Moduli of Framed Sheaves on Projective Surfaces

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Abstract. We show that there exists a fine moduli space for torsion-free sheaves on a projective surface which have a "good framing" on a big and nef divisor. This moduli space is a quasi-projective scheme. This is accomplished by showing that such framed sheaves may be considered as stable pairs in the sense of Huybrechts and Lehn. We characterize the obstruction to the smoothness of the moduli space and discuss some examples on rational surfaces.

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1. Introduction

There has been recently some interest in the moduli spaces of framed sheaves. One reason is that they are often smooth and provide desingularizations of the moduli spaces of ideal instantons, which in turn are singular [17, 19, 18]. For this reason, their equivariant cohomology under suitable toric actions is relevant to the computation of partition functions, and more generally expectation values of quantum observables in topological quantum field theory [20, 2, 19, 6, 3]. On the other hand, these moduli spaces can be regarded as higher-rank generalizations of Hilbert schemes of points, and as such they have interesting connections with integrable systems [12, 1], representation theory [26], etc.

While it is widely assumed that such moduli spaces exist and are well behaved, an explicit analysis, showing that they are quasi-projective schemes and are fine moduli spaces, is missing in the literature. In the present paper we provide such a construction for the case of framed sheaves on smooth projective surfaces under some mild conditions. We show that if D is a big and nef curve in a smooth projective surface X, there is a fine quasi-projective moduli space for

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sheaves that have a "good framing" on D (Theorem 3.1). The point here is that the sheaves under consideration are not assumed a priori to be semistable, and the basic idea is to show that there exists a stability condition making all of them stable, so that our moduli space is an open subscheme of the moduli space of stable pairs in the sense of Huybrechts and Lehn [8, 9].

In the papers [21, 22] T. Nevins constructed a scheme structure for these moduli spaces, however we obtain a stronger result, showing that these schemes are quasi-projective, and in particular are separated and of finite type. Moreover we compute the obstruction to the smoothness of these moduli spaces (Theorem 4.3). In fact, the tangent space is well known, but we provide a more precise description of the obstruction space than the one given by Lehn [14]. We show that it lies in the kernel of the trace map, thus extending a previous result of Lübke [15] to the non-locally free case.

In some cases there is another way to give the moduli spaces $\mathfrak{M}(r,c,n)$ a structure of algebraic variety, namely, by using ADHM data. This was done for vector bundles on $\mathbb{P}^2$ by Donaldson [5], while (always in the locally free case) the case of the blowup of $\mathbb{P}^2$ at a point is studied in A. King's thesis [13], and $\mathbb{P}^2$ blown-up at an arbitrary number of points was analyzed by Buchdahl [4]. The general case (i.e., including torsion-free sheaves) is studied by C. Rava for Hirzebruch surfaces [24] and A.A. Henni for multiple blowups of $\mathbb{P}^2$ at distinct points [7]. The equivalence between the two approaches follows from the fact that in both cases one has *fine* moduli spaces. On the ADHM side, this is shown by constructing a universal monad on the moduli space [23, 7, 25].

In the final section we discuss some examples, i.e. framed bundles on Hirzebruch surfaces with "minimal invariants", and rank 2 framed bundles on the blowup of $\mathbb{P}^2$ at one point.

In the present article, all the schemes we consider are separated and are of finite type over $\mathbb{C}$, and "a variety" is a reduced irreducible scheme of finite type over $\mathbb{C}$. A "sheaf" is always coherent, the term "(semi)stable" always means "μ-(semi)stable", and the prefix μ- will be omitted. Framed sheaves are always assumed to be torsion-free.

2. Framed sheaves

Let us characterize the objects that we shall study.

Definition 2.1. *Let X be a scheme over $\mathbb{C}$, $D \subset X$ an effective Weil divisor, and $\mathcal{E}_D$ a sheaf on D. We say that a sheaf $\mathcal{E}$ on X is $(D,\mathcal{E}_D)$-framable if $\mathcal{E}$ is torsion-free and there is an epimorphism $\mathcal{E} \to \mathcal{E}_D$ of $\mathcal{O}_X$-modules inducing an isomorphism $\mathcal{E}_{|D} \xrightarrow{\sim} \mathcal{E}_D$. An isomorphism $\phi\colon \mathcal{E}_{|D} \xrightarrow{\sim} \mathcal{E}_D$ will be called a $(D,\mathcal{E}_D)$-framing of $\mathcal{E}$. A framed sheaf is a pair $(\mathcal{E},\phi)$ consisting of a $(D,\mathcal{E}_D)$-framable sheaf $\mathcal{E}$ and a framing ϕ. Two framed sheaves $(\mathcal{E},\phi)$ and $(\mathcal{E}',\phi')$ are isomorphic if there is an isomorphism $f\colon \mathcal{E} \to \mathcal{E}'$ and a nonzero constant $\lambda \in \mathbb{C}$ such that $\phi' \circ f_{|D} = \lambda\phi$.*

Let us remark that our notion of framing is the same as the one used in [14, 22, 21], but is more restrictive than that of [8], where a framing is any homomorphism $\alpha : \mathcal{E} \to \mathcal{E}_D$ of $\mathcal{O}_X$-modules, not necessarily factoring through an *isomorphism* $\mathcal{E}_{|D} \xrightarrow{\sim} \mathcal{E}_D$. To distinguish between the two definitions, we will call such a pair $(\mathcal{E}, \alpha)$ a *framed pair*, whilst the term *framed sheaf* will refer to the notion introduced in Definition 2.1

Our strategy to show that framed sheaves on a projective variety make up "good" moduli spaces will consist in proving that, under some conditions, the framed sheaves $(\mathcal{E}, \phi)$ are stable according to a notion of stability introduced by Huybrechts and Lehn [8, 9]. The definition of stability for framed pairs depends on the choice of a polarization H on X and a positive real number δ (in our notation, δ is the leading coefficient of the polynomial δ in the definition of (semi)stability in [8]).

Definition 2.2 ([8, 9]). *A framed pair $(\mathcal{E}, \alpha)$ on an n-dimensional projective variety X, consisting of a torsion-free sheaf $\mathcal{E}$ and its framing $\alpha : \mathcal{E} \to \mathcal{E}_D$, is said to be (H, δ)-stable, if for any subsheaf $\mathcal{G} \subset \mathcal{E}$ with $0 < \operatorname{rk} \mathcal{G} \le \operatorname{rk} \mathcal{E}$, the following inequalities hold:*

(1) $\dfrac{c_1(\mathcal{G}) \cdot H^{n-1}}{\operatorname{rk}(\mathcal{G})} < \dfrac{c_1(\mathcal{E}) \cdot H^{n-1} - \delta}{\operatorname{rk}(\mathcal{E})}$ *when $\mathcal{G}$ is contained in $\ker \alpha$;*

(2) $\dfrac{c_1(\mathcal{G}) \cdot H^{n-1} - \delta}{\operatorname{rk}(\mathcal{G})} < \dfrac{c_1(\mathcal{E}) \cdot H^{n-1} - \delta}{\operatorname{rk}(\mathcal{E})}$ *otherwise.*

Remark, that according to this definition, any rank-1 framed sheaf is (H, δ)-stable for any ample H and any $0 \le \delta < \deg D$.

For any sheaf $\mathcal{F}$ on X, $P^H_{\mathcal{F}}$ denotes the Hilbert polynomial $P^H_{\mathcal{F}}(k) = \chi(\mathcal{F} \otimes \mathcal{O}_X(kH))$. For a non-torsion sheaf $\mathcal{F}$ on X, μ^H denotes the slope of $\mathcal{F}$: $\mu^H(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot H^{n-1}}{\operatorname{rk} \mathcal{F}}$.

Theorem 2.3 ([8, 9]). *Let X be a smooth projective variety, H an ample divisor on X and δ a positive real number. Let $D \subset X$ be an effective divisor, and $\mathcal{E}_D$ a sheaf on D. Then there exists a fine moduli space $\mathfrak{M} = \mathfrak{M}^H_X(P)$ of (H, δ)-stable $(D, \mathcal{E}_D)$-framed sheaves $(\mathcal{E}, \phi)$ with fixed Hilbert polynomial $P = P^H_{\mathcal{E}}$, and this moduli space is a quasi-projective scheme.*

Since we are using slope stability, and a more restrictive definition of framing with respect to that of [8, 9], our moduli space $\mathfrak{M}^H_X(P)$ is actually an open subscheme of the moduli space constructed by Huybrechts and Lehn.

Another general result on framed sheaves we shall need is a boundedness theorem due to M. Lehn. Given $X, D, \mathcal{E}_D$ as above, a set $\mathcal{M}$ of $(D, \mathcal{E}_D)$-framed pairs $(\mathcal{E}, \phi)$ is bounded if there exists a scheme of finite type S over $\mathbb{C}$ together with a family $(\boldsymbol{\mathcal{G}}, \boldsymbol{\phi})$ of $(D, \mathcal{E}_D)$-framed pairs over S such that for any $(\mathcal{E}, \phi) \in \mathcal{M}$, there exist $s \in S$ and an isomorphism $(\boldsymbol{\mathcal{G}}_s, \boldsymbol{\phi}_{|D \times s}) \simeq (\mathcal{E}, \phi)$.

Definition 2.4. *Let X be a smooth projective variety. An effective divisor D on X is called a good framing divisor if we can write $D = \sum n_i D_i$, where*

D_i are prime divisors and $n_i > 0$, and there exists a nef and big divisor of the form $\sum a_i D_i$ with $a_i \geq 0$. For a sheaf $\mathcal{E}_D$ on D, we shall say that $\mathcal{E}_D$ is a good framing sheaf, if it is locally free and there exists a real number A_0, $0 \leq A_0 < \frac{1}{r} D^2 \cdot H^{n-2}$, such that for any locally free subsheaf $\mathcal{F} \subset \mathcal{E}_D$ of constant positive rank, $\frac{1}{\operatorname{rk} \mathcal{F}} \deg c_1(\mathcal{F}) \leq \frac{1}{\operatorname{rk} \mathcal{E}_D} \deg c_1(\mathcal{E}_D) + A_0$.

Theorem 2.5. *Let X be a smooth projective variety of dimension $n \geq 2$, H an ample divisor on X, $D \subset X$ an effective divisor, and $\mathcal{E}_D$ a vector bundle on D. Assume that D is a good framing divisor. Then for every polynomial P with coefficients in $\mathbb{Q}$, the set of torsion-free sheaves $\mathcal{E}$ on X that satisfy the conditions $P^H_{\mathcal{E}} = P$ and $\mathcal{E}_{|D} \simeq \mathcal{E}_D$ is bounded.*

This is proved in [14], Theorem 3.2.4, for locally free sheaves, but the proof goes through also in the torsion-free case, provided that $\mathcal{E}_D$ is locally free, as we are assuming.

3. Quasi-projective moduli spaces

Using the notions introduced in the previous section, we now can state the main existence result for quasi-projective moduli spaces:

Theorem 3.1. *Let X be a smooth projective surface, $D \subset X$ a big and nef curve, and $\mathcal{E}_D$ a good framing sheaf on D. Then for any $c \in H^*(X, \mathbb{Q})$, there exists an ample divisor H on X and a real number $\delta > 0$ such that all the $(D, \mathcal{E}_D)$-framed sheaves $\mathcal{E}$ on X with Chern character $\operatorname{ch}(\mathcal{E}) = c$ are (H, δ)-stable, so that there exists a quasi-projective scheme $\mathfrak{M}_X(c)$ which is a fine moduli space for these framed sheaves.*

Proof. Let us fix an ample divisor C on X. Set $\mathcal{O}_X(k) = \mathcal{O}_X(kC)$ and $\mathcal{E}(k) = \mathcal{E} \otimes \mathcal{O}_X(k)$ for any sheaf $\mathcal{E}$ on X and for any $k \in \mathbb{Z}$. Recall that the Castelnuovo-Mumford regularity $\rho(\mathcal{E})$ of a sheaf $\mathcal{E}$ on X is the minimal integer m such that $h^i(X, \mathcal{E}(m-i)) = 0$ for all $i > 0$. According to Lehn's Theorem (Theorem 2.5), the family $\mathcal{M}$ of all the sheaves $\mathcal{E}$ on X with $\operatorname{ch}(\mathcal{E}) = c$ and $\mathcal{E}_{|D} \simeq \mathcal{E}_D$ is bounded. Hence $\rho(\mathcal{E})$ is uniformly bounded over all $\mathcal{E} \in \mathcal{M}$. By Grothendieck's Lemma (Lemma 1.7.9 in [10]), there exists $A_1 \geq 0$, depending only on $\mathcal{E}_D$, c and C, such that $\mu^C(\mathcal{F}) \leq \mu^C(\mathcal{E}) + A_1$ for all $\mathcal{E} \in \mathcal{M}$ and for all nonzero subsheaves $\mathcal{F} \subset \mathcal{E}$.

For $n > 0$, denote by H_n the ample divisor $C + nD$. We shall verify that there exists a positive integer n such that the range of positive real numbers δ, for which all the framed sheaves $\mathcal{E}$ from $\mathcal{M}$ are (H_n, δ)-stable, is nonempty.

Let $\mathcal{F} \subset \mathcal{E}$, $0 < r' = \operatorname{rk} \mathcal{F} \leq r = \operatorname{rk} \mathcal{E}$. Assume first that $\mathcal{F} \not\subset \ker\left(\mathcal{E} \to \mathcal{E}_{|D}\right)$. Then we may only consider the case $r' < r$, and the (H_n, δ)-stability condition for $\mathcal{E}$ reads:

$$\mu^{H_n}(\mathcal{F}) < \mu^{H_n}(\mathcal{E}) + \left(\frac{1}{r'} - \frac{1}{r}\right) \delta. \tag{1}$$

Saturating $\mathcal{F}$, we make $\mu^{H_n}(\mathcal{F})$ bigger, so we may assume that $\mathcal{F}$ is a saturated subsheaf of $\mathcal{E}$, and hence that it is locally free. Then $\mathcal{F}_{|D} \subset \mathcal{E}_{|D}$ and we have:

$$\mu^{H_n}(\mathcal{F}) = \frac{n}{r'} \deg c_1(\mathcal{F}_{|D}) + \mu^C(\mathcal{F}) \leq \mu^{H_n}(\mathcal{E}) + nA_0 + A_1. \tag{2}$$

Thus we see that (2) implies (1) whenever

$$\frac{rr'}{r - r'}(nA_0 + A_1) < \delta. \tag{3}$$

Assume now that $\mathcal{F}$ is a saturated, and hence a locally free subsheaf of $\ker\big(\mathcal{E} \to \mathcal{E}_{|D}\big) \simeq \mathcal{E}(-D)$. Then the (H_n, δ)-stability condition for $\mathcal{E}$ is

$$\mu^{H_n}(\mathcal{F}) < \mu^{H_n}(\mathcal{E}) - \frac{1}{r}\delta, \tag{4}$$

and the inclusion $\mathcal{F}(D) \subset \mathcal{E}$ yields:

$$\mu^{H_n}(\mathcal{F}) < \mu^{H_n}(\mathcal{E}) - H_n D + nA_0 + A_1 = \mu^{H_n}(\mathcal{E}) - (D^2 - A_0)n + A_1 - DC. \tag{5}$$

We see that (5) implies (4) whenever

$$\delta < r[(D^2 - A_0)n - A_1 + DC]. \tag{6}$$

The inequalities (3), (6) for all $r' = 1, \ldots, r-1$ have a nonempty interval of common solutions δ if

$$n > \max\left\{\frac{rA_1 - CD}{D^2 - rA_0}, 0\right\}.$$

□

Remark 3.2. Grothedieck's Lemma is stated in [10] in terms of the so called $\hat{\mu}$ slope. However, for torsion-free sheaves, the $\hat{\mu}$ slope and the usual slope differ by constants depending only on $(X, \mathcal{O}_X(1))$, see Definition 1.6.8 in [10] and the following remark. △

Note that up to isomorphism, the quasi-projective structure making $\mathfrak{M}_X(c)$ a fine moduli space is unique, which follows from the existence of a universal family of framed sheaves over it.

If D is a smooth and irreducible curve and $D^2 > 0$, then our definition of a good framing sheaf with $A_0 = 0$ is just the definition of semistability. The following is thus an immediate consequence of the theorem:

Corollary 3.3. *Let X be a smooth projective surface, $D \subset X$ a smooth, irreducible, big and nef curve, and $\mathcal{E}_D$ a semistable vector bundle on D. Then for any $c \in H^*(X, \mathbb{Q})$, there exists a quasi-projective scheme $\mathfrak{M}_X(c)$ which is a fine moduli space of $(D, \mathcal{E}_D)$-framed sheaves on X with Chern character c.*

4. Infinitesimal study

Let X be a smooth projective variety, D an effective divisor on X, $\mathcal{E}_D$ a vector bundle on D. We shall consider sheaves $\mathcal{E}$ on X framed to $\mathcal{E}_D$ on D. We recall the notion of a simplifying framing bundle introduced by Lehn.

Definition 4.1. *$\mathcal{E}_D$ is simplifying if for any two vector bundles $\mathcal{E}$, $\mathcal{E}'$ on X such that $\mathcal{E}_{|D} \simeq \mathcal{E}'_{|D} \simeq \mathcal{E}_D$, the group $H^0(X, \mathcal{H}om(\mathcal{E}, \mathcal{E}')(-D))$ vanishes.*

An easy sufficient condition for $\mathcal{E}_D$ to be simplifying is $H^0(D, \mathcal{E}nd(\mathcal{E}_D) \otimes \mathcal{O}_X(-kD)_{|D}) = 0$ for all $k > 0$.

Lehn [14] proved that if D is good and $\mathcal{E}_D$ is simplifying, there exists a fine moduli space $\mathfrak{M}$ of $(D, \mathcal{E}_D)$-framed vector bundles on X in the category of separated algebraic spaces. Lübke [15] proved a similar result: if X is a compact complex manifold, D a smooth hypersurface (not necessarily "good") and if $\mathcal{E}_D$ is simplifying, then the moduli space $\mathfrak{M}$ of $(D, \mathcal{E}_D)$-framed vector bundles exists as a Hausdorff complex space. In both cases the tangent space $T_{[\mathcal{E}]}\mathfrak{M}$ at a point representing the isomorphism class of a framed bundle $\mathcal{E}$ is naturally identified with $H^1(X, \mathcal{E}nd(\mathcal{E})(-D))$, and the moduli space is smooth at $[\mathcal{E}]$ if $H^2(X, \mathcal{E}nd(\mathcal{E})(-D)) = 0$. Lübke gives a more precise statement about smoothness: $[\mathcal{E}]$ is a smooth point of $\mathfrak{M}$ if $H^2(X, \mathcal{E}nd_0(\mathcal{E})(-D)) = 0$, where $\mathcal{E}nd_0$ denotes the traceless endomorphisms. Huybrechts and Lehn in [9] define the tangent space and give a smoothness criterion for the moduli space of stable pairs that are more general objects than our framed sheaves. In this section, we adapt Lübke's criterion to our moduli space $\mathfrak{M}_X(c)$, parametrizing not only vector bundles, but also some non-locally-free sheaves. When we work with stable framed sheaves, we do not need the assumption that $\mathcal{E}_D$ is simplifying.

We shall use the notions of the trace map and traceless exts, see Definition 10.1.4 from [10]. Assuming X is a smooth algebraic variety, $\mathcal{F}$ any (coherent) sheaf on it, and $\mathcal{N}$ a locally free sheaf (of finite rank), the trace map is defined

$$\mathrm{tr}\,:\mathrm{Ext}^i(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}) \to H^i(X, \mathcal{N})\,, \quad i \in \mathbb{Z}, \tag{7}$$

and the traceless part of the ext-group, denoted by $\mathrm{Ext}^i(\mathcal{F}, \mathcal{F} \otimes \mathcal{N})_0$, is the kernel of this map.

We shall need the following property of the trace:

Lemma 4.2. *Let $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{E} \to 0$ be an exact triple of sheaves and $\mathcal{N}$ a locally free sheaf. Then there are two long exact sequences of ext-functors giving rise to the natural maps*

$$\mu_i : \mathrm{Ext}^i(\mathcal{F}, \mathcal{E} \otimes \mathcal{N}) \to \mathrm{Ext}^{i+1}(\mathcal{E}, \mathcal{E} \otimes \mathcal{N})\ ,$$
$$\tau_i : \mathrm{Ext}^i(\mathcal{F}, \mathcal{E} \otimes \mathcal{N}) \to \mathrm{Ext}^{i+1}(\mathcal{F}, \mathcal{F} \otimes \mathcal{N})\ ,$$

and we have $\mathrm{tr} \circ \mu_i = (-1)^i \mathrm{tr} \circ \tau_i$ *as maps* $\mathrm{Ext}^i(\mathcal{F}, \mathcal{E} \otimes \mathcal{N}) \to H^{i+1}(X, \mathcal{N})$.

Proof. This is a particular case of the graded commutativity of the trace with respect to cup-products on Homs in the the derived category (see Section

V.3.8 in [11]): if $\xi \in \mathrm{Hom}(\mathcal{F}, \mathcal{E} \otimes \mathcal{N}[i])$, $\eta \in \mathrm{Hom}(\mathcal{E}, \mathcal{F}[j])$, then $\mathrm{tr}\,(\xi \circ \eta) = (-1)^{ij}\mathrm{tr}\,((\eta \otimes \mathrm{id}_{\mathcal{N}}) \circ \xi)$. This should be applied to $\xi \in \mathrm{Hom}(\mathcal{F}, \mathcal{E} \otimes \mathcal{N}[i])$ and $\eta = \partial \in \mathrm{Hom}(\mathcal{E}, \mathcal{F}[1])$, where ∂ is the connecting homomorphism in the distinguished triangle associated to the given exact triple:

$$\mathcal{E}[-1] \xrightarrow{-\partial} \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{E} \xrightarrow{\partial} \mathcal{F}[1].$$

□

THEOREM 4.3. *Let X be a smooth projective surface, $D \subset X$ an effective divisor, $\mathcal{E}_D$ a locally free sheaf on D, and $c \in H^*(X, \mathbb{Q})$ the Chern character of a $(D, \mathcal{E}_D)$-framed sheaf $\mathcal{E}$ on X. Assume that there exists an ample divisor H on X and a positive real number δ such that $\mathcal{E}$ is (H, δ)-stable, and denote by $\mathfrak{M}_X(c)$ the moduli space of $(D, \mathcal{E}_D)$-framed sheaves on X with Chern character c which are (H, δ)-stable. Then the tangent space to $\mathfrak{M}_X(c)$ is given by*

$$T_{[\mathcal{E}]}\mathfrak{M}_X(c) = \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_X(-D)),$$

and $\mathfrak{M}_X(c)$ is smooth at $[\mathcal{E}]$ if the traceless ext-group

$$\mathrm{Ext}^2(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_X(-D))_0 = \ker\left[\ \mathrm{tr}\ : \mathrm{Ext}^2(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_X(-D)) \to H^2(X, \mathcal{O}(-D))\ \right]$$

vanishes.

Proof. We prove this result by a combination of arguments of Huybrechts-Lehn and Mukai, so we just give a sketch, referring to [9, 16] for details. As in Section 4.iv) of [9], the smoothness of $\mathfrak{M} = \mathfrak{M}_X(c)$ follows from the T^1-lifting property for the complex $\mathcal{E} \to \mathcal{E}_D$.

Let $A_n = k[t]/(t^{n+1})$, $X_n = X \times \mathrm{Spec}A_n$, $D_n = D \times \mathrm{Spec}A_n$, $\mathcal{E}_{D_n} = \mathcal{E}_D \boxtimes A_n$, and let $\mathcal{E}_n \xrightarrow{\alpha_n} \mathcal{E}_{D_n}$ be an A_n-flat lifting of $\mathcal{E} \to \mathcal{E}_D$ to X_n. Then the infinitesimal deformations of α_n over $k[\epsilon]/(\epsilon^2)$ are classified by the hyper-ext $\mathbb{E}\mathrm{xt}^1(\mathcal{E}_n, \mathcal{E}_n \xrightarrow{\alpha_n} \mathcal{E}_{D_n})$, and one says that the T^1-lifting property is verified for $\mathcal{E} \to \mathcal{E}_D$ if all the natural maps

$$T^1_n : \mathbb{E}\mathrm{xt}^1(\mathcal{E}_n, \mathcal{E}_n \xrightarrow{\alpha_n} \mathcal{E}_{D_n}) \to \mathbb{E}\mathrm{xt}^1(\mathcal{E}_{n-1}, \mathcal{E}_{n-1} \xrightarrow{\alpha_{n-1}} \mathcal{E}_{D_{n-1}})$$

are surjective whenever $(\mathcal{E}_n, \alpha_n) \equiv (\mathcal{E}_{n-1}, \alpha_{n-1}) \mod (t^n)$. In loc. cit., the authors remark that there is an obstruction map ob on the target of T^1_n which embeds the cokernel of T^1_n into $\mathbb{E}\mathrm{xt}^2(\mathcal{E}, \mathcal{E} \to \mathcal{E}_D)$, so that if the latter vanishes, the T^1-lifting property holds.

In our case, $\mathcal{E}$ is locally free along D, so the complex $\mathcal{E} \to \mathcal{E}_D$ is quasi-isomorphic to $\mathcal{E}(-D)$ and $\mathbb{E}\mathrm{xt}^i(\mathcal{E}, \mathcal{E} \to \mathcal{E}_D) = \mathrm{Ext}^i(\mathcal{E}, \mathcal{E}(-D))$. It remains to prove that the image of ob is contained in the traceless part of $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}(-D))$. This is done by a modification of Mukai's proof in the non-framed case.

First we assume that $\mathcal{E}$ is locally free. Then the elements of $\mathrm{Ext}^1(\mathcal{E}_{n-1}, \mathcal{E}_{n-1}(-D_{n-1}))$ can be given by Čech 1-cocycles with values in $\mathcal{E}nd(\mathcal{E}_{n-1})(-D_{n-1})$ for some open covering of X, and the image of such a 1-cocycle (a_{ij}) under the obstruction map $\mathrm{Ext}^1(\mathcal{E}_{n-1}, \mathcal{E}_{n-1}(-D_{n-1})) \to$

$\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}(-D))$ is a Čech 2-cocycle (c_{ijk}) with values in $\mathcal{E}nd(\mathcal{E})(-D)$. A direct calculation shows that $(\operatorname{tr} c_{ijk})$ is a Čech 2-cocycle with values in $\mathcal{O}_X(-D)$ which is the obstruction to the lifting of the infinitesimal deformation of the framed line bundle $\det \mathcal{E}_{n-1}$ from A_{n-1} to A_n. As we know that the moduli space of line bundles, whether framed or not, is smooth, this obstruction vanishes, so the cocycle $(\operatorname{tr} c_{ijk})$ is cohomologous to 0.

Now consider the case when $\mathcal{E}$ is not locally free. Replacing $\mathcal{E}, \mathcal{E}_D$ by their twists $\mathcal{E}(n)$, $\mathcal{E}_D(n)$ for some $n > 0$, we may assume that $H^i(X, \mathcal{E}) = H^i(X, \mathcal{E}(-D)) = 0$ for $i = 1, 2$ and that $\mathcal{E}$ is generated by global sections. Then we get the exact triple of framed sheaves

$$0 \to (\mathcal{G}, \gamma) \to (H^0(X, \mathcal{E}) \otimes \mathcal{O}_X, \beta) \to (\mathcal{E}, \alpha) \to 0,$$

where $\mathcal{G}$ is locally free (at this point it is essential that $\dim X = 2$ and X is smooth). Then we verify the T^1-lifting property for the exact triples

$$0 \to (\mathcal{G}_n, \gamma_n) \to (\mathcal{O}_{X_n}^N, \beta_n) \to (\mathcal{E}_n, \alpha_n) \to 0.$$

The infinitesimal deformations of such exact triples are classified by $\mathrm{Hom}(\mathcal{G}_n, \mathcal{E}_n(-D_n))$, and the obstructions lie in $\mathrm{Ext}^1(\mathcal{G}, \mathcal{E}(-D))$. We have two connecting homomorphisms $\mu_1 : \mathrm{Ext}^1(\mathcal{G}, \mathcal{E}(-D)) \to \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}(-D))$ and $\tau_1 : \mathrm{Ext}^1(\mathcal{G}, \mathcal{E}(-D)) \to \mathrm{Ext}^2(\mathcal{G}, \mathcal{G}(-D))$. Our hypotheses on $\mathcal{E}$ imply that: 1) every infinitesimal deformation of $(\mathcal{E}_n, \alpha_n)$ lifts to that of the triple, and 2) μ_1 is an isomorphism, that is, the infinitesimal deformation of $\mathcal{E}_n$ is unobstructed if and only if that of the triple is. By Lemma 4.2, $\operatorname{tr}(\mu_1(\xi)) = -\operatorname{tr}(\tau_1(\xi))$ in $H^2(X, \mathcal{O}_X(-D))$. As in 1.10 of [16], $\tau_1(\xi)$ is the obstruction $\mathrm{ob}(G_{n-1}, \gamma_{n-1})$ to lifting (G_{n-1}, γ_{n-1}) from A_{n-1} to A_n. As G_{n-1} is locally free, we can use the Čech cocycles as above and see that $\operatorname{tr}(\tau_1(\xi)) \in H^2(X, \mathcal{O}_X(-D))$ is the obstruction to lifting $(\det G_{n-1}, \det \gamma_{n-1})$, hence it is zero and we are done. □

The following Corollary describes a situation where the moduli space $\mathfrak{M}_X(c)$ is smooth (hence, every connected component is a smooth quasi-projective variety).

Corollary 4.4. *In addition to the hypothesis of Theorem 4.3, let us assume that D is irreducible, that $(K_X + D) \cdot D < 0$, and choose the framing bundle to be trivial. Then the moduli space $\mathfrak{M}_X(c)$ is smooth.*

This happens for instance when X is a Hirzebruch surface, or the blowup of $\mathbb{P}^2$ at a number of distinct points, taking for D the inverse image of a generic line in $\mathbb{P}^2$ via the birational morphism $X \to \mathbb{P}^2$. In this case one can also compute the dimension of the moduli space, obtaining $\dim \mathfrak{M}_X(c) = 2rn$, with $r = \mathrm{rk}(\mathcal{E})$ and

$$c_2(\mathcal{E}) - \frac{r-1}{2r} c_1(\mathcal{E})^2 = n\,\varpi,$$

where ϖ is the fundamental class of X. When X is the p-th Hirzebruch surface $\mathbb{F}_p$ we shall denote this moduli space by $\mathfrak{M}^p(r, k, n)$ if $c_1(\mathcal{E}) = kC$, where C is the unique curve in $\mathbb{F}_p$ having negative self-intersection.

The next example shows that the moduli space may be nonsingular even if the group $\operatorname{Ext}^2(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_X(-D))$ does not vanish.

Example 4.5. For $r = 1$ the moduli space $\mathfrak{M}(1, 0, n)$ is isomorphic to the Hilbert scheme $X_0^{[n]}$ parametrizing length n 0-cycles in $X_0 = X \backslash D$. Of course this space is a smooth quasi-projective variety of dimension $2n$. Indeed in this case the trace morphism $\operatorname{Ext}^2(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_X(-D)) \to H^2(X, \mathcal{O}(-D))$ is an isomorphism.

5. Examples

5.1. Bundles with small invariants on Hirzebruch surfaces. Let X be the p-th Hirzebruch surface $\mathbb{F}_p$, and normalize the Chern character by twisting by powers of the line bundle $\mathcal{O}_{\mathbb{F}_p}(C)$ so that $0 \leq k \leq r - 1$. It has been shown in [3] that the moduli space $\mathfrak{M}^p(r, k, n)$ is nonempty if and only if the bound

$$n \geq N = \frac{pk}{2r}(r - k)$$

is satisfied. The moduli spaces $\mathfrak{M}^p(r, k, N)$ can be explicitly characterized: $\mathfrak{M}^p(r, k, N)$ is a rank $k(r-k)(p-1)$ vector bundle on the Grassmannian $G(k, r)$ of k-planes in $\mathbb{C}^r$ [25]; in particular, $\mathfrak{M}^1(r, k, N) \simeq G(k, r)$, and $\mathfrak{M}^2(r, k, N)$ is isomorphic to the tangent bundle of $G(k, r)$. This is consistent with instanton counting, which shows that the spaces $\mathfrak{M}^p(r, k, N)$ have the same Betti numbers as $G(k, r)$ [3].

5.2. Rank 2 vector bundles on $\mathbb{F}_1$. We study in some detail the moduli spaces $\mathfrak{M}^1(2, k, n)$. As [27] and [28] show, the non-locally free case turns out to be very complicated as soon as the value of n exceeds the rank. So we consider only locally free sheaves. To simplify notation we call this moduli space $\hat{M}(k, n)$, where n denotes now the second Chern class. We normalize k so that it will assume only the values 0 and -1. Moreover we shall denote by $M(n)$ the moduli space of rank 2 bundles on $\mathbb{P}^2$, with second Chern class n, that are framed on the "line at infinity" $\ell_\infty \subset \mathbb{P}^2$ (which we identify with the image of D via the blow-down morphism $\pi \colon \mathbb{F}_1 \to \mathbb{P}^2$).

Let us start with the case $k = -1$. We introduce a stratification on $\hat{M}(-1, n)$ according to the splitting type of the bundles it parametrizes on the exceptional line $E \subset \mathbb{F}_1$

$$\hat{M}(-1, n) = Z_0(-1, n) \supset Z_1(-1, n) \supset Z_2(-1, n) \supset \dots$$

defined as follows: if $Z_k^0(-1, n) = Z_k(-1, n) \setminus Z_{k+1}(-1, n)$ then

$$Z_k^0(-1, n) = \{\mathcal{E} \in \hat{M}(-1, n) \,|\, \mathcal{E}_{|E} \simeq \mathcal{O}_E(-k) \oplus \mathcal{O}_E(k + 1)\}\,.$$

Proposition 5.1. *There is a map*

$$F_1 \colon \hat{M}(-1, n) \to \coprod_{k=0}^{n} M(n - k)$$

which restricted to the subset $Z_k^0(-1,n)$ *yields a morphism*

$$Z_k^0(-1,n) \to M(n-k)$$

whose fibre is an open set in $\operatorname{Hom}(\sigma^*\mathcal{E}_{|E}, \mathcal{O}_E(k))/\mathbb{C}^* \simeq \mathbb{P}^{2k+1}$*, made by* k*-linear forms that have no common zeroes on the exceptional line.*

Proof. We start by considering $Z_0^0(-1,n)$. The morphism $Z_0^0(-1,n) \to M(n)$ is given by $\mathcal{E}_1 \mapsto \mathcal{E} = (\pi_*\mathcal{E})^{**}$. The fibre of this morphism includes a $\mathbb{P}^1$. To show that this is indeed a $\mathbb{P}^1$-fibration we need to check that $\mathcal{E}_1$ has no other deformations than those coming from the choice of a point in $M(n)$ and a point in this $\mathbb{P}^1$. This follows from the equalities

$$\dim \operatorname{Ext}^1(\mathcal{E}_1, \mathcal{E}_1(-E)) = \dim \operatorname{Ext}^1(\mathcal{E}, \mathcal{E}(-\ell_\infty) + 1$$

$$\operatorname{Ext}^2(\mathcal{E}_1, \mathcal{E}_1(-E)) = 0$$

Note that this result is compatible with the isomorphism $\mathfrak{M}^1(r,k,N) \simeq G(k,r)$ mentioned in Section 5.1.

In general, if $\mathcal{E}_1 \in Z_k^0(-1,n)$ with $k \geq 1$, so that $\mathcal{E}_{1|E} \simeq \mathcal{O}_E(k+1) \oplus \mathcal{O}_E(-k)$, the direct image $\pi_*(\mathcal{E}_1(kE))$ is locally free. This defines the morphism $Z_k^0(-1,n) \to M(n-k)$. $\square$

We consider now the case $k = 0$. One has $Z_0^0(0,n) \simeq M(n)$. We study the other strata by reducing to the odd case. If $\mathcal{E}_1 \in Z_k^0(0,n)$, there is a unique surjection $\alpha\colon \mathcal{E}_1 \to \mathcal{O}_E(-k)$; let $\mathcal{F}$ be the kernel. Restricting $0 \to \mathcal{F} \to \mathcal{E}_1 \to \mathcal{O}_E(-k) \to 0$ we get an exact sequence

$$0 \to \mathcal{O}_E(1-k) \to \mathcal{F}_{|E} \to \mathcal{O}_E(k) \to 0$$

so that

$$\mathcal{F}_{|E} \simeq \mathcal{O}_E(a+1) \oplus \mathcal{O}_E(-a) \qquad \text{with} \qquad -k \leq a \leq k-1.$$

A detailed analysis shows that $a = k-1$. As a result we have:

Proposition 5.2. *For all* $k \geq 1$ *there is a morphism*

$$Z_k^0(0,n) \to M(n-2k+1)$$

whose fibres have dimension $2k-1$.

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