Theta Series and Function Field Analogue of Gross Formula

Fu-Tsun Wei and Jing Yu[1]

Received: October 14, 2010
Revised: July 6, 2011

Communicated by Peter Schneider

Abstract. Let $k = \mathbb{F}_q(t)$, with q odd. In this article we introduce "definite" (with respect to the infinite place of k) Shimura curves over k, and establish Hecke module isomorphisms between their Picard groups and the spaces of Drinfeld type "new" forms of corresponding level. An important application is a function field analogue of Gross formula for the central critical values of Rankin type L-series coming from automorphic cusp forms of Drinfeld type.

2010 Mathematics Subject Classification: 11R58 11G18 11F41 11F67
Keywords and Phrases: function field, quaternion algebra, Shimura curve, automorphic form, Hecke operator, special value of L-series.

Introduction

We present here a theory of "definite" quaternion algebras over the rational function field $k := \mathbb{F}_q(t)$ with q odd, "definite" means that the place ∞ at infinity ramifies for the quaternion algebra in question. Following Gross [8], we first give a geometric translation of Eichler's arithmetic theory of definite quaternion algebra by introducing the so-called "definite" Shimura curves. The geometry of these curves is simple and easy to manipulate. Basing on Eichler's trace computation, one is lead (via Jacquet-Langlands) to an explicit Hecke module isomorphism between the Picard groups of definite Shimura curves and spaces of automorphic forms of Drinfeld type over the function field k.

[1]Both authors were supported by National Science Council grant (100)-2119-M002-007. The first author was also supported by the Graduate Students Study Abroad Program of National Science Council.

Automorphic forms of Drinfeld type are very useful tools for function fields arithmetic (cf. [7], [12] and [17] for more details and applications), which can be viewed as an analogue of classical modular forms of weight 2. To illustrate our approach to quaternion algebras over function field, we give an application to the study of central critical values of certain L-series of "Rankin type" in the global function field setting. These L-series include, among others, L-series coming from elliptic curves over k with square free conductor supported at even number of places and having split multiplicative reduction at ∞. Having the extensive calculations done in [12], we obtain in particular a function field analogue of Gross formula for the central critical values of these L-series over "imaginary" quadratic extensions of k (with respect to ∞).

The structure of this article is modelled on [8]. Let $\mathcal{D}$ be a "definite" quaternion algebra over k and let N_0 be the product of finite ramified primes of $\mathcal{D}$. We introduce the definite Shimura curve $X = X_{N_0}$ over k (for maximal orders) in §1, which is a finite union of genus zero curves. Also introduced are the Gross points, which are special points on these curves associated to orders in imaginary quadratic extensions of k. With a natural choice of basis on the Picard group $\mathrm{Pic}(X)$, the Hecke correspondences can be expressed by Brandt matrices.

From the entries of Brandt matrices we introduce certain theta series. Taking into account the Gross height pairing on the $\mathrm{Pic}(X)$ (defined in §1.2), we then have at hand a construction of automorphic forms of Drinfeld type for the congruence subgroup $\Gamma_0(N_0)$ of $\mathrm{GL}_2(\mathbb{F}_q[t])$. The main theorem of this article in §2.3 is:

Theorem. *There is a map* $\Phi : \mathrm{Pic}(X) \times \mathrm{Pic}(X)^\vee \longrightarrow M^{\mathrm{new}}(\Gamma_0(N_0))$ *such that for all monic polynomials* m *of* $\mathbb{F}_q[t]$

$$T_m\Phi(e,e') = \Phi(t_m e, e') = \Phi(e, t_m e').$$

Here $\mathrm{Pic}(X)^\vee$ *is the dual group* $\mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z})$*,* $M^{\mathrm{new}}(\Gamma_0(N_0))$ *is the space of Drinfeld type "new" forms for* $\Gamma_0(N_0)$*,* t_m *are Hecke correspondences on* X*, and* T_m *are Hecke operators on* $M^{\mathrm{new}}(\Gamma_0(N_0))$*. Moreover, this map induces an isomorphism* (*as Hecke modules*)

$$(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}) \otimes_{\mathbb{T}_{\mathbb{C}}} (\mathrm{Pic}(X)^\vee \otimes_{\mathbb{Z}} \mathbb{C}) \cong M^{\mathrm{new}}(\Gamma_0(N_0)).$$

This theorem in fact tells us that all automorphic "new" forms of Drinfeld type come from our theta series. The special case of our theorem when N_0 is single prime is also obtained in Papikian [10] §3, by a different geometric method using Néron models of jacobians of Drinfeld modular curve $X_0(N_0)$. In our proof of the above theorem, we use the explicit construction of theta series and claim the equality of the trace of the m-th Brandt matrix $B(m)$ and the trace of the Hecke operator T_m on $M^{\mathrm{new}}(\Gamma_0(N_0))$ for each monic polynomial m in $\mathbb{F}_q[t]$. This claim is essentially the *Jacquet-Langlands correspondence* (cf.

[9]) between automorphic representations of quaternion algebras over k and automorphic cuspidal representations of GL_2 over k. Another crucial step in the proof is to show that the Hecke module $M^{\mathrm{new}}(\Gamma_0(N_0))$ is free of rank one, which follows from the multiplicity one theorem (cf. [3]). For the sake of completeness, we recall these results in Appendix.

Let D be an irreducible polynomial in $\mathbb{F}_q[t]$ such that $K = k(\sqrt{D})$ is imaginary and P is inert in K if the prime P divides N_0. For each ideal class $\mathcal{A}$ of $\mathbb{F}_q[t][\sqrt{D}] = O_K$, we construct in §2.4 an automorphic form $g_{\mathcal{A}}$ of Drinfeld type with its Fourier coefficients worked out. In §3.1 we recall Rankin product of L-series $\Lambda(f, \mathcal{A}, s)$ associated to Drinfeld type new form f for $\Gamma_0(N_0)$ and partial zeta function $\zeta_{\mathcal{A}}$. In §3.2 we express the central critical value $\Lambda(f, \mathcal{A}, 0)$ as the Petersson inner product of f and $g_{\mathcal{A}}$. Furthermore, when f is a "normalized" Hecke eigenform and χ is a character of ideal class group $\mathrm{Pic}(O_K)$ of O_K, we give the twisted critical value $\Lambda(f, \chi, 0)$ explicitly in terms of the Gross height of a special divisor class $e_{f,\chi}$ on the definite Shimura curve X_{N_0}. This is our analogue of Gross formula.

Let E be an elliptic curves over k with conductor $N_0\infty$ and split multiplicative reduction at ∞. From the work of Weil, Jacquet-Langlands, and Deligne, it is well known that there exists a Drinfeld type cusp form f_E such that

$$L(E/k, s+1) = L(f_E, s).$$

Here $L(E/k, s)$ is the Hasse-Weil L-series of E over k. After doing base change to the quadratic field K, one gets

$$L(E/K, s+1) = \Lambda(f, \mathbf{1}_D, s)$$

where $\mathbf{1}_D$ is the trivial character of $\mathrm{Pic}(O_K)$. Our formula can certainly be applied to these elliptic curves. An example is given in §3.4.

Notation

We fix the following notations:

k :	the rational function field $\mathbb{F}_q(t)$, $q = p^{\ell_0}$ where p is an odd prime.
A :	the polynomial ring $\mathbb{F}_q[t]$.
∞ :	the infinite place, which corresponds to degree valuation v_∞.
π_∞ :	t^{-1}, a fixed uniformizer of ∞.
k_∞ :	$\mathbb{F}_q((t^{-1}))$, i.e. the completion of k at ∞.
$\mathcal{O}_\infty$:	$\mathbb{F}_q[[t^{-1}]]$, i.e. the valuation ring in k_∞.
P :	a finite prime (place) of k.
k_P :	the completion of k at the finite prime P.

A_P : the closure of A in k_P.
$\mathbb{A}_k$: the adele ring of k.
$\hat{k}$: $\prod'_P k_P$, the finite adele ring of k.
$\hat{A}$: $\prod_P A_P$.
ψ_∞ : a fixed additive character on k_∞: for $y = \sum_i a_i \pi_\infty^i \in k_\infty$, we define $\psi_\infty(y) := \exp\left(\frac{2\pi\sqrt{-1}}{p} \cdot \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(-a_1)\right)$.

We identify non-zero ideals of A with the monic polynomials in A by using the same notation.

1 Definite Shimura curves

Let $\mathcal{D}$ be a quaternion algebra over k ramified at ∞ (call $\mathcal{D}$ "definite"). Before introducing the definite Shimura curve for $\mathcal{D}$, we start with a genus 0 curve Y over k associated with the quaternion algebra $\mathcal{D}$, which is defined by the following: the points of Y over any k-algebra M are

$$Y(M) = \{x \in \mathcal{D} \otimes_k M : x \neq 0, \mathrm{Tr}(x) = \mathrm{Nr}(x) = 0\}/M^\times,$$

where the action of $M^\times$ on $\mathcal{D} \otimes_k M$ is by multiplication on M, Tr and Nr are respectively the reduced trace and the reduced norm of $\mathcal{D}$. More precisely, if $\mathcal{D} = k+ku+kv+kuv$ where $u^2 = \alpha$, $v^2 = \beta$, α and β are in $k^\times$, and $uv = -vu$, then Y is just the conic

$$\alpha y^2 + \beta z^2 = \alpha\beta w^2$$

in the projective plane $\mathbb{P}^2$. The group $\mathcal{D}^\times$ acts on Y (from the right) by conjugation. If K is a quadratic extension of k, $Y(K)$ is canonically identified with the set $\mathrm{Hom}(K, \mathcal{D})$ of embeddings: for each embedding $f : K \to \mathcal{D}$, let $y = y_f$ be the image of the unique K-line on the quadric $\{x \in \mathcal{D} \otimes_k K : \mathrm{Tr}(x) = \mathrm{Nr}(x) = 0\}$ on which conjugation by $f(K^\times)$ acts by multiplication by the character $a \mapsto a/\bar{a}$. Note that y_f is one of the two fixed points of $f(K^\times)$ acting on $Y(K)$; another one is the image of the line where conjugation acts by the character $a \mapsto \bar{a}/a$.
Let N_0 be the product of the finite ramified primes of $\mathcal{D}$. Choose a maximal A-order R of $\mathcal{D}$. For any finite prime P let $R_P := R \otimes_A A_P$, $\mathcal{D}_P := \mathcal{D} \otimes_k k_P$, and

$$\hat{R} := R \otimes_A \hat{A},\ \hat{\mathcal{D}} := \mathcal{D} \otimes_k \hat{k}.$$

Definition 1.1. (cf. [2] and [8]) The *definite Shimura curve* X_{N_0} is defined as

$$X_{N_0} = \left(\hat{R}^\times \backslash \hat{\mathcal{D}}^\times \times Y\right)/\mathcal{D}^\times.$$

We will use the notation X instead of X_{N_0} when N_0 is fixed.

Lemma 1.2. *X_{N_0} is a finite union of curves of genus* 0.

Proof. Let $g_1, ..., g_n$ be representatives for the finite double coset space $\hat{R}^\times \backslash \hat{\mathcal{D}}^\times / \mathcal{D}^\times$, i.e.

$$\hat{\mathcal{D}}^\times = \coprod_{i=1}^{n} \hat{R}^\times g_i \mathcal{D}^\times .$$

Then each coset of X_{N_0} has a representative $(\hat{R}^\times g_i, y) \mod \mathcal{D}^\times$ and the map

$$\begin{array}{ccc} X_{N_0} & \longrightarrow & \coprod_{i=1}^{n} Y/\Gamma_i \\ (\hat{R}^\times g_i, y) & \longmapsto & y \bmod \Gamma_i \end{array}$$

is a bijection, where $\Gamma_i = g_i^{-1} \hat{R}^\times g_i \cap \mathcal{D}^\times$ is a finite group for $i = 1, ..., n$. □

Definition 1.3. Let K be an imaginary quadratic extension of k (i.e. ∞ is not split in K). We call

$$x = (g, y) \in \text{Image}\left[\hat{R}^\times \backslash \hat{\mathcal{D}}^\times \times Y(K) \to X_{N_0}(K)\right]$$

a *Gross point on X_{N_0} over K.*

Let $f : K \to \mathcal{D}$ be the embedding corresponding to y. Then

$$f(K) \cap g^{-1} \hat{R} g = f(O_d)$$

for some quadratic order $O_d := A[\sqrt{d}]$ where d is an element in A with $d \notin k_\infty^2$. In this case, we say x is *of discriminant d.* Note that the discriminant of a Gross point is well-defined up to multiplying with elements in $(\mathbb{F}_q^\times)^2$. Set $X_i := Y/\Gamma_i$. If the component g of a Gross point x is congruent to g_i in $\hat{R}^\times \backslash \hat{\mathcal{D}}^\times / \mathcal{D}^\times$, then x lies on the component $X_i(K) = (Y/\Gamma_i)(K)$.

1.1 Actions on Gross points

Let $a \in \hat{K}^\times$ where $\hat{K} := K \otimes_k \hat{k}$ and $x = (g, y)$ be a Gross point of discriminant d. Let $f : K \to \mathcal{D}$ be the embedding corresponding to y. This induces a homomorphism $\hat{f} : \hat{K} \to \hat{\mathcal{D}}$ and we define

$$x_a := (g\hat{f}(a), y).$$

Note that x_a is also of discriminant d, and it is easy to check that $x = x_a$ if and only if $a \in \hat{O}_d^\times K^\times$ where $\hat{O}_d := O_d \otimes_A \hat{A}$. Hence $\hat{O}_d^\times \backslash \hat{K}^\times / K^\times \cong \text{Pic}(O_d)$ acts freely on the set G_d of Gross points of discriminant d.

The orbit space $G_d / \text{Pic}(O_d)$ is identified with the space of double cosets

$$\hat{R}^\times \backslash \mathcal{E} / \hat{f}(\hat{K}^\times),$$

where $f : K \to \mathcal{D}$ is a fixed embedding (if any exist) and

$$\mathcal{E} := \{g \in \hat{\mathcal{D}}^\times : f(K) \cap g^{-1}\hat{R}g = f(O_d)\}.$$

Note that

$$\hat{R}^\times \backslash \mathcal{E}/\hat{f}(\hat{K}^\times) = \prod_P R_P^\times \backslash \mathcal{E}_P/f(K_P^\times)$$

where $\mathcal{E}_P := \{g_P \in \mathcal{D}_P^\times : f(K_P) \cap g_P^{-1} R_P g_P = f(O_{d,P})\}$ and $O_{d,P}$ is the closure of O_d in $K_P := K \otimes_k k_P$.

LEMMA 1.4. (cf. [16] or [17])

$$\#(R_P^\times \backslash \mathcal{E}_P/f(K_P^\times)) = \begin{cases} 1 & \textit{if } P \nmid N_0, \\ 1 - \left\{\frac{d}{P}\right\} & \textit{if } P \mid N_0. \end{cases}$$

Here $\left\{\frac{d}{P}\right\}$ is the *Eichler quadratic symbol*, i.e.

$$\left\{\frac{d}{P}\right\} = \begin{cases} 1 & \text{if } P^2 | d \text{ or } d \bmod P \in \left((A/P)^\times\right)^2, \\ -1 & \text{if } d \bmod P \in (A/P)^\times - \left((A/P)^\times\right)^2, \\ 0 & \text{if } P | d \text{ but } P^2 \nmid d. \end{cases}$$

Remark. The above lemma tells us that the number $\#(G_d)$ is equal to

$$h(d) \prod_{P | N_0} \left(1 - \left\{\frac{d}{P}\right\}\right)$$

where $h(d)$ is the class number of O_d.

There is a natural action of $\mathrm{Gal}(K/k)$ on Gross points in the following way: let $x = (g, y)$ be a Gross point and $f_y : K \hookrightarrow \mathcal{D}$ be the embedding corresponding to y. Define

$$x^\sigma = (g, y)^\sigma = (g, y_\sigma)$$

where $\sigma \in \mathrm{Gal}(K/k)$ and y_σ corresponds to the embedding $f_y \circ \sigma$. If x is a Gross point of discriminant d in X_i then so is x^σ. Moreover, let $a \in \hat{O}_d^\times \backslash \hat{K}^\times / K^\times \cong \mathrm{Pic}(O_d)$ and $\sigma \in \mathrm{Gal}(K/k)$ one has

$$(x^\sigma)_a = (x_{\sigma(a)})^\sigma.$$

Therefore we have an action of $\mathrm{Pic}(O_d) \rtimes \mathrm{Gal}(K/k)$ on the set G_d of Gross points of discriminant d.

1.2 Hecke correspondences and Gross height pairing

Let P be a prime of A. Let $\mathcal{T}$ be the Bruhat-Tits tree of $\mathrm{PGL}_2(k_P)$ as defined in [14]. The vertices are the equivalence classes of A_P-lattices L in k_P^2, and two such vertices $[L]$ and $[L']$ are adjacent if there exists an integer r such that

$$P^{r+1}L \subsetneq L' \subsetneq P^r L.$$

This is a tree where each vertex has degree $q^{\deg P}+1$. For a vertex v, the Hecke correspondence t_P sends v to the formal sum of its $q^{\deg P}+1$ neighbors in the tree. Identifying $\mathrm{PGL}_2(A_P)\backslash \mathrm{PGL}_2(k_P)$ with the Bruhat-Tits tree, we can write the Hecke correspondence for $g \in \mathrm{PGL}_2(A_P)\backslash \mathrm{PGL}_2(k_P)$:

$$t_P(g) := \sum_{\deg(u)<\deg P} \begin{pmatrix} 1 & u \\ 0 & P \end{pmatrix} g + \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} g.$$

Note that X_{N_0} can be written as

$$\left(\hat{R}^\times \backslash \hat{\mathcal{D}}^\times / \hat{k}^\times\right) \times Y/\mathcal{D}^\times$$

and

$$\hat{R}^\times \backslash \hat{\mathcal{D}}^\times / \hat{k}^\times = \prod_P{}' R_P^\times \backslash \mathcal{D}_P^\times / k_P^\times.$$

When $(P, N_0) = 1$,

$$R_P^\times \backslash \mathcal{D}_P^\times / k_P^\times \cong \mathrm{PGL}_2(A_P)\backslash \mathrm{PGL}_2(k_P)$$

and so we have the Hecke correspondence t_P on X_{N_0}.

Now suppose P divides N_0, then $R_P^\times \backslash \mathcal{D}_P^\times / k_P^\times$ has two elements and define the *Atkin-Lehner involution*

$$w_P(g, y) := (g', y)$$

where g' is another double coset in $R_P^\times \backslash \mathcal{D}_P^\times / k_P^\times$.

From the construction, these correspondences commute with each other. Therefore we can define Hecke correspondence t_m for every non-zero ideal (m) of A in the following way:

$$\begin{cases} t_{mm'} = t_m t_{m'} & \text{if } m \text{ and } m' \text{ are relatively prime,} \\ t_{P^\ell} = t_{P^{\ell-1}} t_P - q^{\deg P} t_{P^{\ell-2}} & \text{for } P \nmid N_0, \\ t_{P^\ell} = w_P^\ell & \text{for } P \mid N_0. \end{cases}$$

Note that $X = X_{N_0} = \coprod_{i=1}^n X_i$, where n is the left ideal class number of R. Consider the Picard group $\mathrm{Pic}(X)$, which is isomorphic to $\mathbb{Z}^n$ and is generated by the classes e_i of degree 1 corresponding to the component X_i. Then the correspondences t_m induce endomorphisms of the group $\mathrm{Pic}(X)$. In fact, with respect to the basis $\{e_1, ..., e_n\}$, these endomorphisms can be represented by *Brandt matrices*.

Let $\{I_1, ..., I_n\}$ be a set of left ideals of R representing the distinct ideal classes, with $I_1 = R$. Let $w_i := \#(R_i^\times)/(q-1)$ where R_i is the right order of I_i. Consider $M_{ij} := I_j^{-1}I_i$, which is a left ideal of R_j with right order R_i. Choose a generator $N_{ij} \in k$ of the reduced ideal norm $\mathrm{Nr}(M_{ij})(:=<\mathrm{Nr}(b) : b \in M_{ij} >_A)$ of M_{ij}. For each monic polynomial m in A, define

$$B_{ij}(m) := \frac{\#\{b \in M_{ij} : (\mathrm{Nr}(b)/N_{ij}) = (m)\}}{(q-1)w_j}$$

and the *m-th Brandt matrix*

$$B(m) := \Big(B_{ij}(m)\Big)_{1\leq i,j\leq n} \in \mathrm{Mat}_n(\mathbb{Z}).$$

Proposition 1.5. *For all non-zero ideal* (m) *in* A *and* $i = 1, 2, ..., n$,

$$t_m e_i = \sum_{j=1}^{n} B_{ij}(m)e_j.$$

Proof. From the definition of t_m and the recurrence relations of $B(m)$ (cf. [16]), we can reduce the proof to the case when $m = P$ is a prime.

From the following bijection

$$\begin{array}{ccc} \hat{R}^\times\backslash\hat{\mathcal{D}}^\times & \cong & \{\text{left ideals of } R\} \\ \hat{R}^\times g & \leftrightarrow & I_g := \hat{R}g \cap \mathcal{D}, \end{array}$$

for any element g in $\hat{\mathcal{D}}^\times$ we can identify the following set

$$\left\{\hat{R}^\times \begin{pmatrix} 1 & u \\ 0 & P \end{pmatrix} g : \deg u < \deg P\right\} \cup \left\{\hat{R}^\times \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} g\right\}$$

with

$$\{\text{left ideal } I \text{ of } R \text{ contained in } I_g \text{ with } \mathrm{Nr}(I) = P\,\mathrm{Nr}(I_g)\}.$$

According to the definition of t_P, $t_P e_i = \sum_j \alpha_j e_j$ where α_j is the number of left ideals I of R equivalent to I_j which are contained in I_i with $\mathrm{Nr}(I) = P\,\mathrm{Nr}(I_i)$. It is easy to see that $\alpha_j = B_{ij}(P)$ and so the proposition holds. □

We define the *Gross height pairing* $< \cdot, \cdot >$ on $\mathrm{Pic}(X)$ with values in $\mathbb{Z}$ by setting

$$\begin{cases} < e_i, e_j > := 0 & \text{if } i \neq j, \\ < e_i, e_i > := w_i, & \end{cases}$$

and extending bi-additively. Therefore $\mathrm{Pic}(X)^\vee := \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z})$ can be viewed as a subgroup of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with basis $\{\check{e}_i := e_i/w_i : i = 1, ..., n\}$ via this pairing. Since $w_j B_{ij}(m) = w_i B_{ji}(m)$, one has the following proposition.

PROPOSITION 1.6. *For all classes e and e' in* $\mathrm{Pic}(X)$*, we have*

$$< t_m e, e' >=< e, t_m e' > .$$

Proof. Since $w_j B_{ij}(m) = w_i B_{ji}(m)$, we have

$$< t_m e_i, e_j >=< e_i, t_m e_j > .$$

for all i, j and the result holds. □

Let $d \in A$ with $d \notin k_\infty^2$. Assume every prime factor P of N_0 is not split in K and P^2 does not divides d (i.e. the set G_d of Gross points of discriminant d is not empty). For any prime $P \mid N_0$, one has $w_P(G_d) = G_d$. Suppose $P_1, \ldots, P_r$ are primes dividing N_0 and inert in K. We have in fact a free action of $\mathrm{Pic}(O_d) \times \prod_{i=1}^r \langle w_{P_i} \rangle$ on G_d. Since w_{P_i} are of order 2 for all i, $\mathrm{Pic}(O_d) \times \prod_{i=1}^r \langle w_{P_i} \rangle$ acts simply transitively on G_d.

Let $a \in A$ with $a \notin k_\infty^2$. Consider the rational divisor

$$c_a := \sum_{a=df^2, f \text{ monic}} \frac{1}{2u(d)} \sum_{x_d \in G_d} x_d.$$

Here $u(d) = \#(O_d^\times)$. By calculation one has

$$\deg(c_a) = \frac{1}{2} \sum_{a=df^2, f \text{ monic}} \left[\frac{h(d)}{u(d)} \cdot \prod_{P \mid N_0} (1 - \left\{ \frac{d}{P} \right\}) \right].$$

Let $e_a \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the class of the divisor c_a. It can be shown that

PROPOSITION 1.7. *The class e_a lies in* $\mathrm{Pic}(X)^\vee$*, which is considered as a subgroup of* $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Note that we can extend the Gross height pairing to $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ which is linear in the first term and conjugate linear in the second. In the next section this pairing gives us a construction of automorphic forms of Drinfeld type.

2 AUTOMORPHIC FORMS OF DRINFELD TYPE AND MAIN THEOREM

2.1 AUTOMORPHIC FORMS OF DRINFELD TYPE

Consider the open compact subgroup $\mathcal{K}_0(N\infty) := \prod_P \mathcal{K}_{0,P} \times \Gamma_\infty$ of $\mathrm{GL}_2(\mathbb{A}_k)$, where

$$\mathcal{K}_{0,P} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A_P) : c \in NA_P \right\}$$

for finite prime P, and

$$\Gamma_\infty := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\infty) : c \in \pi_\infty \mathcal{O}_\infty \right\}.$$

An *automorphic form f on* $\mathrm{GL}_2(\mathbb{A}_k)$ *for* $\mathcal{K}_0(N\infty)$ (*with trivial central character*) is a $\mathbb{C}$-valued function on the double coset space

$$\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}_k)/\mathcal{K}_0(N\infty)k_\infty^\times.$$

Note that by strong approximation theorem (cf. [16]) we have the following bijection

$$\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}_k)/K_0(N\infty)k_\infty^\times \cong \Gamma_0(N)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$$

where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) : c \equiv 0 \bmod N \right\}.$$

Therefore f can be viewed as a $\mathbb{C}$-valued function on $\Gamma_0(N)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$. From now on, we call f an *automorphic form for* $\Gamma_0(N)$ if f is a function on the space of double cosets $\Gamma_0(N)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$. An automorphic form f for $\Gamma_0(N)$ is called a *cusp form* if for every $g_\infty \in \mathrm{GL}_2(k_\infty)$ and $\gamma \in \mathrm{GL}_2(A)$

$$\int_{A\backslash k_\infty} f\left(\gamma \begin{pmatrix} 1 & h_\gamma x \\ 0 & 1 \end{pmatrix} g_\infty\right) dx = 0.$$

Here du is a Haar measure with $\int_{A\backslash k_\infty} du = 1$ and h_γ is a generator of the ideal of A which is maximal for the property that

$$\gamma \begin{pmatrix} 1 & h_\gamma A \\ 0 & 1 \end{pmatrix} \gamma^{-1} \subset \Gamma_0(N).$$

Note that the coset space $\mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$ can be represented by the two disjoint sets

$$\mathcal{T}_+ := \left\{ \begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} : r \in \mathbb{Z}, u \in k_\infty/\pi_\infty^r \mathcal{O}_\infty \right\}$$

and

$$\mathcal{T}_- := \left\{ \begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix} : r \in \mathbb{Z}, u \in k_\infty/\pi_\infty^r \mathcal{O}_\infty \right\}.$$

Definition 2.1. An automorphic form f on $\mathrm{GL}_2(k_\infty)$ is *of Drinfeld type* if it satisfies the following *harmonic properties*: for any $g_\infty \in \mathrm{GL}_2(k_\infty)$ we have

$$\tilde{f}(g_\infty) := f(g_\infty \begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}) = -f(g_\infty) \text{ and } \sum_{\kappa \in \mathrm{GL}_2(\mathcal{O}_\infty)/\Gamma_\infty} f(g_\infty \kappa) = 0.$$

Suppose a function $f : \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times \to \mathbb{C}$ is given. Recall the *Fourier expansion of* f (cf. [18]): for $r \in \mathbb{Z}$ and $u \in k_\infty$,

$$f\begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} = \sum_{\lambda \in A} f^*(r,\lambda)\psi_\infty(\lambda u)$$

where

$$f^*(r,\lambda) := \int_{A\backslash k_\infty} f\begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} \psi_\infty(-\lambda u) du.$$

Here ψ_∞ is the fixed additive character on k_∞ in the notation table. Since $f(g\gamma_\infty) = f(g)$ for all $\gamma_\infty \in \Gamma_\infty$, $f^*(r,\lambda) = 0$ if $\deg \lambda + 2 > r$. Moreover, if f satisfies harmonic properties, then

$$f^*(r,\lambda) = q^{-r+\deg\lambda+2} f^*(\deg\lambda+2,\lambda)$$

if $\deg\lambda + 2 \leq r$.

2.1.1 Example: Theta series

Fix a definite quaternion algebra $\mathcal{D} = \mathcal{D}_{(N_0)}$ where N_0 is the product of finite ramified primes of $\mathcal{D}$. Let R be a maximal order and n be the class number. With representatives of left ideal classes fixed in §1.2, we have introduced for each (i,j), the ideal M_{ij} of $\mathcal{D}$ and chose a generator N_{ij} of the fractional ideal $\mathrm{Nr}(M_{ij})$. For $1 \leq i,j \leq n$ and $(x,y) \in k_\infty^\times \times k_\infty$, define

$$\theta_{ij}(x,y) := \sum_{b \in M_{ij}} \phi_\infty(\frac{\mathrm{Nr}(b)}{N_{ij}} x t^2) \cdot \psi_\infty(\frac{\mathrm{Nr}(b)}{N_{ij}} y),$$

where ϕ_∞ is the characteristic function of $\mathcal{O}_\infty$. It is easy to obtain the following properties:
(1)

$$\theta_{ij}(x,y) = \sum_{\lambda\in A, \deg\lambda+2\leq v_\infty(x)} B'_{ij}(\lambda)\psi_\infty(\lambda y)$$

where for each $\lambda \in A$,

$$B'_{ij}(\lambda) = \#\{b \in M_{ij} : \mathrm{Nr}(b)/N_{ij} = \lambda\}.$$

(2) $\theta_{ij}(x, y+h) = \theta_{ij}(x,y)$ for $h \in A$.
(3) $\theta_{ij}(\alpha x, \beta x + y) = \theta_{ij}(x,y)$ for $\alpha \in \mathcal{O}_\infty^\times$, $\beta \in \mathcal{O}_\infty$.

Basing on Poisson summation formula, we have the following transformation law for θ_{ij} (cf. Appendix B):

Proposition 2.2. *Let* $(x,y) \in k_\infty^\times \times k_\infty$ *and* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(A)$. *Suppose* $v_\infty(cx) > v_\infty(cy+d)$ *and* $c \equiv 0 \bmod N_0$. *Then for* $1 \leq i,j \leq n$,

$$\theta_{ij}\left(\frac{x}{(cy+d)^2}, \frac{ay+b}{cy+d}\right) = q^{-2v_\infty(cy+d)} \cdot \theta_{ij}(x,y).$$

For $g_\infty \in \mathrm{GL}_2(k_\infty)$, write g_∞ as $\gamma \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \gamma_\infty z_\infty$, where γ is in $\Gamma_0(N_0)$, (x,y) is in $k_\infty^\times \times k_\infty$, γ_∞ is in Γ_∞, and z_∞ is in $k_\infty^\times$. We introduce the theta series Θ_{ij} for M_{ij}:

$$\begin{aligned} \Theta_{ij}(g_\infty) &:= \frac{1}{(q-1)w_j} \cdot q^{-v_\infty(x)} \cdot \left(\sum_{\epsilon \in \mathbb{F}_q^\times} \theta_{ij}(x, \epsilon y)\right) \\ &= q^{-v_\infty(x)} \cdot \left[\frac{1}{w_j} + \sum_{\substack{m \in A \text{ monic}, \\ \deg m + 2 \leq v_\infty(x)}} B_{ij}(m) \Big(\sum_{\epsilon \in \mathbb{F}_q^\times} \psi_\infty(\epsilon m y)\Big)\right]. \end{aligned}$$

The last equality follows from $B'_{ij}(0) = 1$ and for each monic polynomial $m \in A$,

$$(q-1)w_j \cdot B_{ij}(m) = \sum_{\epsilon \in \mathbb{F}_q^\times} B'_{ij}(\epsilon m).$$

The transformation law of θ_{ij} tells us that

Lemma 2.3. Θ_{ij} *is a well-defined* $\mathbb{Q}$*-valued function on the double coset space* $\Gamma_0(N_0)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$.

Proof. Let g_∞ be an element in $\mathrm{GL}_2(k_\infty)$. Suppose

$$g_\infty = \gamma_1 \begin{pmatrix} x_1 & y_1 \\ 0 & 1 \end{pmatrix} \gamma_{\infty,1} z_1 = \gamma_2 \begin{pmatrix} x_2 & y_2 \\ 0 & 1 \end{pmatrix} \gamma_{\infty,2} z_2,$$

where for $i = 1,2$, $\gamma_i \in \Gamma_0(N_0)$, $(x_i, y_i) \in k_\infty \times k_\infty^\times$, $\gamma_{\infty,i} \in \Gamma_\infty$, $z_i \in k_\infty^\times$. We need to show that

$$q^{-v_\infty(x_1)} \cdot \left(\sum_{\epsilon \in \mathbb{F}_q^\times} \theta_{ij}(x_1, \epsilon y_1)\right) = q^{-v_\infty(x_2)} \cdot \left(\sum_{\epsilon \in \mathbb{F}_q^\times} \theta_{ij}(x_2, \epsilon y_2)\right).$$

Set $\gamma = \gamma_2^{-1}\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $z = z_1^{-1} z_2$, and $\gamma_\infty = \gamma_{\infty,1}^{-1}\gamma_{\infty,2}$. Then one has $v_\infty(cx_1) > v_\infty(cy_1 + d)$ and

$$\begin{aligned}\gamma\begin{pmatrix} x_1 & y_1 \\ 0 & 1\end{pmatrix} &= \begin{pmatrix} \frac{\det\gamma\cdot x_1}{(cy_1+d)^2} & \frac{ay_1+b}{cy_1+d} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{cx_1}{cy_1+d} & 1\end{pmatrix}\begin{pmatrix} cy_1+d & 0 \\ 0 & cy_1+d\end{pmatrix} \\ &= \begin{pmatrix} x_2 & y_2 \\ 0 & 1\end{pmatrix}\gamma_\infty z.\end{aligned}$$

Therefore $v_\infty(x_2) = v_\infty(x_1) - 2v_\infty(cy_1+d)$, and the properties of θ_{ij} implies

$$\theta_{ij}(x_2, \epsilon y_2) \quad = \quad \theta_{ij}\left(\frac{\det\gamma\cdot x_1}{(cy_1+d)^2}, \epsilon\frac{ay_1+b}{cy_1+d}\right).$$

for each $\epsilon \in \mathbb{F}_q^\times$. Hence the transformation law of θ_{ij} in Proposition 2.2 shows

$$\begin{aligned} q^{-v_\infty(x_2)}\cdot\left(\sum_{\epsilon\in\mathbb{F}_q^\times}\theta_{ij}(x_2,\epsilon y_2)\right) &= q^{-v_\infty(x_1)}\cdot\left(\sum_{\epsilon\in\mathbb{F}_q^\times}\theta_{ij}(\det\gamma\cdot x_1, \epsilon\det\gamma\cdot y_1)\right) \\ &= q^{-v_\infty(x_1)}\cdot\left(\sum_{\epsilon\in\mathbb{F}_q^\times}\theta_{ij}(x_1,\epsilon y_1)\right).\end{aligned}$$

□

The Fourier coefficients of Θ_{ij} can be easily read off from Brandt matrices: for each $r \in \mathbb{Z}$ and $\lambda \in A$ with $\deg\lambda + 2 \leq r$ the Fourier coefficients

$$\Theta_{ij}^*(r,\lambda) = \begin{cases} q^{-r}B_{ij}(m) & \text{if } (\lambda) = (m) \neq 0, \\ q^{-r}/w_j & \text{if } \lambda = 0.\end{cases}$$

Therefore $\Theta_{ij}^*(r+1,\lambda) = q^{-1}\Theta_{ij}^*(r,\lambda)$ for all $\lambda \in A$ with $\deg\lambda + 2 \leq r$.

In fact, Θ_{ij} are of Drinfeld type for all $1 \leq i, j \leq n$. To show the harmonicity of Θ_{ij}, by [6] Lemma 2.13, it is enough to prove that for all $g_\infty \in \mathrm{GL}_2(k_\infty)$

$$\tilde{\Theta}_{ij}(g_\infty) = -\Theta_{ij}(g_\infty).$$

Let $\pi_\infty^r \in k_\infty^\times$ and $u \in k_\infty$. Choose $c, d \in A$ with $c \equiv 0 \bmod N_0$, $(c,d) = 1$, $v_\infty(u + \frac{d}{c}) \geq r+1$, and find $a, b \in A$ with $ad - bc = 1$. Then for $\ell \in \mathbb{Z}$ with $\ell \leq r+1$ the following two matrices:

$$\begin{pmatrix} \pi_\infty^\ell & u \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0\end{pmatrix} \text{ and } \begin{pmatrix} d & -b \\ -c & a\end{pmatrix}\begin{pmatrix} \frac{\pi_\infty^{1-\ell}}{c^2} & \frac{a}{c} \\ 0 & 1\end{pmatrix}$$

represent the same coset in $\mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$. Using this fact for $\ell = r$ and $\ell = r+1$ one obtains

$$\begin{aligned}
&\tilde{\Theta}_{ij}\begin{pmatrix}\pi_\infty^r & u\\ 0 & 1\end{pmatrix} - q^{-1}\tilde{\Theta}_{ij}\begin{pmatrix}\pi_\infty^{r+1} & u\\ 0 & 1\end{pmatrix}\\
=& \sum_{\deg\mu+2=1-r+2\deg c} \Theta_{ij}^*(1-r+2\deg c,\mu)\psi_\infty(\mu\frac{a}{c}).
\end{aligned}$$

Set $u_\epsilon := -\frac{d}{c}+\epsilon\pi_\infty^r$ for $\epsilon\in\mathbb{F}_q^\times$. From the identity

$$\frac{a}{c}-\frac{1}{c^2\epsilon\pi_\infty^r} = \frac{au_\epsilon+b}{cu_\epsilon+d},$$

and summing over all ϵ we get:

$$\begin{aligned}
&(q-1)\tilde{\Theta}_{ij}\begin{pmatrix}\pi_\infty^r & u\\ 0 & 1\end{pmatrix} - \sum_{\epsilon\in\mathbb{F}_q^\times}\Theta_{ij}\begin{pmatrix}\frac{\pi_\infty^{1-r}}{c^2} & \frac{au_\epsilon+b}{cu_\epsilon+d}\\ 0 & 1\end{pmatrix}\\
=\ & q\sum_{\deg\mu+2=1-r+2\deg c} \Theta_{ij}^*(1-r+2\deg c,\mu)\psi_\infty(\mu\frac{a}{c}).
\end{aligned}$$

Note that

$$\begin{pmatrix}\frac{\pi_\infty^{1-r}}{c^2} & \frac{au_\epsilon+b}{cu_\epsilon+d}\\ 0 & 1\end{pmatrix} \text{ and } \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\pi_\infty^{r+1} & u_\epsilon\\ 0 & 1\end{pmatrix}$$

represent the same coset in $\mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$. Thus one has

$$\tilde{\Theta}_{ij}\begin{pmatrix}\pi_\infty^{r+1} & u\\ 0 & 1\end{pmatrix} - \tilde{\Theta}_{ij}\begin{pmatrix}\pi_\infty^r & u\\ 0 & 1\end{pmatrix} = \sum_{\epsilon\in\mathbb{F}_q^\times}\Theta_{ij}\begin{pmatrix}\pi_\infty^{r+1} & u+\epsilon\pi_\infty^r\\ 0 & 1\end{pmatrix}.$$

From the Fourier expansion of $\tilde{\Theta}_{ij}$ and Θ_{ij} we have that for $\lambda\in A$ with $\deg\lambda+2\leq r$,

$$\tilde{\Theta}_{ij}^*(r+1,\lambda)-\tilde{\Theta}_{ij}^*(r,\lambda) = (q-1)\Theta_{ij}^*(r+1,\lambda),$$

and for $\deg\lambda+2 = r+1$,

$$\tilde{\Theta}_{ij}^*(\deg\lambda+2,\lambda) = -\Theta_{ij}^*(r+1,\lambda).$$

Therefore $\tilde{\Theta}_{ij}^*(r,\lambda) = -\Theta_{ij}^*(r,\lambda)$ for $\lambda\in A$ with $\lambda\neq 0$ and $r\geq\deg\lambda+2$.

To compute $\tilde{\Theta}_{ij}^*(r,0)$, note that

$$\begin{aligned}
\tilde{\Theta}_{ij}\begin{pmatrix}\pi_\infty^r & 0\\ 0 & 1\end{pmatrix} &= \sum_{\deg\lambda\leq r-2}\tilde{\Theta}_{ij}^*(r,\lambda)\\
&= \tilde{\Theta}_{ij}^*(r,0)+\sum_{\lambda\neq 0,\deg\lambda\leq r-2} -\Theta_{ij}^*(r,\lambda).
\end{aligned}$$

On the other hand, for any $\epsilon \in \mathbb{F}_q^\times$ and $\ell \geq 0$ the following two matrices

$$\begin{pmatrix} \pi_\infty^{\deg N_0+\ell} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}, \begin{pmatrix} \epsilon^{-1} & -1 \\ -t^\ell N_0 & \epsilon(t^\ell N_0+1) \end{pmatrix} \begin{pmatrix} \frac{\pi_\infty^{1-\deg N_0-\ell}}{(t^\ell N_0)^2} & \frac{\epsilon(t^\ell N_0+1)}{t^\ell N_0} \\ 0 & 1 \end{pmatrix}$$

represent the same coset in $\mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$. Therefore

$$\tilde{\Theta}_{ij} \begin{pmatrix} \pi_\infty^{\deg N_0+\ell} & 0 \\ 0 & 1 \end{pmatrix} = \sum_{\deg \lambda \leq \deg N_0+\ell-1} \Theta_{ij}^*(\deg N_0+\ell+1, \lambda)\psi_\infty(\lambda \frac{\epsilon}{t^\ell N_0})$$

$$= \sum_{\deg \lambda \leq \deg N_0+\ell-2} \Theta_{ij}^*(\deg N_0+\ell+1, \lambda) - \frac{1}{q-1} \sum_{\deg \lambda = \deg N_0+\ell-1} \Theta_{ij}^*(\deg N_0+\ell+1, \lambda).$$

This gives

$$\begin{aligned} &\tilde{\Theta}_{ij}^*(\deg N_0+\ell, 0) = \\ &= \Bigg(\Theta_{ij}^*(\deg N_0+\ell+1, 0) + (1+q) \sum_{\lambda \neq 0, \deg \lambda \leq \deg N_0+\ell-2} \Theta_{ij}^*(\deg N_0+\ell+1, \lambda) \\ &\qquad - \frac{1}{q-1} \sum_{\deg \lambda = \deg N_0+\ell-1} \Theta_{ij}^*(\deg N_0+\ell+1, \lambda) \Bigg) \\ &= -\Theta_{ij}^*(\deg N_0+\ell, 0) + \frac{1}{q-1} \cdot \left[q\Theta_{ij} \begin{pmatrix} \pi_\infty^{\deg N_0+\ell} & 0 \\ 0 & 1 \end{pmatrix} - \Theta_{ij} \begin{pmatrix} \pi_\infty^{\deg N_0+\ell+1} & 0 \\ 0 & 1 \end{pmatrix} \right] \end{aligned}$$

Using the fact that M_{ij} is discrete and cocompact in $\mathcal{D}_\infty = \mathcal{D} \otimes_k k_\infty$, it can be deduced that for sufficiently large s one has

$$q\Theta_{ij} \begin{pmatrix} \pi_\infty^s & 0 \\ 0 & 1 \end{pmatrix} = \Theta_{ij} \begin{pmatrix} \pi_\infty^{s+1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus from the equality $\tilde{\Theta}_{ij}^*(r+1, 0) - \tilde{\Theta}_{ij}^*(r, 0) = (q-1)\Theta_{ij}^*(r+1, 0)$ for all $r \in \mathbb{Z}$ one has

$$\tilde{\Theta}_{ij}^*(r, 0) = -\Theta_{ij}^*(r, 0).$$

Comparing the Fourier coefficients we obtain $\tilde{\Theta}_{ij} = -\Theta_{ij}$ and hence Θ_{ij} is of Drinfeld type for any $1 \leq i, j \leq n$.

2.2 HECKE OPERATORS

Let f be an automorphic form on $\mathrm{GL}_2(k_\infty)$ for $\Gamma_0(N)$. For each prime P of A, the *Hecke operator* T_P is defined by:

$$\begin{aligned} T_P f(g) &:= \sum_{\deg u < \deg P} f(\begin{pmatrix} 1 & u \\ 0 & P \end{pmatrix} \cdot g) + f(\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \cdot g) \quad \text{if } P \nmid N, \\ T_P f(g) &:= \sum_{\deg u < \deg P} f(\begin{pmatrix} 1 & u \\ 0 & P \end{pmatrix} \cdot g) \quad \text{if } P | N. \end{aligned}$$

Note that the Fourier coefficients of $T_P f$ are of the form:

$$\begin{aligned} (T_P f)^*(r, \lambda) &= q^{\deg(P)} \cdot f^*(r + \deg(P), P\lambda) + f^*(r - \deg(P), \tfrac{\lambda}{P}) \quad \text{if } P \nmid N, \\ (T_P f)^*(r, \lambda) &= q^{\deg(P)} \cdot f^*(r + \deg(P), P\lambda) \quad \text{if } P | N. \end{aligned}$$

Here $f^*(\pi_\infty^r, \frac{\lambda}{P}) = 0$ if $P \nmid \lambda$. Since T_P and $T_{P'}$ commute,we can define Hecke operators T_m for monic polynomial m in A as follows:

$$\begin{cases} T_{mm'} = T_m T_{m'} & \text{if } m \text{ and } m' \text{ are relatively prime,} \\ T_{P^\ell} = T_{P^{\ell-1}} T_P - q^{\deg P} T_{P^{\ell-2}} & \text{for } P \nmid N, \\ T_{P^\ell} = T_P^\ell & \text{for } P \mid N. \end{cases}$$

We point out that if f is of Drinfeld type, then so is $T_m f$ for any monic polynomial m (cf. [7] Section 4.9).

When T_m acts on Θ_{ij}, we get:

PROPOSITION 2.4. *For any monic polynomial m in A,*

$$T_m \Theta_{ij} = \sum_\ell B_{i\ell}(m) \Theta_{\ell j} = \sum_\ell B_{\ell j}(m) \Theta_{i\ell}.$$

Proof. The second identity will follow from the first, as

$$w_j \Theta_{ij} = w_i \Theta_{ji} \text{ and } w_\ell B_{i\ell}(m) = w_i B_{\ell i}(m).$$

Note that the Hecke operators T_m satisfy the same relations as the matrices $B(m)$. Moreover, from the recurrence relations of Brandt matrices (cf. [16]) we have

$$\begin{aligned} \sum_\ell B_{i\ell}(P) B_{\ell j}(m) &= B_{ij}(mP) + q^{\deg(P)} B_{ij}(m/P) \text{ if } P \nmid N_0, \\ \sum_\ell B_{i\ell}(P) B_{\ell j}(m) &= B_{ij}(mP) \quad \text{if } P \mid N_0. \end{aligned}$$

Comparing the Fourier coefficients the result holds. □

Remark. Let $\mathcal{E}_{N_0} := \sum_{j=1}^{n} \Theta_{ij}$ (which is independent of the choice of i). For $r \in \mathbb{Z}$ and $\lambda \in A$ with $\deg \lambda + 2 \leq r$ the Fourier coefficients are

$$\mathcal{E}^*_{N_0}(r, \lambda) = q^{-r}\sigma(\lambda)_{N_0}$$

where

$$\sigma(\lambda)_{N_0} = \sum_{\substack{m \mid \lambda \text{ monic} \\ (m, N_0)=1}} q^{\deg m},$$

and

$$\mathcal{E}^*_{N_0}(r, 0) = q^{-r} \sum_{j=1}^{n} \frac{1}{w_j}.$$

Moreover, from Proposition 2.4 we have

$$T_m \mathcal{E}_{N_0} = \sigma(m)_{N_0} \mathcal{E}_{N_0}$$

for all monic polynomials m in A. This tell us that the function $\mathcal{E}_{N_0}$, which is an analogue of Eisenstein series, generates a one-dimensional eigenspace for all Hecke operators. We point out that suppose $N_0 = \prod_{i=1}^{\ell} P_i$, by comparing the Fourier coefficients one gets

$$q^2 \mathcal{E}_{N_0}(g_\infty) = E(g_\infty) + \sum_{i=1}^{\ell} (-1)^i \left[\sum_{1 \leq j_1 < \ldots < j_i \leq \ell} E\left(\begin{pmatrix} P_{j_1} \cdots P_{j_i} & 0 \\ 0 & 1 \end{pmatrix} g_\infty \right) \right]$$

for $g_\infty \in \mathrm{GL}_2(k_\infty)$ where E is the *improper Eisenstein series* introduced in [6].

For each non-zero ideal N of A, recall the *Petersson inner product*, which is a non-degenerate pairing on the finite dimensional $\mathbb{C}$-vector space $S(\Gamma_0(N))$ of automorphic cusp forms of Drinfeld type for $\Gamma_0(N)$,

$$(f, g) := \int_{G_0(N)} f \cdot \overline{g}.$$

Here $G_0(N) = \Gamma_0(N) \backslash \mathrm{GL}_2(k_\infty) / \Gamma_\infty k_\infty^\times$. The measure on $G_0(N)$ is taken by counting the size of the stablizer of an element (cf. [7] §4.8). More precisely, let Γ be a congruence subgroup and $e \in \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$. We denote $\mathrm{Stab}_\Gamma(e)$ the stabilizer of e in Γ, which is a finite subgroup in Γ. One takes the measure $d([e])$ of each double coset $[e]$ in $\Gamma \backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$ where

$$d([e]) := \frac{\#(Z(\Gamma))}{\#(\mathrm{Stab}_\Gamma(e))}.$$

Here $Z(\Gamma)$ is the subgroup of scalar matrices in Γ. When $\Gamma = \Gamma_0(N)$, for f and g in $S(\Gamma_0(N))$,

$$(f, g) = \sum_{[e] \in G_0(N)} f(e)\overline{g(e)} d([e]).$$

Definition 2.5. An *old form* is a linear combinations of forms

$$f'\left(\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} g_\infty\right)$$

for $g_\infty \in \mathrm{GL}_2(k_\infty)$, where f' is an automorphic cusp form of Drinfeld type for $\Gamma_0(M)$, $M|N$, $M \neq N$, and $d|(N/M)$. An automorphic cusp form f of Drinfeld type for $\Gamma_0(N)$ is called a *new form* if for any old form f' one has

$$(f, f') = 0.$$

If f is a new form which is also a Hecke eigenform, then f is called a *newform*.

It is known that the dimension of Drinfeld type cusp forms for $\Gamma_0(N_0)$ is equal to the genus of the Drinfeld modular curve $X_0(N_0)$ (cf. [7]). Let $S^{\mathrm{new}}(\Gamma_0(N_0))$ be the space of new forms for $\Gamma_0(N_0)$ and h_{N_0} be the number of left ideal classes of the maximal order R. As in the classical case, we can deduce that

$$h_{N_0} = \frac{1}{q^2-1}\prod_{P|N_0}(q^{\deg P}-1) + \frac{q}{2(q+1)}\prod_{P|N_0}(1-(-1)^{\deg P}).$$

From the genus formula of $X_0(N_0)$ in [5], the dimension of $S^{\mathrm{new}}(\Gamma_0(N_0))$ is equal to $h_{N_0} - 1$.
In the next subsection we will give our main theorem, which is essentially a construction of the space $S^{\mathrm{new}}(\Gamma_0(N_0))$ of new forms for $\Gamma_0(N_0)$ via the theta series Θ_{ij}.

2.3 Main theorem

Consider the definite Shimura curve $X = X_{N_0}$ introduced in §1. Recall the height pairing

$$< e, e' >= \sum_i a_i a_i'$$

where $e \in \mathrm{Pic}(X)$ with $e = \sum_i a_i e_i$ and $e' \in \mathrm{Pic}(X)^\vee$ with $e' = \sum_i a_i' \check{e}_i$.

Let $M(\Gamma_0(N_0))$ be the space of automorphic forms of Drinfeld type for $\Gamma_0(N_0)$. Define $\Phi : \mathrm{Pic}(X) \times \mathrm{Pic}(X)^\vee \to M(\Gamma_0(N_0))$ by

$$\Phi(e, e') := q^2 \sum_{i,j} a_i a_j' \Theta_{ij}$$

for any $e \in \mathrm{Pic}(X)$ with $e = \sum_i a_i e_i$ and $e' \in \mathrm{Pic}(X)^\vee$ with $e' = \sum_i a_i' \check{e}_i$. Then for $r \in \mathbb{Z}$ and $u \in k_\infty$ we have the following Fourier expansion

$$\Phi(e, e')\begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} = q^{-r+2}\left(\deg e \cdot \deg e' + \sum_{\substack{m \text{ monic,} \\ \deg m \leq r-2}} < e, t_m e' > \sum_{\epsilon \in \mathbb{F}_q^\times} \psi_\infty(\epsilon m u)\right).$$

Since

$$< t_m e, e' > = < e, t_m e' >$$

for any monic polynomial $m \in A$, by Proposition 2.4 one has

$$T_m\big(\Phi(e,e')\big) = \Phi(t_m e, e') = \Phi(e, t_m e').$$

In fact, the image of Φ is in $M^{\text{new}}(\Gamma_0(N_0)) := S^{\text{new}}(\Gamma_0(N_0)) \oplus \mathbb{C}\mathcal{E}_{N_0}$. To see this, we need the following claim.

Claim: for any monic m in A, consider t_m as in $\operatorname{End}(\operatorname{Pic}(X))$ and restrict T_m to the subspace $M^{\text{new}}(\Gamma_0(N_0))$. We have

$$Tr\, t_m = Tr\, T_m.$$

This claim tells us that the $\mathbb{C}$-algebra $\mathbb{T}_{\mathbb{C}}$ generated by all t_m is isomorphic to the $\mathbb{C}$-algebra generated by all Hecke operators T_m. Moreover, $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ and $M^{\text{new}}(\Gamma_0(N_0))$ are isomorphic as $\mathbb{T}_{\mathbb{C}}$-modules.
According to multiplicity one theorem, which will be recalled in the Appendix §A.2, $M^{\text{new}}(\Gamma_0(N_0))$ is a free rank one $\mathbb{T}_{\mathbb{C}}$-module. More precisely, $M^{\text{new}}(\Gamma_0(N_0))$ is generated by the element f whose Fourier coefficients are

$$f^*(r,\lambda) = q^{-r+2} \cdot Tr(T_m)$$

for all $0 \neq \lambda \in A$, $(\lambda) = (m)$, $\deg \lambda + 2 \leq r$. Therefore $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ is also a free rank one $\mathbb{T}_{\mathbb{C}}$-module. This shows

$$\dim_{\mathbb{C}} M^{\text{new}}(\Gamma_0(N_0)) = \dim_{\mathbb{C}} \Big[\big(\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}\big) \otimes_{\mathbb{T}_{\mathbb{C}}} \big(\operatorname{Pic}(X)^\vee \otimes_{\mathbb{Z}} \mathbb{C}\big)\Big].$$

Moreover, since

$$\sum_{i=1}^n < e_i, t_m \check{e}_i > = Tr(B(m)) = Tr(t_m),$$

we get $\sum_{i=1}^n \Phi(e_i, \check{e}_i) = f$, which generates $M^{\text{new}}(\Gamma_0(N_0))$. This also tells us that $\sum_{i=1}^n e_i \otimes \check{e}_i$ is a generator of the cyclic $\mathbb{T}_{\mathbb{C}}$-module $\big(\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}\big) \otimes_{\mathbb{T}_{\mathbb{C}}} \big(\operatorname{Pic}(X)^\vee \otimes_{\mathbb{Z}} \mathbb{C}\big)$.

The above argument gives us the main result:

Theorem 2.6. *There is a map* $\Phi : \operatorname{Pic}(X) \times \operatorname{Pic}(X)^\vee \longrightarrow M^{\text{new}}(\Gamma_0(N_0))$ *satisfying that for* $r \in \mathbb{Z}$ *and* $u \in k_\infty$

$$\Phi(e,e')\begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} = q^{-r+2}\Bigg(\deg e \cdot \deg e' + \sum_{\substack{m \text{ monic,} \\ \deg m \leq r-2}} < e, t_m e' > \sum_{(\lambda)=(m)} \psi_\infty(\lambda u)\Bigg),$$

and for all monic polynomials m in A

$$T_m\Phi(e,e') = \Phi(t_m e, e') = \Phi(e, t_m e').$$

Moreover, this map induces an isomorphism

$$(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}) \otimes_{\mathbb{T}_{\mathbb{C}}} (\mathrm{Pic}(X)^{\vee} \otimes_{\mathbb{Z}} \mathbb{C}) \cong M^{\mathrm{new}}(\Gamma_0(N_0))$$

as $\mathbb{T}_{\mathbb{C}}$-modules.

Remark. 1. When N_0 is a prime, $M^{\mathrm{new}}(\Gamma_0(N_0)) = M(\Gamma_0(N_0))$ and so the theta series Θ_{ij} gives us a construction of all automorphic forms of Drinfeld type for $\Gamma_0(N_0)$. This case was proven by Papikian [10] via a geometric approach.
2. Since the theta series Θ_{ij} are $\mathbb{Q}$-valued, the map Φ in Theorem 2.6 in fact induces an isomorphism

$$(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{T}_{\mathbb{Q}}} (\mathrm{Pic}(X)^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}) \cong M^{\mathrm{new}}(\Gamma_0(N_0), \mathbb{Q})$$

where $\mathbb{T}_{\mathbb{Q}}$ is the $\mathbb{Q}$-algebra generated by t_m for all monic m in A and $M^{\mathrm{new}}(\Gamma_0(N_0), \mathbb{Q})$ is the space of $\mathbb{Q}$-valued functions in $M^{\mathrm{new}}(\Gamma_0(N_0))$.
3. The CLAIM above is essentially *Jacquet-Langlands correspondence* over the function field k, which will be recalled in the Appendix §A.1.

2.4 EXAMPLE: THE FUNCTION $g_{\mathcal{A}}$

Having Theorem 2.6, we exhibit automorphic forms of Drinfeld type with nice arithmetic properties. Let $D \in A - k_\infty^2$ be a square-free element with the *quadratic Legendre symbol* $\left(\frac{D}{P}\right) \neq 1$ for all $P \mid N_0$. Let K be the imaginary quadratic field $k(\sqrt{D})$ and O_K be the integral closure of A in K. Recall that in §1.1 one has a free action of $\mathrm{Pic}(O_K)$ on the set G_D of Gross points of discriminant D in the definite Shimura curve $X = X_{N_0}$:

$$\begin{array}{ccc} G_D \times \mathrm{Pic}(O_K) & \longrightarrow & G_D \\ (x, \mathcal{A}) & \longmapsto & x_{\mathcal{A}}. \end{array}$$

Suppose a Gross point x of discriminant D in X is given. For each ideal class $\mathcal{A}$ in $\mathrm{Pic}(O_K)$, denote $e_{\mathcal{A}}$ to be the divisor class $(x_{\mathcal{A}})$ in $\mathrm{Pic}(X)$. Define

$$g_{\mathcal{A}} := \sum_{\mathcal{B} \in \mathrm{Pic}(O_K)} \Phi(e_{\mathcal{B}}, e_{\mathcal{A}\mathcal{B}}).$$

We have a nice formula for the Fourier coefficients of $g_{\mathcal{A}}$ in terms of Hecke actions: for monic $m \in A$ with $\deg m + 2 \leq r$,

$$g_{\mathcal{A}}^*(r,m) = q^{-r+2} \cdot \sum_{\mathcal{B} \in \mathrm{Pic}(O_K)} < e_{\mathcal{B}}, t_m e_{\mathcal{A}\mathcal{B}} >,$$
$$g_{\mathcal{A}}^*(r,0) = q^{-r+2} \cdot h_{O_K}.$$

Here $h_{O_K} = \#\,\mathrm{Pic}(O_K)$. Note that $g_{\mathcal{A}}$ is independent of the choice of the Gross point x.

From now on we assume D is irreducible with $\left(\frac{D}{P}\right) = -1$ for all primes $P \mid N_0$. According to Dirichlet's theorem there exists a monic irreducible polynomial Q prime to N_0 and $\epsilon_0 \in \mathbb{F}_q^\times - \mathbb{F}_q^2$ such that $\deg N_0QD$ is odd and $\epsilon_0 N_0 Q \equiv 1 \bmod D$. Then there exists $j \in \mathcal{D}$ with $j^2 = \epsilon_0 N_0 Q$ so that $\mathcal{D} = K + Kj$ and $j^{-1}\alpha j = \bar{\alpha}$ for $\alpha \in K$.
Let $\mathfrak{d} = (\sqrt{D})$ be the different of O_K, which is a prime ideal in O_K. Since $\epsilon_0 N_0 Q \equiv 1 \bmod D$, one has $\left(\frac{\epsilon N_0 Q}{D}\right) = 1$. From the reciprocity law we get $\left(\frac{D}{Q}\right) = 1$ and so the prime ideal (Q) is split in K. Suppose $(Q) = \mathfrak{q}\bar{\mathfrak{q}}$ and set

$$R := \{\alpha + \beta j : \alpha \in \mathfrak{d}^{-1}, \beta \in \mathfrak{d}^{-1}\mathfrak{q}^{-1}, \alpha - \beta \in O_{\mathfrak{d}}\}.$$

Here $O_{\mathfrak{d}}$ is the localization of O_K at $\mathfrak{d}$. It is clear that R is an A-lattice in $\mathcal{D}$ containing 1. In fact, R is a maximal A-order and $K \cap R = O_K$. To show R is an A-order, let $\alpha_1 + \beta_1 j$ and $\alpha_2 + \beta_2 j$ be two elements in R. Then

$$(\alpha_1 + \beta_1 j)(\alpha_2 + \beta_2 j) = (\alpha_1\alpha_2 + \beta_1\bar{\beta}_2\epsilon_0 N_0 Q) + (\alpha_1\beta_2 + \beta_1\bar{\alpha}_2)j.$$

For $i = 1, 2$, write β_i as $\alpha_i + \delta_i$ with $\delta_i \in O_{\mathfrak{d}}$. Then

$$\alpha_1\alpha_2 + \beta_1\bar{\beta}_2\epsilon_0 N_0 Q = \alpha_1(\alpha_2 + \bar{\alpha}_2) + (\delta_1\bar{\beta}_2 + \beta_1\bar{\delta}_2 + \delta_1\bar{\delta}_2)\epsilon_0 N_0 Q.$$

Since $\alpha_2 \in \mathfrak{d}^{-1} = A + \sqrt{D^{-1}}A$, one has

$$\alpha_2 + \bar{\alpha}_2 \in A.$$

Hence

$$\alpha_1\alpha_2 + \beta_1\bar{\beta}_2\epsilon_0 N_0 Q \in \mathfrak{d}^{-2} \cap \sqrt{D}^{-1} O_{\mathfrak{d}} = \mathfrak{d}^{-1}.$$

Similarly,

$$\alpha_1\beta_2 + \beta_1\bar{\alpha}_2 \in \mathfrak{d}^{-2}\mathfrak{q}^{-1} \cap \sqrt{D}^{-1} O_{\mathfrak{d}} = \mathfrak{d}^{-1}\mathfrak{q}^{-1}.$$

From the condition that $\epsilon_0 N_0 Q \equiv 1 \bmod D$, one can check that

$$\alpha_1\alpha_2 + \beta_1\bar{\beta}_2\epsilon_0 N_0 Q - (\alpha_1\beta_2 + \beta_1\bar{\alpha}_2) \in O_{\mathfrak{d}}.$$

Therefore R is an A-order. The discriminant of R is $(N_0)^2$, which can be checked locally. This implies that R is maximal.

Let x be the Gross point in the definite Shimura curve $X = X_{N_0}$ which corresponds to the trivial ideal R and the embedding $K \hookrightarrow \mathcal{D}$. Then x is of discriminant D. Using this particular Gross point we can get an explicit formula for the Fourier coefficients of $g_{\mathcal{A}}$.

Note that there is a one-to-one correspondence between the irreducible components of X and the left ideal classes of R. Let $\mathfrak{a} \in \mathcal{A}$, $\mathfrak{b} \in \mathcal{B}$. Then $R\mathfrak{a}$ and $R\mathfrak{a}\mathfrak{b}$ are representatives of the left ideal classes of R corresponding to $e_{\mathcal{A}}$ and $e_{\mathcal{A}\mathcal{B}}$ respectively. Therefore

$$< e_{\mathcal{B}}, t_m e_{\mathcal{A}\mathcal{B}} >= \frac{1}{q-1} \#\{b \in \mathfrak{b}^{-1} R \mathfrak{b}\mathfrak{a} := (\mathrm{Nr}(b))/\operatorname{Nr}(\mathfrak{a}) = (m)\}.$$

Assume $N_0\mathfrak{d}$ and $\mathfrak{a}$ are relatively prime. Then

$$\mathfrak{b}^{-1}R\mathfrak{b}\mathfrak{a} \quad = \quad \{\alpha + \beta j : \alpha \in \mathfrak{d}^{-1}\mathfrak{a}, \beta \in \mathfrak{d}^{-1}\mathfrak{b}^{-1}\bar{\mathfrak{b}}\mathfrak{q}^{-1}\bar{\mathfrak{a}}, \alpha - (-1)^{\mathrm{ord}_{\mathfrak{d}}(\mathfrak{b})}\beta \in O_{\mathfrak{d}}\}.$$

We can express the Fourier coefficients of $g_{\mathcal{A}}$ in terms of sums of the counting numbers

$$r_{\mathcal{A}}((\lambda)) := \#\{\mathfrak{a} \in \mathcal{A} : \mathfrak{a} \text{ integral with } \operatorname{Nr}(\mathfrak{a}) = (\lambda)\},$$

for ideals (λ) of A, by the following proposition:

Proposition 2.7. *Suppose $D \in A - k_\infty^2$ is irreducible with $\left(\frac{D}{P}\right) = -1$ for all primes $P \mid N_0$. Then for any monic polynomial m in A,*

$$\sum_{\mathcal{B} \in \mathrm{Pic}(O_K)} < e_{\mathcal{B}}, t_m e_{\mathcal{A}\mathcal{B}} >= \frac{1}{2(q-1)} \Bigg[2r_{\mathcal{A}}((mD))(q-1)h_{O_K}$$
$$+ \sum_{\substack{\mu \in A, \mu \neq 0 \\ \deg(\mu N_0) \leq \deg(mD)}} r_{\mathcal{A}}((\mu N_0 - mD))(t(\mu, D) + 1)(1 - \delta_{\mu N_0(\mu N_0 - mD)}) \sum_{c \mid \mu} \left(\frac{D}{c}\right) \Bigg].$$

Here $t(\mu, D) = 1$ if D divides μ and 0 otherwise, and δ_z is the norm residue symbol of z for $z \in k_\infty^\times$: $\delta_z = 1$ if $z \in \operatorname{Nr}(K_\infty^\times)$ and -1 otherwise.

Proof. Let $\mathfrak{a} \in \mathcal{A}$ which is a proper ideal of O_K and prime to $N_0\mathfrak{d}$. Fix a generator λ_0 of $\operatorname{Nr}(\mathfrak{a}) = \mathfrak{a}\bar{\mathfrak{a}}$. Given $\mathcal{B} \in \operatorname{Pic}(O_K)$. Let $\mathfrak{b} \in \mathcal{B}$. For $b = \alpha + \beta j \in \mathfrak{b}^{-1}R\mathfrak{b}\mathfrak{a}$, i.e. $\alpha \in \mathfrak{d}^{-1}\mathfrak{a}$, $\beta \in \mathfrak{d}^{-1}\mathfrak{q}^{-1}\mathfrak{b}^{-1}\bar{\mathfrak{b}}\bar{\mathfrak{a}}$, $\alpha - (-1)^{\mathrm{ord}_{\mathfrak{d}}\,\mathfrak{b}}\beta \in O_{\mathfrak{d}}$, define:

(1) $\mathfrak{c} := (\beta)\mathfrak{d}\mathfrak{q}\bar{\mathfrak{b}}^{-1}\mathfrak{b}\bar{\mathfrak{a}} \in [\mathfrak{q}]\mathcal{B}^2\mathcal{A}$,
(2) $\nu := -\operatorname{Nr}(\alpha)D\lambda_0^{-1} \in A$,
(3) $\mu := -\epsilon_0 \operatorname{Nr}(\beta)DQ\lambda_0^{-1} \in A$.

Here $[\mathfrak{q}] \in \operatorname{Pic}(O_K)$ is the ideal class containing $\mathfrak{q}$. Then $\mathfrak{c}$ is integral and

$$\operatorname{Nr}(\alpha + \beta j) = \operatorname{Nr}(\alpha) - \epsilon_0 N_0 Q \operatorname{Nr}(\beta) = (-\nu + N_0\mu)D^{-1}\lambda_0.$$

Thus $(\operatorname{Nr}(\alpha + \beta j)) = (m\lambda_0)$ if and only if $\nu = N_0\mu - \epsilon m D$ for a uniquely determined $\epsilon \in \mathbb{F}_q^\times$.

Since $\beta = 0$ if and only if $b = \alpha \in \mathfrak{a}$, one has

$$\begin{aligned} & \#\{b \in \mathfrak{b}^{-1}R\mathfrak{b}\mathfrak{a} : \mathrm{Nr}(b) = (m\lambda_0)\} \\ = \quad & \#\{b = \alpha + \beta j \in \mathfrak{b}^{-1}R\mathfrak{b}\mathfrak{a} : \beta \neq 0, \mathrm{Nr}(b) = (m\lambda_0)\} \\ & + \#\{\alpha \in \mathfrak{a} : \mathrm{Nr}(\alpha) = (m\lambda_0)\}. \end{aligned}$$

It can be shown that $\#\{\alpha \in \mathfrak{a} : \mathrm{Nr}(\alpha) = (m\lambda_0)\} = (q-1)r_{\mathcal{A}}((mD))$. Note that $\beta \neq 0$ if and only if $\mu \neq 0$. In this case, β is uniquely determined by the integral ideal $\mathfrak{c}$ up to multiplying elements in $O_K^\times$.

Conversely, given $0 \neq \mu \in A$ and $\epsilon \in \mathbb{F}_q^\times$ and set $\nu = N_0\mu - \epsilon mD$. The number of elements $\alpha \in \mathfrak{d}^{-1}\mathfrak{a}$ with $\mathrm{Nr}(\alpha) = -\nu D^{-1}\lambda_0$ is $r_{\mathfrak{a},\lambda_0}(N_0\mu - \epsilon mD)$. Here

$$r_{\mathfrak{a},\lambda_0}(\lambda) := \#\{a \in \mathfrak{a} : \mathrm{Nr}(a) = \lambda\lambda_0\} \text{ for } \lambda \in A.$$

In the case of $r_{\mathfrak{a},\lambda_0}(N_0\mu - \epsilon mD) \neq 0$, choose an element $\alpha \in \mathfrak{d}^{-1}\mathfrak{a}$ with $\mathrm{Nr}(\alpha) = -\nu D^{-1}\lambda_0$. Let $\mathfrak{c}$ be an integral ideal which lies in a class differing from the ideal class $\mathcal{A}[\mathfrak{q}]$ by a square $[\mathfrak{b}]^2$ in the class group $\mathrm{Pic}(O_K)$ and with ideal norm (μ). Then

$$\mathfrak{c} \cdot \mathfrak{b}^{-1}\bar{\mathfrak{b}}\bar{\mathfrak{a}}\mathfrak{q}^{-1}\mathfrak{d}^{-1} = (\beta)$$

for some $\beta \in K^\times$. Suppose we can find β so that $\mu = -\epsilon_0 \mathrm{Nr}(\beta) DQ\lambda_0^{-1} \in A$. Since $\epsilon_0 N_0 Q \equiv 1 \bmod D$, the equality $\epsilon m\lambda_0 = \mathrm{Nr}(\alpha) - \epsilon_0 N_0 Q \mathrm{Nr}(\beta) \in A$ implies

$$\alpha \pm \beta \in O_{\mathfrak{d}}.$$

Choose $\ell \in \{0,1\}$ and replace $\mathfrak{b}$ by $\mathfrak{b}\mathfrak{d}^\ell$ so that $\alpha - (-1)^{\mathrm{ord}_{\mathfrak{d}}(\mathfrak{b})}\beta \in O_{\mathfrak{d}}$. Therefore $b = \alpha + \beta j \in \mathfrak{b}^{-1}R\mathfrak{b}\mathfrak{a}$ with $\mathrm{Nr}(b) = \epsilon m\lambda_0$. Note that if β is not in $O_{\mathfrak{d}}$ (i.e. $D \nmid \mu$), then ℓ is uniquely determined. If $\beta \in O_{\mathfrak{d}}$ (i.e. $D \mid \mu$), then we have two choices $\pm\beta$. The existence of β is equivalent to that $-\epsilon_0^{-1}D\mu Q^{-1}\lambda_0$ is in $\mathrm{Nr}(K^\times)$. Since $\mathrm{Nr}(\sqrt{D}) = -D$ and $(\epsilon_0^{-1}\mu Q^{-1}\lambda_0) = \mathrm{Nr}(\mathfrak{c}\mathfrak{q}^{-1}\bar{\mathfrak{a}})$, we have $\epsilon_0^{-1}\mu Q^{-1}\lambda_0 \in \mathrm{Nr}(K^\times)$ if and only if $\delta_{\epsilon_0^{-1}\mu Q^{-1}\lambda_0} = 1$. Therefore combining the above arguments we have

$$\begin{aligned} & \sum_{\mathcal{B} \in \mathrm{Pic}(O_K)} \#\{b = \alpha + \beta j \in \mathfrak{b}^{-1}R\mathfrak{b}\mathfrak{a} : \beta \neq 0, \mathrm{Nr}(b) = (m\lambda_0)\} \\ = & \sum_{0 \neq \mu \in A} \sum_{\epsilon \in \mathbb{F}_q^\times} r_{\mathfrak{a},\lambda_0}(N_0\mu - \epsilon mD) \cdot (t(\mu, D) + 1) \cdot \mathcal{R}_{\{\mathcal{A}[\mathfrak{q}]\}}((\mu)) \cdot \frac{1 + \delta_{\epsilon_0^{-1}\mu Q^{-1}\lambda_0}}{2}. \end{aligned}$$

Here $\mathcal{R}_{\{\mathcal{A}[\mathfrak{q}]\}}((\mu))$ is the number of integral ideals $\mathfrak{c}$, which lie in a class differing from the class $\mathcal{A}[\mathfrak{q}]$ by a square in the class group $\mathrm{Pic}(O_K)$ and with ideal norm (μ). Following the proof of Lemma 3.4.9 in [12] one has

Lemma 2.8. *For $0 \neq \mu \in A$,*

$$\mathcal{R}_{\{\mathcal{A}[\mathfrak{q}]\}}((\mu)) \cdot \frac{1 + \delta_{\epsilon_0^{-1}\mu Q^{-1}\lambda_0}}{2} = \frac{1}{q-1} \sum_{c \mid \mu} \left(\frac{D}{c}\right) \cdot \frac{1 + \delta_{\epsilon_0^{-1}\mu Q^{-1}\lambda_0}}{2}.$$

Since $\delta_{\epsilon_0^{-1}\mu Q^{-1}\lambda_0} = 1$ if and only if $\delta_{N_0\mu\lambda_0} = -1$, with Lemma 2.6 we have

$$\sum_{\mathcal{B}\in \mathrm{Pic}(O_K)} \#\{b = \alpha + \beta j \in \mathfrak{b}^{-1}R\mathfrak{b}\mathfrak{a} : \beta \neq 0, \mathrm{Nr}(b) = (m\lambda_0)\}$$
$$= \sum_{0\neq\mu\in A}\sum_{\epsilon\in\mathbb{F}_q^\times} r_{\mathfrak{a},\lambda_0}(N_0\mu - \epsilon m D)(t(\mu,D)+1)\cdot\frac{1-\delta_{N_0\mu\lambda_0}}{2}\cdot\frac{1}{q-1}\sum_{c|\mu}\left(\frac{D}{c}\right)$$
$$= \sum_{\substack{\mu\in A,\mu\neq 0\\ \deg(\mu N_0)\leq\deg(mD)}} r_{\mathcal{A}}((\mu N_0 - mD))(t(\mu,D)+1)\cdot\frac{1-\delta_{\mu N_0(\mu N_0 - mD)}}{2}\cdot\sum_{c|\mu}\left(\frac{D}{c}\right).$$

Therefore

$$\sum_{\mathcal{B}\in\mathrm{Pic}(O_K)} < e_{\mathcal{B}}, t_m e_{\mathcal{A}\mathcal{B}} >= \frac{1}{2(q-1)}\Bigg[2r_{\mathcal{A}}((mD))(q-1)h_{O_K}$$
$$+\sum_{\substack{\mu\in A,\mu\neq 0\\ \deg(\mu N_0)\leq\deg(mD)}} r_{\mathcal{A}}((\mu N_0 - mD))(t(\mu,D)+1)(1-\delta_{\mu N_0(\mu N_0-mD)})\sum_{c|\mu}\left(\frac{D}{c}\right)\Bigg].$$

□

3 Special values of L-series

3.1 Rankin product

To an automorphic cusp form f of Drinfeld type for $\Gamma_0(N)$ one can attach an L-series $L(f,s)$: let $\mathfrak{m}$ be an effective divisor of k, which can be written as $\mathrm{div}(\lambda)_0 + (r - \deg\lambda)\infty$ for a nonzero polynomial λ $(= \lambda(\mathfrak{m}))$ in A, with

$$\mathrm{div}(\lambda)_0 := \sum_{\text{finite prime } P} \mathrm{ord}_P(\lambda)P.$$

Denote

$$f^*(\mathfrak{m}) := \int_{A\backslash k_\infty} f\begin{pmatrix}\pi_\infty^{r+2} & u\\ 0 & 1\end{pmatrix}\psi_\infty(-\lambda u)du = f^*(r+2,\lambda).$$

The L-series $L(f,s)$ attached to f is

$$L(f,s) := \sum_{\mathfrak{m}\geq 0} f^*(\mathfrak{m})q^{-\deg(\mathfrak{m})s},\ \ \mathrm{Re}\, s > 1.$$

Let $D \in A - k_\infty^2$ be a square-free element. Consider the imaginary field $K = k(\sqrt{D})$. Let O_K be the integral closure of A in K and $\mathrm{Pic}(O_K)$ be the ideal class group of O_K. Given an ideal class $\mathcal{A} \in \mathrm{Pic}(O_K)$ and a polynomial λ in

A. The number of integral ideals $\mathfrak{a}$ in the class $\mathcal{A}$ with $N_{K/k}(\mathfrak{a}) = (\lambda)$ leads to the partial zeta function attached to $\mathcal{A}$:

$$\zeta_{\mathcal{A}}(s) := \sum_{\mathfrak{m} \geq 0} r_{\mathcal{A}}(\mathfrak{m}) q^{-\deg(\mathfrak{m})s}, \quad \operatorname{Re} s > 1.$$

Here for each effective divisor $\mathfrak{m} = \operatorname{div}(\lambda)_0 + (r - \deg \lambda)\infty$,

$$r_{\mathcal{A}}(\mathfrak{m}) := \#\{\mathfrak{a} \in \mathcal{A} : \mathfrak{a} \text{ integral with } N_{K/k}(\mathfrak{a}) = (\lambda)\}.$$

Let f be an automorphic cusp form of Drinfeld type for $\Gamma_0(N)$. For each ideal class $\mathcal{A} \in \operatorname{Pic}(O_K)$, we are interested in the *Rankin product*:

$$L(f, \mathcal{A}, s) := \sum_{\mathfrak{m} \geq 0} f^*(\mathfrak{m}) r_{\mathcal{A}}(\mathfrak{m}) q^{-\deg(\mathfrak{m})s}, \quad \operatorname{Re}(s) > 1.$$

To study the analytic continuation and the functional equation of $L(f, \mathcal{A}, s)$, consider the function $\Lambda(f, \mathcal{A}, s)$ which is defined by:

$$\Lambda(f, \mathcal{A}, s) := \begin{cases} L^{(N,D)}(2s+1) L(f, \mathcal{A}, s) & \text{when } \deg D \text{ is odd,} \\ \dfrac{1}{1+q^{-s-1}} L^{(N,D)}(2s+1) L(f, \mathcal{A}, s) & \text{when } \deg D \text{ is even.} \end{cases}$$

Here $L^{(N,D)}(s)$ is the following L-series indexed by effective divisors supported outside ∞

$$L^{(N,D)}(s) := \frac{1}{q-1} \sum_{d \in A, (d,N)=1} \left(\frac{D}{d}\right) q^{-s \deg d}, \quad \operatorname{Re}(s) > 1,$$

where $\left(\frac{D}{d}\right)$ denotes the Legendre symbol for the polynomial ring A. Note that

$$L^{(N,D)}(s) = L_D(s) \cdot \prod_{\text{prime ideals } P | N} (1 - \left(\frac{D}{P}\right) q^{-s \deg P})^{-1}$$

where $L_D(s)$ is the Dirichlet L-series:

$$L_D(s) := \frac{1}{q-1} \sum_{d \in A, d \neq 0} \left(\frac{D}{d}\right) q^{-s \deg d}, \ \operatorname{Re}(s) > 1.$$

It is known that $L_D(s)$ can be extended to a polynomial in q^{-s} with the functional equation (cf. [1]):

$$L_D(2s+1) = q^{s(-2\deg D + 2) - \frac{1}{2}\deg D + \frac{1}{2}} L_D(-2s)$$

if $\deg D$ is odd, and

$$L_D(-2s+1) = \frac{1+q^{1-2s}}{1+q^{2s}} q^{\deg D (2s - \frac{1}{2})} L_D(2s)$$

if $\deg D$ is even.
When f is a new form and D is irreducible, Rück and Tipp ([12]) prove the following functional equation of $\Lambda(f, \mathcal{A}, s)$:

$$\Lambda(f, \mathcal{A}, s) = -\left(\frac{D}{N}\right) q^{(5-2\deg D - 2\deg N)s} \Lambda(f, \mathcal{A}, -s)$$

when $\deg D$ is odd, and

$$\Lambda(f, \mathcal{A}, s) = -\left(\frac{D}{N}\right) q^{(6-2\deg D - 2\deg N)s} \Lambda(f, \mathcal{A}, -s)$$

when $\deg D$ is even.

3.2 Central critical values of $\Lambda(f, \mathcal{A}, s)$

We are interested in the special value of $\Lambda(f, \mathcal{A}, s)$ at $s = 0$. Note that if $\left(\frac{D}{N}\right) = 1$, then $\Lambda(f, \mathcal{A}, s)$ has a zero at $s = 0$. We focus here on the special case when $\left(\frac{D}{P}\right) = -1$ for all primes $P \mid N_0$. Adapting Rankin's method (cf. [12]), we can establish the following theorem.

Theorem 3.1. *Let f be a Drinfeld type new form for $\Gamma_0(N_0)$ and let D be an irreducible polynomial in $A - k_\infty^2$ with $\left(\frac{D}{P}\right) = -1$ for all primes $P \mid N_0$. One has*

$$\Lambda(f, \mathcal{A}, 0) = \begin{cases} \dfrac{(f, g_{\mathcal{A}})}{q^{\frac{1}{2}(\deg D + 1)}} & \text{when } \deg D \text{ is odd,} \\[2ex] \dfrac{(f, g_{\mathcal{A}})}{2q^{\frac{1}{2}\deg D}} & \text{when } \deg D \text{ is even.} \end{cases}$$

Here $(\cdot, \cdot)$ is the Petersson inner product and $g_{\mathcal{A}}$ is the Drinfeld type automorphic form for $\Gamma_0(N_0)$ canonically attached to $\mathcal{A}$ in §2.4.

3.2.1 Review of Rankin's method

Given $\mathcal{A} \in \mathrm{Pic}(O_K)$. Choose $\mathfrak{a}_0 \in \mathcal{A}^{-1}$ and $\lambda_0 \in k$ such that $N_{K/k}(\mathfrak{a}_0) = (\lambda_0)$ Recall the counting number

$$r_{\mathfrak{a}_0, \lambda_0}(\lambda) = \#\{\mu \in \mathfrak{a}_0 : N_{K/k}(\mu) = \lambda_0 \lambda\}.$$

Note that $r_{\mathfrak{a}_0, \lambda_0}(\lambda) = r_{\mathfrak{a}_0^{-1}, \lambda_0^{-1}}(\lambda)$, and for effective divisor $\mathfrak{m} = \mathrm{div}(\lambda)_0 + (\deg \mathfrak{m} - \deg \lambda)\infty$ we have

$$r_{\mathcal{A}}(\mathfrak{m}) = \frac{1}{q-1} \sum_{\epsilon \in \mathbb{F}_q^\times} r_{\mathfrak{a}_0, \lambda_0}(\epsilon \lambda).$$

We consider the following theta series $\theta_{\mathfrak{a}_0, \lambda_0}$ (introduced in [11]) defined on $k_\infty^\times \times k_\infty$:

$$\theta_{\mathfrak{a}_0, \lambda_0}(\pi_\infty^r, u) := \sum_{\deg \lambda + 2 \leq r} r_{\mathfrak{a}_0, \lambda_0}(\lambda) \psi_\infty(\lambda u).$$

It satisfies the following transformation law:

$$\theta_{\mathfrak{a}_0,\lambda_0}\left(\frac{\pi_\infty^r}{(cu+d)^2}, \frac{au+b}{cu+d}\right) = \delta_{cu+d}\left(\frac{d}{D}\right) q^{-v_\infty(cu+d)}\theta_{\mathfrak{a}_0,\lambda_0}(\pi_\infty^r, u)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(1)}(N) := \Gamma_0(N) \cap \mathrm{SL}_2(A)$ with $v_\infty(c\pi_\infty^r) > v_\infty(cu+d)$. Here δ is the local norm symbol at ∞, i.e. $\delta_z = 1$ if $z \in k_\infty^\times$ is a norm of an element in $K_\infty = k_\infty(\sqrt{D})$ and -1 otherwise.

Viewing $\theta_{\mathfrak{a}_0,\lambda_0}$ as a function on

$$\mathbb{H}_\infty := \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \Big\backslash \begin{pmatrix} k_\infty^\times & k_\infty \\ 0 & 1 \end{pmatrix} \Big/ \begin{pmatrix} \mathcal{O}_\infty^\times & \mathcal{O}_\infty \\ 0 & 1 \end{pmatrix},$$

one can write

$$\begin{aligned} L(f,\mathcal{A},s) &= \frac{q}{q-1}\sum_{r=2}^{\infty}\left[\sum_{u\in\pi_\infty O_\infty/\pi_\infty^r O_\infty} f\cdot\overline{\theta_{\mathfrak{a}_0,\lambda_0}}\begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} q^{-r(s+1)+2s}\right] \\ &= \frac{q}{q-1}\int_{\mathbb{H}_\infty} f(h)\overline{\theta_{\mathfrak{a}_0,\lambda_0}(h) q^{-r(\bar{s}+1)+2\bar{s}}}dh. \end{aligned}$$

For every monic polynomial M in A, the canonical map

$$\mathbb{H}_\infty \longrightarrow G(M) := \Gamma_0^{(1)}(M)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$$

is surjective. Following [12], we consider the "Eisenstein series"

$$E_s\begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} := \sum_{\substack{c,d\in A, c\equiv 0 \bmod D \\ v_\infty(c\pi_\infty^r)>v_\infty(cu+d)}} \left(\frac{d}{D}\right)\delta_{cu+d}q^{v_\infty(cu+d)(2s+1)}$$

and let $H_s\begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} :=$

$$\begin{cases} q^{-r(s+1)+2s}E_s\begin{pmatrix} N\pi_\infty^r & Nu \\ 0 & 1 \end{pmatrix} & \text{when } \deg D \text{ is odd,} \\ \left(\frac{(-1)^{r-\deg\lambda_0}+1}{2}\right)\cdot q^{-r(s+1)+2s}E_s\begin{pmatrix} N\pi_\infty^r & Nu \\ 0 & 1 \end{pmatrix} & \text{when } \deg D \text{ is even.} \end{cases}$$

Then $\theta_{\mathfrak{a}_0,\lambda_0}H_{\bar{s}}$ can be viewed as a function on $G(ND)$. By [12] Proposition 2.2.2 and Proposition 2.3.2

$$\Lambda(f,\mathcal{A},s) = \frac{q}{2(q-1)}\int_{G(ND)} f\cdot\overline{\theta_{\mathfrak{a}_0,\lambda_0}H_{\bar{s}}}.$$

Given $M \in A$. Let $\mathcal{F}(M)$ be the space of functions on $G(M)$. The trace map from $\mathcal{F}(ND)$ to $\mathcal{F}(N)$ is given by

$$f \longrightarrow Tr_N^{ND} f(g) := \sum_{\gamma \in \Gamma_0^{(1)}(ND)\backslash \Gamma_0^{(1)}(N)} f(\gamma g).$$

Set $\Phi_s := Tr_N^{ND}(\theta_{\mathfrak{a}_0,\lambda_0} H_{\bar{s}})$. Then

$$\Lambda(f,\mathcal{A},s) = \frac{q}{2(q-1)} \int_{G(N)} f \cdot \overline{\Phi_{\bar{s}}}.$$

From the harmonicity of f one has

$$\Lambda(f,\mathcal{A},s) = \frac{q}{4(q-1)} \int_{G(N)} f \cdot \overline{F_{\bar{s}}}$$

where for $g \in \mathrm{GL}_2(k_\infty)$,

$$F_s(g) := \frac{q}{q+1}\big(\Phi_s(g) - \tilde{\Phi}_s(g)\big) - \frac{1}{q+1} \sum_{\substack{\beta \in \mathrm{GL}_2(O_\infty)/\Gamma_\infty, \\ \beta \neq 1}} \big(\Phi_s(g\beta) - \tilde{\Phi}_s(g\beta)\big).$$

Note that F_s depends on the choice of $\mathfrak{a}_0$ and λ_0.

3.2.2 Proof of Theorem 3.1

Let Ψ be the average map from functions F on $G(N)$ to functions on $G_0(N)$:

$$\Psi(F)(g) := \frac{1}{q-1} \sum_{\epsilon \in \mathbb{F}_q^\times} F(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} g).$$

Define

$$\Psi_{\mathcal{A}} := \Psi(F_0).$$

Note that $\Psi_{\mathcal{A}}$ now depends only on $\mathcal{A}$.

Taking the formulas in Proposition 2.7.2 and Proposition 2.7.5 in [12] and specializing at $s=0$ we deduce that for any $\lambda \in A$ with $\deg \lambda + 2 \leq r$

$$\Psi_{\mathcal{A}}^*(r,\lambda) = \frac{3-(-1)^{\deg D}}{4} \cdot q^{-r+1-\lceil \frac{\deg D}{2} \rceil} \cdot \Bigg[2 r_{\mathcal{A}}((\lambda D))(q-1) L_D(0)$$
$$+ \sum_{\substack{\mu \in A, \mu \neq 0 \\ \deg(\mu N) \leq \deg(\lambda D)}} r_{\mathcal{A}}((\mu N - \lambda D))(t(\mu, D)+1)(1-\delta_{\mu N(\mu N - \lambda D)}) \sum_{c \mid \mu} \left(\frac{D}{c}\right) \Bigg].$$

Moreover, one has

Proposition 3.2.

$$\Lambda(f,\mathcal{A},0)=\frac{q}{2(q-1)}\int_{G_0(N)} f\cdot\overline{\Psi_{\mathcal{A}}}.$$

Let $N=N_0$. Note that $L_D(0)=h_{O_K}$. Comparing the Fourier coefficients of $\Psi_{\mathcal{A}}$ with that of $g_{\mathcal{A}}$ we obtain

$$\Psi_{\mathcal{A}}=g_{\mathcal{A}}\cdot\begin{cases} q^{-\frac{1}{2}\deg D+\frac{1}{2}}\cdot q^{-2}\cdot(q-1)\cdot 2 & \text{when } \deg D \text{ is odd,}\\ q^{-1}\cdot q^{-\frac{1}{2}\deg D}\cdot(q-1) & \text{when } \deg D \text{ is even.}\end{cases}$$

Therefor Theorem 3.1 holds. □

3.3 A function field analogue of Gross formula

Now given a character $\chi:\mathrm{Pic}(O_K)\to\mathbb{C}^\times$, define

$$\Lambda(f,\chi,s):=\sum_{\mathcal{A}\in\mathrm{Pic}(O_K)}\chi(\mathcal{A})\Lambda(f,\mathcal{A},s).$$

When χ is the trivial character and f is a newform which is "normalized" so that the Fourier coefficient $f^*(0)=1$, one has

$$\Lambda(f,\chi,s)=L(f,s)L(f\otimes\varepsilon_D,s)$$

where ε_D is the following quadratic character on divisors of k:

$$\varepsilon_D(P)=\left(\frac{D}{P}\right)\ \text{and}\ \varepsilon_D(\infty)=\begin{cases}-1 & \text{if } \deg D \text{ is even,}\\ 0 & \text{if } \deg D \text{ is odd;}\end{cases}$$

and $L(f\otimes\varepsilon_D,s)$ is the *twisted L-series of f by ε_D*:

$$L(f\otimes\varepsilon_D,s):=\sum_{\mathfrak{m}\geq 0} f^*(\mathfrak{m})\varepsilon_D(\mathfrak{m})q^{-\deg\mathfrak{m}s}.$$

From the definition of $\Lambda(f,\chi,s)$ and Theorem 3.1 one has

$$\Lambda(f,\chi,0)=\left(\sum_{\mathcal{A}\in\mathrm{Pic}(O_K)}\chi(\mathcal{A})(f,g_{\mathcal{A}})\right)\cdot\begin{cases}\dfrac{1}{q^{\frac{1}{2}(\deg D+1)}} & \text{if } \deg D \text{ is odd,}\\[2ex] \dfrac{1}{2q^{\frac{1}{2}\deg D}} & \text{if } \deg D \text{ is even.}\end{cases}$$

Note that

$$\begin{aligned}\sum_{\mathcal{A}\in\mathrm{Pic}(O_K)}\chi(\mathcal{A})^{-1}g_{\mathcal{A}} &= \sum_{\mathcal{A}\in\mathrm{Pic}(O_K)}\left(\sum_{\mathcal{B}\in\mathrm{Pic}(O_K)}\chi(\mathcal{A})^{-1}\Phi(e_{\mathcal{B}},e_{\mathcal{A}\mathcal{B}})\right)\\ &= \Phi(e_\chi,e_\chi)\end{aligned}$$

where Φ is the map in Theorem 2.6 and

$$e_\chi = \sum_{\mathcal{A} \in \mathrm{Pic}(O_K)} \chi(\mathcal{A}) e_{\mathcal{A}}.$$

Suppose f is a normalized newform. Then from Theorem 2.6 f corresponds to a particular element $e_f \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ such that

$$f = \Phi(e_f, e_f).$$

Let $e_{f,\chi}$ be the projection of e_χ to the e_f-isotypical component in the space $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ with respect to the Gross height pairing. Then the f-eigencomponent of $\Phi(e_\chi, e_\chi)$ is equal to

$$\Phi(e_{f,\chi}, e_\chi) = \Phi(e_{f,\chi}, e_{f,\chi}) =< e_{f,\chi}, e_{f,\chi} > f.$$

The last equality holds as f is normalized (i.e. $f^*(0) = 1$) and the Fourier coefficient $\Phi(e_{f,\chi}, e_{f,\chi})^*(0) =< e_{f,\chi}, e_{f,\chi} >$. Therefore we obtain

Theorem 3.3. *Let f be an automorphic cusp form of Drinfeld type for $\Gamma_0(N_0)$ which is also a normalized newform. Then*

$$\Lambda(f, \chi, 0) = \begin{cases} \dfrac{(f,f)}{q^{\frac{1}{2}(\deg D+1)}} \cdot < e_{f,\chi}, e_{f,\chi} > & \text{if } \deg D \text{ is odd,} \\ \dfrac{(f,f)}{2q^{\frac{1}{2}\deg D}} \cdot < e_{f,\chi}, e_{f,\chi} > & \text{if } \deg D \text{ is even.} \end{cases}$$

Remark. 1. If χ is non-trivial, then $\deg e_\chi = 0$ and so $\Phi(e_\chi, e_\chi)$ is a cusp form.
2. When χ is trivial, then

$$\sum_{\text{monic } m \mid N_0} t_m e_\chi = 2e_D$$

where e_D is the divisor class introduced in Proposition 1.7.
3. The special case when N_0 is a prime and $\deg D$ is odd, the above formula coincides with the result in [10] §4 (be aware of the different choices of measures for the Petersson inner product).
4. When irreducible $D \in A - k_\infty^2$ satisfies $\left(\frac{D}{N_0}\right) = 1$, the derivative of $\Lambda(f, \chi, s)$ at $s = 0$ is given by Néron-Tate height of Heegner points on the Drinfeld modular curve $X_0(N_0)$, and an analogue of Gross-Zagier formula has been proved by Rück and Tipp in the case D is irreducible (cf. [12]).

3.4 Example and application to elliptic curves

Let E be a non-iso-trivial elliptic curve over k (i.e. E is not defined over the constant field $\mathbb{F}_q$). From the work of Weil, Jacquet-Langlands, and Deligne, one knows that there exists an automorphic cusp form f_E such that

$$L(E/k, s+1) = L(f_E, s).$$

Here $L(E/k,s)$ is the Hasse-Weil L-series of E over k. Suppose the conductor of E is $N_0\infty$, and E has split multiplicative reduction at ∞. Then the automorphic form f_E is of Drinfeld type for $\Gamma_0(N_0)$, which is a normalized newform (cf. [7]).

Consider the Hasse-Weil L-series $L(E/K,s)$ of E over the imaginary quadratic field $K = k(\sqrt{D})$ where $D \in A$ with $\left(\frac{D}{P}\right) = -1$ for all primes $P \mid N_0$. One has

$$L(E/K, s+1) = L(f_E, s)L(f_E \otimes \varepsilon_D, s)$$

where $L(f_E, \otimes\varepsilon_D, s)$ is the twisted L-series of f_E by the quadratic character ε_D. Since

$$L(f_E, s)L(f_E \otimes \varepsilon_D, s) = \Lambda(f_E, \mathbf{1}_D, s)$$

where $\mathbf{1}_D$ is the trivial character on $\mathrm{Pic}(O_K)$, from Theorem 3.3 we obtain a formula for the special value of $L(E/K,s)$ at $s = 1$ when D is irreducible.

Now, let $k = \mathbb{F}_3(t)$ (i.e. $q = 3$). Let E be the following elliptic curve over k:

$$E : y^2 = x^3 + (t^2+1)x^2 + t^2x = x(x+1)(x+t^2).$$

The conductor of E is $(t)(t+1)(t-1)\infty$. More precisely, E has split multiplicative reduction at (t) and ∞, and has non-split multiplicative reduction at $(t+1)$ and $(t-1)$. Let $N_0 = t(t+1)(t-1) = t^3 - t$. Let f_E be the normalized Drinfeld type cusp form for $\Gamma(N_0)$ associated to E. Since the L-series $L(E/k,s)$ of E over k is a polynomial in q^{-s} of degree $(\deg N_0 + 1) - 4$ with constant term 1, this implies that $L(E/k,s) = L(f_E, s-1) = 1$.
Let $D = t^3 - t - 1$ and $K = k(\sqrt{D})$. Then

$$\left(\frac{D}{t}\right) = \left(\frac{D}{t+1}\right) = \left(\frac{D}{t-1}\right) = -1.$$

The *twist* E_D *of* E *by* D is the following elliptic curve over k:

$$y^2 = x^3 + (t^2+1)Dx^2 + t^2D^2x.$$

The conductor of E_D is $(D)^2(t)(t+1)(t-1)\infty^2$, and the L-series $L(E_D/k,s)$ is

$$1 + q^{-s} + 4q^{-2s} + 108q^{-5s} + 243q^{-6s} + 2187q^{-7s}.$$

Since $L(E/K,s) = L(E/k,s) \cdot L(E_D/k,s)$, we have

$$L(E/K,s) = 1 + q^{-s} + 4q^{-2s} + 108q^{-5s} + 243q^{-6s} + 2187q^{-7s}$$

and $L(E/K,1) = \frac{32}{9}$.

On the other hand, from a formula of Gekeler (cf. [13] Theorem 1.1) we immediately get

$$(f_E, f_E) = 32.$$

We point out that our choice of the measure is twice of the one in [13]. Such computation can be also checked via the algorithm in [15].

The only remaining term is the Gross height of the corresponding point e_{f_E} in $\mathrm{Pic}(X_{N_0}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\mathcal{D}$ be the definite quaternion algebra over k ramified at (t), $(t+1)$, and $(t-1)$. Then

$$\mathcal{D} = k + k\alpha + k\beta + k\alpha\beta$$

where $\alpha^2 = -1$, $\beta^2 = N_0 = t^3 - t$, and $\beta\alpha = -\alpha\beta$. Let $R = A + A\alpha + A\beta + A\alpha\beta$, which is a maximal order in $\mathcal{D}$. The cardinality of $R^\times$ is 8, and the class number (of left ideal classes of R) is 4. We choose the following representatives of left ideal classes of R:

$$\begin{array}{rcl} I_1 & = & R, \\ I_2 & = & At + At\alpha + A\beta + A\alpha\beta, \\ I_3 & = & A(t+1) + A(t+1)\alpha + A\beta + A\alpha\beta, \\ I_4 & = & A(t-1) + A(t-1)\alpha + A\beta + A\alpha\beta. \end{array}$$

Note that these ideals are in fact two-sided, and the norm form on each of them can be easily written down. We calculate the following Brandt matrices:

$$B(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, B(t+1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, B(t-1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since we have $T_t f_E = f_E$, $T_{t+1} f_E = -f_E$, $T_{t-1} f_E = -f_E$, and the Gross height $< e_{f_E}, e_{f_E} >= f_E^*(0) = 1$, the corresponding point e_{f_E} in $\mathrm{Pic}(X_{N_0}) \otimes_{\mathbb{Z}} \mathbb{Q}$ can only be

$$\pm[1/4, 1/4, -1/4, -1/4].$$

The class number of $O_K (= A[\sqrt{D}])$ is 1. Choose the Gross point x in the first component of X_{N_0} corresponding to the embedding $K \hookrightarrow \mathcal{D}$ which maps $\sqrt{D}$ to $\alpha + \beta$. Then $e_x = [1, 0, 0, 0]$ in $\mathrm{Pic}(X_{N_0}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore

$$< e_{f_E, \mathbf{1}_D}, e_{f_E, \mathbf{1}_D} >=< e_{f_E}, e_x >^2 = (4 \cdot 1/4)^2 = 1$$

and

$$\frac{(f_E, f_E)}{q^{\frac{1}{2}(\deg D + 1)}} < e_{f_E, \mathbf{1}_D}, e_{f_E, \mathbf{1}_D} >= \frac{32}{9} = L(E/K, 1).$$

Appendix

A Jacquet-Langlands correspondence and multiplicity one theorem

Let ϖ be a Hecke character on $k^\times\backslash\mathbb{A}_k^\times$. Let $\mathcal{D}$ be a quaternion algebra over k and set $\mathcal{D}_{\mathbb{A}_k} := \mathcal{D}\otimes_k \mathbb{A}_k$. We embed $\mathbb{A}_k$ into $\mathcal{D}_{\mathbb{A}_k}$ by $a \longmapsto 1\otimes a$. A $\mathbb{C}$-valued function f on $\mathcal{D}^\times\backslash\mathcal{D}_{\mathbb{A}_k}^\times$ is called an *automorphic form on* $\mathcal{D}_{\mathbb{A}_k}^\times$ *(for* $\mathcal{K}$*) with central character* ϖ if f is a function on the double coset space

$$\mathcal{D}^\times\backslash\mathcal{D}_{\mathbb{A}_k}^\times/\mathcal{K}$$

for an open compact subgroup $\mathcal{K}$ of $\mathcal{D}_{\mathbb{A}_k}^\times$ satisfying that for all g in $\mathcal{D}_{\mathbb{A}_k}^\times$ and a in $\mathbb{A}_k^\times$

$$f(ag) = \varpi(a)f(g).$$

Suppose $\mathcal{D} = \mathrm{Mat}_2(k)$. Then $\mathcal{D}^\times = \mathrm{GL}_2(k)$ and $\mathcal{D}_{\mathbb{A}_k}^\times = \mathrm{GL}_2(\mathbb{A}_k)$. f is called a *cusp form* if for all g in $\mathrm{GL}_2(\mathbb{A}_k)$

$$\int_{k\backslash\mathbb{A}_k} f\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right) du = 0.$$

We denote $\mathbf{A}_0(\varpi)$ to be the space of automorphic cusp forms on $\mathrm{GL}_2(\mathbb{A}_k)$ with central character ϖ.

We recall Jacquet-Langlands correspondence in §A.1 and use newform theory to explain the claim in §2.3. In §A.2 we use multiplicity one theorem to show that the space $M^{\mathrm{new}}(\Gamma_0(N_0))$ in §2.3 is a free $\mathbb{T}_{\mathbb{C}}$-module of rank one.

A.1 Jacquet-Langlands correspondence

Let $\mathcal{D} = \mathcal{D}_{(N_0)}$ be a definite quaternion algebra over k where N_0 is the product of finite ramified primes of $\mathcal{D}$. Let $\mathbf{A}'(\varpi)$ be the space of automorphic forms on $\mathcal{D}_{\mathbb{A}_k}^\times$ with central character ϖ. Jacquet-Langlands correspondence describes the connection between $\mathbf{A}'(\varpi)$ and $\mathbf{A}_0(\varpi)$:

([9] Chapter 3, Theorem 14.4 and Theorem 16.1) *If an irreducible admissible representation $\rho' = \otimes_v \rho'_v$ is a constituent of $\mathbf{A}'(\varpi)$ and ρ'_P is infinite dimensional for all finite primes P which are prime to N_0, then there exist an irreducible admissible representation $\rho(=: \rho'^{JL})$ which is a constituent of $\mathbf{A}_0(\varpi)$ so that*

$$L(s, \varpi'\otimes\rho) = L(s, \varpi'\otimes\rho')$$

for all Hecke characters ϖ'.
Note that $\rho = \otimes_v\rho_v$ where $\rho_v = \rho'_v$ for finite primes v not dividing N_0. Moreover, for the ramified primes v of $\mathcal{D}$, ρ_v is determined from ρ'_v via theta correspondence.
Conversely, suppose $\rho = \otimes_v\rho_v$ is a constituent of $\mathbf{A}_0(\varpi)$. If for every ramified primes v of $\mathcal{D}$ the representation ρ_v is special or supercuspidal, then there is a

constituent $\rho' = \otimes \rho'_v$ *of* $\mathbf{A}'(\varpi)$ *such that* $\rho_v = \rho'^{JL}_v$. *In particular,* ρ'_v *is one dimensional for ramified prime* v *if and only if* ρ_v *is special.*

Let R be a fixed maximal order of $\mathcal{D}$. From Jacquet-Langlands correspondence one has an isomorphism Ψ between

$$\{\ \mathbb{C}\text{-valued non-constant functions on } \hat{R}^\times \backslash \hat{\mathcal{D}}^\times / \mathcal{D}^\times\}$$

and

$$\left\{\text{Drinfeld type new forms on } \Gamma_0(N_0)\backslash \operatorname{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times\right\}$$

which satisfies

$$\Psi(t_m f) = T_m \Psi(f)$$

for all non-constant functions f on $\hat{R}^\times \backslash \hat{\mathcal{D}}^\times / \mathcal{D}^\times$ and monic polynomials m in A. We briefly recall the argument in the following and refer the reader to [9] for further details.

Fix $\varpi = \otimes_v \varpi_v$ to be the TRIVIAL Hecke character on $k^\times \backslash \mathbb{A}_k^\times$. Let v be a prime of k, $\mathcal{O}_v$ be the valuation ring in k_v, and π_v a uniformizer in $\mathcal{O}_v$. Recall that an irreducible admissible infinite-dimensional representation (ρ_v, V_v) of $\operatorname{GL}_2(k_v)$ with central character ϖ_v has *conductor* $v^{c(v)}$ if $\pi_v^{c(v)}\mathcal{O}_v$ is the largest ideal of $\mathcal{O}_v$ such that the space of elements $u \in V_v$ with

$$\rho_v(g_v)u = u \text{ for all } g_v \in \mathcal{K}_0^{c(v)}$$

is non-empty. In fact, it is one dimensional. Here

$$\mathcal{K}_0^{c(v)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{GL}_2(\mathcal{O}_v) : c \in \pi_v^{c(v)}\mathcal{O}_v \right\}.$$

It is known that

$$c(v) = \begin{cases} 0 & \text{if } \rho_v \text{ is an unramified principal series,} \\ 1 & \text{if } \rho_v \text{ is an unramified special representation,} \\ \geq 2 & \text{if } \rho_v \text{ is supercuspidal or ramified.} \end{cases}$$

Let $(\rho, V) = \bigotimes'_v (\rho_v, V_v)$ be a constituent of $\mathcal{A}_0(\varpi)$. The *conductor of* ρ is:

$$\prod_v v^{c(v)}.$$

The space of elements $f \in V$ with

$$\rho(g)f = f \text{ for all } g \in \prod_v \mathcal{K}_0^{c(v)}$$

is one dimensional, and called the space of *new-forms* of ρ. Any new-form f of ρ is a Hecke eigenform, i.e. $T_v f = a_v f$ for all v where $a_v \in \mathbb{C}$.
Recall that $L(s,\rho) = \prod_v L(s,\rho_v)$, where

$$L(s,\rho_v) = \left(1 - \chi_{v,1}(\pi_v) q^{-s \deg v}\right)^{-1} \cdot \left(1 - \chi_{v,2}(\pi_v) q^{-s \deg v}\right)^{-1}$$

if ρ_v is an unramified principal series $\pi(\chi_{v,1},\chi_{v,2})$;

$$L(s,\rho_v) = \left(1 - \chi_v(\pi_v) q^{-(s+1/2)\deg v}\right)^{-1}$$

if ρ_v is an unramified special representation $\operatorname{sp}(\chi_v|\cdot|_v^{1/2}, \chi_v|\cdot|_v^{-1/2})$;

$$L(s,\rho_v) = 1$$

if ρ_v is supercuspidal or ramified. Here $\chi_{v,1}$, $\chi_{v,2}$, and χ_v are unramified characters of $k_v^\times$ with $\chi_{v,1}\cdot\chi_{v,2} = 1 = \chi_v^2$. It is known that

$$a_v = \begin{cases} q^{\frac{1}{2}\deg v}(\chi_{v,1}(\pi_v) + \chi_{v,2}(\pi_v)) & \text{if } \rho_v \cong \pi(\chi_{v,1},\chi_{v,2}), \\ \chi_v(\pi_v) & \text{if } \rho_v \cong \operatorname{sp}(\chi_v|\cdot|_v^{1/2}, \chi_v|\cdot|_v^{-1/2}). \end{cases}$$

Suppose $\rho = \otimes_v \rho_v$ is of conductor $N_0\infty$ and $\rho_\infty \cong \operatorname{sp}(|\cdot|_\infty^{1/2}, |\cdot|_\infty^{-1/2})$. Then new-forms of ρ are functions on

$$\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}_k)/\mathcal{K}_0(N_0\infty)k_\infty^\times.$$

From the bijection in §2.1

$$\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}_k)/\mathcal{K}_0(N_0\infty)k_\infty^\times \cong \Gamma_0(N_0)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times,$$

new-forms of such ρ can be viewed as newforms of Drinfeld type for $\Gamma_0(N_0)$. In fact, the space $S^{\mathrm{new}}(\Gamma_0(N_0))$ of Drinfeld type new forms for $\Gamma_0(N_0)$ is spanned by the new-forms of such ρ with conductor $N_0\infty$.

Since ρ is of conductor $N_0\infty$, $\rho_P \cong \operatorname{sp}(\chi_P|\cdot|_P^{1/2}, \chi_P|\cdot|_P^{-1/2})$ for all $P \mid N_0$ where χ_P is an unramified character of $k_P^\times$ with $\chi_P^2 = 1$. By Jacquet-Langlands correspondence we can find an irreducible constituent $(\rho', V') = \otimes_v \rho'_v$ of $\mathbf{A}'(\varpi)$ so that $\rho = \rho'^{\mathrm{JL}}$. In this case, $\rho'_P = \chi_P \circ \mathrm{Nr}$ for $P \mid N_0$ and ρ'_∞ is the trivial representation. Therefore we can find a subspace of elements $f' \in V'$ which are non-constant functions on

$$\mathcal{D}^\times \backslash \hat{\mathcal{D}}^\times / \hat{R}^\times.$$

This subspace is also one dimensional, called the space of *new-forms of* ρ'. Any new-form f' of ρ' is also a Hecke eigenform, i.e. $t_v f' = a'_v f'$, where a'_v appears in the local factor $L_v(s,\rho'_v)$. Since for any place v

$$L(s,\rho_v) = L(s,\rho'_v),$$

we have $a_v = a'_v$.

In fact, the space of non-constant functions on $\mathcal{D}^\times\backslash\hat{\mathcal{D}}^\times/\hat{R}^\times$ is generated by new-forms such that $\rho' = \otimes_v \rho'_v$ where ρ'_∞ is trivial and for $P \mid N_0$, $\rho'_P = \chi_P \circ \mathrm{Nr}$ for an unramified character χ_P of $k_P^\times$ with $\chi_P^2 = 1$. By taking congugate, we identify functions on $\mathcal{D}^\times\backslash\hat{\mathcal{D}}^\times/\hat{R}^\times$ with functions on $\hat{R}^\times\backslash\hat{\mathcal{D}}^\times/\mathcal{D}^\times$. From the dimension formula at the end of §2.2 we have a bijective map Ψ from

$$\{\ \mathbb{C}\text{-valued non-constant functions on } \hat{R}^\times\backslash\hat{\mathcal{D}}^\times/\mathcal{D}^\times\}$$

to

$$\left\{\text{Drinfeld type new forms on } \Gamma_0(N_0)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times\right\}$$

so that for each monic polynomial m in A,

$$\Psi(t_m f) = T_m \Psi(f).$$

Since constant functions on $\hat{R}^\times\backslash\hat{\mathcal{D}}^\times/\mathcal{D}^\times$ are eigenfunctions of t_m with eigenvalue $\sigma(m)_{N_0}$, we extend Ψ by mapping constant functions into the one dimensional subspace $\mathbb{C}\mathcal{E}_{N_0}$ of $M^{\mathrm{new}}(\Gamma_0(P_0))$.

Consider the definite Shimura curve $X = X_{N_0}$. We have a canonical bijection between components of X and ideal classes of R and this gives the canonical isomorphism

$$\{\ (\mathbb{C}\text{-valued) functions on } \hat{R}^\times\backslash\hat{\mathcal{D}}^\times/\mathcal{D}^\times\} \cong \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{C}) \cong \mathrm{Pic}(X)^\vee \otimes_{\mathbb{Z}} \mathbb{C}.$$

Therefore one has:

Theorem A.1. $\Psi : \mathrm{Pic}(X)^\vee \otimes_{\mathbb{Z}} \mathbb{C} \cong M^{\mathrm{new}}(\Gamma_0(N_0))$ *is an isomorphism so that* $\Psi(t_m f) = T_m \Psi(f)$ *for any monic polynomial* m *in* A. *Moreover,*

$$Tr(t_m) = Tr(T_m)$$

and so the $\mathbb{C}$*-algebra* $\mathbb{T}_{\mathbb{C}}$ *generated by Hecke correspondences* t_m *on* X *is isomorphic to the* $\mathbb{C}$*-algebra generated by Hecke operators* T_m *on* $M^{\mathrm{new}}(\Gamma_0(N_0))$.

A.2 Multiplicity one theorem

Let $\varpi : \mathbb{A}_k^\times/k^\times$ be a Hecke character. Let $\rho_1 = \otimes_v \rho_{1,v}$ and $\rho_2 = \otimes_v \rho_{2,v}$ be two irreducible admissible representations which are constituents of $\mathbf{A}_0(\varpi)$. The *multiplicity one theorem* (cf. [3]) tells us that $\rho_1 = \rho_2$ if and only if

$$\rho_{1,v} \cong \rho_{2,v}$$

for all place v.
Fix ϖ to be trivial. Choose two irreducible admissible representations $\rho_1 = \otimes_v \rho_{1,v}$ and $\rho_2 = \otimes_v \rho_{2,v}$ of conductor $N_0\infty$ which are constituents of $\mathbf{A}_0(\varpi)$ satisfying

$$\rho_{1,\infty} \cong \rho_{2,\infty} \cong \mathrm{sp}(|\cdot|_\infty^{1/2}, |\cdot|_\infty^{-1/2})$$

and $\rho_{1,P}$ and $\rho_{2,P}$ are unramified special representations for $P \mid N_0$. Let f_1 and f_2 be new-forms of ρ_1 and ρ_2 respectively. Then $T_P f_i = a_{P,i} f_i$ where $a_{P,i} \in \mathbb{C}$ for $i = 1, 2$ and all prime P in A. If $a_{P,1} = a_{P,2}$ for all P, then $L_P(s, \rho_{1,P}) = L_P(s, \rho_{2,P})$ and so

$$\rho_{1,P} \cong \rho_{2,P}$$

for all P. By multiplicity one theorem we have $\rho_1 = \rho_2$ and so f_1, f_2 are linearly dependent.

Recall that $M^{\text{new}}(\Gamma_0(N_0)) = S^{\text{new}}(\Gamma_0(N_0)) \oplus \mathbb{C}\mathcal{E}_{N_0}$. for $\Gamma_0(P_0)$. As a $\mathbb{T}_{\mathbb{C}}$-module, the space $M^{\text{new}}(\Gamma_0(N_0))$ is a direct sum $(\oplus_i \mathbb{C} f_i) \oplus \mathbb{C}\mathcal{E}_{N_0}$ of one dimensional submodules and each f_i is a new-form of an irreducible admissible representation $\rho_i = \otimes_v \rho_{i,v}$ which is a constituent of $\mathbf{A}_0(\varpi)$ with

$$\rho_{i,\infty} \cong \mathrm{sp}(|\cdot|_\infty^{1/2}, |\cdot|_\infty^{-1/2})$$

and $\rho_{i,P}$ is an unramified special representation for $P \mid N_0$. According to multiplicity one theorem, each pair of these one dimensional submodules are non-isomorphic. Therefore $M^{\text{new}}(\Gamma_0(N_0))$ is a cyclic $\mathbb{T}_{\mathbb{C}}$-module, which is generated by $\mathcal{E}_{N_0} + \sum_i f_i$. Viewing $\mathbb{T}_{\mathbb{C}}$ as a subring of $\mathrm{End}_{\mathbb{C}}\big(M^{\text{new}}(\Gamma_0(N_0))\big)$, we have

$$\dim_{\mathbb{C}} \mathbb{T}_{\mathbb{C}} \leq \dim_{\mathbb{C}} M^{\text{new}}(\Gamma_0(N_0)).$$

Therefore

Proposition A.2. *The space $M^{\text{new}}(\Gamma_0(N_0))$ is a free $\mathbb{T}_{\mathbb{C}}$-module of rank one.*

B Transformation law of theta series

Fix a definite quaternion algebra $\mathcal{D} = \mathcal{D}_{(N_0)}$ where N_0 is the product of finite ramified primes of $\mathcal{D}$. Let R be a maximal order and n be the class number. In this section we deduce the transformation law of the theta series θ_{ij} for $1 \leq i, j \leq n$ introduced in §2.1.1. Recall that for each (i, j), theta series θ_{ij} is a function on $k_\infty^\times \times k_\infty$:

$$\theta_{ij}(x, y) = \sum_{b \in M_{ij}} \phi_\infty\Big(\frac{\mathrm{Nr}(b)}{N_{ij}} x t^2\Big) \cdot \psi_\infty\Big(\frac{\mathrm{Nr}(b)}{N_{ij}} y\Big),$$

where ϕ_∞ is the characteristic function of $\mathcal{O}_\infty$ and ψ_∞ is the fixed additive character on k_∞.

B.1 Fourier Transform

Let $\mathcal{D}_\infty = \mathcal{D} \otimes_k k_\infty$,. For $\alpha, \beta \in k_\infty^\times$ with $v_\infty(\alpha) > v_\infty(\beta) - 2$, let

$$\begin{array}{rccc} \Phi_{\alpha,\beta} : & \mathcal{D}_\infty & \longrightarrow & \mathbb{C} \\ & w & \longmapsto & \phi_\infty\big(\mathrm{Nr}(w)\alpha\big)\psi_\infty\big(\mathrm{Nr}(w)\beta\big). \end{array}$$

Define $[\cdot,\cdot] : \mathcal{D}_\infty \times \mathcal{D}_\infty \to \mathbb{C}^\times$ by $[w,w^*] := \psi_\infty\big(\mathrm{Tr}(ww^*)\big)$. The *Fourier transform* of $\Phi_{\alpha,\beta}$ is given by:

$$\Phi^*_{\alpha,\beta}(w^*) := \int_{\mathcal{D}_\infty} \Phi_{\alpha,\beta}(w)[w,w^*]dw, \text{ for all } w^* \text{ in } k_\infty$$

where dw is a Haar measure on $\mathcal{D}_\infty$.

We define

$$S(\alpha,\beta,dw) := \int_{\mathcal{D}_\infty} \phi_\infty\big(\mathrm{Nr}(w)\alpha)\psi_\infty(\mathrm{Nr}(w)\beta\big)dw.$$

Then $\Phi^*_{\alpha,\beta}(w^*)$ is equal to

$$S(\alpha,\beta,dw)\phi_\infty\big(\mathrm{Nr}(w^*)\frac{\alpha}{\beta^2}\big)\psi_\infty\big(\mathrm{Nr}(w^*)\frac{-1}{\beta}\big).$$

More generally, take $h \in k_\infty^\times$, $\rho \in \mathcal{D}_\infty$. For $\alpha,\beta \in k_\infty^\times$ with $v_\infty(\alpha) > v_\infty(\beta)-2$, let $\Psi_{\alpha,\beta}(w) := \Phi_{\alpha,\beta}(\rho + hw)$. Then $\Psi^*_{\alpha,\beta}(w^*)$ is equal to

$$q^{4v_\infty(h)} \cdot S(\alpha,\beta,dw)\phi_\infty\big(\mathrm{Nr}(\frac{w^*}{h})\frac{\alpha}{\beta^2}\big)\psi_\infty\big(\mathrm{Nr}(\frac{w^*}{h})\frac{-1}{\beta}\big)\psi_\infty\big(\mathrm{Tr}(-\frac{\rho w^*}{h})\big).$$

B.2 Poisson summation

Let $\mathcal{O}_{\mathcal{D}_\infty}$ be the maximal order of $\mathcal{D}_\infty$. For $v_\infty(\alpha) > v_\infty(\beta)-2$, we have

$$S(\alpha,\beta,dw) = -q^{2v_\infty(\beta)-3} \cdot dw(\mathcal{O}_{\mathcal{D}_\infty}).$$

For the pair (i,j), $1 \leq i,j \leq n$, we choose Haar measure dw with $dw(\mathcal{D}_\infty/M_{ij}) = 1$ and denote the integral $S(\alpha,\beta,dw)$ by $S(\alpha,\beta,M_{ij})$. Then

$$S(\alpha,\beta,M_{ij}) = -q^{2v_\infty(\beta)-\deg(N_0)} \cdot q^{2v_\infty(N_{ij})}.$$

Let $\tilde{M}_{ij}$ be the dual lattice of M_{ij}, i.e.,

$$\tilde{M}_{ij} = \{w \in \mathcal{D}_\infty : \mathrm{Tr}(w\mu) \in A \text{ for all } \mu \in M_{ij}\}.$$

We apply the Poisson summation formula

$$\sum_{\mu \in M_{ij}} \Psi_{\alpha,\beta}(\mu) = \sum_{\mu^* \in \tilde{M}_{ij}} \Psi^*_{\alpha,\beta}(\mu^*)$$

and get

PROPOSITION B.1. *Let $\alpha,\beta \in k_\infty^*$ with $v_\infty(\alpha) > v_\infty(\beta)-2$, $h \in k_\infty^\times$, $\rho \in \mathcal{D}_\infty$. Then*

$$\begin{aligned} &\sum_{\mu \in M_{ij}} \phi_\infty\big(\mathrm{Nr}(\rho + h\mu)\alpha\big)\psi_\infty\big(\mathrm{Nr}(\rho + h\mu)\beta\big) \\ = \;& q^{4v_\infty(h)} S(\alpha,\beta,M_{ij}) \sum_{\mu^* \in \tilde{M}_{ij}} \phi_\infty\big(\mathrm{Nr}(\frac{\mu^*}{h})\frac{\alpha}{\beta^2}\big)\psi_\infty\big(\mathrm{Nr}(\frac{\mu^*}{h})\frac{-1}{\beta}\big)\psi_\infty\big(\mathrm{Tr}(\frac{\rho\mu^*}{h})\big).\end{aligned}$$

Let $x \in k_\infty^\times$, $y \in k_\infty$, $M \subset \mathcal{D}_\infty$ a discrete A-lattice, $N_M \in k$ such that $N_M \cdot A$ is the fractional ideal of A generated by $\mathrm{Nr}(\mu)$ for $\mu \in M$. For $h \in A$ with $h \neq 0$, $\rho \in M$, define "partial theta" series:

$$\theta(x,y,M,N_M,h,\rho) := \sum_{\mu \in M, \mu \equiv \rho \bmod hM} \phi_\infty\left(\frac{\mathrm{Nr}(\mu)xt^2}{N_M h}\right)\psi_\infty\left(\frac{\mathrm{Nr}(\mu)y}{N_M h}\right).$$

Note that $\theta_{ij}(x,y) = \theta(x,y,M_{ij},N_{ij},1,0)$, and

$$\theta(x,y,M,N_M,h,\rho) = \sum_{\mu \in M} \phi_\infty\big(\mathrm{Nr}(\rho+h\mu)\alpha\big)\psi_\infty\big(\mathrm{Nr}(\rho+h\mu)\beta\big)$$

where $\alpha = \frac{xt^2}{N_M h}$, $\beta = \frac{y}{N_M h}$.

PROPOSITION B.2. *Let $x, y \in k_\infty^\times$, $v_\infty(x) > v_\infty(y)$, $0 \neq h \in A$, $\kappa \in \tilde{M}_{ij}$. Then*

$$\begin{aligned}
&\theta\big(\frac{x}{y^2}, \frac{-1}{y}, \tilde{M}_{ij}, N_{ij}^{-1}N_0^{-1}, h, \kappa\big) \\
= \;& S\big(\frac{xt^2}{N_{ij}N_0 h}, \frac{y}{N_{ij}N_0 h}, M_{ij}\big)^{-1} \sum_{\rho \in M_{ij}/hM_{ij}} \psi_\infty\big(\mathrm{Tr}(\frac{\rho\kappa}{h})\big)\theta\big(\frac{x}{N_0}, \frac{y}{N_0}, M_{ij}, N_{ij}, h, \rho\big).
\end{aligned}$$

Proof. By Proposition B.1 we have

$$\begin{aligned}
&\theta(x,y,M_{ij},N_{ij},h,\rho) \\
= \;& q^{4v_\infty(h)}S(\alpha,\beta,M_{ij}) \sum_{\mu^* \in \tilde{M}_{ij}} \phi_\infty\big(\mathrm{Nr}(\frac{\mu^*}{h})\frac{\alpha}{\beta^2}\big)\psi_\infty\big(\mathrm{Nr}(\frac{\mu^*}{h})\frac{-1}{\beta}\big)\psi_\infty\big(\mathrm{Tr}(\frac{-\rho\mu^*}{h})\big).
\end{aligned}$$

Multiply this by $\psi_\infty\big(\mathrm{Tr}(\frac{\rho\kappa}{h})\big)$ for $\kappa \in \tilde{M}_{ij}$ and sum over $\rho \in M_{ij}/hM_{ij}$, we obtain

$$\begin{aligned}
&\sum_{\rho \in M_{ij}/hM_{ij}} \psi_\infty(\mathrm{Tr}(\frac{\rho\kappa}{h})) \cdot \theta(x,y,M_{ij},N_{ij},h,\rho) \\
= \;& q^{4v_\infty(h)}S(\alpha,\beta,M_{ij}) \sum_{\mu^* \in \tilde{M}_{ij}} \Bigg\{\phi_\infty\big(\mathrm{Nr}(\frac{\mu^*}{h})\frac{\alpha}{\beta^2}\big)\psi_\infty\big(\mathrm{Nr}(\frac{\mu^*}{h})\frac{-1}{\beta}\big) \\
&\cdot\Big[\sum_{\rho \in M_{ij}/hM_{ij}} \psi_\infty\Big(\mathrm{Tr}\big(\frac{\rho}{h}(\kappa-\mu^*)\big)\Big)\Big]\Bigg\}.
\end{aligned}$$

Since

$$\sum_{\rho \in M_{ij}/hM_{ij}} \psi_\infty(\mathrm{Tr}(\frac{\rho}{h}(\kappa-\mu^*))) = \begin{cases} 0 & \text{if } \mu^*-\kappa \notin h\tilde{M}_{ij}, \\ q^{-4v_\infty(h)} & \text{if } \mu^*-\kappa \in h\tilde{M}_{ij}. \end{cases}$$

The proposition follows by replacing x with $\frac{x}{N_0}$, and y with $\frac{y}{N_0}$. ◻

B.3 Transformation law

Let $(x,y) \in k_\infty^\times \times k_\infty$. Suppose a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A)$ is given such that $cy+d \neq 0$. We define

$$\gamma \circ (x,y) := \left(\frac{x(ad-bc)}{(cy+d)^2}, \frac{ay+b}{cy+d}\right).$$

LEMMA B.3. *Suppose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(A)$, $c \equiv 0 \bmod N_0$, $v_\infty(x) > v_\infty(y)$, and $v_\infty(cx) > v_\infty(cy+d)$. Let $1 \leq i,j, \leq n$. Then*

$$\begin{aligned}\theta_{ij}\big(\gamma\circ(x,y)\big) &= S(\frac{N_{ij}xt^2}{y^2}, \frac{-N_{ij}(cy+d)}{dy}, \tilde{M}_{ij})^{-1} \cdot S(\frac{xt^2}{N_{ij}}, \frac{y}{N_{ij}}, M_{ij})^{-1} \\ &\quad \cdot \left(\sum_{\kappa \in M_{ij}/dM_{ij}} \psi_\infty(\frac{\mathrm{Nr}(\kappa)b}{N_{ij}d})\right)\theta_{ij}(x,y).\end{aligned}$$

Proof. Put $u = \frac{x}{y^2}$, $v = \frac{-1}{y}$. Then

$$\begin{aligned}\theta_{ij}\big(\gamma\circ(x,y)\big) &= \theta\big(\frac{u}{(c-dv)^2}, \frac{b}{d} + \frac{1}{d(c-dv)}, M_{ij}, N_{ij}, 1, 0\big) \\ &= \sum_{\kappa \in M_{ij}/dM_{ij}} \theta\big(\frac{du}{(c-dv)^2}, b + \frac{1}{c-dv}, M_{ij}, N_{ij}, d, \kappa\big) \\ &= \sum_{\kappa \in M_{ij}/dM_{ij}} \psi_\infty(\frac{\mathrm{Nr}(\kappa)b}{N_{ij}d})\theta\big(\frac{du}{(dv-c)^2}, \frac{-1}{dv-c}, M_{ij}, N_{ij}, d, \kappa\big).\end{aligned}$$

Since $v_\infty(cx) > v_\infty(cy+d)$, we have $v_\infty(du) > v_\infty(dv-c)$ and

$$\begin{aligned}\theta_{ij}\big(\gamma\circ(x,y)\big) &= S(N_{ij}ut^2, N_{ij}(v-c/d), \tilde{M}_{ij})^{-1} \cdot \sum_{\kappa \in M_{ij}/dM_{ij}} \Bigg[\psi_\infty(\frac{\mathrm{Nr}(\kappa)b}{N_{ij}d}) \\ &\quad \cdot \sum_{\rho \in \tilde{M}_{ij}/d\tilde{M}_{ij}} \psi_\infty\big(\mathrm{Tr}(\frac{\rho\kappa}{d})\big)\theta\big(\frac{du}{N_0}, \frac{dv-c}{N_0}, \tilde{M}_{ij}, N_{ij}^{-1}N_0^{-1}, d, \rho\big)\Bigg].\end{aligned}$$

Since $-c/N_0 \in A$, we have

$$\begin{aligned}\theta_{ij}(\gamma\circ(x,y)) &= S\big(N_{ij}ut^2, N_{ij}(v-c/d), \tilde{M}_{ij}\big)^{-1} \\ &\quad \cdot \sum_{\rho \in \tilde{M}_{ij}/d\tilde{M}_{ij}} \Bigg[\theta\big(\frac{du}{N_0}, \frac{dv}{N_0}, \tilde{M}_{ij}, N_{ij}^{-1}N_0^{-1}, d, \rho\big) \\ &\quad \cdot \sum_{\kappa \in M_{ij}/dM_{ij}} \psi_\infty(\frac{\mathrm{Nr}(\kappa)b}{N_{ij}d} + \frac{\mathrm{Tr}(\rho\kappa)}{d} - \frac{\mathrm{Nr}(\rho)cN_{ij}}{d})\Bigg].\end{aligned}$$

Note that $cN_{ij}\bar{\rho} \in M_{ij}$. Replacing κ by $\kappa + cN_{ij}\bar{\rho}$ the last summand equals to

$$\frac{\mathrm{Nr}(\kappa)b}{N_{ij}d} + a\,\mathrm{Tr}(\rho\kappa) + N_{ij}ac\,\mathrm{Nr}(\rho).$$

Since $a\,\mathrm{Tr}(\rho\kappa) + N_{ij}ac\,\mathrm{Nr}(\rho) \in A$, we have

$$\begin{aligned}\theta_{ij}(\gamma \circ (x,y)) \quad &= \quad S(N_{ij}ut^2, N_{ij}(v - c/d), \tilde{M}_{ij})^{-1} \cdot \left(\sum_{\kappa \in M_{ij}/dM_{ij}} \psi_\infty(\frac{\mathrm{Nr}(\kappa)b}{N_{ij}d}) \right) \\ &\quad \cdot \theta(\frac{u}{N_0}, \frac{v}{N_0}, \tilde{M}_{ij}, N_{ij}^{-1}N_0^{-1}, 1, 0).\end{aligned}$$

Recall that $u = \frac{x}{y^2}$, $v = \frac{-1}{y}$. By Proposition B.2 we have

$$\begin{aligned}\theta_{ij}\big(g \circ (x,y)\big) \quad &= \quad S\big(\frac{N_{ij}xt^2}{y^2}, \frac{-N_{ij}(cy+d)}{dy}, \tilde{M}_{ij}\big)^{-1} \cdot S\big(\frac{xt^2}{N_{ij}}, \frac{y}{N_{ij}}, M_{ij}\big)^{-1} \\ &\quad \cdot \left(\sum_{\kappa \in M_{ij}/dM_{ij}} \psi_\infty(\frac{\mathrm{Nr}(\kappa)b}{N_{ij}d}) \right) \theta_{ij}(x,y).\end{aligned}$$

$\square$

Note that

$$S\big(\frac{N_{ij}xt^2}{y^2}, \frac{-N_{ij}(cy+d)}{dy}, \tilde{M}_{ij}\big) \cdot S\big(\frac{xt^2}{N_{ij}}, \frac{y}{N_{ij}}, M_{ij}\big) = q^{2v_\infty(cy+d)+2\deg d}.$$

By standard argument we get $\sum_{\kappa \in M_{ij}/dM_{ij}} \psi_\infty(\frac{\mathrm{Nr}(\kappa)b}{N_{ij}d}) = q^{2\deg(d)}$. Since $\theta_{ij}(x,y) = \theta_{ij}(x, y+h)$ for any $h \in A$, we can drop the assumption $v_\infty(x) > v_\infty(y)$ and obtain the transformation law of θ_{ij}:

THEOREM B.4. *For $1 \le i, j \le n$. Let $x \in k_\infty^\times$, $y \in k_\infty$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(A)$. Assume $v_\infty(cx) > v_\infty(cy+d)$, and $c \equiv 0 \bmod N_0$. Then*

$$\theta_{ij}(\gamma \circ (x,y)) = q^{-2v_\infty(cy+d)} \cdot \theta_{ij}(x,y).$$

ACKNOWLEDGEMENTS

We would like to thank the referee for referring us to the work of Papikian [10].

References

[1] Artin, E., *Quadratische Körper der komplexen Multiplikation,* Enzyklopädie der Math. Wiss. Band I, 2. Teil, Heft 10, Teil II.

[2] Bertolini, M. & Darmon, H., *Heegner points on Mumford-Tate curves,* Inv. Math., 126 (1996) 413-456.

[3] Bump, D., *Automorphic Forms and Representations,* Cambridge studies in advanced mathematics 55, (1996).

[4] Eichler, M., *Zur Zahlentheorie der Quaternionen-Algebren,* Crelle J. 195 (1955), 127-151.

[5] Gekeler, E.-U., *Invariants of Some Algebraic Curves Related to Drinfeld Modular Curves,* J. Number Theory 90 (2001) 166-183.

[6] Gekeler, E.-U., *Improper Eisenstein Series on Bruhat-Tits Trees*, manuscripta math. 86 (1995), 367-391.

[7] Gekeler, E.-U. & Reversat, M., *Jacobians of Drinfeld modular curves,* J. reine angew. Math. 476 (1996), 27-93.

[8] Gross, B. H., *Heights and the Special Values of L-series,* CMS Conference Proceedings, H. Kisilevsky, J. Labute, Eds., 7 (1987) 116-187.

[9] Jacquet, H. & Langlands, R., *Automorphic Forms on* GL(2)*,* LNM 114, Springer 1970.

[10] Papikian, M., *On the variation of Tate-Shafarevich groups of elliptic curves over hyperelliptic curves,* J. Number Theory 115 (2005) 249-283.

[11] Rück, H.-G., *Theta Series of Imaginary Quadratic Function Fields,* manuscripta math. 88 (1995), 387-407.

[12] Rück, H.-G. & Tipp, U., *Heegner Points and L-series of Automorphic Cusp Forms of Drinfeld Type,* Documenta Mathematica 5 (2000) 365-444.

[13] Schweizer, A., *Strong Weil curves over* $\mathbb{F}_q(T)$ *with small conductor,* J. Number Theory 131 (2011) 285-299.

[14] Serre, J.-P., *Tree,* Springer, Berlin-Heidelberg-New York 1980.

[15] Tan, K.-S. & Reckmore, D., *Computation of L-series for elliptic curves over function fields,* J. reine angew. Math. 424 (1992), 107-135.

[16] Vignéras, M.-F., *Arithmétique des Algèbres de Quaternions,* LNM 800, Springer 1980.

[17] Wei, F.-T., *On arithmetic of curves over function fields.* Ph.D. thesis, National Tsing Hua University, 2010.

[18] Weil, A., *Dirichlet Series and Automorphic Forms,* LNM 189, Springer 1971.

Fu-Tsun Wei
Department of Mathematics
National Tsing Hua University
Hsinchu 30013
Taiwan

Jing Yu
TIMS and the
Department of Mathematics
National Taiwan University
Taipei 10617
Taiwan