The Singularity Category of an Algebra with Radical Square Zero

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Abstract. To an artin algebra with radical square zero, a regular algebra in the sense of von Neumann and a family of invertible bimodules over the regular algebra are associated. These data describe completely, as a triangulated category, the singularity category of the artin algebra. A criterion on the Hom-finiteness of the singularity category is given in terms of the valued quiver of the artin algebra.

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1. Introduction

Let R be a commutative artinian ring. All algebras, categories and functors are R-linear. We recall that an R-linear category is *Hom-finite* provided that all the Hom sets are finitely generated R-modules.

Let A be an artin R-algebra. Denote by A-mod the category of finitely generated left A-modules, and by $\mathbf{D}^b(A\text{-mod})$ the bounded derived category. Following [14], the *singularity category* $\mathbf{D}_{\rm sg}(A)$ is the quotient triangulated category of $\mathbf{D}^b(A\text{-mod})$ with respect to the full subcategory formed by perfect complexes; see also [3, 12, 10, 15, 2] and [13]. Here, we recall that a complex in $\mathbf{D}^b(A\text{-mod})$ is *perfect* provided that it is isomorphic to a bounded complex consisting of finitely generated projective modules.

The singularity category measures the homological singularity of an algebra in the sense that an algebra A has finite global dimension if and only if its singularity category $\mathbf{D}_{\rm sg}(A)$ vanishes. In the meantime, the singularity category captures the stable homological features of an algebra ([3]). A fundamental result of Buchweitz and Happel states that for a Gorenstein algebra A, the

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singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is triangle equivalent to the stable category of (maximal) Cohen-Macaulay A-modules ([3, 10]). This implies in particular that the singularity category of a Gorenstein algebra is Hom-finite and has Auslander-Reiten triangles. We point out that Buchweitz and Happel's result specializes to Rickard's result ([15]) on self-injective algebras. However, for non-Gorenstein algebras, not much is known about their singularity categories ([4]).

Our aim is to describe the singularity category of an algebra with radical square zero. We point out that such algebras are usually non-Gorenstein ([5]). In what follows, we describe the results in this paper.

We denote by $\mathbf{r}$ the Jacobson radical of A. The algebra A is said to be with *radical square zero* provided that $\mathbf{r}^2 = 0$. In this case, $\mathbf{r}$ has a natural $A/\mathbf{r}$-$A/\mathbf{r}$-bimodule structure. Set $\mathbf{r}^{\otimes 0} = A/\mathbf{r}$ and $\mathbf{r}^{\otimes i+1} = \mathbf{r} \otimes_{A/\mathbf{r}} (\mathbf{r}^{\otimes i})$ for $i \geq 0$. Then there are obvious algebra homomorphisms $\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i}) \to \mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i+1})$ induced by $\mathbf{r} \otimes_{A/\mathbf{r}} -$. We denote by $\Gamma(A)$ the direct limit of this chain of algebra homomorphisms. It is a regular algebra ([7, 8]) in the sense of von Neumann. We call $\Gamma(A)$ the *associated regular algebra* of A. In most cases, the algebra $\Gamma(A)$ is not semisimple.

For $n \in \mathbb{Z}$ and $i \geq \max\{0, n\}$, $\mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i}, \mathbf{r}^{\otimes i-n})$ has a natural $\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i-n})$-$\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i})$-bimodule structure. Set $K^n(A)$ to be the direct limit of the chain of maps $\mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i}, \mathbf{r}^{\otimes i-n}) \to \mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i+1}, \mathbf{r}^{\otimes i+1-n})$, which are induced by $\mathbf{r} \otimes_{A/\mathbf{r}} -$. Then $K^n(A)$ is naturally a $\Gamma(A)$-$\Gamma(A)$-bimodule for each $n \in \mathbb{Z}$. Observe that $K^0(A) = {}_{\Gamma(A)}\Gamma(A)_{\Gamma(A)}$ as bimodules, and that composition of maps induces $\Gamma(A)$-$\Gamma(A)$-bimodule morphisms $\phi^{n,m} \colon K^n(A) \otimes_{\Gamma(A)} K^m(A) \to K^{n+m}(A)$ for all $n, m \in \mathbb{Z}$. These bimodules $K^n(A)$ are called the *associated bimodules* of A.

Recall that for an algebra Γ, a Γ-Γ-bimodule K is *invertible* provided that the functor $K \otimes_\Gamma -$ induces an auto-equivalence on the category of left Γ-modules.

Theorem A. *Let A be an artin algebra with radical square zero. Use the above notation. Then the associated $\Gamma(A)$-$\Gamma(A)$-bimodules $K^n(A)$ are invertible and the maps $\phi^{n,m}$ are isomorphisms of bimodules.*

Since the algebra $\Gamma(A)$ is regular, the category $\mathrm{proj}\ \Gamma(A)$ of finitely generated right projective $\Gamma(A)$-modules is a semisimple abelian category. The invertible bimodule $K^1(A)$ induces an auto-equivalence

$$\Sigma_A = - \otimes_{\Gamma(A)} K^1(A) \colon \mathrm{proj}\ \Gamma(A) \longrightarrow \mathrm{proj}\ \Gamma(A).$$

We observe that the category $\mathrm{proj}\ \Gamma(A)$ has a unique triangulated structure with Σ_A its shift functor; see Lemma 3.4. This unique triangulated category is denoted by $(\mathrm{proj}\ \Gamma(A), \Sigma_A)$.

The following result describes the singularity category of an artin algebra with radical square zero, which is based on a result by Keller and Vossieck ([12]).

Theorem B. *Let A be an artin algebra with radical square zero. Use the above notation. Then we have a triangle equivalence*

$$\mathbf{D}_{\mathrm{sg}}(A) \simeq (\mathrm{proj}\, \Gamma(A), \Sigma_A).$$

We are interested in the Hom-finiteness of singularity categories. For this, we recall the notion of valued quiver of an artin algebra A. Choose a complete set of representatives of pairwise non-isomorphic simple A-modules $\{S_1, S_2, \cdots, S_n\}$. Set $\Delta_i = \mathrm{End}_A(S_i)$; they are division algebras. Observe that $\mathrm{Ext}^1_A(S_i, S_j)$ has a natural Δ_j-Δ_i-bimodule structure. The *valued quiver* Q_A of A is defined as follows: its vertex set is $\{S_1, S_2, \cdots, S_n\}$, here we identify each simple module S_i with its isoclass; there is an arrow from S_i to S_j whenever $\mathrm{Ext}^1_A(S_i, S_j) \neq 0$, in which case the arrow is endowed with a *valuation* $(\dim_{\Delta_j} \mathrm{Ext}^1_A(S_i, S_j), \dim_{{\Delta_i}^{\mathrm{op}}} \mathrm{Ext}^1_A(S_i, S_j))$; here ${\Delta_i}^{\mathrm{op}}$ denotes the opposite algebra of Δ_i. We say that the valuation of Q_A is *trivial* provided that all the valuations are $(1, 1)$. Recall that a vertex in a valued quiver is a source (*resp.* sink) provided that there is no arrows ending (*resp.* starting) at it. For a valued quiver, to adjoin a (new) source (*resp.* sink) is to add a vertex together with some valued arrows starting (*resp.* ending) at this vertex. For details, we refer to [1, III.1].

The following result characterizes when the singularity category is Hom-finite, using the valued quivers.

Theorem C. *Let A be an artin algebra with radical square zero. Then the following statements are equivalent:*

(1) *the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is Hom-finite;*
(2) *the associated regular algebra $\Gamma(A)$ is semisimple;*
(3) *the valued quiver Q_A is obtained from a disjoint union of oriented cycles with the trivial valuation by repeatedly adjoining sources or sinks.*

The paper is structured as follows. In Section 2, we collect some facts on singularity categories and recall a basic result of Keller and Vossieck. We prove Theorem A and B in Section 3, where an explicit example is presented. In Section 4, we prove that one-point extensions and coextensions of algebras preserve their singularity categories. We introduce the notion of cyclicization of an algebra, which is used in the proof of Theorem C in Section 5.

For artin algebras, we refer to [1]. For triangulated categories, we refer to [11] and [9].

2. Preliminaries

In this section, we collect some facts on singularity categories of artin algebras. We recall a basic result due to Keller and Vossieck ([12]), which is applied to Ω^∞-finite algebras.

Let A be an artin algebra over a commutative artinian ring R. Recall that A-mod denotes the category of finitely generated left A-modules. We denote by A-proj the full subcategory formed by projective modules, and by A-$\underline{\text{mod}}$ the stable category of A-mod modulo projective modules ([1, p.104]). The morphism space $\underline{\text{Hom}}_A(M,N)$ of two modules M and N in A-$\underline{\text{mod}}$ is defined to be $\text{Hom}_A(M,N)/\mathbf{p}(M,N)$, where $\mathbf{p}(M,N)$ denotes the R-submodule formed by morphisms that factor through projective modules.

Recall that for an A-module M, its syzygy $\Omega(M)$ is the kernel of its projective cover $P \to M$. This gives rise to the *syzygy functor* $\Omega\colon A\text{-}\underline{\text{mod}} \to A\text{-}\underline{\text{mod}}$ ([1, p.124]). Set $\Omega^0(M) = M$ and $\Omega^{i+1}(M) = \Omega^i(\Omega(M))$ for $i \geq 0$. Denote by $\Omega^i(A\text{-mod})$ the full subcategory of A-mod formed by modules of the form $P \oplus \Omega^i(M)$ for some module M and projective module P. Then an A-module X belongs to $\Omega^i(A\text{-mod})$ if and only if there is an exact sequence $0 \to X \to P^{1-i} \to \cdots \to P^{-1} \to P^0$ with each P^j projective.

Recall that $\mathbf{D}^b(A\text{-mod})$ denotes the bounded derived category of A-mod, whose shift functor is denoted by [1]. For $n \in \mathbb{Z}$, $[n]$ denotes the n-th power of [1]. The module category A-mod is viewed as a full subcategory of $\mathbf{D}^b(A\text{-mod})$ by identifying an A-module with the corresponding stalk complex concentrated at degree zero ([11, Proposition I.4.3]). Recall that a complex in $\mathbf{D}^b(A\text{-mod})$ is *perfect* provided that it is isomorphic to a bounded complex consisting of projective modules; these complexes form a full triangulated subcategory $\text{perf}(A)$. Recall that, via an obvious functor, $\text{perf}(A)$ is triangle equivalent to the bounded homotopy category $\mathbf{K}^b(A\text{-proj})$; compare [3, 1.1-1.2].

Following [14], we call the quotient triangulated category

$$\mathbf{D}_{\text{sg}}(A) = \mathbf{D}^b(A\text{-mod})/\text{perf}(A)$$

the *singularity category* of A. Denote by $q\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}_{\text{sg}}(A)$ the quotient functor.

The following two results are known; compare [14, Lemma 1.11] and [3, Lemma 2.2.2].

Lemma 2.1. *Let $X^\bullet$ be a complex in $\mathbf{D}_{\text{sg}}(A)$ and $n_0 > 0$. Then for any n large enough, there exists a module M in $\Omega^{n_0}(A\text{-mod})$ such that $X^\bullet \simeq q(M)[n]$.*

Proof. Take a quasi-isomorphism $P^\bullet \to X^\bullet$ with $P^\bullet$ a bounded above complex of projective modules ([11, Lemma I.4.6]). Take $n \geq n_0$ such that $H^i(X^\bullet) = 0$ for all $i < n_0 - n$, where $H^i(X^\bullet)$ denotes the i-th cohomology of $X^\bullet$. Consider the good truncation $\sigma^{\geq -n}P^\bullet = \cdots \to 0 \to M \to P^{1-n} \to P^{2-n} \to \cdots$ of $P^\bullet$, which is quasi-isomorphic to $P^\bullet$. Then the cone of the obvious chain map $\sigma^{\geq -n}P^\bullet \to M[n]$ is perfect, which becomes an isomorphism in $\mathbf{D}_{\text{sg}}(A)$. This shows that $X^\bullet \simeq q(M)[n]$. We observe that M lies in $\Omega^{n_0}(A\text{-mod})$. $\square$

Lemma 2.2. *Let $0 \to M \to P^{1-n} \to \cdots \to P^0 \to N \to 0$ be an exact sequence with each P^i projective. Then we have an isomorphism $q(N) \simeq q(M)[n]$ in $\mathbf{D}_{\text{sg}}(A)$. In particular, for an A-module M, we have a natural isomorphism $q(\Omega^n(M)) \simeq q(M)[-n]$.*

Proof. The stalk complex N is quasi-isomorphic to $\cdots \to 0 \to M \to P^{1-n} \to \cdots \to P^0 \to 0 \to \cdots$. This gives rise to a morphism $N \to M[n]$ in $\mathbf{D}^b(A\text{-mod})$, whose cone is $\cdots \to 0 \to P^{1-n} \to \cdots \to P^0 \to 0 \to \cdots$ with P^0 at degree -1; it is perfect. Then the morphism $N \to M[n]$ becomes an isomorphism in $\mathbf{D}_{\rm sg}(A)$. $\square$

Consider the composite $q' \colon A\text{-mod} \hookrightarrow \mathbf{D}^b(A\text{-mod}) \xrightarrow{q} \mathbf{D}_{\rm sg}(A)$; it vanishes on projective modules. Then it induces uniquely a functor $A\text{-}\underline{\rm mod} \to \mathbf{D}_{\rm sg}(A)$, which is still denoted by q'. Then Lemma 2.2 yields, for each $n \geq 0$, the following commutative diagram

$$\begin{tikzcd} A\text{-}\underline{\rm mod} \arrow[r, "\Omega^n"] \arrow[d, "q'"] & A\text{-}\underline{\rm mod} \arrow[d, "q'"] \\ \mathbf{D}_{\rm sg}(A) \arrow[r, "{[-n]}"] & \mathbf{D}_{\rm sg}(A). \end{tikzcd}$$

We refer to [3, Lemma 2.2.2] for a similar statement.
The functor q' induces a natural map

$$\Phi^0 \colon \underline{\rm Hom}_A(M, N) \to {\rm Hom}_{\mathbf{D}_{\rm sg}(A)}(q(M), q(N))$$

for any modules M, N. Let $n \geq 1$. Lemma 2.2 yields a natural isomorphism $\theta_M \colon q(M) \xrightarrow{\sim} q(\Omega^n(M))[n]$. Then we have a map

$$\Phi^n \colon \underline{\rm Hom}_A(\Omega^n(M), \Omega^n(N)) \to {\rm Hom}_{\mathbf{D}_{\rm sg}(A)}(q(M), q(N))$$

given by $\Phi^n(f) = (\theta^n_N)^{-1} \circ (\Phi^0(f)[n]) \circ \theta^n_M$.
Consider the chain of maps

$$\underline{\rm Hom}_A(\Omega^n(M), \Omega^n(N)) \to \underline{\rm Hom}_A(\Omega^{n+1}(M), \Omega^{n+1}(N))$$

induced by the syzygy functor. It is routine to verify that Φ^n are compatible with this chain of maps. Then we have an induced map

$$\Phi \colon \varinjlim \underline{\rm Hom}_A(\Omega^n(M), \Omega^n(N)) \longrightarrow {\rm Hom}_{\mathbf{D}_{\rm sg}(A)}(q(M), q(N)).$$

We recall the following basic result.

Proposition 2.3. (Keller-Vossieck) *Let M, N be A-modules as above. Then the map Φ is an isomorphism.*

Proof. The statement follows from [12, Exemple 2.3]. We refer to [2, Corollary 3.9(1)] for a detailed proof. $\square$

Recall that an additive category $\mathcal{A}$ is *idempotent split* provided that each idempotent $e \colon X \to X$ splits, that is, it admits a factorization $X \xrightarrow{u} Y \xrightarrow{v} X$ with $u \circ v = {\rm Id}_Y$. For example, a Krull-Schmidt category is idempotent split ([6, Appendix A]). In particular, for an artin algebra A, the stable category $A\text{-}\underline{\rm mod}$ is idempotent split.

Corollary 2.4. *The singularity category $\mathbf{D}_{\rm sg}(A)$ of an artin algebra A is idempotent split.*

Proof. By Lemma 2.1 it suffices to show that for each module M, an idempotent $e\colon q(M) \to q(M)$ splits in $\mathbf{D}_{\rm sg}(A)$. The above proposition implies that for a large n, there is an idempotent $e^n\colon \Omega^n(M) \to \Omega^n(M)$ in A-$\underline{\rm mod}$ which is mapped by Φ to e. The idempotent e^n splits as $\Omega^n(M) \stackrel{u}{\to} Y \stackrel{v}{\to} \Omega^n(M)$ with $u \circ v = {\rm Id}_Y$ in A-$\underline{\rm mod}$. Then the idempotent e splits as $q(M) \stackrel{(q(u)[n])\circ\theta_M^n}{\longrightarrow} q(Y)[n] \stackrel{(\theta_M^n)^{-1}\circ(q(v)[n])}{\longrightarrow} q(M)$. $\square$

Let $\mathcal{A}$ be an additive category. For a subcategory $\mathcal{C}$, denote by add $\mathcal{C}$ the full subcategory of $\mathcal{A}$ formed by direct summands of finite direct sums of objects in $\mathcal{C}$. For any algebra Γ, denote by proj Γ the category of finitely generated right projective Γ-modules. We observe that proj $\Gamma = $ add Γ_Γ.

An artin algebra A is called Ω^∞*-finite* provided that there exists a module E and $n \geq 0$ such that $\Omega^n(A\text{-mod}) \subseteq$ add $(A \oplus E)$. In this case, we call E an Ω^∞*-generator* of A.

Proposition 2.5. *Let A be an Ω^∞-finite algebra with an Ω^∞-generator E. Then we have $\mathbf{D}_{\rm sg}(A) =$ add $q(E)$. Consequently, we have an equivalence of categories*

$$\mathbf{D}_{\rm sg}(A) \simeq \text{proj } {\rm End}_{\mathbf{D}_{\rm sg}(A)}(q(E)),$$

which sends $q(E)$ to ${\rm End}_{\mathbf{D}_{\rm sg}(A)}(q(E))$.

Proof. Observe that $\Omega^{n+1}(A\text{-mod}) \subseteq \Omega^n(A\text{-mod})$. Then we may assume that add $(A \oplus E) \supseteq$ add $\Omega^{n_0}(A\text{-mod}) =$ add $\Omega^{n_0+1}(A\text{-mod}) = \cdots$ for n_0 large enough.

For the first statement, it suffices to show that each object $X^\bullet$ in $\mathbf{D}_{\rm sg}(A)$ belongs to add $q(E)$. By Lemma 2.1, $X^\bullet \simeq q(M)[n_1]$ for a module $M \in \Omega^{n_0}(A\text{-mod})$ and $n_1 > 0$. Since add $\Omega^{n_0}(A\text{-mod}) =$ add $\Omega^{n_0+n_1}(A\text{-mod})$, we may assume that $M \oplus N \in \Omega^{n_0+n_1}(A\text{-mod})$ for some module N. Take an exact sequence $0 \to M \oplus N \to P^{1-n_1} \to \cdots \to P^0 \to L \to 0$ with each P^i projective and $L \in \Omega^{n_0}(A\text{-mod})$. By Lemma 2.2, $q(L) \simeq q(M \oplus N)[n_1]$ and then $X^\bullet$ is a direct summand of $q(L)$. Observing that $L \in$ add $(A \oplus E)$, we are done with the first statement.

The second statement follows from the projectivization; see [1, Proposition II.2.1]. The functor is given by ${\rm Hom}_{\mathbf{D}_{\rm sg}(A)}(q(E), -)$. We point out that Corollary 2.4 is needed here. $\square$

3. Algebras with radical square zero

In this section, we study the singularity category of an algebra with radical square zero, and prove Theorem A and B. An explicit example is given at the end.

Let A be an artin algebra. Denote by $\mathbf{r}$ the Jacobson radical of A. The algebra A is said to be with *radical square zero* provided that $\mathbf{r}^2 = 0$. In this case, $\mathbf{r}$ has an $A/\mathbf{r}$-$A/\mathbf{r}$-bimodule structure, which is induced from the multiplication of A.

Denote by A-$\underline{\mathrm{ssmod}}$ the full subcategory of A-$\underline{\mathrm{mod}}$ formed by semisimple modules. We observe that $\mathbf{r}\otimes_{A/\mathbf{r}} S = 0$ for a simple projective module S. Then the functor $\mathbf{r}\otimes_{A/\mathbf{r}} -\colon A\text{-}\underline{\mathrm{ssmod}} \to A\text{-}\underline{\mathrm{ssmod}}$ is well defined. We observe that the syzygy functor Ω sends semisimple modules to semisimple modules, and then we have the restricted functor $\Omega\colon A\text{-}\underline{\mathrm{ssmod}} \to A\text{-}\underline{\mathrm{ssmod}}$.
The following result is implicitly contained in the proof of [1, Lemma X.2.1].

LEMMA 3.1. *There is a natural isomorphism* $\Omega \simeq \mathbf{r}\otimes_{A/\mathbf{r}} -$ *of functors on* A-$\underline{\mathrm{ssmod}}$.

Proof. Let X be a semisimple A-module. Take a projective cover $P \to X$. Tensoring P with the natural exact sequence of A-A-bimodules $0 \to \mathbf{r} \to A \to A/\mathbf{r} \to 0$ yields $\Omega(X) \simeq \mathbf{r}\otimes_A P$. Using isomorphisms $\mathbf{r}\otimes_A P \simeq \mathbf{r}\otimes_{A/\mathbf{r}} P/\mathbf{r}P$ and $P/\mathbf{r}P \simeq X$, we get an isomorphism $\Omega(X) \simeq \mathbf{r}\otimes_{A/\mathbf{r}} X$. It is routine to verify that this isomorphism is natural in X. □

Recall that an algebra Γ is *regular* in the sense of von Neumann provided that for each element a there exists a' such that $aa'a = a$. For example, a semisimple algebra is regular. Then a direct limit of semisimple algebras is regular. For details, we refer to [7, Theorem and Definition 11.24].
Recall that for an artin algebra A with radical square zero, there is a chain of algebra homomorphisms $\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i}) \to \mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i+1})$ induced by $\mathbf{r}\otimes_{A/\mathbf{r}} -$. Here, $\mathbf{r}^{\otimes 0} = A/\mathbf{r}$ and $\mathbf{r}^{\otimes i+1} = \mathbf{r}\otimes_{A/\mathbf{r}} (\mathbf{r}^{\otimes i})$. We set $\Gamma(A)$ to be the direct limit of this chain. Since each algebra $\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i})$ is semisimple, the algebra $\Gamma(A)$ is regular. It is called the *associated regular algebra* of A. We refer to [8, 19.26B, Example] for a related construction.
We recall the *associated* $\Gamma(A)$-$\Gamma(A)$-*bimodules* $K^n(A)$ of A, $n \in \mathbb{Z}$. For $i \geq \max\{0, n\}$, $\mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i}, \mathbf{r}^{\otimes i-n})$ has a natural $\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i-n})$-$\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i})$-bimodule structure. Consider a chain of maps $\mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i}, \mathbf{r}^{\otimes i-n}) \to \mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i+1}, \mathbf{r}^{\otimes i+1-n})$, which are induced by $\mathbf{r}\otimes_{A/\mathbf{r}} -$, and define $K^n(A)$ to be its direct limit. Then $K^n(A)$ is naturally a $\Gamma(A)$-$\Gamma(A)$-bimodule for each $n \in \mathbb{Z}$. Observe that $K^0(A) = {}_{\Gamma(A)}\Gamma(A)_{\Gamma(A)}$ as $\Gamma(A)$-$\Gamma(A)$-bimodules.

PROPOSITION 3.2. *Let A be an artin algebra with radical square zero. Then there is a natural isomorphism*

$$K^n(A) \simeq \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}), q(A/\mathbf{r})[n])$$

for each $n \in \mathbb{Z}$.

Proof. Consider the case $n \leq 0$ first. In this case, by Lemmas 2.2 and 3.1 we have $q(A/\mathbf{r})[n] \simeq q(\Omega^{-n}(A/\mathbf{r})) \simeq q(\mathbf{r}^{\otimes -n})$. Then Proposition 2.3 yields an isomorphism $\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}), q(A/\mathbf{r})[n]) \simeq \varinjlim \underline{\mathrm{Hom}}_A(\Omega^i(A/\mathbf{r}), \Omega^i(\mathbf{r}^{\otimes -n}))$. By Lemma 3.1 again we have $\Omega^i(A/\mathbf{r}) \simeq \mathbf{r}^{\otimes i}$ and $\Omega^i(\mathbf{r}^{\otimes -n}) = \mathbf{r}^{\otimes i-n}$. Then we have a surjective map $\psi\colon K^n(A) \to \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}), q(A/\mathbf{r})[n])$. On the other hand, every morphism $f\colon \mathbf{r}^{\otimes i} \to \mathbf{r}^{\otimes i-n}$ that is zero in A-$\underline{\mathrm{mod}}$ necessarily factors through a semisimple projective module. However, the functor $\mathbf{r}\otimes_{A/\mathbf{r}} -$ vanishes on semisimple projective modules. Then $\mathbf{r}\otimes_{A/\mathbf{r}} f$ is zero. This forces that ψ is injective. We are done in this case.

For the case $n > 0$, we observe that $\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}), q(A/\mathbf{r})[n])$ is isomorphic to $\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r})[-n], q(A/\mathbf{r}))$, and by the same argument as above, it is isomorphic to $\varinjlim \underline{\mathrm{Hom}}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i+n}, \mathbf{r}^{\otimes i})$. Then we get a surjective map $K^n(A) \to \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}), q(A/\mathbf{r})[n])$. Similarly as above, we have that this map is injective. $\square$

Remark 3.3. In the case $n = 0$, the above isomorphism is an isomorphism $K^0(A) = \Gamma(A) \simeq \mathrm{End}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}))$ of algebras. Then for an arbitrary n, the above isomorphism becomes an isomorphism of $\Gamma(A)$-$\Gamma(A)$-bimodules.

Recall that an abelian category $\mathcal{A}$ is *semisimple* provided that each short exact sequence splits. For example, for a regular algebra Γ, the category proj Γ of finitely generated right projective Γ-modules is a semisimple abelian category. Here, we use the fact that all finitely presented Γ-modules are projective; see [7, Theorem and Definition 11.24(a)].
The following observation is well known.

Lemma 3.4. *Let $\mathcal{A}$ be a semisimple abelian category, and let Σ be an auto-equivalence on $\mathcal{A}$. Then there is a unique triangulated structure on $\mathcal{A}$ with Σ the shift functor.*

The obtained triangulated category in this lemma will be denoted by $(\mathcal{A}, \Sigma)$.

Proof. We use the fact that each morphism in $\mathcal{A}$ is isomorphic to a direct sum of morphisms of the forms $K \to 0$, $I \stackrel{\mathrm{Id}_I}{\to} I$ and $0 \to C$. Then all possible triangles are a direct sum of the following trivial triangles $K \to 0 \to \Sigma(K) \stackrel{\mathrm{Id}_{\Sigma(K)}}{\to} \Sigma(K)$, $I \stackrel{\mathrm{Id}_I}{\to} I \to 0 \to \Sigma(I)$ and $0 \to C \stackrel{\mathrm{Id}_C}{\to} C \to \Sigma(0)$. $\square$

Proposition 3.5. *Let A be an artin algebra with radical square zero, and let $\Gamma(A)$ be its associated regular algebra. Then there is a triangle equivalence*

$$\Psi\colon \mathbf{D}_{\mathrm{sg}}(A) \simeq (\mathrm{proj}\ \Gamma(A), \Sigma)$$

for some auto-equivalence Σ on proj $\Gamma(A)$*, which sends $q(A/\mathbf{r})$ to $\Gamma(A)$.*

Proof. We observe that for any A-module M, its syzygy $\Omega(M)$ is semisimple. Hence we have $\Omega^1(A\text{-mod}) \subseteq \mathrm{add}\ (A \oplus A/\mathbf{r})$. We apply Proposition 2.5 to obtain an equivalence of categories $\mathbf{D}_{\mathrm{sg}}(A) \simeq \mathrm{proj}\ \mathrm{End}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}))$. By Proposition 3.2 this yields an equivalence of categories $\mathbf{D}_{\mathrm{sg}}(A) \simeq \mathrm{proj}\ \Gamma(A)$.
By transport of structures, the shift functor [1] on $\mathbf{D}_{\mathrm{sg}}(A)$ corresponds to an auto-equivalence Σ on proj $\Gamma(A)$, and then proj $\Gamma(A)$ becomes a triangulated category. However, by Lemma 3.4 the semisimple abelian category proj $\Gamma(A)$ has a unique triangulated structure with Σ the shift functor. Then this structure necessarily coincides with the transported one. Then we are done. $\square$

We are interested in the auto-equivalence Σ above. The following result characterizes it using the bimodules $K^n(A)$.

Lemma 3.6. *Use the notation as above. Then for each $n \in \mathbb{Z}$, the auto-equivalence Σ^n is isomorphic to $- \otimes_{\Gamma(A)} K^n(A)\colon \mathrm{proj}\ \Gamma(A) \to \mathrm{proj}\ \Gamma(A)$.*

Proof. Recall that the above equivalence Ψ is given by $\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}), -)$, which sends $q(A/\mathbf{r})$ to $\Gamma(A)$. The auto-equivalence Σ^n corresponds, via Ψ, to $[n]$ on $\mathbf{D}_{\mathrm{sg}}(A)$. Then by Proposition 3.2 we have an isomorphism $\phi\colon K^n(A) \xrightarrow{\sim} \Sigma^n(\Gamma(A))$ of right $\Gamma(A)$-modules. Recall that $\Sigma^n(\Gamma(A))$ has a natural $\Gamma(A)$-$\Gamma(A)$-bimodule structure such that Σ^n is isomorphic to $- \otimes_{\Gamma(A)} \Sigma^n(\Gamma(A))$. Thanks to Remark 3.3, the isomorphism ϕ is an isomorphism of bimodules. This proves the lemma. □

Recall that for an algebra Γ, a Γ-Γ-bimodule K is *invertible* provided that the functor $-\otimes_\Gamma K$ induces an auto-equivalence on the category of right Γ-modules. For details, we refer to [7, Definition and Proposition 12.13].

We recall that for an artin algebra A with radical square zero, the associated $\Gamma(A)$-$\Gamma(A)$-bimodules $K^n(A)$ are defined to be $\varinjlim \mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i}, \mathbf{r}^{\otimes i-n})$, where $i \geq \max\{0, n\}$. Then composition of maps between the $A/\mathbf{r}$-modules $\mathbf{r}^{\otimes j}$ yields morphisms

$$\phi^{n,m}\colon K^n(A) \otimes_{\Gamma(A)} K^m(A) \longrightarrow K^{n+m}(A)$$

of $\Gamma(A)$-$\Gamma(A)$-bimodules, for all $n, m \in \mathbb{Z}$. More precisely, let $f \in K^n(A)$ and $g \in K^m(A)$ be represented by $f'\colon \mathbf{r}^{\otimes j-m} \to \mathbf{r}^{\otimes j-m-n}$ and $g'\colon \mathbf{r}^{\otimes j} \to \mathbf{r}^{\otimes j-m}$ for some large j, respectively. Then $\phi^{n,m}(f \otimes g)$ is represented by the composite $f' \circ g'$.

The following result is Theorem A.

Theorem 3.7. *Let A be an artin algebra with radical square zero. Use the above notation. Then for all $n, m \in \mathbb{Z}$, the $\Gamma(A)$-$\Gamma(A)$-bimodules $K^n(A)$ are invertible and the morphisms $\phi^{n,m}$ are isomorphisms.*

Proof. By Lemma 3.6 the functor $- \otimes_{\Gamma(A)} K^n(A)\colon \mathrm{proj}\ \Gamma(A) \to \mathrm{proj}\ \Gamma(A)$ is an auto-equivalence for each $n \in \mathbb{Z}$. This functor extends naturally to an auto-equivalence on the category of all right $\Gamma(A)$-modules. Then $K^n(A)$ is an invertible bimodule. The second statement follows from Lemma 3.6 and the fact that $\Sigma^m\Sigma^n$ is isomorphic to Σ^{n+m}. Here, we use [7, Proposition 12.9] implicitly. □

We now have Theorem B. Denote the functor $- \otimes_{\Gamma(A)} K^1(A)\colon \mathrm{proj}\ \Gamma(A) \to \mathrm{proj}\ \Gamma(A)$ by Σ_A.

Theorem 3.8. *Let A be an artin algebra with radical square zero. Use the above notation. Then we have a triangle equivalence*

$$\mathbf{D}_{\mathrm{sg}}(A) \simeq (\mathrm{proj}\ \Gamma(A), \Sigma_A),$$

which sends $q(A/\mathbf{r})$ to $\Gamma(A)$.

Proof. It follows from Proposition 3.5 and Lemma 3.6. □

Let A be an artin algebra with radical square zero. For each $n \geq 1$, we consider the artin algebra $G^n = A/\mathbf{r} \oplus \mathbf{r}^{\otimes n}$, which is the *trivial extension* of the $A/\mathbf{r}$-$A/\mathbf{r}$-bimodule $\mathbf{r}^{\otimes n}$ ([1, p.78]). All these algebras G^n have radical square zero. The following observation seems to be of independent interest.

Proposition 3.9. *Use the above notation. Then for each $n \geq 1$, we have a triangle equivalence*

$$\mathbf{D}_{\rm sg}(G^n) \simeq (\text{proj}\,\Gamma(A), \Sigma_A^n).$$

In particular, we have a triangle equivalence $\mathbf{D}_{\rm sg}(A) \simeq \mathbf{D}_{\rm sg}(G^1)$.

Proof. Write $G^n = A'$. Then from the very definition, we have a natural identification $\Gamma(A') = \Gamma(A)$. Moreover, the $\Gamma(A')$-$\Gamma(A')$-bimodule $K^1(A')$ corresponds to the $\Gamma(A)$-$\Gamma(A)$-bimodule $K^n(A)$. Then by Lemma 3.6 $\Sigma_{A'}$ corresponds to Σ_A^n. Then the result follows from Theorem 3.8 immediately. □

Remark 3.10. We point out that for $n \geq 2$, $\mathbf{D}_{\rm sg}(G^n)$ might not be triangle equivalent to $\mathbf{D}_{\rm sg}(A)$, although the underlying categories are equivalent.

We conclude this section with an example.

Example 3.11. *Let k be a field and let $n \geq 1$. Consider the algebra $A = k[x_1, x_2, \cdots, x_n]/(x_ix_j, 1 \leq i,j \leq n)$, which is with radical square zero. We identify $A/\mathbf{r}$ with k, and $\mathbf{r}$ with the n-dimensional k-space $V = kx_1 \oplus kx_2 \oplus \cdots \oplus kx_n$. Consequently, for each $i \geq 0$, the algebra $\text{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i})$ is isomorphic to $\text{End}_k(V^{\otimes i})$, which is identified with the $n^i \times n^i$ total matrix algebra $M_{n^i}(k)$. Then the associated regular algebra $\Gamma(A)$ is isomorphic to the direct limit of the following chain of algebra embeddings*

$$k \longrightarrow M_n(k) \longrightarrow M_{n^2}(k) \longrightarrow M_{n^3}(k) \longrightarrow \cdots$$

Here, for each algebra B, $B \to M_n(B)$ is the algebra embedding sending b to bI_n with I_n the $n \times n$ identity matrix.

We observe that $\Gamma(A)$ is a simple algebra. We point out that this construction is classical; see [8, 19.26 B, Example]. *The algebra A is non-noetherian for $n \geq 2$, while for $n = 1$, it is isomorphic to k.*

Let $1 \leq r, s \leq n$. Define $E_{rs}\colon V \to V$ to be the linear map such that $E_{rs}(x_i) = \delta_{i,s}x_r$, where δ is the Kronecker symbol. Consider, for all $i \geq 0$, the linear maps $- \otimes_k E_{rs}\colon \text{End}_k(V^{\otimes i}) \to \text{End}_k(V^{\otimes i+1})$. Taking the limit, we have the induced linear map $- \otimes_k E_{rs}\colon \Gamma(A) \to \Gamma(A)$ for each pair of r, s. Then we have an isomorphism $\sigma\colon M_n(\Gamma(A)) \to \Gamma(A)$ of algebras, which sends an $n \times n$ matrix (a_{ij}) to $\sum_{1\leq i,j\leq n} a_{ij} \otimes_k E_{ij}$.

The associated $\Gamma(A)$-$\Gamma(A)$-bimodule $K^1(A)$ is described as follows. As a k-space, $K^1(A) = \Gamma(A) \oplus \Gamma(A) \oplus \cdots \oplus \Gamma(A)$ with n copies of $\Gamma(A)$. The left action is given by $a(a_1, a_2, \cdots, a_n) = (aa_1, aa_2, \cdots, aa_n)$, while the right action is given by $(a_1, a_2, \cdots, a_n)a = (a_1, a_2, \cdots, a_n)\sigma^{-1}(a)$.

We remark that the regular algebra $\Gamma(A)$ is related to a quotient abelian category studied in [16], *which might relate to the singularity category $\mathbf{D}_{\rm sg}(A)$ via a version of Koszul duality.*

4. One-point (co)extensions and cyclicizations of algebras

In this section, we prove that one-point extensions and coextensions of algebras preserve their singularity categories. We then introduce the notion of cyclicization of an algebra, which is a repeated operation to remove sources and sinks on the valued quiver. The obtained results will be used in the proof of Theorem C.

Let A be an artin algebra. Let D be a simple artin algebra, and let ${}_AM_D$ be an A-D-bimodule, on which R acts centrally. The *one-point extension* of A by M is the upper triangular matrix algebra $A[M] = \begin{pmatrix} A & M \\ 0 & D \end{pmatrix}$. A left $A[M]$-module is denoted by a column vector $\begin{pmatrix} X \\ V \end{pmatrix}_\phi$, where X and V are a left A-module and D-module, respectively, and that $\phi\colon M \otimes_D V \to X$ is a morphism of A-modules. We sometimes suppress the morphism ϕ, when it is clearly understood. For details, we refer to [1, III.2].

Recall from [1, III.1] the notion of *valued quiver* Q_A for an artin algebra A. We observe that for the unique simple D-module S, the corresponding $A[M]$-module $\begin{pmatrix} 0 \\ S \end{pmatrix}$ is simple injective, which corresponds to a source in the valued quiver $Q_{A[M]}$ of the one-point extension $A[M]$. Indeed, this valued quiver is obtained from Q_A by adding this source together with some valued arrows starting at it.

One-point extensions of algebras preserve singularity categories. Observe the natural exact embedding $i\colon A\text{-mod} \to A[M]\text{-mod}$, which sends ${}_AX$ to $i(X) = \begin{pmatrix} X \\ 0 \end{pmatrix}$.

Proposition 4.1. *Let $A[M]$ be the one-point extension of A as above. Then the exact embedding $i\colon A\text{-mod} \to A[M]\text{-mod}$ induces a triangle equivalence*

$$\mathbf{D}_{\mathrm{sg}}(A) \simeq \mathbf{D}_{\mathrm{sg}}(A[M]).$$

Proof. The exact functor i extends naturally to a triangle functor $i_*\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(A[M]\text{-mod})$. We observe that $i(A)$ is projective, and then i_* sends perfect complexes to perfect complexes. Then it induces a triangle functor $\bar{i_*}\colon \mathbf{D}_{\mathrm{sg}}(A) \to \mathbf{D}_{\mathrm{sg}}(A[M])$. We claim that $\bar{i_*}$ is an equivalence.

For the claim, recall that the functor i admits a left adjoint $j\colon A[M]\text{-mod} \to A\text{-mod}$ which sends $\begin{pmatrix} X \\ V \end{pmatrix}_\phi$ to $X/\mathrm{Im}\phi$. Observe that the corresponding counit $ji \xrightarrow{\sim} \mathrm{Id}_{A\text{-mod}}$ is an isomorphism. One checks that the cohomological dimension ([11, p.57]) of the functor j is at most one. In particular, the left derived functor $\mathbf{L}^b j\colon \mathbf{D}^b(A[M]\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ is defined. Moreover, we have the adjoint pair $(\mathbf{L}^b j, i_*)$, and that the counit is an isomorphism. Since the functor j sends projective modules to projective modules, the functor $\mathbf{L}^b j$ preserves perfect complexes. Then it induces a triangle functor $\mathbf{L}^b\bar{j}\colon \mathbf{D}_{\mathrm{sg}}(A[M]) \to \mathbf{D}_{\mathrm{sg}}(A)$. Moreover, we have the induced adjoint pair $(\mathbf{L}^b\bar{j}, \bar{i_*})$, whose counit $(\mathbf{L}^b\bar{j})\bar{i_*} \xrightarrow{\sim} \mathrm{Id}_{\mathbf{D}_{\mathrm{sg}}(A)}$ is an isomorphism; see [14, Lemma 1.2]. In particular, the functor $\bar{i_*}$ is fully faithful.

It remains to show the denseness of $\bar{i_*}$. We now view the essential image $\mathrm{Im}\,\bar{i_*}$ of $\bar{i_*}$ as a full triangulated subcategory of $\mathbf{D}_{\mathrm{sg}}(A[M])$. It suffices to show that for each $A[M]$-module $\begin{pmatrix} X \\ V \end{pmatrix}$, its image in $\mathbf{D}_{\rm sg}(A[M])$ lies in $\mathrm{Im}\ \bar{i_*}$; see Lemma 2.1. Observe that $\Omega(\begin{pmatrix} 0 \\ V \end{pmatrix})$ lies in $\mathrm{Im}\ i$, and then by Lemma 2.2, $q(\begin{pmatrix} 0 \\ V \end{pmatrix})$ lies in $\mathrm{Im}\ \bar{i_*}$. The following natural exact sequence induces a triangle in $\mathbf{D}_{\rm sg}(A[M])$

$$0 \longrightarrow i(X) \longrightarrow \begin{pmatrix} X \\ V \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ V \end{pmatrix} \longrightarrow 0.$$

This triangle implies that $q(\begin{pmatrix} X \\ V \end{pmatrix})$ lies in $\mathrm{Im}\ \bar{i_*}$. Then we are done. □

Let D be a simple artin algebra, and let ${}_DN_A$ be a D-A-bimodule, on which R acts centrally. The *one-point coextension* of A by N is the upper triangular matrix algebra $[N]A = \begin{pmatrix} D & N \\ 0 & A \end{pmatrix}$. A left $[N]A$-module is written as $\begin{pmatrix} V \\ X \end{pmatrix}_\phi$, where V and X are a left D-module and A-module, respectively, and that $\phi\colon M \otimes_A X \to V$ is a morphism of D-modules. The valued quiver $Q_{[N]A}$ is obtained from Q_A by adding a sink together with some valued arrows ending at it, where the sink corresponds to the simple projective $[N]A$-module $\begin{pmatrix} S \\ 0 \end{pmatrix}$ for a simple D-module S.

For the one-point coextension $[N]A$, we have an exact embedding $i\colon A\text{-mod} \to [N]A\text{-mod}$, which sends ${}_AX$ to $i(X) = \begin{pmatrix} 0 \\ X \end{pmatrix}$.

The following result is similar to Proposition 4.1, while the proof is simpler. This result is closely related to [4, Theorem 4.1(1)].

Proposition 4.2. *Let $[N]A$ be the one-point coextension as above. Then the embedding $i\colon A$-mod $\to [N]A$-mod induces a triangle equivalence*

$$\mathbf{D}_{\rm sg}(A) \simeq \mathbf{D}_{\rm sg}([N]A).$$

Proof. We observe that $i(A)$ has projective dimension at most one. Then the obviously induced functor $i_*\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b([N]A\text{-mod})$ preserves perfect complexes, and it induces the required functor $\bar{i_*}\colon \mathbf{D}_{\rm sg}(A) \to \mathbf{D}_{\rm sg}([N]A)$.

The functor i admits an exact left adjoint $j\colon [N]A\text{-mod} \to A\text{-mod}$, which sends $\begin{pmatrix} V \\ X \end{pmatrix}$ to X; moreover, j preserves projective modules. Then it induces a triangle functor $\bar{j_*}\colon \mathbf{D}_{\rm sg}([N]A) \to \mathbf{D}_{\rm sg}(A)$, which is left adjoint to $\bar{i_*}$. Then as in the proof of Proposition 4.1, we have that $\bar{i_*}$ is fully faithful. The denseness of $\bar{i_*}$ follows from the natural exact sequence

$$0 \longrightarrow \begin{pmatrix} V \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} V \\ X \end{pmatrix} \longrightarrow i(X) \longrightarrow 0,$$

for each $[N]A$-module $\begin{pmatrix} V \\ X \end{pmatrix}$, and that the $[N]A$-module $\begin{pmatrix} V \\ 0 \end{pmatrix}$ is projective. We omit the details. □

We use the above two propositions to reduce the study of the singularity category of arbitrary artin algebras to cyclic-like ones.
Let A be an artin algebra. Consider the valued quiver Q_A. A vertex e is called *cyclic* provided that there is an oriented cycle containing it, and the corresponding simple A-module is called *cyclic*. More generally, a vertex e is called *cyclic-like* provided that there is a path through e, which starts with a cyclic vertex and ends at a cyclic vertex, while the corresponding simple A-module is called *cyclic-like*. An artin algebra A is called *cyclic-like* provided that its valued quiver Q_A is cyclic-like. This is equivalent to that A has neither simple projective nor simple injective modules.
For an artin algebra A, its *cyclicization* is an artin algebra A_c which is either simple or cyclic-like, such that there is a sequence $A_c = A_0, A_1, \cdots, A_r = A$ satisfying that each A_{i+1} is a one-point (co)extension of A_i.
The following is an immediate consequence of the definition.

Lemma 4.3. *Let A be an artin algebra with its cyclicization A_c. Then we have a triangle equivalence*

$$\mathbf{D}_{\rm sg}(A_c) \simeq \mathbf{D}_{\rm sg}(A).$$

Proof. Apply Propositions 4.1 and 4.2, repeatedly. □

The following result seems to be well known.

Proposition 4.4. *The following statements hold.*

(1) *Each artin algebra has a cyclicization.*
(2) *Let A_c and $A_{c'}$ be two cyclicizations of A. Then if A_c is simple, so is $A_{c'}$. Otherwise, we have an isomorphism $A_c \simeq A_{c'}$ of algebras.*

Proof. (1) It follows from the well-known fact that the existence of a simple injective (*resp.* projective) module of A implies that A is a one-point extension (*resp.* coextension) of A'. Moreover, the valued quiver $Q_{A'}$ of A' is obtained from the one of A by deleting the relevant source (*resp.* sink).
(2) The first statement follows from the observation that passing from A to A' in (1), the set of cyclic-like vertices stays the same.
For the isomorphism of algebras, it suffices to observe that A_c-mod is equivalent to the smallest Serre subcategory ([7, Chapter 15]) of A-mod containing the cyclic-like simple A-modules S; moreover, the multiplicity of $P_{A_c}(S)$ in the indecomposable decomposition of A_c equals the multiplicity of $P(S)$ in the one of A. Here, $P(S)$ and $P_{A_c}(S)$ denote the projective cover of S as an A-module and A_c-module, respectively. □

5. Hom-finiteness of singularity categories

In this section, we study the Hom-finiteness of the singularity category of an artin algebra with radical square zero, and prove Theorem C.
Throughout, A is an artin R-algebra such that its Jacobson radical $\mathbf{r}$ satisfies $\mathbf{r}^2 = 0$. Recall that in this case, the syzygy $\Omega(X)$ of any A-module X is semisimple.

Lemma 5.1. *Suppose that A is cyclic-like. Then we have*

(1) *each simple A-module has infinite projective dimension;*
(2) *for each $i \geq 0$, the algebra homomorphism $\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i}) \to \mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i+1})$ induced by $\mathbf{r} \otimes_{A/\mathbf{r}} -$ is injective.*

Proof. (1) Recall that a cyclic-like algebra does not have simple projective modules. Then the statement follows from the observation that for a simple module S with finite projective dimension, we have that proj.dim $\Omega(S) =$ proj.dim $S - 1$.
(2) We recall that A-$\underline{\mathrm{ssmod}}$ is the full subcategory of A-$\underline{\mathrm{mod}}$ consisting of semisimple modules. Then by (1), A-$\underline{\mathrm{ssmod}}$ is naturally equivalent to A-ssmod, and the syzygy functor $\Omega\colon A$-$\underline{\mathrm{ssmod}} \to A$-$\underline{\mathrm{ssmod}}$ is faithful. Now the result follows from Lemma 3.1. □

Recall that the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is naturally R-linear. We are interested in the problem when it is Hom-finite, that is, all the Hom sets are finitely generated R-modules.

Theorem 5.2. *Let A be an artin algebra with radical square zero. Then the following statements are equivalent:*

(1) *the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is Hom-finite;*
(2) *the associated regular algebra $\Gamma(A)$ is semisimple;*
(3) *the cyclicization A_c of A is either simple or isomorphic to a finite product of self-injective algebras.*

We point out that the cyclicization A_c of A is necessarily with radical square zero. Recall that an indecomposable non-simple artin algebra with radical square zero is self-injective if and only if its valued quiver is an oriented cycle with the trivial valuation; see [1, Proposition IV.2.16] or the proof of [5, Corollary 1.3]. Then the statement (3) above is equivalent to the corresponding one in Theorem C.

Proof. Recall from Proposition 3.2 the isomorphism $\Gamma(A) \simeq \mathrm{End}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}))$. Then we have the implication "(1)⇒(2)", since an artin regular algebra is necessarily semisimple.
For "(2)⇒(1)", consider the cyclicization A_c of A, whose Jacobson radical is denoted by $\mathbf{r}_c$. Then by Lemma 4.3 we have an equivalence $\mathbf{D}_{\mathrm{sg}}(A_c) \simeq \mathbf{D}_{\mathrm{sg}}(A)$. Applying Proposition 3.5 we have an equivalence proj $\Gamma(A_c) \simeq$ proj $\Gamma(A)$, that is, $\Gamma(A_c)$ and $\Gamma(A)$ are Morita equivalent. Then $\Gamma(A_c)$ is also semisimple. Recall that $\Gamma(A_c) = \varinjlim \mathrm{End}_{A_c/\mathbf{r}_c}(\mathbf{r}_c^{\otimes i})$. By Lemma 5.1 all the canonical maps $\mathrm{End}_{A_c/\mathbf{r}_c}(\mathbf{r}_c^{\otimes i}) \to \Gamma(A_c)$ are injective. Recall that for a semisimple algebra, the number of pairwise orthogonal idempotents is bounded. Then the R-lengths of the algebras $\mathrm{End}_{A_c/\mathbf{r}_c}(\mathbf{r}_c^{\otimes i})$ are uniformly bounded. Consequently, the algebra $\Gamma(A)$ is an artin R-algebra. By Proposition 3.5 the singularity category $\mathbf{D}_{\mathrm{sg}}(A_c)$ is Hom-finite. Then we are done by Lemma 4.3.

Recall from [15, Theorem 2.1] that the singularity category of a self-injective algebra is equivalent to its stable category. In particular, it is Hom-finite. Then the implication "(3)⇒(1)" follows from Lemma 4.3.
It remains to show "(1)⇒(3)". Without loss of generality, we assume that the algebra A is cyclic-like such that $\mathbf{D}_{\rm sg}(A)$ is Hom-finite. We will show that A is self-injective.
We claim that the sysygy $\Omega(S)$ of any cyclic simple A-module S is simple. Then there is only one arrow starting at S in Q_A, which is valued by $(1, b)$ for some natural number b. Since Q_A is cyclic-like, this forces that Q_A is a disjoint union of oriented cycles. In each oriented cycle, every arrow has valuation $(1, b_i)$ for some b_i. Then the symmetrization condition implies that all these b_i's are necessarily one; compare the proof of [1, Proposition VIII. 6.4]. As we point out above, this implies that A is self-injective.
We prove the claim. Since by Corollary 2.4 $\mathbf{D}_{\rm sg}(A)$ is idempotent split, we have that $\mathbf{D}_{\rm sg}(A)$ is a Krull-Schmidt category ([6, Appendix A]). In particular, each object is uniquely decomposed as a direct sum of finitely many indecomposable objects. We observe that for each semisimple module X, $lX \leq l\Omega(X)$. Here, l denotes the composition length. Consider a cyclic simple A-module S, and take a path $S = S_1 \to S_2 \to \cdots \to S_r \to S_{r+1} = S$ in Q_A. Assume on the contrary that $l\Omega(S) \geq 2$. Then we have $\Omega(S) = S_2 \oplus X$ for some nonzero semisimple module X. Observe that S is a direct summand of $\Omega^{r-1}(S_2)$, and then we have $\Omega^r(S) = S \oplus X'$ for a nonzero semisimple module X'. Consequently, we have $\Omega^{nr}(S) = S \oplus X' \oplus \Omega^r(X') \oplus \cdots \oplus \Omega^{(n-1)r}(X')$. Then the lengths of the semisimple modules $\Omega^{nr}(S)$ tend to the infinity, when n goes to the infinity. By Lemma 5.1(1), $q(T)$ is not zero for any simple A-module T. Recall from Lemma 2.2 that $q(S) \simeq q(\Omega^{nr}(S))[nr]$, and then $q(S) \simeq q(S)[nr] \oplus q(X')[nr] \oplus q(\Omega^r(X'))[nr] \oplus \cdots \oplus q(\Omega^{(n-1)r}(X'))[nr]$ for each $n \geq 1$. This contradicts to the Krull-Schmidt property of $\mathbf{D}_{\rm sg}(A)$, and we are done with the claim. □

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