Jumps and Monodromy of Abelian Varieties

Lars Halvard Halle and Johannes Nicaise[1]

Received: October 10, 2010
Revised: August 19, 2011

Communicated by Peter Schneider

Abstract. We prove a strong form of the motivic monodromy conjecture for abelian varieties, by showing that the order of the unique pole of the motivic zeta function is equal to the maximal rank of a Jordan block of the corresponding monodromy eigenvalue. Moreover, we give a Hodge-theoretic interpretation of the fundamental invariants appearing in the proof.

2010 Mathematics Subject Classification: MSC2000: 11G10, 14D05, 14D07

1. Introduction

Let K be a henselian discretely valued field with algebraically closed residue field k, and let A be a tamely ramified abelian K-variety of dimension g. In [14], we introduced the *motivic zeta function* $Z_A(T)$ of A. It is a formal power series over the localized Grothendieck ring of k-varieties $\mathcal{M}_k$, and it measures the behaviour of the Néron model of A under tame base change. We showed that $Z_A(\mathbb{L}^{-s})$ has a unique pole, which coincides with Chai's *base change conductor* $c(A)$ of A, and that the order of this pole equals $1+t_{\mathrm{pot}}(A)$, where $t_{\mathrm{pot}}(A)$ denotes the *potential toric rank* of A. Moreover, we proved that for every embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$, the value $\exp(2\pi c(A)i)$ is an eigenvalue of the tame monodromy transformation on the ℓ-adic cohomology of A in degree g. The main ingredient of the proof is Edixhoven's filtration on the special fiber of the Néron model of A [12].

As we've explained in [14], this result is a global version of Denef and Loeser's motivic monodromy conjecture for hypersurface singularities in characteristic zero. Denef and Loeser's conjecture relates the poles of the motivic zeta function of the singularity to monodromy eigenvalues on the nearby

[1]The first author was partially supported by the Fund for Scientific Research-Flanders (G.0318.06) and by DFG under grant Hu 337/6-1. The second author was partially supported by the Fund for Scientific Research - Flanders (G.0415.10) and ANR-06-BLAN-0183.

cohomology. It is a motivic generalization of a conjecture of Igusa's for the p-adic zeta function, which relates certain arithmetic properties of polynomials f in $\mathbb{Z}[x_1, \ldots, x_n]$ (namely, the asymptotic behaviour of the number of solutions of the congruence $f \equiv 0$ modulo powers of a prime p) to the structure of the singularities of the complex hypersurface defined by the equation $f = 0$. The conjectures of Igusa and Denef-Loeser have been solved, for instance, in the case $n = 2$ [16][23], but the general case remains wide open. We refer to [20] for a survey.

There also exists a stronger form of Igusa's conjecture, which says that the real parts of the poles of the p-adic zeta function of f are roots of the Bernstein polynomial $b_f(s)$ of f, and that the order of each pole is at most the multiplicity of the corresponding root of $b_f(s)$. This stronger conjecture also has a motivic generalization, replacing the p-adic zeta function by the motivic zeta function, and taking for f any polynomial with coefficients in a field of characteristic zero (or, more generally, any regular function on a smooth algebraic variety over a field of characteristic zero).

It is well-known that, for every complex polynomial f and every root α of $b_f(s)$, the value $\alpha' = \exp(2\pi i\alpha)$ is a monodromy eigenvalue on the nearby cohomology $R\psi_f(\mathbb{C})_x$ of f at some closed point x of the zero locus H_f of f [15][17]. Moreover, if H_f has an isolated singularity at x, then the multiplicity m_α of α as a root of the local Bernstein polynomial $b_{f,x}(s)$ of f at x is closely related to the maximal size $m_{\alpha'}$ of the Jordan blocks with eigenvalue α' of the monodromy transformation on $R^{n-1}\psi_f(\mathbb{C})_x$. In particular, $m_\alpha \leq m_{\alpha'}$ if $\alpha \notin \mathbb{Z}$ [17, 7.1].

The aim of the present paper is twofold. First, we prove a strong form of the motivic monodromy conjecture for abelian varieties. There is no good notion of Bernstein polynomial in our setting, but we can look at the size of the Jordan blocks. We show that the order $1 + t_{\mathrm{pot}}(A)$ of the unique pole $s = c(A)$ of the motivic zeta function $Z_A(\mathbb{L}^{-s})$ is equal to the maximal rank of a Jordan block of the corresponding monodromy eigenvalue on the degree g cohomology of A (Theorem 5.9). Next, we use the theory of Néron models of variations of Hodge structures to give a Hodge-theoretic interpretation of the jumps in Edixhoven's filtration. This is done in Theorems 6.2 and 6.3. In [14, 2.7], we speculated on a generalization of the monodromy conjecture to Calabi-Yau varieties over $\mathbb{C}((t))$ (i.e., smooth, proper, geometrically connected $\mathbb{C}((t))$-varieties with trivial canonical sheaf); we hope that the translation of Edixhoven's invariants to Hodge theory will help to extend the proof of the monodromy conjecture to that setting.

In order to obtain these results, we divide Edixhoven's jumps into three types: toric, abelian, and dual abelian. The basic properties of these types are discussed in Section 3. Not all of these results are used in the proofs of the main results of the paper. We include them for the sake of completeness and because we believe that they are of independent interest. The reader who is only interested in Theorems 4.4, 5.9, 6.2 and 6.3 may skip Lemma 3.2, Proposition 3.4 and all the results in Section 3 after Proposition 3.5.

The different types of jumps are related to the monodromy transformation on the Tate module of A in Section 4. Toric jumps correspond to monodromy eigenvalues with Jordan block of size two, and the abelian and dual abelian jumps to monodromy eigenvalues with Jordan block of size one (Theorem 4.4). If $K = \mathbb{C}((t))$, then the abelian and dual abelian jumps can be distinguished by looking at the Hodge type in the limit mixed Hodge structure (Theorem 6.3).

Acknowledgements. We are grateful to the referee for carefully reading the manuscript and suggesting several improvements to the paper. The second author is indebted to Antoine Chambert-Loir for helpful suggestions concerning Section 5.1.

2. Preliminaries and notation

We denote by R a Henselian discrete valuation ring, with quotient field K and algebraically closed residue field k. We denote by K^a an algebraic closure of K, by K^s the separable closure of K in K^a, and by K^t the tame closure of K in K^s. We fix a topological generator σ of the tame monodromy group $G(K^t/K)$. We denote by p the characteristic exponent of k, and by $\mathbb{N}'$ the set of integers $d > 0$ prime to p. We denote by

$$(\cdot)_s : (R - \text{Schemes}) \to (k - \text{Schemes}) : \mathcal{X} \mapsto \mathcal{X}_s = \mathcal{X} \times_R k$$

the special fiber functor.
For every abelian variety B over a field F, we denote its dual abelian variety by $B^\vee$. For every abelian K-variety A with Néron model $\mathcal{A}$, we denote by $t(A)$, $u(A)$ and $a(A)$ the reductive, resp. unipotent, resp. abelian rank of $\mathcal{A}_s^o$. We call these values the *toric*, resp. *unipotent*, resp. *abelian* rank of A. Obviously, their sum equals the dimension of A.
By Grothendieck's semi-stable reduction theorem, there exists a finite extension K' of K in K^s such that $A \times_K K'$ has semi-abelian reduction [2, IX.3.6]. This means that the identity component of the special fiber of its Néron model is a semi-abelian k-variety; equivalently, $u(A \times_K K') = 0$. The value $t_{\mathrm{pot}}(A) = t(A \times_K K')$ is called the *potential toric rank* of A, and the value $a_{\mathrm{pot}}(A) = a(A \times_K K')$ the *potential abelian rank*. It follows from [2, IX.3.9] that these values are independent of the choice of K'. We say that A is *tamely ramified* if we can take for K' a *tame* finite extension of K (since k is algebraically closed, this means that the degree $[K' : K]$ is prime to p) .
For every scheme S, every S-group scheme G and every integer $n > 0$, we denote by $n_G : G \to G$ the multiplication by n, and by ${}_nG$ its kernel.
If $\mathcal{S}$ is a set, and $g : \mathcal{S} \to \mathbb{R}$ a function with finite support, we set

$$\|g\| = \sum_{s \in \mathcal{S}} g(s).$$

We denote the support of g by $\mathrm{Supp}(g)$.

Definition 2.1. *For every function*

$$f : \mathbb{Q}/\mathbb{Z} \to \mathbb{R}$$

we define its reflection

$$f^* : \mathbb{Q}/\mathbb{Z} \to \mathbb{R}$$

by

$$f^*(x) = f(-x).$$

We call f complete *if for every $x \in \mathbb{Q}/\mathbb{Z}$, the value $f(x)$ only depends on the order of x in the group $\mathbb{Q}/\mathbb{Z}$. We say that f is* semi-complete *if $f + f^*$ is complete.*

Consider a function

$$f : \mathbb{Q}/\mathbb{Z} \to \mathbb{N}$$

and assume that there exists an element e of $\mathbb{Z}_{>0}$ such that $\mathrm{Supp}(f)$ is contained in $((1/e)\mathbb{Z})/\mathbb{Z}$. Let F be any algebraically closed field such that e is prime to the characteristic exponent p' of F. For each generator ζ of $\mu_e(F)$, we put

$$P_{f,\zeta}(t) = \prod_{i \in ((1/e)\mathbb{Z})/\mathbb{Z}} (t - \zeta^{i \cdot e})^{f(i)}$$

in $F[t]$. For each integer $d > 0$, we denote by $\Phi_d(t)$ the cyclotomic polynomial whose roots are the primitive d-th roots of unity. We say that $\Phi_d(t)$ is F-tame if d is prime to p'.

Lemma 2.2. *The function f is complete if and only if for some generator ζ of $\mu_e(F)$, the polynomial $P_{f,\zeta}(t)$ is the image of a product $Q_f(t)$ of F-tame cyclotomic polynomials under the unique ring morphism*

$$\rho : \mathbb{Z}[t] \to F[t]$$

mapping t to t. If f is complete, then $P_{f,\zeta}(t)$ is independent of ζ and e, and $Q_f(t)$ is unique. In that case, if we choose a primitive e-th root of unity ξ in an algebraic closure $\mathbb{Q}^a$ of $\mathbb{Q}$, then

$$Q_f(t) = \prod_{i \in ((1/e)\mathbb{Z})/\mathbb{Z}} (t - \xi^{i \cdot e})^{f(i)}.$$

Proof. First, assume that f is complete, and put

$$Q_f(t) = \prod_{i \in ((1/e)\mathbb{Z})/\mathbb{Z}} (t - \xi^{i \cdot e})^{f(i)}$$

for some primitive e-th root of unity ξ in $\mathbb{Q}^a$. Then $Q_f(t)$ is a product of F-tame cyclotomic polynomials, because f is complete and e is prime to p'. There is a unique ring morphism

$$\widetilde{\rho} : \mathbb{Z}[\xi][t] \to F[t]$$

that maps ξ to ζ and t to t. We clearly have $\widetilde{\rho}(Q_f(t)) = P_{f,\zeta}(T)$. Since $Q_f(t)$ belongs to $\mathbb{Z}[t]$, it follows that $\rho(Q_f(t)) = P_{f,\zeta}(T)$ so that $P_{f,\zeta}(t)$ does not depend on ζ. Uniqueness of $Q_f(t)$ follows from [14, 5.10].

Conversely, if $P_{f,\zeta}(t)$ is the image under ρ of a product $Q(t)$ of F-tame cyclotomic polynomials, then it is easily seen that f is complete. $\square$

3. Toric and abelian multiplicity

3.1. Galois action on Néron models. Let A be a tamely ramified abelian K-variety of dimension g, and let K' be a finite extension of K in K^t such that $A' = A \times_K K'$ has semi-abelian reduction. We denote by R' the integral closure of R in K', and by $\mathfrak{m}'$ the maximal ideal of R'. We put $d = [K' : K]$.
We denote by μ the Galois group $G(K'/K)$, and we let μ act on K' from the left. The action of $\zeta \in \mu$ on $\mathfrak{m}'/(\mathfrak{m}')^2$ is multiplication by $\iota(\zeta)$, for some element $\iota(\zeta)$ in the group $\mu_d(k)$ of d-th roots of unity in k, and the map

$$\mu \to \mu_d(k) : \zeta \mapsto \iota(\zeta)$$

is an isomorphism.
We denote by $\mathcal{A}$ and $\mathcal{A}'$ the Néron models of A, resp. A'. By the universal property of the Néron model, there exists a unique morphism of R'-group schemes

$$h : \mathcal{A} \times_R R' \to \mathcal{A}'$$

that extends the canonical isomorphism between the generic fibers. It induces an injective morphism of free rank g R'-modules

$$\mathrm{Lie}(h) : \mathrm{Lie}(\mathcal{A} \times_R R') \to \mathrm{Lie}(\mathcal{A}').$$

Definition 3.1 (Chai [5]). *The* base change conductor $c(A)$ *of A is* $[K' : K]^{-1}$ *times the length of the cokernel of* $\mathrm{Lie}(h)$.

The definition does not require that A is tamely ramified. The base change conductor is a positive rational number, independent of the choice of K'. It vanishes if and only if A has semi-abelian reduction.
The right μ-action on A' extends uniquely to a right μ-action on $\mathcal{A}'$ such that the structural morphism

$$\mathcal{A}' \to \mathrm{Spec}\, R'$$

is μ-equivariant. We denote by

$$0 \to T \to (\mathcal{A}'_s)^o \to B \to 0 \tag{3.1}$$

the Chevalley decomposition of $(\mathcal{A}'_s)^o$, with T a k-torus and B an abelian k-variety. There exist unique right μ-actions on T, resp. B, such that (3.1) is μ-equivariant. The right μ-action on B induces a left μ-action on the dual abelian variety $B^\vee$.

Lemma 3.2. *(1) The complex*

$$0 \to T^\mu \to ((\mathcal{A}'_s)^o)^\mu \to B^\mu, \tag{3.2}$$

obtained from (3.1) by taking μ-invariants, is an exact complex of smooth group schemes over k.
Taking identity components, we get a complex

$$0 \to (T^\mu)^o \to ((\mathcal{A}'_s)^\mu)^o \to (B^\mu)^o \to 0 \tag{3.3}$$

of smooth group schemes over k that is exact at the left and at the right. The quotient

$$B' = ((\mathcal{A}'_s)^\mu)^o/(T^\mu)^o$$

is an abelian k-variety, and the natural morphism $f : B' \to (B^\mu)^o$ is a separable isogeny.
(2) If we denote by h the unique morphism

$$h : \mathcal{A} \times_R R' \to \mathcal{A}'$$

extending the natural isomorphism between the generic fibers, then the k-morphism $h_s : \mathcal{A}_s \to \mathcal{A}'_s$ factors through a morphism

$$g : \mathcal{A}_s \to (\mathcal{A}'_s)^\mu.$$

The morphism g is smooth and surjective, and its kernel is a connected unipotent smooth algebraic k-group. The identity component $((\mathcal{A}'_s)^\mu)^o$ is semi-abelian.

Proof. (1) It follows from [12, 3.4] that the group schemes in (3.2) are smooth over k. Exactness of (3.2) is clear. The morphism

$$\alpha : (\mathcal{A}'_s)^o \to B$$

is smooth, since T is smooth over k [1, $\mathrm{VI_B}$.9.2]. It follows from [12, 3.5] that

$$\alpha^\mu : ((\mathcal{A}'_s)^o)^\mu \to B^\mu$$

is smooth, as well.
Taking identity components in (3.2), we get a complex

$$(T^\mu)^o \xrightarrow{\ \beta\ } ((\mathcal{A}'_s)^\mu)^o \xrightarrow{\gamma=(\alpha^\mu)^o} (B^\mu)^o$$

of smooth group schemes over k. It is obvious that β is a monomorphism, and thus a closed immersion [1, VI_B.1.4.2]. Surjectivity of γ follows from [1, $\mathrm{VI_B}$.3.11], since γ is smooth, and thus flat. We put

$$B' = ((\mathcal{A}'_s)^\mu)^o/(T^\mu)^o$$

This is a connected smooth algebraic k-group, by [1, $\mathrm{VI_A}$.3.2 and $\mathrm{VI_B}$.9.2]. Consider the natural morphism $f : B' \to (B^\mu)^o$. It is surjective, because γ is surjective. The dimension of B' is equal to

$$\dim (\mathcal{A}'_s)^\mu - \dim T^\mu,$$

which is at most $\dim B^\mu$ by exactness of (3.2). Surjectivity of f then implies that B' and B^μ must have the same dimension, so that f has finite kernel. Thus f is an isogeny and B' is an abelian variety. The kernel of f is canonically isomorphic to

$$\ker(\gamma)/(T^\mu)^o.$$

Since γ is smooth, we know that $\ker(\gamma)$ is smooth over k, so that $\ker(f)$ is smooth over k, by [1, $\mathrm{VI_B}$.9.2]. Hence, f is a separable isogeny.
(2) Since h is μ-equivariant, and μ acts trivially on the special fiber $\mathcal{A}_s$ of $\mathcal{A} \times_R R'$, the morphism h_s factors through a morphism $g : \mathcal{A}_s \to (\mathcal{A}'_s)^\mu$. By [12, 5.3], the morphism g is smooth and surjective, and its kernel is a connected

unipotent smooth algebraic k-group. By [12, 3.4], $((\mathcal{A}'_s)^\mu)^o$ is a connected smooth closed k-subgroup scheme of the semi-abelian k-group scheme $(\mathcal{A}'_s)^o$, so that $((\mathcal{A}'_s)^\mu)^o$ is semi-abelian by [14, 5.2]. $\square$

3.2. Multiplicity functions. Fix an element $e \in \mathbb{N}'$. For every finite dimensional k-vector space V with a right $\mu_e(k)$-action

$$*: V \times \mu_e(k) \to V : (v, \zeta) \mapsto v * \zeta$$

and for every integer i in $\{0, \ldots, e-1\}$, we denote by $V[i]$ the maximal subspace of V such that

$$v * \zeta = \zeta^i \cdot v$$

for all $\zeta \in \mu_e(k)$ and all $v \in V[i]$. We define the *multiplicity function*

$$m_{V,\mu_e(k)} : \mathbb{Q}/\mathbb{Z} \to \mathbb{N}$$

by

$$\begin{cases} m_{V,\mu_e(k)}(i/e) = \dim(V[i]) & \text{for } i \in \{0, \ldots, e-1\} \\ m_{V,\mu_e(k)}(x) = 0 & \text{if } x \notin ((1/e)\mathbb{Z})/\mathbb{Z} \end{cases}$$

Note that $m_{V,\mu_e(k)}$ determines the $k[\mu_e(k)]$-module V up to isomorphism, since the order of $\mu_e(k)$ is invertible in k.

In an analogous way, we define the multiplicity function $m_{\mu_e(k),W}$ for a finite dimensional k-vector space W with left $\mu_e(k)$-action. The inverse of the right $\mu_e(k)$-action on V is the left action

$$\mu_e(k) \times V \to V : (\zeta, v) \mapsto v * \zeta^{-1}.$$

Its multiplicity function $m_{\mu_e(k),V}$ is equal to the reflection $(m_{V,\mu_e(k)})^*$ of the multiplicity function $m_{V,\mu_e(k)}$.

Let A be a tamely ramified abelian K-variety. We adopt the notations of Section 3.1. In the set-up of (3.1), the group $\mu \cong \mu_d(k)$ acts on the k-vector spaces $\mathrm{Lie}(T)$, $\mathrm{Lie}(\mathcal{A}'_s)$ and $\mathrm{Lie}(B)$ from the right, and on $\mathrm{Lie}(B^\vee)$ from the left (via the dual action of μ on $B^\vee$). Hence, we can state the following definitions.

Definition 3.3. *We define the toric multiplicity function m_A^{tor} of A by*

$$m_A^{\mathrm{tor}} = m_{\mathrm{Lie}(T),\mu}.$$

We define the abelian multiplicity function m_A^{ab} of A by

$$m_A^{\mathrm{ab}} = m_{\mathrm{Lie}(B),\mu}.$$

We define the dual abelian multiplicity function $\breve{m}_A^{\mathrm{ab}}$ of A by

$$\breve{m}_A^{\mathrm{ab}} = m_{\mu,\mathrm{Lie}(B^\vee)}.$$

Finally, we define the multiplicity function m_A of A by

$$m_A = m_A^{\mathrm{tor}} + m_A^{\mathrm{ab}} = m_{\mathrm{Lie}(\mathcal{A}'_s),\mu}.$$

Using [2, IX.3.9], it is easily checked that these definitions only depend on A, and not on the choice of K'.

Proposition 3.4. *For every tamely ramified abelian K-variety A, we have*

$$\breve{m}_A^{\mathrm{ab}} = (m_{A^\vee}^{\mathrm{ab}})^*.$$

Proof. We adopt the notations of Section 3.1. We set $(A^\vee)' = A^\vee \times_K K'$ and we denote its Néron model by $(\mathcal{A}^\vee)'$. The identity component of $(\mathcal{A}^\vee)'_s$ is a semi-abelian k-variety [2, IX.2.2.7]. We denote by C its abelian part.

As explained in Section 3.1, the left Galois action of μ on K' induces a right action of μ on C. The canonical divisorial correspondence on $A\times_K A^\vee$ induces a divisorial correspondence on $B\times_k C$ that identifies C with the dual abelian variety of B [2, IX.5.4]. It suffices to show that the right μ-action on C is the inverse of the dual of the right μ-action on B. To this end, we take a closer look at the construction of the divisorial correspondence on $B\times_k C$. Here we need the language of biextensions [2, VII and VIII]. We note that the following proof does not use the assumption that A is tamely ramified and that K' is a tame extension of K.

The canonical divisorial correspondence on $A\times_K A^\vee$ can be interpreted as a *Poincaré biextension* $\mathscr{P}$ of $(A, A^\vee)$ by $\mathbb{G}_{m,K}$ [2, VII.2.9.5], which is defined up to isomorphism. It induces a biextension $\mathscr{P}'$ of $(A', (A^\vee)')$ by $\mathbb{G}_{m,K'}$ by base change. By [2, VIII.7.1], the biextension $\mathscr{P}'$ extends uniquely to a biextension of $((\mathcal{A}')^o, ((\mathcal{A}^\vee)')^o)$ by $\mathbb{G}_{m,R'}$, which restricts to a biextension $\mathscr{P}'_s$ of $((\mathcal{A}')^o_s, ((\mathcal{A}^\vee)')^o_s)$ by $\mathbb{G}_{m,k}$. By [2, VIII.4.8], $\mathscr{P}'_s$ induces a biextension $\mathscr{Q}$ of (B, C) by $\mathbb{G}_{m,k}$ that is characterized (up to isomorphism) by the fact that its pullback to $((\mathcal{A}')^o_s, ((\mathcal{A}^\vee)')^o_s)$ is isomorphic to $\mathscr{P}'_s$. The theorem in [2, IX.5.4] asserts that $\mathscr{Q}$ is a Poincaré biextension.

For every element ζ of μ, we denote by r_ζ the right multiplication by ζ on B and C. Since $\mathscr{P}'$ is obtained from the biextension $\mathscr{P}$ over K by base change to K', it follows easily from the construction that the pullback of the biextension $\mathscr{Q}$ through the k-morphisms

$$\begin{aligned} r_\zeta &: B\to B \\ r_\zeta &: C\to C \end{aligned}$$

is isomorphic to $\mathscr{Q}$. Interpreting $\mathscr{Q}$ as an isomorphism

$$i: B\to C^\vee$$

in the way of [2, VIII.3.2.2], this means that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\ i\ } & C^\vee \\ {\scriptstyle r_\zeta}\downarrow & & \uparrow{\scriptstyle (r_\zeta)^\vee} \\ B & \xrightarrow{\ i\ } & C^\vee \end{array}$$

commutes, which is what we wanted to show. $\square$

In the following proposition, we see how the multiplicity functions of a tamely ramified abelian K-variety A are related to Edixhoven's jumps and Chai's elementary divisors of A. These jumps and elementary divisors are rational numbers in $[0,1[$ that measure the behaviour of the Néron model of A under tame ramification of the base field K. For the definition of Edixhoven's jumps, we refer to [12, 5.4.5]. The terminology we use is the one from [14, 4.12]; in particular, the multiplicity of a jump is defined there. For Chai's elementary divisors, we refer to [5, 2.4]. By definition, the base change conductor $c(A)$ of A is equal to the sum of the elementary divisors.

Proposition 3.5. *Let A be a tamely ramified abelian K-variety. The functions m_A, m_A^{tor}, m_A^{ab} and $\breve{m}_A^{\mathrm{ab}}$ are supported on*

$$((1/e)\mathbb{Z})/\mathbb{Z},$$

with e the degree of the minimal extension of K where A acquires semi-abelian reduction.
If we identify $[0,1[\cap\mathbb{Q}$ with $\mathbb{Q}/\mathbb{Z}$ via the bijection

$$[0,1[\cap\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}:x\mapsto x \bmod \mathbb{Z}$$

then for every $x\in[0,1[\cap\mathbb{Q}$, the value $m_A(x)$ is equal to the multiplicity of x as a jump in Edixhoven's filtration for A. In particular, the support of m_A is the set of jumps in Edixhoven's filtration. The value $m_A(x)$ is also equal to the number of Chai's elementary divisors of A that are equal to x, and the base change conductor $c(A)$ of A is given by

$$c(A)=\sum_{x\in[0,1[\cap\mathbb{Q}}(m_A(x)\cdot x).$$

Proof. See [12, 5.3 and 5.4.5] and [14, 4.8 and 4.13 and 4.18]. $\square$

Proposition 3.6. *We have the following equalities:*

$$\begin{array}{lllllll}\|m_A\| & = & \dim(A), & \|m_A^{\mathrm{ab}}\| & = & \|\breve{m}_A^{\mathrm{ab}}\| & = & a_{\mathrm{pot}}(A),\\ \|m_A^{\mathrm{tor}}\| & = & t_{\mathrm{pot}}(A), & m_A^{\mathrm{ab}}(0) & = & \breve{m}_A^{\mathrm{ab}}(0) & = & a(A),\\ m_A^{\mathrm{tor}}(0) & = & t(A). & & & & & \end{array}$$

Moreover, we have

$$\sum_{x\in(\mathbb{Q}/\mathbb{Z})\setminus\{0\}}m_A(x)=u(A).$$

Proof. We adopt the notations of Section 3.1. It follows immediately from the definitions that

$$\begin{array}{rcccccc} & & \|m_A\| & = & \dim(\mathrm{Lie}(\mathcal{A}'_s)) & = & \dim(A),\\ & & \|m_A^{\mathrm{tor}}\| & = & \dim(\mathrm{Lie}(T)) & = & t_{\mathrm{pot}}(A),\\ \|\breve{m}_A^{\mathrm{ab}}\| & = & \|m_A^{\mathrm{ab}}\| & = & \dim(\mathrm{Lie}(B)) & = & a_{\mathrm{pot}}(A).\end{array}$$

By Lemma 3.2, the abelian, resp., reductive rank of $\mathcal{A}_s^o$ is equal to the abelian, resp., reductive rank of the semi-abelian k-variety $((\mathcal{A}'_s)^{\mu})^o$. In the notation of Lemma 3.2, the Chevalley decomposition of $((\mathcal{A}'_s)^{\mu})^o$ is given by

$$0\to(T^{\mu})^o\to((\mathcal{A}'_s)^{\mu})^o\to B'\to 0$$

and there exists a separable isogeny $f : B' \to (B^{\mu})^o$. By [12, 3.2], the natural morphisms

$$\begin{array}{c} \mathrm{Lie}(T^{\mu}) \to \mathrm{Lie}(T)^{\mu} = \mathrm{Lie}(T)[0] \\ \mathrm{Lie}(B^{\mu}) \to \mathrm{Lie}(B)^{\mu} = \mathrm{Lie}(B)[0] \end{array}$$

are isomorphisms. Since $\mathrm{Lie}(f)$ is also an isomorphism, we find

$$\begin{aligned} m_A^{\mathrm{tor}}(0) &= t(A), \\ m_A^{\mathrm{ab}}(0) &= a(A). \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{x \in (\mathbb{Q}/\mathbb{Z}) \setminus \{0\}} m_A(x) &= \|m_A\| - m_A^{\mathrm{tor}}(0) - m_A^{\mathrm{ab}}(0) \\ &= \dim(A) - t(A) - a(A) \\ &= u(A). \end{aligned}$$

By Proposition 3.4, we have

$$\breve{m}_A^{\mathrm{ab}}(0) = m_{A^{\vee}}^{\mathrm{ab}}(0)$$

and we've just seen that this value is equal to the abelian rank $a(A^{\vee})$ of $A^{\vee}$. But the abelian ranks of A and $A^{\vee}$ coincide, by [2, 2.2.7], so that

$$\breve{m}_A^{\mathrm{ab}}(0) = a(A).$$

□

Lemma 3.7. *If A_1 and A_2 are tamely ramified abelian K-varieties, then*

$$\begin{aligned} m_{A_1 \times_K A_2}^{\mathrm{tor}} &= m_{A_1}^{\mathrm{tor}} + m_{A_2}^{\mathrm{tor}}, \\ m_{A_1 \times_K A_2}^{\mathrm{ab}} &= m_{A_1}^{\mathrm{ab}} + m_{A_2}^{\mathrm{ab}}, \\ \breve{m}_{A_1 \times_K A_2}^{\mathrm{ab}} &= \breve{m}_{A_1}^{\mathrm{ab}} + \breve{m}_{A_2}^{\mathrm{ab}}. \end{aligned}$$

Proof. If we denote by $\mathcal{A}_1$ and $\mathcal{A}_2$ the Néron models of A_1, resp. A_2, then it follows immediately from the universal property of the Néron model that $\mathcal{A}_1 \times_R \mathcal{A}_2$ is a Néron model for $A_1 \times_K A_2$. Since the Chevalley decomposition of a connected smooth algebraic k-group commutes with finite fibered products over k and $\mathrm{Lie}(G_1 \times_k G_2)$ is canonically isomorphic to $\mathrm{Lie}(G_1) \oplus \mathrm{Lie}(G_2)$ for any pair of algebraic k-groups G_1, G_2, the result follows. □

Proposition 3.8. *Let A be a tamely ramified abelian K-variety. Let L be a finite tame extension of K of degree e, and put $A_L = A \times_K L$. Then for each $x \in \mathbb{Q}/\mathbb{Z}$, we have*

$$\begin{aligned} m_{A_L}^{\mathrm{tor}}(x) &= \sum_{y \in \mathbb{Q}/\mathbb{Z},\, e \cdot y = x} m_A^{\mathrm{tor}}(y) \\ m_{A_L}^{\mathrm{ab}}(x) &= \sum_{y \in \mathbb{Q}/\mathbb{Z},\, e \cdot y = x} m_A^{\mathrm{ab}}(y) \\ \breve{m}_{A_L}^{\mathrm{ab}}(x) &= \sum_{y \in \mathbb{Q}/\mathbb{Z},\, e \cdot y = x} \breve{m}_A^{\mathrm{ab}}(y) \end{aligned}$$

Proof. We adopt the notations of Section 3.1. Since the multiplicity functions do not depend on the choice of the field K' where A acquires semi-abelian reduction, we may assume that L is contained in K'. If ζ is a generator of $\mu = G(K'/K)$, then the Galois group $G(K'/L)$ is generated by ζ^e. Now the result easily follows from the definition of the multiplicity functions. □

Proposition 3.9. *If $f : A_1 \to A_2$ is an isogeny of tamely ramified abelian K-varieties and if the degree* $\deg(f)$ *of f is prime to p, then*

$$m^{\mathrm{ab}}_{A_1} = m^{\mathrm{ab}}_{A_2} \quad \textit{and} \quad \breve{m}^{\mathrm{ab}}_{A_1} = \breve{m}^{\mathrm{ab}}_{A_2}.$$

Proof. We put $n = \deg(f)$. Since n is invertible in K, the morphism f is separable, so that $\ker(f)$ is étale over k. Thus $\ker(f)$ is a finite étale K-group scheme of rank n. Every finite group scheme over a field is killed by its rank. (See [1, VII$_A$.8.5]; in our case, the result is elementary, because $\ker(f)$ is étale, so that we can reduce to the case of a constant group by base change to an algebraic closure of K. As an aside, we recall that Deligne has shown that every *commutative* finite group scheme over an arbitrary base scheme is killed by its rank [28, p.4].) It follows that $\ker(f)$ is contained in ${}_n(A_1)$. Hence, there exists an isogeny $g : A_2 \to A_1$ such that $g \circ f = n_{A_1}$.
Let K' be a tame finite extension of K such that A_1 and A_2 acquire semi-abelian reduction over K', and denote by R' the integral closure of R in K'. We denote the Néron model of $(A_i) \times_K K'$ by $\mathcal{A}'_i$, for $i = 1, 2$. The morphisms $f \times_K K'$ and $g \times_K K'$ extend uniquely to morphisms of R'-group schemes

$$f' : \mathcal{A}'_1 \to \mathcal{A}'_2$$
$$g' : \mathcal{A}'_2 \to \mathcal{A}'_1.$$

For $i = 1, 2$, we denote by B_i the abelian part of the semi-abelian k-variety $(\mathcal{A}'_i)^o_s$. By functoriality of the Chevalley decomposition, f'_s induces a morphism of k-group schemes $f'_B : B_1 \to B_2$. Likewise, g'_s induces a morphism of k-group schemes $g'_B : B_2 \to B_1$. Since $g' \circ f'$ is multiplication by n, the same holds for $g'_B \circ f'_B$. In particular, f'_B is an isogeny of degree prime to p. Thus f'_B is a μ-equivariant separable isogeny, so that $\mathrm{Lie}(f'_B) : \mathrm{Lie}(B_1) \to \mathrm{Lie}(B_2)$ is a μ-equivariant isomorphism, and $m^{\mathrm{ab}}_{A_1} = m^{\mathrm{ab}}_{A_2}$.
By [19, p. 143], the dual morphism $(f'_B)^\vee$ is again an isogeny, and its kernel is the Cartier dual of the kernel of f'_B. In particular, f'_B and $(f'_B)^\vee$ have the same degree, so that $(f'_B)^\vee$ is separable. Since it is also equivariant for the left μ-action on $B^\vee$, we find that $\breve{m}^{\mathrm{ab}}_{A_1} = \breve{m}^{\mathrm{ab}}_{A_2}$. □

Remark 3.10. The same proof shows that m^{tor}_A is invariant under isogenies of degree prime to p. We'll see in Corollary 4.5 that, more generally, the functions m^{tor}_A and $m^{\mathrm{ab}}_A + \breve{m}^{\mathrm{ab}}_A$ are invariant under *all* isogenies.

Corollary 3.11. *Let A be a tamely ramified abelian K-variety. If k has characteristic zero, or A is principally polarized, then*

$$m^{\mathrm{ab}}_A = m^{\mathrm{ab}}_{A^\vee}$$

and

$$\breve{m}_A^{\mathrm{ab}} = (m_A^{\mathrm{ab}})^*.$$

Proof. The first equality follows from Proposition 3.9. Together with Proposition 3.4, it implies the second equality. □

We will see in Theorem 6.3 that, when R is the ring of germs of holomorphic functions at the origin of $\mathbb{C}$, the equality

$$\breve{m}_A^{\mathrm{ab}} = (m_A^{\mathrm{ab}})^*$$

expresses that the monodromy eigenvalues on the $(-1,0)$-component of a certain limit mixed Hodge structure associated to A are the complex conjugates of the monodromy eigenvalues on the $(0,-1)$-component. Corollary 3.11 generalizes this Hodge symmetry.

Question 3.12. Is it true that

$$\breve{m}_A^{\mathrm{ab}} = (m_A^{\mathrm{ab}})^*$$

for every tamely ramified abelian K-variety A?

4. Jumps and monodromy

Proposition 4.1. *Let B be an abelian k-variety, and let T be an algebraic k-torus. Fix an element $e \in \mathbb{N}'$, and assume that $\mu_e(k)$ acts on B, resp. T from the right. We consider the dual left action of $\mu_e(k)$ on $B^\vee$. The functions $m_1 := m_{\mathrm{Lie}(T),\mu_e(k)}$ and*

$$m_2 := m_{\mathrm{Lie}(B),\mu_e(k)} + m_{\mu_e(k),\mathrm{Lie}(B^\vee)}$$

are complete.
Moreover, for every prime ℓ invertible in k and for each generator ζ of $\mu_e(k)$, the characteristic polynomial $P_1(t)$ of ζ on the ℓ-adic Tate module

$$\mathscr{V}_\ell T \cong \mathscr{T}_\ell T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is equal to $Q_{m_1}(t)$ (in the notation of Lemma 2.2). Likewise, the characteristic polynomial $P_2(t)$ of ζ on $\mathscr{V}_\ell B$ is equal to $Q_{m_2}(t)$.

Proof. We denote by

$$\rho : \mathbb{Z}[t] \to k[t]$$

the unique ring morphism that maps t to t. It is well-known that the characteristic polynomials $P_1(t)$ and $P_2(t)$ belong to $\mathbb{Z}[t]$. For $P_1(t)$, this follows from the canonical isomorphism

$$\mathscr{V}_\ell T \cong \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Q}_\ell(1)), \tag{4.1}$$

where $X(T)$ denotes the character module of T. For $P_2(t)$, it follows from [19, § 19, Thm.4].
Since e is invertible in k, $P_1(t)$ and $P_2(t)$ are products of k-tame cyclotomic polynomials. Thus, by Lemma 2.2 (and using the notation introduced there), we only have to show the following claims.

Claim 1: The image of $P_1(t)$ under ρ equals $P_{m_1,\zeta}(t)$. Note that, by definition of the function m_1, the polynomial $P_{m_1,\zeta}(t)$ is the characteristic polynomial of the automorphism induced by ζ on $\mathrm{Lie}(T)$. Thus Claim 1 is an immediate consequence of (4.1) and the canonical isomorphism

$$\mathrm{Lie}(T) \cong \mathrm{Hom}_{\mathbb{Z}}(X(T), k).$$

Claim 2: The image of $P_2(t)$ under ρ equals $P_{m_2,\zeta}(t)$. By definition of the function m_2, the polynomial $P_{m_2,\zeta}(t)$ is the product of the characteristic polynomials of the automorphism induced by ζ on $\mathrm{Lie}(B)$ and the dual automorphism on $\mathrm{Lie}(B^\vee)$. By [21, 5.1], the Hodge-de Rham spectral sequence of B degenerates at E_1. This yields a natural short exact sequence

$$0 \to H^0(B, \Omega^1_B) \to H^1_{dR}(B) \to H^1(B, \mathcal{O}_B) \to 0$$

where $H^1_{dR}(B)$ is the degree one de Rham cohomology of B. We have natural isomorphisms

$$\begin{aligned} H^0(B, \Omega^1_B) &\cong \mathrm{Lie}(B)^\vee, \\ H^1(B, \mathcal{O}_B) &\cong \mathrm{Lie}(B^\vee). \end{aligned}$$

(The first isomorphism follows from [4, 4.2.2]; the second isomorphism from [19, § 13, Cor.3]). Thus it suffices to show that the image of $P_2(t)$ under ρ is equal to the characteristic polynomial of ζ on $H^1_{dR}(B)$. As explained in the proof of [14, 5.12], this can be deduced from the fact that étale cohomology is a Weil cohomology, as well as de Rham cohomology (if k has characteristic zero) and crystalline cohomology (if k has characteristic $p > 0$), so that the characteristic polynomials of ζ on the respective cohomology spaces must coincide. $\square$

For every $n \in \mathbb{Z}_{>0}$ and every $a \in \mathbb{C}$, we denote by $\mathrm{Diag}_n(a)$ the rank n diagonal matrix with diagonal $(a, \ldots, a)$, and by $\mathrm{Jord}_n(a)$ the rank n Jordan matrix with diagonal $(a, \ldots, a)$ and subdiagonal $(1, \ldots, 1)$. For any two complex square matrices M and N, of rank m, resp. n, we denote by $M \oplus N$ the rank $m + n$ square matrix

$$M \oplus N = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}.$$

For every integer $q > 0$, we put

$$\oplus^q M = \underbrace{M \oplus \cdots \oplus M}_{q \text{ times}}.$$

Definition 4.2. *For $i = 1, 2$, let*

$$m_i : \mathbb{Q}/\mathbb{Z} \to \mathbb{N}$$

be a function with finite support. The Jordan matrix $\mathrm{Jord}(m_1, m_2)$ *associated to the couple (m_1, m_2) is the complex square matrix of rank $\|m_1\| + 2 \cdot \|m_2\|$*

given by

$$\begin{aligned}\mathrm{Jord}(m_1,m_2) &= \bigoplus_{x\in \mathrm{Supp}(m_1)} \Big(\mathrm{Diag}_{m_1(x)}(\exp(2\pi i x))\Big) \\ &\oplus \bigoplus_{y\in \mathrm{Supp}(m_2)} \Big(\oplus^{m_2(y)}\mathrm{Jord}_2(\exp(2\pi i y))\Big)\end{aligned}$$

where we ordered the set $\mathbb{Q}/\mathbb{Z}$ using the bijection $\mathbb{Q}\cap[0,1[\to \mathbb{Q}/\mathbb{Z}$ and the usual ordering on $[0,1[$.

LEMMA 4.3. *Let V be a finite dimensional vector space over an algebraically closed field F of characteristic zero, and let M be an endomorphism of V. Let $d>0$ be an integer such that M^d is unipotent. Set*

$$W=\{v\in V\,|\,M^d(v)=v\}$$

and assume that M^d acts trivially on V/W and that there exists an M-equivariant isomorphism between $(V/W)^\vee$ and an M-stable subspace W' of W.
Then the endomorphism M of V has the following Jordan form: for every eigenvalue α of M on W' (counted with multiplicities), there is a Jordan block of size two with eigenvalue α, and for every eigenvalue β of M on W/W' (counted with multiplicities), there is a Jordan block of size one with eigenvalue β.

Proof. Since M^d acts trivially on W and V/W, we know that $(M^d-\mathrm{Id})^2=0$ on V, so that the minimal polynomial of M divides $(t^d-1)^2$ and the Jordan blocks of M have size at most two. The subspace W of V is generated by all the eigenvectors of M. Thus the dimension of V/W is equal to the number of Jordan blocks of M of size two, and the eigenvalues of these Jordan blocks are precisely the eigenvalues of M on V/W, or, equivalently, W'. It follows that the eigenvalues of M on V corresponding to a Jordan block of size one are the eigenvalues of M on W/W'. □

THEOREM 4.4. *We fix an embedding $\mathbb{Q}_\ell\hookrightarrow\mathbb{C}$. If A is a tamely ramified abelian K-variety, then the monodromy action of σ on $H^1(A\times_K K^t,\mathbb{Q}_\ell)$ has Jordan form*

$$\mathrm{Jord}(m_A^{\mathrm{ab}}+\breve{m}_A^{\mathrm{ab}},m_A^{\mathrm{tor}}).$$

Moreover, the functions m_A^{tor} and $m_A^{\mathrm{ab}}+\breve{m}_A^{\mathrm{ab}}$ are complete.

Proof. We adopt the notations of Section 3.1. We denote by $\mathscr{T}_\ell A$ the ℓ-adic Tate module of A. We put $I=G(K^s/K)$ and $I'=G(K^s/K')$. Recall that there exists a canonical I-equivariant isomorphism

$$H^1(A\times_K K^s,\mathbb{Q}_\ell)\cong Hom_{\mathbb{Z}_\ell}(\mathscr{T}_\ell A,\mathbb{Q}_\ell)\tag{4.2}$$

(see [18, 15.1]). Since A is tamely ramified, the wild inertia subgroup $P\subset I$ acts trivially on $H^1(A\times_K K^s,\mathbb{Q}_\ell)$ and $\mathscr{T}_\ell A$, so that the I-action on these modules factors through an action of $I/P=G(K^t/K)$.

Since P is a p-group and p is prime to ℓ, there exists for every K-variety X and every integer $i \geq 0$ a canonical $G(K^t/K)$-equivariant isomorphism

$$H^i(X \times_K K^t, \mathbb{Q}_\ell) \cong H^i(X \times_K K^s, \mathbb{Q}_\ell)^P$$

(see [2, I.2.7.1]). In our case, this yields a canonical $G(K^t/K)$-equivariant isomorphism

$$H^1(A \times_K K^s, \mathbb{Q}_\ell) = H^1(A \times_K K^s, \mathbb{Q}_\ell)^P \cong H^1(A \times_K K^t, \mathbb{Q}_\ell). \tag{4.3}$$

By (4.2) and (4.3), it suffices to show that the action of σ on

$$\mathscr{V}_\ell A = \mathscr{T}_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

has Jordan form

$$\mathrm{Jord}(m_A^{\mathrm{ab}} + \breve{m}_A^{\mathrm{ab}}, m_A^{\mathrm{tor}})$$

and that $m_A^{\mathrm{ab}} + \breve{m}_A^{\mathrm{ab}}$ and m_A^{tor} are complete.
Consider the filtration

$$(\mathscr{T}_\ell A)^{\mathrm{et}} \subset (\mathscr{T}_\ell A)^{\mathrm{ef}} \subset \mathscr{T}_\ell A \tag{4.4}$$

from [2, IX.4.1.1], with $(\mathscr{T}_\ell A)^{\mathrm{ef}}$ the *essentially fixed part* of the Tate module $\mathscr{T}_\ell A$, and $(\mathscr{T}_\ell A)^{\mathrm{et}}$ the *essentially toric part*. By definition,

$$(\mathscr{T}_\ell A)^{\mathrm{ef}} = (\mathscr{T}_\ell A)^{I'}$$

and $(\mathscr{T}_\ell A)^{\mathrm{et}}$ is stable under the action of I on $\mathscr{T}_\ell A$. We denote by

$$(\mathscr{V}_\ell A)^{\mathrm{et}} \subset (\mathscr{V}_\ell A)^{\mathrm{ef}} = (\mathscr{V}_\ell A)^{I'} \subset \mathscr{V}_\ell A \tag{4.5}$$

the filtration obtained from (4.4) by tensoring with $\mathbb{Q}_\ell$. By [2, IX.4.1.2] there exists an I-equivariant isomorphism

$$\mathscr{V}_\ell A/(\mathscr{V}_\ell A)^{\mathrm{ef}} \cong ((\mathscr{V}_\ell A)^{\mathrm{et}})^\vee. \tag{4.6}$$

In particular, I' acts trivially on $\mathscr{V}_\ell A/(\mathscr{V}_\ell A)^{\mathrm{ef}}$. It follows that the I'-action on $\mathscr{V}_\ell A$ is unipotent of level ≤ 2 (this means that for every element γ of I', the endomorphism $(\gamma - \mathrm{Id})^2$ of $\mathscr{V}_\ell A$ is zero), and that the I-action on $(\mathscr{V}_\ell A)^{\mathrm{et}}$ and $\mathscr{V}_\ell A/(\mathscr{V}_\ell A)^{\mathrm{ef}}$ factors through an action of $I/I' \cong \mu = \mu_d(k)$, where $d = [K' : K]$. We denote by $\overline{\sigma}$ the image of σ under the projection $G(K^t/K) \to \mu$.
The element σ^d is a topological generator of I'/P. Combining (4.5) and (4.6) and applying Lemma 4.3 to the σ-action on $\mathscr{V}_\ell A$, we see that it suffices to prove the following claims:

(1) the $\overline{\sigma}$-action on $(\mathscr{V}_\ell A)^{\mathrm{et}}$ has Jordan form $\mathrm{Jord}(m_A^{\mathrm{tor}}, 0)$, and m_A^{tor} is complete,
(2) the $\overline{\sigma}$-action on $(\mathscr{V}_\ell A)^{\mathrm{ef}}/(\mathscr{V}_\ell A)^{\mathrm{et}}$ has Jordan form $\mathrm{Jord}(m_A^{\mathrm{ab}} + \breve{m}_A^{\mathrm{ab}}, 0)$, and $m_A^{\mathrm{ab}} + \breve{m}_A^{\mathrm{ab}}$ is complete.

Since $\overline{\sigma}^d$ is the identity, the Jordan forms of the $\overline{\sigma}$-actions in (1) and (2) are diagonal matrices. By [2, IX.4.2.7 and IX.4.2.9] there exist μ-equivariant isomorphisms

$$\begin{array}{rcl} (\mathscr{V}_\ell A)^{\mathrm{et}} & \cong & \mathscr{V}_\ell T \\ (\mathscr{V}_\ell A)^{\mathrm{ef}}/(\mathscr{V}_\ell A)^{\mathrm{et}} & \cong & \mathscr{V}_\ell B \end{array}$$

so that claims (1) and (2) follow from Proposition 4.1. □

Corollary 4.5. *The functions $m_A^{\mathrm{ab}} + \breve{m}_A^{\mathrm{ab}}$ and m_A^{tor} are invariant under isogeny. In particular, $m_A^{\mathrm{tor}} = m_{A^\vee}^{\mathrm{tor}}$, and*

$$m_A^{\mathrm{ab}} + \breve{m}_A^{\mathrm{ab}} = m_{A^\vee}^{\mathrm{ab}} + \breve{m}_{A^\vee}^{\mathrm{ab}}.$$

The multiplicity function m_A, the jumps of A (with their multiplicities), the elementary divisors of A and the base change conductor $c(A)$ are invariant under isogenies of degree prime to p.

Proof. This follows from Propositions 3.5 and 3.9, and Theorem 4.4. □

Remark 4.6. Beware that the base change conductor, and thus the function m_A^{ab}, of a tamely ramified abelian K-variety A can change under isogenies of degree p, if $p > 0$; see [5, 6.10.2] for an example.

Corollary 4.7. *Using the notations of Proposition 3.5, the base change conductor of a tamely ramified abelian K-variety A is given by*

$$c(A) = \frac{1}{2}(t_{\mathrm{pot}}(A) - t(A)) + \sum_{x \in]0,1[\cap \mathbb{Q}} m_A^{\mathrm{ab}}(x)x.$$

In particular, if A has potential purely multiplicative reduction (which means that $a_{\mathrm{pot}}(A) = 0$), then

$$c(A) = \frac{u(A)}{2}.$$

Proof. By Proposition 3.5, we know that

$$c(A) = \sum_{x \in]0,1[\cap \mathbb{Q}} m_A^{\mathrm{tor}}(x)x + \sum_{x \in]0,1[\cap \mathbb{Q}} m_A^{\mathrm{ab}}(x)x.$$

Since m_A^{tor} is complete, we have that

$$\begin{aligned}
\sum_{x \in]0,1[\cap \mathbb{Q}} m_A^{\mathrm{tor}}(x)x &= \frac{1}{2}\left(\sum_{x \in]0,1[\cap \mathbb{Q}} m_A^{\mathrm{tor}}(x)x + \sum_{x \in]0,1[\cap \mathbb{Q}} m_A^{\mathrm{tor}}(x)(1-x)\right) \\
&= \frac{1}{2}\left(\sum_{x \in]0,1[\cap \mathbb{Q}} m_A^{\mathrm{tor}}(x)\right) \\
&= \frac{1}{2}(\|m_A^{\mathrm{tor}}\| - m_A^{\mathrm{tor}}(0)) \\
&= \frac{1}{2}(t_{\mathrm{pot}}(A) - t(A))
\end{aligned}$$

where the last equality follows from Proposition 3.6. If $a_{\mathrm{pot}}(A) = 0$, then it follows from Proposition 3.6 that $m_A^{\mathrm{ab}} = 0$ and $a(A) = 0$, so that

$$c(A) = \frac{1}{2}(t_{\mathrm{pot}}(A) - t(A)) = \frac{1}{2}(\dim(A) - t(A)) = \frac{u(A)}{2}.$$

□

Remark 4.8. If A has potential purely multiplicative reduction, then Corollary 4.7 can be viewed as a special case of Chai's result that for *every* abelian K-variety A (not necessarily tamely ramified) with potential purely multiplicative reduction, the base change conductor $c(A)$ equals one fourth of the Artin conductor of $\mathscr{V}_\ell A$ [5, 5.2]. If A is tamely ramified, then this Artin conductor is simply the dimension of $\mathscr{V}_\ell A/((\mathscr{V}_\ell A)^{ss})^I$, where $I = G(K^s/K)$ and $(\mathscr{V}_\ell A)^{ss}$ is the semi-simplification of the ℓ-adic I-representation $\mathscr{V}_\ell A$. This value is precisely the number of eigenvalues of σ (counted with multiplicities) that are different from one. Combining Proposition 3.6 with Theorem 4.4, we find that the Artin conductor of $\mathscr{V}_\ell A$ equals

$$2\dim(A) - m_A^{\mathrm{ab}}(0) - \breve{m}_A^{\mathrm{ab}}(0) - 2m_A^{\mathrm{tor}}(0) = 2u(A).$$

Corollary 4.9. *Let A be a tamely ramified abelian K-variety, and let e be the degree of the minimal extension of K where A acquires semi-abelian reduction. Fix a primitive e-th root of unity ξ in an algebraic closure $\mathbb{Q}^a$ of $\mathbb{Q}$. The characteristic polynomial*

$$P_\sigma(t) = \det(t\cdot \mathrm{Id} - \sigma \,|\, H^1(A\times_K K^t, \mathbb{Q}_\ell))$$

of σ on $H^1(A\times_K K^t,\mathbb{Q}_\ell)$ is given by

$$P_\sigma(t) = \prod_{i\in((1/e)\mathbb{Z})/\mathbb{Z}} (t-\xi^{e\cdot i})^{m_A^{\mathrm{ab}}(i)+\breve{m}_A^{\mathrm{ab}}(i)+2m_A^{\mathrm{tor}}(i)} \quad \in \mathbb{Z}[t].$$

Proof. This is an immediate consequence of Theorem 4.4. □

Corollary 4.10. *Let A be a tamely ramified abelian K-variety. Assume either that k has characteristic zero, or that A is principally polarized. Then m_A^{ab} and $\breve{m}_A^{\mathrm{ab}}$ are semi-complete, and the monodromy action of σ on $H^1(A\times_K K^t,\mathbb{Q}_\ell)$ has Jordan form*

$$\mathrm{Jord}(m_A^{\mathrm{ab}} + (m_A^{\mathrm{ab}})^*, m_A^{\mathrm{tor}}).$$

In the notation of Corollary 4.9, we have

$$P_\sigma(t) = \prod_{i\in((1/e)\mathbb{Z})/\mathbb{Z}} (t-\xi^{e\cdot i})^{m_A^{\mathrm{ab}}(i)+m_A^{\mathrm{ab}}(-i)+2m_A^{\mathrm{tor}}(i)} \quad \in \mathbb{Z}[t].$$

Proof. Semi-completeness of m_A^{ab} and $\breve{m}_A^{\mathrm{ab}}$ follows from Corollary 3.11 and Theorem 4.4. The remainder of the statement is a combination of Corollaries 3.11 and 4.9. □

5. Potential toric rank and Jordan blocks

5.1. The weight filtration associated to a nilpotent operator. Throughout this section, we fix a field F of characteristic zero and a finite dimensional vector space V over F. For every endomorphism M on V, we consider its Jordan-Chevalley decomposition

$$M = M_s + M_n$$

with M_s the semi-simple part of M and M_n its nilpotent part.
We recall the following well-known property.

PROPOSITION 5.1. *Let N be a nilpotent endomorphism of V. Let w be an integer. There exists a unique finite ascending filtration $W = (W_i,\, i \in \mathbb{Z})$ on V such that*

(1) *$NW_i \subset W_{i-2}$ for all i in $\mathbb{Z}$,*
(2) *the morphism of vector spaces*

$$\mathrm{Gr}^W_{w+\alpha} V \to \mathrm{Gr}^W_{w-\alpha} V$$

induced by N^α is an isomorphism for every integer $\alpha > 0$.

The filtration W is called the weight filtration *centered at w associated to the nilpotent operator N.*

Proof. See, for instance, [10, 1.6.1]. □

It is clear from the definition that the weight filtration centered at another integer w' is the shifted filtration $W'_\bullet = W_{\bullet - w' + w}$. We define the *amplitude* of the filtration W in Proposition 5.1 as the smallest integer $n \geq 0$ such that $W_{w+n} = V$. This value does not depend on the choice of the central weight w. The amplitude is related to sizes of Jordan blocks in the following way.

PROPOSITION 5.2. *Let M be an endomorphism of V. We denote by a the amplitude of the weight filtration W associated to M_n (centered at any weight $w \in \mathbb{Z}$). Then $a+1$ is the maximum of the ranks of the Jordan blocks of M.*

Proof. We may assume that $w = 0$ and that $M = M_n$. Then the proposition is an immediate consequence of the explicit description of the weight filtration in [10, 1.6.7]. □

PROPOSITION 5.3. *Let N be a nilpotent endomorphism of V, and denote by W the associated weight filtration centered at $w \in \mathbb{Z}$.*

(1) *If N' is another nilpotent operator on V such that*

$$(N - N')W_i \subset W_{i-3}$$

for all $i \in \mathbb{Z}$, then the weight filtration associated to N' centered at w coincides with W.
(2) *The weight filtration W does not change if we multiply N with an automorphism S of V that commutes with N.*

Proof. (1) This follows immediately from the definition of the weight filtration in Proposition 5.1.
(2) Clearly, the filtration $(S(W_i),\, i \in \mathbb{Z})$ on V also satisfies properties (1) and (2) in Proposition 5.1, so that $S(W_i) = W_i$ for all i in $\mathbb{Z}$ by uniqueness of the weight filtration. This implies at once that W coincides with the weight filtration associated to NS centered at w. □

Definition 5.4. *Let $W = (W_i,\, i \in \mathbb{Z})$ be an ascending filtration on V. The dual filtration $W^\vee$ on $V^\vee$ is the ascending filtration defined by*

$$(W^\vee)_i = (W_{-i-1})^\perp$$

for all i in $\mathbb{Z}$.

For every integer $j \geq 0$, the degree j exterior power of W is the ascending filtration $\wedge^j W$ on $\wedge^j V$ given by

$$(\wedge^j W)_i = \sum_{i_1+\ldots+i_j=i} W_{i_1} \wedge \cdots \wedge W_{i_j}.$$

If V' is another finite dimensional vector space over F, endowed with an ascending filtration W', then the tensor product of W and W' is the ascending filtration $W \otimes W'$ on $V \otimes V'$ given by

$$(W \otimes W')_i = \sum_{i_1+i_2=i} W_{i_1} \otimes W'_{i_2}.$$

Proposition 5.5. *Let V and V' be finite dimensional vector spaces over F, endowed with nilpotent operators N and N', respectively. We denote by W and W' the associated weight filtrations on V and V', centered at integers w and w'.*

1. *The weight filtration on $V^\vee$ centered at $-w$ associated to the nilpotent operator $-N^\vee$ is the dual $W^\vee$ of the weight filtration W.*
2. *The weight filtration on $V \otimes V'$ centered at $w + w'$ associated to the nilpotent operator $N \otimes \mathrm{Id} + \mathrm{Id} \otimes N'$ is the tensor product $W \otimes W'$ of the weight filtrations W and W'.*
3. *For every integer $j > 0$, the weight filtration on $\wedge^j V$ centered at $w \cdot j$ associated to the nilpotent operator*

$$N^{(\wedge j)} = N \wedge \mathrm{Id} \wedge \cdots \wedge \mathrm{Id} + \mathrm{Id} \wedge N \wedge \mathrm{Id} \wedge \cdots \wedge \mathrm{Id} + \ldots + \mathrm{Id} \wedge \cdots \wedge \mathrm{Id} \wedge N$$

is the exterior power $\wedge^j W$ of the weight filtration W.

Proof. Point (1) and (2) are proven in [10, 1.6.9], using the theorem of Jacobson-Morosov. Point (3) can be proven in exactly the same way, since the morphism of linear groups

$$h : \mathrm{GL}(V) \to \mathrm{GL}(\wedge^j V) : M \mapsto \wedge^j M$$

induces a morphism of Lie algebras

$$\mathrm{Lie}(h) : \mathrm{End}(V) \to \mathrm{End}(\wedge^j V)$$

that sends N to $N^{(\wedge j)}$. □

Corollary 5.6. *Let M be an automorphism of V, and consider an integer w and an integer $j > 0$. We denote by W the weight filtration centered at w associated to the nilpotent operator M_n on V.*

(1) *Let V' be another finite dimensional vector space, endowed with an automorphism M'. If W' is the weight filtration associated to M'_n centered at $w' \in \mathbb{Z}$, then $W \otimes W'$ is the weight filtration centered at $w + w'$ associated to the nilpotent operator $(M \otimes M')_n$ on $V \otimes V'$.*
(2) *The exterior power filtration $\wedge^j W$ is the weight filtration centered at $w \cdot j$ associated to the nilpotent operator $(\wedge^j M)_n$ on $\wedge^j V$.*

Proof. (1) By Proposition 5.3(2), we may assume that M and M' are unipotent, since multiplying these operators with M_s^{-1} and $(M'_s)^{-1}$, respectively, has no influence on the weight filtrations that we want to compare. Then $M = \mathrm{Id} + M_n$ and $M' = \mathrm{Id} + M'_n$, so that

$$(M \otimes M')_n - \mathrm{Id} \otimes M'_n - M_n \otimes \mathrm{Id} = M_n \otimes M'_n.$$

It follows that

$$((M \otimes M')_n - \mathrm{Id} \otimes M'_n - M_n \otimes \mathrm{Id})(W \otimes W')_i \subset (W \otimes W')_{i-4}$$

for all $i \in \mathbb{Z}$. The result now follows from Propositions 5.3(1) and 5.5(2).
(2) The proof is completely similar to the proof of (1): one reduces to the case where M is unipotent, and one shows by direct computation that

$$(\wedge^j M)_n - (M_n)^{(\wedge j)}$$

shifts weights by at least -4. □

The following lemma and proposition will allow us to compute the maximal size of certain Jordan blocks of monodromy on the cohomology of a tamely ramified abelian K-variety (Theorem 5.9).

Lemma 5.7. *Let F be an algebraically closed field of characteristic zero, and let $V \neq \{0\}$ be a finite dimensional vector space over F. Let M be an automorphism of V, with Jordan form*

$$\mathrm{Jord}_m(\xi)$$

where $m \in \mathbb{Z}_{>0}$ and $\xi \in F^\times$. Then for every integer j in $[1, m]$ and every integer w, the weight filtration centered at w associated to the nilpotent operator $(\wedge^j M)_n$ on $\wedge^j V$ has amplitude $m(m - j)$.

Proof. We may assume that $w = 0$. Denote by W the weight filtration associated to M_n centered at 0. By Corollary 5.6, the weight filtration associated to $(\wedge^j M)_n$ centered at 0 coincides with the exterior power filtration $\wedge^j W$.
By the explicit description of the weight filtration in [10, 1.6.7], the dimension of $\mathrm{Gr}^W_\alpha V$ is one if α is an integer in $[1 - m, m - 1]$ such that $\alpha - m$ is odd, and zero in all other cases. This easily implies that $\wedge^j W$ has amplitude

$$(m - 1) + (m - 3) + \ldots + (m - 2j + 1) = m(m - j).$$

□

Proposition 5.8. *Let F be an algebraically closed field of characteristic zero, and let $V \neq \{0\}$ be a finite dimensional vector space over F. Let M be an automorphism of V, with Jordan form*

$$\mathrm{Jord}_{m_1}(\xi_1) \oplus \cdots \oplus \mathrm{Jord}_{m_q}(\xi_q)$$

where $q \in \mathbb{Z}_{>0}$, $m \in (\mathbb{Z}_{>0})^q$ and $\xi_i \in F^\times$ for $i = 1, \ldots, q$.
We fix an integer $j > 0$. For every element ζ of F, we denote by $\max_\zeta$ *the maximum of the ranks of the Jordan blocks of $\wedge^j M$ on $\wedge^j V$ with eigenvalue ζ. If we denote by $\mathscr{S}$ the set of tuples $s \in \mathbb{N}^q$ such that $\|s\| = j$ and $s_i \leq m_i$ for each $i \in \{1, \ldots, q\}$, then*

$$\max_\zeta = \max\{1 + \sum_{i=1}^{q} s_i(m_i - s_i) \,|\, s \in \mathscr{S},\ \prod_{i=1}^{q} (\xi_i)^{s_i} = \zeta\}$$

for every $\zeta \in F$, with the convention that $\max \emptyset = 0$.

Proof. We can write

$$V = V_1 \oplus \cdots \oplus V_q$$

such that $M(V_i) \subset V_i$ for each i and such that the restriction M_i of M to V_i has Jordan form $\mathrm{Jord}_{m_i}(\xi_i)$. If we put

$$V_s = (\wedge^{s_1} V_1) \otimes \cdots \otimes (\wedge^{s_q} V_q)$$

for each $s \in \mathscr{S}$, then we have a canonical isomorphism

$$\wedge^j V \cong \bigoplus_{s \in \mathscr{S}} V_s$$

such that every summand V_s is stable under $\wedge^j M$ and the restriction of $\wedge^j M$ to V_s equals

$$(\wedge^{s_1} M_1) \otimes \cdots \otimes (\wedge^{s_q} M_q).$$

The automorphism $\wedge^j M$ has a unique eigenvalue on V_s, which is equal to

$$\prod_{i=1}^{q} (\xi_i)^{s_i}.$$

By Proposition 5.2, Corollary 5.6(1) and Lemma 5.7, the maximal rank of a Jordan block of $\wedge^j M$ on V_s equals

$$1 + \sum_{i=1}^{q} s_i(m_i - s_i).$$

This yields the desired formula for $\max_\zeta$. $\square$

5.2. THE STRONG FORM OF THE MONODROMY CONJECTURE.

THEOREM 5.9. *Let A be a tamely ramified abelian K-variety of dimension g. For every embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$, the value $\alpha = \exp(2\pi i c(A))$ is an eigenvalue of σ on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$. Each Jordan block of σ on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$ has rank at most $t_{\mathrm{pot}}(A)+1$, and σ has a Jordan block with eigenvalue α on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$ with rank $t_{\mathrm{pot}}(A)+1$.*

Proof. Since A is tamely ramified, we have a canonical $G(K^t/K)$-equivariant isomorphism of $\mathbb{Q}_\ell$-vector spaces

$$H^g(A \times_K K^t, \mathbb{Q}_\ell) \cong \bigwedge^g H^1(A \times_K K^t, \mathbb{Q}_\ell).$$

By Theorem 4.4, the monodromy operator σ has precisely $\|m_A^{\mathrm{tor}}\|$ Jordan blocks of size 2 on $H^1(A \times_K K^t, \mathbb{Q}_\ell)$, and no larger Jordan blocks. It follows from Proposition 5.8 that the size of the Jordan blocks of σ on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$ is bounded by $1 + \|m_A^{\mathrm{tor}}\|$. By Proposition 3.6, we know that $\|m_A^{\mathrm{tor}}\| = t_{\mathrm{pot}}(A)$. By Proposition 3.5, the image in $\mathbb{Q}/\mathbb{Z}$ of the base change conductor $c(A)$ equals

$$\sum_{x \in \mathbb{Q}/\mathbb{Z}} ((m_A^{\mathrm{tor}}(x) + m_A^{\mathrm{ab}}(x)) \cdot x)$$

and by Proposition 3.6, we have

$$\sum_{x \in \mathbb{Q}/\mathbb{Z}} (m_A^{\mathrm{tor}}(x) + m_A^{\mathrm{ab}}(x)) = g.$$

Hence, by Theorem 4.4 and Proposition 5.8, σ has a Jordan block of size $1 + t_{\mathrm{pot}}(A)$ with eigenvalue α on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$. □

6. LIMIT MIXED HODGE STRUCTURE

Let A be a tamely ramified abelian K-variety of dimension g. Theorem 4.4 shows that the couple of functions $(m_A^{\mathrm{tor}}, m_A^{\mathrm{ab}} + \check{m}_A^{\mathrm{ab}})$ and the Jordan form of σ on the tame degree one cohomology of A determine each other. It does not tell us how to recover m_A^{ab} and $\check{m}_A^{\mathrm{ab}}$ individually from the cohomology of A.
In this section, we assume that A is obtained by base change from a family of abelian varieties over a smooth complex curve. We will give an interpretation of the multiplicity functions m_A^{ab}, $\check{m}_A^{\mathrm{ab}}$ and m_A^{tor} in terms of the limit mixed Hodge structure of the family.

6.1. LIMIT MIXED HODGE STRUCTURE OF A FAMILY OF ABELIAN VARIETIES. Let $\overline{S}$ be a connected smooth complex algebraic curve, let s be a closed point on $\overline{S}$, and choose a local parameter t on $\overline{S}$ at s. We put $K = \mathbb{C}((t))$, $R = \mathbb{C}[[t]]$ and $S = \overline{S} \setminus \{s\}$. Let

$$f : X \to S$$

be a smooth projective family of abelian varieties over S, of relative dimension g, and put

$$A = X \times_S \operatorname{Spec} K.$$

We choose an extension of f to a flat projective morphism

$$\overline{f}:\overline{X}\to\overline{S},$$

and we denote by $\overline{X}_s$ the fiber of $\overline{f}$ over the point s.
We denote by $(\cdot)^{\mathrm{an}}$ the complex analytic GAGA functor, but we will usually omit it from the notation if no confusion can occur. For instance, when we speak of the sheaf $R^if_*(\mathbb{Z})$, it should be clear that we mean $R^if_*^{\mathrm{an}}(\mathbb{Z})$.
For every $i\in\mathbb{N}$, we consider the degree i limit cohomology, resp. homology,

$$\begin{aligned} H^i(X_\infty,\mathbb{Z}) &:= \mathbb{H}^i(\overline{X}_s(\mathbb{C}),R\psi_{\overline{f}}(\mathbb{Z}))\\ H_i(X_\infty,\mathbb{Z}) &:= \mathbb{H}^{2g-i}(\overline{X}_s(\mathbb{C}),R\psi_{\overline{f}}(\mathbb{Z}))(g)=H^{2g-i}(X_\infty,\mathbb{Z})(g)\end{aligned}$$

of $\overline{f}$ at s. Here

$$R\psi_{\overline{f}}(\mathbb{Z})\in D^b_c(\overline{X}_s(\mathbb{C}),\mathbb{Z})$$

denotes the complex of nearby cycles associated to $\overline{f}^{\mathrm{an}}$. For every i in $\mathbb{N}$, the $\mathbb{Z}$-module $H^i(X_\infty,\mathbb{Z})$ is non-canonically isomorphic to $H^i(X_z(\mathbb{C}),\mathbb{Z})$, where z is any point of $S(\mathbb{C})$ and X_z is the fiber of f over z [3, XIV.1.3.3.2]. Likewise, by Poincaré duality, $H_i(X_\infty,\mathbb{Z})$ is non-canonically isomorphic to $H_i(X_z(\mathbb{C}),\mathbb{Z})$. The limit cohomology and homology are independent of the chosen compactification $\overline{f}$, as can be deduced from [11, 4.2.11] by dominating two compactifications by a third one.
We set

$$\begin{aligned} H^i(X_\infty,\mathbb{Q}) &:= H^i(X_\infty,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}\\ H^i(X_\infty,\mathbb{C}) &:= H^i(X_\infty,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}\end{aligned}$$

and we use similar notations for the limit homology $H_i(X_\infty,*)$. For all $i\in\mathbb{N}$, we denote by M the monodromy operators on $H_i(X_\infty,\mathbb{Z})$ and $H^i(X_\infty,\mathbb{Z})$, and by N the logarithm of the unipotent part M_u of M.
The $\mathbb{Z}$-modules $H^i(X_\infty,\mathbb{Z})$ and $H_i(X_\infty,\mathbb{Z})$ carry natural mixed Hodge structures, which are the limits at s of the variations of Hodge structures

$$R^if_*(\mathbb{Z}),\ \text{resp. } R^{2g-i}f_*(\mathbb{Z})(g),$$

on S. The existence of these limit mixed Hodge structures was conjectured by Deligne, and they were constructed by Schmid [25] and Steenbrink [26]. The weight filtrations on $H^i(X_\infty,\mathbb{Q})$ and $H_i(X_\infty,\mathbb{Q})$ are the weight filtrations centered at i, resp. $-i$, associated to the nilpotent operator N. We will briefly recall Schmid's construction of the limit Hodge filtration below. The action of the semi-simple part M_s of M on $H_i(X_\infty,\mathbb{Q})$ and $H^i(X_\infty,\mathbb{Q})$ is a morphism of rational mixed Hodge structures, by [27, 2.13].
For every $i\in\mathbb{N}$, there exists an isomorphism of $\mathbb{Q}_\ell$-vector spaces

$$H^i(X_\infty,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}_\ell\cong H^i(A\times_K K^a,\mathbb{Q}_\ell) \tag{6.1}$$

such that the monodromy action on the left hand side corresponds to the action of the canonical topological generator of $G(K^a/K)\cong\widehat{\mathbb{Z}}(1)(\mathbb{C})$ on the right hand side. This follows from Deligne's comparison theorem for ℓ-adic versus complex analytic nearby cycles [3, XIV.2.8]. Thus, if K' is a finite extension of K such that $A \times_K K'$ has semi-abelian reduction and if we set $d = [K' : K]$, then $(M_s)^d$ is the identity on $H^i(X_\infty, \mathbb{Q})$ and $H_i(X_\infty, \mathbb{Q})$ for all $i \geq 0$. Identifying M_s with the canonical generator $\exp(2\pi i/d)$ of $\mu_d(\mathbb{C})$, we obtain an action of $\mu_d(\mathbb{C})$ on the rational mixed Hodge structures $H^i(X_\infty, \mathbb{Q})$ and $H_i(X_\infty, \mathbb{Q})$, for all $i \geq 0$.

For the definition of the dual and the exterior powers of a mixed Hodge structure, we refer to [22, 3.2].

Proposition 6.1. (1) *For every $i \in \mathbb{N}$, there exists a natural isomorphism of mixed Hodge structures*

$$\wedge^i_{\mathbb{Z}} H^1(X_\infty, \mathbb{Z}) \to H^i(X_\infty, \mathbb{Z})$$

that is compatible with the action of M on the underlying $\mathbb{Z}$-modules.

(2) *For every $i \in \mathbb{N}$, there exists a natural isomorphism of mixed Hodge structures*

$$H^i(X_\infty, \mathbb{Z})^\vee \to H_i(X_\infty, \mathbb{Z})$$

that is compatible with the action of M on the underlying $\mathbb{Z}$-modules.

Proof. (1) The cup product defines a morphism

$$\wedge^i_{\mathbb{Z}} R^1 f_*(\mathbb{Z}) \to R^i f_*(\mathbb{Z}) \tag{6.2}$$

of sheaves on S. This is an isomorphism on every fiber, by the proper base change theorem and [19, p. 3], and thus an isomorphism of sheaves. Moreover, it is an isomorphism of variations of Hodge structures, because the cup product defines a morphism of pure Hodge structures on the cohomology of every fiber of f [22, 5.45].

Looking at Schmid's construction of the limit mixed Hodge structure in [25], one checks in a straightforward way that taking the limit of a variation of Hodge structures commutes with taking exterior powers. Compatibility of the Hodge filtrations is easy, since the exterior power defines a holomorphic map between the relevant classifying spaces. The compatibility of the weight filtrations follows from Corollary 5.6.

Thus, taking the limit at s of the isomorphism (6.2), we obtain an isomorphism of mixed Hodge structures

$$\wedge^i_{\mathbb{Z}} H^1(X_\infty, \mathbb{Z}) \to H^i(X_\infty, \mathbb{Z})$$

that is compatible with the action of M.

(2) For every $i \geq 0$, Poincaré duality yields a natural isomorphism of sheaves of $\mathbb{Z}$-modules

$$\alpha : R^i f_*(\mathbb{Z})^\vee \to R^{2g-i} f_*(\mathbb{Z})(g) \tag{6.3}$$

(note that Poincaré duality holds with coefficients in $\mathbb{Z}$ because $R^i f_*(\mathbb{Z})$ is a locally free sheaf of $\mathbb{Z}$-modules for all $i \geq 0$). By [22, 6.19], this isomorphism respects the Hodge structure on every fiber of $R^i f_*(\mathbb{Z})^\vee$ and $R^{2g-i} f_*(\mathbb{Z})(g)$. Thus (6.3) is an isomorphism of variations of Hodge structures on S. As in (1), one checks in a straightforward way that the limit of $R^i f_*(\mathbb{Z})^\vee$ at s is the dual of the limit of $R^i f_*(\mathbb{Z})$, using Proposition 5.5 to verify the compatibility of the weight filtrations. Thus the isomorphism (6.3) induces an isomorphism of mixed Hodge structures

$$H^i(X_\infty,\mathbb{Z})^\vee \to H_i(X_\infty,\mathbb{Z}).$$

□

By Proposition 6.1, in order to describe the limit mixed Hodge structure on $H_i(X_\infty,\mathbb{Q})$ and $H^i(X_\infty,\mathbb{Z})$ for all $i\geq 0$, it suffices to determine the limit mixed Hodge structure on $H_1(X_\infty,\mathbb{Z})$.

6.2. DESCRIPTION OF THE MIXED HODGE STRUCTURE ON $H_1(X_\infty,\mathbb{Z})$. We denote by

$$\mathscr{V} \to S^{\mathrm{an}}$$

the polarized variation of Hodge structures

$$R^{2g-1}f_*(\mathbb{Z})(g)$$

of type $\{(0,-1),(-1,0)\}$ [8, 4.4.3]. We denote by $\mathscr{V}_{\mathbb{Z}}$, $\mathscr{V}_{\mathbb{Q}}$ and $\mathscr{V}_{\mathbb{C}}$ the integer, resp. rational, resp. complex component of $\mathscr{V}$. The sheaf $\mathscr{V}_{\mathbb{Z}}$ is a locally free sheaf of $\mathbb{Z}$-modules on S^{an} of rank $2g$. The fiber of $\mathscr{V}$ over a point z of S^{an} is canonically isomorphic to the weight -1 Hodge structure

$$H^{2g-1}(X_z(\mathbb{C}),\mathbb{Z})(g),$$

where X_z denotes the fiber of f over z. By Poincaré duality, there is a canonical isomorphism of $\mathbb{Z}$-modules

$$H^{2g-1}(X_z(\mathbb{C}),\mathbb{Z}) \cong H_1(X_z(\mathbb{C}),\mathbb{Z}).$$

The limit of $\mathscr{V}$ at the point s is a mixed Hodge structure that was constructed by Schmid [25]. In our notation, this limit is precisely the mixed Hodge structure $H_1(X_\infty,\mathbb{Z})$. For a quick introduction to limit mixed Hodge structures, we refer to [13] and [22, §10 and §11]. Here we will only briefly sketch the main ingredients of the construction.

We consider a small disc Δ around s in S^{an} and we denote by Δ^* the punctured disc $\Delta\setminus\{s\}$. It follows from the definition of the nearby cycles functor that $H_1(X_\infty,\mathbb{Z})$ is the $\mathbb{Z}$-module of global sections of the pullback of $\mathscr{V}_{\mathbb{Z}}$ to a universal cover of Δ^* [3, XIV.1.3.3]. By the comparison isomorphism (6.1) and [2, IX.3.5], the action of the monodromy operator M^d on $H_1(X_\infty,\mathbb{Q})$ is unipotent of level ≤ 2 (meaning that $(M^d-\mathrm{Id})^2=0$). The level of unipotency can also be deduced from the Monodromy Theorem [25, 6.1] and the fact that the fibers of $\mathscr{V}$ are of type $\{(-1,0),(0,-1)\}$.

The *weight filtration* W on $H_1(X_\infty,\mathbb{Q})$ is the weight filtration with center -1 associated to the nilpotent operator N (recall that N is the logarithm of the unipotent part M_u of the monodromy operator M). Since $M_u^d=M^d$, we have

$$dN=\log(M_u^d)=M^d-\mathrm{Id}$$

so that $N^2=0$ and the weight filtration is of the form

$$\{0\}\subset W_{-2}\subset W_{-1}\subset W_0=H_1(X_\infty,\mathbb{Q}).$$

Explicitly, we have $W_{-1} = \ker(N)$ and $W_{-2} = \mathrm{im}(N)$. This filtration induces a weight filtration on the lattice $H_1(X_\infty, \mathbb{Z})$ in $H_1(X_\infty, \mathbb{Q})$. Note that

$$W_{-1}H_1(X_\infty, \mathbb{Z}) := W_{-1}H_1(X_\infty, \mathbb{Q}) \cap H_1(X_\infty, \mathbb{Z})$$

is stable under the action of M_s, since $H_1(X_\infty, \mathbb{Z})$ is stable under M and M is semi-simple on $W_{-1}H_1(X_\infty, \mathbb{Q}) = \ker(M^d - \mathrm{Id})$.

Consider the finite covering

$$\widetilde{\Delta} \to \Delta$$

obtained by taking a d-th root t' of the coordinate t on Δ. This covering is totally ramified over the origin s of Δ. With a slight abuse of notation, we denote again by s the unique point of the open disc $\widetilde{\Delta}$ that lies over $s \in \Delta$. We denote by $\widetilde{\Delta}^*$ the punctured disc $\widetilde{\Delta} \setminus \{s\}$.

Pulling back $\mathscr{V}$ to a variation of Hodge structures $\mathscr{V}'$ on $\widetilde{\Delta}^*$ has the effect of raising the monodromy operator M to the power d. This has no influence on the associated weight filtration on $H_1(X_\infty, \mathbb{Q})$, since the logarithm of $(M_u)^d = M^d$ equals dN. Pulling back $\mathscr{V}$ to $\widetilde{\Delta}^*$ is the first step in the construction of the limit Hodge filtration F on $H_1(X_\infty, \mathbb{Q})$. The important point is that the monodromy M^d of the variation $\mathscr{V}'$ is unipotent.

Schmid considers a complex manifold $\check{D}$ that parameterizes descending filtrations

$$F^1 = H_1(X_\infty, \mathbb{C}) \supset F^0 \supset \{0\}$$

on $H_1(X_\infty, \mathbb{C})$ that satisfy a certain compatibility relation with the polarization on $H_1(X_\infty, \mathbb{C})$ and such that F^0 has dimension g. "Untwisting" the monodromy action on the fibers of $\mathscr{V}'$, he constructs a map

$$\widetilde{\Psi} : \widetilde{\Delta}^* \to \check{D}.$$

The Nilpotent Orbit Theorem (for one variable) in [25, 4.9] guarantees that $\widetilde{\Psi}$ extends to a holomorphic map

$$\Psi : \widetilde{\Delta} \to \check{D}.$$

The point $\Psi(s)$ of $\check{D}$ corresponds to a descending filtration

$$F^{-1} = H_1(X_\infty, \mathbb{C}) \supset F^0 \supset \{0\}$$

that is called the *limit Hodge filtration* on $H_1(X_\infty, \mathbb{C})$. Schmid's fundamental result [25, 6.16] states that the weight filtration W and the limit Hodge filtration F define a polarized mixed Hodge structure on $H_1(X_\infty, \mathbb{Z})$, which is called the *limit mixed Hodge structure* of the variation of Hodge structures $\mathscr{V}'$.

We see from the shape of the weight filtration and the limit Hodge filtration that the limit mixed Hodge structure on $H_1(X_\infty, \mathbb{Z})$ is of type

$$\{(0,0),\ (-1,0),\ (0,-1),\ (-1,-1)\}.$$

Moreover, since

$$\mathrm{Gr}^W_{-1}H_1(X_\infty, \mathbb{Z})$$

is polarizable, the mixed Hodge structure

$$(H_1(X_\infty, \mathbb{Z}), W, F)$$

is a mixed Hodge 1-motive in the sense of [9, § 10].
The construction of the limit Hodge filtration can be reformulated in terms of the *Deligne extension* or *canonical extension*. Consider the holomorphic vector bundle

$$(\mathscr{V}')^h = \mathscr{V}'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{\widetilde{\Delta}^*}$$

on the punctured disc $\widetilde{\Delta}^*$. The locally constant subsheaf $\mathscr{V}'_{\mathbb{C}}$ of this vector bundle defines a connection ∇ on $(\mathscr{V}')^h$, called the Gauss-Manin connection. The vector bundle $(\mathscr{V}')^h$ extends in a unique way to a vector bundle $\widehat{\mathscr{V}'}$ on $\widetilde{\Delta}$ such that ∇ extends to a logarithmic connection on $\widehat{\mathscr{V}'}$ whose residue at s is nilpotent [7, 5.2]. We call $\widehat{\mathscr{V}'}$ the *Deligne extension* of the variation of Hodge structures $\mathscr{V}'$. The fiber of the Deligne extension over the origin s of $\widetilde{\Delta}$ can be identified with $H_1(X_\infty, \mathbb{C})$, by [22, XI-8] (this identification depends on the choice of a local coordinate on $\widetilde{\Delta}$; we take the coordinate t').
The Hodge flags F^0 on the fibers of $\mathscr{V}'$ glue to a holomorphic subbundle $F^0(\mathscr{V}')^h$ of $(\mathscr{V}')^h$, which extends uniquely to a holomorphic subbundle $F^0\widehat{\mathscr{V}'}$ of $\widehat{\mathscr{V}'}$ [22, 11.10]. Taking fibers at s, we obtain a descending filtration

$$(\widehat{\mathscr{V}'})_s = H_1(X_\infty, \mathbb{C}) \supset (F^0\widehat{\mathscr{V}'})_s \supset \{0\}$$

and this is precisely the limit Hodge filtration on $H_1(X_\infty, \mathbb{C})$ [22, 11.10].
Schmid's construction of the limit mixed Hodge structure works for abstract variations of Hodge structures, which need not necessarily come from geometry. If the variation of Hodge structures comes from the cohomology of a proper and smooth family over S (such as in the case that we are considering), the Deligne extension and the extension of the Hodge bundels can be constructed explicitly using a relative logarithmic de Rham complex associated to a suitable compactification $\overline{f}$. This is Steenbrink's construction [26]. We will not need this approach in this paper.

Theorem 6.2. *We apply the terminology of Section 3.1 to the abelian K-variety A and define in this way the degree d extension K' of K, as well as the torus T and the abelian variety B over $\mathbb{C}$, endowed with a right action of $\mu \cong \mu_d(\mathbb{C})$. There exist canonical μ-equivariant isomorphisms of pure Hodge structures*

$$\begin{aligned}
\mathrm{Gr}^W_0(H_1(X_\infty, \mathbb{Q})) &\cong H_1(T(\mathbb{C}), \mathbb{Q})(-1)\\
\mathrm{Gr}^W_{-1}(H_1(X_\infty, \mathbb{Z})) &\cong H_1(B(\mathbb{C}), \mathbb{Z})\\
\mathrm{Gr}^W_{-2}(H_1(X_\infty, \mathbb{Z})) &\cong H_1(T(\mathbb{C}), \mathbb{Z}).
\end{aligned}$$

Proof. We denote by $\mathbb{C}(S')$ the algebraic closure of the function field $\mathbb{C}(S)$ in K', and we consider the normalization

$$\overline{S}' \to \overline{S}$$

of $\overline{S}$ in $\mathbb{C}(S')$. This is a ramified Galois covering, obtained by taking a d-th root of the local parameter t. Its Galois group is canonically isomorphic to μ.

With abuse of notation, we denote again by s the unique point of the inverse image of s in $\overline{S}'$, and we put $S' = \overline{S}' \setminus \{s\}$. Then

$$f' : X' = X \times_S S' \to S'$$

is a smooth projective family of abelian varieties, and we have a canonical isomorphism

$$A' = A \times_K K' \cong X' \times_{S'} \operatorname{Spec} K'.$$

As we've argued above, the fact that A' has semi-abelian reduction implies that the variation of Hodge structures

$$\mathscr{V}' = \mathscr{V} \times_S S' \cong R^{2g-1} f'_*(\mathbb{Z})(g)$$

has unipotent monodromy around s.
We denote by $\mathcal{X}'$ the Néron model of X' over $\overline{S}'$, and by $\mathcal{A}'$ the Néron model of A' over R', where R' denotes the integral closure of R in K'. Note that there is a canonical isomorphism of R'-schemes

$$\mathcal{A}' \cong \mathcal{X}' \times_{\overline{S}'} \operatorname{Spec} R'.$$

The analytic family of abelian varieties

$$(f')^{\mathrm{an}} : (X')^{\mathrm{an}} \to (S')^{\mathrm{an}}$$

is canonically isomorphic to the Jacobian

$$J(\mathscr{V}') \to (S')^{\mathrm{an}}$$

of the variation of Hodge structures $\mathscr{V}'$ [24, 2.10.1]. We will now explain the relation between the complex semi-abelian variety $(\mathcal{A}')^o_s$ and the limit mixed Hodge structure $H_1(X_\infty, \mathbb{Z})$ of $\mathscr{V}'$ at the point s. To simplify notation, we put $H_C = H_1(X_\infty, C)$ for $C = \mathbb{Z}, \mathbb{Q}, \mathbb{C}$, and we denote by H the mixed Hodge structure

$$(H_{\mathbb{Z}}, W_\bullet H_{\mathbb{Q}}, F^\bullet H_{\mathbb{C}}).$$

By [24, 4.5(i)], $(\mathcal{X}')^{\mathrm{an}}$ is canonically isomorphic to Clemens' Néron model of $\mathscr{V}'$ over $\overline{S}'$; see [6] and [24, 2.5] for a definition. In [24, 2.5], Clemens' Néron model is constructed by gluing copies of the Zucker extension $J^Z_{\overline{S}'}(\mathscr{V}')$ of $\mathscr{V}'$, which is defined in [29] and [24, 2.1]. It follows immediately from this construction that the identity component

$$((\mathcal{X}')^o)^{\mathrm{an}}$$

of Clemens' Néron model is canonically isomorphic to the Zucker extension $J^Z_{\overline{S}'}(\mathscr{V}')$.
The Zucker extension is given explicitly by

$$J^Z_{\overline{S}'}(\mathscr{V}') = j_* \mathscr{V}'_{\mathbb{Z}} \backslash \widehat{\mathscr{V}'} / F^0 \widehat{\mathscr{V}'}$$

where $\widehat{\mathscr{V}'}$ is the Deligne extension of $\mathscr{V}'$ to $\overline{S}'$, j is the open immersion of S' into $\overline{S}'$, and $F^0 \widehat{\mathscr{V}'}$ is the unique extension of the holomorphic vector bundle

$$F^0(\mathscr{V}'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{(S')^{\mathrm{an}}})$$

to a holomorphic subbundle of $\widehat{\mathscr{V}'}$. We can describe the fiber

$$J^{Z}_{\overline{S'}}(\mathscr{V}')_s \cong ((\mathcal{X}')^o_s)^{\mathrm{an}} \cong ((\mathcal{A}')^o_s)^{\mathrm{an}}$$

of $J^{Z}_{\overline{S'}}(\mathscr{V}')$ at s in terms of the mixed Hodge structure H, as follows.

As we've explained above, the fiber of $\widehat{\mathscr{V}'}$ over s is isomorphic to $H_{\mathbb{C}}$, and $F^0(\widehat{\mathscr{V}'})_s$ coincides with the degree zero part of the Hodge filtration on $H_{\mathbb{C}}$. Moreover, the fiber of $j_*\mathscr{V}'_{\mathbb{Z}}$ at s is the $\mathbb{Z}$-module of elements in $H_{\mathbb{Z}}$ that are invariant under the monodromy action of M^d. By definition, the weight filtration on $H_{\mathbb{Q}}$ is the filtration centered at -1 defined by the logarithm N of the unipotent part M_u of M, or, equivalently, the logarithm $N' = dN$ of M^d. Since $(M^d - \mathrm{Id})^2 = 0$, we have $N' = M^d - \mathrm{Id}$ and $(N')^2 = 0$, and we see that

$$(j_*\mathscr{V}'_{\mathbb{Z}})_s = \ker(N') = W_{-1}H_{\mathbb{Z}}.$$

Thus, we find canonical isomorphisms

$$\begin{aligned} ((\mathcal{A}')^o_s)^{\mathrm{an}} &\cong J^{Z}_{\overline{S'}}(\mathscr{V}')_s \\ &\cong W_{-1}H_{\mathbb{Z}}\backslash H_{\mathbb{C}}/F^0H_{\mathbb{C}} \\ &\cong W_{-1}H_{\mathbb{Z}}\backslash W_{-1}H_{\mathbb{C}}/(F^0H_{\mathbb{C}} \cap W_{-1}H_{\mathbb{C}}). \end{aligned}$$

The last isomorphism is obtained as follows: since $\mathrm{Gr}^W_0 H$ is purely of type $(0,0)$, we have

$$F^0\mathrm{Gr}^W_0 H_{\mathbb{C}} = \mathrm{Gr}^W_0 H_{\mathbb{C}} = H_{\mathbb{C}}/W_{-1}H_{\mathbb{C}},$$

so that the morphism

$$W_{-1}H_{\mathbb{C}} \to H_{\mathbb{C}}/F^0H_{\mathbb{C}}$$

induced by the inclusion of $W_{-1}H_{\mathbb{C}}$ in $H_{\mathbb{C}}$ is surjective. Its kernel is precisely $F^0H_{\mathbb{C}} \cap W_{-1}H_{\mathbb{C}}$.

By [9, 10.1], we have an extension

$$0 \to J(\mathrm{Gr}^W_{-2}H) \to ((\mathcal{A}')^o_s)^{\mathrm{an}} \to J(\mathrm{Gr}^W_{-1}H) \to 0 \tag{6.4}$$

where

$$J(\mathrm{Gr}^W_{-2}H) = \mathrm{Gr}^W_{-2}H_{\mathbb{C}}/\mathrm{Gr}^W_{-2}H_{\mathbb{Z}}$$

is a torus, and

$$J(\mathrm{Gr}^W_{-1}H) = H_{\mathbb{Z}}\backslash \mathrm{Gr}^W_{-1}H_{\mathbb{C}}/F^0\mathrm{Gr}^W_{-1}H_{\mathbb{C}}$$

an abelian variety, because the Hodge structure $\mathrm{Gr}^W_{-1}H$ is polarizable. By [9, 10.1.3.3], the extension (6.4) must be the analytification of the Chevalley decomposition

$$0 \to T \to (\mathcal{A}')^o_s \to B \to 0.$$

Hence, there exist canonical isomorphisms of pure Hodge structures

$$\mathrm{Gr}^W_{-1}(H) \cong H_1(B(\mathbb{C}),\mathbb{Z}), \tag{6.5}$$

$$\mathrm{Gr}^W_{-2}(H) \cong H_1(T(\mathbb{C}),\mathbb{Z}). \tag{6.6}$$

Moreover, by definition of the weight filtration on $H_{\mathbb{Q}}$, the operator N defines a μ-equivariant isomorphism of $\mathbb{Q}$-Hodge structures

$$\mathrm{Gr}^W_0(H)\otimes_{\mathbb{Z}}\mathbb{Q} \to \mathrm{Gr}^W_{-2}(H)(-1)\otimes_{\mathbb{Z}}\mathbb{Q}.$$

It remains to show that the isomorphisms (6.5) and (6.6) are μ-equivariant. It is enough to prove that the Galois action of μ on

$$\mathscr{V}' \to S'$$

extends analytically to the Zucker extension

$$J^Z_{\overline{S}'}(\mathscr{V}') \to \overline{S}'$$

in such a way that the action of the canonical generator of $\mu = \mu_d(\mathbb{C})$ on

$$J^Z_{\overline{S}'}(\mathscr{V}')_s = W_{-1}H_{\mathbb{Z}} \backslash H_{\mathbb{C}}/F^0 H_{\mathbb{C}}$$

coincides with the semi-simple part M_s of the monodromy action. This follows easily from the constructions. □

6.3. Multiplicity functions and limit mixed Hodge structure.

Theorem 6.3. *We keep the notations of Section 6.1.*

(1) *The potential toric rank* $t_{\mathrm{pot}}(A)$ *is equal to the largest integer* α *such that* $\exp(2\pi c(A)i)$ *is an eigenvalue of* M_s *on* $\mathrm{Gr}^W_{g+\alpha}H^g(X_\infty,\mathbb{Q})$.

(2) *The Jordan form of* M_s *on*

$$\begin{array}{lll}
(\mathrm{Gr}^W_{-1}H_1(X_\infty,\mathbb{Q}))^{1,0} & \textit{is} & \mathrm{Jord}(m^{\mathrm{ab}}_A,0),\\
(\mathrm{Gr}^W_{-1}H_1(X_\infty,\mathbb{Q}))^{0,1} & \textit{is} & \mathrm{Jord}(\breve{m}^{\mathrm{ab}}_A,0),\\
\mathrm{Gr}^W_{-2}H_1(X_\infty,\mathbb{Q}) & \textit{is} & \mathrm{Jord}(m^{\mathrm{tor}}_A,0),\\
\mathrm{Gr}^W_{0}H_1(X_\infty,\mathbb{Q}) & \textit{is} & \mathrm{Jord}(m^{\mathrm{tor}}_A,0).
\end{array}$$

Proof. We denote by M_u the unipotent part of the monodromy operator M, and by N its logarithm. By definition, the weight filtration on $H^g(X_\infty,\mathbb{Q})$ is the filtration with center g associated to the nilpotent operator N. Thus (1) follows from Proposition 5.2, Theorem 5.9 and the isomorphism (6.1).
Point (2) follows from Theorem 6.2 and the canonical μ-equivariant isomorphisms

$$\begin{array}{lll}
H_1(B(\mathbb{C}),\mathbb{C})^{1,0} & \cong & \mathrm{Lie}(B)\\
H_1(B(\mathbb{C}),\mathbb{C})^{0,1} & \cong & \mathrm{Lie}(B^\vee)^\vee\\
H_1(T(\mathbb{C}),\mathbb{C}) & \cong & \mathrm{Lie}(T)
\end{array}$$

(see [19, pp. 4 and 86] for the dual isomorphisms on the level of cohomology). □

References

[1] *Schémas en groupes. I: Propriétés générales des schémas en groupes.* Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 151. Springer-Verlag, Berlin, 1970.

[2] *Groupes de monodromie en géométrie algébrique. I.* Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim. Lecture Notes in Mathematics, Vol. 288. Springer-Verlag, Berlin, 1972.

[3] *Groupes de monodromie en géométrie algébrique. II*, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz. Lecture Notes in Mathematics, Vol. 340. Springer-Verlag, Berlin, 1973.

[4] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, 1990.

[5] C.-L. Chai. Néron models for semiabelian varieties: congruence and change of base field. *Asian J. Math.*, 4(4):715–736, 2000.

[6] H. Clemens. The Néron model for families of intermediate Jacobians acquiring "algebraic" singularities. *Inst. Hautes Études Sci. Publ. Math.* 58:5–18, 1983.

[7] P. Deligne. *Équations différentielles à points singuliers réguliers*, volume 163 of *Lecture Notes in Mathematics*. Berlin: Springer-Verlag, 1970.

[8] P. Deligne. Théorie de Hodge: II. *Inst. Hautes Études Sci. Publ. Math.*, 40:5–57, 1971.

[9] P. Deligne. Théorie de Hodge: III. *Inst. Hautes Études Sci. Publ. Math.*, 44:5–77, 1974.

[10] P. Deligne. La conjecture de Weil: II. *Inst. Hautes Études Sci. Publ. Math.*, 52:137-252, 1980.

[11] A. Dimca. *Sheaves in topology.* Universitext. Springer-Verlag, Berlin, 2004.

[12] B. Edixhoven. Néron models and tame ramification. *Compos. Math.*, 81:291–306, 1992.

[13] R. Hain. Periods of limit mixed Hodge structures. In: *Current developments in mathematics*, pages 113-133, Int. Press, Somerville, MA, 2003.

[14] L.H. Halle and J. Nicaise. Motivic zeta functions of abelian varieties, and the monodromy conjecture. *Adv. Math.*, 227:610–653, 2011.

[15] M. Kashiwara. Holonomic systems of linear differential equations with regular singularities and related topics in topology. In: *Algebraic varieties and analytic varieties, Proc. Symp., Tokyo 1981*, volume 1 of *Adv. Stud. Pure Math.*, pages 49–54, North-Holland, Amsterdam-New York, 1983.

[16] F. Loeser. Fonctions d'Igusa p-adiques et polynômes de Bernstein. *Am. J. of Math.*, 110:1–22, 1988.

[17] B. Malgrange. Polynômes de Bernstein-Sato et cohomologie évanescente. *Astérisque*, 101/102:243–267, 1983.

[18] J.S. Milne. *Abelian varieties.* In: G. Cornell and J.H. Silverman (eds.), *Arithmetic geometry.* Springer Verlag, 1986.

[19] D. Mumford. *Abelian varieties. With appendices by C. P. Ramanujam and Yuri Manin. 2nd ed.*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics.* London: Oxford University Press, 1974.

[20] J. Nicaise. An introduction to p-adic and motivic zeta functions and the monodromy conjecture. In: G. Bhowmik, K. Matsumoto and H. Tsumura (eds.), *Algebraic and analytic aspects of zeta functions and L-functions*, volume 21 of *MSJ Memoirs*, Mathematical Society of Japan, pages 115-140, 2010.

[21] T. Oda. The first de Rham cohomology group and Dieudonné modules. *Ann. Sci. École Norm. Sup. (4)*, 2(1):63-135, 1969.

[22] C. Peters and J.H.M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Berlin: Springer, 2008.

[23] B. Rodrigues. On the monodromy conjecture for curves on normal surfaces. *Math. Proc. Camb. Philos. Soc.*, 136(2):313–324, 2004.

[24] M. Saito. Admissible normal functions. *J. Algebraic Geom.* 5, 235–276, 1996.

[25] W. Schmid. Variation of Hodge structure: the singularities of the period mapping.*Invent. Math.* 22:211-319, 1973.

[26] J.H.M. Steenbrink. Limits of Hodge structures. *Invent. Math.*, 31:229–257, 1976.

[27] J.H.M. Steenbrink. Mixed Hodge structure on the vanishing cohomology. In: *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, pages 525–563. Alphen aan den Rijn: Sijthoff and Noordhoff, 1977.

[28] J. Tate and F. Oort. Group schemes of prime order. *Ann. Sci. École Norm. Sup.* (4) 3:1-21, 1970.

[29] S. Zucker. Generalized intermediate Jacobians and the theorem on normal functions. *Invent. Math.* 33(3):185–222, 1976.

Lars Halvard Halle
Institut für
Algebraische Geometrie
Gottfried-Wilhelm-Leibniz-
Universität
30167 Hannover
Germany
and
Matematisk Institutt
Universitetet i Oslo
Postboks 1053
Blindern
0316 Oslo
Norway
larshhal@math.uio.no

Johannes Nicaise
KULeuven
Department of Mathematics
Celestijnenlaan 200B
3001 Heverlee
Belgium
johannes.nicaise@
wis.kuleuven.be