for each positive $a \in (A_\omega)_{\mathbb{C}}$, we have $0 \leq D(a) \leq D(a + \Phi(a)) = 0$. So there is no dimension function on $(A_\omega)_{\mathbb{C}}$ and therefore, by Theorem II.2.2 of [5], no 2-quasitrace. Therefore $A_{\mathbb{C}}$

is weakly purely infinite by Theorem 4.8 of [28]. By Corollary 4.16 of [28] it is therefore purely infinite. □

Corollary 3.10. (1) *If A and B are stably isomorphic real C*-algebras, and if A is purely infinite and c-simple then so is B.*

(2) *Any inductive limit of real purely infinite c-simple C*-algebras is again purely infinite and c-simple.*

(3) *If A and B are purely infinite and c-simple, then so is $A \otimes_{min} B$.*

Proof. These results follow immediately from Theorem 3.9 and the same results in the complex case (see Proposition 4.1.8 of [39]). □

We now work toward showing that the K_0 and K_1 groups of a real purely infinite algebra can be described in a similar way to the complex case. The next two lemmas provide the required modification of Lemma 1.7 of [19].

Lemma 3.11. *Let A be a real c-simple purely infinite unital C*-algebra and let $u \in \mathcal{U}(A)$ and let $\lambda \in \sigma(u)$. For any $\varepsilon > 0$ there exists $v \in \mathcal{U}(A)$ such that $\|u - v\| < \varepsilon$ and*

(1) *if $\lambda = \lambda^*$ then $v = v_0 + \lambda p$ where p is a non-zero projection in A and $v_0 \in \mathcal{U}(p^\perp A p^\perp)$.*

(2) *if $\lambda \neq \lambda^*$ then $v = v_0 + \lambda p_1 + \lambda^* p_2$ where p_1 and p_2 are orthogonal non-zero orthogonal projections in $A_{\mathbb{C}}$ satisfying $\Phi(p_1) = p_2$ and $v_0 \in \mathcal{U}((p_1+p_2)^\perp A (p_1+p_2)^\perp)$.*

Proof. First assume that $\lambda = \lambda^*$. Let h be a positive function on $\sigma(u)$ such that $\text{supp}(h) \subset N_{\varepsilon_0}(\lambda)$ and $h(z^*) = h(z)$ for all $z \in \sigma(u)$. Then $h(u) \in A$ and let p be a non-zero projection in $\overline{h(u)Ah(u)}$. As in the proof of Lemma 1.7 of [19], we have $\|u - (p^\perp u p^\perp + \lambda p)\| \leq 3\varepsilon_0$. Then the polar decomposition of $(p^\perp u p^\perp + \lambda p)$ yields a unitary v of the required form that, if ε_0 is sufficiently small, will satisfy $\|u - v\| < \varepsilon$.

Now assume $\lambda \neq \lambda^*$. Choose ε_0 small enough so that $N_{\varepsilon_0}(\lambda) \cap N_{\varepsilon_0}(\lambda^*) = \emptyset$. Let h_1 be a positive function on $\sigma(u)$ such that $\text{supp}(h_1) \subset N_{\varepsilon_0}(\lambda)$. By Theorem 3.9, $A_{\mathbb{C}}$ is purely infinite so there is a non-zero projection p_1 in $B = \overline{h_1(u)A_{\mathbb{C}}h_1(u)}$.

Define $p_2 = \Phi(p_1) \in \Phi(B)$ and $p = p_1 + p_2$. Now $\Phi(h_1(u)) = h_2(u)$ where h_2 is the continuous function on $\sigma(u)$ defined by $h_2(z) = h_1(z^*)$. Since $\text{supp}(h_2) \subset N_{\varepsilon_0}(\lambda^*)$, we have $h_1(u)h_2(u) = 0$. Thus p_1 and p_2 are orthogonal projections and $p \in A$.

As in Lemma 1.7 of [19], we have $\|up_1 - \lambda p_1\| \leq \varepsilon_0$ and $\|up_2 - \lambda^* p_2\| \leq \varepsilon_0$ from which it follows that $\|u - (p^\perp u p^\perp + \lambda p_1 + \lambda^* p_2)\| \leq 8\varepsilon_0$. The required unitary v is obtained by taking the polar decomposition of $p^\perp u p^\perp + \lambda p_1 + \lambda^* p_2$ in A. □

Lemma 3.12. *Let A and u be as above. Then there is a projection p in A and a unitary v in $\mathcal{U}(p^\perp A p^\perp)$ such that $u \sim v + p$.*

Proof. If $1 \in \sigma(u)$ then using Lemma 3.11, approximate u by an element of the form $v + p$. If the approximation is close enough, then the two unitaries will be in the same path component.
If $\lambda \in \sigma(u)$ where $\lambda \neq \lambda^*$, use Lemma 3.11 to approximate u by $v + \lambda p_1 + \lambda^* p_2$. Then we can easily find a path from $\lambda p_1 + \lambda^* p_2$ to $p_1 + p_2$ in $(p_1 + p_2)A(p_1 + p_2)$.
The only possibility left is $u = -1$. In that case, find two orthogonal projections q_1 and q_2 and a partial isometry s such that $ss^* = q_1$ and $s^*s = q_2$. Let $p = q_1 + q_2$. The projection p can be rotated to $-p$ within the 2×2 matrix algebra generated by q_1, q_2 and s. Hence the unitary $-1 = -(p^\perp) + -p$ can be connected to the unitary $-(p^\perp) + p$. □

Proposition 3.13. *Let A be a c-simple purely infinite real C*-algebra. Then*

(1) $K_0(A) = \{[p] \mid p \text{ is a non-zero projection in } A\}$
(2) $K_1(A) = \mathcal{U}(A)/\mathcal{U}_0(A)$ *(for A unital).*

Proof. In the complex case, these results are proven in Section 1 of [19]. The proofs of those results as well as the proofs of the preliminary lemmas carry over to the real case, with two modifications. The first is to the proof of Lemma 1.7 of [19], which we already addressed with the proof of Lemma 3.12 above.
Secondly, in the proof of Lemma 1.1 of [19] the author uses an element of the form
$$\widetilde{w} = w + w^* + (1 - w^*w - ww^*), \qquad (\text{where } w^2 = 0)$$
that is a unitary lying in the finite dimensional C*-algebra generated by w. In the complex case it follows that $\widetilde{w} \in \mathcal{U}_0(A)$, whereas in the real case unitary groups of finite dimensional C*-algebras are not connected in general.
However, if instead we take $\widetilde{w} = w - w^* + (1 - w^*w - ww^*)$ then $\widetilde{w}$ is in the connected component of the identity, as it corresponds to a matrix of the form $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The proof of Lemma 1.1 of [19] can be completed without change using this alternative $\widetilde{w}$. □

We note that part (1) of Proposition 3.13 appeared as Proposition 11 of [10].

4. Exponential Rank

Definition 4.1. An element a in a real C*-algebra A is *skew-adjoint* if $a^* = -a$. The set of skew-adjoint elements is denoted by A_{sk}.

If a is skew-adjoint, then $\sigma(a) = -\sigma(a) \subseteq i\mathbb{R}$ and the real unital C*-algebra generated by a is isomorphic to
$$\{f \in C(\sigma(a), \mathbb{C}) \mid f(it)^* = f(-it)\} .$$
Furthermore, if a is a skew-adjoint element in a real unital C*-algebra A, then $\exp(a)$ is a unitary in A.

Lemma 4.2. *Let A and B be unital real C*-algebras.*

(1) $\mathcal{U}_0(A) = \{\prod_{i=1}^n \exp(k_i) \mid k_i \in A_{sk}, n \in \mathbb{N}\}$.
(2) *If $\alpha\colon A \to B$ is unital and surjective, then $\alpha(\mathcal{U}_0(A)) = \mathcal{U}_0(B)$.*

Proof. Suppose first that u is a unitary element in A with $\|u - 1\| < 2$. Then $-1 \notin \sigma(u)$. We define a continuous function $f\colon \mathbb{T} \setminus \{0\} \to i(-\pi, \pi)$ by $f(\exp(it)) = it$ for $t \in (-\pi, \pi)$. Then $f(u)$ is in the real C*-algebra generated by u, is skew-adjoint, and satisfies $\exp(f(u)) = u$.
More, generally, if $u \in \mathcal{U}_0(A)$ then there exists a chain $u = u_0, u_1, u_2, \ldots, u_n = 1$ with $\|u_{i-1} - u_i\| < 2$ for all $i \in \{1, 2, \ldots, n\}$. Then applying the previous paragraph we have $u_{i-1}u_i^* = \exp(k_i)$ for all i with $1 \le i \le n$. Then $u = \prod_{i=1}^n \exp(k_i)$.
Conversely, if $\{k_i\}_{i=1}^n$ is any collection of skew-adjoint elements, then $u(t) = \prod_{i=1}^n \exp(tk_i)$ for $0 \le t \le 1$ is a continuous path of unitaries from 1_A to $\prod_{i=1}^n \exp(k_i)$. This proves (1).
For (2), the inclusion $\alpha(\mathcal{U}_0(A)) \subseteq \mathcal{U}_0(B)$ is immediate. Let $u \in \mathcal{U}_0(B)$. Then $u = \prod_{i=1}^n \exp(k_i)$ for some skew-adjoint elements $k_i \in B$. Let $l_i \in A$ be elements such that $\alpha(l_i) = k_i$. We may assume that l_i is skew-adjoint for all i, by replacing with $\frac{1}{2}(l_i - l_i^*)$ if necessary. Then $u = \alpha\left(\prod_{i=1}^n \exp(l_i)\right)$. □

Let $\mathcal{E} = \{\exp(k) \mid k \in A_{sk}\}$ and let $\mathcal{E}^n$ be the set of all products of at most n elements of $\mathcal{E}$. Thus $\mathcal{U}_0(A) = \cup_{n=1}^\infty \mathcal{E}^n$. The argument in the proof above also implies that the set $\mathcal{E}^{n+1}$ contains the topological closure of $\mathcal{E}^n$ so that we have the an increasing sequence

$$\mathcal{E} \subseteq \overline{\mathcal{E}} \subseteq \mathcal{E}^2 \subseteq \overline{(\mathcal{E})^2} \subseteq \cdots \subseteq (\mathcal{E})^n \subseteq \overline{(\mathcal{E}^n)} \subseteq (\mathcal{E})^{n+1} \subseteq \ldots$$

similar to that in [37], motivating the following definition.
Definition 4.3.

(1) The *exponential rank* of A, written $\mathrm{cer}(A)$, is equal to the integer n if $\mathcal{E}^n$ is the smallest set in this sequence to be equal to $\mathcal{U}_0(A)$ and is equal to the symbol $n + \varepsilon$ if $\overline{\mathcal{E}^n}$ is the smallest set to be equal to $\mathcal{U}_0(A)$. If $\mathcal{E}^n \neq \mathcal{U}_0(A)$ for all n then $\mathrm{cer}(A) = \infty$.
(2) The *exponential length* of A, written $\mathrm{cel}(A)$, is equal to the smallest number $0 < \mathrm{cel}(A) \le \infty$ such that every unitary u in $\mathcal{U}_0(A)$ can be written in the form

$$u = \exp(k_1)\exp(k_2)\cdots\exp(k_n)$$

where $k_i \in A_{sk}$ and

$$\|k_1\| + \|k_2\| + \cdots + \|k_n\| \le \mathrm{cel}(A)\ .$$

With these definitions, the proofs of Section 2 of [37] can be applied with minimal modification to prove the following results.

Lemma 4.4. *Let A be a real unital C*-algebra and let n be a positive integer.*

(1) *If $\mathrm{cel}(A) < n\pi$ then $\mathrm{cer}(A) \le n$.*
(2) *If $\mathrm{cel}(A) \le n\pi$ then $\mathrm{cer}(A) \le n + \varepsilon$.*

Lemma 4.5. *Let A be a real unital C*-algebra. If every unitary $u \in \mathcal{U}_0(A)$ can be connected to the identity by a rectifiable path of length no more than M, then $\mathrm{cel}(A) \le M$.*

Definition 4.6. A real C*-algebra A has *real skew rank zero* if the elements of A_{sk} with finite spectrum are dense in A_{sk}.

In the case of a complex C*-algebra A there is a bicontinuous bijection $A_{sa} \to A_{sk}$ given by multiplication by i, showing that A has skew rank zero if and only it has real rank zero. However, in the case of real C*-algebras things are more subtle. For example the condition of being skew-rank zero is not equivalent (in the unital case) to the condition that the invertible elements of A_{sk} are dense. Indeed, all finite dimensional real C*-algebras have real skew rank zero, but the invertibles of $(M_n)_{sk}$ are dense only if n is even.

Proposition 4.7. *Let A be a real unital c-simple purely infinite C*-algebra satisfying $[1] \in 2K_0(A)$. Then the invertibles of A_{sk} are dense in A_{sk} and A has real skew rank zero.*

Proof. Let A be a real purely infinite C*-algebra such that $[1] \in 2K_0(A)$. Let $a \in A_{sk}$ and let $\varepsilon > 0$ be given. Define functions $g\colon i\mathbb{R} \to \mathbb{R}$ and $f\colon i\mathbb{R} \to i\mathbb{R}$ by

$$g(it) = \max\{\varepsilon - |t|, 0\} \qquad \text{and} \qquad f(it) = \begin{cases} i(t+\varepsilon) & t \le -\varepsilon \\ 0 & |t| < \varepsilon \\ i(t-\varepsilon) & t \ge \varepsilon \,. \end{cases}$$

Then $g(a) \in A_+$ and $f(a) \in A_{sk}$.
Since A is purely infinite, there is a projection $p \in \overline{g(a)Ag(a)}$ with $2[p] = [1] \in K_0(A)$. Then $[1-p] = [p]$ so there is a partial isometry s such that $s^*s = 1-p$ and $ss^* = p$. Since $f(a)g(a) = 0$ we have $f(a) = (1-p)f(a)(1-p)$.
Let $b = f(a) + \varepsilon(s - s^*)$. In matrix form under the decomposition indicated by the projection sum $1 = (1-p) + p$ we have

$$b = \begin{pmatrix} f(a) & -\varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

whence b is invertible. This proves the first statement.
For the second statement, again let $a \in A_{sk}$ and let $\varepsilon > 0$ be given. By the first part of the theorem, we may assume that a is invertible, hence $\sigma(a) \subset i\mathbb{R} \setminus \{0\}$. Write $a = a_1 + a_2$ where the elements $a_i \in A_{\mathbb{C}}$ satisfy $\sigma(a_1) \subset i(0,\infty)$ and $\sigma(a_2) \subset i(-\infty, 0)$. Note also that $\Phi(a_1) = a_2$.
Since $A_{\mathbb{C}}$ is simple and purely infinite it has real rank zero, so there exists $b_1 \in (A_{\mathbb{C}})_{sk}$ such that $\sigma(b_1)$ is a finite subset of $i\mathbb{R}^+$ and $\|a_1 - b_1\| < \varepsilon/2$. Let $b_2 = \Phi(b_1)$ and let $b = b_1 + b_2$. Then b is a skew-adjoint element of A with finite spectrum and $\|a - b\| < \varepsilon$. □

Lemma 4.8. *Let A be a real c-simple unital C*-algebra such that $[1] \in 2K_0(A)$. Let $u \in \mathcal{U}(A)$ be a unitary such that $\sigma(u) \neq S^1$. Then for every $\varepsilon > 0$ there is a unitary v with finite spectrum such that $\|u - v\| < \varepsilon$.*

Proof. If $-1 \notin \sigma(u)$, then there is a continuous function $f\colon \sigma(u) \to i[-\pi, \pi]$ that is a right inverse to the function $it \mapsto \exp(it)$ and that satisfies $f(z^*) = f(z)^*$. Then $f(u) \in A_{sk}$ can be approximated within δ by a skew adjoint

element b with finite spectrum by Proposition 4.7. For an appropriate choice of δ, this implies that $\exp(b) \in \mathcal{U}(A)$ approximates u within ε.
Similarly, if $1 \notin \sigma(u)$, then there is a continuous function $f\colon \sigma(u) \to i[-\pi, \pi]$ that is a right inverse to the function $it \mapsto -\exp(-it)$ and that satisfies $f(z^*) = f(z)^*$.
In the general case, suppose that $\lambda \notin \sigma(u)$ for some $\lambda \in S^1$. Let $\sigma_1 = \{w \in \sigma(u) \mid \mathrm{Re}(w) > \mathrm{Re}(\lambda)\}$ and let $\sigma_2 = \{w \in \sigma(u) \mid \mathrm{Re}(w) < \mathrm{Re}(\lambda)\}$. Then $\sigma = \sigma_1 \cup \sigma_2$. Let $u_i = u_i E_u(\sigma_i)$, where $E_u(\sigma_i)$ denotes the spectral projection of u associated with the clopen subset σ_i of σ. Then $1 \notin \sigma(u_2)$ and $-1 \notin \sigma(u_1)$. Using the results from the first two paragraphs, let v_i be a unitary that approximates u_i in $E_u(\sigma_i) A E_u(\sigma_i)$ within ε. Then since $u = u_1 + u_2$ we have that $v = v_1 + v_2$ is a unitary that approximates u within ε. □

Lemma 4.9. *Let A be a real unital simple purely infinite C*-algebra let $u \in \mathcal{U}(A)$ and let $\{\lambda_1, \dots, \lambda_n\}$ be a subset of $\sigma(u)$ that is closed under conjugation. For any $\varepsilon > 0$ there exist $v \in \mathcal{U}(A)$ and orthogonal projections $p_1, \dots, p_n \in A_{\mathbb{C}}$ such that $\|u - v\| < \varepsilon$ and $v = v_0 + \lambda_1 p_1 + \dots + \lambda_n p_n$ with $v_0 \in \mathcal{U}((p_1 + \dots + p_n)^{\perp} A (p_1 + \dots + p_n)^{\perp})$.*
Furthermore, the elements $\sum_{i=1}^n p_i$ and $\sum_{i=1}^n \lambda_i p_i$ are both in A.

Proof. Use the constructions of Lemma 3.11 above as in the proof of Lemma 6 of [34]. □

Lemma 4.10. *Let A be a real unital C*-algebra and let $u \in \mathcal{U}(A)$. For any $\varepsilon > 0$ there exists an $h \in M_2(A)_{sk}$ such that $\|u \oplus u^* - \exp(h)\| < \varepsilon$.*

Proof. As in the proof of Corollary 5 of [34], there exists a continuous path $v(t)$ of unitaries in $M_2(A)$ with $v(0) = 1$ and $v(\pi/2) = u \oplus u^*$ such that $-1 \notin \sigma(v(t))$ for $0 \le t < \pi/2$. Thus we can find a t close enough to $\pi/2$ such that $\|u \oplus u^* - v(t)\| < \varepsilon$ and $v(t) = \exp(h)$ for a skew-adjoint h. □

Lemma 4.11. *Let A be a real unital c-simple purely infinite C*-algebra such that $[1] \in 2K_0(A)$. Let e_1, e_2, e_3, e_4 be nonzero orthogonal projections in A that sum to 1. Let a be a partial isometry such that $a^*a = e_2$ and $aa^* = e_3$. Let $u \in \mathcal{U}(e_1 A e_1)$ and $v \in \mathcal{U}(e_2 A e_2)$ be unitaries with $\sigma(u) = S^1$. Then for all $\varepsilon > 0$ there is a unitary $z \in \mathcal{U}(A)$ and a unitary $w \in \mathcal{U}(e_4 A e_4)$ with finite spectrum such that*

$$\|z^*(u + 1 - e_1)z - (u + v + av^*a^* + w)\| < \varepsilon\ .$$

Proof. This proof closely follows that of Lemma 7 of [34]. By Lemma 4.10 there is a unitary in $(e_2 + e_3)A(e_2 + e_3)$ that is arbitrarily close to $v + av^*a^*$ and that has the form $\exp h$ for $h \in A_{sk}$. This in turn can be approximated by a unitary that has finite spectrum by Proposition 4.7. The general form of such a unitary is

$$\sum_{k=1}^{n} (\lambda_k q_{k1} + \lambda_k^* q_{k2}) + 1 q_{01} + (-1) q_{02}$$

where $\lambda_k^* \neq \lambda_k$, the nonzero projections $q_{ki} \in A_{\mathbb{C}}$ satisfy $\Phi(q_{k1}) = q_{k2}$ for $1 \leq k \leq n$, and the (possibly zero) projections q_{0i} are in A. Furthermore, the q_{ki} are orthogonal and sum to $e_2 + e_3$. Without loss of generality, we assume that $v + av^*a^*$ has this form. With an obvious choice of coefficients λ_{ki} we can write this as

$$v + av^*a^* = \sum_{k=0}^{n} \sum_{i=1}^{2} \lambda_{ki} q_{ki} = \sum \lambda_{ki} q_{ki} \ .$$

(Henceforth in this proof will use an undecorated $\sum$ to represent a double sum indexed as $\sum_{k=0}^{n} \sum_{i=1}^{2}$.)
Now we replace u by a nearby element of the form given by Lemma 4.9. Specifically, there are orthogonal projections $p_{ki} \in e_1 A_{\mathbb{C}} e_1$ and, setting $p = e_1 - \sum p_{ki} \in A$, there is a unitary $u_0 \in pAp$ such that

$$u = u_0 + \sum \lambda_{ki} p_{ki}$$

(where the projection $p_{0i} = 0$ if and only if $q_{0i} = 0$).
For each $k \in \{1, \ldots, n\}$ let $c_{k1} \in A_{\mathbb{C}}$ be a partial isometry such that $c_{k1}^* c_{k1} = p_{k1}$ and $c_{k1} c_{k1}^* < p_{k1}$. Then $c_{k2} = \Phi(c_{k1})$ satisfies $c_{k2}^* c_{k2} = p_{k2}$ and $c_{k2} c_{k2}^* < p_{k2}$ and $c_k = c_{k1} + c_{k2} \in A$ satisfies $c_k^* c_k = p_{k1} + p_{k2}$ and $c_k c_k^* < p_{k1} + p_{k2}$. For $k = 0$ we obtain partial isometries $c_{0k} \in A$ such that $c_{0i}^* c_{0i} = p_{0i}$ and $c_{ki} c_{ki}^* < p_{ki}$. Then $c = p + \sum c_{ki} \in A$ satisfies

$$c^*c = e_1, \qquad cc^* = e_1 - \sum (p_{ki} - c_{ki} c_{ki}^*), \qquad \text{and}$$

$$cuc^* = u_0 + \sum \lambda_{ki} c_{ki} c_{ki}^* \ .$$

Similarly we can find a collection of partial isometries d_{ki} with domain projection q_{ki} and range projection a subprojection of $p_{ki} - c_{ki}^* c_{ki}$ that also satisfy $\Phi(d_{k1}) = d_{k2}$ for $k \neq 0$ and $\Phi(d_{ki}) = d_{ki}$ for $k = 0$. Then the partial isometry $d = \sum d_{ki} \in A$ satisfies

$$d^*d = e_2 + e_3, \qquad dd^* \leq \sum (p_{ki} - c_{ki} c_{ki}^*), \qquad \text{and}$$

$$d \left(\sum \lambda_{ki} q_{ki} \right) d^* = \sum \lambda_{ki} d_{ki} d_{ki}^* \ .$$

Now, choose a partial isometry b such that

$$b^*b < e_4, \qquad bb^* = \sum (p_{ki} - c_{ki} c_{ki}^* - d_{ki} d_{ki}^*)$$

and define

$$w_0 = \sum \lambda_{ki} b^* (p_{ki} - c_{ki} c_{ki}^* - d_{ki} d_{ki}^*) b \ .$$

Then $z_0 = b + c + d$ is a partial isometry with $z_0^* z_0 = e_1 + e_2 + e_3 + b^*b$ and $z_0 z_0^* = e_1$. So in $K_0(A)$ we have $[e_1] = [e_1 + e_2 + e_3 + b^*b]$, which implies $[1 - e_1] = [e_4 - b^*b]$. By Proposition 11 of [10], there is a partial isometry $z_1 \in A$ such that $z_1 z_1^* = 1 - e_1$ and $z_1^* z_1 = e_4 - b^*b$. Then $w = w_0 + e_4 - b^*b$ is a unitary with finite spectrum in $e_4 A e_4$ and $z = z_0 + z_1$ is a unitary in A that satisfies $z^*(u + 1 - e_1) z = u + \sum \lambda_{ki} q_{ki} + w$. □

Theorem 4.12. *Let A be a real unital c-simple purely infinite C*-algebra such that $[1] \in 2K_0(A)$. For every $u \in \mathcal{U}_0(A)$ and every $\varepsilon > 0$ there is a unitary v with finite spectum such that $\|u - v\| < \varepsilon$.*

Proof. With the lemmas that we have developed, the proof is now the same as that of the unital case of Theorem 1 and Corollary 2 of [34], except that wherever there is an element of the form $\exp(ih)$ where h is self-adjoint, we use $\exp(k)$ where k is skew-adjoint. □

As in the complex case, we have the following corollary concerning exponential length.

Corollary 4.13. *Let A be a real unital c-simple purely infinite C*-algebra such that $[1] \in 2K_0(A)$. Then $\mathrm{cel}(A) \leq 4$.*

Proof. By Theorem 4.12, every unitary $u \in \mathcal{U}_0(A)$ can be approximated within ε by a unitary v with finite spectrum. For ε sufficiently small, $\|v^*u - 1\| < \varepsilon$ implies there exists a skew-adjoint k_2 such that $v^*u = \exp(k_2)$ with $\|k_2\| \leq 4 - \pi$. As v has finite spectrum, there exists a skew-adjoint k_1 such that $v = \exp(k_1)$ and $\|k_1\| \leq \pi$. Then $u = \exp(k_1)\exp(k_2)$ and $\|k_1\| + \|k_2\| \leq 4$. □

5. Homomorphisms from $\mathcal{O}_n^{\mathbb{R}}$

The following theorem gives the real version of the Rokhlin property of the Bernoulli shift, established in [15] and summarized in [39]. Let $M_{2^\infty} = \lim_{k\to\infty} M_{2^k}$ be the real CAR algebra and let $\mathbb{H}$ be the real C*-algebra of quaternions.

Proposition 5.1. *Let σ be the one-sided Bernoulli shift on M_{2^∞}. For each $\varepsilon > 0$ and for each $r \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and projections $e_0, e_1, \dots, e_{2^r} = e_0 \in M_{2^k}$ such that $\sum_{j=1}^{2^r} e_j = 1$ and $\|\sigma(e_j) - e_{j+1}\| < \varepsilon$ for all $j = 0, 1, 2, \dots, 2^r - 1$.*

Proof. Let $A_k = M_{2^k}$ and let $A = M_{2^\infty}$. Using the notation of Proposition 5.1.3 of [39], let S denote the unilateral shift on $\ell^2(\mathbb{N}, \mathbb{C})$, let $\omega_k = \exp(2\pi i/2^k)$ for each $k \geq 0$ and, given $\delta > 0$, let

$$f_0 = \frac{1}{\sqrt{n_0}}(1, 1, \dots, 1, 0, 0, \dots) \in \ell^2(\mathbb{N}, \mathbb{R})$$

be a unit vector with $\|Sf_0 - f_0\| < \delta$ and let

$$f_1 = \frac{1}{\sqrt{n_1}}(0, 0, \dots 0, 1, -1, 1, -1, \dots, -1, 0, 0, \dots)$$

be a unit vector in $\ell^2(\mathbb{N}, \mathbb{R})$, orthogonal to f_0, with $\|Sf_1 + f_1\| < \delta$. Then, for $r \in \mathbb{N}$, let $f_2, \dots, f_r \in \ell^2(\mathbb{N}, \mathbb{C})$ be defined by

$$f_j = \frac{1}{\sqrt{n_j}}(0, 0, \dots 0, 1, \omega_j, \omega_j^2, \dots, \omega_j^{n_j - 1}, 0, 0, \dots)$$

where there are sufficiently many initial zeros to make f_j orthogonal to its predecessors and where n_j is chosen so that

$$\langle f_j, \overline{f_j} \rangle = 1 + \omega_j^2 + \dots \omega_j^{2(n_j - 1)} = 0$$

and $\|Sf_j - \omega_j f_j\| < \delta$. If $f_j = g_j + ih_j$ with $g_j, h_j \in \ell^2(\mathbb{N}, \mathbb{R})$ then, from the orthogonality of f_j and $\overline{f_j}$, $\|g_j\| = \|h_j\| = 1/\sqrt{2}$.
Let $a : \ell^2(\mathbb{N}, \mathbb{C}) \to A_{\mathbb{C}}$ be the map described in [15] and [39] satisfying the canonical anticommutation relations and observe that a maps $\ell^2(\mathbb{N}, \mathbb{R})$ into A. Let $v_1 = w_1 = a(f_1)(a(f_0) + a(f_0)^*)$ and, for each $2 \leq j \leq r$ let $v_{2j-2} = a(f_j)(a(f_0) + a(f_0)^*)$, $v_{2j-1} = a(\overline{f_j})(a(f_0) + a(f_0)^*)$, $w_{2j-2} = a(\sqrt{2}g_j)(a(f_0) + a(f_0)^*) = (v_{2j-2} + v_{2j-1})/\sqrt{2}$ and $w_{2j-1} = a(\sqrt{2}h_j)(a(f_0) + a(f_0)^*) = -i(v_{2j-2} - v_{2j-1})/\sqrt{2}$. Note that $\{w_1, w_2, \ldots, w_{2r-1}\} \subset A_k$ for all sufficiently large k.
It is noted in the proof of Proposition 4.1 of [15] that the elements v_i for $1 \leq i \leq 2r-1$ satisfy the relations $v_i v_j + v_j v_i = 0$ and $v_i v_j^* + v_j^* v_i = \delta_{ij} 1$. It follows from this that the elements w_i for $1 \leq i \leq 2r-1$ satisfy the same relations. Therefore, using the matrix units described in the proof of Proposition 4.1 of [15], the real C*-algebra B generated by $w_1, \ldots, w_{2r-1}$ is isomorphic to $M_{2^{2r-1}}$.
Slightly varying the proof of Proposition 4.1 of [15], let β be the automorphism of the complexification of B determined by $\beta(v_1) = -v_1$, $\beta(v_{2j}) = \omega_j v_{2j}$ and $\beta(v_{2j+1}) = \overline{\omega_j} v_{2j+1}$ for each $1 \leq j \leq r-1$. Note that $\beta(w_{2j}) = \frac{1}{2}(\omega_j + \overline{\omega_j}) w_{2j} + \frac{i}{2}(\omega_j - \overline{\omega_j}) w_{2j+1}$ and $\beta(w_{2j+1}) = -\frac{i}{2}(\omega_j - \overline{\omega_j}) w_{2j} + \frac{1}{2}(\omega_j + \overline{\omega_j}) w_{2j+1}$, so that β leaves the real algebra B invariant. Identifying B with $M_{2^{2r-1}}$, there is an orthogonal matrix W implementing β. By standard linear algebra, described for example in Section 81 of [22], W is orthogonally conjugate to an orthogonal matrix consisting of diagonal elements ± 1 and diagonal 2×2 rotation matrices, determined by the eigenvalues of W.
As in [39], on the complexification of B, identified with $M_{2^{2r-1}}(\mathbb{C})$, β is implemented by a diagonal unitary with entries $1, \omega_r, \omega_r^2, \ldots, \omega_r^{2^r-1}$, each repeated 2^{r-1} times. (The unitary arises as the tensor product of one diagonal unitary with entries $1, \omega_r, \omega_r^2, \ldots, \omega_r^{2^r-1}$ and another with entries $1, \overline{\omega_r}, \overline{\omega_r}^2, \ldots, \overline{\omega_r}^{2^{r-1}-1}$.) On $B \cong M_{2^{2r-1}}$ the orthogonal matrix W implementing β is therefore conjugate to an orthogonal matrix with 2×2 diagonal blocks $\operatorname{diag}(1, -1), R, R^2, \ldots, R^{2^{r-1}-1}$, each repeated 2^{r-1} times, where

$$R = \begin{pmatrix} \cos(\pi/2^{r-1}) & -\sin(\pi/2^{r-1}) \\ \sin(\pi/2^{r-1}) & \cos(\pi/2^{r-1}) \end{pmatrix}.$$

The cyclic shift on M_{2^r} is implemented by the unitary

$$V = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

which is orthogonally conjugate to $\operatorname{diag}\left(\operatorname{diag}(1, -1), R, R^2, \ldots, R^{2^{r-1}-1}\right)$. It follows that the orthogonal element W implementing β on B is orthogonally conjugate to a direct sum of 2^{r-1} copies of V and thus that β is conjugate to a direct sum of 2^{r-1} cyclic shifts. It follows that there are 2^r orthogonal projections $e_0, e_1, \dots, e_{2^r} = e_0$ in B (each of rank 2^{r-1}) that are cyclically permuted by β. As in the proof of Proposition 4.1 of [15], a suitable choice of δ at the start of the proof ensures that $\|\sigma(e_j) - \beta(e_j)\| < \varepsilon$ for each j and therefore the projections $e_0, e_1, \dots, e_{2^r} = e_0$ have the required properties. □

Theorem 5.2. *Let D be a real unital C*-algebra satisfying*

(i) *the canonical homomorphism $\mathcal{U}(D)/\mathcal{U}_0(D) \to K_1(D)$ is an isomorphism, and*
(ii) $\mathrm{cel}(D) < \infty$.

Let n be an even integer, let ϕ, ψ be unital homomorphisms from $\mathcal{O}_n^{\mathbb{R}}$ to D, let λ be the endomorphism of D defined by $\lambda(a) = \sum_{j=1}^n \phi(s_j) a \phi(s_j)^*$ and let $u \in \mathcal{U}(D)$ be defined by $u = \sum_{j=1}^n \psi(s_j)\phi(s_j)^*$, where $s_1, \dots, s_n$ are the canonical generators of $\mathcal{O}_n^{\mathbb{R}}$. Then the following are equivalent:

(1) $u \in \overline{\{v\lambda(v)^* \mid v \in \mathcal{U}(D)\}}$,
(2) $[u] \in (n-1)K_1(D)$
(3) $[\phi] = [\psi] \in KK_0(\mathcal{O}_n^{\mathbb{R}}, D)$,
(4) *ϕ and ψ are approximately unitarily equivalent.*

In particular, these statements are equivalent if D is a real unital purely infinite c-simple C-algebra.*

Proof of Theorem 5.2. The proof of the equivalence of the four statements, assuming (i) and (ii), is similar to that of the complex case in Sections 3 and 4 of [38], modified only by the use of unitaries of the form $\exp(h)$ with $h \in A_{sk}$ in the proof of the real version of Lemma 4.6 of [38]. We note that in the proof of the real version of Lemma 3.7 of [38], the required result from [19] holds, as was observed already in the proof of Proposition 3.13 above.
Suppose D is a real unital purely infinite c-simple C*-algebra. Then condition (i) holds for D by Proposition 3.13. Since $K_0(\mathcal{O}_n^{\mathbb{R}}) = \mathbb{Z}_{n-1}$ and n is even, we have $[1_{\mathcal{O}_n^{\mathbb{R}}}] \in 2K_0(\mathcal{O}_n^{\mathbb{R}})$. Using the unital homomorphism ϕ (or ψ) we obtain $[1_D] \in 2K_0(D)$. Then condition (ii) holds by Corollary 4.13. □

Corollary 5.3.

(1) *Let A be a real unital purely infinite c-simple C*-algebra. Any two unital homomorphisms $\phi, \psi\colon \mathcal{O}_2^{\mathbb{R}} \to A$ are approximately unitarily equivalent.*
(2) *Any inductive limit of the form $\mathcal{O}_2^{\mathbb{R}} \to \mathcal{O}_2^{\mathbb{R}} \to \mathcal{O}_2^{\mathbb{R}} \to \dots$, with unital connecting homomorphisms, is isomorphic to $\mathcal{O}_2^{\mathbb{R}}$.*
(3) $\mathcal{O}_2^{\mathbb{R}} \otimes \mathcal{O}_2^{\mathbb{R}} \cong \mathcal{O}_2^{\mathbb{R}}$.
(4) $\bigotimes_{n=1}^{\infty} \mathcal{O}_2^{\mathbb{R}} \cong \mathcal{O}_2^{\mathbb{R}}$.
(5) $\mathcal{O}_2^{\mathbb{R}} \otimes M_{2^\infty} \cong \mathcal{O}_2^{\mathbb{R}}$.
(6) $\mathcal{O}_2^{\mathbb{R}} \otimes \mathbb{H} \cong \mathcal{O}_2^{\mathbb{R}}$.

Proof. We know that $K^{CRT}(\mathcal{O}_2^{\mathbb{R}}) = 0$ from Section 5 of [8] so the universal coefficient theorem (Theorem 4.1 of [9]) implies that $KK_0(\mathcal{O}_2^{\mathbb{R}}, D) = 0$. Then part (1) follows immediately from Theorem 5.2.

Parts (2) and (3) can be proven in the same way as in the complex case. See Corollary 5.1.5 and Theorem 5.2.1 in [39]. Then part (4) follows from parts (2) and (3).
There is an isomorphism $\mathcal{O}_2^{\mathbb{R}} \cong M_2(\mathcal{O}_2^{\mathbb{R}})$, established as in the complex case: if s_1 and s_2 are generators of $\mathcal{O}_2^{\mathbb{R}}$ satisfying the canonical relations $s_i^* s_j = \delta_{ij} 1_{\mathcal{O}_2^{\mathbb{R}}}$ and $\sum_{i=1}^2 s_i s_i^* = 1$, then

$$S_1 = \left(\begin{smallmatrix} s_1 & s_2 \\ 0 & 0 \end{smallmatrix}\right) \qquad \text{and} \qquad S_2 = \left(\begin{smallmatrix} 0 & 0 \\ s_1 & s_2 \end{smallmatrix}\right)$$

satisfy the same relations and generate $M_2(\mathcal{O}_2^{\mathbb{R}})$. Using that isomorphism, part (5) follows from part (2).
Finally, part (6) follows from (5) and the formula $M_{2^\infty} \otimes \mathbb{H} \cong M_{2^\infty}$, which follows from Theorem 10.1 of [21] or from Theorem 4.8 of [42]). □

6. Tensor Product Theorems

In this section, we reproduce for real C*-algebras some standard results regarding tensor products with $\mathcal{O}_2^{\mathbb{R}}$ and $\mathcal{O}_\infty^{\mathbb{R}}$.

Definition 6.1.

(1) A real (resp. complex) C*-algebra A is *amenable* if for all $\varepsilon > 0$ and all finite subsets $F \subset A$, there is a finite dimensional real (resp. complex) C*-algebra B and contractive completely positive linear maps $\phi\colon A \to B$ and $\psi\colon B \to A$ such that

$$\|\psi \circ \phi(a) - a\| < \varepsilon \quad \text{for all } a \in F.$$

(2) A real (resp. complex) C*-algebra A is *nuclear* if for all real (resp. complex) C*-algebras B the algebraic tensor product $A \otimes_{\mathbb{R}} B$ (resp. $A \otimes_{\mathbb{C}} B$) has a unique C*-norm.

(3) A real (resp. complex) C*-algebra A is *exact* if the tensor product functor $B \mapsto A \otimes_{\min} B$ is exact. Here the tensor product is over $\mathbb{R}$ (resp. $\mathbb{C}$) and B can be any real (resp. complex) C*-algebra.

Lemma 6.2. *Let A be a real C*-algebra. Then*

(1) *A is amenable if and only if $A_{\mathbb{C}}$ is amenable.*
(2) *A is nuclear if and only if $A_{\mathbb{C}}$ is nuclear.*
(3) *A is exact if and only if $A_{\mathbb{C}}$ is exact.*

Consequently, A is amenable if and only if it is nuclear; and in this case it is also exact.

Proof. Part (1) can be found in Proposition 3 of [25] and the preceding text.
We claim that there is a one-to-one correspondence between C*-norms on the algebraic tensor product $A \otimes_{\mathbb{R}} B$ and those on $A_{\mathbb{C}} \otimes_{\mathbb{C}} B_{\mathbb{C}}$. Let γ be a C*-norm on $A \otimes_{\mathbb{R}} B$, and let $A \otimes_\gamma B$ be the real C*-algebra obtained by completion. Then the complexification $(A \otimes_\gamma B)_{\mathbb{C}}$ has a unique C*-norm extending that on $A \otimes_{\mathbb{R}} B$. Thus every C*-norm on the algebraic tensor product $A \otimes_{\mathbb{R}} B$ extends uniquely to a C*-norm on $A_{\mathbb{C}} \otimes_{\mathbb{C}} B_{\mathbb{C}}$. Part (2) follows immediately from this claim.

It also follows that the restriction of the minimal C*-norm on $A_{\mathbb{C}} \otimes_{\mathbb{C}} B_{\mathbb{C}}$ gives the minimal C*-norm on $A \otimes_{\mathbb{R}} B$. This fact, plus the fact that the complexification functor $A \mapsto A_{\mathbb{C}}$ is exact, implies (3).
The final statement then follows from the corresponding statement for complex C*-algebras. See Theorem 6.1.3 of [39] and Theorem 6.5.2 of [32]. □

PROPOSITION 6.3. *Let A be a real separable C*-algebra A. Then A is exact if and only if there is an injective homomorphism $\iota \colon A \to \mathcal{O}_2^{\mathbb{R}}$. If A is unital then ι can be chosen to be unital.*

Proof. Suppose that A is exact. Then $A_{\mathbb{C}}$ is separable and exact. Thus, by Theorem 6.3.11 of [39], there is an injective homomorphism $\iota_{\mathbb{C}} \colon A_{\mathbb{C}} \to \mathcal{O}_2$ (which is unital if $A_{\mathbb{C}}$ is unital). Then we can take ι to be the composition

$$A \hookrightarrow A_{\mathbb{C}} \xrightarrow{\iota_{\mathbb{C}}} \mathcal{O}_2 \hookrightarrow M_2(\mathcal{O}_2^{\mathbb{R}}) \cong \mathcal{O}_2^{\mathbb{R}} \ .$$

Conversely, if there is an injective homomorphism $\iota \colon A \to \mathcal{O}_2^{\mathbb{R}}$ then the complexification yields an injective homomorphism from $A_{\mathbb{C}}$ to $\mathcal{O}_2$. By Theorem 6.3.11 of [39] this implies that $A_{\mathbb{C}}$ is exact, hence A is exact. □

LEMMA 6.4. *Let A be a real purely infinite c-simple nuclear unital C*-algebra. Then all unital endomorphisms on $A \otimes \mathcal{O}_2^{\mathbb{R}}$ are approximately unitarily equivalent.*

Proof. In the complex case, this result is found as Theorem 6.3.8 of [39]. We will use that result to prove the real version.
By Corollary 5.3, Part (5) it suffices to show that any unital homomorphism

$$\gamma \colon A \otimes \mathcal{O}_2^{\mathbb{R}} \otimes M_{2^\infty} \to A \otimes \mathcal{O}_2^{\mathbb{R}} \otimes M_{2^\infty}$$

is approximately unitarily equivalent to the identity. We write $A' = A \otimes \mathcal{O}_2^{\mathbb{R}}$ and let

$$\alpha_{\ell,k} \colon A' \otimes M_{2^k} \hookrightarrow A' \otimes M_{2^\ell} \quad \text{for } k < \ell$$
$$\alpha_k \colon A' \otimes M_{2^k} \hookrightarrow A' \otimes M_{2^\infty}$$

be the canonical injections. Then we use the commutative diagram

$$\begin{array}{ccccccccc}
A' & \to \cdots \to & A' \otimes M_{2^k} & \xrightarrow{\alpha_{k+1,k}} & A' \otimes M_{2^{k+1}} & \to \cdots \to & A' \otimes M_{2^\infty} \\
\downarrow c & & \downarrow c & & \downarrow c & & \downarrow c \\
A'_{\mathbb{C}} & \to \cdots \to & (A' \otimes M_{2^k})_{\mathbb{C}} & \xrightarrow{\alpha_{k+1,k}} & (A' \otimes M_{2^{k+1}})_{\mathbb{C}} & \to \cdots \to & (A' \otimes M_{2^\infty})_{\mathbb{C}}
\end{array}$$

By Theorem 6.3.8 of [39], there is a sequence of unitaries $u_n \in (A' \otimes M_{2^\infty})_{\mathbb{C}}$ such that

$$\|u_n a u_n^* - \gamma(a)\| \to 0 \quad \text{for all } a \in A' \otimes M_{2^\infty} \ .$$

For each n find an integer $k(n)$ and a unitary $v_n \in (A' \otimes M_{2^{k(n)}})_{\mathbb{C}}$ such that $\|\alpha_{k(n)}(v_n) - u_n\| < 1/n$. Let $w_n = r(v_n) \in A' \otimes M_{2^{k(n)+1}}$, where r is induced by the realification map $M_{2^{k(n)}} \otimes \mathbb{C} \to M_{2^{k(n)+1}}$. We may assume that the sequence $\{k(n)\}_{n=1}^\infty$ is increasing.
Let $a \in A' \otimes M_{2^\infty}$ be given such that $\|a\| = 1$ and let $\varepsilon > 0$. Then find an integer N large enough so that, for all $n \geq N$,

- $\|\alpha_{k(n)}(v_n) - u_n\| < \varepsilon$,
- $\|u_n a u_n^* - \gamma(a)\| < \varepsilon$,
- there exist $a_n, b_n \in A' \otimes M_{2^{k(n)}}$ such that

$$\|a - \alpha_{k(n)}(a_n)\| < \varepsilon \qquad \text{and} \qquad \|\gamma(a) - \alpha_{k(n)}(b_n)\| < \varepsilon \,.$$

Then a calculation shows that, for all $n \geq N$,

$$\|v_n a_n v_n^* - b_n\| = \|\alpha_{k(n)}(v_n)\alpha_{k(n)}(a_n)\alpha_{k(n)}(v_n)^* - \alpha_{k(n)}(b_n)\| < 5\varepsilon.$$

Now for any element $x \in A' \otimes M_{2^{k(n)}}$ we have

$$\alpha_{k(n)+1} rc(x) = \alpha_{k(n)+1}\alpha_{k(n)+1,k(n)}(x) = \alpha_{k(n)}(x) \,.$$

It follows that

$$\begin{aligned}
&\|\alpha_{k(n)+1}(w_n) a \alpha_{k(n)+1}(w_n)^* - \gamma(a)\| \\
<&\|\alpha_{k(n)+1}(w_n)\alpha_{k(n)}(a_n)\alpha_{k(n)+1}(w_n)^* - \alpha_{k(n)}(b_n)\| + 2\varepsilon \\
=&\|\alpha_{k(n)+1} r(v_n)\alpha_{k(n)+1} rc(a_n)\alpha_{k(n)+1} r(v_n)^* - \alpha_{k(n)+1} rc(b_n)\| + 2\varepsilon \\
=&\|v_n c(a_n) v_n^* - c(b_n)\| + 2\varepsilon \\
=&\|v_n a_n v_n^* - b_n\| + 2\varepsilon < 7\varepsilon.
\end{aligned}$$

□

Theorem 6.5. *Let A be a real C*-algebra. Then A is c-simple, separable, unital, and nuclear if and only if $A \otimes \mathcal{O}_2^{\mathbb{R}} \cong \mathcal{O}_2^{\mathbb{R}}$*

Proof. Suppose that A is c-simple, separable, unital, and nuclear. There is a unital homomorphim $\gamma\colon \mathcal{O}_2^{\mathbb{R}} \to A \otimes \mathcal{O}_2^{\mathbb{R}}$ given by $x \mapsto 1 \otimes x$ and there is a unital homomorphism $\kappa\colon A \otimes \mathcal{O}_2^{\mathbb{R}} \to \mathcal{O}_2^{\mathbb{R}}$ by Lemma 6.2 and Proposition 6.3. Then by Theorem 5.2 we have $\kappa \circ \gamma \approx_u 1_{\mathcal{O}_2^{\mathbb{R}}}$ and by Lemma 6.4 we have $\gamma \circ \kappa \approx_u 1_{A \otimes \mathcal{O}_2^{\mathbb{R}}}$. Therefore, by (the real analog of) Corollary 2.3.4 of [39], $A \otimes \mathcal{O}_2^{\mathbb{R}} \cong \mathcal{O}_2^{\mathbb{R}}$.
Conversely, if the isomorphism $A \otimes \mathcal{O}_2^{\mathbb{R}} \cong \mathcal{O}_2^{\mathbb{R}}$ holds for a real C*-algebra A, then we have $A_{\mathbb{C}} \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ which implies by Theorem 7.1.2 of [39] that $A_{\mathbb{C}}$ is simple, separable, unital, and nuclear. Therefore A is c-simple, separable, unital, and nuclear. □

We note that the hypothesis above requiring that A be c-simple cannot be relaxed, as the result does not hold for $A = \mathcal{O}_2$ (considered as a real C*-algebra).
Theorem 6.6.

(1) *Any two unital homomorphisms ϕ, ψ from $\mathcal{O}_\infty^{\mathbb{R}}$ into a real, unital, purely infinite, nuclear, c-simple C*-algebra A are approximately unitarily equivalent.*
(2) *Let A be a real c-simple, separable, and nuclear C*-algebra. Then A is isomorphic to $A \otimes \mathcal{O}_\infty^{\mathbb{R}}$ if and only if A is purely infinite.*
(3) $\mathcal{O}_\infty^{\mathbb{R}} \cong \bigotimes_{n=1}^{\infty} \mathcal{O}_\infty^{\mathbb{R}}$.

Proof. As in Section 7.2 of [39]. □

Corollary 6.7. *Let A and B be real, c-simple, separable, nuclear C*-algebras. If A or B is purely infinite, then $A \otimes B$ is purely infinite.*

Proof. From part (2) of Theorem 6.6. $\square$

7. Homomorphisms from $\mathcal{O}_\infty^{\mathbb{R}}$

The goal of this section is to prove the following theorem, analogous to Proposition 2.2.7 of [35].

Theorem 7.1. *Let D be a real unital purely infinite simple C*-algebra, and let $\phi, \psi \colon \mathcal{O}_\infty^{\mathbb{R}} \to D$ be unital homomorphisms. Then ϕ is asymptotically unitarily equivalent to ψ.*

The proof of Theorem 7.1 will be the same as that in [35]. However, there are a couple of background topics that need to be addressed in the context of real C*-algebras.

We begin with a discussion of approximately divisible real C*-algebras, following [6]. It is sufficient to consider only separable unital C*-algebras. Also, we skirt the general topic of completely noncommutative C*-algebras by taking into account Definition 2.6 of [6] and the subsequent comment.

Definition 7.2. A separable unital real C*-algebra A is *approximately divisible* if for all $x_1, x_2, \dots, x_n \in A$ and $\varepsilon > 0$, there is a unital subalgebra B isomorphic to M_2, M_3, or $M_2 \oplus M_3$ such that $\|x_i y - y x_i\| < \varepsilon$ for all $i = 1, 2, \dots, n$ and all y in the unit ball of B.

The following theorem is the real version of Corollary 2.1.6 of [35].

Lemma 7.3. *The tensor product $\mathcal{O}_\infty^{\mathbb{R}} \otimes D$ is approximately divisible for any real separable unital C*-algebra D. In particular, every c-simple, separable, nuclear, purely infinite, unital real C*-algebra is approximately divisible.*

Proof. Let $A = \mathcal{O}_\infty^{\mathbb{R}} \otimes D$. Using the isomorphism $\mathcal{O}_\infty^{\mathbb{R}} \cong \bigotimes_{n=1}^\infty \mathcal{O}_\infty^{\mathbb{R}}$ of Theorem 6.6 we obtain a sequence of mutually commuting unital homomophisms $\phi_n \colon \mathcal{O}_\infty^{\mathbb{R}} \to A$ such that $\|\phi_n(a)b - b\phi_n(a)\| \to 0$ for all $a \in \mathcal{O}_\infty^{\mathbb{R}}$ and all $b \in A$. Choose a unital map $\gamma \colon M_2 \oplus M_3 \to \mathcal{O}_\infty^{\mathbb{R}}$ and let $\psi_n = \phi_n \circ \gamma$. Then for large enough n, the subalgebra $B = \psi_n(M_2 \oplus M_3)$ works.

The second statement follows from part (2) of Theorem 6.6. $\square$

Lemma 7.4. *Let p and q be full projections in $M_\infty(A)$ where A is a real, separable, unital, approximately divisible C*-algebra. Then $p \sim q$ if and only if $[p] = [q]$ in $K_0(A)$.*

Proof. The proof is the same as the proof of (the first part of) Proposition 3.10 in [6] in complex case. That proof relies on a progression of results from Section 2 of [6] which can all be proven in the real case in the same way with one minor caveat. The proof of Proposition 2.1 of [6] (which in that paper was left to the reader) relies on the fact that a complex C*-algebra is spanned by its unitaries. While this fact is not true in general for real C*-algebras, it can

easily be shown to be true for finite dimensional real C*-algebras, which is the relevant case.
The proof of Proposition 3.10 in [6] also relies on Theorem 3.1.4 of [3], which is a ring-theoretic result stated in enough generality to apply to real C*-algebras. □

We remark that a more direct proof of Lemma 7.4 can be achieved in the special case (which is sufficient for our purposes) that $A = \mathcal{O}_\infty^\mathbb{R} \otimes D$ where D is separable and unital. In that case, we write $A = \bigotimes_{i=1}^\infty \mathcal{O}_\infty^\mathbb{R} \otimes D$ and let $A_n = \bigotimes_{i=1}^n \mathcal{O}_\infty^\mathbb{R} \otimes D$ be the unital subalgebra of A consisting of the first n factors in the tensor product. Then for each n and each k, it is easy to find a unital subalgebra $B_n \subset A_n' \cap A$ that is isomorphic to $M_{2^k} \oplus M_{3^k}$. Thus we achieve the result of Corollary 2.10 of [6] without having to recheck all the earlier material of Section 2 of [6] in the real case.

Lemma 7.5. *Let D be a unital real C*-algebra and let p, q be any two full projections in $\mathcal{K}^\mathbb{R} \otimes \mathcal{O}_\infty^\mathbb{R} \otimes D$. Then p is Murray-von Neumann equivalent to a subprojection of q. Furthermore, p is homotopic to q if and only if they represent the same class in $K_0(\mathcal{K}^\mathbb{R} \otimes \mathcal{O}_\infty^\mathbb{R} \otimes D) \cong K_0(D)$.*

Proof. With our Lemmas 7.3 and 7.4, as well as Theorem 3.6 of [11], the proof is the same as that of Lemma 2.1.8 of [35]. □

Proof of Theorem 7.1. With these preliminary definitions and results, the proof is the same as the proof of Proposition 2.2.7 of [35] including all of the lemmas and intermediate results in Sections 2.1 and 2.2 of [35]. We note that in [35], the proofs of Propositions 2.1.9 and 2.1.10 (having to do with exact stability of the relations defining $\mathcal{O}_m^\mathbb{R}$ and $E_m(\delta)$) are referred back to the proofs of parts (1) and (2) of Lemma 1.3 of [30]. The proof given there for part (2) produces isometries w_j that live in the real algebra $E_n(\delta)$. Therefore the homomorphisms $\phi_\delta^{(m)}$ constructed in the complex case restrict to homomorphisms between the real algebras. The same will be true for the analogous proof of part (1).
We also note that the proofs for the real versions of Lemmas 2.2.1 and 2.2.3 of [35] rely on our Theorem 5.2 which is only established for n even. Hence for real C*-algebras, we need to take m to be even in Lemma 2.2.1 and n to be even in Lemma 2.2.3. This is however, sufficient for all subsequent arguments. □

8. Asymptotic Morphisms

We appropriate the following definition of an asymptotic morphism from Section 25.1 of [4]. The other definitions in this section and the next are adapted from [35].

Definition 8.1. Let A and B be real C*-algebras. An *asymptotic morphism* ϕ from A to B is a family $\{\phi_t\}_{t\in[0,\infty)}$ of maps $\phi_t \colon A \to B$ such that

(1) the map $t \mapsto \phi_t(a)$ is continuous for each $a \in A$, and

(2) for all $a, b \in A$ and all $\lambda \in \mathbb{R}$, the following functions vanish in norm as $t \to \infty$:
   (a) $\phi_t(a+b) - \phi_t(a) - \phi_t(b)$,
   (b) $\phi_t(\lambda a) - \lambda\phi_t(a)$,
   (c) $\phi_t(ab) - \phi_t(a)\phi_t(b)$,
   (d) $\phi_t(a^*) - \phi_t(a)^*$.

We say that two asymptotic morphisms ϕ_t and ψ_t from A to B are *equivalent* if $\|\phi_t(a) - \psi_t(a)\|$ vanishes as $t \to \infty$ for all $a \in A$. We say that ϕ_t and ψ_t are *homotopic* if there is an asymptotic morphism Φ_t from A to $C([0,1], B)$ such that $\Phi_t(a)(0) = \phi_t(a)$ and $\Phi_t(a)(1) = \psi_t(a)$ for all $a \in A$. Equivalent asymptotic morphisms are homotopy equivalent (see Remark 25.1.2 of [4]).
We leave the easy proof of the next lemma to the reader.

Lemma 8.2. *If A and B are real C^*-algebras and ϕ is an asymptotic morphism from A to B, then there is an asymptotic morphism $\phi_{\mathbb{C}} \colon A_{\mathbb{C}} \to B_{\mathbb{C}}$ defined by $(\phi_{\mathbb{C}})_t(a+ib) = \phi_t(a) + i\phi_t(b)$.*

It can be proven, then, from the same result in the complex case, that for any asymptotic morphism ϕ we have $\limsup_{t\to\infty} \|\phi_t(a)\| \leq \|a\|$ for all $a \in A$ (see Proposition 25.1.3 of [4]). Thus, an asymptotic morphism $\{\phi_t\}$ gives rise to a unique homomorphism

$$\phi \colon A \to C_b([0,\infty), B)/C_0([0,\infty), B)$$

defined in the natural way; and every such homomorphism represents an asymptotic morphism, unique up to equivalence.

Lemma 8.3. *Let A be separable and nuclear. Every asymptotic morphism from A to B is equivalent to one that is completely positive and contractive. Furthermore, if ϕ and ψ are homopic completely positive and contractive asymptotic morphisms from A to B, then in fact there is a homotopy from ϕ to ψ consisting of completely positive and contractive asymptotic morphisms.*

Proof. Let ϕ be an asymptotic morphism from A to B. Then by Proposition 1.1.5 of [35], the complexification $\phi_{\mathbb{C}}$ is equivalent to an asymptotic morphism ψ that is completely positive and contractive. The map $\alpha \colon B_{\mathbb{C}} \to B$ defined by $\alpha(a+ib) = a$ is completely positive and contractive. Then the restriction of $\alpha \circ \psi$ to A is a completely positive, contractive asymptotic morphism from A to B and is equivalent to ϕ.
The same construction can be applied to a homotopy to prove the second statement. □

Definition 8.4. Let ϕ and ψ be asymptotic morphisms from A to $\mathcal{K}^{\mathbb{R}} \otimes D$. We define an asymptotic morphism $\phi \oplus \psi$, also from A to $\mathcal{K}^{\mathbb{R}} \otimes D$, as follows. Choose an isomorphism $\delta \colon M_2(\mathcal{K}^{\mathbb{R}}) \to \mathcal{K}^{\mathbb{R}}$ and define

$$(\phi \oplus \psi)_t(a) = (\delta \otimes 1_D) \begin{pmatrix} \phi_t(a) & 0 \\ 0 & \psi_t(a) \end{pmatrix} .$$

Lemma 8.5. *The asymptotic morphism $\phi \oplus \psi$ is well defined up to unitary equivalence, as well as up to homotopy.*

Proof. As in the complex case every automorphism of $\mathcal{K}^{\mathbb{R}}$ is implemented by a unitary in $\mathcal{U}(B(\mathcal{H}^{\mathbb{R}}))$ (the proof in, for example, Lemma V.6.1 of [20] works in the real case). Furthermore, by [36], $\mathcal{U}(B(\mathcal{H}^{\mathbb{R}}))$ is path connected. (In fact, by Theorem 3 of [29], it is contractible.) $\square$

Definition 8.6. Let $\phi\colon A \to B$ be an asymptotic morphism of real C*-algebras and let $p \in A$ be a projection. A *tail projection* for $\phi(p)$ is a continuous path p_t of projections for $t \in [0, \infty)$ such that $\lim_{t\to\infty} \|\phi_t(p) - p_t\| = 0$.
We say that ϕ is *full* if there is a full projection $p \in A$ such that $\phi(p)$ has a full tail projection.

Definition 8.7. Let A and B be real C*-algebras. Two asymptotic morphisms ϕ and ψ from A and B are *asymptotically unitarily equivalent* if there is a continuous family of unitary elements $u_t \in \widetilde{B}$ such that $\lim_{t\to\infty} \|u_t\phi_t(a)u_t^* - \psi_t(a)\| = 0$ for all $a \in A$.

With these definitions, all the results of Sections 1.2 and 1.3 of [35] hold for real C*-algebras.

Definition 8.8. Let A and D be real C*-algebras. An asymptotic morphism $\phi\colon A \to D$ has a *standard factorization* through $\mathcal{O}_\infty^{\mathbb{R}} \otimes A$ if there is an asymptotic morphism $\psi\colon \mathcal{O}_\infty^{\mathbb{R}} \otimes A \to D$ such that the asymptotic morphisms $\phi(a)$ and $\psi(1 \otimes a)$ (both from A to D) are asymptotically unitarily equivalent.
Similarly, ϕ is *asymptotically trivially factorizable* if there is an asymptotic morphism $\psi\colon \mathcal{O}_2^{\mathbb{R}} \otimes A \to D$ such that $\phi(a)$ and $\psi(1 \otimes a)$ are asymptotically unitarily equivalent.

Theorem 8.9 (Theorem 2.3.7 of [35]). *Let A be a separable, nuclear, unital, and c-simple. Let D_0 be a unital C*-algebra, and let $D = \mathcal{O}_\infty^{\mathbb{R}} \otimes D_0$. Then two full asymptotic morphisms from A to $\mathcal{K}^{\mathbb{R}} \otimes D$ are asymptotically unitarily equivalent if and only they are homotopic.*

Proof. The proof of Theorem 2.3.7 in [35] as well as the proofs of all of the preceeding lemmas in Section 2.3 of [35] can be proven in the real case with the same proofs, with some extra attention paid to the issue of connectedness of unitary groups.
In a few places Phillips uses the fact that the unitary group of $\mathcal{O}_2$ is connected. It is also true that $\mathcal{O}_2^{\mathbb{R}}$ is connected since $K_1(\mathcal{O}_2^{\mathbb{R}}) \cong 0$. However, on page 85 of [35], Phillips also uses the fact that the unitary group of a corner algebra of $\mathcal{O}_\infty$ is connected. The corresponding statement in the real case is not true since $K_1(\mathcal{O}_\infty^{\mathbb{R}}) \cong \mathbb{Z}_2$. We will show how to adjust the proof so that it works in the real case.
At this point in the proof we are (using Phillips' notation) trying to find a path of partial isometries from $w_n + f_{n+2}$ to $v_{n+1} + w_{n+1}$ (these are partial isometries from $f_{n+1} + f_{n+2}$ to $f_{n+2} + e$). If the unitaries $(w_n + f_{n+2})^*(v_{n+1} + w_{n+1})$ and $f_{n+1} + f_{n+2}$ are not in the same connected component of $(f_{n+1} +$

$f_{n+2})\mathcal{O}_\infty^{\mathbb{R}}(f_{n+1} + f_{n+2})$, then this can be changed by by multiplying w_{n+1} on the right by a suitable unitary in $f_{n+2}\mathcal{O}_\infty^{\mathbb{R}} f_{n+2}$. Thus by re-choosing the w_n's inductively, we can be sure that there is an appropriate path of partial isometries at each step. □

9. Groups of Asymptotic Morphisms

Definition 9.1. Let A be a real, separable, nuclear, unital, c-simple C*-algebra and let D be unital. We define $E_A(D)$ to be the the set of homotopy classes of full asymptotic morphisms from A to $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D$. That is,

$$E_A(D) = [[A, \mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D]]_+ \ .$$

More generally, for D unital or not, we define

$$\widetilde{E}_A(D) = \ker\left(E_A(D^+) \to E_A(\mathbb{R})\right) \ .$$

Proposition 9.2. *Let A be real, separable, nuclear, unital, and c-simple. Then $\widetilde{E}_A(-)$ is a functor from the category of separable real C*-algebras with homotopy classes of asymptotic morphisms to abelian groups, that is homotopy invariant, stable, half exact, and split exact.*

Proof. In the complex case, these results are proven in Section 3.1 of [35]. In the real case, they are proven the same way. Note that split exactness follows from homotopy invariance and half exactness by Corollary 3.5 of [12]. □

Lemma 9.3. *Let A and B be C*-algebras (real or complex). Let $\phi\colon A \to B$ be an asymptotic morphism. If p, q are projections in A with $p \leq q$, then there are tail projections p_t (for $\phi(p)$) and q_t (for $\phi(q)$) in B with $p_t \leq q_t$ for all t.*

Proof. Let $\widetilde{p_t}$ and q_t be arbitrary tail projections corresponding to $\phi(p)$ and $\phi(q)$, respectively (these exist as in Remark 1.2.2 of [35]). One can easily show that

$$\lim_{t\to\infty} \|\widetilde{p_t} - q_t\widetilde{p}_t q_t\| = 0 \ .$$

For each t, the element $q_t\widetilde{p}_t q_t$ is a self adjoint and asymptotically idempotent element of $q_t B q_t$. Therefore, there is a continuous path of projections $p_t \in q_t B q_t$ such that

$$\lim_{t\to\infty} \|q_t\widetilde{p}_t q_t - p_t\| = 0 \ .$$

The tail projections p_t and q_t have the desired properties. □

We note that if A and D are complex C*-algebras there are two groups one might consider: we let $\widetilde{E}_A^{\mathbb{C}}(D)$ denote the functor of [35] that is based on complex asymptotic morphisms. On the other hand, according to the notation established in Definition 9.1, the asymptotic morphisms comprising $\widetilde{E}_A(D)$ are only required to be asymptotically linear over $\mathbb{R}$ (thus the complex structures of A and D are forgotten). The following theorem relates the two groups.

Proposition 9.4. *If A is a real C*-algebra satisfying the hypotheses of Definition 9.1 and D is a complex unital C*-algebra, then there is a isomorphism*

$$\widetilde{E}_A(D) \cong \widetilde{E}^{\mathbb{C}}_{A_{\mathbb{C}}}(D)$$

which is natural with respect to complex homomorphisms.

Proof. We show that for a real unital C*-algebra A and a complex C*-algebra B, there is a bijection

$$[[A,B]]_+ \cong [[A_{\mathbb{C}},B]]^{\mathbb{C}}_+$$

of equivalence classes of full asymptotic morphisms.
Given a complex asymptotic morphism ϕ from $A_{\mathbb{C}}$ to B, then we let $\Gamma(\phi)$ be the restriction of ϕ to A. If ϕ is full, then we claim that $\Gamma(\phi)$ is full. Since ϕ is full, there is a full projection $p \in A_{\mathbb{C}}$ and a full tail projection $r_t \in B$ such that $\|\phi_t(p) - r_t\| \to 0$. Applying Lemma 9.3 to $p \leq 1$ we obtain tail projections p_t and q_t for p and 1, respectively, such that $p_t \leq q_t$ for all t. Since the tail projections p_t and r_t are asymptotically equal, it must be that p_t are full projections. It follows that q_t are also full projections; and since they are tail projections for the full projection 1_A in A, it follows that $\Gamma(\phi)$ is full.
Given a real asymptotic morphism ψ from A to B, then

$$\Delta(\psi)_t(a+ib) = \psi_t(a) + i\psi_t(b)$$

defines a complex asymptotic morphism from $A_{\mathbb{C}}$ to B. Suppose that ψ is full. Let p be a full projection in A and let $q_t \in B$ be a full tail projection for $\psi(p)$. Then clearly p is full in $A_{\mathbb{C}}$ and q_t is a full tail projection for $\Delta(\psi(p))$. Hence $\Delta(\psi)$ is full.
It is immediate that Δ is a two-sided inverse for Γ. Furthermore, in the case that B is stable, it is easy to see that Γ preserves the semigroup operation of Definition 8.4. Therefore, under the hypotheses of the theorem, there is an group isomorphism $\widetilde{E}_A(D) \cong \widetilde{E}^{\mathbb{C}}_{A_{\mathbb{C}}}(D)$. □

Proposition 9.5. *Let A be a separable, nuclear, c-simple unital, real C*-algebra. Let B be a separable real C*-algebra. Then there is a natural isomophism $KK(A,B) \cong \widetilde{E}_A(B)$.*

The proof in the complex case takes place in Section 3.2 of [35]. Rather than reconstructing all of the arguments in the real case, we give a proof that uses results from [12] to reduce the real case to the complex case.

Proof of Proposition 9.5. Fix A satisfying the hypotheses above. Let e be a rank one projection in $\mathcal{K}^{\mathbb{R}}$ and let $\iota_A \colon A \to \mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}^{\mathbb{R}}_\infty \otimes A$ be the homomorphism defined by $\iota_A(a) = e \otimes 1 \otimes a$. Let $[[\iota_A]]$ be the induced element of $\widetilde{E}_A(A)$. Let $[1_A] \in KK(A,A)$ be the class of the identity. By Corollary 3.3 of [12], there is a unique natural transformation α from $KK(A,-)$ to $\widetilde{E}_A(-)$ such that $\alpha([1_A]) = [[\iota_A]]$. We will show that

$$\alpha \colon KK(A,B) \to \widetilde{E}_A(B)$$

is an isomorphism for all separable real C*-algebras B. By Theorem 3.9 of [12] it suffices to show that α is an isomorphism when B is complex.
In the complex case we have the element $[1_{A_\mathbb{C}}] \in KK^\mathbb{C}(A_\mathbb{C}, A_\mathbb{C})$ and the homomorphism

$$(\iota_A)_\mathbb{C} \colon A_\mathbb{C} \to \mathcal{K}^\mathbb{R} \otimes \mathcal{O}_\infty^\mathbb{R} \otimes A_\mathbb{C} \cong \mathcal{K} \otimes \mathcal{O}_\infty \otimes A_\mathbb{C} \ .$$

By Theorem 3.7 of [24] there is a unique natural transformation $\alpha^\mathbb{C}$ from $KK^\mathbb{C}(A_\mathbb{C}, -)$ to $\widetilde{E}^\mathbb{C}_{A_\mathbb{C}}(-)$ such that $\alpha^\mathbb{C}([1_{A_\mathbb{C}}]) = [[\iota_A^\mathbb{C}]]$. A special case of Theorem 3.2.6 of [35] shows that $\alpha^\mathbb{C}$ is an isomorphism for all separable complex C*-algebras B.
Consider the following diagram for a complex C*-algebra B,

$$\begin{tikzcd} KK^\mathbb{C}(A_\mathbb{C}, B) \arrow[r, "\alpha^\mathbb{C}"] \arrow[d, "\nu"] & \widetilde{E}^\mathbb{C}_{A_\mathbb{C}}(B) \arrow[d, "\mu"] \\ KK(A, B) \arrow[r, "\alpha"] & \widetilde{E}_A(B) \end{tikzcd}$$

where μ is the isomorphism of Proposition 9.4 above and ν is the isomorphism of Lemma 4.3 of [9]. To complete the proof, we only need to show that the diagram commutes. Since the homomorphism $\alpha^\mathbb{C}$ is characterized by the value of $\alpha^\mathbb{C}([1_{A_\mathbb{C}}])$ it suffices to consider the case $B = A_\mathbb{C}$ as in the diagram

$$\begin{tikzcd} KK^\mathbb{C}(A_\mathbb{C}, A_\mathbb{C}) \arrow[r, "\alpha^\mathbb{C}"] \arrow[d, "\nu"] & \widetilde{E}^\mathbb{C}_{A_\mathbb{C}}(A_\mathbb{C}) \arrow[d, "\mu"] \\ KK(A, A_\mathbb{C}) \arrow[r, "\alpha"] & \widetilde{E}_A(A_\mathbb{C}) \end{tikzcd}$$

and to show that $\alpha^\mathbb{C}([1_{A_\mathbb{C}}]) = (\mu^{-1} \circ \alpha \circ \nu)([1_{A_\mathbb{C}}])$ or, equivalently, $(\mu \circ \alpha^\mathbb{C})([1_{A_\mathbb{C}}]) = (\alpha \circ \nu)([1_{A_\mathbb{C}}])$.
From the construction of ν in the proof of Lemma 4.3 of [9] it is apparent that $\nu([1_{A_\mathbb{C}}]) = [c_A] = (c_A)_*([1_A])$ where $c_A \colon A \to A_\mathbb{C}$ is the real C*-algebra homomorphism induced by the unital inclusion $c \colon \mathbb{R} \hookrightarrow \mathbb{C}$. Thus

$$(\alpha \circ \nu)([1_{A_\mathbb{C}}]) = \alpha((c_A)_*([1_A])) = (c_A)_*(\alpha([1_A])) = (c_A)_*([[\iota_A]]) = [[c_A]] \ .$$

On the other hand, it is apparent from the construction of μ in the proof of Proposition 9.4 above that $\mu([[\iota_A^\mathbb{C}]]) = [[c_A]]$. Thus

$$(\mu \circ \alpha^\mathbb{C})([1_{A_\mathbb{C}}]) = \mu(\alpha^\mathbb{C}([1_{A_\mathbb{C}}])) = \mu([[\iota_A^\mathbb{C}]]) = [[c_A]] \ .$$

□

The following is the real version of Theorems 4.1.1 and 4.1.3 of [35].

Theorem 9.6. *Let A be a real separable unital nuclear c-simple C*-algebra and let D be a separable unital C*-algebra. Then the following groups are naturally isomorphic, via the obvious maps.*

(1) $KK(A, D)$

(2) *The set of asymptotic unitary equivalence classes of full homomorphisms from A to $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D$.*
(3) *The set of homotopy classes of full homomorphisms from A to $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D$.*
(4) *The set of asymptotic unitary equivalence classes of full homomorphisms from $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes A$ to $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D$.*
(5) *The set of homotopy classes of full homomorphisms from $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes A$ to $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D$.*

Proof. The proof of the isomorphism of (1), (2), and (3) is the same as the proof of Theorem 4.1.1 in [35]. The proof of the isomorphism of (1), (4), and (5) relies on Lemma 9.7 below (which is the real version of Lemma 4.1.2 of [35]). Once that lemma is established, the proof of the isomorphism of (1), (4), and (5) is the same as the proof of Theorem 4.1.3 of [35]. ☐

Lemma 9.7. *Let A be separable, nuclear, unital, and c-simple; let D_0 be separable and unital; and let $D = \mathcal{O}_\infty^{\mathbb{R}} \otimes D_0$. Let $t \mapsto \phi_t$, for $t \in [0, \infty)$, be a continuous path of full homomorphisms from $\mathcal{K}^{\mathbb{R}} \otimes A$ to $\mathcal{K}^{\mathbb{R}} \otimes D$, and let $\psi \colon \mathcal{K}^{\mathbb{R}} \otimes A \to \mathcal{K}^{\mathbb{R}} \otimes D$ be a full homomorphism. Assume that $[\phi_0] = [\psi]$ in $KK_0(A, D)$. Then there is an asymptotic unitary equivalence from ϕ to ψ that consists of unitaries in $\mathcal{U}_0((\mathcal{K}^{\mathbb{R}} \otimes D)^+)$.*

The proof will be essentially the same as the proof of Lemma 4.1.2 of [35]. However, that proof has an error in the third paragraph. The element w_t introduced there does not seem to be a unitary as purported. Also, the order of the product in the definition of z_t seems wrong. Fortunately, there is an easy fix and most of the proof can be left as it is. For clarity and completeness we present the entire proof, but the only significant difference is the unitary w in the third paragraph and following. In places where the proof does not change (such as the entire first and second paragraphs, and most of the final paragraph), we use exactly the same language as in [35], except for the references to previous results in the present paper.

Proof of Lemma 9.7. Let $\{e_{ij}\}$ be a system of matrix units for $\mathcal{K}^{\mathbb{R}}$. Identify A with the subalgebra $e_{11} \otimes A$ of $\mathcal{K}^{\mathbb{R}} \otimes A$. Define $\psi_t^{(0)}$ and $\psi^{(0)}$ to be the restrictions of ϕ_t and ψ to A. Then $[\phi_0^{(0)}] = [\psi^{(0)}]$ in $KK_0(A, D)$. It follows from (the equivalence of (1) and (3) of) Theorem 9.6 that $\phi_0^{(0)}$ is homotopic to $\psi^{(0)}$. Therefore $\phi_0^{(0)}$ and $\psi^{(0)}$ are homotopic as asymptotic morphisms, and Theorem 8.9 provides an asymptotic unitary equivalence $t \mapsto u_t$ in $\mathcal{U}((\mathcal{K}^{\mathbb{R}} \otimes D)^+)$ from $\phi^{(0)}$ to $\psi^{(0)}$. Let $c \in \mathcal{U}((\mathcal{K}^{\mathbb{R}} \otimes D)^+)$ be a unitary with $c\psi^{(0)}(1) = \psi^{(0)}(1)c = \psi^{(0)}(1)$ and such that c is homotopic to u_0^{-1}. Then c commutes with every $\psi^{(0)}(a)$. Replacing u_t by cu_t, we obtain an asymptotic unitary equivalence, which we again call $t \mapsto u_t$, from $\phi^{(0)}$ to $\psi^{(0)}$ which is in $\mathcal{U}_0(\mathcal{K}^{\mathbb{R}} \otimes D)^+)$.
Define $\overline{e}_{ij} = e_{ij} \otimes 1$. Then in particular $u_t \phi_t(\overline{e}_{11}) u_t^* \to \psi(\overline{e}_{11})$ as $t \to \infty$. Therefore there is a continuous path $t \to z_t^{(1)} \in \mathcal{U}_0((\mathcal{K}^{\mathbb{R}} \otimes D)^+)$ such that $z_t^{(1)} \to$

1 and $z_t^{(1)}u_t\phi_t(\overline{e}_{11})u_t^*(z_t^{(1)})^* = \psi(\overline{e}_{11})$ for all t. We still have $z_t^{(1)}u_t\phi_t(e_{11} \otimes a)u_t^*(z_t^{(1)})^* \to \psi(e_{11} \otimes a)$ for $a \in A$.

For convenience, set $f_{ijt}^{(1)} = z_t^{(1)}u_t\phi_t(\overline{e}_{ij})u_t^*(z_t^{(1)})^*$ and set $g_{ij} = \psi(\overline{e}_{ij})$. For each fixed t, the $f_{ijt}^{(1)}$ are matrix units for $\mathcal{K}^{\mathbb{R}}$ as are the g_{ij}. Also, we have $f_{11t}^{(1)} = g_{11}$. The projections $f_{11t}^{(1)}+f_{22t}^{(1)}$ and $g_{11}+g_{22}$ represent the same element of $K_0(D)$ so (using Lemma 7.4) there is a continuous path of partial isometries $x_t^{(1)}$ in $\mathcal{K}^{\mathbb{R}}\otimes D$ such that $x_t^{(1)}(x_t^{(1)})^* = 1-g_{11}-g_{22}$ and $(x_t^{(1)})^*x_t^{(1)} = 1-f_{11t}^{(1)}-f_{22t}^{(1)}$. Set $w_t^{(1)} = g_{11}+g_{21}f_{12t}^{(1)}+x_t^{(1)} \in \mathcal{U}((\mathcal{K}^{\mathbb{R}}\otimes D)^+)$. Then one checks that $w_t^{(1)}f_{ijt}^{(1)}(w_t^{(1)})^* = g_{ij}$ for all t and for $1 \leq i,j \leq 2$. Choose $c^{(1)} \in \mathcal{U}((\mathcal{K}^{\mathbb{R}} \otimes D)^+)$ with

$$c^{(1)}(g_{11}+g_{22}) = (g_{11}+g_{22})c^{(1)} = g_{11}+g_{22}$$

$$\text{and} \qquad c^{(1)}w_1^{(1)} \in \mathcal{U}_0((\mathcal{K}^{\mathbb{R}} \otimes D)^+)\ .$$

Set $z^{(2)} = c^{(1)}w_t^{(1)}$ for $t \geq 1$ and extend $z_t^{(2)}$ continuously over $[0,1]$ through unitaries so that $z_0^{(2)} = 1$, retaining the property that $z_t^{(2)}g_{11} = g_{11}z_t^{(2)} = g_{11}$. This gives $z_t^{(2)} = 1$ for $t = 0$, $z_t^{(2)}g_{11} = g_{11}z_t^{(2)} = g_{11}$ for all t, and

$$z_t^{(2)}z_t^{(1)}u_t\phi_t(\overline{e}_{ij})u_t^*(z_t^{(1)})^*(z_t^{(2)})^* = \psi(\overline{e}_{ij})$$

for $t \geq 1$ and $1 \leq i,j \leq 2$.

Set $p^{(m)} = \sum_{k=1}^m g_{kk}$ for all postive integers m. For the induction step, assume that we have continuous paths unitaries $z_t^{(1)}, z_t^{(2)}, \ldots, z_t^{(n)}$ defined on $[0,\infty)$ such that

- $z_t^{(n)} = 1$ for $0 \leq t \leq n-2$,
- $z_t^{(n)}p^{(n-1)} = p^{(n-1)}z_t^{(n)} = p^{(n-1)}$ for all $t \geq 0$,
- $z_t^{(n)}\cdots z_t^{(1)}u_t\phi_t(\overline{e}_{ij})u_t^*(z_t^{(1)})^*\cdots(z_t^{(n)})^* = \psi(\overline{e}_{ij})$ for $t \geq n-1$ and $1 \leq i,j \leq n$.

We must construct a $z_t^{(n+1)}$ with the corresponding properties. Initially, working with $t \in [n,\infty)$, set

$$f_{ijt}^{(n)} = z_t^{(n)}\ldots z_t^{(1)}u_t\phi_t(\overline{e}_{ij})u_t^*(z_t^{(1)})^*\ldots(z_t^{(n)})^*$$

and let $x_t^{(n)}$ be a continuous path of partial isometries such that $x_t^{(n)}(x_t^{(n)})^* = 1-\sum_{k=1}^n g_{kk} = 1-p^{(n)}$ and $(x_t^{(n)})^*x_t^{(n)} = 1-\sum_{k=1}^n f_{kkt}$. Set $w_t^{(n)} = p^{(n)} + g_{(n+1)\,1}f_{1\,(n+1)\,t}^{(n)} + x_t^{(n)}$. This continuous path of unitaries satisfies $w_t^{(n)}p^{(n)} = p^{(n)}w_t^{(n)} = p^{(n)}$ and $w_t^{(n)}f_{ijt}^{(n)}(w_t^{(n)})^* = g_{ij}$ for all $t \geq n$ and all $1 \leq i,j \leq n+1$. As above, we can find a unitary $c^{(n)}$ such that $z_t^{(n+1)} = c^{(n)}w_t^{(n)}$ is in the connected component of the identity and $c^{(n)}p^{(n+1)} = p^{(n+1)}c^{(n)} = c^{(n)}$. Then extend $z_t^{(n+1)}$ so that it is defined for all $t \geq 0$ and $z_t^{(n+1)} = 1$ for $0 \leq t \leq n-1$. Check that this $z^{(n+1)}$ satisfies the corresponding properties listed above.

Now define

$$z_t = \left(\lim_{n\to\infty} z_t^{(n)}\cdots z_t^{(2)}z_t^{(1)}\right)u_t\ .$$

In a neighborhood of each t, all but finitely many of the $z_t^{(k)}$ are equal to 1, so this limit of products yields a continuous path of unitaries of $\mathcal{U}_0((\mathcal{K}^{\mathbb{R}} \otimes D)^+)$. Moreover, $z_t \phi_t(\overline{e}_{ij}) z_t^* = \psi(\overline{e}_{ij})$ whenever $t \geq i, j$, so that $\lim_{t\to\infty} z_t \phi_t(\overline{e}_{ij}) z_t^* = \psi(\overline{e}_{ij})$ for all i and j, while

$$\lim_{t\to\infty} z_t \phi_t(e_{11} \otimes a) z_t^* = \lim_{t\to\infty} z_t^{(1)} u_t \phi_t(e_{11} \otimes a) u_t^* (z_t^{(1)})^* = \psi(e_{11} \otimes a)$$

for all $a \in A$. Since the $\overline{e}_{ij}$ and $e_{11} \otimes a$ generate $\mathcal{K}^{\mathbb{R}} \otimes A$, this shows that $t \mapsto z_t$ is an asymptotic unitary equivalence. □

10. Classification of Real Kirchberg Algebras

We now present our main classification theorems for real Kirchberg algebras, analogous to the results of Section 4.2 of [35].

Theorem 10.1. *Let A and B be unital separable nuclear purely infinite c-simple C^*-algebras.*

1. *Let η be an invertible element in $KK(A, B)$. Then there is an isomorphism $\phi\colon \mathcal{K}^{\mathbb{R}} \otimes A \to \mathcal{K}^{\mathbb{R}} \otimes B$ such that $[\phi] = \eta$.*
2. *Let η be an invertible element in $KK(A, B)$ such that $[1_A] \times \eta = [1_B]$. Then there is an isomorphism $\phi\colon A \to B$ such that $[\phi] = \eta$.*

Proof. As in the proofs of Theorem 4.2.1 and Corollary 4.2.2 of [35]. □

Theorem 10.2. *Let A and B be unital separable nuclear purely infinite c-simple C^*-algebras that satisfy the universal coefficient theorem.*

1. *The stable C^*-algebras $\mathcal{K}^{\mathbb{R}} \otimes A$ and $\mathcal{K}^{\mathbb{R}} \otimes B$ are isomorphic if and only if $K^{CRT}(A)$ and $K^{CRT}(B)$ are isomorphic CRT-modules.*
2. *The unital C^*-algebras A and B are isomorphic if and only if the invariants $(K^{CRT}(A), [1_A])$ and $(K^{CRT}(B), [1_B])$ are isomorphic.*
3. *The stable C^*-algebras $\mathcal{K}^{\mathbb{R}} \otimes A$ and $\mathcal{K}^{\mathbb{R}} \otimes B$ are isomorphic if and only if $K^{CR}(A)$ and $K^{CR}(B)$ are isomorphic CR-modules.*
4. *The unital C^*-algebras A and B are isomorphic if and only if the invariants $(K^{CR}(A), [1_A])$ and $(K^{CR}(B), [1_B])$ are isomorphic.*

Proof. Parts (1) and (2) are proven as in the proof of Theorem 4.2.4 of [35], using Proposition 2.3. Parts (3) and (4) then follow by Proposition 2.5. □

Corollary 10.3.

1. *The functor $A \mapsto K^{CRT}(A)$ is a bijection from isomorphism classes of real stable separable nuclear purely infinite c-simple C^*-algebras that satisfy the universal coefficient theorem to isomorphism classes of countable acyclic CRT-modules.*
2. *The functor $A \mapsto (K^{CRT}(A), [1_A])$ is a bijection from isomorphism classes of real unital separable nuclear purely infinite c-simple C^*-algebras that satisfy the universal coefficient theorem to isomorphism classes of countable acyclic CRT-modules M with distinguished element $m \in M_0^O$.*

Proof. Combine Theorem 10.2 above with Theorem 1 of [10]. □

Definition 10.4.

(1) Let A be a complex C*-algebra. A *real form* of A is a real C*-algebra B such that $B_{\mathbb{C}} \cong A$.
(2) Let $G_* = (G_0, G_1)$ be a pair of groups. A *real form* of G_* is an acyclic *CRT*-module such that $M^U_* \cong G_*$.
(3) Let $G_* = (G_0, G_1, g)$ be a pair of groups with a distinguished element $g \in G_0$. A *real form* of G_* is a pair (M, m) where M is an acyclic *CRT*-module and m is a distinguished element of M^O_0 such that $(M^U_0, M^U_1, c(m)) \cong (G_0, G_1, g)$.

Corollary 10.5. *Let A be a complex unital separable nuclear purely infinite simple C*-algebra satisfying the universal coefficient theorem.*

(1) *The functor $B \mapsto K^{CRT}(B)$ is a bijection from isomorphism classes of real forms of $\mathcal{K}^{\mathbb{R}} \otimes A$ to isomorphism classes of real forms of $K_*(A)$.*
(2) *The functor $B \mapsto (K^{CRT}(B), [1_B])$ is a bijection from isomorphism classes of real forms of A to isomorphism classes of real forms of $(K_*(A), [1_A])$.*

Proof. If B is a real form of $\mathcal{K}^{\mathbb{R}} \otimes A$, then B is necessarily stable separable nuclear purely infinite and c-simple. Then $KU_*(B) = K_*(B_{\mathbb{C}}) \cong K_*(A)$, so $K^{CRT}(B)$ is a real form of $K_*(A)$. Conversely, suppose M is a real form of $K_*(A)$. Since $K_*(A)$ is countable, the exact sequences of Section 2.3 of [14] imply that M is countable. Then by Corollary 10.3, $M \cong K^{CRT}(B)$ for some real stable separable nuclear purely infinite c-simple C*-algebra satisfying the universal coefficient theorem. Since $K_*(B_{\mathbb{C}}) \cong K_*(A)$, it follows from Theorem 4.2.4 of [35] that $B_{\mathbb{C}} \cong A$ hence B is a real form of A. Furthermore, Corollary 10.3 also implies that B is unique up to isomorphism.
In the unital case, suppose that B is a real form of A. As there is a isomorphism $B_{\mathbb{C}} \cong A$ and the unit of $B_{\mathbb{C}}$ is $c(1_B)$, there is an isomorphism $\phi \colon KU_*(B) \to K_*(A)$ such that $\phi_*(c([1_B])) = [1_A]$. Thus $(K^{CRT}(B), [1_B])$ is a real form of $(K_*(A), [1_A])$. Conversely, if (M, m) is a real form of $K_*(A)$, then let B be a real unital separable nuclear purely infinite c-simple C*-algebra such that $(K^{CRT}(B), [1_B]) \cong (M, m)$. Again, Theorem 4.2.4 of [35], implies that B is a real form of A. □

11. Real Forms of Cuntz Algebras

In this section, we use Corollary 10.5 to give a complete description of all real forms of the complex Cuntz algebras $\mathcal{O}_n$ for $n \in \{2, \ldots, \infty\}$. The natural real form of $\mathcal{O}_n$ is the real Cuntz algebra $\mathcal{O}^{\mathbb{R}}_n$, but we will find that there are others when n is odd. For reference, we show in Table 1 the groups making up $K^{CRT}(\mathcal{O}^{\mathbb{R}}_n)$. In the case of $n = \infty$ this arises from the isomorphism $K^{CRT}(\mathbb{R}) \cong K^{CRT}(\mathcal{O}^{\mathbb{R}}_\infty)$ of Proposition 2.2; while for finite n, these *CRT*-modules were computed in Section 5.1 of [8].

Table 1

$K^{CRT}(\mathcal{O}_\infty^{\mathbb{R}})$

	0	1	2	3	4	5	6	7	8
KO_*	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
KU_*	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
KT_*	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$

$K^{CRT}(\mathcal{O}_n^{\mathbb{R}})$ for n even

	0	1	2	3	4	5	6	7	8
KO_*	$\mathbb{Z}_{n-1}$	0	0	0	$\mathbb{Z}_{n-1}$	0	0	0	$\mathbb{Z}_{n-1}$
KU_*	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$
KT_*	$\mathbb{Z}_{n-1}$	0	0	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$	0	0	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$

$K^{CRT}(\mathcal{O}_n^{\mathbb{R}})$ for $n-1 \equiv 2 \pmod 4$

	0	1	2	3	4	5	6	7	8
KO_*	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	0	0	0	$\mathbb{Z}_{n-1}$
KU_*	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$
KT_*	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$

$K^{CRT}(\mathcal{O}_n^{\mathbb{R}})$ for $n-1 \equiv 0 \pmod 4$

	0	1	2	3	4	5	6	7	8
KO_*	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	0	0	0	$\mathbb{Z}_{n-1}$
KU_*	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$
KT_*	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$

Theorem 11.1. (1) *For n even or $n = \infty$, there is up to isomorphism only one real form of $\mathcal{O}_n$: the real Cuntz algebra $\mathcal{O}_n^{\mathbb{R}}$.*
(2) *For n odd, there are up to isomorphism two real forms of $\mathcal{O}_n$: the real Cuntz algebra $\mathcal{O}_n^{\mathbb{R}}$ and an exotic real form $\mathcal{E}_n$.*

Proof. First check that for odd integers n, $n \geq 3$, the groups and operations shown in Table 2 form an acyclic *CRT*-module. Using Corollary 10.3 (that is, Theorem 1 of [10]), let $\mathcal{E}_n$ be the unique real unital separable nuclear c-simple purely infinite C*-algebra satisfying the universal coefficient theorem with united K-theory as shown in Table 2 and such that $[1_{\mathcal{E}_n}]$ corresponds to a generator of the group in the real part in degree 0.
By Corollary 10.5, the problem of classifying real forms of $\mathcal{O}_n$ (for $n \in \{2, 3, \ldots, \infty\}$) reduces to the algebraic problem of classifying real forms of $(K_*(\mathcal{O}_n), [1_{\mathcal{O}_n}])$. Suppose that (M, m) is such a real form. For n even (respectively $n = \infty$) we will show that (M, m) is isomorphic to $(K^{CRT}(\mathcal{O}_n^{\mathbb{R}}), [1_{\mathcal{O}_n^{\mathbb{R}}}])$ (respectively $(K^{CRT}(\mathcal{O}_\infty^{\mathbb{R}}), [1_{\mathcal{O}_\infty^{\mathbb{R}}}])$). For n odd we will show that (M, m) is either isomorphic to $(K^{CRT}(\mathcal{O}_n^{\mathbb{R}}), [1_{\mathcal{O}_n^{\mathbb{R}}}])$ or to $(K^{CRT}(\mathcal{E}_n), [1_{\mathcal{E}_n}])$. Furthermore, by

TABLE 2. $K^{\scriptscriptstyle CRT}(\mathcal{E}_n)$, for n odd and $n \geq 3$.

	0	1	2	3	4	5	6	7	8
KO_*	$\mathbb{Z}_{2(n-1)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_{(n-1)/2}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{2(n-1)}$
KU_*	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$
KT_*	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$
c_*	1	0	0	0	2	0	$\frac{n-1}{2}$	0	1
r_*	2	0	1	0	1	0	0	0	2
ε_*	1	1	0	0	2	0	1	$\frac{n-1}{2}$	1
ζ_*	1	0	$\frac{n-1}{2}$	0	1	0	$\frac{n-1}{2}$	0	1
$(\psi_U)_*$	1	0	-1	0	1	0	-1	0	1
$(\psi_T)_*$	1	1	1	-1	1	1	1	-1	1
γ_*	1	0	1	0	1	0	1	0	1
τ_*	1	1	0	1	0	0	1	2	1

Proposition 2.5 it suffices to restrict our attention to the *CR*-module consisting of the real and complex parts of M.

Since (M, m) is a real form of $(K_*(\mathcal{O}_n), [1_{\mathcal{O}_n}])$ we know that $M_0^U \cong \mathbb{Z}_{n-1}$ (respectively $M_0^U \cong \mathbb{Z}$ when $n = \infty$), $M_1^U = 0$, and $m \in M_0^O$. We further suppose that $c_0(m) \in M_0^U$ is a generator (corresponding to the class of the unit in $K_0(\mathcal{O}_n)$).

We will compute the real part of M (and the behavior of the operations $\eta_O, \xi, r, c, \psi_U$) using the long exact sequence

$$\cdots \to M_n^O \xrightarrow{\eta_O} M_{n+1}^O \xrightarrow{c} M_{n+1}^U \xrightarrow{r\beta_U^{-1}} M_{n-1}^O \to \dots$$

and the *CRT*-relations described in Section 2.

Since $M_k^U = 0$ for k odd it follows that $(\eta_O)_k$ is injective for k odd and surjective for k even. Furthermore, our hypothesis that $c_0(m)$ generates M_0^U implies that c_0 is surjective, which implies that $r_{-2} = 0$ and that $(\eta_O)_{-2}$ is injective. Thus $(\eta_O)_{-2} \colon M_{-2}^O \to M_{-1}^O$ is an isomorphism and $\eta_O^3 \colon M_{-3}^O \to M_0^O$ is injective. Then the relations $\eta_O^3 = 0$ and $2\eta_O = 0$ imply that $M_{-3}^O = 0$ and that M_{-2}^O consists only of 2-torsion.

Suppose first that $M_{-2}^O \cong M_{-1}^O = 0$. Then using the long exact sequence above and the relation $rc = 2$, the rest of the groups of M^O can be easily computed; except that in the case that n is odd we encounter an extension problem wherein M_2^O is either isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. In that case, the same argument as in the computation of $K^{\scriptscriptstyle CRT}(\mathcal{O}_n^{\mathbb{R}})$ in Section 5.1 of [8] shows that $M_2^O \cong \mathbb{Z}_4$ exactly when $n - 1 \equiv 0 \pmod 4$ and $M_2^O \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ exactly when $n - 1 \equiv 2 \pmod 4$. Thus we find that the real and complex parts of M (as well as the operations η_O ξ, r, c, ψ_U) are isomorphic to the real and complex parts of $K^{\scriptscriptstyle CRT}(\mathcal{O}_n^{\mathbb{R}})$ (respectively $K^{\scriptscriptstyle CRT}(\mathcal{O}_\infty^{\mathbb{R}})$).

For the remaining case, suppose that $M^O_{-2} \neq 0$. Since this leads to the exotic *CRT*-module $K^{\scriptscriptstyle CRT}(\mathcal{E}_n)$, we will include all of the details of this computation. Since c_0 is surjective, the relation $\psi_{\scriptscriptstyle U} c = c$ implies that $(\psi_{\scriptscriptstyle U})_0 = 1$. Then the relation $\beta_{\scriptscriptstyle U} \psi_{\scriptscriptstyle U} = -\psi_{\scriptscriptstyle U} \beta_{\scriptscriptstyle U}$ implies that $\psi_{\scriptscriptstyle U} = 1$ in degrees congruent to $0 \pmod 4$ and $\psi_{\scriptscriptstyle U} = -1$ in degrees congruent to $2 \pmod 4$.

From $M^O_{-3} = 0$ it follows that c_{-2} is injective. But the only non-trivial 2-torsion subgroup of M^U_{-2} is isomorphic to $\mathbb{Z}_2$, and that occurs only when n is finite and odd. Thus $M^O_{-2} \cong M^O_{-1} \cong \mathbb{Z}_2$ and the complexification map $c_{-2} \colon \mathbb{Z}_2 \to \mathbb{Z}_{n-1}$ is multiplication by $(n-1)/2$ (in terms of chosen generators). The map r_{-4} is surjective and has kernel equal to $((n-1)/2)\mathbb{Z}_{n-1} \cong \mathbb{Z}_2$ so $M^O_{-4} \cong \mathbb{Z}_{(n-1)/2}$. The relation $c_{-4} r_{-4} = 1 + (\psi_{\scriptscriptstyle U})_{-4} = 2$ implies that the map $c_{-4} \colon \mathbb{Z}_{(n-1)/2} \to \mathbb{Z}_{n-1}$ is multiplication by 2.

Continuing to work our way down, the fact that c_{-4} is injective implies that $M^O_{-5} = 0$. The fact that the image of c_{-4} is $2\mathbb{Z}_{n-1}$ implies that $M^O_{-6} \cong \mathbb{Z}_2$ and r_{-6} is surjective. The relation $c_{-6} r_{-6} = 1 + (\psi_{\scriptscriptstyle U})_{-6} = 0$ implies that $c_{-6} = 0$ from which we see that η_{-7} is an isomorphism. Thus $M^O_{-7} \cong \mathbb{Z}_2$.

Finally, we compute $M^O_{-8} \cong M^O_0$. The exact sequence indicates that it is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_{n-1}$. We will prove that it is isomorphic to $\mathbb{Z}_{2(n-1)}$. If not, then $M^O_0 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{n-1}$ and we can arrange the direct sum decomposition so that $\eta_{-1} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $c_0 = \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)$. Then the relation $rc = 2$ implies that $r_0 = \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)$. But then there is no isomorphism from $M^O_0/\mathrm{image}(r_0) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ to $M^O_1 \cong \mathbb{Z}_2$ as required by the long exact sequence.

Thus in the case that $M^O_{-2} \neq 0$ it must be that n is odd and it must be that the real and complex parts of M are isomorphic to the real and complex parts of $K^{\scriptscriptstyle CRT}(\mathcal{E}_n)$ as in Table 2, completing the proof. □

We remark that the above result can instead be obtained using the analysis of acyclic *CRT*-modules in [23]. Indeed, let M be an acyclic *CRT*-module such that M^U_0 is isomorphic to $\mathbb{Z}_{k-1}$ or $\mathbb{Z}$, $M^U_1 = 0$, and $c_0 \colon M^O_0 \to M^U_0$ is surjective (hence $(\psi_{\scriptscriptstyle U})_0 = 1$). By Lemma 8.3.1, Proposition 8.3.2, and Theorem 8.3.3 of [23], there are isomorphisms

$$h_k(M) := \ker(1 - (\psi_{\scriptscriptstyle U})_k)/\mathrm{image}(1 + (\psi_{\scriptscriptstyle U})_k) \cong \eta_{\scriptscriptstyle O} M^O_k \oplus \eta_{\scriptscriptstyle O} M^O_{k+4}$$

and, furthermore, M is determined up to isomorphism by M^U, $\psi_{\scriptscriptstyle U}$, and the resulting decompositions of $h_k(M)$ for $k = 0$ and $k = 2$. Using $(\psi_{\scriptscriptstyle U})_0 = 1$ and $(\psi_{\scriptscriptstyle U})_2 = -1$, we obtain

$$(h_0(M), h_2(M)) = \begin{cases} (\mathbb{Z}_2, 0) & \text{if } M^U_0 = \mathbb{Z} \\ (0, 0) & \text{if } M^U_0 = \mathbb{Z}_{n-1} \text{ with } n \text{ even} \\ (\mathbb{Z}_2, \mathbb{Z}_2) & \text{if } M^U_0 = \mathbb{Z}_{n-1} \text{ with } n \text{ odd.} \end{cases}$$

The resulting possibilities for M are realized by the united K-theory of $\mathcal{O}^{\mathbb{R}}_\infty$ and $\mathbb{H} \otimes \mathcal{O}^{\mathbb{R}}_\infty$ in the first case; by that of $\mathcal{O}^{\mathbb{R}}_n$ in the second case; and by that of $\mathcal{O}^{\mathbb{R}}_n$, $\mathbb{H} \otimes \mathcal{O}^{\mathbb{R}}_n$, $\mathcal{E}_n$, and $\mathbb{H} \otimes \mathcal{E}_n$ in the third case. The assumption that c_0 is surjective reduces the possibilities to the united K-theory of $\mathcal{O}^{\mathbb{R}}_\infty$, $\mathcal{O}^{\mathbb{R}}_n$, or $\mathcal{E}_n$.

References

1. D. W. Anderson, *A new cohomology theory*, Ph.D. thesis, University of California at Berkeley, 1964.
2. B. V. Rajarama Bhat, George A. Elliott, and Peter A. Fillmore (eds.), *Lectures on operator theory*, Fields Institute Monographs, vol. 13, American Mathematical Society, Providence, RI, 1999.
3. Bruce Blackadar, *Rational C^*-algebras and nonstable K-theory*, Proceedings of the Seventh Great Plains Operator Theory Seminar (Lawrence, KS, 1987), vol. 20, 1990, pp. 285–316.
4. ______, *K-theory for operator algebras*, second ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998.
5. Bruce Blackadar and David Handelman, *Dimension functions and traces on C^*-algebras*, J. Funct. Anal. 45 (1982), no. 3, 297–340.
6. Bruce Blackadar, Alexander Kumjian, and Mikael Rørdam, *Approximately central matrix units and the structure of noncommutative tori*, K-Theory 6 (1992), no. 3, 267–284.
7. Bruce E. Blackadar and Joachim Cuntz, *The structure of stable algebraically simple C^*-algebras*, Amer. J. Math. 104 (1982), no. 4, 813–822.
8. Jeffrey L. Boersema, *Real C^*-algebras, united K-theory, and the Künneth formula*, K-Theory 26 (2002), no. 4, 345–402.
9. ______, *Real C^*-algebras, united KK-theory, and the universal coefficient theorem*, K-Theory 33 (2004), no. 2, 107–149.
10. ______, *The range of united K-theory*, J. Funct. Anal. 235 (2006), no. 2, 701–718.
11. Jeffrey L. Boersema and Efren Ruiz, *Stability of real C^*-algebras*, Canad. Math. Bull., to appear.
12. ______, *Axiomatic K-theory for real C^*-algebras*, arXiv.org arXiv:0909.0972v1 (2009).
13. Jeffrey L. Boersema and Peter J. Stacey, *Correction to the paper: "Real structure in purely infinite C^*-algebras"*, J. Operator Theory 53 (2005), no. 2, 441–442.
14. A. K. Bousfield, *A classification of K-local spectra*, J. Pure Appl. Algebra 66 (1990), no. 2, 121–163.
15. Ola Bratteli, Erling Størmer, Akitaka Kishimoto, and Mikael Rørdam, *The crossed product of a UHF algebra by a shift*, Ergodic Theory Dynam. Systems 13 (1993), no. 4, 615–626.
16. Lawrence G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pacific J. Math. 71 (1977), no. 2, 335–348.
17. Joachim Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. 57 (1977), no. 2, 173–185.
18. ______, *The structure of multiplication and addition in simple C^*-algebras*, Math. Scand. 40 (1977), no. 2, 215–233.

19. ______, *K-theory for certain C^*-algebras*, Ann. of Math. (2) 113 (1981), no. 1, 181–197.
20. Kenneth R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996.
21. Thierry Giordano, *A classification of approximately finite real C^*-algebras*, J. Reine Angew. Math. 385 (1988), 161–194.
22. Paul Halmos, *Finite dimensional vector spaces*, second ed., Van Nostrand, 1958.
23. Beatrice Hewitt, *On the homotopical classification of KO-module spectra*, Ph.D. thesis, University of Illinois at Chicago, 1996.
24. Nigel Higson, *A characterization of KK-theory*, Pacific J. Math. 126 (1987), no. 2, 253–276.
25. Nhan Bao Ho, *Amenability for real C^*-algebras*, Bull. Aust. Math. Soc. 77 (2008), no. 3, 509–514.
26. Eberhard Kirchberg, *the classification purely infinite of C^*-algebras using kasparov's theory*, preprint (1994).
27. Eberhard Kirchberg and Mikael Rørdam, *Non-simple purely infinite C^*-algebras*, Amer. J. Math. 122 (2000), no. 3, 637–666.
28. ______, *Infinite non-simple C^*-algebras: absorbing the Cuntz algebras O_∞*, Adv. Math. 167 (2002), no. 2, 195–264.
29. Nicolaas H. Kuiper, *The homotopy type of the unitary group of Hilbert space*, Topology 3 (1965), 19–30.
30. Hua Xin Lin and N. Christopher Phillips, *Approximate unitary equivalence of homomorphisms from O_∞*, J. Reine Angew. Math. 464 (1995), 173–186.
31. Hua Xin Lin and Shuang Zhang, *On infinite simple C^*-algebras*, J. Funct. Anal. 100 (1991), no. 1, 221–231.
32. Gerard Murphy, *C^*-algebras and operator theory*, Academic Press, 1990.
33. Gert K. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs, vol. 14, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1979.
34. N. Christopher Phillips, *Approximation by unitaries with finite spectrum in purely infinite C^*-algebras*, J. Funct. Anal. 120 (1994), no. 1, 98–106.
35. ______, *A classification theorem for nuclear purely infinite simple C^*-algebras*, Doc. Math. 5 (2000), 49–114 (electronic).
36. Calvin R. Putnam and Aurel Wintner, *The connectedness of the orthogonal group in Hilbert space*, Proc. Nat. Acad. Sci. U. S. A. 37 (1951), 110–112.
37. John R. Ringrose, *Exponential length and exponential rank in C^*-algebras*, Proc. Roy. Soc. Edinburgh Sect. A 121 (1992), no. 1-2, 55–71.
38. Mikael Rørdam, *Classification of inductive limits of Cuntz algebras*, J. Reine Angew. Math. 440 (1993), 175–200.
39. ______, *Classification of nuclear C^*-algebras.*, Encyclopaedia of Mathematical Sciences, vol. 126, Springer-Verlag, Berlin, 2002, Operator Algebras and Non-commutative Geometry, 7.

40. Jonathan Rosenberg and Claude Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor*, Duke Math. J. 55 (1987), no. 2, 431–474.
41. Claude Schochet, *Topological methods for C^*-algebras. II. Geometric resolutions and the Künneth formula*, Pacific J. Math. 98 (1982), no. 2, 443–458.
42. Peter J. Stacey, *Real structure in direct limits of finite-dimensional C^*-algebras*, J. London Math. Soc. (2) 35 (1987), no. 2, 339–352.
43. ———, *Real structure in purely infinite C^*-algebras*, J. Operator Theory 49 (2003), no. 1, 77–84.

Jeffrey L. Boersema
Department of Mathematics
Seattle University
Seattle, WA
USA
boersema@seattleu.edu

Jeffrey L. Boersema
Department of Mathematics
University of Hawaii
Hilo, HI
USA
ruize@hawaii.edu

P. J. Stacey
Department of Mathematics
LaTrobe University
Melbourne, VIC
Australia
P.Stacey@latrobe.edu.au

A Simple Criterion for Extending Natural Transformations to Higher K-Theory

Gonçalo Tabuada[1]

Received: January 26, 2011

Communicated by Max Karoubi

Abstract. In this article we introduce a very simple an widely applicable criterion for extending natural transformations to higher K-theory. More precisely, we prove that every natural transformation defined on the Grothendieck group and with values in an additive theory admits a unique extension to higher K-theory. As an application, the higher trace maps and the higher Chern characters originally constructed by Dennis and Karoubi, respectively, can be obtained in an elegant, unified, and conceptual way from our general results.

2010 Mathematics Subject Classification: 19D55, 18G55
Keywords and Phrases: Higher K-theory, higher trace maps, higher Chern characters, non-commutative motives

Introduction

In his foundational work, Grothendieck [10] introduced a very simple and elegant construction K_0, the *Grothendieck group*, in order to formulate a far-reaching generalization of the Riemann-Roch theorem. Since then, this versatile construction spawned well-beyond the realm of algebraic geometry to become one of the most important (working) tools in mathematics.
Latter, through revolutionary topological techniques, Quillen [23] extended the Grothendieck group to a whole family of higher K-theory groups $K_n, n \geq 0$. However, in contrast with K_0, these higher K-theory groups are rather misterious and their computation is often out of reach. In order to capture some of its flavour, Connes, Dennis, Karoubi, and others, constructed natural transformations towards simpler theories E making use of a variety of highly involved techniques; see [6, 7, 15]. Typically, the construction of a natural transformation $K_0 \Rightarrow E_0$ is very simple, while its extension $K_n \Rightarrow E_n$ to higher K-theory is a real "tour-de-force". For example, the trace map $K_0 \Rightarrow HH_0$ consists simply in taking the trace of an idempotent, while its extension

[1]The author was partially supported by the grant FCT-Portugal `PTDC/MAT/098317/2008`.

$K_n \Rightarrow HH_n$ makes use of an array of tools (Hurewicz maps, group homology, assembly maps, etc) coming from topology, algebra, representation theory, etc. These phenomena motivate the following general questions:

Questions: *Given a natural transformation $K_0 \Rightarrow E_0$, is it possible to extend it to higher K-theory $K_n \Rightarrow E_n$? If so, is such an extension unique ?*

In this article we prove that if E verifies three very simple conditions, not only such an extension exists, but it is moreover unique. The precisely formulation of our results makes use of the language of Grothendieck derivators, a formalism which allows us to state and prove precise universal properties; see Appendix A.

1. Statement of results

A *differential graded (=dg) category*, over a fixed commutative base ring k, is a category enriched over cochain complexes of k-modules (morphisms sets are such complexes) in such a way that composition fulfills the Leibniz rule: $d(f \circ g) = (df) \circ g + (-1)^{\deg(f)} f \circ (dg)$. Dg categories extend the classical notion of (dg) k-algebra and solve many of the technical problems inherent to triangulated categories; see Keller's ICM address [16]. In non-commutative algebraic geometry in the sense of Bondal, Drinfeld, Kaledin, Kontsevich, Van den Bergh, and others, they are considered as differential graded enhancements of (bounded) derived categories of quasi-coherent sheaves on a hypothetic non-commutative space; see [1, 8, 9, 14, 17, 18].

Let $E : \mathsf{dgcat} \to \mathsf{Spt}$ be a functor, defined on the category of dg categories, and with values in the category of spectra [2]. We say that E is an *additive functor* if it verifies the following three conditions:

(i) filtered colimits of dg categories are mapped to filtered colimits of spectra;

(ii) *derived Morita equivalences* (*i.e.* dg functors which induce an equivalence on the associated derived categories; see [16, §4.6]) are mapped to weak equivalences of spectra;

(iii) *split exact sequences* (*i.e.* sequences of dg categories which become split exact after passage to the associated derived categories; see [24, §13]) are mapped to direct sums

$$0 \longrightarrow \mathcal{A} \rightleftarrows \mathcal{B} \rightleftarrows \mathcal{C} \longrightarrow 0 \qquad \mapsto \qquad E(\mathcal{A}) \oplus E(\mathcal{C}) \simeq E(\mathcal{B})$$

in the homotopy category of spectra.

Examples of additive functors include Hochschild homology (HH), cyclic homology (HC), and algebraic K-theory (K); see [16, §5]. Recall from [25] that the category dgcat carries a Quillen model structure whose weak equivalences are the derived Morita equivalences. Given an additive functor E, we obtain then an induced morphism of derivators $\mathbb{E} : \mathsf{HO}(\mathsf{dgcat}) \to \mathsf{HO}(\mathsf{Spt})$. Associated to E, we have also the composed functors

$$E_n : \mathsf{dgcat} \xrightarrow{E} \mathsf{Spt} \xrightarrow{\pi_n^s} \mathsf{Ab} \qquad n \geq 0 \,,$$

where π_n^s denotes the n^{th} stable homotopy group functor and Ab the category of abelian groups. Our answer to the questions stated in the Introduction is:

Theorem 1.1. *For any additive functor E, the natural map*

$$\mathrm{Nat}(\mathbb{K}, \mathbb{E}) \xrightarrow{\sim} \mathrm{Nat}(K_0, E_0) \tag{1.2}$$

is bijective. In particular, every natural transformation $\phi : K_0 \Rightarrow E_0$ admits a canonical extension $\phi_n : K_n \Rightarrow E_n$ to all higher K-theory groups.

Intuitively speaking, Theorem 1.1 show us that all the information concerning a natural transformation is encoded on the Grothendieck group. Its proof relies in an essential way on the theory of *non-commutative motives*, a subject envisioned by Kontsevich [17, 19] and whose development was initiated in [3, 4, 24, 25, 27, 28]. In the next section we illustrate the potential of this general result by explaining how the highly involved constructions of Dennis and Karoubi can be obtained as simple instantiations of the above theorem. Due to its generality and simplicity, we believe that Theorem 1.1 will soon be part of the toolkit of any mathematician whose research comes across the above conditions (i)-(iii).

2. Applications

2.1. Higher trace maps. Recall from [16, §5.3] the construction of the Hochschild homology complex $HH(\mathcal{A})$ of a dg category $\mathcal{A}$. This construction is functorial in $\mathcal{A}$ and so by promoting it to spectra we obtain a well-defined functor

$$HH : \mathsf{dgcat} \longrightarrow \mathsf{Spt}\,. \tag{2.1}$$

As explained in *loc. cit.*, this functor verifies conditions (i)-(iii) and hence it is additive. Now, given a k-algebra A, recall from [20, Example 8.3.6] the construction of the classical trace map

$$K_0(A) \to HH_0(A) = A/[A, A]\,.$$

Roughly, it is the map induced by sending an idempotent matrix to the image of its trace (*i.e.* the sum of the diagonal entries) in the quocient $A/[A, A]$. This construction extends naturally from k-algebras to dg categories (see [26]) giving rise to a natural transformation

$$K_0 \Rightarrow HH_0\,. \tag{2.2}$$

Proposition 2.3. *In Theorem 1.1 let E be the additive functor* (2.1) *and let ϕ be the natural transformation* (2.2). *Then, for every k-algebra A, the canonical extension $\phi_n : K_n(A) \to HH_n(A)$ of ϕ agrees with the n^{th} trace map constructed originally by Dennis (see* [20, §8.4 and §11.4]*).*

2.2. HIGHER CHERN CHARACTERS. Recall also from [16, §5.3] the construction of the cyclic homology complex $HC(\mathcal{A})$ of a dg category $\mathcal{A}$. By promoting this construction to spectra we obtain a functor

$$HC : \mathsf{dgcat} \longrightarrow \mathsf{Spt} \tag{2.4}$$

which verifies conditions (i)-(iii). Given a k-algebra A, recall from [20, Theorem 8.3.4] the construction of the Chern characters

$$ch_{0,i} : K_0(A) \longrightarrow HC_{2i}(A) \qquad i \geq 0\,.$$

Morally, these are the non-commutative analogues of the classical Chern character with values in even dimensional de Rham cohomology. As shown in [26] this construction extends naturally from k-algebras to dg categories giving rise to natural transformations

$$K_0 \Rightarrow HC_{2i} \qquad i \geq 0\,.$$

PROPOSITION 2.5. *In Theorem 1.1 let E be the additive functor $\Omega^{2i}HC$ (obtained by composing (2.4) with the $(2i)^{\text{th}}$-fold looping functor on* Spt*) and let ϕ be the natural transformation $K_0 \Rightarrow (\Omega^{2i}HC)_0 = HC_{2i}$. Then, for every k-algebra A, the canonical extension $\phi_n : K_n(A) \to (\Omega^{2i}HC)_n(A) = HC_{n+2i}(A)$ of ϕ agrees with the higher Chern character $ch_{n,i}$ constructed originally by Karoubi (see* [15, §2.27-2.36]*).*

3. PROOF OF THEOREM 1.1

We start by describing the natural map (1.2). As mentioned in §1, the category dgcat carries a Quillen model structure whose weak equivalences are the derived Morita equivalences; see [25, Theorem 5.3]. Let us write Hmo for the associated homotopy category and $l : \mathsf{dgcat} \to \mathsf{Hmo}$ for the localization functor. According to our notation the map (1.2) sends a natural transformation $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{E})$ to the natural transformation $\pi_0^s \circ \Phi(e) \circ l \in \mathrm{Nat}(K_0, E_0)$. Pictorially, we have:

$$\begin{tikzcd} \mathsf{dgcat} \arrow[r, "l"] & \mathsf{Hmo} \arrow[r, bend left, "\mathbb{K}(e)"] \arrow[r, bend right, "\mathbb{E}(e)"'] \arrow[r, phantom, "\Downarrow \Phi(e)"] & \mathsf{Ho}(\mathsf{Spt}) \arrow[r, "\pi_0^s"] & \mathsf{Ab}\,. \end{tikzcd} \tag{3.1}$$

The functors $K, E : \mathsf{dgcat} \to \mathsf{Spt}$ are additive and so the following diagrams

$$\begin{tikzcd} \mathsf{dgcat} \arrow[r, "K"] \arrow[d, "l"'] & \mathsf{Spt} \arrow[d] \\ \mathsf{Hmo} \arrow[r, "\mathbb{K}(e)"'] & \mathsf{Ho}(\mathsf{Spt}) \end{tikzcd} \qquad \begin{tikzcd} \mathsf{dgcat} \arrow[r, "E"] \arrow[d, "l"'] & \mathsf{Spt} \arrow[d] \\ \mathsf{Hmo} \arrow[r, "\mathbb{E}(e)"'] & \mathsf{Ho}(\mathsf{Spt}) \end{tikzcd}$$

are commutative. Moreover, the 0^{th} stable homotopy group functor π_0^s descends to the homotopy category Ho(Spt). These facts show us that the composed horizontal functors in the above diagram (3.1) are in fact K_0 and E_0.
We now study the set $\mathrm{Nat}(K_0, E_0)$. Recall from [16, §5.1] the notion of *additive invariant*. Intutively, it consists of a functor defined on dgcat and with values

in an additive category which verifies conditions similar to (ii)-(iii). Since by hypothesis E is additive, the composed functor

$$E_0 : \mathsf{dgcat} \xrightarrow{l} \mathsf{Hmo} \xrightarrow{\mathbb{E}(e)} \mathsf{Ho}(\mathsf{Spt}) \xrightarrow{\pi_0^s} \mathsf{Ab}$$

is an additive invariant. Hence, as proved in [26, Proposition 4.1], we have the following natural bijection

$$\text{(3.2)} \qquad \mathrm{Nat}(K_0, E_0) \xrightarrow{\sim} E_0(\underline{k}) \qquad \eta \mapsto \eta(\underline{k})([k])\,.$$

Some explanations are in order: $\underline{k}$ denotes the dg category naturally associated to the base ring k, *i.e.* the dg category with only one object and with k as the dg algebra of endomorphisms (concentrated in degree zero); the symbol $[k]$ stands for the class of k (as a module over itself) in the Grothendieck group $K_0(\underline{k}) = K_0(k)$.

Let us now turn our attention to $\mathrm{Nat}(\mathbb{K}, \mathbb{E})$. Recall from [24, §15] the notion of *additive invariant of dg categories*. Roughly speaking, it consists of a morphism of derivators defined on $\mathsf{HO}(\mathsf{dgcat})$ and with values in a triangulated derivator which verifies conditions analogous to (i)-(iii). Since the functor E is additive, the induced morphism of derivators

$$\mathbb{E} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathsf{Spt})$$

is an additive invariant of dg categories. Following [3, Theorem 8.1] we have then a natural bijection[2]

$$\text{(3.3)} \qquad \mathrm{Nat}(\mathbb{K}, \mathbb{E}) \xrightarrow{\sim} \pi_0^s\, \mathbb{E}(\underline{k}) = E_0(\underline{k})\,.$$

A careful inspection of the proof of [3, Theorem 8.1] show us that (3.3) sends a natural transformation $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{E})$ to the element $\pi_0^s(\Phi(e)(\underline{k}))([k])$ of the abelian group $E_0(\underline{k})$. Note that this element is simply the image of $[k]$ by the abelian group homomorphism

$$K_0(k) = \pi_0^s(\mathbb{K}(e)(\underline{k})) \xrightarrow{\pi_0^s(\Phi(e)(\underline{k}))} \pi_0^s(\mathbb{E}(e)(\underline{k})) = E_0(\underline{k})\,.$$

We now prove that the following diagram

$$\text{(3.4)} \qquad \begin{tikzcd} \mathrm{Nat}(\mathbb{K}, \mathbb{E}) \arrow[r, "(1.2)"] \arrow[dr, "(3.3)"'] & \mathrm{Nat}(K_0, E_0) \arrow[d, "(3.2)"] \\ & E_0(\underline{k}) \end{tikzcd}$$

commutes. Let $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{E})$. On the one hand, we observe that the composed map (3.2) $\circ$ (1.2) sends Φ to the element $(\pi_0^s \circ \Phi(e) \circ l)(\underline{k})([k])$ of the abelian group $E_0(\underline{k})$. On the other hand, the following equalities hold:

$$(\Phi(e) \circ l)(\underline{k}) = \Phi(e)(\underline{k}) \qquad (\pi_0^s \circ \Phi(e))(\underline{k}) = \pi_0^s(\Phi(e)(\underline{k}))\,.$$

[2] In [3] this bijection was established for a localizing invariant $\mathbb{E}$. However, the arguments in the additive case are completely similar.

Therefore, we have

$$(\pi_0^s \circ \Phi(e) \circ l)(\underline{k})([k]) = \pi_0^s(\Phi(e)(\underline{k}))([k])\,.$$

Finally, since the right-hand side in this latter equality coincides with the image of Φ by the map (3.3), we conclude that (3.3) = (3.2) $\circ$ (1.2).
Theorem 1.1 now follows from diagram (3.4) and the fact that both maps (3.2) and (3.3) are bijective. The canonical extension $\phi_n : K_n \Rightarrow E_n$ of $\phi : K_0 \Rightarrow E_0$ is then the composition $\pi_n^s \circ \Phi(e) \circ l$, where Φ is the unique natural transformation associated to ϕ under the bijection (1.2).

4. Proof of Proposition 2.3

The essence of the proof consists in describing the unique natural transformation $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{HH})$ which corresponds to (2.2) under the bijection (1.2). Recall from [24, §15] the construction of the universal additive invariant of dg categories

$$\mathcal{U}_{\mathsf{A}} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathrm{Mot}_{\mathsf{A}}\ .$$

Given any Quillen model category $\mathcal{M}$ we have an induced equivalence of categories

$$(\mathcal{U}_{\mathsf{A}})^* : \underline{\mathsf{Hom}}_!(\mathrm{Mot}_{\mathsf{A}}, \mathsf{HO}(\mathcal{M})) \xrightarrow{\sim} \underline{\mathsf{Hom}}_{\mathsf{A}}(\mathsf{HO}(\mathsf{dgcat}), \mathsf{HO}(\mathcal{M}))\,, \tag{4.1}$$

where the left-hand side denotes the category of homotopy colimit preserving morphisms of derivators and the right-hand side the category of additive invariants of dg categories. The algebraic K-theory functor K is additive and so the induced morphism $\mathbb{K}$ is an additive invariant of dg categories. Thanks to equivalence (4.1), it factors then uniquely through $\mathcal{U}_{\mathsf{A}}$. Recall from [24, Theorem 15.10] that for every dg category $\mathcal{A}$ we have a weak equivalence of spectra

$$\mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{A}}(\underline{k}), \mathcal{U}_{\mathsf{A}}(\mathcal{A})) \simeq K(\mathcal{A})\,,$$

where $\mathbb{R}\mathsf{Hom}(-,-)$ denotes the spectral enrichment of $\mathrm{Mot}_{\mathsf{A}}$ (see [3, §A.3]). Therefore, we conclude that $\mathbb{K}$ can be expressed as the following composition

$$\mathsf{HO}(\mathsf{dgcat}) \xrightarrow{\mathcal{U}_{\mathsf{A}}} \mathrm{Mot}_{\mathsf{A}} \xrightarrow{\mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{A}}(\underline{k}),-)} \mathsf{HO}(\mathsf{Spt})\,. \tag{4.2}$$

The Hochschild homology functor, with values in the projective Quillen model category $\mathcal{C}(k)$ of cochain complexes of k-modules (see [12, Theorem 2.3.11]), verifies conditions (i)-(iii). Hence, it gives rise to an additive additive invariant of dg categories which we denote by

$$\mathbb{H}H : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathcal{C}(k))\,.$$

Note that, according to our notation, $\mathbb{HH}$ can be expressed as the following composition

$$\mathsf{HO}(\mathsf{dgcat}) \xrightarrow{\mathbb{H}H} \mathsf{HO}(\mathcal{C}(k)) \xrightarrow{\mathbb{R}\mathsf{Hom}(k,-)} \mathsf{HO}(\mathsf{Spt})\,. \tag{4.3}$$

Equivalence (4.1) provide us then the following commutative diagram

$$\begin{tikzcd} \mathsf{HO}(\mathsf{dgcat}) \arrow[r, "\mathbb{H}H"] \arrow[d, "\mathcal{U}_{\mathsf{A}}"'] & \mathsf{HO}(\mathcal{C}(k)) \\ \mathrm{Mot}_{\mathsf{A}} \arrow[ur, "\overline{\mathbb{H}H}"'] & \end{tikzcd} .$$

By construction, the morphism $\overline{\mathbb{H}H}$ maps $\mathcal{U}_{\mathsf{A}}(\underline{k})$ to $\mathbb{H}H(\underline{k}) = k$. Hence, by making use of the above factorizations (4.2) and (4.3), we conclude that it induces a natural transformation $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{H}\mathbb{H})$. We now show that the image of this natural transformation Φ by the map (1.2) is the natural transformation (2.2). By taking $E = HH$ in bijection (3.2) we obtain:

$$\mathrm{Nat}(K_0, HH_0) \simeq HH_0(\underline{k}) \simeq k \qquad \eta \mapsto \eta(\underline{k})([k])\,. \tag{4.4}$$

Under this bijection, the natural transformation (2.2) corresponds to the unit of the base ring k; see [26, Theorem 1.3]. Hence, it suffices to show that the same holds for the natural transformation $\pi_0^s \circ \Phi(e) \circ l$ associated to Φ. The class $[k]$ of k (as a module over itself) in the Grothendieck group $K_0(k)$ corresponds to the identity morphism in

$$\mathsf{Hom}_{\mathrm{Mot}_{\mathsf{A}}(e)}(\mathcal{U}_{\mathsf{A}}(\underline{k}), \mathcal{U}_{\mathsf{A}}(\underline{k})) \simeq K_0(\underline{k}) \simeq K_0(k)\,.$$

By functoriality, $\overline{\mathbb{H}H}(e)$ maps this identity morphism to the identity morphism in $\mathsf{Hom}_{\mathcal{D}(k)}(\mathbb{H}H(\underline{k}), \mathbb{H}H(\underline{k}))$. Under the natural isomorphisms

$$\mathsf{Hom}_{\mathcal{D}(k)}(\mathbb{H}H(\underline{k}), \mathbb{H}H(\underline{k})) \simeq \mathsf{Hom}_{\mathcal{D}(k)}(k,k) \simeq HH_0(k) \simeq k$$

the identity morphism corresponds to the unit of the base ring k and so we conclude that $\pi_0^s \circ \Phi(e) \circ l$ agrees with (2.2). This implies that Φ is in fact the unique natural transformation which corresponds to (2.2) under the bijection (1.2).

Finally, let A be a k-algebra. As proved in [27, Theorem 2.8], the canonical extension $\phi_n : K_n(A) \to HH_n(A)$ of ϕ (*i.e.* the abelian group homomorphism $(\pi_n^s \circ \Phi(e) \circ l)(A)$) agrees with the n^{th} trace map constructed by Dennis and so the proof is finished.

5. Proof of Proposition 2.5

We prove first the particular case $(i = 0)$. Let us start by describing the unique natural transformation $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{H}\mathbb{C})$ which corresponds to $\phi : K_0 \Rightarrow HC_0$ under the bijection (1.2). Observe that $\mathbb{H}\mathbb{C}$ can be expressed as the following composition

$$\mathsf{HO}(\mathsf{dgcat}) \xrightarrow{\mathbb{M}} \mathsf{HO}(\mathcal{C}(\Lambda)) \xrightarrow{\mathbb{P}} \mathsf{HO}(k[u]\text{-Comod}) \xrightarrow{\mathbb{U}} \mathsf{HO}(\mathcal{C}(k)) \xrightarrow{\mathbb{R}\mathsf{Hom}(k,-)} \mathsf{HO}(\mathsf{Spt})\,.$$

Some explanations are in order: $\mathcal{C}(\Lambda)$ is the projective Quillen model category of mixed complexes and $\mathbb{M}$ the morphism induced by the mixed complex construction[3] (see [4, Example 7.10]); $k[u]$-Comod is the Quillen model category of $k[u]$-comodules (where $k[u]$ is the Hopf algebra of polynomials in one variable

[3]Denoted by C in [4, Example 7.10].

of degree 2) and $\mathbb{P}$ the morphism induced by the perioditization construction (see [4, Example 7.11]); $\mathbb{U}$ is the morphism induced by the natural forgetful construction. Moreover, as explained in [4, Examples 8.10 and 8.11], negative cyclic homology and periodic cyclic homology admit the following factorizations:

$$\mathbb{HC}^- : \mathsf{HO}(\mathsf{dgcat}) \xrightarrow{\mathbb{M}} \mathsf{HO}(\mathcal{C}(\Lambda)) \xrightarrow{\mathbb{R}\mathsf{Hom}(k,-)} \mathsf{HO}(\mathsf{Spt}) \tag{5.1}$$

$$\mathbb{HP} : \mathsf{HO}(\mathsf{dgcat}) \xrightarrow{(\mathbb{P}\circ\mathbb{M})} \mathsf{HO}(k[u]\text{-Comod}) \xrightarrow{\mathbb{R}\mathsf{Hom}(k[u],-)} \mathsf{HO}(\mathsf{Spt})\,.$$

Therefore, since $\mathbb{P}$ maps k to $k[u]$ and $\mathbb{U}$ maps $k[u]$ to k, we obtain the classical natural transformations

$$\mathbb{HC}^- \Rightarrow \mathbb{HP} \Rightarrow \mathbb{HC} \tag{5.2}$$

between the cyclic homology variants; see [20, §5.1]. The mixed complex morphism $\mathbb{M}$ is an additive invariant of dg categories and so by equivalence (4.1) it factors uniquely through $\mathcal{U}_\mathsf{A}$. We obtain then a commutative diagram

$$\begin{tikzcd} \mathsf{HO}(\mathsf{dgcat}) \arrow[r, "\mathbb{M}"] \arrow[d, "\mathcal{U}_\mathsf{A}"'] & \mathsf{HO}(\mathcal{C}(\Lambda)) \\ \mathrm{Mot}_\mathsf{A} \arrow[ur, "\overline{\mathbb{M}}"'] & \end{tikzcd} .$$

By construction, the morphism $\overline{\mathbb{M}}$ maps $\mathcal{U}_\mathsf{A}(\underline{k})$ to $\mathbb{M}(\underline{k}) = k$. Therefore, making use of the factorizations (4.2) and (5.1), we conclude that $\overline{\mathbb{M}}$ induces a natural transformation $\Phi_1 : \mathbb{K} \Rightarrow \mathbb{HC}^-$. Its composition with (5.2) gives rise to a natural transformation which we denote by $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{HC})$. We now show that the image of Φ by the map (1.2) is the natural transformation $\phi : K_0 \Rightarrow HC_0$. Recall from [26, Theorem 1.7(ii)] that ϕ admits the following factorization

$$K_0 \overset{ch_0^-}{\Rightarrow} HC_0^- \Rightarrow HP_0 \Rightarrow HC_0\,,$$

where ch_0^- is the negative Chern character and the other natural transformations are the ones associated to (5.2). Hence, it suffices to show that the natural transformation $\pi_0^s \circ \Phi_1(e) \circ l$, associated to $\Phi_1 : \mathbb{K} \Rightarrow \mathbb{HC}^-$, agrees with ch_0^-. This fact is proved in [28, Proposition 4.2] and so we conclude that Φ is the unique natural transformation which corresponds to ϕ under the bijection (1.2). Now, let A be a k-algebra. As explained in [20, §11.4.3], Karoubi's Chern character $ch_{n,0}(A)$ can be expressed as the following composition

$$K_n(A) \xrightarrow{ch_n^-(A)} HC_n^-(A) \longrightarrow HP_n(A) \longrightarrow HC_n(A)\,.$$

Note that the right-hand side maps coincide the ones associated to (5.2). Therefore, it suffices to show that the abelian group homomorphism

$$(\pi_n^s \circ \Phi_1(e) \circ l)(A) : K_n(A) \longrightarrow HC_n^-(A)$$

agrees with $ch_n^-(A)$. This latter fact is proved in [27, Theorem 2.8] and so the proof of the particular case $(i = 0)$ is finished.

We now prove the case ($i > 0$). Recall from [13, §1] that for any dg category $\mathcal{A}$ we have a natural periodicity map $S : \Omega^2\mathbb{M}(\mathcal{A}) \to \mathbb{M}(\mathcal{A})$ in the category $\mathcal{C}(\Lambda)$ of mixed complexes. This construction is natural in $\mathcal{A}$ and so by iterating it we obtain an infinite sequence of maps

$$\cdots \longrightarrow \Omega^{2i}\mathbb{M}(\mathcal{A}) \longrightarrow \cdots \longrightarrow \Omega^2\mathbb{M}(\mathcal{A}) \longrightarrow \mathbb{M}(\mathcal{A})\,. \tag{5.3}$$

Under the natural equivalences

$$\begin{array}{rcl} \mathbb{R}\mathsf{Hom}(k, \Omega^{2i}\mathbb{M}(-)) & \simeq & \Omega^{2i}\mathbb{HC}^- \\ \mathbb{R}\mathsf{Hom}(k[u], \mathbb{P}(\Omega^{2i}\mathbb{M}(-))) & \simeq & \Omega^{2i}\mathbb{HP} \\ \mathbb{R}\mathsf{Hom}(k, \mathbb{U}(\mathbb{P}(\Omega^{2i}\mathbb{M}(-)))) & \simeq & \Omega^{2i}\mathbb{HC}\,, \end{array}$$

the above sequence of maps (5.3) gives rise to the following commutative diagram of natural transformations

$$\begin{tikzcd} \mathbb{K} \arrow[r, Rightarrow, "\Phi_1"] & \mathbb{HC}^- \arrow[r, Rightarrow] & \mathbb{HP} \arrow[r, Rightarrow] & \mathbb{HC} \\ & \Omega^2\mathbb{HC}^- \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \Omega^2\mathbb{HP} \arrow[r, Rightarrow] \arrow[u, Rightarrow, "\simeq"] & \Omega^2\mathbb{HC} \arrow[u, Rightarrow] \\ & \vdots \arrow[u, Rightarrow] & \vdots \arrow[u, Rightarrow, "\simeq"] & \vdots \arrow[u, Rightarrow] \\ & \Omega^{2i}\mathbb{HC}^- \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \Omega^{2i}\mathbb{HP} \arrow[r, Rightarrow] \arrow[u, Rightarrow, "\simeq"] & \Omega^{2i}\mathbb{HC} \arrow[u, Rightarrow] \\ & \vdots \arrow[u, Rightarrow] & \vdots \arrow[u, Rightarrow, "\simeq"] & \vdots \arrow[u, Rightarrow] \end{tikzcd} \tag{5.4}$$

The periodicity map S becomes invertible in periodic cyclic homology and so the middle column in (5.4) consists of natural isomorphisms. Hence, we obtain the classical sequence of natural transformations

$$\mathbb{HP} \Rightarrow \cdots \Rightarrow \Omega^{2i}\mathbb{HC} \Rightarrow \cdots \Rightarrow \Omega^2\mathbb{HC} \Rightarrow \mathbb{HC}$$

which relates periodic cyclic homology with the even dimensional loopings of cyclic homology; see [20, §5.1.8]. Let us then take for Φ the composed natural transformation

$$\mathbb{K} \stackrel{\Phi_1}{\Rightarrow} \mathbb{HC}^- \Rightarrow \mathbb{HP} \Rightarrow \Omega^{2i}\mathbb{HC}\,.$$

The fact that its image by the map (1.2) is the natural transformation $\phi : K_0 \Rightarrow HC_{2i}$ is now an immediate consequence of the following factorization

$$\phi : K_0 \stackrel{ch_0^-}{\Rightarrow} HC_0^- \Rightarrow HP_0 \Rightarrow HC_{2i}\,,$$

see [26, Theorem 1.7(ii)], and the particular case ($i = 0$). Similarly, the fact that the canonical extension $\phi_n : K_n(A) \to HC_{n+2i}(A)$ agrees with Karoubi's higher Chern character $ch_{n,i}(A)$ follows from the following factorization

$$ch_{n,i}(A) : K_n(A) \stackrel{ch_n^-(A)}{\longrightarrow} HC_n^-(A) \longrightarrow HP_n(A) \longrightarrow HC_{n+2i}(A)\,,$$

see [20, §11.4.3], and the particular case ($i = 0$). This achieves the proof.

Appendix A. Grothendieck Derivators

In order to make this article more self-contained we give a brief introduction to Grothendieck's theory of derivators [11]; this language can easily be acquired by skimming through [21], [5, §1] or [3, 4, Appendix A].
Derivators originate in the problem of higher homotopies in derived categories. Given a triangulated category $\mathcal{T}$ and a small category I, it essentially never happens that the diagram category $\mathrm{Fun}(I^{\mathsf{op}}, \mathcal{T})$ remains triangulated; this already fails for the category of arrows in $\mathcal{T}$. However, our triangulated category $\mathcal{T}$ often appears as the homotopy category $\mathcal{T} = \mathsf{Ho}(\mathcal{M})$ of some Quillen model category $\mathcal{M}$ (see [22]). In this case we can consider the category $\mathrm{Fun}(I^{\mathsf{op}}, \mathcal{M})$ of diagrams in $\mathcal{M}$ whose homotopy category $\mathsf{Ho}(\mathrm{Fun}(I^{\mathsf{op}}, \mathcal{M}))$ is triangulated and provides a reasonable approximation to $\mathrm{Fun}(I^{\mathsf{op}}, \mathcal{T})$. More importantly, one can let I vary. This "nebula" of categories $\mathsf{Ho}(\mathrm{Fun}(I^{\mathsf{op}}, \mathcal{M}))$, indexed by small categories I, and the various (adjoint) functors between them is what Grothendieck formalized into the concept of a *derivator*.
A derivator consists of a strict contravariant 2-functor, from the 2-category of small categories to the 2-category of all categories, subject to five natural conditions. We shall not list these conditions here for it would be too long; see [5, §1]. The essential example to keep in mind is the (triangulated) derivator $\mathsf{HO}(\mathcal{M})$ associated to a (stable) Quillen model category $\mathcal{M}$ and defined for every small category I by

$$\mathsf{HO}(\mathcal{M})(I) := \mathsf{Ho}(\mathrm{Fun}(I^{\mathsf{op}}, \mathcal{M}))\,.$$

We will write e for the 1-point category with one object and one identity morphism. Note that $\mathsf{HO}(\mathcal{M})(e)$ is the homotopy category $\mathsf{Ho}(\mathcal{M})$. Given Quillen model categories $\mathcal{M}_1$ and $\mathcal{M}_2$ and weak equivalence preserving functors $E, F : \mathcal{M}_1 \to \mathcal{M}_2$, we will denote by $\mathbb{E}, \mathbb{F} : \mathsf{HO}(\mathcal{M}_1) \longrightarrow \mathsf{HO}(\mathcal{M}_2)$ the induced morphisms of derivators and by $\mathrm{Nat}(\mathbb{E}, \mathbb{F})$ the set of natural transformations from $\mathbb{E}$ to $\mathbb{F}$; see [5, §5]. Note that given $\Phi \in \mathrm{Nat}(\mathbb{E}, \mathbb{F})$, $\Phi(e)$ is a natural transformation between the induced functors $\mathbb{E}(e), \mathbb{F}(e) : \mathsf{Ho}(\mathcal{M}_1) \to \mathsf{Ho}(\mathcal{M}_2)$.

Acknowledgments: The author is very grateful to Bruno Kahn for motivating questions, and to Paul Balmer, Denis-Charles Cisinski and Christian Haesemeyer for useful discussions. He would also like to thank the Departments of Mathematics of UCLA, MIT and UIC for its hospitality and the Clay Mathematics Institute, the Midwest Topology Network, and the Calouste Gulbenkian Foundation for financial support.

References

[1] A. Bondal and M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry.* Moscow Math. J. 3 (2003), 1–36.

[2] A. K. Bousfield and D. Kan, *Homotopy limits, completions, and localizations*. Lecture notes in Mathematics 304 (1972).
[3] D.-C. Cisinski and G. Tabuada, *Non-connective K-theory via universal invariants*. Compositio Mathematica 147 (2011), 1281–1320.
[4] ———, *Symmetric monoidal structure on Non-commutative motives*. Available at arXiv:1001.0228v2. To appear in Journal of K-theory.
[5] D.-C. Cisinski and A. Neeman, *Additivity for derivator K-theory*. Adv. Math. 217 (2008), no. 4, 1381–1475.
[6] A. Connes, *Non-commutative differential geometry*. Publ. Math. IHES 62 (1985), 257–360.
[7] K. Dennis, *In search of a new homology theory*. Unpublished manuscript (1976).
[8] V. Drinfeld, *DG quotients of DG categories*. J. Algebra 272 (2004), 643–691.
[9] ———, *DG categories*. University of Chicago geometric Langlands seminar 2002. Notes available at www.math.utexas.edu/users/benzvi/GRASP/lectures/Langlands.html.
[10] A. Grothendieck, *Classes de faisceaux et théoréme de Riemann-Roch (SGA6)*. Lecture notes in Mathematics 225 (1971), 20–77.
[11] ———, *Les Dérivateurs*. Available at http://people.math.jussieu.fr/maltsin/groth/Derivateurs.html.
[12] M. Hovey, *Model categories*. Mathematical Surveys and Monographs 63. American Mathematical Society (1999).
[13] J.D.S. Jones and C. Kassel, *Bivariant cyclic theory*. K-theory 3 (1989), 339–365.
[14] D. Kaledin, *Motivic structures in non-commutative geometry*. Available at arXiv:1003.3210. To appear in the Proceedings of the ICM 2010.
[15] M. Karoubi, *Homologie cyclique et K-théorie*. Astérisque 149 (1987).
[16] B. Keller, *On differential graded categories*, International Congress of Mathematicians (Madrid), Vol. II, 151–190. Eur. Math. Soc., Zürich (2006).
[17] M. Kontsevich, *Non-commutative motives*. Talk at the Institute for Advanced Study on the occasion of the 61st birthday of Pierre Deligne, October 2005. Video available at http://video.ias.edu/Geometry-and-Arithmetic.
[18] ———, *Triangulated categories and geometry*. Course at the École Normale Supérieure, Paris, 1998. Notes available at www.math.uchicago.edu/mitya/langlands.html
[19] ———, *Notes on motives in finite characteristic*. Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, 213–247, Progr. Math., 270, Birkhäuser Boston, MA, 2009.
[20] J.-L. Loday, *Cyclic homology*. Grundlehren der Mathematischen Wissenschaften 301 (1992). Springer-Verlag, Berlin.
[21] G. Maltsiniotis, *Introduction à la théorie des dérivateurs (d'après Grothendieck)*. Available at www.math.jussieu.fr/maltsin/textes.html.

[22] D. Quillen, *Homotopical algebra.* Lecture notes in Mathematics 43, Springer-Verlag (1967).

[23] ———, *Higher algebraic K-theory I.* Lecture notes in Mathematics 341 (1973), 85–147.

[24] G. Tabuada, *Higher K-theory via universal invariants.* Duke Math. J. 145 (2008), 121–206.

[25] ———, *Additive invariants of dg categories.* Int. Math. Res. Not. 53 (2005), 3309–3339.

[26] ———, *A universal characterization of the Chern character maps.* Proc. Amer. Math. Soc. 139 (2011), 1263–1271.

[27] ———, *Products, multiplicative Chern characters, and finite coefficients via Non-commutative motives.* Available at arXiv:1101.0731v2.

[28] ———, *Bivariant cyclic cohomology and Connes' bilinear pairings in Non-commutative motives.* Available at arXiv:1005.2336v2.

Gonçalo Tabuada
Department of Mathematics
MIT
Cambridge, MA 02139
USA
tabuada@math.mit.edu

Dep. de Matemática
FCT-UNL
Quinta da Torre, 2829-516
Caparica
Portugal
tabuada@fct.unl.pt