Asymptotic Behavior of Word Metrics on Coxeter Groups

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Abstract. We study the geometry of tessellation defined by the walls in the Moussong complex $\mathcal{M}_W$ of a Coxeter group W. It is proved that geodesics in $\mathcal{M}_W$ can be approximated by geodesic galleries of the tessellation. A formula for the translation length of an element of W is given. We prove that the restriction of the word metric on the W to any free abelian subgroup A is Hausdorff equivalent to a regular norm on A.

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Introduction

For any Coxeter system (W, S), Moussong constructed a certain piecewise Euclidean complex $\mathcal{M}_W$ on which W acts properly and cocompactly by isometries [Mou88]. This complex is complete, contractible, has nonpositive curvature and the Cayley graph $\mathcal{C}_W$ of W (with respect to S) is isomorphic to the 1-skeleton of $\mathcal{M}_W$. A wall in $\mathcal{M}$ is the fixed-point set of a reflection in W. It turns out that the walls are totally geodesic subspaces in $\mathcal{M}_W$ and each wall divides $\mathcal{M}_W$ into two path components. The set of all walls defines a wall tessellation of $\mathcal{M}$. The set of all tiles (=chambers) of this tessellation together with an appropriate adjacency relation is isomorphic to the Cayley graph $\mathcal{C}_W$. We shall prove that geodesics in $\mathcal{M}_W$ can be uniformly approximated by geodesic galleries of the wall tessellation (= geodesic paths in $\mathcal{C}_W$) (Theorem 3.3.2). This approximation result allows us to prove that for any "generic" element $w \in W$ of infinite order there is a conjugate v which is straight i.e., $\ell(v^n) = n\ell(v)$ for all $n \in \mathbb{N}$, where $\ell(v)$ is the word length on W (Theorem 4.1.5). There is a constant $c = c(W)$, such that for any $w \in W$ of infinite order there is a conjugate v of w^c, which is straight (Theorem 4.1.6). The restriction of the word metric on W to any free abelian subgroup A is Hausdorff equivalent to a regular norm on A (Theorem 4.3.2).

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1 Preliminaries on Moussong complexes

To any Coxeter system (W, S) one can canonically associate the Moussong complex $\mathcal{M} = \mathcal{M}_W$, which is a piecewise Euclidean complex with W as the set of vertices. Their cells are Euclidean polyhedra, which are the convex hulls of sets, naturally bijected with the spherical cosets of W. In particular, the 1-cells of $\mathcal{M}$ are in bijective correspondence with the sets $\{w, ws\}$, where $w \in W$ and $s \in S$. Hence the 1-skeleton of $\mathcal{M}$ is nothing but a modified Cayley graph of W with respect to S (the modification consists in identifying an edge $w \xrightarrow{s} ws$ with its inverse $ws \xrightarrow{s} w$). W acts cellularly and isometrically on $\mathcal{M}_W$ and this induces the standard W-action on the Cayley graph of W. In the next subsections we carry out in detail the construction of $\mathcal{M}_W$ following the thesis of D. Krammer [Kra94].

1.1 Coxeter groups

A Coxeter system is a pair (W, S) where W is a group and where S is a finite set of involutions in W such that W has the following presentation:

$$\langle s : s \in S | (ss')^{m_{ss'}} = 1 \text{ when } m_{ss'} < \infty \rangle,$$

where $m_{ss'} \in \{1, 2, 3, \ldots, \infty\}$ is the order of ss', and $m_{ss'} = 1$ if and only if $s = s'$. We refer to W itself as a Coxeter group when the presentation is understood. The number of elements of S is called its rank. The Coxeter system (W, S) is called spherical if W has finite order. A subgroup of W is called special if it is generated by a subset of S. For each $T \subseteq S$, W_T denotes the special subgroup generated by T. Any conjugate of such a subgroup will be called parabolic. A remarkable feature of Coxeter systems is that for any subset $T \subseteq S$ the pair (W_T, T) is a Coxeter system in its right and moreover a presentation of W_T is defined by the numbers $m_{tt'}, t, t' \in T$. If (W_S, S) is a Coxeter system of finite rank then we write V_S for the real vector space with a basis of elements (e_s) for $s \in S$. Put a symmetric bilinear form B on V_S by requiring:

$$B(e_s, e_{s'}) = -\cos(\pi / m_{ss'}).$$

(This expression is interpreted to be -1 in case $m(s,s') = \infty$.) Evidently $B(e_s, e_s) = 1$, while $B(e_s, e_{s'}) \leq 0$ if $s \neq s'$. Since e_s is non-isotropic, the subspace $H_s = e_s^{\perp}$ orthogonal to e_s is complementary to the line $\mathbb{R}e_s$. Associated to $s \in S$ is an automorphism a_s of B acting as the reflection $v \mapsto v - 2(v, e_s)e_s$ in the hyperplane $e_s^{\perp}$. The result by Tits asserts that the correspondence $s \mapsto a_s$ extends to a faithful representation of W as a group of automorphisms of the form B. (cf. [Bou], Ch.V, s.4).

1.2 Trading Coxeter cells

The Coxeter group W is finite if and only if the form $B(e_s, e_{s'})$ is positive definite. We call a set $J \subseteq S$ spherical if W_J is finite or, equivalently, the restriction of the form B to the subspace $V_J = \sum_{j\in J} \mathbb{R}\, e_j$ is positive definite. Let $J \subseteq S$ be spherical. Since V_J is non-degenerate, there exists a unique basis $\{f_j^J | j \in J\}$ of V_J dual to $\{e_j : j \in J\}$ with respect to B. A space V_J that comes equipped with a positive definite inner product $B|V_J$ will be denoted by E_J and called the Euclidean space associated to J. Define the Coxeter cell X_J to be the convex hull of the W_J-orbit:

$$X_J = \mathrm{Ch}(W_J x_J)$$

where

$$x_J = \sum_{j\in J} f_j^J \in E_J.$$

For convenience we define $W_\emptyset = \{1\}$ and $X_\emptyset = \{0\}$– the origin of E_J. More generally, for any spherical K and any $J \subseteq K$ we consider the faces of the polyhedron $X_K = \mathrm{Ch}(W_K x_K)$ of the form

$$X_{JK} = \mathrm{Ch}(W_J x_K).$$

We do not exclude the case $J = \emptyset$, where $X_{\emptyset K} = \{x_K\}$. We call the extremal points of the cell X_J the vertices.
For spherical $J \subseteq S$, let $p_J : V_S \to E_J$ denote the orthogonal projection. It is well defined since the quadratic form on E_J is non-degenerate.

Lemma 1.2.1 ([Kra94], B.2.2.) *The dimension of the cell X_J equals the cardinality of J. For spherical subsets $J \subseteq K$ of S we have $p_J x_K = x_J$. Moreover, $p_J|X_{JK} : X_{JK} \to X_J$ is a W_J-equivariant isometry of cells. The nonempty faces of X_K are precisely those of the form wX_{JK} ($J \subseteq K, w \in W_K$). In particular, the vertex set of X_J is precisely $W_J x_J$.*

Example 1.2.2 1) If $J = \{j\}$ then $f_j^J = e_j$ and $X_J = \mathrm{Ch}(e_j, -e_j)$ is a line segment. 2) Let $J = \{s, s'\}$ be spherical, so $w = ss'$ has finite order $m_{ss'}$. Set $V_{s,s'} = \mathbb{R}e_s + \mathbb{R}e_{s'}$. The restriction of B to $V_{s,s'}$ is positive definite and both s and s' act as orthogonal reflections in the lines $\mathbb{R}f_s, \mathbb{R}f_{s'}$ respectively. Since $B(e_s, e_{s'}) = -\cos(\pi/m_{ss'}) = \cos(\pi - (\pi/m_{ss'}))$, the angle between the rays $\mathbb{R}^+e_s$ and $\mathbb{R}^+e_{s'}$ is equal to $\pi - (\pi/m_{ss'})$, forcing the angle between the reflecting lines $\mathbb{R}f_s, \mathbb{R}f_{s'}$ to be equal $\pi/m_{ss'}$. The vectors $f_s, f_{s'}$ are of the same length, lie in the cone $\mathbb{R}^+e_s + \mathbb{R}^+e_{s'}$; moreover, $f_s + f_{s'}$ is a bisectrix between the reflecting lines $\mathbb{R}f_s, \mathbb{R}f_{s'}$ hence the convex hull of the orbit $W_J(f_s + f_{s'})$ is a regular $2m_{ss'}-$gon.

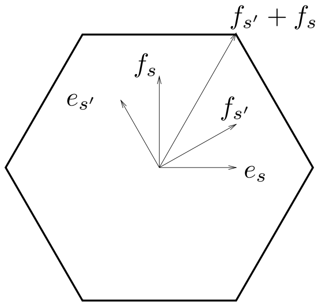

Figure 1: The cell X_J for $J = \{s, s'\}, m_{ss'} = 3$.

1.3 Gluing the Moussong complex

Now we build the Moussong complex of $W = W_S$ as follows. Take the union

$$U = \bigcup \{(w, X_J) : w \in W, J \subseteq S \text{ spherical}\}.$$

Introduce an equivalence relation R on U, generated by the following gluing relations:

1. $(wu, x) \sim (w, u^{-1}x)$, whenever $w \in W, u \in W_J, x \in X_J$,
2. The cells $(w, X_K), (w, X_L)$ are glued along the face $(w, X_J), J = K \cap L$, which is embedded into each of them (by the map p_J) as (w, X_{JK}) and (w, X_{JL}) respectively .

The quotient space of U modulo R is called the Moussong complex of W and is denoted by $\mathcal{M}_W$. The group W acts on U by $u(w, x) = (uw, x)$. This action respects the relation R and hence induces a cellular action of W on $\mathcal{M}_W$. With some abuse in notation we will denote the natural image of $(1, X_J)$ in $\mathcal{M}$ by X_J, so any cell in $\mathcal{M}$ is of the form wX_J for some $w \in W, J \subseteq S$. We call J the type of the cell wX_J. There is a distinguished vertex $x_0 = X_\emptyset$ in $\mathcal{M}$. Note that $x_J = x_0$ for any spherical J (by condition (2)).
It can be shown that the inclusion maps of the cells are injective, see [Kra94]. The canonical metric in each cell allows to measure the lengths of finite polygonal paths in $\mathcal{M}$. The *path metric* d on $\mathcal{M}$ is defined by setting the distance

between $x, y \in \mathcal{M}$ to be the infimum of the lengths of polygonal paths joining x to y.
We summarize the main properties of $\mathcal{M}$ in the following theorem.

Theorem 1.3.1 ([Kra94],[Mou88]) *Relative to the path metric $\mathcal{M}$ is a contractible, complete, proper CAT(0) space. The Coxeter group W acts on $\mathcal{M}$ cellularly and this action is isometric, proper and cocompact. This action is simply transitive on the set of vertices $\mathcal{M}^{(0)}$ of $\mathcal{M}$, in particular $\mathcal{M}^{(0)}$ coincides with Wx_0.*

For the convenience of the reader we repeat the relevant definitions. A geodesic, or geodesic segment, in a metric space (X, d) is a subset isometric to a closed interval of real numbers. Similarly, a loop $S^1 \to X$ is a closed geodesic if it is an isometric embedding. (Here S^1 denotes the standard circle equipped with its arc metric, possibly rescaled so that its length can be arbitrary). We say that X is a geodesic metric space if any two points of X can be connected by a geodesic. We denote by $[x, y]$ any geodesic joining x and y. We will always parameterize $[x, y]$ by $t \mapsto p_t (0 \leq t \leq 1)$, where $d(x, p_t) = td(x, y)$ for all t. Given three points x, y, z in X, the triangle inequality implies that there is a comparison triangle in the Euclidean plane $\mathbb{R}^2$, whose vertices $\overline{x}, \overline{y}, \overline{z}$ have the same pairwise distances as x, y, z. Given a geodesic $[x, y]$ and a point $p = p_t \in [x, y]$, there is a corresponding point $\overline{p} = \overline{p_t}$ on the line segment $[\overline{x}, \overline{y}]$ in $\mathbb{R}^2$. A geodesic metric space X is called a $\mathsf{CAT}(0)$ space if for any x, y in X there is a geodesic $[x, y]$ with the following property: For all $p \in [x, y]$ and all $z \in X$, we have

$$d(z, p) \leq d_{\mathbb{R}^2}(\overline{z}, \overline{p}),$$

with $\overline{z}$ and $\overline{p}$ as above. Let X be a CAT(0) space. Then there is a unique geodesic segment joining each pair of points $x, y \in X$ and this geodesic segment varies continuously with its endpoints. Every local geodesic in X is a geodesic. For the proof see [BH99], Chapter II.1 , Prop. 1.4.

Examples 1.3.2 If W is a finite Coxeter group of rank n then $\mathcal{M}_W$ is isometric to an n-dimensional convex polyhedron. If, for example, W is the dihedral group of order $2m$, then $\mathcal{M}$ is a regular $2m$-gon with the usual W-action. If W is an affine Coxeter group of rank n then $\mathcal{M}_W$ is a tessellation of the $n-1$-dimensional Euclidean space E. This tessellation is dual to the tessellation, given by the structure of a Coxeter complex on E. Let, for example, W be an affine Coxeter group generated by the reflections s_1, s_2, s_3 in the sides of an equilateral triangle C in the Euclidean plane. Then $\mathcal{M}_W$ is the tessellation of the plane by hexagons, dual to the tessellation consisting of the images of C under W. If W is a product of $n \geq 2$ copies of $\mathbb{Z}/2$ (that is $m_{ss'} = \infty$ for $s \neq s'$), then $\mathcal{M}_W$ is an infinite n-regular tree with edges of length 2. $\square$

Lemma 1.3.3 *Any cell of a $CAT(0)$ piecewise Euclidean complex X is isometrically embedded into X. In view of uniqueness of geodesics this is equivalent to the convexity of a cell.*

Proof. We have to show is that for any two points a, b of a cell C the Euclidean arc α in C between them is a global geodesic. We may assume that C is of minimal dimension. For any two points x and y in the interior of α the closed subarc $\beta \subset \alpha$ between x and y lies in the interior of C. Clearly there is an $\epsilon > 0$, such that for any cell C', having C as a face, the distance from β to the set $\partial C' - C$ is $\geq \epsilon$. Let us cover β by intervals of radius $\epsilon/2$. Each such an interval is geodesic. Indeed, a geodesic γ connecting the points of the interval can not cross ∂C, hence it lies in the union U of cells, having C as a face. For any cell C', having C as a face, γ can not cross $\partial C' - C$ since it has to pass a distance at least ϵ. Hence it lies in only one such C' and thus coincides with the interval. It follows from the considerations above that β is a local geodesic, and therefore a global geodesic since X is CAT(0).
Now let γ be a path in X joining a to b. For any positive $\epsilon < d_C(a,b)/2$ we may choose points x and y in the interior of α such that $d(a,x) = \epsilon = d(y,b)$ A path from x to y obtained by traveling along α to a then along γ to b has length $length(\gamma) + 2\epsilon$, while a geodesic from x to y has length $d_C(a,b) - 2\epsilon$, so $d_C(a,b) \leq length(\gamma) + 4\epsilon$. Since this is true for any sufficiently small $\epsilon > 0$, we conclude that $d_C(a,b) \leq length(\gamma)$, and so α is a geodesic from a to b. $\square$

1.4 The action of reflections on cells

We refer to the notation of §1.3.

Lemma 1.4.1 *An element $w \in W$ leaves the cell uX_K invariant if and only if $u^{-1}wu \in W_K$. In the latter case w acts on the X_K-coordinate of $ux \in uX_K$ as the element $u^{-1}wu \in W_K$.*

Proof. Indeed, the cell uX_K is uniquely determined by its set of vertices uW_Kx_0 and it is w-invariant if and only if uW_K is w-invariant under left translation. The latter happens if and only if $wuW_K = uW_k \Leftrightarrow u^{-1}wu \in W_K$. The second assertion follows from the equality $w(ux) = u(u^{-1}wux)$. $\square$

Lemma 1.4.2 *(An "overcell" of invariant cell is invariant too.) If $C \subseteq C'$ are cells and $wC = C$ for some $w \in W$, then $wC' = C'$.*

Proof. Writing $C = uX_J$ with $w \in W, J \subseteq S$ we can represent C' in the form $C' = uX_K, J \subseteq K$. By Lemma 1.4.1 $wC = C$ implies $u^{-1}wu \in W_J$ and thus $u^{-1}wu \in W_K$. Again by the same lemma $wC' = C'$. $\square$

Definition 1.4.3 *Let $(W; S)$ be a Coxeter system. A* reflection *in W is an element that is conjugate in W to an element of S.*

Lemma 1.4.4 *For any cell C of $\mathcal{M}$ and any reflection $w \in W$ either $C \cap wC$ is empty or else w acts as a reflection on C.*

Proof. Suppose that the cell $C \cap wC$ is nonempty. Then it is invariant under the action of w. Since it is a face of C, by Lemma 1.4.2 we conclude that $wC = C$. Now by Lemma 1.4.1 $w \in W$ acts as a reflection on C. $\square$

1.5 Angles and geodesics in $\mathcal{M}$

The notion of angle in an arbitrary piecewise Euclidean complex can be defined in terms of the link distance, see e.g. [BB97]. Namely, let X be a piecewise Euclidean complex, $x \in X$ and let A be a Euclidean cell of X containing x. The link $\mathrm{lk}_x A$ of is the set of unit tangent vectors ξ at x such that a nontrivial line segment with the initial direction ξ is contained in A. We define the link $\mathrm{lk}_x X$ by $\mathrm{lk}_x X = \cup_{A \ni x} \mathrm{lk}_x A$, where the union is taken over all closed cells containing x.

Recall that the CAT(0)– condition for X is equivalent to the following (see e.g. [BB97]):

1. X is 1-connected and
2. The length of any geodesic loop in the link of any vertex of X is greater or equal to 2π.

A path $\alpha : [a, b] \to X$ is geodesic if it is an isometric embedding: $d(\alpha(s), \alpha(t)) = |s-t|$, for any $s, t \in [a, b]$. Similarly, a loop $\alpha : S^1 \to X$ is a closed geodesic if it is an isometric embedding. Here S^1 denotes the standard circle equipped with its arc metric (possibly rescaled so that its length can be arbitrary). Angles in $\mathrm{lk}_x A$ induce a natural length metric d_x on $\mathrm{lk}_x S$, which turns $\mathrm{lk}_x S$ into a piecewise spherical complex. For $\xi, \eta \in \mathrm{lk}_x X$ define $\angle(\xi, \eta) = \min(d_x(\xi, \eta), \pi)$. Now any two segments σ_1, σ_2 in X with the same endpoint x have the natural projection image in $\mathrm{lk}_x X$ and we define $\angle_x(\sigma_1, \sigma_2)$ to be the angle between these two projections.

We will use the following criterion of geodesicity:

Lemma 1.5.1 ([BB97]) *Let X be a piecewise Euclidean CAT(0)-complex. If each of the segments σ_1, σ_2 is contained in a cell and $\sigma_1 \cap \sigma_2 = \{x\}$, where x is an endpoint of each of the segments, then the union $\sigma_1 \cup \sigma_2$ is geodesic if and only if $\angle_x(\sigma_1, \sigma_2) = \pi$.*

An m- chain from x to y is an $(m+1)$-tuple $T = (x_0, x_1, \ldots, x_m)$ of points in X such that $x = x_0, y = x_m$ and each consecutive pair of points is contained in a cell. Every m-chain determines a polygonal path in X, given by the concatenation of the line segments $[x_i, x_{i+1}], i = 0, ..., i = m$. An m-taut chain from x to y is an m-chain such that

1. there is no triple of consecutive points contained in a cell and
2. (2) the union of two subsequent segments is geodesic in the union of cells, containing these segments.

(The union is equipped with its path metric). Note that if a chain is taut then only its first and last entries lie in the interior of a top dimensional simplex of X. The result of M. Bridson asserts that if X is a piecewise Euclidean complex then X with its path metric is a geodesic space and the geodesics are the paths determined by taut chains [BH99, Theorem. 7.21].

2 Walls in the Moussong complex

The notion of wall in the Moussong complex (as well as in the Coxeter complex) can be defined as the fixed-point set of reflection from the underlying Coxeter group. On the other hand they can be defined as the equivalence classes of "midplanes" (which are the fixed-point sets of stabilizers of cells). Both points of view are useful. Note that in contrast to the situation with Coxeter complexes, the walls in the Moussong complex are not subcomplexes.

2.1 Midplanes and blocks in cells

Let (W_J, J) be a finite Coxeter group and V_J the Euclidean vector space on which W_J acts. We summarize here the basic properties of a Coxeter complex of $W = W_J$. For more about them see [Hum90] or [Bro96]. We define a reflection in W_J to be a conjugate of element of J. The reflecting hyperplanes H_w of reflection $w \in W_J$ cut V_J into polyhedral pieces, which turn out to be cones over simplices. In this way one obtains a simplicial complex $\mathcal{C} = \mathcal{C}(W)$ which triangulates the unit sphere in V_J. This is called the Coxeter complex associated with W_J. The group W_J acts simplicially on $\mathcal{C}$ and this action is simply transitive on the set of maximal simplices (=chambers). Moreover the closure of any chamber C is a fundamental domain of the action of W on $\mathcal{C}$, i.e., each $x \in V$ is conjugated under W to one and only one point in $\mathcal{C}$. Two chambers are adjacent if they have a common codimension one face. For any two adjacent chambers there is a unique reflection in W_J interchanging these two chambers.
A similar picture we have for the Coxeter cell X_J. By a midplane in X_J we mean the intersection $H_w \cap X_J$, where $w \in W_J$ is a reflection and H_w its reflecting hyperplane. We denote this midplane by $M(J, w)$. By equivariance we define the notion of a midplane in any cell of $\mathcal{M}_W$. Each midplane M defines a unique cell in $\mathcal{M}_W$, the cell of least dimension in $\mathcal{M}_W$ which contains M, and we will denote this by $C(M)$.

Lemma 2.1.1 *Every cell X_J contains an open neighborhood of the origin of V_J. In particular midplanes in X_J have dimension $|J| - 1$ and there is one-to-one correspondence between reflecting hyperplanes and midplanes.*

Proof. Note first that the ray $\mathbb{R}^+ x_J$ lies in the interior of the chamber $C = \{x \in V_J : B(x, e_s) > 0 \ \forall s \in S\}$. Hence in each chamber $wC, w \in W_J$ there is a vertex wx_J of X_J. Now suppose that X_J does not contain the origin in the interior, then there is a hyperplane H through the origin such that X_J is contained in one of the closed half-spaces defined by H, say in H_+. This implies that each chamber has an interior point, lying in H_+. Take an arbitrary closed chamber D. If D lies entirely in H_+ then $-D$ lies in the opposite half-space H_- and hence there is no interior point in it belonging H_+ – contradiction. If D does not lie entirely in H_+ then H separates some codimension one face F of D from the remaining vertex x of D. Let D' be the chamber, adjacent to

D in a face F, then D' lies entirely either in H_+ or in H_- and the previous argument works. $\square$

Definitions 2.1.2 It follows from Lemma 2.1.1 that the midplanes $M(J, w)$ also cut X_J into (relatively open) polyhedral pieces of dimension $|J|$ – blocks. Two blocks are adjacent if they have a common codimension one face. There is a canonical one-to-one correspondence between blocks in X_J, chambers of the Coxeter complex $\mathcal{C}(W_J)$ and vertices of X_J. This correspondence clearly preserves the adjacency relation. Each block contains a unique vertex of X_J since a closed block B is a fundamental domain of the action of W on X_J, i.e., each $x \in X_J$ is conjugated under W to one and only one point in B. The group W_J acts on the set of blocks and this action is simply transitive. For a block B the intersection of the closed block B with a midplane is called by internal face of B.

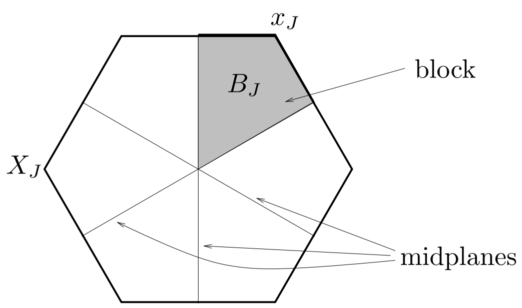

Figure 2: The cell X_J for $J = \{s, s'\}, m_{ss'} = 3$ divided into blocks by midplanes.

Lemma 2.1.3 *The only faces of a cell X_K having nonempty intersection with midplane $M(K, s)$, $s \in S$ are those wX_{JK} with $w^{-1}sw \in W_J$. In particular $M(K, s)$ contains no vertices of X_K. More generally a face of X_K has nonempty intersection with midplane $M(K, usu^{-1})$, $s \in S, u \in W$ iff it is of the form uwX_{JK} with $w^{-1}sw \in W_J$.*

Proof. If $w^{-1}sw \in W_J$ then $swW_J = wW_J$, that is s leaves the vertex set of wX_{JK} invariant and hence it leaves invariant the cell itself and has a nonempty fixed-point set in this cell. Conversely if $M(K, s) \cap wX_{JK}$ is nonempty then there a face F of the cell wX_{JK} such that $M(K, s) \cap F$ contains an interior point of F. But then s leaves F invariant hence by Lemma 1.4.2 it also leaves any "overcell" invariant in particular wX_{JK} and this implies that $w^{-1}sw \in W_J$. To deduce the second statement from the first, one need only to note that $M(K, wsw^{-1}) \cap wX_{JK} = w(M(K, s) \cap X_{JK})$. $\square$

Lemma 2.1.4 *If $w \in W_J$ leaves invariant some midplane M in X_J then it fixes this midplane pointwise.*

Proof. Indeed, w leaves invariant the ambient face C and we can apply Lemma 1.4.1. □

Lemma 2.1.1 *For any cell X_K the following hold:*

1. *The intersection of a midplane of X_K with any of its face is again a midplane.*
2. *Any midplane of any face of X_K is an intersection with this face of a precisely one midplane of X_K.*

Proof. 1) We may assume that a given midplane M is of the form $M(K, s)$ and the face of X_K is $X_{JK}, J \subseteq K$. Since s belongs to W_J, it leaves X_{JK} invariant and its fixed-point set X^s_{JK} bijects onto the fixed-point set X^s_J by a W_J-equivariant isometry $p_J|X_{JK} : X_{JK} \to X_J$. The general assertion follows by equivariance.
2) We may assume that the face is of the form X_{JK} for $J \subseteq K$. Let M_{JK} be a midplane of X_{JK}, then by definition $M_{JK} = (p_J|X_{JK})^{-1}(M(J, w))$ for some $w \in W_J$. Hence, by W_J–equivariance, w is identical on M_{JK} thus $M_{JK} = M(K, w) \cap X_{JK}$. Furthermore, $w \in J$ by Lemma 2.1.3. Hence the segment $\sigma = [wx_J, x_J]$ is an edge of the face X_{JK}, flipped by w. The intersection $M_{JK} \cap \sigma = \{m\}$ is a midpoint of σ and M_{JK} is orthogonal to σ. Now if M any midplane with the same intersection with X_{JK} as M_K, then the reflection in M flips the edge σ and hence this edge is orthogonal to M and thus $M = M_K$. □

Lemma 2.1.5 1)*For every $x \in M(K, s) \cap X_{JK}$ there is a nondegenerate segment of the form $[y, sy], y \in X_{JK}$ with x as a midpoint.* 2) *The segment $[y, sy]$ is orthogonal to midplane $M(K, s)$.* 3) *For any midplane M in X_K there is an edge of X_K, intersected by M in the midpoint.*

Proof. 1) Since $M(K, s) \cap X_{JK}$ is nonempty, it follows from Lemma 2.1.3 that $s \in J$. Let

$$U = \{u \in W_J ; x_k \text{ and } ux_K \text{ are on the same side of } M(K, s).\}$$

Clearly $W_J = U \cup sU, U \cap sU = \emptyset$ and the sets Ux_K, sUx_K lie entirely on the different sides of the midplane $M(K, s)$. Since $X_{JK} = \mathrm{Ch}(W_J x_K)$, we have

$$x = \sum_{u \in U} (\lambda_u ux_K + \mu_u sux_K),$$

where $\sum_{u \in U} (\lambda_u + \mu_u) = 1$ and all coefficients are nonnegative. Since x is fixed by s, applying s to both parts of the equality above we get

$$x = \sum_{u \in U} (\mu_u ux_K + \lambda_u sux_K),$$

We conclude from these two equalities that $x = 1/2(y + sy)$, where $y = \sum_{u\in U}(\lambda_u + \mu_u)ux \in X_{JK}$.
2) The segment $[y, sy]$ is orthogonal to $M(K, s)$ since it is flipped by an orthogonal transformation s.
3) If $M = M(K, s)$, then the edge $[sx_K, x_K]$ of X_K is intersected by M in the midpoint. □

We will call the segment $[y, sy]$ from the lemma above to be a perpendicular to $M(K, s)$ in the point x.

Lemma 2.1.6 *Let $x \in M(K, s), z \in X_K, x \neq z$ and let $[y, sy]$ be a perpendicular to $M(K, s)$ in the point x. Then either $[x, z] \subset M(K, s)$ or one of the angles $\angle_x([x, z], [x, y])$, $\angle_x([x, z], [x, y])$ is strictly less than $\pi/2$.*

Proof. It follows from the fact that the tangent space in x is orthogonal sum of a the tangent space of $M(K, s)$ and a tangent space of the segment $[y, sy]$. □

2.2 Walls as equivalence classes of midplanes

We assume that $\mathcal{M} = \mathcal{M}_W$ is the Moussong complex of a Coxeter group W. The following definition mimics the definition of a hyperplane in a cube complex given in [NR98].

Definitions 2.2.1 For midplanes M_1 and M_2 of the cells $C_1 = C(M_1)$ and $C_2 = C(M_2)$ respectively we write $M_1 \sim M_2$ if $M_1 \cap M_2$ is again a midplane (and then of course it is a midplane of $C_1 \cap C_2$). The transitive closure of this symmetric relation is an equivalence relation, and the union of all midplanes in an equivalence class is called a wall in $\mathcal{M}$. Clearly the equivalences above are generated by those of the form $M_1 \sim M_2, C_1 \leq C_2$ or $C_2 \leq C_1$. Thus to prove some property $\mathcal{P}$ for midplanes of a wall H it is enough to prove this property for some midplane in H and then show that the validity of $\mathcal{P}$ is preserved under equivalences just mentioned. If M is a midpoint of a 1-cell(=edge) in $\mathcal{M}$ then the wall spanned by M will be called a dual wall of e and denoted by $H(e)$. We denote by $\mathcal{H}_M$ the union of midplanes in the equivalence class of a midplane M.

It follows immediately from Lemma 2.1.5 that

Lemma 2.2.2 *Any wall H of $\mathcal{M}$ has the form $H(e)$ for some edge e.*

Clearly W acts on the set of midplanes, preserving the equivalence relation and hence acts on the set of walls. For any wall H we denote by $\tilde{H}$ the complex obtained from the disjoint union of midplanes in H by gluing any two midplanes in H along their common submidplane in $\mathcal{M}$ (if such one exists). One can easily see that $\tilde{H}$ is nonpositively curved, i. e. satisfies the link condition. Namely, the link of any cell C of $\mathcal{M}$ is isometric to the product $C \times [-\pi, \pi]$.

Lemma 2.2.3 *Let $p : \tilde{H} \to \mathcal{M}$ be the natural map which sends each midplane in $\tilde{H}$ to its image in $\mathcal{M}$. Then p is an isometry of $\tilde{H}$ onto H. As a consequence of the above walls are convex in $\mathcal{M}$.*

Proof. It is similar to the proof of lemma 2.6 in [NR98]. Clearly, p is an isometry on each midplane. By result of M. Gromov ([Gro87], Section 4) it is enough to show that p is a local isometry, that is if $x \in \tilde{H}$, then there is a neighborhood U of x such that $p|_U$ is an isometry. Clearly p bijects the star $St(x)$ onto the union U of all midplanes, containing $p(x)$. This union is the fixed-point set of some reflection from W (see Lemmas 1.4.1, 1.4.2, 1.4.4). Hence U is convex, and p maps $St(x)$ isometrically onto U. □

Lemma 2.2.4 *Each wall in $\mathcal{M}_W$ is the fixed-point set of a precisely one reflection in W. Conversely, the fixed-point set of a reflection in W is a wall.*

Proof. Let H_M be the wall, spanned by a midplane M of the cell C. From the description of cells and that of the action of W we know that M is the fixed point set of a reflection from the stabilizer S_C of C in W. We will show that H_M coincides with the fixed-point set H_w of w.

Any reflection w fixing a midplane M pointwise fixes also H_M pointwise, i.e., $H_M \subseteq H_w$. We have to show that the claimed property is invariant under equivalence relation of midplanes, see §2.2.1. If $M_1 \sim M_2$ are midplanes in $C_1 = C(M_1), C_2 = C(M_2)$ respectively, $C_1 \leq C_2$, and w fixes M_1 then w leaves C_1 invariant, hence by Lemma 1.4.2 it leaves C_2 invariant and by Lemma 2.1.1 it leaves M_2 invariant and finally by Lemma 2.1.4 it fixes M_2 pointwise. In case $M_1 \sim M_2, C_1 \geq C_2$, and w fixes M_1 pointwise it is clear that w fixes M_2 pointwise.

Every wall H is the fixed-point set of a unique reflection in W. Write H as the dual wall $H = H(e)$ of some edge e of $\mathcal{M}$. If there were two reflections w, w' with the same reflection wall H then their difference $w^{-1}w'$ would fix e pointwise. But W acts simply transitively on the vertices of $\mathcal{M}$ hence $w = w'$. Now any $w \in W$ fixing at least one cell pointwise is an identity. Indeed the set of cells fixing by w pointwise is nonempty and containing with each cell C every its "overcell" $C' \supset C$ because by Lemma 1.4.2 $wC' = C'$ and since the stabilizer of C' acts fixed point free on the cell we conclude that $w = 1$.

H_w coincides with H_M. Suppose, to the contrary, that there is a w-fixed point x outside H_M. Take any $y \in H_M$, then w fixes the endpoints x, y of the geodesic $[x, y]$ hence, by uniqueness, it fixes the whole geodesic. Shortening $[x, y]$ if necessary we may assume that $[x, y)$ is outside H_M. Take $z \in [x, y], z \neq y$ such that the open segment (z, y) is contained entirely in the interior of some cell C. Since w fixes (z, y), it leaves C invariant. As far as $y \in H_M \cap C$, the point y is contained in some midplane $M' \subset H_M$ of C. Because w fixes M' and the segment (z, y), lying entirely outside H_M, we conclude that w fixes C pointwise - contradiction.

For the converse, let w be a reflection in W. Note first that H_w contains at least one midplane. Indeed, since any reflection in W is conjugate to some $s, s \in$

S, we may assume that $w = s$. Take $J = \{s\}$, then the cell $X_J = \mathrm{Ch}(x_J, sx_J)$ is a segment on which s acts as a reflection thereby fixing its midpoint M.
We conclude that H_w contains H_M for some midplane M. Therefore, as was proved above, H_w coincides with H_M. □

Lemma 2.2.5 *The edge path in $\mathcal{M}^{(1)}$ is geodesic if and only if it crosses each wall at most once.*

Proof. If an edge path $p = e_1e_2 \cdots e_k$ crosses a wall H twice, say distinct edges $e_i, e_j, i < j$ cross H, then we delete the subpath $e_i \cdots e_j$ and instead insert the path $w(e_{i+1} \cdots e_{j-1})$, where w is the reflection in the wall H. The resulting path is strictly shorter than p but connects the same vertices. Conversely, suppose that an edge path p from x to y crosses each wall at most once. Let $\mathcal{H}_H$ be the set of all walls crossing by p. Since x and y are at the different sides of each wall from $\mathcal{H}_H$, we conclude that any path from x to y should cross than that of p. □

Any wall in the Moussong complex is "totally geodesic" in the following sense

Lemma 2.2.6 *Any geodesic in $\mathcal{M}$ having nondegenerate piece in a wall H, lies entirely in H.*

Proof. Suppose the lemma is false, then there are nondegenerate segments $\sigma_1 = [x, x_1], \sigma_2 = [x, x_2]$, cells C_1, C_2, and midplanes M_1, M_2 of C_1, C_2 respectively such that
1) $M_1 \sim M_2$,
2) $x \in M_1 \cap M_2$,
3) $\sigma_1 \subset M_1, x_2 \in C_2 - M_2$,
4) $\sigma_1 \cup \sigma_2$ is geodesic.
It follows from Lemma 2.1.5 that there is a reflection $w \in W$ and a segment $[y, wy]$ with x as a midpoint and orthogonal to both M_1 and M_2. Write $\sigma = [x, y], \sigma' = [x, wy]$. Since, by 3), $x_2 \in (C_2 - M_2)$, it follows from Lemma 2.1.6 that one of the angles $\angle_x(\sigma_2, \sigma)$, $\angle_x(\sigma_2, \sigma')$ is strictly less than $\pi/2$ and $\angle_x(\sigma_1, \sigma') = \angle_x(\sigma_2, \sigma) = \pi/2$. Hence the angle between the segments σ_1, σ_2 in the point x is strictly less than π, thus $\sigma_1 \cup \sigma_2$ can not be geodesic by criterion of Lemma 1.5.1.

2.3 Separation properties

Lemma 2.3.1 *Every wall in $\mathcal{M}$ separates $\mathcal{M}$ into exactly two connected components.*

Proof. First, we claim that H separates $\mathcal{M}$ into at least two components. We know from Lemma 2.2.2 that $H = H(e)$ – the dual wall of some edge $e = [x, y]$. We will show that x, y belong to different connectedness components. Suppose, to the contrary, that x, y are in the same connectedness component. Then there is a closed edge path α in $\mathcal{M}^{(1)}$ crossing H only once. (Clearly any edge either intersects H in a midpoint or does not intersects H at all.) Since $\mathcal{M}$ is contractible this path can be contracted to a constant path by a sequence of combinatorial contractions in cells. By Lemma 2.1.1 any cell C either has an empty intersection with H_M or $H_M \cap C$ is a midplane of C. This implies that each combinatorial contraction of the edge path in the cell does not change the number of intersections with H_M modulo 2. Since this number is 0 for the final constant path, it cannot be 1 for the initial path.

To prove that the H cuts out $\mathcal{M}$ into exactly two components, we proceed as in [NR98], lemma 2.3 (preprint version.) Notice first that H is 2-sided, that is there exists a neighborhood of H in $\mathcal{M}$ which is homeomorphic $H \times I, I = [0, 1]$. Indeed, by Lemma 2.1.5, in each cell there is a neighborhood which is fibered as $M \times I$: the fibrations can be chosen to agree on face maps so this induces an I-bundle structure on some neighborhood N over H.

Since H itself is CAT(0) it is contractible so the bundle is trivial. It follows that N has two disjoint components, $\{-1/2\} \times H$ and $\{1/2\} \times H$. Any point in the complement of H can be joined to one of these boundary components by a path in the complement of H, and therefore $X - H$ has exactly 2 components as required. □

Lemma 2.3.1 *For any wall H both components of the complement $\mathcal{M} - H$ are convex.*

Proof. Suppose that x_1, x_2 lie on the same side of H, say H^+. We claim that $[x_1, x_2]$ lies entirely in H^+. Suppose the contrary, then by Lemma 2.2.6 the intersection $[x_1, x_2] \cap H$ consists of precisely one point, say x. Similar to the proof of Lemma 2.2.6 there are segments $\sigma_1 \subset [x, x_1], \sigma_2 \subset [x, x_2]$, cells C_1, C_2, and midplanes M_1, M_2 of C_1, C_2 respectively such that

1) $M_1, M_2 \subset H$,

2) $x \in \sigma_1 \cap \sigma_2$,

3) $\sigma_1 \subset C_1, \sigma_2 \subset C_2$,

4) $\sigma_1 \cup \sigma_2$ is geodesic.

5) The interiors of σ_1, σ_2 are contained entirely in H^+.

Then it follows from Lemma 2.1.5 that there exists a reflection $w \in W$ and a segment $[y, wy]$ such that the segment has x as a midpoint and is orthogonal to both M_1 and M_2. By interchanging the roles of y and wy if necessary we may assume that $y \in H^+$. Denote $\sigma = [x, y], \sigma' = [x, wy]$. It follows from Lemma 2.1.6 that the angles $\angle_x(\sigma_2, \sigma)$, $\angle_x(\sigma_2, \sigma')$ are both strictly less than $\pi/2$. But a small nonzero move of x along σ would strictly shorten the length of $\sigma_1 \cup \sigma_2$ contradicting the assumption 4) above. □

3 Chambers and galleries

3.1 Chambers

Since the complex $\mathcal{M}$ is locally finite and there are only finite number of midplanes in each cell, we conclude that the set of all walls $\mathcal{H}$ in $\mathcal{M}$ is locally finite, in the sense that every point of $\mathcal{M}$ has a neighborhood which meets only finitely many $H \in \mathcal{H}$.

Definition 3.1.1 By Lemma 2.3.1 the walls $H \in \mathcal{H}$ yield a partition of $\mathcal{M}$ into open convex sets, which are the connected components of the complement $\mathcal{M} - (\cup_{\mathcal{H}} H)$. We call these sets chambers.

To distinguish chambers from cells, we will denote them by letter D, possibly with indices, dashes, etc.

Lemma 3.1.2 *For any two distinct chambers $D(x), D(y), x, y \in \mathcal{M}^{(0)}$ there is a wall H separating them.*

Proof. Consider a geodesic edge path $p = e_1 e_2 \cdots e_k$ from x to y, then by Lemma 2.2.5 $H(e_1)$ separates x from y and hence separates $D(x)$ from $D(y)$. □

Lemma 3.1.3 *Each chamber contains precisely one vertex of $\mathcal{M}$.*

Proof. Since W acts simply transitively on the set of vertices of $\mathcal{M}$ and each vertex is contained in some chamber we deduce that each chamber contains at least one vertex. Now, if x, y are distinct vertices in a chamber C, we connect them by a geodesic path p in $\mathcal{M}^{(1)}$. Then by criterion of geodesicity any wall crossed by p separates x from y, contradicting the definition of chamber. □

In view of this lemma we will write $D(x)$ for the chamber containing the vertex x of $\mathcal{M}$.

Definitions 3.1.4 Recall from §2.1.2 that midplanes of any cell C in $\mathcal{M}$ yield a partition of C into convex (open) blocks. (Blocks are open in C, not in $\mathcal{M}$.) A maximal block is a block in a maximal cell. Two maximal blocks are adjacent if they are contained in the same maximal cell and share a codimension one face. Two chambers D, D' are adjacent if there are maximal blocks $B \subset D, B' \subset D'$ which are adjacent. A wall H is a wall of a chamber D if there is a maximal cell C such that $H \cap C$ contains a codimension one face F of a maximal block B of D.

Lemma 3.1.5 1) *Every chamber is uniquely determined by any of its maximal blocks.* 2) *Every chamber is a union of maximal blocks, and it contains at most one maximal block from each maximal cell.*

Proof. 1) Indeed, the interior of a maximal block is open in $\mathcal{M}$ and does not intersect any wall, consequently there is only one chamber containing this block.
2) Since $\mathcal{M}$ is a union of maximal cells, any chamber is a union of maximal blocks. Take a chamber D, then

$$D = \cup\{D \cap C : C \text{ is a maximal Moussong cell }\}.$$

The intersection $D \cap C$ is a union of maximal blocks because $D \cap C$ is an intersection of open half-cells in M. Next, if D contains two maximal blocks B_1, B_2 from one cell, then there is a midplane M separating B_1 from B_2 and the ambient wall H also separates B_1 from B_2 contradicting the definition of D. $\square$

Lemma 3.1.6 *Let B, B' be maximal adjacent blocks and let D, D' be corresponding ambient chambers. Let H be a wall separating B from B'. Then H is the only wall that separates D from D'.*

Proof. Let C be a maximal cell containing B, B', then B, B' are adjacent in this cell and clearly there is only one midplane separating them. But the wall is uniquely determined by any of its midplanes, whence the lemma. $\square$

Lemma 3.1.7 *Let D, D' be chambers such that their closures $\overline{D}, \overline{D'}$ have a nonempty intersection. Let H be a wall, separating D from D'. Then H contains the intersection $\overline{D} \cap \overline{D'}$.*

Proof. Suppose, to the contrary, that there is $b \in \overline{D} \cap \overline{D'}$ which is not contained in H. Since H is closed a small neighborhood of b does not intersect H. But this neighborhood contains points both from D and D', which thus belong to one halfspace of H, contradicting the separation hypothesis. $\square$

Lemma 3.1.8 *Two distinct chambers $D(x), D(y)$ $(x, y \in \mathcal{M}^{(0)})$ are adjacent if and only if the vertices x, y are adjacent in $\mathcal{M}^{(1)}$. For any two adjacent chambers there is a reflection in W, permuting these chambers and fixing the intersection of their closures pointwise.*

Proof. The lemma is about Coxeter cell, thus it follows from the description of its structure as a Coxeter complex. $\square$

Definition 3.1.9 The base chamber D_0 of $\mathcal{M}$ is the chamber, containing the base vertex x_0 of $\mathcal{M}$. For each s from the generating set S of W, we denote by H_s^- those open halfspace of the wall H_s, which contains the base vertex x_0.

Lemma 3.1.10 $D_0 = \cap\{H_-^s : s \in S\}$.

Proof. Since $D = \cap\{H^s_- : s \in S\}$ contains x_0, it contains also D_0. Let B_J be a block of a maximal cell X_J, containing $x_J = x_0$. Then $B_J \subset D_0$ – indeed it follows from the description of the chambers in the Coxeter complex that B_J is bounded by the hyperplanes $H_s = e_s^{\perp}, s \in J$. Suppose now that D strictly contains D_0 and let $x \in D - D_0$. Since D is convex, the whole segment $[x, x_0]$ lies in D. Let $T = (x_0, x_1, \dots, x_m)$ be a taut chain from x_0 to $x_m = x$. The first piece $[x_0, x_1]$ lies entirely in some maximal cell of the form X_K and we know that the block $B_K = D_0 \cap X_K$ is the maximal block in X_K and it is bounded by the hyperplanes $H_s^-, s \in K$.
If x_1 is a vertex of X_K, then it is separated by some $H^s, s \in K$ from x_0. If x_1 is not a vertex of X_K, then x_1 is the boundary point of X_K and hence it is contained in the interior of some face F of X_K. If F contains x_0, then all three points x_0, x_1, x_2 lie in some cell contradicting to the choice. Hence F does not contain x_0 and thus the open interval (x_0, x_1) lies entirely in the interior of X_J and hence crosses some wall $H^s, s \in J$ – contradiction. □

3.2 Galleries

Definitions 3.2.1 A gallery is a sequence of chambers $\Gamma = D_1 D_2 \cdots D_k$ such that any two consecutive ones are adjacent.

Recall that the chambers are in one-to-one correspondence with the vertices of $\mathcal{M}$ and chambers are adjacent if and only if the correspondent vertices are adjacent in the 1-skeleton of $\mathcal{M}$. It follows immediately that the following lemma is true.

Lemma 3.2.2 *1) Any two chambers D, D' can be connected by a gallery of length $d(D, D')$. 2) A gallery is geodesic if and only if and only if it does not cross any wall more than once. 3) Given $s_1, \dots, s_d \in S$, there is a gallery of the form $D_0(s_1 D_0)(s_1 s_2 D_0) \cdots (s_1 s_2 \cdots s_d D_0)$. Conversely, any gallery starting at C has this form. 4) The action of W is simply transitive on the set of chambers.*

□

Lemma 3.2.3 *There is a constant $c(\mathcal{M})$ such that for any two distinct chambers D, D' with nonempty intersection $\overline{D} \cap \overline{D'}$, there is a geodesic gallery $\Gamma = D_0 D_1 \cdots D_k$ from $D_0 = D$ to $D_k = D'$ whose length k does not exceed $c(\mathcal{M})$.*

Proof. Let $\mathcal{H}_0$ be the set of walls separating D from D'. In view of Lemma 3.2.2, it is enough to bound the cardinality of $\mathcal{H}_0$. According to Lemma 3.1.7 each $H \in \mathcal{H}_0$ contains $\overline{D} \cap \overline{D'}$. Let $x \in \overline{D} \cap \overline{D'}$. Clearly the number of cells containing x is uniformly bounded and for each such a cell C the number of midplanes in C containing x is also uniformly bounded. Since a wall is uniquely determined by any of its midplanes, this proves the lemma. □

3.3 Approximation property

Definition 3.3.1 Let (X, G) be a pair consisting of a geodesic metric space X and a graph G embedded into X. We say that (X, G) satisfies the approximation property if X-geodesics between the vertices of G can be uniformly approximated by geodesics in G. This means that there is a constant δ such that for any X-geodesic α_X between the vertices of G there is a G-geodesic α_G between the same vertices such that both α_X and α_G lie entirely in the δ-neighborhoods of each other. We will express this by saying that α_X, α_G are δ-close to each other.

Of particular interest is the case when G is the embedded Cayley graph of a group acting on X.

Theorem 3.3.2 *Let (W, S) be a Coxeter group and let $\mathcal{M}$ be its Moussong complex. Embed the Cayley graph $\mathcal{C}_W$ as an orbit Wx_0 for a point x_0 in a base chamber D_0 of $\mathcal{M}$. Then the pair $(\mathcal{M}_W, \mathcal{C}_W)$ satisfies the approximation property.*

Proof. Let $\sigma = [a, b]$ be a nondegenerate segment in $\mathcal{M}$ and $\mathcal{H}_\sigma$ be the set of walls having a nonempty intersection with the interior (a, b). Since the family of all walls is locally finite and the walls are totally geodesic, we have a partition

$$\mathcal{H}_\sigma = \mathcal{H}'_\sigma \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots \cup \mathcal{H}_n,$$

where the walls from $\mathcal{H}'_\sigma$ contain σ and the walls from $\mathcal{H}_i$ cross σ precisely in the point $a_i, i = 1, \cdots, n$, and $a = a_0 < a_1 < \cdots < a_n < a_{n+1} = b$.
Now we define a gallery Γ along the geodesic $\sigma = [a, b]$ as the gallery

$$\Gamma = D_1\Gamma_1 D_2\Gamma_2 D_3 \cdots D_n\Gamma_n D_{n+1}$$

such that
1) $\overline{D_i} \cap [a, b] = [a_{i-1}, a_i]$ $(i = 1, 2 \ldots, n + 1)$,
2) Each spherical piece $D_i\Gamma_i D_{i+1}$ is a geodesic gallery and the lengths of spherical pieces are bounded from above by the constant $c(\mathcal{M})$ from Lemma 3.2.3,
3) Each spherical piece $D_i\Gamma_i D_{i+1}, i = 1, \ldots, n$ crosses the walls only from the set $\mathcal{H}'_\sigma \cup \mathcal{H}_i$.

Lemma 3.3.3 *For any geodesic $\sigma = [a, b]$ in $\mathcal{M}$ there is a geodesic gallery along σ.*

Proof of the lemma. By construction of the sequence $\{a_i\}$, for each $i = 1, \ldots, n + 1$ there is a chamber D_i such that $\overline{D_i} \cap [a, b] = [a_{i-1}, a_i]$. The corresponding sequence of chambers $D_1, D_2, \ldots D_n, D_1 = D, D_n = D'$ is the first approximation to the required gallery. In general, this sequence is not a gallery, since two consecutive chambers are not necessarily adjacent. For each $1 \leq i \leq n$, the intersection of neighbors $\overline{D_i} \cap \overline{D_{i+1}}$ contains the point a_i.

Application of Lemmas 3.1.7, 3.2.3 enables us to inscribe a spherical geodesic subgallery of bounded length between these neighbors and get a gallery

$$\Gamma = D_1\Gamma_1 D_2\Gamma_2 \cdots D_{n-1}\Gamma_{n-1}D_n$$

such that the spherical pieces $D_i\Gamma_i D_{i+1}$ are geodesic galleries of uniformly bounded length satisfying condition 3) from the definition above. We will show that Γ can be modified to a geodesic gallery along $[a, b]$. If Γ is not geodesic then by Lemma 3.2.2 it crosses some wall H at least twice. Clearly $H \in \mathcal{H}'_\sigma$ i.e., H contains σ. Then there are indices $i+1 < j$ and subgalleries Γ_1, Γ_2 each of length 1 such that
a) Γ_1, Γ_2 belong to i-th and $j-$th spherical piece respectively,
b) Γ_1, Γ_2 cross H and moreover there are no crossing subgalleries in between.
Let $\Gamma_1 = DD', \Gamma_2 = D''D'''$. In particular the chambers D and D''' lie on the same side of H, say H^-, and the subgallery Γ' of Γ, joining D' with D'' lies on the opposite side, say H^+.
Let $w \in W$ be the reflection in the wall H. If we modify Γ by applying w to the portion $\tilde{\Gamma}$, we obtain the gallery from D to D''' that is strictly shorter than $D\Gamma'D'''$. Replacing $D\Gamma'D'''$ by $w(\tilde{\Gamma})$ we get the gallery Γ' that is strictly shorter than Γ but still is the gallery along σ. Repeating the previous process will construct a geodesic gallery along σ. This proves Lemma 3.3.3.
The theorem now follows easily from Lemma 3.3.3. Namely, given two chambers D, D' we take the points d, d' inside them and build a geodesic gallery $\Gamma = D_1 \cdots D_n$ along $[d, d']$. Γ not necessarily joins D to D' but the intersections $\overline{D} \cap \overline{D_1}, \overline{D'} \cap \overline{D_n}$ are nonempty, so we can join D to D_1 and D' to D_2 respectively by the galleries of uniformly bounded length thereby getting the gallery joining D to D' and that is $\delta(\mathcal{M})-$close to σ for some universal constant $\delta(\mathcal{M})$. $\square$

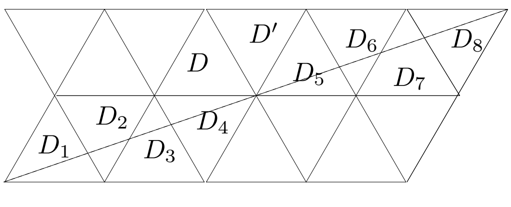

Figure 3: Gallery along geodesic. The spherical piece is $D_4DD'D_5$.

4 Word length on abelian subgroups of a Coxeter group

4.1 Straightness

Definition 4.1.1 Let G be a group with a fixed word metric $x \mapsto \ell(x)$. We say that an element $x \neq 0$ is straight if $\ell(v^n) = n\ell(v)$ for all natural n.

Remark 4.1.2 Straight elements have been studied for Coxeter groups in [Kra94] and for small cancellation groups in [Kap97] (in the last paper they are called periodically geodesic.

Example 4.1.3 (An element that is not straight.) Let W be an affine Coxeter group generated by reflections s_1, s_2, s_3 in the sides of an equilateral triangle C of a Euclidean plane. Let L_1, L_2, L_3 be the corresponding reflecting lines of this triangle. It is easily seen that there is nontrivial translation $u \in W$ with an axis L_1. We assert that nor s_1u neither any of it conjugates $v = ws_1uw^{-1}$ are straight. Indeed, the length $|ws_1uw^{-1}|$ is the length of a geodesic gallery Γ from C to $ws_1uw^{-1}C$. Any such a gallery intersects the line wL_1. The concatenation $\Gamma(v\Gamma)$ is a gallery from C to v^2C of length $2|ws_1uw^{-1}|$. But $\Gamma(v\Gamma)$ can not be geodesic, since it intersects wL_1 twice. Hence $|v^2| < 2|v|$. □

Definition 4.1.4 Let $\mathcal{M}$ be the Moussong complex of a Coxeter group W. Recall that $\mathcal{M}$ is a proper complete CAT(0) space and W acts properly and cocompactly on $\mathcal{M}$ by isometries. In particular, any element $w \in W$ of infinite order acts as an axial isometry i.e., there is a geodesic axis A_w in $\mathcal{M}$, isometrical to $\mathbb{R}$, on which w acts as a nonzero translation [Bal95]. We say that w is generic if A_w intersects any wall in at most one point. In view of Lemma 2.2.6, this is equivalent to saying that no nondegenerate segment of A_w is contained in a wall.

Theorem 4.1.5 *Let (W, S) be a Coxeter system of finite type. For any generic element w of W of infinite order there is a conjugate v which is straight, that is $\ell(v^n) = n\ell(v)$ for all $n \in \mathbb{N}$, where $\ell(v)$ is a word length in generators S.*

Proof. We make use of the action of W on the Moussong complex $\mathcal{M}$. Since the family of all walls is locally finite, there is a point a on the axis A_w such that a does not belong to any wall of $\mathcal{M}$. Every point $w^ia (i \in \mathbb{Z})$ also does not belong to any wall of $\mathcal{M}$. Let $\mathcal{H}$ be the set of walls crossed by the segment $[a, wa]$ and let $a < a_1 < a_2 < \cdots < a_k < wa$ be the crossing points, so that $\mathcal{H}$ is a disjoint union $\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_k$ of subsets $\mathcal{H}_i$ crossing $[a, wa]$ in $a_i, i = 1, 2, \ldots, k$. There are the chambers $D_1, D_2, \cdots, D_k$ such that $\overline{D_1} \cap [a, wa] = [a, a_1], \overline{D_i} \cap [a, wa] = [a_{i-1}, a_i] (i = 1, 2, \ldots, k)$. Inscribe into the sequence $D_1, D_2, \cdots, D_k(wD_1)$ subgalleries $\Gamma_1, \ldots, \Gamma_k$, so that the concatenation $\Gamma = D_1\Gamma_1D_2 \cdots \Gamma_{k-1}D_k\Gamma_k(wD_1)$ is a gallery, crossing only the walls from $\mathcal{H}$ and crossing each wall precisely once. In particular this gallery is geodesic. Let $\Gamma_0 = D_1\Gamma_1D_2 \cdots \Gamma_{k-1}D_k\Gamma_k$. Translating by w and concatenating, we get a gallery $\tilde{\Gamma} = \Gamma_0(w\Gamma_0))(w^2\Gamma_0) \cdots (w^{n-1}\Gamma_0)w^nD_1$. The walls that it crosses are precisely those from the union $\mathcal{H} \cup w\mathcal{H} \cup w^2\mathcal{H} \cup \cdots \cup w^{n-1}\mathcal{H}$, and each wall is crossed precisely once. Hence the gallery $\tilde{\Gamma}$ is geodesic. Now let $D_1 = uD_0, u \in W$, where D_0 is the base chamber. Being a geodesic path in the Cayley graph, the gallery $\tilde{\Gamma}$ joins the vertex u to the vertex $w^nu = u(u^{-1}w^nu)$. Hence its length $n\ell(\Gamma_0)$ equals the word length of the element $u^{-1}w^nu \in W$.

We conclude that for $v = u^{-1}wu$ the equality $|v^n| = n|v|$ holds for all $n \in \mathbb{N}$. $\square$

Theorem 4.1.6 *Let (W, S) be a Coxeter group of finite type. There is a constant $c = c(W)$ such that for any element w of W of infinite order there is a conjugate v of w^c which is straight.*

Proof. Let $w \in W$ be of infinite order and let A_w be an axis of w. Let $\mathcal{H}_u = \mathcal{H}_u(A_u)$ denote the set of walls in the Moussong complex $\mathcal{M}_W$, containing A_u. It is easy to see that the cardinality of $\mathcal{H}_w$ is bounded by a constant depending only on W and we take $c = c(W)$ to be the number

$$2 \times \text{l.c.m.} \times (\text{card}\{\mathcal{H}_w : w \in W \text{ is of infinite order }\}).$$

Clearly A_w is an axis of w^c as well. Furthermore, w^c leaves invariant each wall $H \in \mathcal{H}_w$; moreover, it leaves invariant each of the two components of $\mathcal{M}_W - H, H \in \mathcal{H}_w$. It follows that for any chamber D, a geodesic gallery from D to w^cD does not cross a wall H from $\mathcal{H}_w$. Indeed, otherwise D and w^cD would lie in different components of $\mathcal{M}_W - H$ implying that w^c interchanges these components, contradicting the property above. Take a chamber D such that $\overline{D} \cap A_w$ is a nondegenerate segment and fix a point a in the interior of this segment. Let $\mathcal{H}$ denote the set of walls H that are crossed by the segment $[a, w^ca]$ but do not contain it. Clearly any $H \in \mathcal{H}$ separates D from w^cD. And conversely, if H separates, then the points a, w^ca lie in different components of $\mathcal{M}_W - H$ implying that H crosses the segment $[a, w^ca]$ in precisely one point. Let Γ be a geodesic gallery from D to w^cD then the walls that it crosses are precisely those from $\mathcal{H}$, and each wall $H \in \mathcal{H}$ is crossed by Γ precisely once. Iterating we obtain a gallery $\tilde{\Gamma} = \Gamma(w\Gamma))(w^2\Gamma) \cdots (w^{n-1}\Gamma)w^nD(n \in \mathbb{N})$ of the length $n\ell(\Gamma)$. This gallery crosses the walls only from (disjoint) union $\mathcal{H} \cup w^c\mathcal{H} \cup w^{2c}\mathcal{H} \cup \cdots \cup w^{(n-1)c}\mathcal{H}$, each precisely once. Hence the gallery $\tilde{\Gamma}$ is geodesic. Now let $D = uD_0, u \in W$, where D_0 is the base chamber. Being a geodesic path in the Cayley graph, the gallery $\tilde{\Gamma}$ joins the vertex u to the vertex $w^{nc}u = u(u^{-1}w^{nc}u)$. Hence its length $n\ell(\Gamma)$ equals the word length of the element $u^{-1}w^{nc}u \in W$. We conclude that for $v = u^{-1}w^cu$ the equality $|v^n| = n|v|$ holds for all $n \in \mathbb{N}$. $\square$

For elements which are not necessarily generic we have the following

Lemma 4.1.7 *Let (W, S) be a Coxeter group of finite type and let $w \in W$ be an element of infinite order. Fix an axis A_w of w in the Moussong complex $\mathcal{M}_W$. There is a chamber D such that for all $n \in \mathbb{Z}$*

$$d(D, w^nD) = n\ d(D, wD) - n\ \text{card}\ (w^{\mathbb{Z}} \backslash \mathcal{H}_w) + c_n,$$

where $|c_n|$ is bounded by a constant depending only on W and $\mathcal{H}_w$ is the set of all walls H in $\mathcal{M}_W$, containing A_w and such that H separates w^iD from $w^{i+1}D$ for some $i \in \mathbb{Z}$.

Proof. We follow the proof of Theorem 4.1.5. Take a chamber D, such $\overline{D} \cap A_w$ is a nondegenerate segment. Let $\mathcal{H}$ be the set of walls, separating D from wD and do not containing A_w. By total geodesicity, any $H \in \mathcal{H}$ crosses A_w precisely in one point. Let Γ be a geodesic gallery from D to wD then it crosses all $H \in \mathcal{H}$, each precisely once, and some of the walls from $\mathcal{H}_w$. Iterating we get the gallery $\tilde{\Gamma} = \Gamma(w\Gamma))(w^2\Gamma)\cdots(w^{n-1}\Gamma)w^nD$. This gallery crosses the walls from (disjoint) union $\mathcal{H} \cup w\mathcal{H} \cup w^2\mathcal{H} \cup \cdots \cup w^{n-1}\mathcal{H}$, each precisely once. Also, it crosses some walls from $\mathcal{H}_w$. Note that, whenever $\tilde{\Gamma}$ crosses $H \in \mathcal{H}_w$, it crosses it periodically with a period $r_H =$ card $w^{\mathbb{Z}}H$. Hence, the integer part $[n/r_H]$ is the number of times the gallery $\tilde{\Gamma}$ crosses each $H' \in w^{\mathbb{Z}}H$. Hence it crosses the walls from the orbit $w^{\mathbb{Z}}H$ approximately n times, up to a universal constant. Hence, the number $d(D, w^nD)$ of walls, separating D from w^nD, equals $n\, d(D, wD) - n$ card $(w^{\mathbb{Z}} \backslash \mathcal{H}_w) + c_n$, where c_n is uniformly bounded. □

Theorem 4.1.8 *If, under conditions of Lemma 4.1.7, $D = uD_0, u \in W$, where D_0 is the base chamber, then $d(D, wD)$ is the word length of the conjugate $v = u^{-1}wu \in W$ and we get the following formula*

$$\ell(v^n) = n\ell(v) - \text{ card } (w^{\mathbb{Z}} \backslash \mathcal{H}_W) + c_n.$$

From this we get the following formula for a translation length $||w||$ *of w:*

$$||w|| \overset{\text{def}}{=} \lim_{n\to\infty} \frac{\ell(w^n)}{n} = \lim_{n\to\infty} \frac{\ell(v^n)}{n} = \ell(v) - \text{card}(w^{\mathbb{Z}} \backslash \mathcal{H}_w).$$

In particular, translation length of any element of W is rational (even integral).

Remark 4.1.9 The formula for translation length is similar to the one given in [Kra94], where it follows from the classification of roots. It seems unknown whether translation length is rational in an arbitrary "semihyperbolic group".

4.2 Norms and Burago's inequality

Let A be a normed abelian group, so A is equipped with a function $\ell : A \to \mathbb{R}$ satisfying (1) $\ell(a^{-1}) = \ell(a)$, (2) $\ell(ab) \leq \ell(a) + \ell(b)$, and (3) $\ell(a) \geq 0$ with $\ell(a) = 0$ iff $a = 1$, for $a, b \in A$. If (3) is replaced by (3') $\ell(a) \geq 0$ for $a \in A$, we call A a pseudonormed abelian group. Two pseudonorms ℓ and ℓ' on the abelian group A are called Hausdorff equivalent if there is a constant $k > 0$ so that $|\ell(a) - \ell'(a)| \leq k$ for all $a \in A$. The (pseudo)norm ℓ on the abelian group A is called regular if $\ell(a^n) = n\ell(a)$ for all $a \in A$ and all positive natural numbers n. Let ℓ be a norm on the abelian group A. We define the regularization $R\ell$ of ℓ by

$$R\ell(a) = \lim_{n\to\infty} \frac{\ell(a^n)}{n}.$$

By [PS78], p. 23, Exercise 99, this limit always exists, and it is an exercise to see that $R\ell$ is a regular pseudonorm.

Lemma 4.2.1 *The norm ℓ on the abelian group A is regular iff $R\ell = \ell$.*

Proof. If ℓ is regular, then clearly $R\ell = \ell$. Conversely, if $\ell(a^n) < n\ell(a)$ for some positive number n and some $a \in A$, then

$$R\ell(a) = \lim_{m\to\infty} \frac{\ell(a^{mn})}{mn} \leq \frac{\ell(a^n)}{n} < \ell(a),$$

thus the lemma. □

In general positivity of $R\ell$ fails, so it is possible that $R\ell(a) = 0$ but $a \neq 0$. Also it may easily happen that $R\ell$ is not Hausdorff equivalent to ℓ. We give a criterion for positivity and Hausdorff equivalence in terms of Burago's inequality [Gro93].

Definition 4.2.2 We say that a norm ℓ on an abelian group A satisfies the Burago's inequality if there exists a constant $c = c(A) > 0$ such that

$$\ell(a^2) \geq 2\ell(a) - c \quad \text{for all} \quad a \in A.$$

The norm is discrete if for all $n \in \mathbb{N}$ the ball $B_n = \{x \in A : \ell(x) \leq n\}$ is finite. For example any word metric, corresponding to a finite generating set, is discrete.

Lemma 4.2.3 *If a discrete norm ℓ on a torsionfree abelian group A satisfies Burago's inequality then its regularization $R\ell$ is a norm also and, furthermore, $R\ell$ is Hausdorff equivalent to ℓ.*

Proof. By induction from Burago's inequality we deduce that $\ell(a^{2^n}) \geq 2^n\ell(a) - (2^n - 1)c$, for all $a \in A, n \in \mathbb{N}$. This implies that

$$\ell(a) \geq R\ell(a) = \lim_{m\to\infty} \frac{\ell(a^{2^m})}{2^m} \geq \ell(a) - c$$

for all $a \in A$. Thus the regularization $R\ell$ is Hausdorff equivalent to ℓ. As any regularization, this pseudonorm is regular. It remains to prove that $R\ell$ is a norm on A, i.e., it does not vanish on nonzero $a \in A$. If $a \in A$ is such that $\ell(a) \geq 1 + c$, then

$$R\ell(a) = \lim_{m\to\infty} \frac{\ell(a^{2^m})}{2^m} \geq \ell(a) - c \geq 1.$$

Now suppose $a \in A$ is arbitrary nonzero, then by the discreteness assumption $\ell(a^n) \geq 1+c$ for sufficiently large n, and since $R\ell$ is regular, $R\ell(a) = \frac{1}{n}R\ell(a^n) > 0$. □

4.3 Approximation and Burago's inequality

Lemma 4.3.1 *Let Γ be a finitely generated group of isometries of a proper CAT(0) space X, acting cocompactly and properly on X. Suppose that $x_0 \in X$ has a trivial stabilizer so that the Cayley graph $\mathcal{C}$ of Γ can be considered as embedded into X via the orbit map $g \mapsto gx_0 (g \in \Gamma)$. Suppose that the pair $(X, \Gamma x_0)$ satisfies the approximation property. Then the restriction of the word length ℓ on Γ to any finitely generated free abelian subgroup A satisfies the Burago's inequality.*

Proof. By assumption there is a $\delta > 0$ such that for any $g \in \Gamma$ the X-geodesic α_X from x_0 to gx_0 and some $\mathcal{C}$–geodesic $\alpha_{\mathcal{C}}$ from x_0 to gx_0 are δ–close to each other. By the flat torus theorem [Bow95], [Bri95] there is a Euclidean subspace F in X on which A acts by translation. Fix the point $y_0 \in F$ and let a be an arbitrary nontrivial element in A. We will show that ax_0 is contained in a c-neighborhood of $\alpha_{\mathcal{C}}$ for a suitable $c > 0$. Clearly $d_X(a^2x_0, a^2y_0) = d_X(x_0, y_0)$. Parameterize the segments $[x_0, a^2x_0], [y_0, a^2y_0]$ by the segment $[0, 1]$ proportionally to arc length. It follows from the convexity of X–metric that the corresponding points on the segments are distance at most $d_X(x_0, y_0)$ from each other. Let u be the point on $[x_0, a^2x_0]$ corresponding to the point ay_0. By assumption u is distance at most δ from some point v on $\alpha_{\mathcal{C}}$. Hence we have bounded the X-distance from $ax_0 \in \mathcal{C}$ to $v \in \mathcal{C}$. (This key observation is illustrated in Figure 4). Since the Cayley graph $\mathcal{C}$ is quasiisometric to X this bounds the Cayley graph distance also. Thus, there is a constant $c = c(\Gamma, X) > 0$ such that $d_{\mathcal{C}}(ax_0, v) \leq c$. We have $\ell(a^2) = d_{\mathcal{C}}(x_0, v) + d_{\mathcal{C}}(v, a^2x_0) \geq (d_{\mathcal{C}}(x_0, ax_0) - c) + (d_{\mathcal{C}}(ax_0, a^2x_0) - c) = (\ell(a) - c) + (\ell(a) - c) = 2\ell(a) - 2c$, that is the Burago's inequality. □

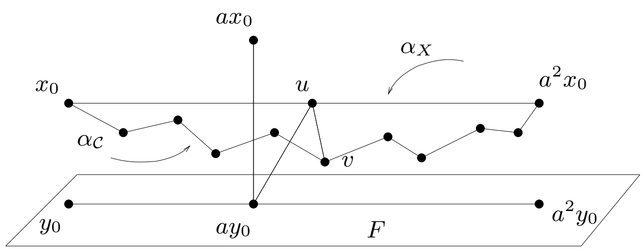

Figure 4: Lemma 4.3.1

Theorem 4.3.2 *Let (W, S) be a Coxeter group and let ℓ be the word length in generators S. Then the restriction of ℓ to any free abelian subgroup A of W is Hausdorff equivalent to a regular norm on A.*

Proof. Consider the pair $(\mathcal{M}_W, \mathcal{C}_W)$ where the Cayley graph $\mathcal{C}_W$ is embedded into the Moussong complex as an orbit Wx_0. By Theorem 3.3.2 $(\mathcal{M}_W, \mathcal{C}_W)$

satisfies the approximation property. Therefore by Lemma 4.3.1 the restriction of the word length ℓ on W to any finitely generated free abelian subgroup A satisfies the Burago's inequality. Finally, by Lemma 4.2.3 ℓ is Hausdorff equivalent to its regularization $R\ell$ and thus $R\ell$ is the required norm on A.

□

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