Euler Characteristics of Categories and Homotopy Colimits

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Received: October 20, 2010

Communicated by Stephan Schwede

Abstract. In a previous article, we introduced notions of finiteness obstruction, Euler characteristic, and L^2-Euler characteristic for wide classes of categories. In this sequel, we prove the compatibility of those notions with homotopy colimits of $\mathcal{I}$-indexed categories where $\mathcal{I}$ is any small category admitting a finite $\mathcal{I}$-CW-model for its $\mathcal{I}$-classifying space. Special cases of our Homotopy Colimit Formula include formulas for products, homotopy pushouts, homotopy orbits, and transport groupoids. We also apply our formulas to Haefliger complexes of groups, which extend Bass–Serre graphs of groups to higher dimensions. In particular, we obtain necessary conditions for developability of a finite complex of groups from an action of a finite group on a finite category without loops.

2010 Mathematics Subject Classification: Primary: 18F30, 19J05; Secondary: 18G10, 19A49, 55U35, 19A22, 46L10.
Keywords and Phrases: finiteness obstruction, Euler characteristic of a category, L^2-Euler characteristic, projective class group, homotopy colimits of categories, Grothendieck construction, spaces over a category, Grothendieck fibration, complex of groups, small category without loops.

0. Introduction and Statement of Results

In our previous paper [16], we presented a unified conceptual framework for Euler characteristics of categories in terms of finiteness obstructions and projective class groups. Many excellent properties of our invariants stem from the homological origins of our approach: the theory of modules over categories and the dimension theory of modules over von Neumann algebras provide us with an array of tools and techniques. In the present paper, we additionally draw upon the homotopy theory of diagrams to prove the compatibility of our invariants with homotopy colimits.

If $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ is a diagram of categories (or more generally a pseudo functor into the 2-category of small categories), then our invariants of the homotopy colimit can be computed in terms of the invariants of the vertex categories $\mathcal{C}(i)$. In particular, our Homotopy Colimit Formula, Theorem 4.1, states

$$\chi\big(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}; R\big) = \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \chi(\mathcal{C}(i_\lambda); R) \tag{0.1}$$

under certain hypotheses. The set Λ_n indexes the $\mathcal{I}$-n-cells of a finite $\mathcal{I}$-CW-model $E\mathcal{I}$ for the $\mathcal{I}$-classifying space of $\mathcal{I}$, that is, we have a functor $E\mathcal{I}\colon \mathcal{I}^{\mathrm{op}} \to \mathsf{SPACES}$ which is inductively built by gluing finitely many cells of the form $\operatorname{mor}_{\mathcal{I}}(-, i_\lambda) \times D^n$ for $\lambda \in \Lambda_n$, and moreover $E\mathcal{I}(i) \simeq *$ for all objects i of $\mathcal{I}$. Similar formulas hold for the finiteness obstruction, the functorial Euler characteristic, the functorial L^2-Euler characteristic, and the L^2-Euler characteristic.

Motivation for such a formula is provided by the classical Inclusion-Exclusion Principle: if A, B, and $A \cap B$ are finite simplicial complexes, then one has

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

However, one cannot expect the Euler characteristic to be compatible with pushouts, even in the simplest cases. The pushout in CAT of the discrete categories

$$\{*\} \leftarrow \{y, z\} \rightarrow \{*'\}$$

is a point, but $\chi(\text{point}) \neq 1+1-2$. On the other hand, their *homotopy* pushout in CAT is the category whose objects and nontrivial morphisms are pictured below.

$$\begin{tikzcd} y \arrow[r] \arrow[d] & *' \\ * & z \arrow[l] \arrow[u] \end{tikzcd}$$

The classifying space of this category has the homotopy type of S^1, so that

$$\chi(\text{homotopy pushout}) = \chi(\{*\}) + \chi(\{*'\}) - \chi(\{y, z\})$$

is true. In fact, the formula for homotopy pushouts is a special case of (0.1): the category $\mathcal{I} = \{1 \leftarrow 0 \rightarrow 2\}$ admits a finite model with $\Lambda_0 = \{1, 2\}$ and $\Lambda_1 = \{0\}$, as constructed in Example 2.6. See Example 5.4 for the homotopy pushout formulas of the other invariants.

The Homotopy Colimit Formula in Theorem 4.1 has many applications beyond homotopy pushouts. Other special cases are formulas for Euler characteristics of products, homotopy orbits, and transport groupoids. Our formulas also have ramifications for the developability of Haefliger's *complexes of groups* in geometric group theory. If a group G acts on an M_κ-polyhedral complex by isometries preserving cell structure, and if each $g \in G$ fixes each cell pointwise that g fixes setwise, then the quotient space is also an M_κ-polyhedral complex, see Bridson–Haefliger [11, page 534]. Let us call the quotient M_κ-polyhedral complex Q. To each face $\overline{\sigma}$ of Q, one can assign the stabilizer G_σ of a chosen representative cell σ. This assignment, along with the various conjugated inclusions of groups obtained from face inclusions, is called the *complex of groups associated to the group action.* It is a pseudo functor from the poset of faces of Q into groups. In the finite case, the Euler characteristic and L^2-Euler characteristic of the homotopy colimit can be computed in terms of the original complex and the order of the group. We prove this in Theorem 8.30. Homotopy colimits of complexes of groups play a special role in Haefliger's theory, see the discussion after Definition 8.9.

In Section 1, we review the notions and results from [16] that we need in this sequel. Explanations of the finiteness obstruction, the functorial Euler characteristic, the Euler characteristic, the functorial L^2-Euler characteristic, the L^2-Euler characteristic, and the necessary theorems are all contained in Section 1 in order to make the present paper self-contained. Section 2 is dedicated to an assumption in the Homotopy Colimit Formula, namely the requirement that a finite $\mathcal{I}$-CW-model exists for the $\mathcal{I}$-classifying space of $\mathcal{I}$. We recall the notion of $\mathcal{I}$-CW-complex, present various examples, and prove that finite models are preserved under equivalences of categories. Homotopy colimits of diagrams of categories are recalled in Section 3. The homotopy colimit construction in CAT is the same as the Grothendieck construction, or the category of elements. Thomason proved that the homotopy colimit construction has the expected properties. We prove our main theorem, the Homotopy Colimit Formula, in Section 4, work out various examples in Section 5, and derive the generalized Inclusion-Exclusion Principle in Section 6. We review the groupoid cardinality of Baez–Dolan and the Euler characteristic of Leinster in Section 7, and compare our Homotopy Colimit Formula with Leinster's compatibility with Grothendieck fibrations in terms of weightings. We apply our results to Haefliger complexes of groups in Section 8 to prove Theorems 8.30 and 8.35, which express Euler characteristics of complexes of groups associated to group actions in terms of the initial data.

Acknowledgements. All three authors were supported by the Sonderforschungsbereich 878 – Groups, Geometry and Actions – and the Leibniz-Preis of Wolfgang Lück.

Thomas M. Fiore was supported at the University of Chicago by NSF Grant DMS-0501208. At the Universitat Autònoma de Barcelona he was supported by grant SB2006-0085 of the Spanish Ministerio de Educación y Ciencia under the Programa Nacional de ayudas para la movilidad de profesores de universidad e investigadores españoles y extranjeros. Thomas M. Fiore also thanks the Centre de Recerca Matemàtica in Bellaterra (Barcelona) for its hospitality during the CRM Research Program on Higher Categories and Homotopy Theory in 2007-2008, where he heard Tom Leinster speak about Euler characteristics. He also thanks the Max Planck Insitut für Mathematik for its hospitality and support during his stay in Summer 2010.

Contents

1. The Finiteness Obstruction and Euler Characteristics

We quickly recall the main definitions and results needed from our first paper [16] in order to make this article as self-contained as possible. See [16] for proofs and more detail.

Throughout this paper, let Γ be a category and R an associative, commutative ring with identity. The first ingredient we need is the theory of modules over categories developed by Lück [23], and recalled in [16]. An *$R\Gamma$-module* is a contravariant functor from Γ into the category of left R-modules. For example, if Γ is a group G viewed as a one-object category, then an $R\Gamma$-module is the same as a right module over the group ring RG. An $R\Gamma$-module P is *projective* if it is projective in the usual sense of homological algebra, that is, for every surjective $R\Gamma$-morphism $p\colon M \to N$ and every $R\Gamma$-morphism $f\colon P \to N$ there exists an $R\Gamma$-morphism $\overline{f}\colon P \to M$ such that $p \circ \overline{f} = f$. An $R\Gamma$-module M is *finitely generated* if there is a surjective $R\Gamma$-morphism $B(C) \to M$ from an $R\Gamma$-module $B(C)$ that is free on a collection C of sets indexed by $\mathrm{ob}(\Gamma)$ such that $\coprod_{x\in \mathrm{ob}(\Gamma)} C_x$ is finite. Explicitly, the *free $R\Gamma$-module on the* $\mathrm{ob}(\Gamma)$*-set* C is

$$B(C) := \bigoplus_{x\in\mathrm{ob}(\Gamma)} \bigoplus_{C_x} R\,\mathrm{mor}_\Gamma(?, x). \tag{1.1}$$

A contravariant $R\Gamma$-module may be tensored with a covariant $R\Gamma$-module to obtain an R-module: if $M\colon \Gamma^{\mathrm{op}} \to R\text{-}\mathsf{MOD}$ and $N\colon \Gamma \to R\text{-}\mathsf{MOD}$ are functors, then the *tensor product* $M \otimes_{R\Gamma} N$ is the quotient of the R-module

$$\bigoplus_{x\in\mathrm{ob}(\Gamma)} M(x) \otimes_R N(x)$$

by the R-submodule generated by elements of the form

$$(M(f)m) \otimes n - m \otimes (N(f)n)$$

where $f : x \to y$ is a morphism in Γ, $m \in M(y)$, and $n \in N(x)$.
Finite projective resolutions of the constant $R\Gamma$-module $\underline{R}$ play a special role in our theory of Euler characteristic for categories. A resolution P_* of an $R\Gamma$-module M is said to be *finite projective* if it has finite length and each P_n is finitely generated and projective. We say that a category Γ *is of type (FP$_R$)* if the constant $R\Gamma$-module $\underline{R}\colon \Gamma^{\mathrm{op}} \to R\text{-}\mathsf{MOD}$ with value R admits a finite projective resolution. Categories in which every endomorphism is an isomorphism, the so-called *EI-categories*, provide important examples. Finite EI-categories in which $|\operatorname{aut}(x)|$ is invertible in R for each object x are of type (FP$_R$). Further examples of categories of type (FP$_R$) include categories Γ which admit a finite Γ-CW-model for the classifying Γ-space $E\Gamma$ (see Section 2 and Examples 2.4, 2.5, 2.6, and 2.7). In fact, such categories Γ are even *of type (FF$_R$)*: the cellular chains on a finite Γ-CW-model for $E\Gamma$ provide a finite free resolution of $\underline{R}$. In general, if a category is of type (FF$_{\mathbb{Z}}$), then it is of type (FF$_R$) for any associative, commutative ring R with identity.
A home for the finiteness obstruction of a category Γ is provided by the *projective class group* $K_0(R\Gamma)$. The generators of this abelian group are the isomorphism classes of finitely generated projective $R\Gamma$-modules and the relations are given by expressions $[P_0] - [P_1] + [P_2] = 0$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective $R\Gamma$-modules.

Definition 1.2 (Finiteness obstruction of a category). Let Γ be a category of type (FP$_R$) and P_* a finite projective resolution of the constant $R\Gamma$-module $\underline{R}$. The *finiteness obstruction of* Γ *with coefficients in* R is

$$o(\Gamma; R) := \sum_{n \geq 0} (-1)^n \cdot [P_n] \ \in K_0(R\Gamma).$$

We also use the notation $[\underline{R}]$, or simply $[R]$, to denote the finiteness obstruction $o(\Gamma; R)$. The finiteness obstruction, when it exists, does not depend on the choice P_* of finite projective resolution of $\underline{R}$.

The finiteness obstruction is compatible with most everything one could hope for. If $F\colon \Gamma_1 \to \Gamma_2$ is a right adjoint, and Γ_1 is of type (FP$_R$), then Γ_2 is of type (FP$_R$) and $F_* o(\Gamma_1; R) = o(\Gamma_2; R)$ (here the group homomorphism F_* is induced by induction with F). Since an equivalence of categories is a right adjoint (and also a left adjoint), a particular instance of the previous sentence is: if $F\colon \Gamma_1 \to \Gamma_2$ is an equivalence of categories, then Γ_1 is of type (FP$_R$) if and only if Γ_2 is, and in this case $F_* o(\Gamma_1; R) = o(\Gamma_2; R)$. The finiteness obstruction is also compatible with finite coproducts of categories, finite products of categories, restriction along admissable functors, and homotopy colimits, as we prove in Theorem 4.1. If G is a finitely presented group of type (FP$_{\mathbb{Z}}$), then Wall's finiteness obstruction $o(BG)$ is the same as $o(\widehat{G}; \mathbb{Z})$, which is the finiteness obstruction of G viewed as a one-object category $\widehat{G}$ with morphisms G.

The finiteness obstruction in Definition 1.2 is a special case of the finiteness obstruction of a finitely dominated $R\Gamma$-chain complex C, denoted $o(C) \in K_0(R\Gamma)$. The image of $o(C)$ in the reduced K-theory $\tilde{K}_0(R\Gamma)$ vanishes if and only if C is $R\Gamma$-homotopy equivalent to a finite free $R\Gamma$-chain complex, see [23, Chapter 11].

We will occasionally work with directly finite categories. A category is called *directly finite* if for any two objects x and y and morphisms $u\colon x \to y$ and $v\colon y \to x$ the implication $vu = \mathrm{id}_x \implies uv = \mathrm{id}_y$ holds. If Γ_1 and Γ_2 are equivalent categories, then Γ_1 is directly finite if and only if Γ_2 is directly finite. Examples of directly finite categories include groupoids, and more generally EI-categories.

A key result in the theory of modules over an EI-category is Lück's splitting of the projective class group of Γ into the projective class groups of the automorphism groups $\mathrm{aut}_\Gamma(x)$, one ach isomorphism class of objects. We next recall the relevant maps and notation. For $x \in \mathrm{ob}(\Gamma)$, we denote $R\,\mathrm{aut}_\Gamma(x)$ by $R[x]$ for simplicity. The *splitting functor at* $x \in \mathrm{ob}(\Gamma)$

$$S_x\colon \mathsf{MOD}\text{-}R\Gamma \to \mathsf{MOD}\text{-}R[x], \tag{1.3}$$

maps an $R\Gamma$-module M to the quotient of the R-module $M(x)$ by the R-submodule generated by all images of R-module homomorphisms $M(u)\colon M(y) \to M(x)$ induced by all non-invertible morphisms $u\colon x \to y$ in Γ. The right $R[x]$-module structure on $M(x)$ induces a right $R[x]$-module structure on S_xM. Note that S_xM is an $R[x]$-module, not an $R\Gamma$-module. The functor S_x respects direct sums, sends epimorphisms to epimorphisms, and sends free modules to free modules. If Γ is directly finite, then S_x also preserves finitely generated and projective. The *extension functor at* $x \in \mathrm{ob}(\Gamma)$

$$E_x\colon \mathsf{MOD}\text{-}R[x] \to \mathsf{MOD}\text{-}R\Gamma \tag{1.4}$$

maps an $R[x]$-module N to the $R\Gamma$-module $N \otimes_{R[x]} R\,\mathrm{mor}_\Gamma(?, x)$. The functor E_x respects direct sums, sends epimorphisms to epimorphisms, sends free modules to free modules, and preserves finitely generated and projective. If Γ is directly finite, and P is a projective $R[x]$-module, then there is a natural isomorphism $P \cong S_xE_xP$ compatible with direct sums.

Theorem 1.5 (Splitting of $K_0(R\Gamma)$ for EI-categories, Theorem 10.34 on page 196 of Lück [23]). *If Γ is an EI-category, then the group homomorphisms*

$$K_0(R\Gamma) \underset{E}{\overset{S}{\rightleftarrows}} \mathrm{Split}\, K_0(R\Gamma) := \bigoplus_{\overline{x} \in \mathrm{iso}(\Gamma)} K_0(R\,\mathrm{aut}_\Gamma(x))$$

defined by

$$S[P] = \{[S_xP] \mid \overline{x} \in \mathrm{iso}(\Gamma)\}$$

and

$$E\{[Q_x] \mid \overline{x} \in \mathrm{iso}(\Gamma)\} = \sum_{\overline{x} \in \mathrm{iso}(\Gamma)} [E_xQ_x],$$

are isomorphisms and inverse to one another. They are covariantly natural with respect to functors between EI-categories.

Remark 1.6. If Γ is not an EI-category, then the splitting homomorphism $S\colon K_0(R\Gamma) \to \operatorname{Split} K_0(R\Gamma)$ may not be an isomorphism. However, S is covariantly natural with respect to functors between directly finite categories, see [16, Lemma 3.15].

The splitting functors S_x allow us to define the notion of $R\Gamma$-rank $\operatorname{rk}_{R\Gamma}$ for finitely generated $R\Gamma$-modules, which in turn allows the definition of the functorial Euler characteristic, as we explain next. We assume a fixed notion of a rank $\operatorname{rk}_R(N) \in \mathbb{Z}$ for finitely generated R-modules N such that $\operatorname{rk}_R(R) = 1$ and $\operatorname{rk}_R(N_1) = \operatorname{rk}_R(N_0) + \operatorname{rk}_R(N_2)$ for any sequence $0 \to N_0 \to N_1 \to N_2 \to 0$ of finitely generated R-modules. If R is a commutative principal ideal domain, we use $\operatorname{rk}_R(N) := \dim_F(F \otimes_R N)$, where F is the quotient field of R. Let $U(\Gamma)$ be the free abelian group on the set of isomorphism classes of objects in Γ, that is $U(\Gamma) := \mathbb{Z}\operatorname{iso}(\Gamma)$. The augmentation homomorphism $\epsilon\colon U(\Gamma) \to \mathbb{Z}$ adds up the components of an element of $U(\Gamma)$.

Definition 1.7 (Rank of a finitely generated $R\Gamma$-module). If M is a finitely generated $R\Gamma$-module M, then its $R\Gamma$*-rank* is

$$\operatorname{rk}_{R\Gamma}(M) := \left\{\operatorname{rk}_R(S_x M \otimes_{R[x]} R) \mid \overline{x} \in \operatorname{iso}(\Gamma)\right\} \quad \in U(\Gamma).$$

Definition 1.8 (The (functorial) Euler characteristic of a category). Suppose that Γ is of type (FP_R). The *functorial Euler characteristic of* Γ *with coefficients in* R is the image of the finiteness obstruction $o(\Gamma; R) \in K_0(R\Gamma)$ under the homomorphism $\operatorname{rk}_{R\Gamma}\colon K_0(R\Gamma) \to U(\Gamma)$, that is

$$\chi_f(\Gamma; R) := \operatorname{rk}_{R\Gamma} o(\Gamma; R) = \left\{\sum_{n\geq 0}(-1)^n \operatorname{rk}_R(S_x P_n \otimes_{R[x]} R) \mid \overline{x} \in \operatorname{iso}(\Gamma)\right\} \in U(\Gamma),$$

where P_* is any finite projective $R\Gamma$-resolution of the constant $R\Gamma$-module $\underline{R}$. The *Euler characteristic of* Γ *with coefficients in* R is the sum of the components of the functorial Euler characteristic, that is,

$$\chi(\Gamma; R) := \epsilon(\chi_f(\Gamma; R)) = \sum_{\overline{x}\in\operatorname{iso}(\Gamma)} \sum_{n\geq 0}(-1)^n \operatorname{rk}_R(S_x P_n \otimes_{R[x]} R).$$

For example, if $\mathcal{G}$ is a finite groupoid, then $\chi_f(\mathcal{G}) \in U(\mathcal{G})$ is $(1, 1, \ldots, 1)$, and $\chi(\mathcal{G})$ counts the isomorphism classes of objects, or equivalently the connected components, of $\mathcal{G}$.

Theorem 1.9 (Theorem 4.20 of Fiore–Lück–Sauer [16]). *Let R be a Noetherian ring and Γ a directly finite category of type (FP_R). Then the Euler characteristic and topological Euler characteristic of Γ agree. That is, $H_n(B\Gamma; R)$ is a finitely generated R-module for every $n \geq 0$, there exists a natural number d*

with $H_n(B\Gamma;R) = 0$ *for all* $n > d$, *and*

$$\chi(\Gamma;R) = \chi(B\Gamma;R) = \sum_{n\geq 0}(-1)^n \cdot \mathrm{rk}_R(H_n(B\Gamma;R)) \in \mathbb{Z}.$$

Here $\chi(\Gamma;R)$ *is defined in Definition 1.8 and* $B\Gamma$ *denotes the geometric realization of the nerve of* Γ.

The functorial Euler characteristic and Euler characteristic have many desirable properties. They are invariant under equivalence of categories and are compatible with finite products and finite coproducts. As we prove in Theorem 4.1, they are also compatible with homotopy colimits.

The L^2-Euler characteristic, which is in some sense the better invariant, can be defined similarly by taking $R = \mathbb{C}$ and using the L^2-rank $\mathrm{rk}_\Gamma^{(2)}$ rather than the $R\Gamma$-rank. For this we need group von Neumann algebras and their dimension theory from Lück [24] and [25], as recalled in our first paper [16] for the purpose of Euler characteristics. If G is a group, its *group von Neumann algebra*

$$\mathcal{N}(G) = \mathcal{B}(l^2(G))^G$$

is the algebra of bounded operators on $l^2(G)$ that are equivariant with respect to the right G-action. If G is finite, $\mathcal{N}(G)$ is the group ring $\mathbb{C}G$. In any case, the group ring $\mathbb{C}G$ embeds as a subring of $\mathcal{N}(G)$ by sending $g \in G$ to the isometric G-equivariant operator $l^2(G) \to l^2(G)$ given by left multiplication with g. In particular, we can view $\mathcal{N}(G)$ as a $\mathbb{C}G$-$\mathcal{N}(G)$-bimodule and tensor $\mathbb{C}G$-modules on the right with $\mathcal{N}(G)$. If G is the automorphism group of an object in Γ, then we write $\mathcal{N}(x)$ for $\mathcal{N}\big(\mathrm{aut}_\Gamma(x)\big)$.

The *von Neumann dimension*, $\dim_{\mathcal{N}(G)}$, is a function that assigns to *every* right $\mathcal{N}(G)$-module M a non-negative real number of ∞. It is the unique such function which satisfies Hattori-Stallings rank, additivity, cofinality, and continuity. If G is a finite group, then $\mathcal{N}(G) = \mathbb{C}G$ and we get for a $\mathbb{C}G$-module M the von Neumann dimension

$$\dim_{\mathcal{N}(G)}(M) = \frac{1}{|G|} \cdot \dim_{\mathbb{C}}(M),$$

where $\dim_{\mathbb{C}}$ is the dimension of M viewed as a complex vector space. A category Γ is said to be *of type* (L^2) if for one (and hence every) projective $\mathbb{C}\Gamma$-resolution P_* of the constant $\mathbb{C}\Gamma$-module $\underline{\mathbb{C}}$ we have

$$\sum_{\overline{x}\in\mathrm{iso}(\Gamma)}\sum_{n\geq 0}\dim_{\mathcal{N}(x)} H_n\big(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\big) < \infty.$$

Note that the projective resolution P_* of $\underline{\mathbb{C}}$ is not required to be of finite length, nor finitely generated. Examples of categories of type (L^2) include finite EI-categories, in particular finite posets and finite groupoids. Infinite categories can also be of type (L^2), for example any (small) groupoid with finite automorphism groups such that

$$\sum_{\overline{x}\in\mathrm{iso}(\mathcal{G})} \frac{1}{|\,\mathrm{aut}_{\mathcal{G}}(x)|} < \infty \tag{1.10}$$

holds is of type (L^2). The condition of type (L^2) is weaker than ($\mathrm{FP}_{\mathbb{C}}$), since any directly finite category of type ($\mathrm{FP}_{\mathbb{C}}$) is also of type (L^2).

Definition 1.11 (The (functorial) L^2-Euler characteristic of a category). Suppose that Γ is of type (L^2). Define

$$U^{(1)}(\Gamma) := \left\{ \sum_{\overline{x}\in\mathrm{iso}(\Gamma)} r_{\overline{x}} \cdot \overline{x} \;\middle|\; r_{\overline{x}} \in \mathbb{R}, \sum_{\overline{x}\in\mathrm{iso}(\Gamma)} |r_{\overline{x}}| < \infty \right\} \subseteq \prod_{\bar{x}\in\mathrm{iso}(\Gamma)} \mathbb{R}.$$

The *functorial L^2-Euler characteristic of* Γ is

$$\chi_f^{(2)}(\Gamma) := \left\{ \sum_{n\geq 0} (-1)^n \dim_{\mathcal{N}(x)} H_n\big(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\big) \mid \bar{x} \in \mathrm{iso}(\Gamma) \right\} \in U^{(1)}(\Gamma),$$

where P_* is any projective $\mathbb{C}\Gamma$-resolution of the constant $\mathbb{C}\Gamma$-module $\underline{\mathbb{C}}$. The *L^2-Euler characteristic of* Γ is the sum over $\bar{x} \in \mathrm{iso}(\Gamma)$ of the components of the functorial Euler characteristic, that is,

$$\chi^{(2)}(\Gamma) := \sum_{\overline{x}\in\mathrm{iso}(\Gamma)} \sum_{n\geq 0} (-1)^n \dim_{\mathcal{N}(x)} H_n\big(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\big).$$

If $\mathcal{G}$ is a groupoid such that (1.10) holds, then the functorial L^2-Euler characteristic $\chi_f^{(2)}(\mathcal{G}) \in \prod_{\overline{x}\in\mathrm{iso}(\mathcal{G})} \mathbb{R}$ has at $\overline{x} \in \mathrm{iso}(\mathcal{G})$ the value $1/|\mathrm{aut}_{\mathcal{G}}(x)|$. The L^2-Euler characteristic is

$$\chi^{(2)}(\mathcal{G}) = \sum_{\overline{x}\in\mathrm{iso}(\mathcal{G})} \frac{1}{|\mathrm{aut}_{\mathcal{G}}(x)|}. \tag{1.12}$$

See Lemma 7.5 for an explicit formula for $\chi^{(2)}(\Gamma)$ in the case of a finite, skeletal EI-category Γ in which the left $\mathrm{aut}_\Gamma(y)$-action on $\mathrm{mor}_\Gamma(x,y)$ is free for every two objects $x, y \in \mathrm{ob}(\Gamma)$.

Definition 1.13 (L^2-rank of a finitely generated $\mathbb{C}\Gamma$-module). Let M be a finitely generated $\mathbb{C}\Gamma$-module M. Its *L^2-rank* is

$$\mathrm{rk}_\Gamma^{(2)}(M) := \big\{\dim_{\mathcal{N}(x)}(S_x M \otimes_{\mathbb{C}[x]} \mathcal{N}(x)) \mid \bar{x} \in \mathrm{iso}(\Gamma)\big\} \in U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{R} = \bigoplus_{\mathrm{iso}(\Gamma)} \mathbb{R}.$$

Theorem 1.14 (Relating the finiteness obstruction and the L^2-Euler characteristic, Theorem 5.22 of Fiore–Lück–Sauer [16]). *Suppose that Γ is a directly finite category of type ($\mathrm{FP}_{\mathbb{C}}$). Then Γ is of type (L^2) and the image of the finiteness obstruction $o(\Gamma;\mathbb{C})$ (see Definition 1.2) under the homomorphism*

$$\mathrm{rk}_\Gamma^{(2)} \colon K_0(\mathbb{C}\Gamma) \to U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{R} = \bigoplus_{\overline{x}\in\mathrm{iso}(\Gamma)} \mathbb{R}$$

is the functorial L^2-Euler characteristic $\chi_f^{(2)}(\Gamma)$.

The L^2-Euler characteristic agrees with the groupoid cardinality of Baez–Dolan [4] and the Euler characteristic of Leinster [21] in certain cases, see Lemma 7.5 and Section 7. In particular, the Baez–Dolan groupoid cardinality of a groupoid satisfying (1.10) is (1.12). However, the Baez–Dolan groupoid cardinality and Leinster's Euler characteristic $\chi_L(\Gamma)$ only depend on the underlying graph of Γ, whereas our invariants truly depend on the category structure. For instance, χ_L is $\frac{1}{2}$ for both the two-element monoid $(\mathbb{Z}/2, \times)$ and the two-element group $(\mathbb{Z}/2, +)$, whereas $\chi^{(2)}$ is 1 respectively $\frac{1}{2}$. The distinction can already be seen on the level of the finiteness obstructions. The Euler characteristic $\chi(-)$ and topological Euler characteristic $\chi(B-)$ can also distinguish categories with the same underlying directed graph as in the following example. For $S = \{1, 2, 3, 4\}$, $G_1 = \langle(1234)\rangle$, $G_2 = \langle(12), (34)\rangle$, and $k = 1, 2$, let Γ_k be the EI-category with objects x and y and $\operatorname{mor}(x, y) := S$, $\operatorname{mor}(x, x) := \{\mathrm{id}_x\}$, $\operatorname{mor}(y, y) := G_k$, and $\operatorname{mor}(y, x) = \emptyset$. Composition in Γ_k is the composition in G_k and the left G_k-action on S, that is, Γ_k is the EI-category associated to the respective G_k-$\{1\}$-biset S as in Subsection 6.4 of Fiore–Lück–Sauer [16]. Then Γ_1 and Γ_2 have the same underlying directed graph but $\chi(\Gamma_1; \mathbb{Q}) = \chi(B\Gamma_1; \mathbb{Q}) = 1$ and $\chi(\Gamma_2; \mathbb{Q}) = \chi(B\Gamma_2; \mathbb{Q}) = 0$ by Theorem 6.23 (iii) of Fiore–Lück–Sauer [16]. An infinite example of categories with the same underlying graph but different Euler characteristics is provided by the groups $\mathbb{Z}$ and $\mathbb{Z} * \mathbb{Z}$, each of which admits a finite Γ-CW-model for its respective Γ-classifying space. The categories $\widehat{\mathbb{Z}}$ and $\widehat{\mathbb{Z} * \mathbb{Z}}$ have the same underlying directed graph, but we have $\chi^{(2)}(\widehat{\mathbb{Z}}) = 0 \neq \chi^{(2)}(\widehat{\mathbb{Z} * \mathbb{Z}})$, and similarly for χ. Typically, the Euler characteristic of a category Γ_{free} free on a directed graph (V, E) is the same as the Euler characteristic of the directed graph (V, E). For the topological Euler characteristic this is clearly true, since $B\Gamma_{\mathrm{free}}$ is homotopy equivalent to the topological realization $|(V, E)|$. If Γ_{free} is directly finite and R is Noetherian, then we also have $\chi(\Gamma_{\mathrm{free}}) = \chi(|(V, E)|)$ by Theorem 1.9. For example for the directed graph with one vertex and one arrow we have $\chi(\widehat{\mathbb{N}}) = 0 = \chi(S^1)$.

The functorial L^2-Euler characteristic and the L^2-Euler characteristic have many desirable properties. They are invariant under equivalence of categories and are compatible with finite products, finite coproducts, and isofibrations and coverings between finite groupoids. We prove in Theorem 4.1 the compatibility with homotopy colimits. In the case of a group G, the L^2-Euler characteristic of $\widehat{G}$ coincides with the classical L^2-Euler characteristic of G, which is $1/|G|$ when G is finite. The L^2-Euler characteristic is also closely related to the geometry and topology of the classifying space for proper G-actions, namely the functorial L^2-Euler characteristic of the proper orbit category $\underline{\mathrm{Or}}(G)$ is equal to the equivariant Euler characteristic of the classifying space $\underline{E}G$ for proper G-actions, whenever $\underline{E}G$ admits a finite G-CW-model.

The question arises: what are sufficient conditions for the Euler characteristic and L^2-Euler characteristic to coincide with the Euler characteristic of the classifying space? This is answered in the following Theorem.

Theorem 1.15 (Invariants agree for directly finite and type ($\mathrm{FF}_{\mathbb{Z}}$), Theorem 5.25 of Fiore–Lück–Sauer [16]). *Suppose Γ is directly finite and of type ($FF_{\mathbb{Z}}$). Then the functorial L^2-Euler characteristic of Definition 1.11 coincides with the functorial Euler characteristic of Definition 1.8 for any associative,* *commutative ring R with identity*

$$\chi_f^{(2)}(\Gamma) = \chi_f(\Gamma; R) \in U(\Gamma) \subseteq U^{(1)}(\Gamma),$$

and thus $\chi^{(2)}(\Gamma) = \chi(\Gamma; R)$ in Definition 1.11 and Definition 1.8.
If R is additionally Noetherian, then

$$\chi(B\Gamma; R) = \chi(\Gamma; R) = \chi^{(2)}(\Gamma). \tag{1.16}$$

Moreover, if Γ is merely of type ($FF_{\mathbb{C}}$) rather than ($FF_{\mathbb{Z}}$), then equation (1.16) *holds for any Noetherian ring R containing $\mathbb{C}$.*

Any category Γ which admits a finite Γ-CW-model in the sense of Section 2 is of type (FF_R) for any ring R, by an application of the cellular R-chain functor. Thus, Theorem 1.15 applies to any directly finite category Γ which admits a finite Γ-CW-model. For example, finite categories without loops are directly finite and admit finite models (Lemma 8.4 and Theorem 8.5), so equation (1.16) holds for instance for $\{j \rightrightarrows k\}$, $\{k \leftarrow j \rightarrow \ell\}$, and finite posets. The monoid $\mathbb{N}$ and group $\mathbb{Z}$, viewed as one-object categories $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{Z}}$, are also directly finite and admit finite models (see Example 2.8), so we have

$$0 = \chi(S^1; R) = \chi(B\widehat{\mathbb{N}}; R) = \chi(\widehat{\mathbb{N}}; R) = \chi^{(2)}(\widehat{\mathbb{N}})$$

and

$$0 = \chi(S^1; R) = \chi(B\widehat{\mathbb{Z}}; R) = \chi(\widehat{\mathbb{Z}}; R) = \chi^{(2)}(\widehat{\mathbb{Z}})$$

($B\widehat{\mathbb{N}} \to B\widehat{\mathbb{Z}} \simeq S^1$ is a homotopy equivalence by Quillen's Theorem A, see Rabrenović [35, Proposition 10]). The equations $\chi(\widehat{\mathbb{N}}; R) = 0 = \chi^{(2)}(\widehat{\mathbb{N}})$ and $\chi(\widehat{\mathbb{Z}}; R) = 0 = \chi^{(2)}(\widehat{\mathbb{Z}})$ also follow from Example 5.3, since the finite models for $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{Z}}$ in Example 2.8 each have one $\mathcal{I}$-0-cell and one $\mathcal{I}$-1-cell.
We may use Theorem 1.15 to obtain an explicit formula for Euler characteristics of finite categories without loops as follows. Let Γ be a finite category without loops, and choose a skeleton Γ'. Let $c_n(\Gamma')$ denote the number of paths

$$i_0 \to i_1 \to i_2 \to \cdots \to i_n$$

of n-many non-identity morphisms in Γ'. Then $c_n(\Gamma')$ is the number of n-cells in the CW-complex $B\Gamma'$, and we have

$$\chi(\Gamma; R) = \chi^{(2)}(\Gamma) = \chi(B\Gamma; R) = \chi(B\Gamma'; R) = \sum_{n \geq 0} (-1)^n c_n(\Gamma'). \tag{1.17}$$

See [21, Corollary 1.5] for a different derivation of this formula for Leinster's Euler characteristic $\chi_L(\Gamma)$ in the case Γ was already skeletal. See also Examples 5.3 and 8.7 where skeletality of $\mathcal{I}$ is not required.

Remark 1.18 (Homotopy Invariance). If $F : \Gamma_1 \to \Gamma_2$ is a functor such that BF is a homotopy equivalence, and both Γ_1 and Γ_2 are of type (FP_R), and if

$$\chi(\Gamma_1; R) = \chi(B\Gamma_1; R) \quad \text{and} \quad \chi(\Gamma_2; R) = \chi(B\Gamma_2; R),$$

then clearly $\chi(\Gamma_1; R) = \chi(\Gamma_2; R)$. However, it is possible for two categories to be homotopy equivalent, one of which is (FP_R) and the other is not, so that one has a notion of Euler characteristic and the other does not. In Section 10 of Fiore–Lück–Sauer [16] such an example is discussed.

2. Spaces over a Category

An important hypothesis in our Homotopy Colimit Formula involves the idea of a space over a category, see Davis–Lück [14]. Namely, we assume that the indexing category $\mathcal{I}$ for the diagram $\mathcal{C}$ of categories admits a finite $\mathcal{I}$-CW-model for its $\mathcal{I}$-classifying space. Essentially this means it is possible to functorially assign a contractible CW-complex $E\mathcal{I}(i)$ to each $i \in \operatorname{ob}(\mathcal{I})$, and moreover, these local CW-complexes are constructed globally by gluing $\mathcal{I}$-n-cells of the form $\operatorname{mor}_{\mathcal{I}}(-, i_\lambda) \times D^n$ onto the already globally constructed $(n-1)$-skeleton $E\mathcal{I}_n$. The Homotopy Colimit Formula then expresses the invariants of the homotopy colimit of $\mathcal{C}$ in terms of the invariants of the categories $\mathcal{C}(i_\lambda)$ at the base objects i_λ for $E\mathcal{I}$.

The gluing described above takes place in the more general category of $\mathcal{I}$-spaces. A *(contravariant) $\mathcal{I}$-space* is a contravariant functor from $\mathcal{I}$ to the category SPACES of (compactly generated) topological spaces. As usual, we will always work in the category of compactly generated spaces (see Steenrod [39]). A *map between $\mathcal{I}$-spaces* is a natural transformation. Given an object $i \in \operatorname{ob}(\mathcal{I})$, we obtain an $\mathcal{I}$-space $\operatorname{mor}_{\mathcal{I}}(?, i)$ which assigns to an object j the discrete space $\operatorname{mor}_{\mathcal{I}}(j, i)$.

The next definition is taken from Davis–Lück [14, Definition 3.2], where an $\mathcal{I}$-CW-complex is called a free $\mathcal{I}$-CW-complex and we will omit the word free here. The more general notion of $\mathcal{I}$-CW-complex was defined by Dror Farjoun [15, 1.16 and 2.1]. See also Piacenza [34].

Definition 2.1 ($\mathcal{I}$-CW-complex). A *(contravariant) $\mathcal{I}$-CW-complex X* is a contravariant $\mathcal{I}$-space X together with a filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots \subset X = \bigcup_{n \geq 0} X_n$$

such that $X = \operatorname{colim}_{n \to \infty} X_n$ and for any $n \geq 0$ the *n-skeleton* X_n is obtained from the $(n-1)$-skeleton X_{n-1} by *attaching $\mathcal{I}$-n-cells*, i.e., there exists a pushout of $\mathcal{I}$-spaces of the form

$$\begin{array}{ccc} \coprod_{\lambda \in \Lambda_n} \operatorname{mor}_{\mathcal{I}}(-, i_\lambda) \times S^{n-1} & \longrightarrow & X_{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{\lambda \in \Lambda_n} \operatorname{mor}_{\mathcal{I}}(-, i_\lambda) \times D^n & \longrightarrow & X_n \end{array}$$

where the vertical maps are inclusions, Λ_n is an index set, and the i_λ-s are objects of $\mathcal{I}$. In particular, $X_0 = \coprod_{\lambda \in \Lambda_0} \operatorname{mor}_{\mathcal{I}}(-, i_\lambda)$.

We refer to the inclusion functor $\operatorname{mor}_{\mathcal{I}}(-, i_\lambda) \times (D^n - S^{n-1}) \to X$ as an *$\mathcal{I}$-n-cell based at i_λ*.

An $\mathcal{I}$-CW-complex has *dimension* $\leq n$ if $X = X_n$. We call X *finite dimensional* if there exists an integer n with $X = X_n$. It is called *finite* if it is finite dimensional and Λ_n is finite for every $n \geq 0$.
The definition of a *covariant* $\mathcal{I}$-CW*-complex* is analogous.

Definition 2.2 (Classifying $\mathcal{I}$-space). A model for *the classifying* $\mathcal{I}$*-space* $E\mathcal{I}$ is an $\mathcal{I}$-CW-complex $E\mathcal{I}$ such that $E\mathcal{I}(i)$ is contractible for all objects i.

The universal property of $E\mathcal{I}$ is that for any $\mathcal{I}$-CW-complex X there is up to homotopy precisely one map of $\mathcal{I}$-spaces from X to $E\mathcal{I}$. In particular two models for $E\mathcal{I}$ are $\mathcal{I}$-homotopy equivalent (see Davis–Lück [14, Theorem 3.4]). A model for the usual *classifying space* $B\mathcal{I}$ is given by $E\mathcal{I} \otimes_{\mathcal{I}} \{\bullet\}$ (see [14, Definition 3.10]), where $\{\bullet\}$ is the constant covariant $\mathcal{I}$-space with value the one point space and $\otimes_{\mathcal{I}}$ denotes the tensor product of a contravariant and a covariant $\mathcal{I}$-space as follows (see [14, Definition 1.4]).

Definition 2.3 (Tensor product of a contravariant and a covariant $\mathcal{I}$-space). Let X be a contravariant $\mathcal{I}$-space and Y a covariant $\mathcal{I}$-space. The *tensor product of* X *and* Y is

$$X \otimes_{\mathcal{I}} Y = \left(\coprod_{i \in \mathcal{I}} X(i) \times Y(i) \right) / \sim$$

where $(X(\phi)(x), y) \sim (x, Y(\phi)y)$ for all morphisms $\phi : i \to j$ in $\mathcal{I}$ and points $x \in X(j)$ and $y \in Y(i)$.

We present some examples of classifying $\mathcal{I}$-spaces for various categories $\mathcal{I}$.

Example 2.4. If $\mathcal{I}$ has a terminal object t, then a finite model for the classifying $\mathcal{I}$-space $E\mathcal{I}$ is simply $\operatorname{mor}_{\mathcal{I}}(-, t)$.

Example 2.5. Let $\mathcal{I} = \{j \rightrightarrows k\}$ be the category consisting of two objects and a single pair of parallel arrows between them. All other morphisms are identity morphisms. We obtain a finite model X for the classifying $\mathcal{I}$-space $E\mathcal{I}$ as follows. The $\mathcal{I}$-CW-space X has a single $\mathcal{I}$-0-cell based at k and a single $\mathcal{I}$-1-cell based at j. The gluing map $\operatorname{mor}_{\mathcal{I}}(-, j) \times S^0 \to \operatorname{mor}_{\mathcal{I}}(-, k)$ is induced by the two parallel arrows $j \rightrightarrows k$. Then $X(j) = D^1 \simeq *$ and $X(k) = *$.

Example 2.6. Let $\mathcal{I} = \{k \leftarrow j \rightarrow \ell\}$ be the category with objects j, k and ℓ, and precisely one morphism from j to k and one morphism from j to ℓ. All other morphisms are identity morphisms. A finite model for $E\mathcal{I}$ is given by the $\mathcal{I}$-CW-complex with precisely two $\mathcal{I}$-0-cells $\operatorname{mor}_{\mathcal{I}}(?, k)$ and $\operatorname{mor}_{\mathcal{I}}(?, \ell)$ and precisely one $\mathcal{I}$-1-cell $\operatorname{mor}_{\mathcal{I}}(?, j) \times D^1$ whose attaching map $\operatorname{mor}_{\mathcal{I}}(?, j) \times S^0 \to \operatorname{mor}_{\mathcal{I}}(?, k) \amalg \operatorname{mor}_{\mathcal{I}}(?, \ell)$ is the disjoint union of the canonical maps $\operatorname{mor}_{\mathcal{I}}(?, j) \to \operatorname{mor}_{\mathcal{I}}(?, k)$ and $\operatorname{mor}_{\mathcal{I}}(?, j) \to \operatorname{mor}_{\mathcal{I}}(?, \ell)$. The value of this 1-dimensional $\mathcal{I}$-CW-complex at the objects k and ℓ is a point and at the object j is D^1. Hence it is a finite model for $E\mathcal{I}$.

Example 2.7. Let $\mathcal{I}$ be the category with objects the non-empty subsets of $[q] = \{0, 1, \ldots, q\}$ and a unique arrow $J \to K$ if and only if $K \subseteq J$. In other words $\mathcal{I}$ is the *opposite* of the poset of non-empty subsets of $[q]$. Then $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model X for the classifying $\mathcal{I}$-space $E\mathcal{I}$ as follows. The functor $X\colon \mathcal{I}^{\mathrm{op}} \to \mathsf{SPACES}$ assigns to L the space $|\Delta[L]|$, which is the geometric realization of the simplicial set which maps $[m]$ to the set of weakly order preserving maps $[m] \to L$. The space $|\Delta[L]|$ is homeomorphic to the standard simplex with $\mathrm{card}(L)$ vertices. The n-skeleton X_n of X sends each L to the n-skeleton of $|\Delta[L]|$. The $\mathcal{I}$-cells of X are attached globally in the following way. The 0-skeleton is

$$X_0 = \coprod_{J\subseteq[q],|J|=1} \mathrm{mor}_{\mathcal{I}}(-,J).$$

For $n \leq q$, we construct X_n out of X_{n-1} as the pushout

$$\begin{tikzcd} \coprod_J \mathrm{mor}_{\mathcal{I}}(-,J) \times |\partial\Delta[n]| \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod_J \mathrm{mor}_{\mathcal{I}}(-,J) \times |\Delta[n]| \arrow[r] & X_n. \end{tikzcd}$$

The disjoint unions are over all $J \subseteq [q]$ with $|J| = n+1$. The J-component of the gluing map is induced by the $(n-1)$-face inclusion

$$|\Delta[K]| \longrightarrow \partial|\Delta[J]| \cong \partial|\Delta[n]|$$

for all $K \subseteq J$ with $|K| = n$. Clearly X is a finite $\mathcal{I}$-CW-complex. For each object L of $\mathcal{I}$, we have $X(L) = |\Delta[L]| \simeq *$, so that X is a finite model for $E\mathcal{I}$.

Example 2.8. Infinite categories may also admit finite models. Let $\mathcal{I} = \widehat{\mathbb{N}}$ be the monoid of natural numbers $\mathbb{N}$ viewed as a one-object category. A finite model X for the $\widehat{\mathbb{N}}$-classifying space has $X_0(*) = \mathrm{mor}_{\widehat{\mathbb{N}}}(*,*) = \mathbb{N}$ and $X_1(*) = [0,\infty)$. This model has a single $\widehat{\mathbb{N}}$-0-cell $\mathrm{mor}_{\widehat{\mathbb{N}}}(-,*)$ and a single $\widehat{\mathbb{N}}$-1-cell $\mathrm{mor}_{\widehat{\mathbb{N}}}(-,*) \times D^1$. The gluing map $\mathbb{N} \times S^0 \to \mathbb{N}$ sends $(n,-1)$ and $(n,1)$ to n and $n+1$ respectively. Similarly, the group of integers $\mathbb{Z}$ viewed as a one object category admits a finite model Y with one $\widehat{\mathbb{Z}}$-0-cell and one $\widehat{\mathbb{Z}}$-1-cell, so that $Y_0(*) = \mathbb{Z}$ and $Y_1(*) = \mathbb{R}$.

Remark 2.9. Suppose a category $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model for $E\mathcal{I}$. Then the cellular R-chains of a finite model provide a finite free resolution of the constant $R\mathcal{I}$-module $\underline{R}$, so $\mathcal{I}$ is of type (FF_R). If $\mathcal{I}$ is additionally directly finite and R is Noetherian, then $\chi(B\mathcal{I};R) = \chi(\mathcal{I};R) = \chi^{(2)}(\mathcal{I})$ by Theorem 1.15.

Remark 2.10 (Bar construction of classifying $\mathcal{I}$-space). There exists a functorial construction $E^{\mathrm{bar}}\mathcal{I}$ of $E\mathcal{I}$ by a kind of bar construction. Namely, the contravariant functor $E^{\mathrm{bar}}\mathcal{I}\colon \mathcal{I} \to \mathsf{SPACES}$ sends an object i to the space $B^{\mathrm{bar}}(i \downarrow \mathcal{I})$, which is the geometric realization of the nerve of the category of objects under i (see Davis–Lück [14, page 230] and also Bousfield–Kan [10, page 327]). An equivalent definition of the bar construction in terms of the tensor product in Definition 2.3 is

$$E^{\mathrm{bar}}\mathcal{I} = \{*\} \otimes_{\mathcal{I}} B^{\mathrm{bar}}(? \downarrow \mathcal{I} \downarrow ??), \tag{2.11}$$

from which we prove that $E^{\mathrm{bar}}\mathcal{I}$ is an $\mathcal{I}$-CW-complex. The $\mathcal{I} \times \mathcal{I}^{\mathrm{op}}$-space $B^{\mathrm{bar}}(? \downarrow \mathcal{I} \downarrow ??)$ is an $\mathcal{I} \times \mathcal{I}^{\mathrm{op}}$-$CW$-complex (see [14, page 228]). For each path

$$i_0 \to i_1 \to i_2 \to \cdots \to i_n$$

of n-many non-identity morphisms in $\mathcal{I}$, $B^{\mathrm{bar}}(? \downarrow \mathcal{I} \downarrow ??)$ has an n-cell based at (i_0, i_n), that is a cell of the form $\mathrm{mor}_{\mathcal{I}}(?, i_0) \times \mathrm{mor}_{\mathcal{I}}(i_n, ??) \times D^n$. By [14, Lemma 3.19 (2)], the tensor product $E^{\mathrm{bar}}\mathcal{I}$ in (2.11) is an $\mathcal{I}$-CW-complex: an $(m+n)$-cell based at i is an n-cell of $B^{\mathrm{bar}}(? \downarrow \mathcal{I} \downarrow ??)$ based at (i,j) and an m-cell of the CW-complex $*(j)$ (see [14, page 229]). More explicitly, for each path of n-many non-identity morphisms

$$i_0 \to i_1 \to i_2 \to \cdots \to i_n \tag{2.12}$$

the $\mathcal{I}$-CW-complex $E^{\mathrm{bar}}\mathcal{I}$ has an n-cell based at i_0.
Though the bar construction is in general not a finite $\mathcal{I}$-CW-complex, it is in certain cases. For example, if $\mathcal{I}$ has only finitely many morphisms, no nontrivial isomorphisms, and no nontrivial endomorphisms, then there are only finitely many paths as in (2.12), and hence only finitely many $\mathcal{I}$-cells in $E^{\mathrm{bar}}\mathcal{I}$.
The bar construction is also compatible with induction. Given a functor $\alpha\colon \mathcal{I} \to \mathcal{D}$, we obtain a map of $\mathcal{D}$-spaces

$$E^{\mathrm{bar}}\alpha\colon \alpha_* E^{\mathrm{bar}}\mathcal{I} \to E^{\mathrm{bar}}\mathcal{D},$$

where α_* denotes induction with the functor α (see [14, Definition 1.8]). If $T\colon \alpha \to \beta$ is a natural transformation of functors $\mathcal{I} \to \mathcal{D}$, we obtain for any $\mathcal{I}$-space X a natural transformation $T_*\colon \alpha_* X \to \beta_* X$ which comes from the map of $\mathcal{I}$-$\mathcal{D}$-spaces $\mathrm{mor}_{\mathcal{D}}(??, \alpha(?)) \to \mathrm{mor}_{\mathcal{D}}(??, \beta(?))$ sending $g\colon ?? \to \alpha(?)$ to $T(?) \circ g\colon ?? \to \beta(?)$.

LEMMA 2.13 (Invariance of finite models under equivalence of categories). *Suppose $\mathcal{I}$ and $\mathcal{J}$ are equivalent categories. Then $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model for $E\mathcal{I}$ if and only if $\mathcal{J}$ admits a finite $\mathcal{J}$-CW-model for $E\mathcal{J}$. More precisely, if $F\colon \mathcal{I} \to \mathcal{J}$ is an equivalence of categories and Y is a finite $\mathcal{J}$-CW-model for $E\mathcal{J}$, then the restriction $\mathrm{res}_F\, Y$ is a finite $\mathcal{I}$-CW-model for $E\mathcal{I}$.*

Proof. For any functor $F\colon \mathcal{I} \to \mathcal{J}$, we have an adjunction

$$\mathrm{ind}_F\colon \mathcal{I}\text{-}\mathsf{SPACES} \rightleftarrows \mathcal{J}\text{-}\mathsf{SPACES}\colon \mathrm{res}_F$$

defined by

$$\mathrm{ind}_F(X) := X(?) \otimes_{\mathcal{I}} \mathrm{mor}_{\mathcal{J}}\big(??, F(?)\big) \qquad \mathrm{res}_F(Y) := Y \circ F(?).$$

The $\mathcal{I}$-space $\mathrm{res}_F(Y)$ is naturally homeomorphic to $Y(?) \otimes_{\mathcal{J}} \mathrm{mor}_{\mathcal{J}}\big(F(??), ?\big)$. But since we are assuming F is an equivalence of categories, it is a left adjoint in an adjoint equivalence (F, G), and we have natural homeomorphisms of $\mathcal{I}$-spaces

$$\begin{aligned} \mathrm{res}_F(Y) &\cong Y(?) \otimes_{\mathcal{J}} \mathrm{mor}_{\mathcal{J}}\big(F(??), ?\big) \\ &\cong Y(?) \otimes_{\mathcal{J}} \mathrm{mor}_{\mathcal{J}}\big(??, G(?)\big) \\ &\cong \mathrm{ind}_G(Y). \end{aligned}$$

Since ind_G is a left adjoint, so is res_F, and res_F therefore preserves pushouts. Note also

$$\mathrm{res}_F \operatorname{mor}_{\mathcal{J}}(?, j) = \operatorname{mor}_{\mathcal{J}}\big(F(?), j\big) \cong \operatorname{mor}_{\mathcal{I}}\big(?, G(j)\big).$$

If Y is a finite $\mathcal{J}$-CW-model for $E\mathcal{J}$ with n-skeleton

$$\begin{array}{ccc} \coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{J}}(-, j_\lambda) \times S^{n-1} & \longrightarrow & Y_{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{J}}(-, j_\lambda) \times D^{n} & \longrightarrow & Y_{n}, \end{array}$$

then $X := \mathrm{res}_F Y$ is a finite $\mathcal{I}$-CW-complex with n-skeleton

$$\begin{array}{ccc} \coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{I}}\big(-, G(j_\lambda)\big) \times S^{n-1} & \longrightarrow & X_{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{I}}\big(-, G(j_\lambda)\big) \times D^{n} & \longrightarrow & X_{n}. \end{array}$$

Clearly, $\mathrm{res}_F Y$ is contractible at each object i, since $\mathrm{res}_F Y(i) = Y(F(i)) \simeq *$. □

3. Homotopy Colimits of Categories

Definition 3.1 (Homotopy colimit for categories). Let $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ be a covariant functor from some (small) index category $\mathcal{I}$ to the category of small categories. Its *homotopy colimit*

$$\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$$

is the following category. Objects are pairs (i, c), where $i \in \mathrm{ob}(\mathcal{I})$ and $c \in \mathrm{ob}\big(\mathcal{C}(i)\big)$. A morphism from (i, c) to (j, d) is a pair (u, f), where $u\colon i \to j$ is a morphism in $\mathcal{I}$ and $f\colon \mathcal{C}(u)(c) \to d$ is a morphism in $\mathcal{C}(j)$. The composition of the morphisms $(u, f)\colon (i, c) \to (j, d)$ and $(v, g)\colon (j, d) \to (k, e)$ is the morphism

$$(v, g) \circ (u, f) = (v \circ u, g \circ \mathcal{C}(v)(f))\colon (i, c) \to (k, e).$$

The identity of (i, c) is given by $(\mathrm{id}_i, \mathrm{id}_c)$.
This homotopy colimit construction for functors is often called the *Grothendieck construction* or the *category of elements*.

In which sense is $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ a homotopy colimit? First, recall from [20] that the nerve functor induces an equivalence of categories $\mathrm{Ho}\,\mathsf{CAT} \to \mathrm{Ho}\,\mathsf{SSET}$, where $\mathrm{Ho}\,\mathsf{CAT}$ denotes the localization of CAT with respect to nerve weak equivalences and $\mathrm{Ho}\,\mathsf{SSET}$ denotes the localization of SSET with respect to the usual weak equivalences. In [40], Thomason proved that $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ in CAT corresponds to the Bousfield–Kan construction in SSET under this equivalence of categories. Consequently, $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ has a universal property in the form of a bijection

$$\mathrm{Ho}\,\mathsf{CAT}(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}, \Gamma) \cong \mathrm{Ho}\,\mathsf{CAT}^{\mathcal{I}}(\mathcal{C}, \underline{\Gamma}), \tag{3.2}$$

for any category Γ. Here $\underline{\Gamma}$ indicates the $\mathcal{I}$-diagram that is constant Γ. In [41], Thomason proved that CAT admits a cofibrantly generated model structure in which the weak equivalences are the nerve weak equivalences, so that the associated projective model structure on $\mathsf{CAT}^{\mathcal{I}}$ exists. The model-theoretic construction of a homotopy colimit of the $\mathcal{I}$-diagram $\mathcal{C}$ in CAT as a colimit of a cofibrant replacement of $\mathcal{C}$ in the projective model structure therefore works. This model-theoretic construction also has the universal property in (3.2), so is isomorphic to $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{C}$ in Ho CAT, i.e. weakly equivalent to $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{C}$ in CAT.[1] A direct proof that $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{C}$ satisfies the universal property (3.2) is in Grothendieck's letter [17], see the article of Maltsiniotis [32, Section 3.1].

Remark 3.3. If $\mathcal{C}$ is merely a pseudo functor, then it of course still has a homotopy colimit. A *pseudo functor* $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ is like an ordinary functor, but only preserves composition and unit up to specified coherent natural isomorphisms $\mathcal{C}_{v,u}\colon \mathcal{C}(v)\circ\mathcal{C}(u) \Rightarrow \mathcal{C}(v\circ u)$ and $\mathcal{C}_i\colon 1_{\mathcal{C}(i)} \Rightarrow \mathcal{C}(\mathrm{id}_i)$. Moreover, $\mathcal{C}_{v,u}$ is required to be natural in v and u. The objects and morphisms of the *homotopy colimit* $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{C}$ are defined as in the strict case of Definition 3.1. The composition in $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{C}$ is defined by the modified rule

$$(v,g)\circ(u,f) = (v\circ u, g\circ(\mathcal{C}(v)(f))\circ\mathcal{C}_{v,u}^{-1}(c))$$

while the identity of the object (i,c) is given by

$$(\mathrm{id}_i, \mathcal{C}_i^{-1}(c)).$$

The homotopy colimit of a pseudo functor $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ is an ordinary 1-category with strictly associative and strictly unital composition.

Remark 3.4. For a fixed category $\mathcal{I}$, the homotopy colimit construction $\mathrm{hocolim}_{\mathcal{I}} -$ is a strict 2-functor from the strict 2-category of pseudo functors $\mathcal{I} \to \mathsf{CAT}$, pseudo natural transformations, and modifications into the strict 2-category CAT.

Example 3.5 (Homotopy colimit of a constant functor). If $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ is a constant functor, say constantly a category also called $\mathcal{C}$, then $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{C} = \mathcal{I}\times\mathcal{C}$.

Example 3.6 (Homotopy colimit for $\mathcal{I}$ with a terminal object). Suppose $\mathcal{I}$ has a terminal object t and $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ is a strict covariant functor. Then $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{C}$ is homotopy equivalent to $\mathcal{C}(t)$ as follows. This is analogous to the familiar fact that $\mathcal{C}(t)$ is a colimit of $\mathcal{C}$. The components of the universal cocone

$$\pi\colon \mathcal{C} \Rightarrow \Delta_{\mathcal{C}(t)} \qquad (3.7)$$

[1]We thank George Maltsiniotis for clarifying these points about homotopy colimits in CAT.

are $\mathcal{C}(i \to t)$. Applying $\operatorname{hocolim}_{\mathcal{I}} -$ to (3.7) and composing with the projection gives us a functor F

$$\begin{tikzcd}
\operatorname{hocolim}_{\mathcal{I}} \mathcal{C} \arrow[r, "\operatorname{hocolim}_{\mathcal{I}} \pi"'] \arrow[rr, bend left, "F"] & \mathcal{I} \times \mathcal{C}(t) \arrow[r, "\mathrm{pr}_{\mathcal{C}(t)}"'] & \mathcal{C}(t) \\
(i,c) \arrow[rr, mapsto] & & \mathcal{C}(i \to t)(c).
\end{tikzcd}$$

The functor $G\colon \mathcal{C}(t) \to \operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$, $G(c) = (t, c)$ is a homotopy inverse, since $F \circ G = \mathrm{id}_{\mathcal{C}(t)}$ and we have a natural transformation $\mathrm{id}_{\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}} \Rightarrow G \circ F$ with components

$$(i \to t, \mathrm{id}_{\mathcal{C}(i \to t)})\colon (i, c) \longrightarrow (t, \mathcal{C}(i \to t)c).$$

Let $\mathcal{H}$ denote the homotopy colimit of the $\mathcal{I}$-diagram of categories $\mathcal{C}$. We now construct an $\mathcal{I}$-diagram of $\mathcal{H}$-spaces $E^{\mathcal{H}}$ with the property that its tensor product with $E\mathcal{I}$ is $\mathcal{H}$-homotopy equivalent to a classifying $\mathcal{H}$-space for $\mathcal{H}$. This $\mathcal{I}$-diagram of $\mathcal{H}$-spaces $E^{\mathcal{H}}$ will play an important role in the inductive proof of the Homotopy Colimit Formula Theorem 4.1.

Construction 3.8 (Construction of $E^{\mathcal{H}}$). Let $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ be a strict covariant functor, and abbreviate $\mathcal{H} = \operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$. Define a functor

$$E^{\mathcal{H}}\colon \mathcal{I} \to \mathcal{H}\text{-}\mathsf{SPACES} \tag{3.9}$$

as follows. Given an object $i \in \mathcal{I}$, we have the functor

$$\alpha(i)\colon \mathcal{C}(i) \to \mathcal{H} \tag{3.10}$$

sending an object c to the object (i, c) and a morphism $f\colon c \to d$ to the morphism (id_i, f). We define

$$E^{\mathcal{H}}(i) = \alpha(i)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big).$$

Consider a morphism $u\colon i \to j$ in $\mathcal{I}$. It induces a natural transformation $T(u)\colon \alpha(i) \to \alpha(j) \circ \mathcal{C}(u)$ from the functor $\alpha(i)\colon \mathcal{C}(i) \to \mathcal{H}$ to the functor $\alpha(j) \circ \mathcal{C}(u)\colon \mathcal{C}(i) \to \mathcal{H}$ by assigning to an object c in $\mathcal{C}(i)$ the morphism

$$(u, \mathrm{id}_{\mathcal{C}(u)(c)})\colon \alpha(i)(c) = (i, c) \to \alpha(j) \circ \mathcal{C}(u)(c) = (j, \mathcal{C}(u)(c)).$$

From Remark 2.10 we obtain a map of $\mathcal{H}$-spaces

$$T(u)_*\colon \alpha(i)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big) \to \alpha(j)_* \mathcal{C}(u)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big)$$

and a map of $\mathcal{C}(j)$-spaces

$$E^{\mathrm{bar}}\big(\mathcal{C}(u)\big)\colon \mathcal{C}(u)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big) \to E^{\mathrm{bar}}\big(\mathcal{C}(j)\big).$$

Finally, for the morphism u in $\mathcal{I}$, we define $E^{\mathcal{H}}(u)\colon E^{\mathcal{H}}(i) \to E^{\mathcal{H}}(j)$ by the composite of the two maps below.

$$\alpha(i)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big) \xrightarrow{T(u)_*} \alpha(j)_* \mathcal{C}(u)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big)$$

$$\alpha(j)_* \mathcal{C}(u)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big) \xrightarrow{\alpha(j)_*(E^{\mathrm{bar}}(\mathcal{C}(u))} \alpha(j)_* E^{\mathrm{bar}}\big(\mathcal{C}(j)\big)$$

Define the homotopy colimit of the covariant functor $E^{\mathcal{H}}$ of (3.9) to be the contravariant $\mathcal{H}$-space

$$\operatorname{hocolim}_{\mathcal{I}} E^{\mathcal{H}} \quad := \quad (i,c) \mapsto E\mathcal{I} \otimes_{\mathcal{I}} \left(E^{\mathcal{H}}(i,c)\right). \tag{3.11}$$

Lemma 3.12. *Consider any model $E\mathcal{I}$ for the classifying $\mathcal{I}$-space of the category $\mathcal{I}$. Then the contravariant $\mathcal{H}$-space $E\mathcal{I} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ of* (3.11) *is $\mathcal{H}$-homotopy equivalent to the classifying $\mathcal{H}$-space $E\mathcal{H}$ of the category $\mathcal{H} := \operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$.*

Proof. We first show that for any object (i,c) in $\mathcal{H}$ the space $E\mathcal{I} \otimes_{\mathcal{I}} \left(E^{\mathcal{H}}(i,c)\right)$ is contractible. The covariant functor $E^{\mathcal{H}}(i,c)\colon \mathcal{I} \to \mathsf{SPACES}$ sends an object j to

$$\begin{aligned}
&\alpha(j)_* \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)(i,c) \\
&\quad = \quad \alpha(j)_* \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)(?) \otimes_{\mathcal{H}} \operatorname{mor}_{\mathcal{H}}\big((i,c),?\big) \\
&\quad = \quad \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)(?) \otimes_{\mathcal{C}(j)} \operatorname{mor}_{\mathcal{H}}\big((i,c),(j,?)\big) \\
&\quad = \quad \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)(?) \otimes_{\mathcal{C}(j)} \left(\coprod_{u \in \operatorname{mor}_{\mathcal{I}}(i,j)} \operatorname{mor}_{\mathcal{C}(j)}\big(\mathcal{C}(u)(c),?\big)\right) \\
&\quad = \quad \coprod_{u \in \operatorname{mor}_{\mathcal{I}}(i,j)} \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)(?) \otimes_{\mathcal{C}(j)} \operatorname{mor}_{\mathcal{C}(j)}\big(\mathcal{C}(u)(c),?\big) \\
&\quad = \quad \coprod_{u \in \operatorname{mor}_{\mathcal{I}}(i,j)} \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)\big(\mathcal{C}(u)(c)\big).
\end{aligned}$$

Since $\left(E^{\mathrm{bar}}\mathcal{C}(j)\right)\big(\mathcal{C}(u)(c)\big)$ is contractible, the projection

$$\coprod_{u \in \operatorname{mor}_{\mathcal{I}}(i,j)} \left(E^{\mathrm{bar}}\mathcal{C}(j)\right)\big(\mathcal{C}(u)(c)\big) \to \operatorname{mor}_{\mathcal{I}}(i,j)$$

is a homotopy equivalence. Hence the collection of these projections for $j \in \operatorname{ob}(\mathcal{I})$ induces a map of $\mathcal{I}$-spaces

$$\operatorname{pr}\colon E^{\mathcal{H}}(i,c) \to \operatorname{mor}_{\mathcal{I}}(i,?)$$

whose evaluation at each object j in $\operatorname{ob}(\mathcal{I})$ is a homotopy equivalence. We conclude from Davis–Lück [14, Theorem 3.11] that

$$E\mathcal{I} \otimes_{\mathcal{I}} \operatorname{pr}\colon E\mathcal{I} \otimes_{\mathcal{I}} E^{\mathcal{H}}(i,c) \xrightarrow{\simeq} E\mathcal{I} \otimes_{\mathcal{I}} \operatorname{mor}_{\mathcal{I}}(i,?).$$

is a homotopy equivalence. Since $E\mathcal{I} \otimes_{\mathcal{I}} \operatorname{mor}_{\mathcal{I}}(i,?) = E\mathcal{I}(i)$ is contractible, this implies that for any object (i,c) in $\mathcal{H}$ the space $E\mathcal{I} \otimes_{\mathcal{I}} \left(E^{\mathcal{H}}(i,c)\right)$ is contractible, as we initially claimed.
It remains to show that $E\mathcal{I} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ has the $\mathcal{H}$-homotopy type of an $\mathcal{H}$-CW-complex. It is actually an $\mathcal{H}$-CW-complex. The following argument, that $E\mathcal{I} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ has the homotopy type of an $\mathcal{H}$-CW-complex, will be used again later.[2]

[2]This is a well-known standard argument, which we present only so that the reader easily sees that it works in the setting of $\mathcal{H}$-spaces.

We have a filtration of $E\mathcal{I}$

$$\emptyset = E\mathcal{I}_{-1} \subseteq E\mathcal{I}_0 \subseteq E\mathcal{I}_1 \subseteq \ldots \subseteq E\mathcal{I}_n \subseteq \ldots \subseteq E\mathcal{I} = \bigcup_{n \geq 0} E\mathcal{I}_n$$

such that

$$E\mathcal{I} = \operatorname{colim}_{n\to\infty} E\mathcal{I}_n$$

and for every $n \geq 0$ there exists a pushout of $\mathcal{I}$-spaces

$$\begin{array}{ccc} \coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{I}}(-, i_\lambda) \times S^{n-1} & \longrightarrow & E\mathcal{I}_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\lambda\in\Lambda_n} \operatorname{mor}_{\mathcal{I}}(-, i_\lambda) \times D^n & \longrightarrow & E\mathcal{I}_n. \end{array} \tag{3.13}$$

Since $- \otimes_{\mathcal{I}} Z$ has a right adjoint [14, Lemma 1.9] we get

$$E\mathcal{I} \otimes_{\mathcal{I}} E^{\mathcal{H}} = \operatorname{colim}_{n\to\infty} E\mathcal{I}_n \otimes_{\mathcal{I}} E^{\mathcal{H}}$$

as a colimit of $\mathcal{H}$-spaces. After an application of $- \otimes_{\mathcal{I}} E^{\mathcal{H}}$ to (3.13), we obtain pushouts of $\mathcal{H}$-spaces

$$\begin{array}{ccc} \coprod_{\lambda\in\Lambda_n} E^{\mathcal{H}}(i_\lambda) \times S^{n-1} & \xrightarrow{f_{n-1}} & E\mathcal{I}_{n-1} \otimes_{\mathcal{I}} E^{\mathcal{H}} \\ \downarrow & & \downarrow \\ \coprod_{\lambda\in\Lambda_n} E^{\mathcal{H}}(i_\lambda) \times D^n & \longrightarrow & E\mathcal{I}_n \otimes_{\mathcal{I}} E^{\mathcal{H}} \end{array} \tag{3.14}$$

where the left vertical arrow and hence the right vertical arrow are cofibrations of $\mathcal{H}$-spaces. By induction we may assume that $E\mathcal{I}_{n-1} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ has the homotopy type of an $\mathcal{H}$-CW-complex. Since the vertical maps are cofibrations, by replacing it with a homotopy equivalent $\mathcal{H}$-CW-complex we do not change the homotopy type of the pushout (the usual proof for spaces goes through for $\mathcal{H}$-spaces). Hence we may assume that $E\mathcal{I}_{n-1} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ is a $\mathcal{H}$-CW-complex. We may also assume that f_{n-1} is cellular: since the vertical maps are cofibrations, by replacing f_{n-1} by a homotopic cellular map, which exists by Davis–Lück [14, cf. Theorem 3.7], we also do not change the homotopy type of the pushout. See Selick [38, Theorem 7.1.8] for a proof of this statement for spaces which translates verbatim to the setting of $\mathcal{H}$-spaces. If f_{n-1} is cellular, diagram (3.14) is a cellular pushout. Hence we completed the induction step, showing that $E\mathcal{I}_n \otimes_{\mathcal{I}} E^{\mathcal{H}}$ has the homotopy type of an $\mathcal{H}$-CW-complex.
It remains to show that $E\mathcal{I} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ has the homotopy type of a $\mathcal{H}$-CW-complex: choose $\mathcal{H}$-CW-complexes Z_n and $\mathcal{H}$-homotopy equivalences $g_n : Z_n \to E\mathcal{I}_n \otimes_{\mathcal{I}} E^{\mathcal{H}}$. By iteratively replacing Z_n by the mapping cylinder of

$$Z_{n-1} \xrightarrow{g_{n-1}} E\mathcal{I}_{n-1} \otimes_{\mathcal{I}} E^{\mathcal{H}} \to E\mathcal{I}_n \otimes_{\mathcal{I}} E^{\mathcal{H}} \xrightarrow{\bar{g}_n} Z_n,$$

where $\bar{g}_n$ is a homotopy inverse of g_n, one finds a new sequence of homotopy equivalences $g'_n : Z_n \to E\mathcal{I}_n \otimes_{\mathcal{I}} E^{\mathcal{H}}$ (with the modified $\mathcal{H}$-CW-complexes Z_n) such that $g'_n|_{Z_{n-1}} = g'_{n-1}$. $\square$

4. Homotopy Colimit Formula for Finiteness Obstructions and Euler Characteristics

In this section we prove the main theorem of this paper: the Homotopy Colimit Formula. It expresses the finiteness obstruction, the Euler characteristic, and the L^2-Euler characteristic of the homotopy colimit of a diagram in CAT in terms of the respective invariants for the diagram entries at the base objects for cells in a finite model for the $\mathcal{I}$-classifying space of $\mathcal{I}$. Analogous formulas for the functorial counterparts of the Euler characteristic and L^2-Euler characteristic are included. The Homotopy Colimit Formula is initially stated and proved for strict functors $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$, but we prove that it also holds for pseudo functors $\mathcal{D}\colon \mathcal{I} \to \mathsf{CAT}$ in Corollary 4.2. The full generality of pseudo functors is needed for the applications to complexes of groups in Section 8.

4.1. Homotopy Colimit Formula.

Theorem 4.1 (Homotopy Colimit Formula). *Let $\mathcal{I}$ be a small category such that there exists a finite $\mathcal{I}$-CW-model for its classifying $\mathcal{I}$-space. Fix such a finite $\mathcal{I}$-CW-model $E\mathcal{I}$. Denote by Λ_n the finite set of n-cells $\lambda = \operatorname{mor}_{\mathcal{I}}(?, i_\lambda) \times D^n$ of $E\mathcal{I}$. Let $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ be a covariant functor. Abbreviate $\mathcal{H} = \operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$. Then:*

(i) *If $\mathcal{I}$ is directly finite, and $\mathcal{C}(i)$ is directly finite for every object $i \in \operatorname{ob}(\mathcal{I})$, then the category $\mathcal{H}$ is directly finite;*

(ii) *If $\mathcal{I}$ is an EI-category, $\mathcal{C}(i)$ is an EI-category for every object $i \in \operatorname{ob}(\mathcal{I})$, and for every automorphism $u\colon i \xrightarrow{\cong} i$ the map $\operatorname{iso}(\mathcal{C}(i)) \to \operatorname{iso}(\mathcal{C}(i))$, $\overline{x} \mapsto \overline{\mathcal{C}(u)(x)}$ is the identity, then the category $\mathcal{H}$ is an EI-category;*

(iii) *If for every object i the category $\mathcal{C}(i)$ is of type (FP_R), then the category $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ is of type (FP_R);*

(iv) *If for every object i the category $\mathcal{C}(i)$ is of type (FF_R), then the category $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ is of type (FF_R);*

(v) *If for every object i the category $\mathcal{C}(i)$ is of type (FP_R), then we obtain for the finiteness obstruction*

$$o(\mathcal{H};R) = \sum_{n\geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha(i_\lambda)_*(o(\mathcal{C}(i_\lambda);R)),$$

where $\alpha(i_\lambda)_*\colon K_0(R\mathcal{C}(i_\lambda)) \to K_0(R\mathcal{H})$ is the homomorphism induced by the canonical functor $\alpha(i_\lambda)\colon \mathcal{C}(i_\lambda) \to \mathcal{H}$ defined in (3.10)*;*

(vi) *Suppose that $\mathcal{I}$ is directly finite and $\mathcal{C}(i)$ is directly finite for every object $i \in \operatorname{ob}(\mathcal{I})$. If for every object i the category $\mathcal{C}(i)$ is additionally of type (FP_R) then we obtain for the functorial Euler characteristic*

$$\chi_f(\mathcal{H};R) = \sum_{n\geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha(i_\lambda)_*(\chi_f(\mathcal{C}(i_\lambda);R)),$$

where $\alpha(i_\lambda)_*\colon U(\mathcal{C}(i_\lambda)) \to U(\mathcal{H})$ is the homomorphism induced by the canonical functor $\alpha(i_\lambda)\colon \mathcal{C}(i_\lambda) \to \mathcal{H}$ defined in (3.10)*. Summing up,* *we also have*

$$\chi\big(\mathcal{H};R\big) = \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \chi(\mathcal{C}(i_\lambda);R).$$

If R is Noetherian, in addition to the direct finiteness and (FP$_R$) hypotheses, we obtain for the Euler characteristics of the classifying spaces

$$\chi\big(B\mathcal{H};R\big) = \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \chi(B\mathcal{C}(i_\lambda);R);$$

(vii) *Suppose that $\mathcal{I}$ is directly finite and $\mathcal{C}(i)$ is directly finite for every object $i \in \mathrm{ob}(\mathcal{I})$. If for every object i the category $\mathcal{C}(i)$ is additionally of type (L^2), then $\mathcal{H}$ is of type (L^2) and we obtain for the functorial L^2-Euler characteristic*

$$\chi_f^{(2)}(\mathcal{H}) = \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \alpha(i_\lambda)_*\big(\chi_f^{(2)}(\mathcal{C}(i_\lambda))\big),$$

where $\alpha(i_\lambda)_*\colon U^{(1)}(\mathcal{C}(i_\lambda)) \to U^{(1)}(\mathcal{H})$ is the homomorphism induced by the canonical functor $\alpha(i_\lambda)\colon \mathcal{C}(i_\lambda) \to \mathcal{H}$ defined in (3.10)*, and we obtain for the L^2-Euler characteristic*

$$\chi^{(2)}(\mathcal{H}) = \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \chi^{(2)}(\mathcal{C}(i_\lambda)).$$

Proof. (i) Consider morphisms $(u,f)\colon (i,c) \to (j,d)$ and $(v,g)\colon (j,d) \to (i,c)$ in $\mathcal{H}$ with $(v,g)\circ(u,f) = \mathrm{id}_{(i,c)}$. This implies $vu = \mathrm{id}_i$ and $g \circ \mathcal{C}(v)(f) = \mathrm{id}_c$. Since $\mathcal{I}$ and $\mathcal{C}(i)$ are by assumption directly finite, we conclude $uv = \mathrm{id}_j$ and $\mathcal{C}(v)(f)\circ g = \mathrm{id}_{\mathcal{C}(v)(d)}$. Hence

$$\begin{aligned}(u,f)\circ(v,g) &= \big(uv, f\circ\mathcal{C}(u)(g)\big) = \big(uv, \mathcal{C}(uv)(f)\circ\mathcal{C}(u)(g)\big)\\ &= \big(uv, \mathcal{C}(u)(\mathcal{C}(v)(f)\circ g)\big) = \big(uv, \mathcal{C}(u)(\mathrm{id}_{\mathcal{C}(v)(d)})\big)\\ &= \big(\mathrm{id}_j, \mathrm{id}_{\mathcal{C}(u)\left(\mathcal{C}(v)(d)\right)}\big) = (\mathrm{id}_j, \mathrm{id}_d).\end{aligned}$$

(ii) Consider an endomorphism $(u,f)\colon (i,c) \to (i,c)$ in $\mathcal{H}$. Since $\mathcal{I}$ is an EI-category, $u\colon i \to i$ is an automorphism. Since $\overline{\mathcal{C}(u)(c)} = \overline{c}$ by assumption, we can choose an isomorphism $g\colon c \xrightarrow{\cong} \mathcal{C}(u)(c)$. Hence fg is an endomorphism in $\mathcal{C}(i)$. Since $\mathcal{C}(i)$ is an EI-category, and g is an isomorphism, f is also an isomorphism. Since u and f are isomorphisms, (u,f) is an isomorphism.

(iii) and (v). We say that an $R\mathcal{H}$-chain complex C_* is of type (FP$_R$) if it admits a *finite projective approximation*, i.e., there is a finite length chain complex P_* of finitely generated, projective $R\mathcal{H}$-modules together with an $R\mathcal{H}$-chain map $f_*\colon P_* \to C_*$ such that $H_n(f_*(i,c))$ is bijective for all $n \geq 0$ and $(i,c) \in \mathrm{ob}(\mathcal{H})$. If C_* is of type (FP$_R$), define its finiteness obstruction

$$o(C_*) := \sum_{n\geq 0}(-1)^n \cdot [P_n] \quad \in K_0(R\mathcal{H})$$

for any choice P_* of finite projective approximation. This is independent of the choice of P_* and the basic properties of it were studied by Lück [23, Chapter 11]. If $0[\underline{R}]$ is the $R\mathcal{H}$-chain complex concentrated in dimension zero and given there by the constant $R\mathcal{H}$-module $\underline{R}$, then $\mathcal{H}$ is of type (FP_R) if and only if $0[\underline{R}]$ is of type (FP_R) and in this case

$$o(\mathcal{H};R) = o(0[\underline{R}]) \in K_0(R\mathcal{H}).$$

Consider a finite $\mathcal{I}$-CW-complex X. We want to show by induction over the dimension of X that the $R\mathcal{H}$-chain complex $C_*(X \otimes_{\mathcal{I}} E^{\mathcal{H}})$ is of type (FP_R) and satisfies

$$o\big(C_*(X \otimes_{\mathcal{I}} E^{\mathcal{H}})\big) = \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \alpha(i_\lambda)_*(o(\mathcal{C}(i_\lambda);R)),$$

where Λ_n denotes the set of n-cells of X and i_λ is the object at which the n-cell $\lambda = \mathrm{mor}_{\mathcal{I}}(?, i_\lambda) \times D^n$ of X is based.
The induction beginning, where X is the empty set, is obviously true. The induction step is done as follows. Let d be the dimension of X. Then X_d is obtained from X_{d-1} by a pushout of $\mathcal{I}$-spaces

$$\begin{array}{ccc}
\coprod_{\lambda\in\Lambda_d} \mathrm{mor}_{\mathcal{C}}(-,i_\lambda) \times S^{d-1} & \longrightarrow & X_{d-1} \\
\big\downarrow & & \big\downarrow \\
\coprod_{\lambda\in\Lambda_d} \mathrm{mor}_{\mathcal{C}}(-,i_\lambda) \times D^{d} & \longrightarrow & X = X_d.
\end{array}$$

Applying $- \otimes_{\mathcal{I}} E^{\mathcal{H}}$ to it yields, because $E^{\mathcal{H}}(i) = \alpha(i)_* E^{\mathrm{bar}}\big(\mathcal{C}(i)\big)$, a pushout of $\mathcal{H}$-spaces with a cofibration as left vertical arrow

$$\begin{array}{ccc}
\coprod_{\lambda\in\Lambda_d} \alpha(i_\lambda)_* E^{\mathrm{bar}}\big(\mathcal{C}(i_\lambda)\big) \times S^{d-1} & \longrightarrow & X_{d-1} \otimes_{\mathcal{I}} E^{\mathcal{H}} \\
\big\downarrow & & \big\downarrow \\
\coprod_{\lambda\in\Lambda_d} \alpha(i_\lambda)_* E^{\mathrm{bar}}\big(\mathcal{C}(i_\lambda)\big) \times D^{d} & \longrightarrow & X \otimes_{\mathcal{I}} E^{\mathcal{H}}.
\end{array}$$

In the sequel we can assume without loss of generality that $X_{d-1} \otimes_{\mathcal{I}} E^{\mathcal{H}}$ and $X \otimes_{\mathcal{I}} E^{\mathcal{H}}$ are $\mathcal{H}$-CW-complexes and the diagram above is a pushout of $\mathcal{H}$-CW-complexes, since this can be arranged by replacing them by homotopy equivalent $\mathcal{H}$-CW-complexes (see the proof of Lemma 3.12). We obtain an exact sequence of $R\mathcal{H}$-chain complexes

$$0 \to C_*(X_{d-1} \otimes_{\mathcal{I}} E^{\mathcal{H}}) \to C_*(X \otimes_{\mathcal{I}} E^{\mathcal{H}}) \to \bigoplus_{\lambda\in\Lambda_d} \Sigma^d C_*\big(\alpha(i_\lambda)_* E^{\mathrm{bar}}\mathcal{C}(i_\lambda)\big) \to 0.$$

Consider $\lambda \in \Lambda_d$. Since $\mathcal{C}(i_\lambda)$ is of type (FP_R), we can find a finite projective $R\mathcal{C}(i_\lambda)$-chain complex P_* whose homology is concentrated in dimension zero and given there by the constant $R\mathcal{C}(i_\lambda)$-module $\underline{R}$. Since $C_*(E^{\mathrm{bar}}\mathcal{C}(i_\lambda))$ is a projective $R\mathcal{C}(i_\lambda)$-chain complex with the same homology, there is an $R\mathcal{C}(i_\lambda)$-chain homotopy equivalence $f_*\colon P_* \xrightarrow{\simeq} C_*\big(E^{\mathrm{bar}}\mathcal{C}(i_\lambda)\big)$ (see Lück [23, Lemma 11.3 on page 213] and

$$o(\mathcal{C}(i_\lambda);R) = o(P_*) = \sum_{n\geq 0}(-1)^n\cdot[P_n] \in K_0(R\mathcal{C}(i_\lambda)).$$

Obviously

$$\alpha(i_\lambda)_*f_*\colon \alpha(i_\lambda)_*P_* \xrightarrow{\simeq} \alpha(i_\lambda)_*C_*\big(E^{\mathrm{bar}}\big(\mathcal{C}(i_\lambda)\big)\big) = C_*\big(\alpha(i_\lambda)_*E^{\mathrm{bar}}\mathcal{C}(i_\lambda)\big)$$

is an $R\mathcal{H}$-chain homotopy equivalence. Hence $C_*(\alpha(i_\lambda)_*E^{\mathrm{bar}}\mathcal{C}(i_\lambda))$ and, by the induction hypothesis, $C_*(X_{d-1}\otimes_{\mathcal{I}}E^{\mathcal{H}})$ are $R\mathcal{H}$-chain complexes of type (FP_R). We conclude from Lück [23, Lemma 11.3 on page 213] that $C_*(X\otimes_{\mathcal{I}}E^{\mathcal{H}})$ is of type (FP_R) and

$$o\big(C_*(X\otimes_{\mathcal{I}}E^{\mathcal{H}})\big) = o\big(C_*(X_{d-1}\otimes_{\mathcal{I}}E^{\mathcal{H}})\big) + \sum_{\lambda\in\Lambda_d} o\big(\Sigma^d\alpha(i_\lambda)_*C_*(E^{\mathrm{bar}}\mathcal{C}(i_\lambda))\big).$$

This implies together with the induction hypothesis applied to X_{d-1}

$$\begin{aligned} & o\big(C_*(X\otimes_{\mathcal{I}}E^{\mathcal{H}})\big) \\ = {} & \sum_{n=0}^{d-1}(-1)^n\cdot\sum_{\lambda\in\Lambda_n}\alpha(i_\lambda)_*(o(\mathcal{C}(i_\lambda);R)) + \sum_{\lambda\in\Lambda_d}(-1)^d\cdot\alpha(i_\lambda)_*(o(\mathcal{C}(i_\lambda);R)) \\ = {} & \sum_{n=0}^{d}(-1)^n\cdot\sum_{\lambda\in\Lambda_n}\alpha(i_\lambda)_*(o(\mathcal{C}(i_\lambda);R)). \end{aligned}$$

This finishes the induction step.
Assertions (iii) and (v) follow by taking $X = E\mathcal{I}$.

(iv) This proof is analogous to that of assertion (iii).

(vi) By (i) and (iii), the category $\mathcal{H}$ is directly finite and of type (FP_R). Then an application of $\mathrm{rk}_{R\mathcal{H}}$ to the formula for $o(\mathcal{H};R)$ in (v) yields the formula for $\chi_f(\mathcal{H};R)$ in (vi) by the naturality of rk_{R-} with respect to the functors $\alpha(i_\lambda)$ between directly finite categories, see Fiore–Lück–Sauer [16, Lemma 4.9].
An application of the augmentation homomorphism $\epsilon\colon U(\mathcal{H})\to\mathbb{Z}$ to the formula for $\chi_f(\mathcal{H};R)$ yields the formula for $\chi(\mathcal{H};R)$. We also use the naturality of the augmentation homomorphism, that is, the commutativity of diagram (4.5) in [16] for $F=\alpha(i_\lambda)$.
If R is additionally Noetherian, then Theorem 1.9 applies, and the Euler characteristics of the categories agree with the Euler characteristics of the classifying spaces.

(vii) The proofs for the functorial L^2-Euler characteristic and the L^2-Euler characteristic are somewhat more complicated since the property (L^2) is more general than (FP_R), and the L^2-Euler characteristic comes from the finiteness obstruction only in the case (FP_R). The proofs are variations of the proofs for assertions (iii) and (v). Instead of using Lück [23, Lemma 11.3 on page 213], we now use the basic properties of L^2-Euler characteristics for chain complexes of modules over group von Neumann algebras [16, Lemma 5.7]. For example, we use [16, Lemma 5.7 (iv)], which says for any injective group homomorphism $i\colon H \to G$ and $\mathcal{N}(H)$-chain complex C_*, we have $\chi^{(2)}(C_*) = \chi^{(2)}(\mathrm{ind}_{i_*} C_*)$, provided the sum of the L^2-Betti numbers of C_* is finite. The injectivity hypothesis is easily verified: for every object $i \in \mathrm{ob}(\mathcal{I})$ and object $x \in \mathcal{C}(i)$ the functor $\alpha(i)\colon \mathcal{C}(i) \to \mathcal{H}$ clearly induces an injection $\mathrm{aut}_{\mathcal{C}(i)}(x) \to \mathrm{aut}_{\mathcal{H}}(i,x)$. This finishes the proof of Theorem 4.1. □

Corollary 4.2. *Theorem 4.1 on homotopy colimits holds for pseudo functors* $\mathcal{D}\colon \mathcal{I} \to \mathsf{CAT}$.

Proof. We first remark that the pseudo functor $\mathcal{D}\colon \mathcal{I} \to \mathsf{CAT}$ is equivalent to a strict functor $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ in the following sense. As usual, we denote by $\mathrm{Hom}(\mathcal{I}, \mathsf{CAT})$ the strict 2-category of pseudo functors $\mathcal{I} \to \mathsf{CAT}$, pseudo natural transformations between them, and modifications. The pseudo functor $\mathcal{D}$ is equivalent to a strict functor $\mathcal{C}$ *as objects of the 2-category* $\mathrm{Hom}(\mathcal{I}, \mathsf{CAT})$. For example, we may take $\mathcal{C}$ to be the strict functor

$$i \mapsto \mathrm{mor}_{\mathrm{Hom}(\mathcal{I},\mathsf{CAT})}(\mathcal{I}(i,-), \mathcal{D}).$$

The equivalence between $\mathcal{C}$ and $\mathcal{D}$ in $\mathrm{Hom}(\mathcal{I}, \mathsf{CAT})$ has two useful consequences. Since

$$\mathrm{hocolim}_{\mathcal{I}}\colon\ \mathrm{Hom}(\mathcal{I}, \mathsf{CAT}) \to \mathsf{CAT}$$

is a strict 2-functor, it sends any equivalence between $\mathcal{C}$ and $\mathcal{D}$ to an equivalence in CAT between the categories $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{C}$ and $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{D}$. Another consequence of the equivalence between $\mathcal{C}$ and $\mathcal{D}$ is that for every $i \in \mathcal{I}$, the categories $\mathcal{C}(i)$ and $\mathcal{D}(i)$ are equivalent. With these observations we reduce Corollary 4.2 to Theorem 4.1.

(i) Suppose $\mathcal{D}(i)$ is directly finite for every $i \in \mathrm{ob}(\mathcal{I})$ and $\mathcal{I}$ is directly finite. Since direct finiteness is preserved under equivalence of categories by Fiore–Lück–Sauer [16, Lemma 3.2], and $\mathcal{C}(i)$ is equivalent to $\mathcal{D}(i)$, we see that $\mathcal{C}(i)$ is directly finite for every $i \in \mathrm{ob}(\mathcal{I})$. Hence $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{C}$ is directly finite by Theorem 4.1 (i). Since $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{D}$ is equivalent to $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{C}$, it is also directly finite, again by [16, Lemma 3.2].

(ii) Suppose that $\mathcal{I}$ is an EI-category, $\mathcal{D}(i)$ is an EI-category for every $i \in \mathrm{ob}(\mathcal{I})$, and for every automorphism $u\colon i \xrightarrow{\cong} i$ the map $\mathrm{iso}(\mathcal{D}(i)) \to \mathrm{iso}(\mathcal{D}(i))$, $\overline{y} \mapsto \overline{\mathcal{D}(u)(y)}$ is the identity. Since EI is preserved under equivalence of categories [16, Lemma 3.11], and $\mathcal{C}(i)$ is equivalent to $\mathcal{D}(i)$, we see $\mathcal{D}(i)$ is an EI-category. We claim that for each automorphism u, the functor $\mathcal{C}(u)$ also induces the identity on isomorphism classes of objects of $\mathcal{C}(i)$. Let $\phi\colon \mathcal{D} \to \mathcal{C}$ be a pseudo equivalence, that is, an equivalence in the 2-category $\mathrm{Hom}(\mathcal{I}, \mathsf{CAT})$. For $x \in \mathcal{C}(i)$, there is a $y \in \mathcal{D}(i)$ and an isomorphism $x \cong \phi_i(y)$. We have isomorphisms

$$\mathcal{C}(u)(x) \cong \mathcal{C}(u)\phi_i(y) \cong \phi_i\mathcal{D}(u)(y) \cong \phi_i(y) \cong x,$$

and $\mathcal{C}(u)$ induces the identity on isomorphism classes of objects of $\mathcal{C}(i)$. Then $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{C}$ is an EI-category by Theorem 4.1 (ii), and so is $\mathrm{hocolim}_{\mathcal{I}}\,\mathcal{D}$, again by [16, Lemma 3.11].

(iii) and (iv) similarly follow from Theorem 4.1 (iii) and (iv), since property (FP_R), property (FF_R), and the finiteness obstruction are all invariant under equivalence of categories [16, Theorem 2.8].

(v) Suppose $\mathcal{D}(i)$ is of type (FP_R) for every $i \in \mathrm{ob}(\mathcal{I})$. Then every $\mathcal{C}(i)$ is also of type (FP_R), since property (FP_R) is invariant under equivalence of categories [16, Theorem 2.8]. As in (3.10), we have for each $i \in \mathcal{I}$ the functor

$$\alpha^{\mathcal{D}}(i)\colon \mathcal{D}(i) \to \mathrm{hocolim}_{\mathcal{I}}\, \mathcal{D}$$

which sends an object d to the object (i, d) and a morphism $g\colon d \to d'$ to the morphism $(\mathrm{id}_i, g \circ \mathcal{D}_i^{-1}(d))$. From a pseudo equivalence $\psi\colon \mathcal{C} \to \mathcal{D}$ we obtain a strictly commutative diagram

$$\begin{tikzcd} \mathcal{C}(i) \arrow[r, "\alpha^{\mathcal{C}}(i)"] \arrow[d, "\psi_i"'] & \mathrm{hocolim}_{\mathcal{I}}\, \mathcal{C} \arrow[d, "\mathrm{hocolim}_{\mathcal{I}}\, \psi"] \\ \mathcal{D}(i) \arrow[r, "\alpha^{\mathcal{D}}(i)"'] & \mathrm{hocolim}_{\mathcal{I}}\, \mathcal{D} \end{tikzcd} \tag{4.3}$$

for each $i \in \mathrm{ob}(\mathcal{I})$. Since the finiteness obstruction is invariant under equivalence of categories [16, Theorem 2.8], we may use Theorem 4.1 (v) for $\mathcal{C}$ to obtain

$$\begin{aligned} o(\mathrm{hocolim}_{\mathcal{I}}\, \mathcal{D}; R) &= (\mathrm{hocolim}_{\mathcal{I}}\, \psi)_*(o(\mathrm{hocolim}_{\mathcal{I}}\, \mathcal{C}; R)) \\ &= (\mathrm{hocolim}_{\mathcal{I}}\, \psi)_* \left(\sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha^{\mathcal{C}}(i_\lambda)_*(o(\mathcal{C}(i_\lambda); R)) \right) \\ &= \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} (\mathrm{hocolim}_{\mathcal{I}}\, \psi)_* \circ \alpha^{\mathcal{C}}(i_\lambda)_*(o(\mathcal{C}(i_\lambda); R)) \\ &= \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha^{\mathcal{D}}(i_\lambda)_* \circ (\psi_{i_\lambda})_*(o(\mathcal{C}(i_\lambda); R)) \\ &= \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \alpha^{\mathcal{D}}(i_\lambda)_*(o(\mathcal{D}(i_\lambda); R)). \end{aligned}$$

(vi) follows from (i), (iii), and (v) in the same way that Theorem 4.1 (vi) follows from Theorem 4.1 (ii), (iii), and (v).

(vii) Suppose that $\mathcal{I}$ is directly finite and $\mathcal{D}(i)$ is directly finite for every object $i \in \mathrm{ob}(\mathcal{I})$. Suppose also for every object $i \in \mathcal{I}$ the category $\mathcal{D}(i)$ is of type (L^2). By the proof of Corollary 4.2 (i) above, the values of the strict functor $\mathcal{C}$ are directly finite categories. If Γ_1 and Γ_2 are equivalent categories, then Γ_1 is both directly finite and of type (L^2) if and only if Γ_2 is both directly finite and of type (L^2) [16, Lemma 5.15 (i)]. Since each $\mathcal{D}(i)$ is directly finite, of type (L^2), and equivalent to $\mathcal{C}(i)$, we see that each $\mathcal{C}(i)$ is also directly finite and of type (L^2). So we may now apply Theorem 4.1 (i) and (vii) to $\mathcal{C}$ and conclude that $\mathrm{hocolim}_{\mathcal{I}}\, \mathcal{C}$ is directly finite and of type (L^2). Again using the preservation of the direct finiteness and (L^2) under equivalence [16, Lemma 5.15 (i)], and the

equivalence of $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ with $\operatorname{hocolim}_{\mathcal{I}} \mathcal{D}$, we see $\operatorname{hocolim}_{\mathcal{I}} \mathcal{D}$ is both directly finite and of type (L^2).
To prove the formulas for $\chi_f^{(2)}$ and $\chi^{(2)}$, we use [16, Lemma 5.15 (ii)], which says: if $F\colon \Gamma_1 \to \Gamma_2$ is an equivalence of categories, and both Γ_1 and Γ_2 are both directly finite and of type (L^2), then $U^{(1)}(F)\chi_f^{(2)}(\Gamma_1) = \chi_f^{(2)}(\Gamma_2)$ and $\chi^{(2)}(\Gamma_1) = \chi^{(2)}(\Gamma_2)$. We apply this to the equivalences ψ_i and $\operatorname{hocolim}_{\mathcal{I}} \psi$, and use the commutativity of diagram (4.3). For readability, we write $(\operatorname{hocolim}_{\mathcal{I}} \psi)_*$ for $U(\operatorname{hocolim}_{\mathcal{I}} \psi)$ and $\alpha(i_\lambda)_*$ for $U^{(1)}(\alpha(i_\lambda))$, et cetera.

$$\begin{aligned}
\chi_f^{(2)}(\operatorname{hocolim}_{\mathcal{I}} \mathcal{D}) &= (\operatorname{hocolim}_{\mathcal{I}} \psi)_* \chi_f^{(2)}(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}) \\
&= (\operatorname{hocolim}_{\mathcal{I}} \psi)_* \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \alpha(i_\lambda)_*\big(\chi_f^{(2)}(\mathcal{C}(i_\lambda))\big) \\
&= \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} (\operatorname{hocolim}_{\mathcal{I}} \psi)_* \circ \alpha^{\mathcal{C}}(i_\lambda)_*\big(\chi_f^{(2)}(\mathcal{C}(i_\lambda))\big) \\
&= \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \alpha^{\mathcal{D}}(i_\lambda)_* \circ (\psi_{i_\lambda})_*\big(\chi_f^{(2)}(\mathcal{C}(i_\lambda))\big) \\
&= \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \alpha^{\mathcal{D}}(i_\lambda)_*\big(\chi_f^{(2)}(\mathcal{D}(i_\lambda))\big).
\end{aligned}$$

The formula for $\chi^{(2)}$ follows by summing up the components of the functorial L^2-Euler characteristics. □

4.2. The Case of an Indexing Category of Type (FP_R). The Homotopy Colimit Formula of Theorem 4.1 can be extended to the case, where $\mathcal{I}$ is of type (FP_R) and not necessarily of type (FF_R) as follows (recall that the existence of a finite $\mathcal{I}$-CW-model for $E\mathcal{I}$ implies $\mathcal{I}$ is of type (FF_R), since cellular chains then provide a finite free resolution of $\underline{R}$.). The evaluation of the covariant functor

$$E^{\mathcal{H}}\colon \mathcal{I} \to \mathcal{H}\text{-}\mathsf{SPACES}$$

of (3.9) at every object $i \in \mathcal{I}$ is an $\mathcal{H}$-CW-complex. Composing it with the cellular chain complex functor yields a covariant functor

$$C_*(E^{\mathcal{H}})\colon \mathcal{I} \to R\mathcal{H}\text{-}\mathsf{CHCOM}$$

whose evaluation at every object in $\mathcal{I}$ is a free $R\mathcal{H}$-chain complexes. Since by assumption $\mathcal{C}(i)$ is of type (FP_R), $C_*(E^{\mathcal{H}})(i)$ is $R\mathcal{H}$-chain homotopy equivalent to a finite projective $R\mathcal{H}$-chain complex for every object $i \in \mathcal{I}$. Since $R\operatorname{mor}_{\mathcal{I}}(?, i) \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})$ is $R\mathcal{H}$-isomorphic to $C_*(E^{\mathcal{H}})$, we conclude for every finitely generated projective $R\Gamma$-module P that $P \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})$ is $R\mathcal{H}$-chain homotopy equivalent to finite projective $R\mathcal{H}$-chain complex and in particular possesses a finiteness obstruction $o\big(P \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}}\big) \in K_0(R\mathcal{H})$ (see Lück [23, Theorem 11.2 on page 212]). Because of Lück [23, Theorem 11.2 on page 212] we obtain a homomorphism

$$\alpha_{\mathcal{C}}\colon K_0(R\mathcal{I}) \to K_0(R\mathcal{H}), \quad [P] \mapsto o\big(P \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})\big).$$

The chain complex version of the proof of Lemma 3.12 shows that the $R\mathcal{H}$-chain complex $C_*(\mathcal{I}) \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})$ is a projective $R\mathcal{H}$-resolution of the constant $R\Gamma$-module $\underline{R}$. Choose a finite projective $R\mathcal{I}$-chain complex P_* and an $R\mathcal{I}$-chain homotopy equivalence $f_* \colon P_* \xrightarrow{\simeq} C_*(\mathcal{I})$. Then $f_* \otimes_{R\mathcal{I}} \mathrm{id} \colon P_* \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}}) \to C_*(\mathcal{I}) \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})$ is an $R\Gamma$-chain homotopy equivalence of $R\Gamma$-chain complexes and $P_* \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})$ is is $R\mathcal{H}$-chain homotopy equivalent to finite projective $R\mathcal{H}$-chain complex by Lück [23, Theorem 11.2 on page 212]. This implies

$$o(\Gamma; R) = o\big(P_* \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})\big).$$

We conclude from [23, Theorem 11.2 on page 212]

$$o\big(P_* \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})\big) = \sum_{n \geq p} (-1)^n \cdot o\big(P_n \otimes_{R\mathcal{I}} C_*(E^{\mathcal{H}})\big)$$

Since $o(\mathcal{I}; R)$ is $\sum_{n \geq p} (-1)^n \cdot [P_n]$, this implies

Theorem 4.4 (The Homotopy Colimit Formula for an indexing category of type (FP_R)). *We obtain under the conditions above*

$$\alpha_{\mathcal{C}}\big(o(\mathcal{I}; R)\big) = o(\mathcal{H}; R).$$

Remark 4.5. See Section 7 for a comparison with Leinster's Euler characteristic and his results.

5. Examples of the Homotopy Colimit Formula

We now present several examples of the Homotopy Colimit Formula Theorem 4.1. These include the cases: $\mathcal{I}$ with a terminal object, the constant functor, the trivial functor, homotopy pushouts, homotopy orbits, and the transport groupoid. For the transport groupoid in the finite case, see also Example 8.33.

Example 5.1 (Homotopy Colimit Formula for $\mathcal{I}$ with a terminal object). Suppose that $\mathcal{I}$ has a terminal object t and $\mathcal{C} \colon \mathcal{I} \to \mathsf{CAT}$ is a functor. Then $\mathrm{mor}_{\mathcal{I}}(-, t)$ is a finite $\mathcal{I}$-CW model for $E\mathcal{I}$. If every category $\mathcal{C}(i)$ is of type (FP_R), then $o(\mathcal{H}; R) = \alpha(t)_*(o(\mathcal{C}(t); R)$. If $\mathcal{I}$ and $\mathcal{C}$ additionally satisfy the hypotheses of Theorem 4.1 (vi), then $\chi_f(\mathcal{H}; R) = \chi_f(\mathcal{C}(t); R)$ and $\chi(\mathcal{H}; R) = \chi(\mathcal{C}(t); R)$, as we anticipated in Example 3.6. Similar statements hold for $\chi_f^{(2)}$ and $\chi^{(2)}$ in the L^2 case.

Example 5.2 (Homotopy Colimit Formula for a constant functor). Consider the situation of Theorem 4.1 in the special case where the covariant functor $\mathcal{C} \colon \mathcal{I} \to \mathsf{CAT}$ is constant $\mathcal{C} \in \mathsf{CAT}$. Suppose that $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model for $E\mathcal{I}$. Then we may draw various conclusions about the homotopy colimit $\mathcal{H} = \mathcal{I} \times \mathcal{C}$. If $\mathcal{I}$ and $\mathcal{C}$ are of type (FP_R), then so is $\mathcal{I} \times \mathcal{C}$. If $\mathcal{I}$ and $\mathcal{C}$ are of type (FF_R), then so is $\mathcal{I} \times \mathcal{C}$. The statements in Theorem 4.1 provide us with formulas in terms of $\mathcal{C}$ for $o(\mathcal{I} \times \mathcal{C}; R)$, $\chi_f(\mathcal{I} \times \mathcal{C}; R)$, $\chi(\mathcal{I} \times \mathcal{C}; R)$, $\chi_f^{(2)}(\mathcal{I} \times \mathcal{C})$, and $\chi^{(2)}(\mathcal{I} \times \mathcal{C})$. We recall that the invariants o, χ_f, χ, $\chi_f^{(2)}$, and $\chi^{(2)}$ are multiplicative, see Fiore–Lück–Sauer [16, Theorems 2.17, 4.22, and 5.17].

Example 5.3 (Homotopy Colimit Formula for the trivial functor). Consider the situation of Theorem 4.1 in the special case where the covariant functor $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ is constantly the terminal category, which consists of a single object and its identity morphism. Then $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ agrees with $\mathcal{I}$, as we see from Example 3.5. Obviously $\mathcal{C}(i)$ is of type (FF_R), its finiteness obstruction is $[R] \in K_0(R) = K_0(R\mathcal{C}(i))$ and both its Euler characteristic and L^2-Euler characteristic equals 1. We obtain from Theorem 4.1

$$\begin{array}{lcll}
o(\mathcal{I};R) & = & \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} [R\operatorname{mor}_{\mathcal{I}}(?,i_\lambda)] & \in K_0(R\mathcal{I});\\
\chi_f(\mathcal{I};R) & = & \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \overline{i_\lambda} & \in U(\Gamma);\\
\chi(\mathcal{I};R) & = & \sum_{n\geq 0}(-1)^n \cdot |\Lambda_n| & \in \mathbb{Z};\\
\chi_f^{(2)}(\mathcal{I}) & = & \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \overline{i_\lambda} & \in U^{(1)}(\mathcal{I});\\
\chi^{(2)}(\mathcal{I}) & = & \sum_{n\geq 0}(-1)^n \cdot |\Lambda_n| & \in \mathbb{R}.
\end{array}$$

Example 5.4 (Homotopy pushout formula). Let $\mathcal{I}$ be the category with objects j, k and ℓ such that there is precisely one morphism from j to k and from j to ℓ and all other morphisms are identity morphisms.

$$\mathcal{I} = \{\ k \xleftarrow{g} j \xrightarrow{h} \ell\ \}$$

By Example 2.6, the category $\mathcal{I}$ admits a finite model for the classifying $\mathcal{I}$-space $E\mathcal{I}$.

A covariant functor $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ is the same as specifying three categories $\mathcal{C}(j)$, $\mathcal{C}(k)$ and $\mathcal{C}(\ell)$ and two functors $\mathcal{C}(g)\colon \mathcal{C}(j) \to \mathcal{C}(k)$ and $\mathcal{C}(h)\colon \mathcal{C}(j) \to \mathcal{C}(\ell)$. Let $\mathcal{H} = \operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ be the homotopy colimit. Let $\alpha(i)\colon \mathcal{C}(i) \to \mathcal{H}$ be the canonical functor for $i = j,k,\ell$. Then we obtain a square of functors which commutes up to natural transformations

$$\begin{tikzcd}
\mathcal{C}(j) \arrow[r, "\mathcal{C}(g)"] \arrow[d, "\mathcal{C}(h)"'] \arrow[dr, "\alpha(j)"] & \mathcal{C}(k) \arrow[d, "\alpha(k)"] \\
\mathcal{C}(\ell) \arrow[r, "\alpha(\ell)"'] & \mathcal{H}.
\end{tikzcd}$$

It induces diagrams which do NOT commute in general

$$\begin{tikzcd}
K_0(R\mathcal{C}(j)) \arrow[r, "\mathcal{C}(g)_*"] \arrow[d, "\mathcal{C}(h)_*"'] \arrow[dr, "\alpha(j)_*"] & K_0(R\mathcal{C}(k)) \arrow[d, "\alpha(k)_*"] \\
K_0(R\mathcal{C}(\ell)) \arrow[r, "\alpha(\ell)_*"'] & K_0(\mathcal{H})
\end{tikzcd}$$

and

$$\begin{tikzcd} U(\mathcal{C}(j)) \arrow[r, "\mathcal{C}(g)_*"] \arrow[d, "\mathcal{C}(h)_*"'] \arrow[dr, "\alpha(j)_*"] & U(R\mathcal{C}(k)) \arrow[d, "\alpha(k)_*"] \\ U(R\mathcal{C}(\ell)) \arrow[r, "\alpha(\ell)_*"'] & U(\mathcal{H}). \end{tikzcd}$$

Suppose that $\mathcal{C}(i)$ is of type (FP_R) for $i = j, k, \ell$. We conclude from Theorem 4.1 (iii) that $\mathcal{H}$ is of type (FP_R) and

$$\begin{array}{lcl} o(\mathcal{H};R) & = & \alpha(k)_*\big(o(\mathcal{C}(k);R)) + \alpha(\ell)_*\big(o(\mathcal{C}(\ell);R)) - \alpha(j)_*\big(o(\mathcal{C}(j;R)) \\ & & \in K_0(R\mathcal{H}); \\ \chi_f(\mathcal{H};R) & = & \alpha(k)_*\big(\chi_f(\mathcal{C}(k);R)\big) + \alpha(\ell)_*\big(\chi_f(\mathcal{C}(\ell);R)\big) - \alpha(j)_*\big(\chi_f(\mathcal{C}(j);R)\big) \\ & & \in U(\mathcal{H}); \\ \chi(\mathcal{H};R) & = & \chi(\mathcal{C}(k);R) + \chi(\mathcal{C}(\ell);R) - \chi(\mathcal{C}(j);R) \\ & & \in \mathbb{Z}; \\ \chi_f^{(2)}(\mathcal{H}) & = & \alpha(k)_*\big(\chi_f^{(2)}(\mathcal{C}(k)\big) + \alpha(\ell)_*\big(\chi_f^{(2)}(\mathcal{C}(\ell))\big) - \alpha(j)_*\big(\chi_f^{(2)}(\mathcal{C}(j))\big) \\ & & \in U^{(1)}(\mathcal{H}); \\ \chi^{(2)}(\mathcal{H}) & = & \chi^{(2)}(\mathcal{C}(k)) + \chi^{(2)}(\mathcal{C}(\ell)) - \chi^{(2)}(\mathcal{C}(j)) \\ & & \in \mathbb{R}. \end{array}$$

Example 5.5 (Homotopy orbit formula). Suppose that a group G acts on a category $\mathcal{C}$ from the left. This can be viewed as a covariant functor $\widehat{G} \to \mathsf{CAT}$ whose source is the groupoid $\widehat{G}$ with one object and G as its automorphism group. Let $\mathcal{H} = \operatorname{hocolim}_{\widehat{G}} \mathcal{C}$ be its homotopy colimit, also called the *homotopy orbit*. Notice that $\mathcal{H}$ and $\mathcal{C}$ have the same set of objects.
Suppose there is a finite model for BG of the group G, or equivalently, a finite model for the $\widehat{G}$-classifying space $E\widehat{G}$ of the category $\widehat{G}$. Let $\chi(BG) \in \mathbb{Z}$ be its Euler characteristic. Let $\alpha\colon \mathcal{C} \to \mathcal{H}$ be the canonical inclusion. Suppose that $\mathcal{C}$ is of type (FP_R). Then we conclude from Theorem 4.1 (iii) that $\mathcal{H}$ is of type (FP_R) and we have

$$\begin{array}{lcll} o(\mathcal{H};R) & = & \chi(BG)\cdot\alpha_*\big(o(\mathcal{C};R)\big) & \in K_0(R\mathcal{H}); \\ \chi_f(\mathcal{H};R) & = & \chi(BG)\cdot\alpha_*\big(\chi_f(\mathcal{C};R)\big) & \in U(\mathcal{H}); \\ \chi(\mathcal{H};R) & = & \chi(BG)\cdot\chi(\mathcal{C};R) & \in \mathbb{Z}; \\ \chi_f^{(2)}(\mathcal{H};R) & = & \chi(BG)\cdot\alpha_*\big(\chi_f^{(2)}(\mathcal{C};R)\big) & \in U^{(1)}(\mathcal{H}); \\ \chi^{(2)}(\mathcal{H};R) & = & \chi(BG)\cdot\chi^{(2)}(\mathcal{C};R) & \in \mathbb{R}. \end{array}$$

Example 5.6 (Transport groupoid). Let G be a group and let S be a left G-set. Its *transport groupoid* $\mathcal{G}^G(S)$ has S as its set of objects. The set of morphisms from s_1 to s_2 is $\{g \in G \mid gs_1 = s_2\}$. The composition is given by the multiplication in G. Denote by $\underline{S}$ the category whose set of objects is S and which has no morphisms besides the identity morphisms. The group G acts from the left on $\underline{S}$. One easily checks that $\mathcal{G}^G(S)$ is the homotopy orbit of $\underline{S}$ defined in Example 5.5.
Recall from Fiore–Lück–Sauer [16, Lemma 6.15 (iv)]: if Γ is a quasi-finite EI-category and for any morphism $f\colon x \to y$ in Γ, the order of the finite group $\{g \in \operatorname{aut}(x) \mid f \circ g = f\}$ is invertible in R, then Γ is of type (FP_R) if and only

if $\mathrm{iso}(\Gamma)$ is finite and for every object $x \in \mathrm{ob}(\Gamma)$ the trivial $R[x]$-module R is of type (FP_R). Thus, category $\underline{S}$ is of type (FP_R) if and only if S is finite. Suppose that $\underline{S}$ is of type (FP_R) and there is a finite model for BG. Obviously $o(\underline{S};R)$ is given in $K_0(R\underline{S}) = \oplus_S K_0(R)$ by the collection $\{[R] \in K_0(R) \mid s \in S\}$.
Suppose for simplicity that G acts transitively on S. Fix an element $s \in S$. Let G_s be its isotropy group. Since S is finite, G_s is a subgroup of G of finite index, namely $[G : G_s] = |S|$. The transport groupoid $\mathcal{G}^G(S)$ is connected and the automorphism group of s is G_s. Hence evaluation at s induces an isomorphism

$$\mathrm{ev}\colon K_0(R\mathcal{G}^G(S)) \xrightarrow{\cong} K_0(R[G_s]).$$

The composition

$$K_0(R\underline{S}) \xrightarrow{\alpha_*} K_0(R\mathcal{G}^G(S)) \xrightarrow{\cong} K_0(R[G_s])$$

sends $o(\underline{S};R)$ to $|S| \cdot [RG_s]$, where $\alpha\colon \underline{S} \to \mathcal{G}^G(S)$ is the obvious inclusion. Hence Example 5.5 implies

$$\mathrm{ev}\big(o(\mathcal{G}^G(S);R)\big) = \chi(BG) \cdot |S| \cdot [RG_S] \quad \in K_0(RG_s).$$

Since BG has a finite model, BG_s as a finite covering of BG has a finite model. The cellular RG_s-chain complex of EG_s yields a finite free resolution of the trivial RG_s-module R. This implies

$$\mathrm{ev}\big(o(\mathcal{G}^G(S);R)\big) = \chi(BG_s) \cdot [RG_s] \quad \in K_0(RG_s).$$

Hence we obtain the equality in $K_0(RG_s)$

$$\chi(BG_s) \cdot [RG_s] = \chi(BG) \cdot |S| \cdot [RG_S] = \chi(BG) \cdot [G : G_s] \cdot [RG_s].$$

This is equivalent to the equality of integers

$$\chi(BG_s) = \chi(BG) \cdot [G : G_s].$$

This equation is compatible with the well-know fact that for a d-sheeted covering $\overline{X} \to X$ of a finite CW-complex X the total space $\overline{X}$ is again a finite CW-complex and we have $\chi(\overline{X}) = d \cdot \chi(X)$.
For the transport groupoid in the finite case, see also Example 8.33.

6. Combinatorial Illustrations of the Homotopy Colimit Formula

The classical Inclusion-Exclusion Principle follows from the Homotopy Colimit Formula Theorem 4.1. We can also easily calculate the cardinality of a coequalizer in SETS in certain cases. These are different proofs of Examples 3.4.d and 3.4.b of Leinster's paper [21].

Example 6.1 (Inclusion-Exclusion Principle). Let X be a set and $S_0, \ldots, S_q$ finite subsets of X. Then

$$|S_0 \cup S_1 \cup \cdots \cup S_q| = \sum_{\emptyset \neq J \subseteq [q]} (-1)^{|J|-1} \cdot \left| \bigcap_{j \in J} S_j \right|.$$

Proof. Let $\mathcal{I}$ be the category in Example 2.7 and consider the finite $\mathcal{I}$-CW-model for its classifying $\mathcal{I}$-space constructed there. We define a functor $\mathcal{C} : \mathcal{I} \to \mathsf{SETS}$ by $\mathcal{C}(J) := \bigcap_{j \in J} S_j$. The functor

$$\operatorname{hocolim}_{\mathcal{I}} \mathcal{C} \longrightarrow \operatorname{colim}_{\mathcal{I}} \mathcal{C} = S_0 \cup S_1 \cup \cdots \cup S_q$$

is an equivalence of categories, since it is surjective on objects and fully faithful. We have

$$\begin{aligned}
|S_0 \cup S_1 \cup \cdots \cup S_q| &= \chi(S_0 \cup S_1 \cup \cdots \cup S_q)\\
&= \chi(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C})\\
&= \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \chi(\mathcal{C}(i_\lambda))\\
&= \sum_{n \geq 0} (-1)^n \cdot \sum_{J \subseteq [q] \text{ and } |J| = n+1} \chi(\mathcal{C}(J))\\
&= \sum_{n \geq 0} (-1)^n \left(\sum_{J \subseteq [q] \text{ and } |J| = n+1} \left| \bigcap_{j \in J} S_j \right| \right)\\
&= \sum_{\emptyset \neq J \subseteq [q]} \left((-1)^{|J|-1} \left| \bigcap_{j \in J} S_j \right| \right).
\end{aligned}$$

□

Example 6.2 (Cardinality of a Coequalizer). Let $\mathcal{I}$ be the category

$$a \underset{g}{\overset{f}{\rightrightarrows}} b$$

and $\mathcal{C} : \mathcal{I} \to \mathsf{SETS}$ a functor such that:

(i) the maps $\mathcal{C}f$ and $\mathcal{C}g$ are injective,
(ii) the images of the maps $\mathcal{C}f$ and $\mathcal{C}g$ are disjoint, and
(iii) the sets $\mathcal{C}a$ and $\mathcal{C}b$ are finite.

Then the coequalizer $\operatorname{colim} \mathcal{C}$ has cardinality $|\mathcal{C}b| - |\mathcal{C}a|$.

Proof. The assumptions that $\mathcal{C}f$ and $\mathcal{C}g$ are injective and have disjoint images imply that the functor

$$\operatorname{hocolim}_{\mathcal{I}} \mathcal{C} \longrightarrow \operatorname{colim}_{\mathcal{I}} \mathcal{C}$$

is fully faithful. Clearly it is also surjective on objects, and hence an equivalence of categories. The category $\mathcal{I}$ has a finite $\mathcal{I}$-CW-model for its classifying $\mathcal{I}$-space, constructed explicitly in Example 2.5. By Theorem 4.1, we have

$$\begin{aligned}
\chi(\operatorname{colim}_{\mathcal{I}} \mathcal{C}) &= \chi(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C})\\
&= \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \chi(\mathcal{C}(i_\lambda))\\
&= \chi(\mathcal{C}b) - \chi(\mathcal{C}a)\\
&= |\mathcal{C}b| - |\mathcal{C}a|.
\end{aligned}$$

□

7. Comparison with Results of Baez–Dolan and Leinster

We recall Baez–Dolan's groupoid cardinality [4] and Leinster's Euler characteristic for certain finite categories [21], compare our Homotopy Colimit Formula with his result on compatibility with Grothendieck fibrations, prove an analogue for indexing categories $\mathcal{I}$ that admit finite $\mathcal{I}$-CW-models for their classifying $\mathcal{I}$-spaces, and finally mention a Homotopy Colimit Formula for Leinster's invariant in a restricted case.

7.1. Review of Leinster's Euler Characteristic. Let Γ be a category with finitely many objects and finitely many morphisms. A *weighting* on Γ is a function $q^\bullet\colon \operatorname{ob}(\Gamma) \to \mathbb{Q}$ such that for all objects $x \in \operatorname{ob}(\Gamma)$, we have

$$\sum_{y\in\operatorname{ob}(\Gamma)} |\operatorname{mor}_\Gamma(x,y)| \cdot q^y = 1.$$

A *coweighting* $q_\bullet$ on Γ is a weighting on Γ^{op}. If a finite category admits both a weighting $q^\bullet$ and a coweighting $q_\bullet$, then $\sum_{y\in\operatorname{ob}(\Gamma)} q^y = \sum_{x\in\operatorname{ob}(\Gamma)} q_x$. For a discusion of which matrices have the form $(|\operatorname{mor}_\Gamma(x,y)|)_{x,y\in\operatorname{ob}(\Gamma)}$ for some finite category Γ, see Allouch [2] and [3].
As proved in [16], free resolutions of the constant $R\Gamma$-module $\underline{R}$ give rise to weightings on Γ.

Theorem 7.1 (Weighting from a free resolution, Theorem 7.6 of Fiore–Lück–Sauer [16]). *Let Γ be a finite category. Suppose that the constant $R\Gamma$-module $\underline{R}$ admits a finite free resolution P_*. If P_n is free on the finite $\operatorname{ob}(\Gamma)$-set C_n, that is*

$$P_n = B(C_n) = \bigoplus_{y\in\operatorname{ob}(\Gamma)} \bigoplus_{C_n^y} R\operatorname{mor}_\Gamma(?,y), \tag{7.2}$$

then the function $q^\bullet\colon \operatorname{ob}(\Gamma) \to \mathbb{Q}$ defined by

$$q^y := \sum_{n\geq 0} (-1)^n \cdot |C_n^y|$$

is a weighting on Γ.

Corollary 7.3 (Construction of a weighting from a finite $\mathcal{I}$-CW-model for the classifying $\mathcal{I}$-space, Corollary 7.8 of Fiore–Lück–Sauer [16]). *Let $\mathcal{I}$ be a finite category. Suppose that $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model X for the classifying $\mathcal{I}$-space. Then the function $q^\bullet\colon \operatorname{ob}(\mathcal{I}) \to \mathbb{Q}$ defined by*

$$q^y := \sum_{n\geq 0} (-1)^n (\textit{number of } n\textit{-cells of } X \textit{ based at } y)$$

is a weighting on $\mathcal{I}$.

As explained in Section 7.5 of [16], we use this Corollary to obtain several of Leinster's weightings in [21] from $\mathcal{I}$-CW-models for $\mathcal{I}$-classifying spaces. If $\mathcal{I}$ has a terminal object, then we obtain from the finite model in Example 2.4 the weighting which is 1 on the terminal object and 0 otherwise. The category $\mathcal{I} = \{j \rightrightarrows k\}$ in Example 2.5 has weighting $(q^j, q^k) = (-1, 1)$. The category $\mathcal{I} = \{k \leftarrow j \rightarrow \ell\}$ in Example 2.6 has weighting $(q^j, q^k, q^\ell) = (-1, 1, 1)$. Lastly, the category in Example 2.7 has weighting $q^J := (-1)^{|J|-1}$.
Weightings and coweightings play a key role in Leinster's notion of Euler characteristic. See also Berger–Leinster [9].

Definition 7.4 (Definition 2.2 of Leinster [21]). A finite category Γ *has an Euler characteristic in the sense of Leinster* if it admits both a weighting and a coweighting. In this case, its *Euler characteristic in the sense of Leinster* is defined as

$$\chi_L(\Gamma) := \sum_{y \in \mathrm{ob}(\Gamma)} q^y = \sum_{x \in \mathrm{ob}(\Gamma)} q_x$$

for any choice of weighting $q^\bullet$ or coweighting $q_\bullet$.

The Euler characteristic of Leinster agrees with the *groupoid cardinality* of Baez–Dolan [4] in the case of a finite groupoid $\mathcal{G}$, namely they are both

$$\sum_{\overline{x} \in \mathrm{iso}(\mathcal{G})} \frac{1}{|\operatorname{aut}_{\mathcal{G}}(x)|}.$$

The Euler characteristic of Leinster agrees with our L^2-Euler characteristic in some cases, as in the following Lemma.

Lemma 7.5 (Lemma 7.3 of Fiore–Lück–Sauer [16]). *Let Γ be a finite EI-category which is skeletal, i.e., if two objects are isomorphic, then they are equal. Suppose that the left $\operatorname{aut}_\Gamma(y)$-action on $\operatorname{mor}_\Gamma(x, y)$ is free for every two objects $x, y \in \mathrm{ob}(\Gamma)$.*
Then Γ is of type (FP$_{\mathbb{C}}$) and of type (L^2), and has an Euler characteristic in the sense of Leinster. Furthermore, the L^2-Euler characteristic $\chi^{(2)}(\Gamma)$ of Definition 1.11 coincides with Leinster's Euler characteristic $\chi_L(\Gamma)$ of Definition 7.4:

$$\chi^{(2)}(\Gamma) = \chi_L(\Gamma).$$

Moreover, these are both equal to

$$\sum_{l \geq 0} (-1)^l \cdot \sum_{x_0, x_l \in \mathrm{ob}(\Gamma)} \sum \frac{1}{|\operatorname{aut}(x_l)| \cdot |\operatorname{aut}(x_{l-1})| \cdot \cdots \cdot |\operatorname{aut}(x_0)|},$$

where the inner sum is over all paths $x_0 \to x_1 \to \cdots \to x_l$ from x_0 to x_l such that $x_0, \ldots, x_l$ are all distinct [16, Example 6.33].

This concludes the review of Leinster's and Baez–Dolan's invariants and how they relate to our L^2-Euler characteristic. Next we turn to a comparison of homotopy colimit results.

7.2. Comparison with Leinster's Proposition 2.8. Leinster's result on homotopy colimits, rephrased in our notation to make the comparison more apparent, is below.

Theorem 7.6 (Proposition 2.8 of Leinster [21]). *Let $\mathcal{I}$ be a category with finitely many objects and finitely many morphisms, and $\mathcal{C}: \mathcal{I} \to \mathsf{CAT}$ a pseudo functor. Assume that $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ has finitely many objects and finitely many morphisms. Let $q^\bullet$ be a weighting on $\mathcal{I}$ and suppose that $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ and all $\mathcal{C}(i)$ have Euler characteristics. Then*

$$\chi_L(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}) = \sum_{i \in \operatorname{ob}(\mathcal{I})} q^i \chi_L(\mathcal{C}(i)).$$

For example, if $\mathcal{I} = \{k \leftarrow j \rightarrow \ell\}$, then $\mathcal{I}$ admits the weighting $(q^j, q^k, q^\ell) = (-1, 1, 1)$ as discussed above. If $\mathcal{C} : \mathcal{I} \to \mathsf{CAT}$ is a pseudo functor, and the homotopy pushout has finitely many objects and finitely many morphisms, and $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ and all $\mathcal{C}(i)$ have Euler characteristics, then Leinster's result says that the homotopy pushout has the Euler characteristic $\chi_L(\mathcal{C}(k)) + \chi_L(\mathcal{C}(\ell)) - \chi_L(\mathcal{C}(j))$.
Leinster's Proposition 2.8 tells us how the Euler characteristic is compatible with Grothendieck fibrations. We can remove the hypothesis of finite from that Proposition, at the expense of requiring a finite model, as in the following theorem for our invariants.

Theorem 7.7. *Let $\mathcal{I}$ be a finite category. Suppose that $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model X for the classifying $\mathcal{I}$-space of $\mathcal{I}$. Let $q^\bullet \colon \operatorname{ob}(\mathcal{I}) \to \mathbb{Q}$ be the $\mathcal{I}$-Euler characteristic of X, namely*

$$q^i := \sum_{n \geq 0} (-1)^n (\textit{number of } n\textit{-cells of } X \textit{ based at } i).$$

Let $\mathcal{C} : \mathcal{I} \to \mathsf{CAT}$ be a functor such that for every object i the category $\mathcal{C}(i)$ is of type (FP$_R$). Suppose that $\mathcal{I}$ is directly finite and $\mathcal{C}(i)$ is directly finite for all $i \in \operatorname{ob}(\mathcal{I})$. Then

$$\chi(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}; R) = \sum_{i \in \operatorname{ob}(\mathcal{I})} q^i \chi(\mathcal{C}(i); R).$$

If each $\mathcal{C}(i)$ is of type (L^2) rather than (FP$_R$), we have

$$\chi^{(2)}(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}) = \sum_{i \in \operatorname{ob}(\mathcal{I})} q^i \chi^{(2)}(\mathcal{C}(i)).$$

Proof. By Theorem 4.1 (vi), we have

$$\begin{aligned}\chi(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}; R) &= \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \chi(\mathcal{C}(i_\lambda); R)\\ &= \sum_{n\geq 0}(-1)^n \cdot \sum_{i\in \operatorname{ob}(\mathcal{I})} (\text{number of } n\text{-cells of } X \text{ at } i)\chi(\mathcal{C}(i); R)\\ &= \sum_{i\in \operatorname{ob}(\mathcal{I})} \sum_{n\geq 0}(-1)^n(\text{number of } n\text{-cells of } X \text{ at } i)\chi(\mathcal{C}(i); R)\\ &= \sum_{i\in \operatorname{ob}(\mathcal{I})} q^i \chi(\mathcal{C}(i); R).\end{aligned}$$

The statement for $\chi^{(2)}$ is proved similarly from Theorem 4.1 (vii). □

Remark 7.8. Whenever $\chi(\operatorname{colim}_{\mathcal{I}} \mathcal{C}; R) = \chi(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}; R)$, Theorem 4.1 and Theorem 7.7 can be used to calculate the Euler characteristic of a colimit. Indeed, the hypotheses of Examples 6.1 and 6.2 guaranteed the equivalence of the colimit and the homotopy colimit, and this equivalence was a crucial ingredient in those proofs. For example, under Leinster's hypothesis of familial representability on $\mathcal{C}$, each connected component of $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ has an initial object, so

$$\chi(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}; R) = \chi(\operatorname{colim}_{\mathcal{I}} \mathcal{C}; R)$$

(recall that $\operatorname{colim}_{\mathcal{I}} \mathcal{C}$ is the set of connected components of $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ whenever $\mathcal{C}$ takes values in SETS). This is the role of familial representability in his Examples 3.4.

As a corollary to our Homotopy Colimit Formula for the L^2-Euler characteristic, we have a Homotopy Colimit Formula for Leinster's Euler characteristic when they agree.

Corollary 7.9 (Homotopy Colimit Formula for Leinster's Euler characteristic). *Let $\mathcal{I}$ be a skeletal, finite, EI-category such that the left $\operatorname{aut}_{\mathcal{I}}(y)$-action on $\operatorname{mor}_{\mathcal{I}}(x, y)$ is free for every two objects $x, y \in \operatorname{ob}(\mathcal{I})$. Assume there exists a finite $\mathcal{I}$-CW-model for the $\mathcal{I}$-classifying space of $\mathcal{I}$. Let $\mathcal{C}\colon \mathcal{I} \to \mathsf{CAT}$ be a covariant functor such that for each $i \in \operatorname{ob}(\mathcal{I})$, the category $\mathcal{C}(i)$ is a skeletal, finite, EI and the left $\operatorname{aut}_{\mathcal{C}(i)}(d)$-action on $\operatorname{mor}_{\mathcal{C}(i)}(c, d)$ is free for every two objects $c, d \in \operatorname{ob}(\mathcal{C}(i))$. Assume for every object $i \in \operatorname{ob}(\mathcal{I})$, for each automorphism $u\colon i \to i$ in $\mathcal{I}$, and each $\overline{x} \in \operatorname{iso}(\mathcal{C}(i))$ we have $\overline{\mathcal{C}(u)(x)} = \overline{x}$.*
Then $\mathcal{H} := \operatorname{hocolim}_{i\in I} \mathcal{C}$ is again a skeletal, finite, EI-category such that the left $\operatorname{aut}_{\mathcal{H}}(h)$-action on $\operatorname{mor}_{\mathcal{H}}(g, h)$ is free for every two objects $g, h \in \operatorname{ob}(\operatorname{hocolim}_{i\in I} \mathcal{C})$, and

$$\chi_L(\mathcal{H}) = \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \chi_L(\mathcal{C}(i_\lambda); R).$$

Proof. The category $\mathcal{H}$ is an EI-category by Theorem 4.1 (ii). Skeletality and finiteness of $\mathcal{H}$ follow directly from the skeletality and finiteness of $\mathcal{I}$ and $\mathcal{C}(i)$, and the definition of $\mathcal{H}$. The hypotheses on $\mathcal{C}(i)$ imply that

$\chi^{(2)}(\mathcal{C}(i)) = \chi_L(\mathcal{C}(i))$ by Theorem 7.5, and similarly $\chi^{(2)}(\mathcal{H}) = \chi_L(\mathcal{H})$. Finally, Theorem 4.1 (vii), which is the Homotopy Colimit Formula for the L^2-Euler characteristic $\chi^{(2)}$, implies the formula is also true for Leinster's Euler characteristic χ_L in the special situation of the Corollary. $\square$

8. Scwols and Complexes of Groups

As an illustration of the Homotopy Colimit Formula, we consider Euler characteristics of small categories without loops (*scwols*) and complexes of groups in the sense of Haefliger [18], [19] and Bridson–Haefliger [11]. One-dimensional complexes of groups are the classical Bass–Serre graphs of groups [37]. For finite scwols, the Euler characteristic, L^2-Euler characteristic, and Euler characteristic of the classifying space all coincide, essentially because finite scwols admit finite models for their classifying spaces. The Euler characteristic of a finite scwol is particularly easy to find: one simply chooses a skeleton, counts the paths of non-identity morphisms of length n, and then computes the alternating sum of these numbers.

Scwols and complexes of groups are combinatorial models for polyhedral complexes and group actions on them. The poset of faces of a polyhedral complex is a scwol. Suppose a group G acts on an M_κ-polyhedral complex by isometries preserving cell structure, and suppose each group element $g \in G$ fixes each cell pointwise that g fixes setwise. In this case, the quotient is also an M_κ-polyhedral complex, say Q, and we obtain a pseudo functor from its scwol of faces into groups. Namely, to a face $\overline{\sigma}$ of Q, one associates the stabilizer G_σ for a selected representative σ of $\overline{\sigma}$. Inclusions of subfaces of Q then correspond to inclusions of stabilizer subgroups up to conjugation. This pseudo functor is the complex of groups associated to the group action.

However, it is sometimes easier to work directly with the combinatorial model rather than with the original M_κ-polyhedral complex, and consider instead appropriate group actions on the associated scwol, as in Definition 8.11. Then the quotient category of a scwol is again a scwol, and the associated pseudo functor on the quotient scwol is called the *complex of groups associated to the group action*. Any group-valued pseudo functor on a scwol that arises in this way is called *developable*.

Our main results in this section concern the Euler characteristics of homotopy colimits of complexes of groups associated to group actions in the sense of Definition 8.11. Theorem 8.30, concludes that the Euler characteristic and L^2-Euler characteristic of the homotopy colimit are $\chi(\mathcal{X}/G)$ and $\chi^{(2)}(\mathcal{X})/|G|$ respectively, G and $\mathcal{X}$ are finite. These formulas provide necessary conditions for developability. That is, if F is a pseudo functor from a scwol $\mathcal{Y}$ to groups, one may ask if there are a scwol $\mathcal{X}$ and a group G such that $\mathcal{Y}$ is isomorphic to $\mathcal{X}/G$ and F is the associated complex of groups. To obtain conditions on $\chi(\mathcal{X})$, $\chi^{(2)}(\mathcal{X})$, and $|G|$, one forms the homotopy colimit of F, calculates its Euler characteristic and L^2-Euler characteristic, and then compares with the formulas of Theorem 8.30. A simple case is illustrated in Example 8.31. Another application of the formulas is the computation of the Euler characteristic and L^2-Euler characteristic for the transport groupoid of a finite left G-set, as in Example 8.33. We finish with Theorem 8.35, which extends Haefliger's formula for the Euler characteristic of the classifying space of the homotopy colimit of a complex of groups in terms of Euler characteristics of lower links and groups.

One novel aspect of our approach is that we do not require scwols to be skeletal. We prove in Theorem 8.24 that any scwol with a G-action in the sense of Definition 8.11 can be replaced by a skeletal scwol with a G-action, and this process preserves quotients, stabilizers, complexes of groups, and homotopy colimits. Moreover, if the initial G-action was free on the object set, then so is the G-action on the object set of the skeletal replacement.

We begin by recalling the notions in Chapter III.$\mathcal{C}$ of Bridson–Haefliger [11], rephrased in the conceptual framework of 2-category theory.

Notation 8.1 (2-Category of groups). We denote by GROUPS the 2-category of groups. Objects are groups and morphisms are group homomorphisms. The 2-cells are given by conjugation: a 2-cell (g, a)

$$\begin{tikzcd} H \arrow[r, bend left, "a"] \arrow[r, bend right, "a'"'] \arrow[r, phantom, "\Downarrow (g,a)"] & G \end{tikzcd}$$

is an element $g \in G$ such that $ga(h)g^{-1} = a'(h)$ for all $h \in H$. The vertical composition is $(g_2, a_2) \odot (g_1, a_1) = (g_2g_1, a_1)$ and the horizontal composition of

$$\begin{tikzcd} H \arrow[r, bend left, "a"] \arrow[r, bend right, "a'"'] \arrow[r, phantom, "\Downarrow (g,a)"] & G \arrow[r, bend left, "b"] \arrow[r, bend right, "b'"'] \arrow[r, phantom, "\Downarrow (k,b)"] & K \end{tikzcd}$$

is $(kb(g), ba)$.

Definition 8.2 (Scwol). A *scwol*[3] is a small category without loops, that is, a small category in which every endomorphism is trivial.

Example 8.3. The categories $\{j \rightrightarrows k\}$ and $\mathcal{I} = \{k \leftarrow j \rightarrow \ell\}$ of Examples 2.5 and 2.6 are scwols. Every partially ordered set is a scwol, for example, the set of simplices of a simplicial complex, ordered by the face relation, is a scwol. The poset of non-empty subsets of $[q]$, and its opposite category in Example 2.7, are scwols. The opposite category of a scwol is also a scwol.

[3]Bridson–Haefliger additionally require scwols to be skeletal [11, page 574]. However, we do not require scwols to be skeletal, since we prove in Theorem 8.24 that general statements about scwols can be reduced to the skeletal case.

Lemma 8.4. *Every scwol is an EI-category and consequently also directly finite.*

Proof. Every endomorphism in a scwol is trivial, and therefore an automorphism, so every scwol is an EI-category. By Fiore–Lück–Sauer [16, Lemma 3.13], every EI-category is also directly finite.
For a direct proof of direct finiteness: if $u\colon x \to y$ and $v\colon y \to x$ are morphisms in a scwol, then vu and uv are automorphisms, and hence both $vu = \mathrm{id}_x$ and $uv = \mathrm{id}_y$ hold automatically. □

Theorem 8.5 (Finite scwols admit finite models). *Suppose $\mathcal{I}$ is a finite scwol. Then $\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model for its $\mathcal{I}$-classifying space in the sense of Definition 2.2.*

Proof. By Lemma 2.13, we may assume that $\mathcal{I}$ is skeletal.
Since $\mathcal{I}$ has only finitely many morphisms, no nontrivial isomorphisms, and no nontrivial endomorphisms, there are only finitely many paths of non-identity morphisms. Thus the bar construction of $E^{\mathrm{bar}}\mathcal{I}$ Remark 2.10 has only finitely many $\mathcal{I}$-cells. □

Corollary 8.6. *Any finite scwol $\mathcal{I}$ is of types (FF_R) and (FP_R) for every associative, commutative ring R with identity. Moreover, any finite scwol is also of type (L^2).*

Proof. The cellular R-chains of the finite model in Theorem 8.5 provide a finite, free resolution of the constant module $\underline{R}$. By Theorem 1.14, any directly finite category of type ($\mathrm{FP}_{\mathbb{C}}$) is of type (L^2). Scwols are directly finite by Lemma 8.4. □

Example 8.7 (Invariants coincide for finite scwols). Let $\mathcal{I}$ be any finite scwol. Then by Corollary 8.6 it is of type (FF_R) for any associative, commutative ring with identity, and by Theorems 1.9 and 1.15, we have

$$\chi(\mathcal{I};R) = \chi(B\mathcal{I};R) = \chi^{(2)}(\mathcal{I}).$$

If Γ is any skeleton of $\mathcal{I}$, then by (1.17),

$$\chi(\Gamma;R) = \sum_{n\geq 0}(-1)^n c_n(\Gamma), \tag{8.8}$$

where $c_n(\Gamma)$ is the number of paths of n-many non-identity morphisms in Γ. But by Fiore–Lück–Sauer [16, Theorem 2.8 and Corollary 4.19], type (FF_R) and the Euler characteristic are invariant under equivalence of categories between directly finite categories, so $\chi(\mathcal{I};R) = \chi(\Gamma;R)$ and all three invariants $\chi(\mathcal{I};R)$, $\chi(B\mathcal{I};R)$, $\chi^{(2)}(\mathcal{I})$ are given by (8.8).

We now arrive at the main notion of this section: a complex of groups. We will apply our Homotopy Colimit Formula to complexes of groups.

Definition 8.9 (Complex of groups). Let $\mathcal{Y}$ be a scwol. A *complex of groups over* $\mathcal{Y}$ is a pseudo functor $F\colon \mathcal{Y} \to \mathsf{GROUPS}$ such that $F(a)$ is injective for every morphism a in $\mathcal{Y}$. For each object σ of $\mathcal{Y}$, the group $F(\sigma)$ is called the *local group at* σ.

In 2.5 and 2.4 of [18] and [19] respectively, Haefliger denotes by $CG(X)$ the homotopy colimit of a complex of groups $G(X)\colon C(X) \to \mathsf{GROUPS}$. Bridson–Haefliger use the notation $CG(\mathcal{Y})$ in [11, III.$\mathcal{C}$.2.8]. The fundamental group of a complex of groups $G(X)$ in the sense of [11, Definition 3.5 on p. 548] equals the fundamental group of the geometric realization of $CG(X)$ [11, Appendix A.12 on p. 578 and Remark A.14 on p. 579]. Categories which are homotopy colimits of complexes of groups are characterized by Haefliger on page 283 of [19]. From the homotopy colimit $CG(X)$, Haefliger reconstructs the category $C(X)$ and the complex of groups $G(X)$ up to a coboundary on pages 282-283 of [19]. Every aspherical realization [19, Definition 3.3.4] of $G(X)$ has the homotopy type of the geometric realization of the homotopy colimit, denoted $BG(X)$ [19, page 296]. The homotopy colimit also plays a role in the homology and cohomology of complexes of groups [19, Section 4]; a left $G(X)$-module is a functor $CG(X) \to \mathsf{ABELIAN\text{-}GROUPS}$.
We return to our recollection of complexes of groups and examples that arise from group actions.

Definition 8.10 (Morphism from a complex of groups to a group). A *morphism from a complex of groups F to a group G* is a pseudo natural transformation $F \Rightarrow \Delta_G$, where Δ_G indicates the constant 2-functor $\mathcal{Y} \to \mathsf{GROUPS}$ with value G.

A typical example of a complex of groups equipped with a morphism to a group G arises from an action of a group G on a scwol, as we now explain.

Definition 8.11 (Group action on a scwol, 1.11 of Bridson–Haefliger [11]). An *action of a group G on a scwol $\mathcal{X}$* is a group homomorphism from G into the group of strictly invertible functors $\mathcal{X} \to \mathcal{X}$ such that

(i) For every nontrivial morphism a of $\mathcal{X}$ and every $g \in G$, we have $gs(a) \neq t(a)$,
(ii) For every nontrivial morphism a of $\mathcal{X}$ and every $g \in G$, if $gs(a) = s(a)$, then $ga = a$.

Example 8.12. The group $G = \mathbb{Z}_2$ acts in the sense of Definition 8.11 on the scwol $\mathcal{X}$ pictured below.

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, "g"'] & z \\ y & x' \arrow[l, "g'"] \arrow[u, "h'"'] \end{tikzcd}$$

The group $\mathbb{Z}_2$ permutes respectively x and x', g and g', and h and h'. The objects y and z are fixed. This action of $\mathbb{Z}_2$ on $\mathcal{X}$ is a combinatorial model for a reflection action on S^1.

Example 8.13. Consider the scwol $\mathcal{X}$ pictured below. The group $G = \{\pm 1\} \ltimes \mathbb{Z}$ acts on $\mathcal{X}$ in the sense of Definition 8.11 where $-1 \cdot m := -m$ and $n \cdot m := m + 2n$.

$$\cdots \longrightarrow -2 \longleftarrow -1 \longrightarrow 0 \longleftarrow 1 \longrightarrow 2 \longleftarrow \cdots$$

This action of $\{\pm 1\} \ltimes \mathbb{Z}$ on $\mathcal{X}$ is a combinatorial model for the reflection and translation action on $\mathbb{R}$.

Lemma 8.14 (Consequences of group action conditions). *If a group G acts on a scwol $\mathcal{X}$ in the sense of Definition 8.11, then the following statements hold.*

(i) *If σ is an object of $\mathcal{X}$ and $g, h \in G$, then $g\sigma \cong h\sigma$ implies $g\sigma = h\sigma$.*
(ii) *If a is a morphism in $\mathcal{X}$ and $g, h \in G$, then $gs(a) = hs(a)$ implies $ga = ha$.*
(iii) *If $\sigma \cong \tau$, then the stabilizers G_σ and G_τ are equal.*

Proof. For statement (i), $g\sigma \cong h\sigma$ implies $\sigma \cong (g^{-1}h)\sigma$, so $\sigma = (g^{-1}h)\sigma$ by Definition 8.11 part (i), and $g\sigma = h\sigma$.
For statement (ii), $gs(a) = hs(a)$ implies $(h^{-1}g)s(a) = s(a)$ and $(h^{-1}g)a = a$ by Definition 8.11 part (ii), and finally $ga = ha$.
For statement (iii), suppose $\sigma \cong \tau$ and $g\sigma = \sigma$. We have

$$\tau \cong \sigma = g\sigma \cong g\tau.$$

Then $\tau = g\tau$ by (i), and $G_\sigma \subseteq G_\tau$. The proof is symmetric, so we also have $G_\tau \subseteq G_\sigma$. □

Definition 8.15 (Quotient of a scwol by a group action). If a scwol $\mathcal{X}$ is equipped with a G-action as above, then the *quotient scwol* $\mathcal{X}/G$ has objects and morphisms

$$\mathrm{ob}(\mathcal{X}/G) := (\mathrm{ob}(\mathcal{X}))/G$$
$$\mathrm{mor}(\mathcal{X}/G) := (\mathrm{mor}(\mathcal{X}))/G.$$

Composition and identities are induced by those of $\mathcal{X}$.

Remark 8.16 (III.$\mathcal{C}$.1.13 of Bridson–Haefliger [11]). The projection functor $p\colon \mathcal{X} \to \mathcal{X}/G$ induces a bijection

$$\{a \in \mathrm{mor}(\mathcal{X}) | sa = x\} \longrightarrow \{b \in \mathrm{mor}(\mathcal{X}/G) | sb = p(x)\} \tag{8.17}$$

for each $x \in \mathcal{X}$. If $G/\mathcal{X}$ is connected and the action of G on $\mathrm{ob}(\mathcal{X})$ is free, then p is a *covering of scwols*. That is, in addition to the bijection (8.17), p induces a bijection

$$\{a \in \mathrm{mor}(\mathcal{X}) | ta = x\} \longrightarrow \{b \in \mathrm{mor}(\mathcal{X}/G) | tb = p(x)\} \tag{8.18}$$

for each $x \in \mathcal{X}$.

Lemma 8.19 (Quotients of skeletal scwols are skeletal). *If $\mathcal{X}$ is a skeletal scwol, and a group G acts on $\mathcal{X}$ in the sense of Definition 8.11, then the quotient scwol $\mathcal{X}/G$ is also skeletal.*

Proof. Suppose $\overline{\sigma}$ is isomorphic to $\overline{\tau}$ in $\mathcal{X}/G$. We show $\overline{\sigma}$ is actually equal to $\overline{\tau}$. If $\overline{a}\colon \overline{\sigma} \to \overline{\tau}$ is an isomorphism with inverse $\overline{b}$, then there are lifts $a\colon \sigma \to \tau$ and $b\colon \tau \to \sigma'$ in $\mathcal{X}$, and an element $g \in G$ such that $g(ba) = \mathrm{id}_\sigma$. Since g fixes the source of ba, the group element g fixes also ba, so $ba = \mathrm{id}_\sigma$ and $\sigma' = \sigma$. Since ab is an endomorphism of τ, it is therefore id_τ. By the skeletality of $\mathcal{X}$, we have $\sigma = \tau$, and also $\overline{\sigma} = \overline{\tau}$. □

Lemma 8.20 (Quotient of path set is set of paths in quotient). *Suppose $\mathcal{X}$ is a scwol equipped with an action of a group G in the sense of Definition 8.11. Let $\Lambda_n(\mathcal{X})$ respectively $\Lambda_n(\mathcal{X}/G)$ denote the set of paths of n-many non-identity composable morphisms in $\mathcal{X}$ respectively $\mathcal{X}/G$. Give $\Lambda_n(\mathcal{X})$ the induced G-action. Then the function*

$$\Lambda_n(\mathcal{X}) \to \Lambda_n(\mathcal{X}/G)$$

$$(a_1, \dots, a_n) \mapsto (\overline{a}_1, \dots, \overline{a}_n)$$

induces a bijection $\Lambda_n(\mathcal{X})/G \to \Lambda_n(\mathcal{X}/G)$.

Proof. Remark 8.16 implies that a path $(a_1, \dots, a_n)$ in $\mathcal{X}$ consists entirely of non-identity morphisms if and only if the projection $(\overline{a}_1, \dots, \overline{a}_n)$ in $\mathcal{X}/G$ consists entirely of non-identity morphisms, so from now on we work only with non-identity morphisms. Note

$$(g_1a_1, g_2a_2, \dots, g_na_n) = (g_1a_1, g_1a_2, \dots, g_1a_n)$$

by Definition 8.11 (ii). For injectivity, we have $(\overline{a}_1, \dots, \overline{a}_n) = (\overline{b}_1, \dots, \overline{b}_n)$ if and only if for some $g_i \in G$

$$(g_1a_1, g_2a_2, \dots, g_na_n) = (b_1, \dots, b_n),$$

which happens if and only if for some $g \in G$

$$(ga_1, ga_2, \dots, ga_n) = (b_1, \dots, b_n),$$

(take $g = g_1$). For the surjectivity, we can lift any path $(\overline{a}_1, \dots, \overline{a}_n)$ by first lifting $\overline{a}_1$ to a_1, then $\overline{a}_2$ to a_2, and so on using Remark 8.16. $\square$

Definition 8.21 (Complex of groups from a group action on a scwol, 2.9 of Bridson–Haefliger [11]). Let G be a group and $\mathcal{X}$ a scwol upon which G acts in the sense of Definition 8.11. Let $p\colon \mathcal{X} \to \mathcal{X}/G$ denote the quotient map.
Haefliger and Bridson–Haefliger define a pseudo functor $F\colon \mathcal{X}/G \to \mathsf{GROUPS}$ as follows. In the procedure choices are made, but different choices lead to isomorphic complexes of groups. For each object $\overline{\sigma}$ of $\mathcal{X}/G$, choose an object σ of $\mathcal{X}$ such that $p(\sigma) = \overline{\sigma}$ (our overline convention is the opposite of that in [11]). Then $F(\overline{\sigma})$ is defined to be G_σ, the isotropy group of σ under the G-action.
If $\overline{a}\colon \overline{\sigma} \to \overline{\tau}$ is a morphism in $\mathcal{X}/G$, then there exists a unique morphism a in $\mathcal{X}$ such that $p(a) = \overline{a}$ and $sa = \sigma$, as in (8.17). For $\overline{a}$ we choose an element $h_{\overline{a}} \in G$ such that $h_{\overline{a}} \cdot ta$ is the object τ of $\mathcal{X}$ chosen above so that $p(\tau) = \overline{\tau}$.
An injective group homomorphism $F(\overline{a})\colon G_\sigma \to G_\tau$ is defined by

$$F(\overline{a})(g) := h_{\overline{a}} g h_{\overline{a}}^{-1}.$$

Suppose $\overline{a}$ and $\overline{b}$ are composable morphisms of $\mathcal{X}/G$. We define a 2-cell in GROUPS

$$F_{\overline{b},\overline{a}}\colon F(\overline{b}) \circ F(\overline{a}) \Rightarrow F(\overline{b} \circ \overline{a})$$

to be $(h_{\overline{b}\overline{a}} h_{\overline{a}}^{-1} h_{\overline{b}}^{-1}, F(\overline{b}) \circ F(\overline{a}))$ as in Notation 8.1.
The pseudo functor $F\colon \mathcal{X}/G \to \mathsf{GROUPS}$ is called the *complex of groups associated to the group action of G on the scwol $\mathcal{X}$*. This complex of groups

comes equipped with a morphism to the group G, that is, a pseudo natural transformation $F \Rightarrow \Delta_G$. The inclusion of each isotropy group $F(\overline{\sigma}) = G_\sigma$ into G provides the components of the pseudo natural transformation.

Example 8.22. The quotient scwols for the actions in Examples 8.12 and 8.13 are both $\{k \leftarrow j \rightarrow \ell\}$, and the associated complexes of groups are both

$$\mathbb{Z}_2 \longleftarrow \{0\} \longrightarrow \mathbb{Z}_2.$$

Remark 8.23. If a group G acts on a scwol in the sense of Definition 8.11, each object stabilizer is finite, and the quotient scwol is finite, then the associated complex of groups $F\colon \mathcal{X}/G \to \mathsf{GROUPS}$ satisfies all of the hypotheses of the Homotopy Colimit Formula in Theorem 4.1 (vii) and in Corollary 4.2 (vii). If, in addition, R is a ring such that the order $|H|$ of each object stabilizer $H \subset G$ is invertible in R, then $F\colon \mathcal{X}/G \to \mathsf{GROUPS}$ also satisfies all of the hypotheses of the Homotopy Colimit Formula in Theorem 4.1 (vi) and in Corollary 4.2 (vi). See Examples 8.12, 8.13, and 8.22.

Even without finiteness assumptions, it is possible to replace scwols with skeletal scwols and preserve much of the accompanying structure, as Theorem 8.24 explains.

Theorem 8.24 (Reduction to skeletal case). *Let G be a group acting on a scwol $\mathcal{X}$ in the sense of Definition 8.11. Let Γ be any skeleton of $\mathcal{X}$, $i\colon \Gamma \to \mathcal{X}$ the inclusion, and $r\colon \mathcal{X} \to \Gamma$ a functor equipped with a natural isomorphism $ir \cong \mathrm{id}_\mathcal{X}$, and satisfying $ri = \mathrm{id}_\Gamma$. Then there is a G-action on the scwol Γ in the sense of Definition 8.11 such that following hold.*

(i) *The functor r is G-equivariant.*
(ii) *The induced functor $\overline{r}$ on quotient categories is an equivalence of categories compatible with the quotient maps, that is, the diagram below commutes.*

(8.25)
$$\begin{tikzcd} \mathcal{X} \arrow[r, "r"] \arrow[d, "p^\mathcal{X}"'] & \Gamma \arrow[d, "p^\Gamma"] \\ \mathcal{X}/G \arrow[r, "\overline{r}"'] & \Gamma/G \end{tikzcd}$$

(iii) *The inclusion $i\colon \Gamma \to \mathcal{X}$ preserves stabilizers, that is $G_{i\gamma} = G_\gamma$ for all $\gamma \in \mathrm{ob}(\Gamma)$. Note that the inclusion may not be G-equivariant.*
(iv) *Choices can be made in the definitions of $F^\mathcal{X}$ and F^Γ (the complexes of groups associated to the G-actions on $\mathcal{X}$ and Γ in Definition 8.21), so that the diagram below strictly commutes.*

(8.26)
$$\begin{tikzcd} \mathcal{X}/G \arrow[rr, "\overline{r}"] \arrow[dr, "F^\mathcal{X}"'] & & \Gamma/G \arrow[dl, "F^\Gamma"] \\ & \mathsf{GROUPS} & \end{tikzcd}$$

(v) *The functor* $(\overline{r}, \mathrm{id})$ *is an equivalence of categories*

$$(\overline{r}, \mathrm{id})\colon\ \mathrm{hocolim}_{\mathcal{X}/G} F^{\mathcal{X}} \longrightarrow \mathrm{hocolim}_{\Gamma/G} F^{\Gamma}.$$

(vi) *If* G *acts freely on* $\mathrm{ob}(\mathcal{X})$*, then* G *acts freely on* $\mathrm{ob}(\Gamma)$*.*

Proof. To define the group action, let $\mathrm{Aut}(\mathcal{X})$ and $\mathrm{Aut}(\Gamma)$ denote the strictly invertible endofunctors on $\mathcal{X}$ and Γ respectively, and consider the monoid homomorphism

$$\varphi\colon\ \mathrm{Aut}(\mathcal{X}) \to \mathrm{End}(\Gamma),\ \ F \mapsto r \circ F \circ i. \tag{8.27}$$

This is strictly multiplicatively because the natural isomorphism of functors

$$\begin{aligned} r \circ G \circ F \circ i &= r \circ G \circ \mathrm{id}_{\mathcal{X}} \circ F \circ i \\ &\cong (r \circ G \circ i) \circ (r \circ F \circ i), \end{aligned}$$

and skeletality of Γ imply $\varphi(GF)$ agrees with $\varphi(G)\varphi(F)$ on objects of Γ, so each component $\varphi(GF)(\gamma) \cong \varphi(G)\varphi(F)(\gamma)$ is an endomorphism in the scwol Γ, and is therefore trivial. By naturality, $\varphi(GF)$ and $\varphi(G)\varphi(F)$ agree on morphisms also. Consequently, φ takes values in $\mathrm{Aut}(\Gamma)$ and is a homomorphism $\varphi\colon\ \mathrm{Aut}(\mathcal{X}) \to \mathrm{Aut}(\Gamma)$.

We define a G-action on Γ as the composite of the action $G \to \mathrm{Aut}(\mathcal{X})$ with φ in (8.27). We indicate the action of g on Γ by $\varphi(g)\gamma$ and the action of g on $\mathcal{X}$ by gx. For simplicity, we suppress i from the notation when indicating the G-action in $\mathcal{X}$ on objects and morphisms of Γ, so for example, if a is morphism in Γ, then $gs(a)$ actually means $gis(a)$ throughout.

To verify Definition 8.11 (i) for Γ, suppose a is a nontrivial morphism in Γ and $\varphi(g)s(a) = t(a)$, that is $rgs(a) = t(a)$. Then $gs(a) \cong t(a)$ in $\mathcal{X}$, but $gs(a) \neq t(a)$ (for if $gs(a) = t(a)$, then a must be trivial by Definition 8.11 (i) for $\mathcal{X}$). Let $b\colon t(a) \to gs(a)$ be an isomorphism in $\mathcal{X}$ and consider the composite $ba\colon s(a) \to t(a) \to gs(a)$. Then $gs(ba) = gs(a) = t(ba)$, so ba must be trivial by Definition 8.11 (i) for $\mathcal{X}$. Consequently $a = b^{-1}$ is a nontrivial *iso*morphism in Γ, and we have a contradiction to either skeletality or the no loops requirement. Thus $\varphi(g)s(a) \neq t(a)$, and Definition 8.11 (i) holds for Γ. The verification of Definition 8.11 (ii) is shorter: if a is a nontrivial morphism in Γ and $\varphi(g)s(a) = s(a)$, that is $rgs(a) = s(a)$, then $gs(a) \cong s(a)$, and $gs(a) = s(a)$ by Lemma 8.14 (i) for $\mathcal{X}$. Finally, $ga = a$ by Definition 8.11 (ii) for $\mathcal{X}$, $rga = a$ as a is in Γ, and $\varphi(g)a = a$. The action of G on Γ satisfies Definition 8.11 and we may form the quotient scwol Γ/G as in Defition 8.15, which is skeletal by Lemma 8.19.

(i) For the G-equivariance of r, let $f\colon x \to y$ be a morphism in $\mathcal{X}$ and consider the naturality diagram.

$$\begin{tikzcd} rgirx \arrow[r, "rgirf=\varphi(g)r(f)"] \arrow[d, "\cong"'] & rgiry \arrow[d, "\cong"] \\ rgx \arrow[r, "rgf"'] & rgy \end{tikzcd}$$

The vertical morphisms must be identities by skeletality of Γ and the no loops condition, so $\varphi(g)r(f) = r(gf)$. Equivariance on objects then follows by taking $f = \mathrm{id}_x$.

(ii) Diagram (8.25) commutes by definition of $\overline{r}$. The functor $\overline{r}$ is surjective on objects because $p^\Gamma r$ and $p^\mathcal{X}$ are. The functor $\overline{r}$ is fully faithful since the equivariant bijection $r(x,y)\colon \operatorname{mor}_\mathcal{X}(x,y) \to \operatorname{mor}_\Gamma(r(x), r(y))$ induces the equivariant bijection $\overline{r}(p^\mathcal{X}x, p^\mathcal{X}y)$.

(iii) Let $\gamma \in \mathrm{ob}(\Gamma)$, and suppose $gi\gamma = i\gamma$. Then

$$\begin{aligned} \varphi(g)\gamma &\stackrel{\text{def}}{=} r(gi\gamma) \\ &= r(i\gamma) \\ &= \gamma \end{aligned}$$

and $G_{i\gamma} \subseteq G_\gamma$. Now suppose $\varphi(g)\gamma = \gamma$. Then $r(gi\gamma) = \gamma$ by definition, and $gi\gamma \cong i\gamma$ in $\mathcal{X}$, which says $g \cdot i\gamma = i\gamma$ by Lemma 8.14 (i), and $G_\gamma \subseteq G_{i\gamma}$.

(iv) We claim that choices can be made in the definitions of the associated complexes of groups $F^\mathcal{X}$ and F^Γ (see Definition 8.21) so that diagram (8.26) strictly commutes. First choose a skeleton $\mathcal{Q}$ of the quotient $\mathcal{X}/G$, define $F^\mathcal{X}$ on object in the skeleton $\mathcal{Q}$, and then extend to all objects in $\mathcal{X}/G$. For every $\overline{q} \in \mathrm{ob}(\mathcal{Q})$, select a $q \in \mathrm{ob}(\mathcal{X})$ such that $p^\mathcal{X}(q) = \overline{q}$ and define $F^\mathcal{X}(\overline{q}) = G_q$. We remain with the choice of the selected preimage q of $\overline{q}$ throughout. If $\overline{\sigma} \in \mathrm{ob}(\mathcal{X}/G)$ and $\overline{a}\colon \overline{q} \cong \overline{\sigma}$ is an isomorphism in $\mathcal{X}/G$, then also define $F^\mathcal{X}(\overline{\sigma}) = G_q$. This is allowed, since $\overline{a}\colon \overline{q} \cong \overline{\sigma}$ implies existence of morphisms $a\colon q \to g_\sigma\sigma$ and $b\colon \sigma \to g_q q$ in $\mathcal{X}$, and the composite

$$q \xrightarrow{a} g_\sigma\sigma \xrightarrow{g_\sigma b} g_\sigma g_q q$$

is trivial by Definition 8.11 (i). The opposite composite is also trivial, as it is a loop, and we have $q \cong g_\sigma\sigma$ in $\mathcal{X}$. Then by Lemma 8.14 (iii), $G_q = G_{g_\sigma\sigma}$ and we may define $F^\mathcal{X}(\overline{\sigma}) = G_q$ because $p^\mathcal{X}(g_\sigma\sigma) = \overline{\sigma}$. In particular, the selected preimage of $\overline{\sigma}$ in $\mathcal{X}$ is $g_\sigma\sigma$ and we select $h_{\overline{a}} = e_G$ for $\overline{a}\colon \overline{q} \cong \overline{\sigma}$ in Definition 8.21, so $F^\mathcal{X}(\overline{a}) = \mathrm{id}_{G_q}$. We remark that the isomorphism $\overline{a}$ is the only morphism $\overline{q} \to \overline{\sigma}$ because there are no loops in $\mathcal{X}/G$, so the element $g_\sigma\sigma$ is uniquely defined as the target of the unique morphism a with source q and $p^\mathcal{X}$-image $\overline{a}$. We next define F^Γ on objects of Γ/G using the equivalence $\overline{r}$ and the definition of $F^\mathcal{X}$ on objects of $\mathcal{Q}$. For $\overline{q} \in \mathrm{ob}(\mathcal{Q})$, we also define $F^\Gamma(\overline{r}(\overline{q})) = G_q$. This is allowed: for $\overline{r}(\overline{q}) = \overline{r(q)}$ we choose $r(q)$ as the selected preimage in $\mathrm{ob}(\Gamma)$, and $ir(q) \cong q$ in $\mathcal{X}$, so $G_{r(q)} = G_{ir(q)} = G_q$ by (iii) and Lemma 8.14 (iii). Every $\overline{\gamma} \in \mathrm{ob}(\Gamma/G)$ is of the form $\overline{r}(\overline{q})$ for a unique $\overline{q} \in \mathcal{Q}$, so F^Γ is now defined on all objects of Γ/G, and we have $F^\Gamma \circ \overline{r} = F^\mathcal{X}$ on all objects of $\mathcal{X}/G$.

We must now define $F^\mathcal{X}$ and F^Γ on morphisms so that $F^\Gamma \circ \overline{r} = F^\mathcal{X}$ for morphisms also. The idea is to first define $F^\mathcal{X}$ on morphisms in the skeleton $\mathcal{Q}$, then extend to all of $\mathcal{X}/G$, and then define F^Γ on morphisms of Γ/G. If $\overline{a}\colon \overline{q}_1 \to \overline{q}_2$ is a morphism in $\mathcal{Q}$, then there is a unique morphism a in $\mathcal{X}$ with source q_1 and $p^\mathcal{X}(a) = \overline{a}$. Select any $h_{\overline{a}}$ such that $h_{\overline{a}}ta = q_2$. Then we define an injective group homomorphism $F(\overline{a})\colon G_{q_1} \to G_{q_2}$ by

$$F(\overline{a})(g) := h_{\overline{a}} g h_{\overline{a}}^{-1}.$$

If $\overline{b}\colon \overline{\sigma}_1 \to \overline{\sigma}_2$ is any morphism in $\mathcal{X}/G$, then there exists a unique $\overline{a}$ in $\mathcal{Q}$ and a unique commutative diagram with vertical isomorphisms as below.

$$\begin{tikzcd} \overline{q}_1 \arrow[r, "\overline{a}"] \arrow[d, "\cong"'] & \overline{q}_2 \arrow[d, "\cong"] \\ \overline{\sigma}_1 \arrow[r, "\overline{b}"'] & \overline{\sigma}_2 \end{tikzcd}$$

Then we choose $h_{\overline{b}}$ to be $h_{\overline{a}}$, and we consequently have $F(\overline{a}) = F(\overline{b})$. If $\overline{c}\colon \overline{r}(\overline{q}_1) \to \overline{r}(\overline{q}_2)$ is a morphism in Γ/G, then there is a unique $\overline{a}\colon \overline{q}_1 \to \overline{q}_2$ in $\mathcal{Q}$ with $\overline{r}(\overline{a}) = \overline{c}$ and we choose $h_{\overline{c}}$ to be $h_{\overline{a}}$. Manifestly, we have $F^\Gamma \circ \overline{r} = F^{\mathcal{X}}$. The coherences of $F^{\mathcal{X}}$ and F^Γ are also compatible, since they are determined by the $h_{\overline{a}}$'s.

(v) From (ii) we know $\overline{r}$ is a surjective-on-objects equivalence of categories and from (iv) we have $F^{\mathcal{X}} = F^\Gamma \circ \overline{r}$. From this, one sees

$$(\overline{r}, \mathrm{id})\colon\ \mathrm{hocolim}_{\mathcal{X}/G} F^{\mathcal{X}} = \mathrm{hocolim}_{\mathcal{X}/G} F^\Gamma \circ \overline{r} \longrightarrow \mathrm{hocolim}_{\Gamma/G} F^\Gamma$$

is an equivalence of categories.

(vi) If the action of G on $\mathrm{ob}(\mathcal{X})$ is free, then for each $\gamma \in \mathrm{ob}(\Gamma)$, the group $G_\gamma = G_{i\gamma}$ (see (iii)) is trivial, and G acts freely on $\mathrm{ob}(\Gamma)$. □

Remark 8.28. In Theorem 8.24, it is even possible to select a skeleton so that the inclusion is G-equivariant, though we will not need this. See Section 9.

In [16, Theorems 5.30 and 5.37], we proved the compatibility of the L^2-Euler characteristic with coverings and isofibrations of finite connected groupoids. Theorem 8.29 is an analogue for scwols (see Remark 8.16).

Theorem 8.29 (Compatibility with free actions on finite scwols). *Let G be a finite group acting on a finite scwol $\mathcal{X}$. If G acts freely on $\mathrm{ob}(\mathcal{X})$, then*

$$\chi(\mathcal{X}/G; R) = \frac{\chi(\mathcal{X}; R)}{|G|} \quad \textit{and} \quad \chi^{(2)}(\mathcal{X}/G) = \frac{\chi^{(2)}(\mathcal{X})}{|G|}.$$

Recall $\chi(-;R)$ and $\chi^{(2)}(-)$ agree for finite scwols by Example 8.7.

Proof. By Theorem 8.24 (i), (ii), and (vi), we may assume $\mathcal{X}$ is skeletal.
A consequence of Definition 8.11 (ii) (independent of skeletality) is that an element $g \in G$ fixes a path $a = (a_1, \ldots, a_n)$ in $\mathcal{X}$ if and only if g fixes sa_1, so $G_{sa_1} = G_a$. Then G acts freely on $\Lambda_n(\mathcal{X})$, since it acts freely on $\mathrm{ob}(\mathcal{X})$.

The scwol $\mathcal{X}/G$ is skeletal by Lemma 8.19, and by Example 8.7 and Lemma 8.20 we have

$$\begin{aligned}
\chi^{(2)}(\mathcal{X}/G) &= \sum_{n\geq 0}(-1)^n c_n(\mathcal{X}/G) \\
&= \sum_{n\geq 0}(-1)^n |\Lambda_n(\mathcal{X}/G)| \\
&= \sum_{n\geq 0}(-1)^n |\Lambda_n(\mathcal{X})/G| \\
&= \sum_{n\geq 0}(-1)^n \frac{|\Lambda_n(\mathcal{X})|}{|G|} \\
&= \frac{1}{|G|}\sum_{n\geq 0}(-1)^n |\Lambda_n(\mathcal{X})| \\
&= \frac{1}{|G|}\sum_{n\geq 0}(-1)^n c_n(\mathcal{X}) \\
&= \frac{\chi^{(2)}(\mathcal{X})}{|G|}.
\end{aligned}$$

□

A complex of groups is called *developable* if it is isomorphic to a complex of groups associated to a group action. A classical theorem of Bass–Serre says that every complex of groups on a scwol with maximal path length 1 is developable. The following gives a necessary condition of developability of a complex of groups from a scwol and group of specified Euler characteristics.

Theorem 8.30 (Euler characteristics of associated complexes of groups). *Let G be a finite group that acts on a finite scwol $\mathcal{X}$ in the sense of Definition 8.11. Let $F\colon \mathcal{X}/G \to \mathsf{GROUPS}$ be the associated complex of groups. Then*

$$\chi^{(2)}(\operatorname{hocolim}_{\mathcal{X}/G} F) = \frac{\chi^{(2)}(\mathcal{X})}{|G|} = \frac{\chi(\mathcal{X};\mathbb{C})}{|G|} = \frac{\chi(B\mathcal{X};\mathbb{C})}{|G|}.$$

If R is a ring such that the orders of subgroups $H \subset G$ are invertible in R, then we also have

$$\chi(\operatorname{hocolim}_{\mathcal{X}/G} F; R) = \chi(\mathcal{X}/G; R).$$

Proof. By Theorem 8.24 (i), (ii), (iv), and (v), we may assume $\mathcal{X}$ is skeletal. Then $\mathcal{X}/G$ is also skeletal by Lemma 8.19.
Let $\Lambda_n(\mathcal{X})$ respectively $\Lambda_n(\mathcal{X}/G)$ denote the set of paths of n-many non-identity composable morphisms in $\mathcal{X}$ respectively $\mathcal{X}/G$. Then by Lemma 8.20, the sets $\Lambda_n(\mathcal{X})/G$ and $\Lambda_n(\mathcal{X}/G)$ are in bijective correspondence.
We will also use the fact that an element $g \in G$ fixes a path $a = (a_1, \ldots, a_n)$ in $\mathcal{X}$ if and only if g fixes sa_1, so $G_{sa_1} = G_a$. This is a consequence of Definition 8.11 (ii).

By Theorem 8.5, $E^{\mathrm{bar}}\mathcal{X}$ and $E^{\mathrm{bar}}(\mathcal{X}/G)$ are finite models for the skeletal scwols $\mathcal{X}$ and $\mathcal{X}/G$, and the n-cells are indexed by $\Lambda_n(\mathcal{X})$ and $\Lambda_n(\mathcal{X}/G)$, respectively. For each path $(a_1,\ldots,a_n)$ in $\mathcal{X}$, there is an n-cell in $E^{\mathrm{bar}}\mathcal{X}$ based at sa_1. A similar statement holds for $\mathcal{X}/G$ and $E^{\mathrm{bar}}(\mathcal{X}/G)$.
Now we may apply the Homotopy Colimit Formula to the associated complex of groups $F:\mathcal{X}/G\to\mathsf{GROUPS}$ by Remark 8.23. For the Euler characteristic, we have

$$\begin{aligned}
\chi(\operatorname{hocolim}_{\mathcal{X}/G}F;R) &= \sum_{n\geq 0}(-1)^n\cdot\left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X}/G)}\chi(F(s\overline{a}_1);R)\right)\\
&= \sum_{n\geq 0}(-1)^n\cdot\left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X}/G)}1\right)\\
&= \sum_{n\geq 0}(-1)^n|\Lambda_n(\mathcal{X}/G)|\\
&= \sum_{n\geq 0}(-1)^n c_n(\mathcal{X}/G)\\
&= \chi(\mathcal{X}/G;R).
\end{aligned}$$

For the L^2-Euler characteristic on the other hand, we have

$$\begin{aligned}
\chi^{(2)}(\operatorname{hocolim}_{\mathcal{X}/G}F) &= \sum_{n\geq 0}(-1)^n\cdot\left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X}/G)}\chi^{(2)}(F(s\overline{a}_1))\right)\\
&= \sum_{n\geq 0}(-1)^n\cdot\left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X}/G)}\frac{1}{|G_{sa_1}|}\right)\\
&= \sum_{n\geq 0}(-1)^n\cdot\left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X})/G}\frac{1}{|G_a|}\right)\\
&= \sum_{n\geq 0}(-1)^n\cdot\left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X})/G}\frac{|\mathrm{orbit}(a)|}{|G|}\right)\\
&= \frac{1}{|G|}\sum_{n\geq 0}(-1)^n\cdot\left(\sum_{\overline{a}\in\Lambda_n(\mathcal{X})/G}|\mathrm{orbit}(a)|\right)\\
&= \frac{1}{|G|}\sum_{n\geq 0}(-1)^n|\Lambda_n(\mathcal{X})|\\
&= \frac{1}{|G|}\sum_{n\geq 0}(-1)^n c_n(\mathcal{X})\\
&= \frac{\chi^{(2)}(\mathcal{X})}{|G|}.
\end{aligned}$$

□

Example 8.31. By the classical theorem of Bass–Serre, any injective group homomorphism

$$G_0 \to G_1 \tag{8.32}$$

is a developable complex of groups. The L^2-Euler characteristic of the homotopy colimit of (8.32) is $1/|G_1|$ by Example 5.1. Theorem 8.30 then says we must have

$$\frac{|G|}{|G_1|} = \chi^{(2)}(\mathcal{X}) = \chi(B\mathcal{X}; \mathbb{C})$$

if (8.32) is to be developable from a scwol $\mathcal{X}$ by an action of G in the sense of Definition 8.11. Thus (8.32) is not developable from any scwol $\mathcal{X}$ whose geometric realization has Euler characteristic 0, such as $\{j \rightrightarrows k\}$. Nor can (8.32) be developed from any scwol $\mathcal{X}$ with $\chi(B\mathcal{X}; \mathbb{C})$ negative. The integer $|G|$ must also be divisible by $|G_1|$, since $\chi(B\mathcal{X}; \mathbb{C})$ is always an integer. Moreover, the Euler characteristic of $\mathcal{X}$ must be less than or equal to $|G|$. This trivial example illustrates how one can find necessary conditions on $\mathcal{X}$ and G if a given complex of groups is to be developable from $\mathcal{X}$ and G.

Example 8.33 (Euler characteristics of transport groupoid in finite case). Let X be a finite set and G a finite group acting on X. Let R be a ring such that the orders of subgroups of G are invertible in R. Considering X as a scwol, we clearly have an action in the sense of Definition 8.11. The associated complex of groups $F : X/G \to \mathsf{GROUPS}$ assigns to orbit(σ) the stabilizer G_σ. The homotopy colimit $\operatorname{hocolim}_{X/G} F$ is equivalent to the transport groupoid $\mathcal{G}^G(X)$ of Example 5.6, so

$$\chi\left(\mathcal{G}^G(X); R\right) = \chi(\operatorname{hocolim}_{X/G} F; R) = \chi(X/G; R) = |X/G|.$$

For the L^2-Euler characteristic, on the other hand, we have

$$\chi^{(2)}\left(\mathcal{G}^G(X)\right) = \chi^{(2)}(\operatorname{hocolim}_{X/G} F) = \frac{\chi^{(2)}(X)}{|G|} = \frac{|X|}{|G|},$$

a formula obtained by Baez–Dolan [4].

We also generalize the following formula of Haefliger for the Euler characteristic of the homotopy colimit of a (not necessarily developable) complex of groups.

Theorem 8.34 (Corollary 3.5.3 of Haefliger [19]). *Let $G(X)$ be a complex of groups over a finite ordered simplicial cell complex X. Assume that each G_σ is the fundamental group of a finite aspherical cell complex. Then $BG(X)$ has the homotopy type of a finite complex and its Euler-Poincaré characteristic is* *given by*[4]

$$\chi(BG(X)) = \sum_{\sigma\in \mathrm{ob}(C(X))} (1-\chi(Lk^\sigma))\chi(G_\sigma).$$

The terms in Haefliger's theorem have the following meanings. An *ordered simplicial cell complex* X is by definition the nerve of a skeletal scwol, denoted $C(X)$. The notation $BG(X)$ signifies the geometric realization of the nerve of the homotopy colimit of the pseudo functor $G(X)\colon C(X) \to \mathsf{GROUPS}$. An *aspherical* cell complex is one for which all homotopy groups beyond the fundamental group vanish. The *lower link Lk^σ of the object* σ is the full subcategory of the scwol $\sigma \downarrow C(X)$ on all objects except 1_σ.

Theorem 8.35 (Extension of Corollary 3.5.3 of Haefliger [19]). *Let $\mathcal{I}$ be a finite skeletal scwol and $F\colon \mathcal{I} \to \mathsf{GROUPS}$ a complex of groups such that for each object i of $\mathcal{I}$, the group $F(i)$ is of type ($FF_\mathbb{Z}$). Then*

$$\chi(B\operatorname{hocolim}_{\mathcal{I}} F) = \sum_{i\in \mathrm{ob}(\mathcal{I})} (1-\chi(BLk^i))\chi(BF(i)),$$

where B indicates geometric realization composed with the nerve functor.

Proof. All hypotheses of Theorem 4.1(vi) are satisfied. The skeletal scwol $\mathcal{I}$ is directly finite by Lemma 8.4 and admits a finite $\mathcal{I}$-CW-model for its $\mathcal{I}$-classifying space by Theorem 8.5. Each group $\mathcal{C}(i)$ is automatically directly finite, and assumed to be of type ($\mathrm{FF}_\mathbb{Z}$). The bar construction model $E^{\mathrm{bar}}\mathcal{I}$ in Remark 2.10 has an n-cell based at i for each path of n-many non-identity morphisms in $\mathcal{I}$

$$i \to i_1 \to i_2 \to \cdots \to i_n.$$

Each such path in $\mathcal{I}$ corresponds uniquely to a path of $(n-1)$-many non-identity morphisms in the scwol Lk^i beginning at the object $i \to i_1$. Thus

$$\begin{aligned}
1-\chi(BLk^i) &= 1-\sum_{m\geq 0}(-1)^m c_m(Lk^i)\\
&= 1-\sum_{m\geq 0}(-1)^m \mathrm{card}\{(m+1)\text{-paths in } \mathcal{I} \text{ beginning at } i\}\\
&= 1-\sum_{n\geq 1}(-1)^{n-1} \mathrm{card}\{n\text{-paths in } \mathcal{I} \text{ beginning at } i\}\\
&= 1+\sum_{n\geq 1}(-1)^{n} \mathrm{card}\{n\text{-paths in } \mathcal{I} \text{ beginning at } i\}\\
&= \sum_{n\geq 0}(-1)^{n} \mathrm{card}\{n\text{-paths in } \mathcal{I} \text{ beginning at } i\}.
\end{aligned}$$

[4]Haefliger's original formula has, instead of the lower link L^σ, the upper link L_σ, which is the full subcategory of the scwol $C(X)\downarrow\sigma$ on all objects except 1_σ. However, this is merely a typo, for if we use the upper link Lk_σ and consider the example $C(X) = \{k \leftarrow j \to \ell\}$ with pseudo functor $G(X)(\ell) := \mathbb{Z}$ and $G(X)(j) := G(X)(k) := \{0\}$, then $\chi(BG(X)) = \chi(S^1) = 0$ but $\sum(1-\chi(Lk_\sigma))\chi(G_\sigma) = 1$.

Then by Theorem 4.1 (i), Theorem 4.1 (iv), Theorem 1.15, and Theorem 4.1 (vi), we have

$$\begin{aligned}
\chi(B\operatorname{hocolim}_{\mathcal{I}} F) &= \chi(\operatorname{hocolim}_{\mathcal{I}} F) \\
&= \sum_{n\geq 0}(-1)^n \cdot \sum_{\lambda\in\Lambda_n} \chi(F(i_\lambda)) \\
&= \sum_{i\in \operatorname{ob}(\mathcal{I})} \big(1-\chi(BLk^i)\big)\cdot \chi(F(i)) \\
&= \sum_{i\in \operatorname{ob}(\mathcal{I})} \big(1-\chi(BLk^i)\big)\cdot \chi(BF(i)).
\end{aligned}$$

□

Remark 8.36. The assumptions in our Theorem 8.35 on the groups $F(i)$ are related to the assumptions in Theorem 8.34 on the groups G_σ in that any finitely presentable group of type $(\mathrm{FF}_{\mathbb{Z}})$ admits a finite model for its classifying space.

9. Appendix

Let G be a group acting on a scwol $\mathcal{X}$ in the sense of Definition 8.11. In connection with Theorem 8.24, we remark that it is possible to choose a skeleton Γ_0 of $\mathcal{X}$, a G-equivariant functor $r\colon \mathcal{X}\to\Gamma_0$, and a natural isomorphism $\eta\colon ir \cong \mathrm{id}_{\mathcal{X}}$ so that

- the inclusion $i_0\colon \Gamma_0\to\mathcal{X}$ is G-equivariant,
- $ri_0 = \mathrm{id}_{\Gamma_0}$, and
- for every object $x\in\operatorname{ob}(\mathcal{X})$ and each $g\in G$, we have $\eta_{gx} = g\eta_x$.

To prove this, we first choose the object set of Γ_0 via an equivariant section of the projection $\pi\colon \operatorname{ob}(\mathcal{X})\to\operatorname{iso}(\mathcal{X})$, which assigns to each object of $\mathcal{X}$ its isomorphism class of objects. Let Θ denote the set of G-orbits of $\operatorname{iso}(\mathcal{X})$. For each G-orbit $\theta\in\Theta$, we use the axiom of choice to select an element $\overline{x}_\theta\in\theta$. For each θ, select then a π-preimage $s(\overline{x}_\theta) := x_\theta$ of $\overline{x}_\theta$. On the orbit of each $\overline{x}_\theta$ we define the section s by $s(g\overline{x}_\theta) := gx_\theta$. This is well defined, for if $g_1\overline{x}_\theta = g_2\overline{x}_\theta$, then $g_1x_\theta\cong g_2x_\theta$, and $g_1x_\theta = g_2x_\theta$ by Lemma 8.14 (i). Define the skeleton Γ_0 to be the full subcategory of $\mathcal{X}$ on the objects in the image of the equivariant section $s\colon \operatorname{iso}(\mathcal{X})\to\operatorname{ob}(\mathcal{X})$.

For each $\overline{x}_\theta$, and each $x\in\overline{x}_\theta$, choose an isomorphism $\eta_x\colon x_\theta\to x$. For gx, we define η_{gx} as $g\eta_x$. Next, we define a functor $r\colon\mathcal{X}\to\Gamma_0$ on objects $x\in\operatorname{ob}(\mathcal{X})$ by $r(x) := s\pi(x)$ and on morphisms $f\colon x\to y$ by $r(f) := \eta_y\circ f\circ\eta_x^{-1}$. Then η is clearly a natural isomorphism, the inclusion $i_0\colon\Gamma_0\to\mathcal{X}$ is G-equivariant, and $ri_0 = \mathrm{id}_{\Gamma_0}$.

References

[1] M. Aigner. *Combinatorial theory.* Springer-Verlag, Berlin, 1979.

[2] S. Allouch. Sur l'existence d'une catégorie ayant une matrice strictement positive donnée. arXiv:0806.0060v1 [math.CT], 2008.

[3] S. Allouch. On the existence of a category with a given matrix. arXiv:1007.2884v1 [math.CT], 2010.

[4] J. Baez and J. Dolan. From finite sets to Feynman diagrams. In *Mathematics Unlimited - 2001 and Beyond*, vol. 1, eds. Björn Engquist and Wilfried Schmid, Springer, Berlin, 2001, pp. 29-50.

[5] A. Bartels and W. Lück. The Borel conjecture for hyperbolic and CAT(0)-groups. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Münster, arXiv:0901.0442v1 [math.GT], 2009.

[6] A. Bartels, W. Lück, and H. Reich. The K-theoretic Farrell-Jones conjecture for hyperbolic groups. *Invent. Math.*, 172(1):29–70, 2008.

[7] A. Bartels, W. Lück, and H. Reich. On the Farrell-Jones Conjecture and its applications. *Journal of Topology*, 1:57–86, 2008.

[8] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and K-theory of group C^*-algebras. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[9] C. Berger and T. Leinster. The Euler characteristic of a category as the sum of a divergent series. *Homology, Homotopy Appl.*, 10(1):41–51, 2008.

[10] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 304.

[11] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*. Springer-Verlag, Berlin, 1999. Die Grundlehren der mathematischen Wissenschaften, Band 319.

[12] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[13] J. Cheeger and M. Gromov. L_2-cohomology and group cohomology. *Topology*, 25(2):189–215, 1986.

[14] J. F. Davis and W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory*, 15(3):201–252, 1998.

[15] E. Dror Farjoun. Homotopy and homology of diagrams of spaces. Algebraic topology, Proc. Workshop, Seattle/Wash. 1985, Lect. Notes Math. 1286, 93-134 (1987).

[16] T. M. Fiore, W. Lück, and R. Sauer. Finiteness obstructions and Euler characteristics of categories. arXiv, 2010. Accepted to *Advances in Mathematics*.

[17] A. Grothendieck. *Pursuing Stacks*. 1983.

[18] A. Haefliger. Complexes of groups and orbihedra. Group theory from a geometrical viewpoint (Trieste, 1990), 504–540, World Sci. Publ., River Edge, NJ, 1991.

[19] A. Haefliger. Extension of complexes of groups. *Ann. Inst. Fourier (Grenoble)*, 42(1-2):275–311, 1992.

[20] L. Illusie. *Complexe cotangent et déformations. II.* Springer-Verlag, Berlin-New York, 1972.

[21] T. Leinster. The Euler characteristic of a category. *Doc. Math.*, 13:21–49, 2008.
[22] W. Lück. Equivariant Eilenberg-Mac Lane spaces $K(G,\mu,1)$ for possibly non-connected or empty fixed point sets. *Manuscripta Math.*, 58:67–75, 1987.
[23] W. Lück. *Transformation groups and algebraic K-theory*. Springer-Verlag, Berlin, 1989. Mathematica Gottingensis.
[24] W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers. I. Foundations. *J. Reine Angew. Math.*, 495:135–162, 1998.
[25] W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers. II. Applications to Grothendieck groups, L^2-Euler characteristics and Burnside groups. *J. Reine Angew. Math.*, 496:213–236, 1998.
[26] W. Lück. The universal functorial Lefschetz invariant. *Fund. Math.*, 161(1-2):167–215, 1999. Algebraic topology (Kazimierz Dolny, 1997).
[27] W. Lück. *L^2-invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.
[28] W. Lück. L^2-invariants from the algebraic point of view. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 295, Münster, arXiv:math.GT/0310489 v1, 2003.
[29] W. Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005.
[30] W. Lück and D. Meintrup. On the universal space for group actions with compact isotropy. In *Geometry and topology: Aarhus (1998)*, pages 293–305. Amer. Math. Soc., Providence, RI, 2000.
[31] W. Lück and H. Reich. The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory. In *Handbook of K-theory. Vol. 1, 2*, pages 703–842. Springer, Berlin, 2005.
[32] G. Maltsiniotis. La théorie de l'homotopie de Grothendieck. *Astérisque* 301, 2005.
[33] D. Meintrup and T. Schick. A model for the universal space for proper actions of a hyperbolic group. *New York J. Math.*, 8:1–7 (electronic), 2002.
[34] R. J. Piacenza. Homotopy theory of diagrams and CW-complexes over a category. *Can. J. Math.* 43(4):814–824, 1991.
[35] M. D. Rabrenović. Classifying spaces of monoids—applications in geometric combinatorics. *Publ. Inst. Math. (Beograd) (N.S.)*, 77(91):21–27, 2005.
[36] J.-P. Serre. *Linear representations of finite groups.* Springer-Verlag, New York, 1977. Translated from the second French edition by Leonard L. Scott, *Graduate Texts in Mathematics*, Vol. 42.

[37] J.-P. Serre. *Trees*. Springer-Verlag, Berlin, 1980. Translated from the French by J. Stillwell.
[38] P. Selick *Introduction to homotopy theory* AMS, Providence, 1997 *Fields Institute Monographs*.
[39] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967.
[40] R. W. Thomason. Homotopy colimits in the category of small categories. *Math. Proc. Camb. Philos. Soc.* 85:91-109, 1979.
[41] R. W. Thomason. Cat as a closed model category. *Cahiers Topologie Géom. Différentielle* 21(3):305-324, 1980.
[42] T. tom Dieck. *Transformation groups and representation theory*. Springer-Verlag, Berlin, 1979.
[43] T. tom Dieck. *Transformation groups*. volume 8 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1987.
[44] C. T. C. Wall. Finiteness conditions for CW-complexes. *Annals of Math. (2)*, 81:56–69, 1965.
[45] C. T. C. Wall. Finiteness conditions for CW-complexes. II. *Proc. Roy. Soc. Ser. A*, 295:129–139, 1966.

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