An Infinite Level Atom coupled to a Heat Bath

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Abstract. We consider a W^*-dynamical system $(\mathfrak{M}_\beta, \tau)$, which models finitely many particles coupled to an infinitely extended heat bath. The energy of the particles can be described by an unbounded operator, which has infinitely many energy levels. We show existence of the dynamics τ and existence of a (β, τ) -KMS state under very explicit conditions on the strength of the interaction and on the inverse temperature β.

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1 Introduction

In this paper, we study a W^*-dynamical system $(\mathfrak{M}_\beta, \tau)$ which describes a system of finitely many particles interacting with an infinitely extended bosonic reservoir or heat bath at inverse temperature β. Here, $\mathfrak{M}_\beta$ denotes the W^*-algebra of observables and τ is an automorphism-group on $\mathfrak{M}_\beta$, which is defined by

$$\tau_t(X) := e^{it\mathcal{L}_Q}\, X\, e^{-it\mathcal{L}_Q},\ X \in \mathfrak{M}_\beta,\ t \in \mathbb{R}. \tag{1}$$

In this context, t is the time parameter. $\mathcal{L}_Q$ is the Liouvillean of the dynamical system at inverse temperature β, Q describes the interaction between particles and heat bath. On the one hand the choice of $\mathcal{L}_Q$ is motivated by heuristic arguments, which allow to derive the Liouvillean $\mathcal{L}_Q$ from the Hamiltonian H of the joint system of particles and bosons at temperature zero. On the other hand we ensure that $\mathcal{L}_Q$ anti-commutes with a certain anti-linear conjugation $\mathcal{J}$, that will be introduced later on. The Hamiltonian, which represents the interaction

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with a bosonic gas at temperature zero, can be the Standard Hamiltonian of the non-relativistic QED, (see or instance [2]), or the Pauli-Fierz operator, which is defined in [7, 2], or the Hamiltonian of Nelson's Model. We give the definition of these Hamiltonians in the sequel of Definition 11.
Our first result is the following:

Theorem 1.1. *$\mathcal{L}_Q$, defined in* (16)*, has a unique self-adjoint realization and $\tau_t(X) \in \mathfrak{M}_\beta$ for all $t \in \mathbb{R}$ and all $X \in \mathfrak{M}_\beta$.*

The proof follows from Theorem 4.2 and Lemma 5.2. The main difficulty in the proof is, that $\mathcal{L}_Q$ is not semi-bounded, and that one has to define a suitable auxiliary operator in order to apply Nelson's commutator theorem.
Partly, we assume that the isolated system of finitely many particles is confined in space. This is reflected in Hypothesis 1, where we assume that the particle Hamiltonian H_{el} possesses a Gibbs state. In the case where H_{el} is a Schrödinger-operator, we give in Remark 2.1 a sufficient condition on the external potential V to ensure the existence of a Gibbs state for H_{el}. Our second theorem is

Theorem 1.2. *Assume Hypothesis 1 and that $\Omega_0^\beta \in \mathrm{dom}(e^{-(\beta/2)(\mathcal{L}_0+Q)})$. Then there exists a (β,τ)-KMS state ω^β on $\mathfrak{M}_\beta$.*

This theorem ensures the existence of an equilibrium state on $\mathfrak{M}_\beta$ for the dynamical system $(\mathfrak{M}_\beta, \tau)$. Its proof is part of Theorem 5.3 below. Here, $\mathcal{L}_0$ denotes the Liouvillean for the joint system of particles and bosons, where the interaction part is omitted. Ω_0^β is the vector representative of the (β,τ)-KMS state for the system without interaction. In a third theorem we study the condition $\Omega_0^\beta \in \mathrm{dom}(e^{-(\beta/2)(\mathcal{L}_0+Q)})$:

Theorem 1.3. *Assume Hypothesis 1 is fulfilled. Then there are two cases,*

1. *If $0 \leqslant \gamma < 1/2$ and $\underline{\eta}_1 (1+\beta) \ll 1$, then $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0+Q)})$.*
2. *If $\gamma = 1/2$ and $(1+\beta)(\underline{\eta}_1 + \underline{\eta}_2) \ll 1$, then $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0+Q)})$.*

Here, $\gamma \in [0, 1/2)$ is a parameter of the model, see (32) and $\underline{\eta}_1, \underline{\eta}_2$ are parameters, which describe the strength of the interaction, see (32). In a last theorem we consider the case where $H_{el} = -\Delta_q + \Theta^2 q^2$ and the interaction Hamiltonian is $\lambda\, q\, \Phi(f)$ at temperature zero for $\lambda \neq 0$. Then,

Theorem 1.4. *Ω_0^β is in $\mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0\,+\,Q)})$ for all $\beta \in (0,\infty)$, whenever*

$$|2\Theta^{-1}\,\lambda|\, \||k|^{-1/2} f\|_{\mathcal{H}_{ph}} < 1.$$

Furthermore, we show that our strategy can not be improved to obtain a result, which ensures existence for all values of λ, see (60).
In the last decade there appeared a large number of mathematical contributions to the theory of open quantum system. Here we only want to mention

some of them [3, 6, 8, 9, 10, 13, 14, 15], which consider a related model, in which the particle Hamilton H_{el} is represented as a finite symmetric matrix and the interaction part of the Hamiltonian is linear in annihilation and creation operators. In this case one can prove existence of a β- KMS without any restriction to the strength of the coupling. (In this case we can apply Theorem 1.3 with $\gamma = 0$ and $\underline{\eta}_1 = 0$). We can show existence of KMS-states for an infinite level atom coupled to a heat bath. Furthermore, in [6] there is a general theorem, which ensures existence of a (β, τ)-KMS state under the assumption, that $\Omega_0^\beta \in \mathrm{dom}(e^{-(\beta/2)Q})$, which implies $\Omega_0^\beta \in \mathrm{dom}(e^{-(\beta/2)(\mathcal{L}_0+Q)})$. In Remark 7.3 we verify that this condition implies the existence of a (β, τ)-KMS state in the case of a harmonic oscillator with dipole interaction $\lambda\, q \cdot \Phi(f)$, whenever $(1+\beta)\lambda\|(1+|k|^{-1/2})f\| \ll 1$.

2 Mathematical Preliminaries

2.1 Fock Space, Field- Operators and Second Quantization

We start our mathematical introduction with the description of the joint system of particles and bosons at temperature zero. The Hilbert space describing bosons at temperature zero is the *bosonic Fock space* $\mathcal{F}_b$, where

$$\mathcal{F}_b \,:=\, \mathcal{F}_b[\mathcal{H}_{ph}] \,:=\, \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_{ph}^{(n)}, \qquad \mathcal{H}_{ph}^{(n)} \,:=\, \bigotimes_{sym}^{n} \mathcal{H}_{ph}.$$

$\mathcal{H}_{ph}$ is either a closed subspace of $L^2(\mathbb{R}^3)$ or $L^2(\mathbb{R}^3 \times \{\pm\})$, being invariant under complex conjugation. If phonons are considered we choose $\mathcal{H}_{ph} = L^2(\mathbb{R}^3)$, if photons are considered we choose $\mathcal{H}_{ph} = L^2(\mathbb{R}^3 \times \{\pm\})$. In the latter case "+" or "-" labels the polarization of the photon. However, we will write $\langle f \,|\, g \rangle_{\mathcal{H}_{ph}} \;:=\; \int \overline{f(k)}\, g(k)\, dk$ for the scalar product in both cases. This is an abbreviation for $\sum_{p=\pm} \int \overline{f(k,p)}\, g(k,p)\, dk$ in the case of photons.

$\mathcal{H}_{ph}^{(n)}$ is the n-fold symmetric tensor product of $\mathcal{H}_{ph}$, that is, it contains all square integrable functions f_n being invariant under permutations π of the variables, i.e., $f_n(k_1, \ldots, k_n) = f_n(k_{\pi(1)}, \ldots, k_{\pi(n)})$. For phonons we have $k_j \in \mathbb{R}^3$ and $k_j \in \mathbb{R}^3 \times \{\pm\}$ for photons. The wave functions in $\mathcal{H}_{ph}^{n}$ are states of n bosons.

The vector $\Omega \,:=\, (1,\, 0,\, \ldots) \in \mathcal{F}_b$ is called the *vacuum*. Furthermore we denote the subspace $\mathcal{F}_b$ of finite sequences with ${\mathcal{F}_b}^{fin}$. On ${\mathcal{F}_b}^{fin}$ the *creation and* *annihilation operators*, $a^*(h)$ and $a(h)$, are defined for $h \in \mathcal{H}_{ph}$ by

$$\begin{aligned}
&(a^*(h)\, f_n)(k_1, \ldots, k_{n+1}) && (2)\\
&\quad = (n+1)^{-1/2} \sum_{i=1}^{n+1} h(k_i)\, f_n(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_{n+1}),\\
&(a(h)\, f_{n+1})(k_1, \ldots, k_n) && (3)\\
&\quad = (n+1)^{1/2} \int \overline{h(k_{n+1})}\, f_{n+1}(k_1, \ldots, k_{n+1})\, dk_{n+1},
\end{aligned}$$

and $a^*(h)\,\Omega = h$, $a(h)\,\Omega = 0$. Since $a^*(h) \subset (a(h))^*$ and $a(h) \subset (a^*(h))^*$, the operators $a^*(h)$ and $a(h)$ are closable. Moreover, the canonical commutation relations (CCR) hold true, i.e.,

$$[a(h)\,,\, a(\widetilde{h})] \,=\, [a^*(h)\,,\, a^*(\widetilde{h})] \,=\, 0, \qquad [a(h)\,,\, a^*(\widetilde{h})] \,=\, \langle\, h \,|\, \widetilde{h}\,\rangle_{\mathcal{H}_{ph}}.$$

Furthermore we define field operator by

$$\Phi(h) \,:=\, 2^{-1/2}\,(a(h) \,+\, a^*(h)), \qquad h \in \mathcal{H}_{ph}.$$

It is a straightforward calculation to check that the vectors in ${\mathcal{F}_b}^{fin}$ are analytic for $\Phi(h)$. Thus, $\Phi(h)$ is essentially self-adjoint on ${\mathcal{F}_b}^{fin}$. In the sequel, we will identify $a^*(h)$, $a(h)$ and $\Phi(h)$ with their closures. The Weyl operators $W(h)$ are given by $W(h) = \exp(i\,\Phi(h))$. They fulfill the CCR-relation for the Weyl operators, i.e.,

$$W(h)\, W(g) \,=\, \exp(i/2\, \mathrm{Im}\, \langle\, h \,|\, g\,\rangle_{\mathcal{H}_{ph}})\, W(g+h),$$

which follows from explicit calculations on ${\mathcal{F}_b}^{fin}$. The Weyl algebra $W(\mathfrak{f})$ over a subspace $\mathfrak{f}$ of $\mathcal{H}_{ph}$ is defined by

$$W(\mathfrak{f}) \,:=\, \mathrm{cl}\, \mathrm{LH}\{W(g) \in \mathcal{B}(\mathcal{F}_b) \,:\, g \in \mathfrak{f}\}. \tag{4}$$

Here, cl denotes the closure with respect to the norm of $\mathcal{B}(\mathcal{F}_b)$, and "LH" denotes the linear hull.
Let $\alpha : \mathbb{R}^3 \to [0, \infty)$ be a locally bounded Borel function and $\mathrm{dom}(\alpha) := \{f \in \mathcal{H}_{ph} \,:\, \alpha f \in \mathcal{H}_{ph}\}$. Note, that $(\alpha f)(k)$ is given by $\alpha(k)\, f(k,p)$ for photons. If $\mathrm{dom}(\alpha)$ is dense subspace of $\mathcal{H}_{ph}$, α defines a self-adjoint multiplication operator on $\mathcal{H}_{ph}$. In this case, the second quantization $d\Gamma(\alpha)$ of α is defined by

$$(d\Gamma(\alpha)\, f_n)(k_1, \ldots, k_n) \,:=\, (\alpha(k_1) + \alpha(k_2) + \ldots + \alpha(k_n))\, f_n(k_1, \ldots, k_n)$$

and $d\Gamma(\alpha)\,\Omega \,=\, 0$ on its maximal domain.

2.2 Hilbert space and Hamiltonian for the particles

Let $\mathcal{H}_{el}$ be a closed, separable subspace of $L^2(X, d\mu)$, that is invariant under complex conjugation. The Hamiltonian H_{el} for the particle is a self-adjoint operator on $\mathcal{H}_{el}$ being bounded from below. We set $H_{el,+} := H_{el} - \inf\sigma(H_{el}) + 1$. Partly, we need the assumption

Hypothesis 1. *Let $\beta > 0$. There exists a small positive constant $\epsilon > 0$, and*

$$\mathrm{Tr}_{\mathcal{H}_{el}}\{e^{-(\beta-\epsilon)\,H_{el}}\} < \infty.$$

The condition implies the existence of a Gibbs state

$$\omega_{el}^{\beta}(A) = \mathcal{Z}^{-1}\,\mathrm{Tr}_{\mathcal{H}_{el}}\{e^{-\beta\,H_{el}}A\}, \quad A \in \mathcal{B}(\mathcal{H}_{el}),$$

for $\mathcal{Z} = \mathrm{Tr}_{\mathcal{H}_{el}}\{e^{-\beta\,H_{el}}\}$.

Remark 2.1. *Let $\mathcal{H}_{el} = L^2(\mathbb{R}^n, d^n x)$ and $H_{el} = -\Delta_x + V_1 + V_2$, where V_1 is a $-\Delta_x$-bounded potential with relative bound $a < 1$ and V_2 is in $L^2_{loc}(\mathbb{R}^n, d^n x)$. Thus H_{el} is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^n)$. Moreover, if additionally*

$$\int e^{-(\beta-\epsilon)\,V_2(x)}\,d^n x < \infty, \tag{5}$$

then one can show, using the Golden-Thompson-inequality, that Hypothesis 1 is satisfied.

2.3 Hilbert space and Hamiltonian for the interacting system

The Hilbert space for the joint system is $\mathcal{H} := \mathcal{H}_{el} \otimes \mathcal{F}_b$. The vectors in $\mathcal{H}$ are sequences $f = (f_n)_{n\in\mathbb{N}_0}$ of wave functions, $f_n \in \mathcal{H}_{el} \otimes \mathcal{H}_{ph}^{(n)}$, obeying

$$\begin{aligned} \underline{k}_n &\mapsto f_n(x, \underline{k}_n) \in \mathcal{H}_{ph}^{(n)} && \text{for } \mu\text{- almost every x} \\ x &\mapsto f_n(x, \underline{k}_n) \in \mathcal{H}_{el} && \text{for Lebesgue - almost every } \underline{k}_n, \end{aligned}$$

where $\underline{k}_n = (k_1, \ldots, k_n)$. The complex conjugate vector is $\overline{f} := (\overline{f}_n)_{n\in\mathbb{N}_0}$. Let $G^j := \{G_k^j\}_{k\in\mathbb{R}^3}$, $H^j := \{H_k^j\}_{k\in\mathbb{R}^3}$ and $F := \{F_k\}_{k\in\mathbb{R}^3}$ be families of closed operators on $\mathcal{H}_{el}$ for $j = 1, \ldots, r$. We assume, that $\mathrm{dom}(F_k^*)$, $\mathrm{dom}(F_k) \supset \mathrm{dom}(H_{el,+}^{1/2})$ and that

$$k \mapsto G_k^j,\ (H_k^j),\ F_k H_{el,+}^{-1/2},\ (F_k)^* H_{el,+}^{-1/2} \in \mathcal{B}(\mathcal{H}_{el})$$

are weakly (Lebesgue-)measurable. For $\phi \in \mathrm{dom}(H_{el,+}^{1/2})$ we assume that

$$k \mapsto (G_k^j\,\phi)(x),\ (H_k^j\,\phi)(x),\ (F_k\,\phi)(x) \in \mathcal{H}_{ph}, \tag{6}$$

$$k \mapsto ((G_k^j)^*\,\phi)(x),\ ((H_k^j)^*\,\phi)(x),\ ((F_k)^*\,\phi)(x) \in \mathcal{H}_{ph},\ \text{for } x \in X. \tag{7}$$

Moreover we assume for $\vec{G} = (G^1, \dots, G^r)$, $\vec{H} := (H^1, \dots, H^r)$ and F, that

$$\|\vec{G}\|_w < \infty, \ \|\vec{H}\|_w < \infty, \ \|F\|_{w,1/2} < \infty,$$

where

$$\|G_j\|_w^2 := \int (\alpha(k) + \alpha(k)^{-1}) \left(\|(G_k^j)^*\|^2_{\mathcal{B}(H_{el})} + \|G_k^j\|^2_{\mathcal{B}(H_{el})} \right) dk$$

$$\|\vec{G}\|_w^2 := \sum_{j=1}^r \|G_j\|_w^2, \qquad \|F\|^2_{w,1/2} := \|F H_{el,+}^{-1/2}\|_w^2 + \|F^* H_{el,+}^{-1/2}\|_w^2.$$

We define for $f = (f_n)_{n=0}^\infty \in \mathrm{dom}(H_{el,+}^{1/2}) \otimes \mathcal{F}_b^{fin}$ the (generalized) creation operator

$$\begin{aligned} &(a^*(F)\, f_n)(x, k_1, \dots, k_{n+1}) \\ &\qquad := (n+1)^{-1/2} \sum_{i=1}^{n+1} (F_{k_i}\, f_n)(x, k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_{n+1}) \end{aligned} \tag{8}$$

and $a(F)\, f_0(x) = 0$. The (generalized) annihilation operator is

$$\begin{aligned} &(a(F)\, f_{n+1})(x, k_1, \dots, k_n) \\ &\qquad := (n+1)^{1/2} \int (F^*_{k_{n+1}}\, f_{n+1})(x, k_1, \dots, k_n, k_{n+1})\, dk_{n+1}. \end{aligned} \tag{9}$$

Moreover, the corresponding (generalized) field operator is $\Phi(F) := 2^{-1/2}(a(F) + a^*(F))$. $\Phi(F)$ is symmetric on $\mathrm{dom}(H_{el,+}^{1/2}) \otimes \mathcal{F}_b^{fin}$. The bounds follow directly from Equations (8) and (9).

$$\begin{aligned} \| a(F) H_{el,+}^{-1/2} f\|^2_{\mathcal{H}} &\leqslant \int |\alpha(k)|^{-1} \|F_k^* H_{el,+}^{-1/2}\|^2_{\mathcal{B}(\mathcal{H}_{el})}\, dk \cdot \|d\Gamma(|\alpha|)^{1/2} f\|^2_{\mathcal{H}} \\ \|a^*(F) H_{el,+}^{-1/2} f\|^2_{\mathcal{H}} &\leqslant \int |\alpha(k)|^{-1} \|F_k H_{el,+}^{-1/2}\|^2_{\mathcal{B}(\mathcal{H}_{el})}\, dk \cdot \|d\Gamma(|\alpha|)^{1/2} f\|^2_{\mathcal{H}} \\ &\quad + \int \|F_k H_{el,+}^{-1/2}\|^2_{\mathcal{B}(\mathcal{H}_{el})}\, dk \cdot \|f\|^2_{\mathcal{H}}. \end{aligned} \tag{10}$$

For $(G_k)^j$, $(H_k)^j \in \mathcal{B}(\mathcal{H}_{el})$, the factor $H_{el,+}^{-1/2}$ can be omitted. The Hamiltonians for the non-interacting, resp. interacting model are

Definition 2.2. *On* $\mathrm{dom}(H_{el}) \otimes \mathrm{dom}(d\Gamma(\alpha)) \cap \mathcal{F}_b^{fin}$ *we define*

$$H_0 := H_{el} \otimes \mathbf{1} + \mathbf{1} \otimes d\Gamma(\alpha), \qquad H := H_0 + W, \tag{11}$$

where $W := \Phi(\vec{G})\,\Phi(\vec{H}) + \mathrm{h.c.} + \Phi(F)$ *and* $\Phi(\vec{G})\,\Phi(\vec{H}) := \sum_{j=1}^r \Phi(G^j)\,\Phi(H^j)$. *The abbreviation "h.c." means the formal adjoint operator of* $\Phi(\vec{G})\,\Phi(\vec{H})$.

We give examples for possible configurations:
Let $\gamma \in \mathbb{R}$ be a small coupling parameter.

▶ The Nelson Model:
$\mathcal{H}_{el} \subset L^2(\mathbb{R}^{3N})$, $H_{el} := -\Delta + V$, $\mathcal{H}_{ph} = L^2(\mathbb{R}^3)$ and $\alpha(k) = |k|$. The form factor is $F_k = \gamma \sum_{\nu=1}^{N} e^{-i\,kx_\nu}\, |k|^{-1/2}\, \mathbf{1}[\,|k| \leqslant \kappa]$, $x_\nu \in \mathbb{R}^3$ and $H^j, G^j = 0$.

▶ The Standard Model of Nonrelativistic QED:
$\mathcal{H}_{el} \subset L^2(\mathbb{R}^{3N})$, $H_{el} := -\Delta + V$, $\mathcal{H}_{ph} = L^2(\mathbb{R}^3 \times \{\pm\})$ and $\alpha(k) = |k|$. The form factors are

$$F_{\mathbf{k}} = 4\gamma^{3/2}\,\pi^{-1/2} \sum_{\nu=1}^{N} (-i\nabla_{x_\nu} \cdot \epsilon(k,p)) e^{-i\,\gamma^{1/2} kx_\nu}\, (2|k|)^{-1/2}\, \mathbf{1}[\,|k| \leqslant \kappa] + \mathrm{h.\,c.},$$

$$G^{i,\,\nu}_{\mathbf{k}} = H^{i,\,\nu}_{\mathbf{k}} = 2\gamma^{3/2}\,\pi^{-1/2}\,\epsilon_i(k,\,p)\, e^{-i\,\gamma^{1/2}\,kx_\nu}\, (2|k|)^{-1/2}\, \mathbf{1}[\,|k| \leqslant \kappa]$$

for $i = 1, 2, 3$, $\nu = 1, \ldots, N$, $x_\nu \in \mathbb{R}^3$ and $\mathbf{k} = (k,p) \in \mathbb{R}^3 \times \{\pm\}$. $\epsilon_i(k,\,\pm) \in \mathbb{R}^3$ are polarization vectors.

▶ The Pauli-Fierz-Model:
$\mathcal{H}_{el} \subset L^2(\mathbb{R}^{3N})$, $H_{el} := -\Delta + V$, $\mathcal{H}_{ph} = L^2(\mathbb{R}^3)$ or $\mathcal{H}_{ph} = L^2(\mathbb{R}^3 \times \{\pm\})$, and $\alpha(k) = |k|$. The form factor is $F_k = \gamma \sum_{\nu=1}^{N} \mathbf{1}[\,|k| \leqslant \kappa]\, k \cdot x_\nu$ and $G^j_k = H^j_k = 0$

3 The Representation π

In order to describe the particle system at inverse temperature β we introduce the algebraic setting. For $\mathfrak{f} = \{f \in \mathcal{H}_{ph} : \alpha^{-1/2} f \in \mathcal{H}_{ph}\}$ we define the algebra of observables by

$$\mathfrak{A} = \mathcal{B}(\mathcal{H}_{el}) \otimes \mathcal{W}(\mathfrak{f}).$$

For elements $A \in \mathfrak{A}$ we define $\tilde{\tau}^0_t(A) := e^{i\,t\,H_0}\, A\, e^{-i\,t\,H_0}$ and $\tilde{\tau}^g_t(A) := e^{i\,t\,H} A e^{-i\,t\,H}$. We first discuss the model without interaction.

3.1 The Representation π_f

The time-evolution for the Weyl operators is given by

$$e^{i\,t\,\check{H}}\, \mathcal{W}(f)\, e^{-i\,t\,\check{H}} = \mathcal{W}(e^{i\,t\,\alpha}\, f).$$

For this time-evolution an equilibrium state ω^β_f is defined by

$$\omega^\beta_f(\mathcal{W}(f)) = \langle\, f\,|\,(1 + 2\,\varrho_\beta)\, f\,\rangle_{\mathcal{H}_{ph}},$$

where $\varrho_\beta(k) = \big(\exp(\beta\,\alpha(k)) - 1\big)^{-1}$. It describes an infinitely extended gas of bosons with momentum density ϱ_β at temperature β. Since ω^β_f is a quasi-free state on the Weyl algebra, the definition of ω^β_f extends to polynomials of creation and annihilation operators. One has

$$\omega_f^\beta(a(f)) = \omega_f^\beta(a^*(f)) = \omega_f^\beta(a(f)\,a(g)) = \omega_f^\beta(a^*(f)\,a^*(g)) = 0,$$
$$\omega_f^\beta(a^*(f)\,a(g)) = \langle\, g \,|\, \varrho_\beta\, f \,\rangle_{\mathcal{H}_{ph}}.$$

For polynomials of higher degree one can apply Wick's theorem for quasi-free states, i.e.,

$$\omega_f^\beta(a^{\sigma_{2m}}(f_{2m})\cdots a^{\sigma_1}(f_1)) = \sum_{P\in\mathcal{Z}_2} \prod_{\substack{\{i,j\}\in P\\ i>j}} \omega_f^\beta\big(a^{\sigma_i}(f_i)\,a^{\sigma_j}(f_j)\big), \tag{12}$$

where $a^{\sigma_k} = a^*$ or $a^{\sigma_k} = a$ for $k = 1, \ldots, 2m$. $\mathcal{Z}_2$ are the pairings, that is

$$P \in \mathcal{Z}_2, \text{ iff } P = \{Q_1, \ldots, Q_m\},\ \#Q_i = 2 \text{ and } \bigcup_{i=1}^m Q_i = \{1, \ldots, 2m\}.$$

The Araki-Woods isomorphism $\pi_f : \mathcal{W}(\mathfrak{f}) \to \mathcal{B}(\mathcal{F}_b \otimes \mathcal{F}_b)$ is defined by

$$\pi_f[\mathcal{W}(f)] := \mathcal{W}_\beta(f) := \exp(i\,\Phi_\beta(f)),$$
$$\Phi_\beta(f) := \Phi((1 + \varrho_\beta)^{1/2} f) \otimes \mathbf{1} + \mathbf{1} \otimes \Phi(\varrho_\beta^{1/2}\,\overline{f}).$$

The vector $\Omega_f^\beta := \Omega \otimes \Omega$ fulfills

$$\omega_f^\beta(\mathcal{W}(f)) = \langle\, \Omega_f^\beta \,|\, \pi_f[\mathcal{W}(f)]\, \Omega_f^\beta \,\rangle. \tag{13}$$

3.2 The representation π^{el}

The particle system without interaction has the observables $\mathcal{B}(\mathcal{H}_{el})$, the states are defined by density operators ρ, i.e., $\rho \in \mathcal{B}(\mathcal{H}_{el}),\ 0 \leqslant \rho,\ \mathrm{Tr}\{\rho\} = 1$. The expectation of $A \in \mathcal{B}(\mathcal{H}_{el})$ in ρ at time t is

$$\mathrm{Tr}\{\,\rho\, e^{i\,t\,H_{el}}\, A\, e^{-i\,t\,H_{el}}\}.$$

Since ρ is a compact, self-adjoint operator, there is an ONB $(\phi_n)_n$ of eigenvectors, with corresponding (positive) eigenvalues $(p_n)_n$. Let

$$\sigma(x, y) = \sum_{n=1}^\infty p_n^{1/2}\, \phi_n(x)\, \overline{\phi_n(y)} \in \mathcal{H}_{el} \otimes \mathcal{H}_{el}. \tag{14}$$

For $\psi \in \mathcal{H}_{el}$ we define $\sigma\,\psi := \int \sigma(x, y)\,\psi(y)\,d\mu(y)$. Obviously, σ is an operator of Hilbert-Schmidt class. Note, $\overline{\sigma}\,\psi := \overline{\sigma\,\overline{\psi}}$ has the integral kernel $\overline{\sigma(x, y)}$. It is a straightforward calculation to verify that

$$\mathrm{Tr}\{\rho\, e^{i\,t\,H_{el}}\, A\, e^{-i\,t\,H_{el}}\} = \langle\, e^{-i\,t\,\mathcal{L}_{el}}\,\sigma \,|\, (A\otimes 1)\, e^{-i\,t\,\mathcal{L}_{el}}\sigma \,\rangle_{\mathcal{H}_{el}\otimes\mathcal{H}_{el}},$$

where $\mathcal{L}_{el} = H_{el}\otimes\mathbf{1} - \mathbf{1}\otimes\overline{H}_{el}$. This suggests the definition of the representation

$$\pi^{el} : \mathcal{B}(\mathcal{H}_{el}) \to \mathcal{B}(\mathcal{H}_{el}\otimes\mathcal{H}_{el}), \quad A \mapsto A \otimes \mathbf{1}.$$

Now, we define the representation map for the joint system by

$$\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K}), \qquad \pi := \pi_{el} \otimes \pi_f,$$

where $\mathcal{K} := \mathcal{H}_{el} \otimes \mathcal{H}_{el} \otimes \mathcal{F}_b \otimes \mathcal{F}_b$. Let $\mathfrak{M}_\beta := \pi[\mathfrak{A}]''$ be the *enveloping* W^*-*algebra*, here $\pi[\mathfrak{A}]'$ denotes the commutant of $\pi[\mathfrak{A}]$, and $\pi[\mathfrak{A}]''$ the bicommutant. We set $\mathcal{D} := U_1 \otimes \overline{U_1} \otimes \mathcal{C}$, where $\mathcal{C}$ is a subspace of vectors in $\mathcal{F}_b^{fin} \otimes \mathcal{F}_b^{fin}$, with compact support, and $U_1 := \cup_{n=1}^{\infty} \operatorname{ran} \mathbb{1}[\mathcal{H}_{el} \leqslant n]$. On $\mathcal{D}$ the operator $\mathcal{L}_0$, given by

$$\begin{aligned} \mathcal{L}_0 \quad &:= \quad \mathcal{L}_{el} \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}_f, \quad \text{on } \mathcal{K}, \\ \mathcal{L}_f \quad &:= \quad d\Gamma(\alpha) \otimes \mathbf{1} - \mathbf{1} \otimes d\Gamma(\alpha), \quad \text{on } \mathcal{F}_b \otimes \mathcal{F}_b, \end{aligned}$$

is essentially self-adjoint and we can define

$$\tau_t^0(X) := e^{i\,t\,\mathcal{L}_0}\, X\, e^{-i\,t\,\mathcal{L}_0} \in \mathfrak{M}_\beta, \quad X \in \mathfrak{M}_\beta, \quad t \in \mathbb{R},$$

It is not hard to see, that

$$\pi[\tilde{\tau}_t^0(A)] = \tau_t^0(\pi[A]), \quad A \in \mathfrak{A}, \quad t \in \mathbb{R}$$

On $\mathcal{K}$ a we introduce a conjugation by

$$\mathcal{J}\,(\phi_1 \otimes \phi_2 \otimes \psi_1 \otimes \psi_2) = \overline{\phi_2} \otimes \overline{\phi_1} \otimes \overline{\psi_2} \otimes \overline{\psi_1}.$$

It is easily seen, that $\mathcal{J}\,\mathcal{L}_0 = -\mathcal{L}_0\,\mathcal{J}$. In this context one has $\mathfrak{M}_\beta' = \mathcal{J}\,\mathfrak{M}_\beta\,\mathcal{J}$, see for example [4]. In the case, where H_{el} fulfills Hypothesis 1, we define the vector representative $\Omega_{el}^\beta \in \mathcal{H}_{el} \otimes \mathcal{H}_{el}$ of the Gibbs state ω_{el}^β as in (14) for $\rho = e^{-\beta H_{el}}\,\mathcal{Z}^{-1}$.

Theorem 3.1. *Assume Hypothesis 1 is fulfilled. Then,* $\Omega_0^\beta := \Omega_{el}^\beta \otimes \Omega_f^\beta$ *is a* cyclic *and* separating *vector for* $\mathfrak{M}_\beta$. $e^{-\beta/2\mathcal{L}_0}$ *is a* modular operator *and* $\mathcal{J}$ *is the* modular conjugation *for* Ω_0^β*, that is*

$$X\Omega_0^\beta \in \operatorname{dom}(e^{-\beta/2\,\mathcal{L}_0}), \quad \mathcal{J}\,X\,\Omega_0^\beta = e^{-\beta/2\,\mathcal{L}_0}\,X^*\,\Omega_0^\beta \tag{15}$$

for all $X \in \mathfrak{M}_\beta$ *and* $\mathcal{L}_0\,\Omega_0^\beta = 0$. *Moreover,*

$$\omega_0^\beta(X) := \langle\, \Omega_0^\beta \,|\, X\,\Omega_0^\beta \,\rangle_{\mathcal{K}}, \qquad X \in \mathfrak{M}_\beta$$

is a (τ^0, β)*-KMS-state for* $\mathfrak{M}_\beta$*, i.e., for all* $X, Y \in \mathfrak{M}_\beta$ *exists* $F_\beta(X, Y, \cdot)$*, analytic in the strip* $S_\beta = \{z \in \mathbb{C} : 0 < Im\,z < \beta\}$*, continuous on the closure and taking the boundary conditions*

$$\begin{aligned} F_\beta(X, Y, t) \quad &= \quad \omega_0^\beta(X\,\tau_t^0(Y)) \\ F_\beta(X, Y, t + i\,\beta) \quad &= \quad \omega_0^\beta(\tau_t^0(Y)\,X) \end{aligned}$$

For a proof see [14].

4 The Liouvillean $\mathcal{L}_Q$

In this and the next section we will introduce the Standard Liouvillean $\mathcal{L}_Q$ for a dynamics τ on $\mathfrak{M}_\beta$, describing the interaction between particles and bosons at inverse temperature β. The label Q denotes the interaction part of the Liouvillean, it can be deduced from the interaction part W of the corresponding Hamiltonian by means of formal arguments, which we will not give here. In a first step we prove self-adjointness of $\mathcal{L}_Q$ and of other Liouvilleans. A main difficulty stems from the fact, that $\mathcal{L}_Q$ and the other Liouvilleans, mentioned before, are not bounded from below. The proof of self-adjointness is given in Theorem 4.2, it uses Nelson's commutator theorem and auxiliary operators which are constructed in Lemma 4.1. The proof, that $\tau_t(X) \in \mathfrak{M}_\beta$ for $X \in \mathfrak{M}_\beta$, is given in Lemma 5.2. Assuming $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2(\mathcal{L}_0+Q)})$ we can ensure existence of a (τ,β)-KMS state $\omega^\beta(X) = \langle \Omega^\beta \,|X\,\Omega^\beta\rangle \cdot \|\Omega^\beta\|^{-2}$ on $\mathfrak{M}_\beta$, where $\Omega^\beta = e^{-\beta/2(\mathcal{L}_0+Q)}\Omega_0^\beta$. Moreover, we can show that $e^{-\beta\mathcal{L}_Q}$ is the modular operator for Ω^β and conjugation $\mathcal{J}$. This is done in Theorem 5.3.

Our proof of 5.3 is inspired by the proof given in [6]. The main difference is that we do not assume, that Q is self-adjoint and that $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta Q})$. For this reason we need to introduce an additional approximation Q_N of Q, which is self-adjoint and affiliated with $\mathfrak{M}_\beta$, see Lemma 5.1.

The interaction on the level of Liouvilleans between particles and bosons is given by Q , where

$$Q \,:=\, \Phi_\beta(\vec{G})\,\Phi_\beta(\vec{H}) \,+\, \mathrm{h.c.} + \Phi_\beta(F), \quad \Phi_\beta(\vec{G})\,\Phi_\beta(\vec{H}) \,:=\, \sum_{j=1}^{r} \Phi_\beta(G^j)\,\Phi_\beta(H^j).$$

For each family $K = \{K_k\}_k$ of closed operators on $\mathcal{H}_{el}$ with $\|K\|_{w,1/2} < \infty$ we set

$$\Phi_\beta(K) \,:=\, \big(a^*((1\,+\,\varrho_\beta)^{1/2}\,K) \otimes \mathbf{1} \,+\, \mathbf{1} \otimes a^*(\varrho_\beta^{1/2}\,K^*)\big) \,+\, \mathrm{h.c.}.$$

Here, K_k acts as $K_k \otimes \mathbf{1}$ on $\mathcal{H}_{el} \otimes \mathcal{H}_{el}$. A Liouvillean, that describes the dynamics of the joint system of particles and bosons is the so-called *Standard Liouvillean*

$$\mathcal{L}_Q\,\phi \,:=\, (\mathcal{L}_0 \,+\, Q \,-\, Q^{\mathcal{J}})\,\phi, \qquad \phi \in \mathcal{D}, \tag{16}$$

which is distinguished by $\mathcal{J}\,\mathcal{L}_Q = -\mathcal{L}_Q\,\mathcal{J}$. For an operator A, acting on $\mathcal{K}$, the symbol $A^{\mathcal{J}}$ is an abbreviation for $\mathcal{J}\,A\,\mathcal{J}$. An important observation is, that $[Q\,,\,Q^{\mathcal{J}}] \,=\, 0$ on $\mathcal{D}$. Next, we define four auxiliary operators on $\mathcal{D}$

$$\begin{aligned}
\mathcal{L}_a^{(1)} \,&:=\, \big(H_{el,+} \otimes \mathbf{1} \,+\, \mathbf{1} \otimes \overline{H}_{el,+}\big) \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}_{f,a} + \mathbf{1} \qquad (17)\\
\mathcal{L}_a^{(2)} \,&:=\, H_{el,+}^Q + (H_{el,+}^Q)^{\mathcal{J}} + c_1\mathbf{1} \otimes \mathcal{L}_{f,a} + c_2\\
\mathcal{L}_a^{(3)} \,&:=\, H_{el,+}^Q + (H_{el,+})^{\mathcal{J}} + c_1\mathbf{1} \otimes \mathcal{L}_{f,a} + c_2\\
\mathcal{L}_a^{(4)} \,&:=\, H_{el,+} \otimes \mathbf{1} + (H_{el,+}^Q)^{\mathcal{J}} + c_1\mathbf{1} \otimes \mathcal{L}_{f,a} + c_2,
\end{aligned}$$

where $\mathcal{L}_{f,a}$ is an operator on $\mathcal{F}_b \otimes \mathcal{F}_b$ and $H^Q_{el,+}$ acts on $\mathcal{K}$. Furthermore,

$$\begin{aligned}\mathcal{L}_{f,a} &= d\Gamma(1+\alpha)\otimes \mathbf{1} + \mathbf{1}\otimes d\Gamma(1+\alpha) + \mathbf{1},\\ \mathcal{L}_{el,a} &= H_{el,+}\otimes \mathbf{1} \,+\, \mathbf{1}\otimes \overline{H}_{el,+} \quad H^Q_{el,+} := H_{el,+}\otimes \mathbf{1} + Q.\end{aligned}$$

Obviously, $\mathcal{L}^{(i)}_a,\ i = 1, 2, 3, 4$ are symmetric operators on $\mathcal{D}$.

Lemma 4.1. *For sufficiently large values of* $c_1, c_2 \geqslant 0$ *we have that* $\mathcal{L}^{(i)}_a,\ i = 1, 2, 3, 4$ *are essentially self-adjoint and positive. Moreover, there is a constant* $c_3 > 0$ *such that*

$$c_3^{-1}\,\|\mathcal{L}^{(1)}_a\,\phi\| \,\leqslant\, \|\mathcal{L}^{(i)}_a\,\phi\| \leqslant c_3\,\|\mathcal{L}^{(1)}_a\,\phi\|, \qquad \phi\in \mathrm{dom}(\mathcal{L}^{(1)}_a). \tag{18}$$

Proof. Let $a, a' \in \{l, r\}$ and $K_i,\ i = 1, 2$ be families of bounded operators with $\|K_i\|_w < \infty$. Let $\Phi_l(K_i) = \Phi(K_i)\otimes\mathbf{1}$ and $\Phi_r(K_i) := \mathbf{1}\otimes\Phi(K_i)$. We have for $\phi\in\mathcal{D}$

$$\begin{aligned}\|\,\Phi_a(\eta K_1)\,\Phi_{a'}(\eta' K_2)\,\phi\,\| &\leqslant \text{const}\,\|\,\mathcal{L}_{f,a}\,\phi\,\| \\ \|\,\Phi_a(\eta F)\,\phi\,\| &\leqslant \text{const}\,\|(\mathcal{L}_{el,a})^{1/2}(\mathcal{L}_{f,a})^{1/2}\,\phi\,\|,\end{aligned} \tag{19}$$

where $\eta, \eta' \in \{(1+\varrho_\beta)^{1/2}, \varrho_\beta^{1/2}\}$. Note, that the estimates hold true, if $\Phi_a(\eta K_i)$ or $\Phi_a(\eta F)$ are replaced by $\Phi_a(\eta K_i)^{\mathcal{J}}$ or $\Phi_a(\eta F)^{\mathcal{J}}$. Thus, we obtain for sufficiently large $c_1 \gg 1$, depending on the form-factors, that

$$\|Q\,\phi\| \,+\, \|Q^{\mathcal{J}}\,\phi\| \,\leqslant\, 1/2\,\big\|\big(\mathcal{L}_{el,a} + c_1\,\mathcal{L}_{f,a}\big)\,\phi\big\|. \tag{20}$$

By the Kato-Rellich-Theorem ([17], Thm. X.12) we deduce that $\mathcal{L}^{(i)}_a$ is self-adjoint on $\mathrm{dom}(\mathcal{L}_{el,a} + c_1\,\mathcal{L}_{f,a})$, bounded from below and that $\mathcal{L}_{el,a} + c_1\,\mathcal{L}_{f,a}$ is $\mathcal{L}^{(i)}_a$-bounded for every $c_2 \geqslant 0$ and $i = 2, 3, 4$. In particular, $\mathcal{D}$ is a core of $\mathcal{L}^{(i)}_a$. The proof follows now from $\|\mathcal{L}^{(i)}_a\,\phi\| \leqslant \|(\mathcal{L}_{el,a}+c_1\,\mathcal{L}_{f,a})\,\phi\| \leqslant c_1\,\|\mathcal{L}^{(1)}_a\,\phi\|$ for $\phi\in\mathcal{D}$. □

Theorem 4.2. *The operators*

$$\mathcal{L}_0, \quad \mathcal{L}_Q = \mathcal{L}_0 + Q - Q^{\mathcal{J}}, \quad \mathcal{L}_0 + Q, \quad \mathcal{L}_0 - Q^{\mathcal{J}}, \tag{21}$$

defined on $\mathcal{D}$*, are essentially self-adjoint. Every core of* $\mathcal{L}^{(1)}_a$ *is a core of the operators in line* (21)*.*

Proof. We restrict ourselves to the case of $\mathcal{L}_Q$. We check the assumptions of Nelson's commutator theorem ([17], Thm. X.37). By Lemma 4.1 it suffices to show $\|\mathcal{L}_Q\phi\| \leqslant \text{const}\,\|\mathcal{L}^{(1)}_a\phi\|$ and $|\langle\,\mathcal{L}_Q\phi\,|\,\mathcal{L}^{(2)}_a\phi\rangle - \langle\,\mathcal{L}^{(2)}_a\phi|\,\mathcal{L}_Q\phi\rangle| \leqslant \text{const}\,\|(\mathcal{L}^{(1)}_a)^{1/2}\phi\|^2$ for $\phi\in\mathcal{D}$. The first inequality follows from Equation (20).

To verify the second inequality we observe

$$
\begin{aligned}
&\big|\big\langle\, \mathcal{L}_Q\, \phi \,\big|\, \mathcal{L}_a^{(2)}\, \phi \,\big\rangle - \big\langle\, \mathcal{L}_a^{(2)}\, \phi \,\big|\, \mathcal{L}_Q\, \phi \,\big\rangle\big| \\
&\quad \leqslant c_1 \Big|\big\langle\, Q\phi \,\big|\, \mathcal{L}_{f,a}\phi \big\rangle - \big\langle\, \mathcal{L}_{f,a}\phi \,\big|\, Q\phi \big\rangle\Big| \\
&\qquad + c_1 \Big|\big\langle\, Q^{\mathcal{J}}\phi \,\big|\, \mathcal{L}_{f,a}\phi \,\big\rangle - \big\langle\, \mathcal{L}_{f,a}\phi \,\big|\, Q^{\mathcal{J}}\phi \,\big\rangle\Big| \\
&\qquad + \Big|\big\langle\, \mathcal{L}_f\phi \big|\, Q\phi \,\big\rangle - \big\langle\, Q\phi \,\big|\, \mathcal{L}_f\phi \,\big\rangle\Big| + \Big|\big\langle\, \mathcal{L}_f\phi \,\big|\, Q^{\mathcal{J}}\phi \,\big\rangle - \big\langle\, Q^{\mathcal{J}}\phi \,\big|\, \mathcal{L}_f\phi \,\big\rangle\Big|,
\end{aligned}
\tag{22}
$$

where we used, that $\big[H^{Q}_{el,+},\, (H^{Q}_{el,+})^{\mathcal{J}}\big] \;=\; 0$. Let $K_i \in \{G_j,\, H_j\}$ and $\eta,\, \eta' \in \{\varrho^{1/2},\, (1\,+\,\varrho)^{1/2}\}$. We remark, that

$$
\begin{aligned}
[\Phi_a(\eta\, K_1)\, \Phi_{a'}(\eta'\, K_2)\,,\, \mathcal{L}_{f,a}] \;&=\; i\, \Phi_a(i\,(1\,+\,\alpha)\,\eta\, K_1)\, \Phi_{a'} v(\eta'\, K_2) \\
&\quad +\; i\, \Phi_a(\eta\, K_1)\, \Phi_{a'}(i\,(1\,+\,\alpha)\,\eta'\, K_2) \\
[\Phi_a(\eta\, F)\,,\, \mathcal{L}_{f,a}] \;&=\; i\, \Phi_a(i\,(1\,+\,\alpha)\,\eta\, F).
\end{aligned}
\tag{23}
$$

Hence, for $\phi \in \mathrm{dom}(\mathcal{L}_a^{(2)})$, we have by means of (10) that

$$
\begin{aligned}
\big|\big\langle\, \phi \,\big|\, [\Phi_a(\eta K_1)\, \Phi_{a'}(\eta' K_2),\; \mathcal{L}_{f,a}]\phi \,\big\rangle\big| \;&\leqslant\; \mathrm{const}\, \|\mathcal{L}_{f,a}^{1/2}\phi\|^2 \\
\big|\big\langle \phi \,\big|\, [\Phi_a(\eta F),\; \mathcal{L}_{f,a}]\phi \,\big\rangle\big| \;&\leqslant\; \mathrm{const}\, \|\mathcal{L}_{f,a}^{1/2}\phi\|\, \|(\mathcal{L}_{el,a})^{1/2}\phi\|.
\end{aligned}
\tag{24}
$$

Thus, (24) is bounded by a constant times $\|(\mathcal{L}_a^{(1)})^{1/2}\phi\|^2$. The essential self-adjointness of $\mathcal{L}_Q$ follows now from estimates analog to (23) and (24), where $\mathcal{L}_{f,a}$ is replaced by $\mathcal{L}_f$ in (23) and in the left side of (24). For $\mathcal{L}_0{+}Q$ and $\mathcal{L}_0{-}Q^{\mathcal{J}}$ one has to consider the commutator with $\mathcal{L}_a^{(3)}$ and $\mathcal{L}_a^{(4)}$, respectively. □

Remark 4.3. *In the same way one can show, that H is essentially self-adjoint on any core of $H_1 := H_{el} \,+\, d\Gamma(1\,+\,\alpha)$, even if H is not bounded from below.*

5 Regularized Interaction and Standard Form of $\mathfrak{M}_\beta$

In this subsection a regularized interaction Q_N is introduced:

$$
Q_N \;:=\; \Big\{\Phi_\beta(\vec{G}_N)\, \Phi_\beta(\vec{H}_N) \,+\, \mathrm{h.c.}\Big\} \,+\, \Phi_\beta(F_N). \tag{25}
$$

The regularized form factors $\vec{G}_N$, $\vec{H}_N$, F_N are obtained by multiplying the finite rank projection $P_N := \mathbf{1}[H_{el} \leqslant N]$ from the left and the right. Moreover, an additional ultraviolet cut-off $\mathbf{1}[\alpha \leqslant N]$, considered as a spectral projection, is added. The regularized form factors are

$$
\begin{gathered}
\vec{G}_N(k) \;:=\; \mathbf{1}[\alpha \,\leqslant\, N]\, P_N\, \vec{G}(k)\, P_N, \qquad \vec{H}_N(k) \;:=\; \mathbf{1}[\alpha \,\leqslant\, N]\, P_N\, \vec{H}(k)\, P_N, \\
F_N(k) \;:=\; \mathbf{1}[\alpha \,\leqslant\, N]\, P_N\, F(k)\, P_N.
\end{gathered}
$$

Lemma 5.1. *i) Q_N is essentially self-adjoint on $\mathcal{D} \subset \mathrm{dom}(Q_N)$. Q_N is affiliated with $\mathfrak{M}_\beta$, i.e,. Q_N is closed and*

$$X'\, Q_N \subset Q_N\, X', \quad \forall\, X' \in \mathfrak{M}_\beta'.$$

ii) $\mathcal{L}_0 + Q_N$, $\mathcal{L}_0 - \mathcal{J} Q_N \mathcal{J}$ and $\mathcal{L}_0 + Q_N - \mathcal{J} Q_N \mathcal{J}$ converges in the strong resolvent sense to $\mathcal{L}_0 + Q$, $\mathcal{L}_0 - \mathcal{J} Q \mathcal{J}$ and $\mathcal{L}_0 + Q - \mathcal{J} Q \mathcal{J}$, respectively.

Proof. Let Q_N be defined on $\mathcal{D}$. With the same arguments as in the proof of Theorem 4.2 we obtain

$$\|Q_N\phi\| \leqslant C\|\mathcal{L}_{f,a}\phi\|, \quad \big|\big\langle\, Q_N\phi \,\big|\, \mathcal{L}_{f,a}\phi\big\rangle \,-\, \big\langle\, \mathcal{L}_{f,a}\phi \,\big|\, Q_N\phi\big\rangle\big| \,\leqslant\, C\big\|(\mathcal{L}_{f,a})^{1/2}\phi\big\|^2,$$

for $\phi \in \mathcal{D}$ and some constant $C > 0$, where we have used that $\|F_N\|_w \,<\, \infty$. Thus, from Theorem 4.2 and Nelson's commutator theorem we obtain that $\mathcal{D}$ is a common core for Q_N, $\mathcal{L}_0 + Q_N$, $\mathcal{L}_0 \,-\, Q_N^{\mathcal{J}}$, $\mathcal{L}_0 \,+\, Q_N \,-\, Q_N^{\mathcal{J}}$ and for the operators in line (21). A straightforward calculation yields

$$\lim_{N\to\infty} Q_N\phi \,=\, Q\phi, \quad \lim_{N\to\infty} \mathcal{J} Q_N \mathcal{J}\phi \,=\, \mathcal{J} Q \mathcal{J}\phi \qquad \forall\, \phi \in \mathcal{D}.$$

Thus statement ii) follows, since it suffices to check strong convergence on the common core $\mathcal{D}$, see [16, Theorem VIII.25 a)].
Let $N_f \,:=\, d\Gamma(1)\otimes\mathbf{1}+\mathbf{1}\otimes d\Gamma(\mathbf{1})$ be the number-operator. Since $\mathrm{dom}(N_f) \supset \mathcal{D}$ and $\mathcal{W}_\beta(f)^{\mathcal{J}} \,:\, \mathrm{dom}(N_f) \,\to\, \mathrm{dom}(N_f)$, see [4], we obtain

$$Q_N(A \otimes \mathbf{1} \otimes \mathcal{W}_\beta(f))^{\mathcal{J}}\phi \,=\, (A \otimes \mathbf{1} \otimes \mathcal{W}_\beta(f))^{\mathcal{J}} Q_N\phi \tag{26}$$

for $A \in \mathcal{B}(\mathcal{H}_{el})$, $f \in \mathfrak{f}$ and $\phi \in \mathcal{D}$. By closedness of Q_N and density arguments the equality holds for $\phi \in \mathrm{dom}(Q_N)$ and $X \in \mathfrak{M}_\beta$ instead of $A \otimes \mathbf{1} \otimes \mathcal{W}_\beta(f)$. Thus Q_N is affiliated with $\mathfrak{M}_\beta$ and therefore $e^{i\,tQ_N} \in \mathfrak{M}_\beta$ for $t \in \mathbb{R}$.

□

Lemma 5.2. *We have for $X \in \mathfrak{M}_\beta$ and $t \in \mathbb{R}$*

$$\tau_t(X) = e^{it(\mathcal{L}_0+Q)}\, X\, e^{it(\mathcal{L}_0+Q)}, \quad \tau_t^0(X) = e^{it(\mathcal{L}_0-Q^{\mathcal{J}})}\, X\, e^{it(\mathcal{L}_0-Q^{\mathcal{J}})} \tag{27}$$

Moreover, $\tau_t(X) \in \mathfrak{M}_\beta$ for all $X \in \mathfrak{M}_\beta$ and $t \in \mathbb{R}$, such as

$$E_Q(t) \,:=\, e^{i\,t\,(\mathcal{L}_0\,+\,Q)}\, e^{-i\,t\,\mathcal{L}_0} = e^{i\,t\,\mathcal{L}_Q}\, e^{-i\,t\,(\mathcal{L}_0-Q^{\mathcal{J}})} \in \mathfrak{M}_\beta.$$

Proof. First, we prove the statement for Q_N, since Q_N is affiliated with $\mathfrak{M}_\beta$ and therefore $e^{itQ_N} \in \mathfrak{M}_\beta$. We set

$$\hat{\tau}_t^N(X) = e^{i\,t\,(\mathcal{L}_0\,+\,Q_N)}\, X\, e^{-i\,t\,(\mathcal{L}_0\,+\,Q_N)}, \quad \hat{\tau}_t(X) = e^{i\,t\,(\mathcal{L}_0\,+\,Q)}\, X\, e^{-i\,t\,(\mathcal{L}_0\,+\,Q)} \tag{28}$$

On account of Lemma 5.1 and Theorem 4.2 we can apply the Trotter product formula to obtain

$$\begin{aligned}\hat{\tau}_t^N(X) &= \text{w-lim}_{n\to\infty} \left(e^{i\frac{t}{n}\mathcal{L}_0} e^{i\frac{t}{n}Q_N}\right)^n X \left(e^{-i\frac{t}{n}Q_N} e^{-i\frac{t}{n}\mathcal{L}_0}\right)^n \\ &= \text{w-lim}_{n\to\infty} \tau^0_{\frac{t}{n}}\left(e^{i\frac{t}{n}Q_N}\cdots\tau^0_{\frac{t}{n}}(e^{i\frac{t}{n}Q_N} X e^{-i\frac{t}{n}Q_N})\cdots e^{-i\frac{t}{n}Q_N}\right).\end{aligned}$$

Since $e^{i\frac{t}{n}Q_N}, X \in \mathfrak{M}_\beta$ and since τ^0 leaves $\mathfrak{M}_\beta$ invariant, $\hat{\tau}_t^N(X)$ is the weak limit of elements of $\mathfrak{M}_\beta$, and hence $\hat{\tau}_t^N(X) \in \mathfrak{M}_\beta$. Moreover,

$$\hat{\tau}_t(X) = \text{w-lim}_{N\to\infty} \hat{\tau}_t^N(X) \in \mathfrak{M}_\beta.$$

For $E_N(t) := e^{it(\mathcal{L}_0+Q_N)}e^{-it\mathcal{L}_0} \in \mathcal{B}(\mathcal{K})$ we obtain

$$\begin{aligned}e^{it(\mathcal{L}_0+Q_N)}e^{-it\mathcal{L}_0} &= \text{s-lim}_{n\to\infty}\left(e^{i\frac{t}{n}\mathcal{L}_0}e^{i\frac{t}{n}Q_N}\right)^n e^{-it\mathcal{L}_0} \\ &= \text{s-lim}_{n\to\infty} \tau^0_{\frac{t}{n}}(e^{i\frac{t}{n}Q_N})\tau^0_{\frac{2t}{n}}(e^{i\frac{t}{n}Q_N})\cdots\tau^0_{\frac{nt}{n}}(e^{i\frac{t}{n}Q_N}) \in \mathfrak{M}_\beta.\end{aligned}$$

By virtue of Lemma 5.1 we get $E_Q(t) := e^{it(\mathcal{L}_0+Q)}e^{-it\mathcal{L}_0} = \text{w-lim}_{N\to\infty} E_N(t) \in \mathfrak{M}_\beta$. Since $\mathcal{J}$ leaves $\mathcal{D}$ invariant and thanks to Lemma 5.1, we deduce, that $\mathcal{D}$ is a core of $\mathcal{J}Q_N\mathcal{J}$. Moreover, we have $e^{-itQ_N^{\mathcal{J}}} = \mathcal{J}e^{itQ_N}\mathcal{J} \in \mathfrak{M}'_\beta$. Since we have shown, that $\hat{\tau}^N$ leaves $\mathfrak{M}_\beta$ invariant, we get

$$\begin{aligned}\tau_t^N(X) &= \text{w-lim}_{n\to\infty}(e^{i\frac{t}{n}(\mathcal{L}_0+Q_N)}e^{i\frac{t}{n}(-Q_N^{\mathcal{J}})})^n X \,(e^{-i\frac{t}{n}(-Q_N^{\mathcal{J}})}e^{-i\frac{t}{n}(\mathcal{L}_0+Q_N)})^n \\ &= \text{w-lim}_{n\to\infty}\hat{\tau}^N_{\frac{t}{n}}\left(e^{-i\frac{t}{n}Q_N^{\mathcal{J}}}\cdots\hat{\tau}^N_{\frac{t}{n}}(e^{-i\frac{t}{n}Q_N^{\mathcal{J}}} X e^{i\frac{t}{n}Q_N^{\mathcal{J}}})\cdots e^{i\frac{t}{n}Q_N^{\mathcal{J}}}\right) \\ &= \hat{\tau}_t^N(X).\end{aligned}$$

Thanks to Lemma 5.1 we also have

$$\tau_t(X) = \text{w-lim}_{n\to\infty}\tau_t^N(X) = \text{w-lim}_{\mathbb{N}\to\infty}\hat{\tau}_t^N(X) = \hat{\tau}_t(X). \tag{29}$$

The proof of $\tau_t^0(X) = e^{it(\mathcal{L}_0-Q^{\mathcal{J}})}\,X\,e^{it(\mathcal{L}_0-Q^{\mathcal{J}})}$ follows analogously. Using the Trotter product formula we obtain

$$\begin{aligned}e^{it(\mathcal{L}_0+Q_N)}\,e^{-it\mathcal{L}_0} &= \text{s-lim}_{n\to\infty}\left(e^{i\frac{t}{n}\mathcal{L}_0}e^{i\frac{t}{n}Q_N}\right)^n e^{-it\mathcal{L}_0} \\ &= \text{s-lim}_{n\to\infty}\tau^0_{\frac{t}{n}}(e^{i\frac{t}{n}Q_N})\tau^0_{\frac{2t}{n}}(e^{i\frac{t}{n}Q_N})\cdots\tau^0_{\frac{nt}{n}}(e^{i\frac{t}{n}Q_N}) \\ &= \text{s-lim}_{n\to\infty}\left(e^{i\frac{t}{n}(\mathcal{L}_0-Q_N{}^{\mathcal{J}})}e^{i\frac{t}{n}Q_N}\right)^n e^{-it(\mathcal{L}_0-Q_N{}^{\mathcal{J}})} \\ &= e^{it(\mathcal{L}_0+Q_N-\mathcal{J}Q_N\mathcal{J})}\,e^{-it(\mathcal{L}_0-Q_N{}^{\mathcal{J}})}.\end{aligned}$$

By strong resolvent convergence we may deduce $E(t) = e^{it\mathcal{L}_Q}\,e^{-it(\mathcal{L}_0-Q^{\mathcal{J}})}$. □

Let $\mathcal{C}$ be the natural positive cone associated with $\mathcal{J}$ and Ω_0^β and let $\mathfrak{M}_\beta^{ana}$ be the τ-analytic elements of $\mathfrak{M}_\beta$, (see [4]).

Theorem 5.3. *Assume Hypothesis 1 and* $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0 + Q)})$. *Let* $\Omega^\beta := e^{-\beta/2\,(\mathcal{L}_0 + Q)}\,\Omega_0^\beta$. *Then*

$$\begin{aligned} \mathcal{J}\,\Omega^\beta \;&=\; \Omega^\beta, \qquad \Omega^\beta \;=\; e^{\beta/2\,(\mathcal{L}_0\, -\, Q^{\mathcal{J}})}\,\Omega_0^\beta, \\ \mathcal{L}_Q\,\Omega^\beta \;&=\; 0, \qquad \mathcal{J}\,X^*\,\Omega^\beta \;=\; e^{-\beta/2\,\mathcal{L}_Q}\,X\,\Omega^\beta, \quad \forall\, X \in \mathfrak{M}_\beta \end{aligned} \tag{30}$$

Furthermore, Ω^β *is separating and cyclic for* $\mathfrak{M}_\beta$, *and* $\Omega^\beta \in \mathcal{C}$. *The state* ω^β *is defined by*

$$\omega^\beta(X) := \|\Omega^\beta\|^{-2}\,\langle \Omega^\beta\,|X\,\Omega^\beta\rangle,\ X \in \mathfrak{M}_\beta$$

is a $(\tau,\,\beta)$*-KMS state on* $\mathfrak{M}_\beta$.

Proof. First, we define $\Omega(z) \,=\, e^{-z\,(\mathcal{L}_0+Q)}\,\Omega_0^\beta$ for $z \in \mathbb{C}$ with $0 \leqslant \mathrm{Re}\, z \leqslant \beta/2$. Since $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0+Q)})$, $\Omega(z)$ is analytic on $\mathcal{S}_{\beta/2} := \{z \in \mathbb{C} \,:\, 0 < \mathrm{Re}\,(z) < \alpha\}$ and continuous on the closure of $\mathcal{S}_{\beta/2}$, see Lemma A.2 below.

► Proof of $\mathcal{J}\,\Omega(\beta/2) \,=\, \Omega(\beta/2)$:
We pick $\phi \in \bigcup_{n\in\mathbb{N}} \mathrm{ran}\,\mathbf{1}[|\mathcal{L}_0| \leqslant n]$. Let $f(z) := \langle\, \phi\,|\,\mathcal{J}\,\Omega(\overline{z})\,\rangle$ and $g(z) := \langle\, e^{-(\beta/2\,-\,\overline{z})\,\mathcal{L}_0}\,\phi\,|\,e^{-z\,(\mathcal{L}_0\,+\,Q)}\,\Omega_0^\beta\,\rangle$. Both f and g are analytic on $\mathcal{S}_{\beta/2}$ and continuous on its closure. Thanks to Lemma 5.2 we have $E_Q(t) \in \mathfrak{M}_\beta$, and hence

$$f(it) \,=\, \langle\, \phi\,|\,\mathcal{J}\,E_Q(t)\,\Omega_0^\beta\,\rangle \,=\, \langle\, \phi\,|\,e^{-\beta/2\,\mathcal{L}_0}\,E_Q(t)^*\,\Omega_0^\beta\,\rangle \,=\, g(i\,t),\ t \in \mathbb{R}.$$

By Lemma A.1, f and g are equal, in particular in $z \,=\, \beta/2$. Note that ϕ is any element of a dense subspace.

► Proof of $\Omega_0^\beta \in \mathrm{dom}(e^{\beta/2\,(\mathcal{L}_0 - Q^{\mathcal{J}})})$ and $\Omega(\beta/2) \,=\, e^{\beta/2\,(\mathcal{L}_0 - Q^{\mathcal{J}})}\,\Omega_0^\beta$:
Let $\phi \in \bigcup_{n\in\mathbb{N}} \mathrm{ran}\,\mathbf{1}[|\mathcal{L}_0 \,-\, Q^{\mathcal{J}}| \leqslant n]$. We set $g(z) := \langle\, e^{\overline{z}(\mathcal{L}_0\, - Q^{\mathcal{J}})}\,\phi\,|\,e^{-z\,\mathcal{L}_0}\,\Omega_0^\beta\,\rangle$. Since $E_Q(t)^{\mathcal{J}} \,=\, e^{i\,t\,(\mathcal{L}_0 - Q^{\mathcal{J}})}\,e^{-i\,t\,\mathcal{L}_0}$, g coincides for $z \,=\, i\,t$ with $f(z) := \langle\, \phi\,|\,\mathcal{J}\,\Omega(\overline{z})\,\rangle$. Hence they are equal in $z \,=\, \beta/2$. The rest follows since $e^{\beta/2\,(\mathcal{L}_0 - Q^{\mathcal{J}})}$ is self-adjoint.

► Proof of $\mathcal{L}_Q\,\Omega(\beta/2) \,=\, 0$:
Choose $\phi \in \bigcup_{n\in\mathbb{N}} \mathrm{ran}\,\mathbf{1}[|\mathcal{L}_Q| \leqslant n]$. We define $g(z) := \langle\, e^{-\overline{z}\mathcal{L}_Q}\phi\,|\,e^{z\,(\mathcal{L}_0 - Q^{\mathcal{J}})}\,\Omega_0^\beta\,\rangle$ and $f(z) := \langle\, \phi\,|\,\Omega(z)\,\rangle$ for z in the closure of $\mathcal{S}_{\beta/2}$. Again both functions are equal on the line $z \,=\, i\,t$, $t \in \mathbb{R}$. Hence f and g are identical, and therefore $\Omega(\beta/2) \in \mathrm{dom}(e^{-\beta/2\,\mathcal{L}_Q})$ and $e^{-\beta/2\mathcal{L}_Q}\,\Omega(\beta/2) \,=\, \Omega(\beta/2)$. We conclude that $\mathcal{L}_Q\,\Omega(\beta/2) \,=\, 0$.

► Proof of $\mathcal{J}\,X^*\,\Omega(\beta/2) \,=\, e^{-\beta/2\mathcal{L}_Q}\,X\,\Omega(\beta/2),\ \forall\, X \in \mathfrak{M}_\beta$:
Fore $A \in \mathfrak{M}_\beta^{ana}$ we have, that

$$\begin{aligned} \mathcal{J}\,A^*\,\Omega(-it) \,&=\, \mathcal{J}\,A^*\,E_Q(t)\,\Omega_0^\beta \,=\, e^{-\beta/2\mathcal{L}_0}\,E_Q(t)^* A\Omega_0^\beta \\ &=\, e^{-(\beta/2\,-\,i\,t)\,\mathcal{L}_0}\,e^{-it(\mathcal{L}_0+Q)}\,A\,\Omega_0^\beta \\ &= e^{-(\beta/2-it)\mathcal{L}_0}\,\tau_{-t}(A)\,e^{-it(\mathcal{L}_0+Q)}\,\Omega_0^\beta. \end{aligned}$$

Let $\phi \in \bigcup_{n\in\mathbb{N}} \operatorname{ran} \mathbf{1}[|\mathcal{L}_0| \leqslant n]$. We define $f(z) = \langle \phi \,|\, \mathcal{J} A^* \,\Omega(\overline{z}) \rangle$ and $g(z) = \langle e^{-(\beta/2-\overline{z})\mathcal{L}_0} \phi \,|\, \tau_{iz}(A)\, \Omega(z) \rangle$. Since f and g are analytic and equal for $z = it$, we have $\mathcal{J} A^*\Omega(\beta/2) = \tau_{i\,\beta/2}(A)\,\Omega(\beta/2)$. To finish the proof we pick $\phi \in \bigcup_{n\in\mathbb{N}} \operatorname{ran} \mathbf{1}[|\mathcal{L}_Q| \leqslant n]$, and set $f(z) := \langle \phi \,|\, \tau_{iz}(A)\,\Omega(\beta/2) \rangle$ and $g(z) := \langle e^{-\overline{z}\mathcal{L}_Q} \phi \,|\, A\Omega(\beta/2) \rangle$. For $z = i\,t$ we see

$$g(it) = \langle \phi \,|\, e^{-it\mathcal{L}_Q}\, A\, e^{it\mathcal{L}_Q}\Omega(\beta/2) \rangle = \langle \phi \,|\, \tau_{-t}(A)\,\Omega(\beta/2) \rangle = f(i\,t).$$

Hence $A\,\Omega(\beta/2) \in \operatorname{dom}(e^{-\beta/2\,\mathcal{L}_Q})$ and $\mathcal{J} A^*\Omega(\beta/2) = e^{-\beta/2\mathcal{L}_Q} A\Omega(\beta/2)$.
Since $\mathfrak{M}_\beta^{ana}$ is dense in the strong topology, the equality holds for all $X \in \mathfrak{M}_\beta$.

► Proof, that Ω^β is separating for $\mathfrak{M}_\beta$:
Let $A \in \mathfrak{M}_\beta^{ana}$. We choose $\phi \in \bigcup_{n\in\mathbb{N}} \operatorname{ran} \mathbf{1}[|(\mathcal{L}_0 + Q)| \leqslant n]$. First, we have

$$\mathcal{J}\, A^*\,\Omega(\beta/2) = \tau_{i\beta/2}(A)\,\Omega(\beta/2).$$

Let $f_\phi(z) = \langle \phi | \tau_z(A)\Omega(\beta/2) \rangle$ and $g_\phi(z) = \langle e^{\overline{z}(\mathcal{L}_0+Q)}\phi \,|\, Ae^{-(\beta/2+z)(\mathcal{L}_0+Q)}\Omega_0^\beta \rangle$ for $-\beta/2 \leqslant \operatorname{Re} z \leqslant 0$. Both functions are continuous and analytic if $-\beta/2 < \operatorname{Re} z < 0$. Furthermore, $f_\phi(i\,t) = g_\phi(i\,t)$ for $t \in \mathbb{R}$. Hence $f_\phi = g_\phi$ and for $z = -\beta/2$

$$\langle \phi \,|\, \mathcal{J}\, A^*\Omega(\beta/2) \rangle = \langle e^{-\beta/2\,(\mathcal{L}_0+Q)}\phi \,|\, A\Omega_0^\beta \rangle.$$

This equation extends to all $A \in \mathfrak{M}_\beta$, we obtain $A\,\Omega_0^\beta \in \operatorname{dom}(e^{-\beta/2\,(\mathcal{L}_0+Q)})$, such as $e^{-\beta/2(\mathcal{L}_0+Q)}\,A\,\Omega_0^\beta = \mathcal{J}\,A^*\Omega(\beta/2)$ for $A \in \mathfrak{M}_\beta$. Assume $A^*\,\Omega(\beta/2) = 0$, then
$e^{-\beta/2\,(\mathcal{L}_0+Q)} A\Omega_0^\beta = 0$ and hence $A\Omega_0^\beta = 0$. Since Ω_0^β is separating, it follows that $A = 0$ and therefore $A^* = 0$.

► Proof of $\Omega^\beta \in \mathcal{C}$, and that Ω^β is cyclic for $\mathfrak{M}_\beta$:
To prove that $\phi \in \mathcal{C}$ it is sufficient to check that $\langle \phi \,|\, A\mathcal{J}A\Omega_0^\beta \rangle \geqslant 0$ for all $A \in \mathfrak{M}_\beta$. We have

$$\begin{aligned} \langle \Omega(\beta/2) |\, A\mathcal{J}A\,\Omega_0^\beta \rangle &= \overline{\langle \mathcal{J}A^*\Omega(\beta/2)\,|A\Omega_0^\beta \rangle} \\ &= \overline{\langle e^{-\beta/2(\mathcal{L}_0+Q)}\, A\Omega_0^\beta \,|\, A\Omega_0^\beta \rangle} \geqslant 0. \end{aligned}$$

The proof follows, since every separating element of $\mathcal{C}$ is cyclic.
► Proof, that ω^β is a $(\tau,\,\beta)$-KMS state:
For $A,\,B \in \mathfrak{M}_\beta$ and $z \in S_\beta$ we define

$$F_\beta(A,\,B,\,z) = c\,\langle e^{-i\overline{z}/2\mathcal{L}_Q} A^*\Omega^\beta \,|\, e^{iz/2\mathcal{L}_Q} B\Omega^\beta \rangle,$$

where $c := \|\Omega^\beta\|^{-2}$. First, we observe

$$\begin{aligned} F_\beta(A,\,B,\,t) &= c\,\langle e^{-it/2\mathcal{L}_Q} A^*\Omega^\beta \,|\, e^{it/2\mathcal{L}_Q} B\Omega^\beta \rangle = c\,\langle \Omega^\beta\,| A\tau_t(B)\Omega^\beta \rangle \\ &= \omega^\beta(A\,\tau_t(B)) \end{aligned}$$

and

$$\begin{aligned}\omega^\beta(\tau_t(B)A) &= c\,\langle\, \tau_t(B^*)\Omega^\beta \,|\, A\Omega^\beta \,\rangle \;=\; c\,\langle\, \mathcal{J} A\Omega^\beta \,|\, \mathcal{J}\tau_t(B^*)\Omega^\beta \,\rangle \\ &= c\,\langle\, e^{-\beta/2\mathcal{L}_Q} A^*\Omega^\beta \,|\, e^{-\beta/2\,\mathcal{L}_Q}\tau_t(B)\Omega^\beta \,\rangle \\ &= c\,\langle\, e^{-i\overline{(i\beta+t)}/2\mathcal{L}_Q} A^*\Omega^\beta \,|\, e^{i\,(i\beta+t)/2\mathcal{L}_Q} B\Omega^\beta \,\rangle \\ &= F_\beta(A,\, B, t+i\beta).\end{aligned}$$

The requirements on the analyticity of $F_\beta(A,\, B,\, \cdot)$ follow from Lemma A.2. $\square$

6 Proof of Theorem 1.3

For $\underline{s}_n := (s_n, \ldots, s_1) \in \mathbb{R}^n$ we define

$$Q_N(\underline{s}_n) \;:=\; Q_N(s_n)\cdots Q_N(s_1), \qquad Q_N(s) \;:=\; e^{-s\mathcal{L}_0} Q_N e^{s\,\mathcal{L}_0},\;\; s\in\mathbb{R} \tag{31}$$

At this point, we check that $Q_N(\underline{s}_n)\Omega_0^\beta$ is well defined, and that it is an analytic vector of $\mathcal{L}_0$, see Equation (25). The goal of Theorem 1.3 is to give explicit conditions on H_{el} and W, which ensure $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0+Q)})$. Let

$$\underline{\eta}_1 \;:=\; \int \big(\|\vec{G}(k)\|^2_{\mathcal{B}(\mathcal{H}_{el})} + \|\vec{H}(k)\|^2_{\mathcal{B}(\mathcal{H}_{el})}\big)(2+4\alpha(k)^{-1})\,dk \tag{32}$$

$$\underline{\eta}_2 \;:=\; \int \big(\|F(k)\,H_{el,+}^{-\gamma}\|^2_{\mathcal{B}(\mathcal{H}_{el})} + \|F(k)^*\,H_{el,+}^{-\gamma}\|_{\mathcal{B}(\mathcal{H}_{el})}\big)(2+4\alpha(k)^{-1})\,dk$$

The idea of the proof is the following. First, we expand $e^{-\beta/2(\mathcal{L}_0+Q_N)}e^{\mathcal{L}_0}$ in a Dyson-series, i.e.,

$$\begin{aligned}&e^{-\beta/2(\mathcal{L}_0+Q_N)}e^{\mathcal{L}_0} \\ &\quad= \mathbf{1} + \sum_{n=1}^\infty (-1)^n \int_{\Delta^n_{\beta/2}} e^{-s_n\mathcal{L}_0} Q_N e^{s_n\,\mathcal{L}_0}\cdots e^{-s_1\mathcal{L}_0}Q_N e^{s_1\,\mathcal{L}_0}\,d\underline{s}_n.\end{aligned} \tag{33}$$

Under the assumptions of Theorem 1.3 we obtain an upper bound, uniform in N, for

$$\begin{aligned}&\langle \Omega_0^\beta \,|\, e^{-\beta(\mathcal{L}_0+Q_N)}\Omega_0^\beta\rangle \\ &\quad= 1 + \sum_{n=1}^\infty (-1)^n \int_{\Delta^n_\beta} \langle \Omega_0^\beta \,|\, e^{-s_n\mathcal{L}_0} Q_N e^{s_n\,\mathcal{L}_0}\cdots e^{-s_1\mathcal{L}_0}Q_N e^{s_1\,\mathcal{L}_0}\Omega_0^\beta\rangle\,d\underline{s}_n.\end{aligned} \tag{34}$$

This is proven in Lemma 6.4 below, which is the most important part of this section. In Lemma 6.1 and Lemma 6.2 we deduce from the upper bound for (34) an upper bound for $\|e^{-(\beta/2)(\mathcal{L}_0+Q_N)}\Omega_0^\beta\|$, which is uniform in N. The proof of Theorem 1.3 follows now from Lemma 6.3, where we show that $\Omega_0^\beta \in \mathrm{dom}(e^{-(\beta/2)(\mathcal{L}_0+Q)})$.

Lemma 6.1. *Assume*

$$\limsup_{n\to\infty}\ \sup_{0\leqslant x\leqslant\beta/2}\Big\|\int_{\Delta_x^n} Q_N(\underline{s}_n)\Omega_0^\beta\, d\underline{s}_n\Big\|^{1/n} < 1.$$

for all $N\in\mathbb{N}$. *Then* $\Omega_0^\beta\in\mathrm{dom}(e^{-x(\mathcal{L}_0+Q_N)}),\ 0<x\leqslant\beta/2$ *and*

$$e^{-x\,(\mathcal{L}_0+Q_N)}\Omega_0^\beta = \Omega_0^\beta+\sum_{n=1}^\infty(-1)^n\int_{\Delta_x^n} Q_N(\underline{s}_n)\Omega_0^\beta\, d\underline{s}_n. \tag{35}$$

In this context $\Delta_x^n=\{(s_1,\dots,s_n)\in\mathbb{R}^n : 0\leqslant s_n\leqslant\dots\leqslant s_1\leqslant x\}$ *is a simplex of dimension* n *and sidelength* x.

Proof. Let $\phi\in\mathrm{ran}\,\mathbf{1}[|\mathcal{L}_0+Q_N|\leqslant k]$ and $0\leqslant x\leqslant\beta/2$ be fixed. An m-fold application of the fundamental theorem of calculus yields

$$\begin{aligned}\langle\, e^{-x\,(\mathcal{L}_0+Q_N)}\,\phi\,|\,e^{x\,\mathcal{L}_0}\,\Omega_0^\beta\,\rangle &= \Big\langle\ \phi\,|\,\Omega_0^\beta+\sum_{n=1}^m(-1)^n\int_{\Delta_x^n} Q_N(\underline{s}_n)\,\Omega_0^\beta\, d\underline{s}_n\Big\rangle\\ &+(-1)^{m+1}\int_{\Delta_x^{m+1}}\langle\, e^{-s_{m+1}\,(\mathcal{L}_0+Q_N)}\phi\,|\,e^{s_{m+1}\,\mathcal{L}_0}\,Q_N(\underline{s}_{m+1})\,\Omega_0^\beta\,\rangle\, d\underline{s}_{m+1}\end{aligned} \tag{36}$$

Since $\mathcal{L}_0\,\Omega_0^\beta=0$ we have for $r(\underline{s}_{m+1}) := (s_m-s_{m+1},\dots,s_1-s_{m+1})$ that

$$e^{s_{m+1}\mathcal{L}_0}\,Q_N(\underline{s}_{m+1})\Omega_0^\beta = Q_N\,Q_N(r(\underline{s}_{m+1}))\Omega_0^\beta,$$

We turn now to the second expression on the right side of Equation (36), after a linear transformation depending on s_{m+1} we get

$$(-1)^{m+1}\int_0^x\Big\langle e^{-s_{m+1}(\mathcal{L}_0+Q_N)}\phi\,|\,Q_N\int_{\Delta^m_{x-s_{m+1}}} Q_N(\underline{r}_m)\Omega_0^\beta\, d\underline{r}_m\Big\rangle\, ds_{m+1}.$$

Since $\|e^{-s_{m+1}\,(\mathcal{L}_0+Q_N)}\,\phi\|\leqslant e^{\beta/2\,k}\,\|\phi\|$, and using that $Q_N(\underline{r}_m)\,\Omega_0^\beta$ is a state with at most $2m$ bosons, we obtain the upper bound

$$\mathrm{const}\,\|\phi\|\,\sqrt{(2m)\,(2m+1)}\ \sup_{0\leqslant x\leqslant\beta/2}\Big\|\int_{\Delta^m_{x-s_{m+1}}} Q_N(\underline{r}_m)\,\Omega_0^\beta\, d\underline{r}_m\Big\|.$$

Hence, for $m\to\infty$ we get

$$\langle\, e^{-x(\mathcal{L}_0+Q_N)}\phi\,|\,\Omega_0^\beta\,\rangle = \Big\langle\,\phi\,|\,\Omega_0^\beta+\sum_{n=1}^\infty(-1)^n\int_{\Delta_x^n} Q_N(\underline{s}_n)\,\Omega_0^\beta\, d\underline{s}_n\Big\rangle.$$

Since $\bigcup_{k=1}^\infty\mathrm{ran}\,\mathbf{1}[|\mathcal{L}_0+Q_N|\leqslant k]$ is a core of $e^{-x(\mathcal{L}_0+Q_N)}$, the proof follows from the self-adjointness of $e^{-x(\mathcal{L}_0+Q_N)}$. ∎

LEMMA 6.2. *Let $0 < x \leqslant \beta/2$. We have the identity*

$$\int_{\Delta^n_{x/2}} \int_{\Delta^m_{x/2}} \langle\, Q_N(\underline{r}_m)\Omega^\beta_0 \,|\, Q_N(\underline{s}_n)\Omega^\beta_0 \,\rangle \, d\underline{r}_m \, d\underline{s}_n \tag{37}$$
$$= \int_{\Delta^{n+m}_\beta} \mathbf{1}[z_m \geqslant \beta - x \geqslant x \geqslant z_{m+1}] \, \langle\, \Omega^\beta_0 \,\big|\, Q_N(\underline{z}_{n+m})\Omega^\beta_0 \,\rangle \, d\underline{z}_{n+m}.$$

For $m = n$ it follows

$$\Big\| \int_{\Delta^n_{x/2}} Q_N(\underline{s}_n)\Omega^\beta_0 \, d\underline{s}_n \Big\|^2 \leqslant \int_{\Delta^{2n}_\beta} \big|\langle\, \Omega^\beta_0 \,|\, Q_N(\underline{s}_{2n})\Omega^\beta_0 \,\rangle\big| \, d\underline{s}_{2n}. \tag{38}$$

Proof. Recall Theorem 3.1 and Lemma 5.1. Since $\mathcal{J}$ is a conjugation we have $\langle\, \phi \,|\, \psi \,\rangle = \langle\, \mathcal{J}\,\psi \,|\, \mathcal{J}\,\phi \,\rangle$, and for every operator X, that is affiliated with $\mathfrak{M}_\beta$, we have $\mathcal{J}\, X \,\Omega^\beta_0 \,=\, e^{-\beta/2\,\mathcal{L}_0}\, X^* \,\Omega^\beta_0$. Thus,

$$\int_{\Delta^n_{x/2}} \int_{\Delta^m_{x/2}} \langle\, Q_N(\underline{r}_m)\Omega^\beta_0 \,|\, Q_N(\underline{s}_n)\,\Omega^\beta_0 \,\rangle \, d\underline{r}_m \, d\underline{s}_n \tag{39}$$
$$= \int_{\Delta^n_{x/2}} \int_{\Delta^m_{x/2}} \big\langle\, e^{-\beta/2\mathcal{L}_0}\, Q_N(\underline{s}_n)^* \,\Omega^\beta_0 \,\big|\, e^{-\beta/2\mathcal{L}_0}\, Q_N(\underline{r}_m)^* \,\Omega^\beta_0 \,\big\rangle \, d\underline{r}_m \, d\underline{s}_n$$

Since $\mathcal{L}_0\,\Omega^\beta_0 \,=\, 0$ we have

$$e^{-\beta\mathcal{L}_0}\, Q_N(\underline{r}_m)^* \,\Omega^\beta_0 \,=\, Q_N(\beta - r_1) \cdots Q_N(\beta - r_m)\,\Omega^\beta_0.$$

Next, we introduce new variables for $\underline{r}$, namely $y_i \,:=\, \beta - r_{m-i+1}$. Let $D^m_{x/2} \,:=\, \{\underline{y}_m \in \mathbb{R}^m \,:\, \beta - x \leqslant y_m \leqslant \ldots \leqslant y_1 \leqslant \beta\}$. Thus the right side of Equation (39) equals

$$\int_{\Delta^n_{x/2}} \int_{D^m_{x/2}} \big\langle\, \Omega^\beta_0 \,\big|\, Q_N(\underline{s}_n)\, Q_N(\underline{y}_m)^* \,\Omega^\beta_0 \,\big\rangle \, d\underline{s}_n \, d\underline{y}_m$$
$$= \int_{\Delta^{n+m}_\beta} \mathbf{1}[z_m \,\geqslant\, \beta \,-\, x \,\geqslant\, x \,\geqslant\, z_{m+1}] \, \big\langle\, \Omega^\beta_0 \,\big|\, Q_N(\underline{z}_{n+m})\,\Omega^\beta_0 \,\big\rangle \, d\underline{z}_{n+m}.$$

The second statement of the Lemma follows by choosing $n = m$. □

LEMMA 6.3. *Assume* $\sup_{N\in\mathbb{N}} \|e^{-x(\mathcal{L}_0+Q_N)}\Omega^\beta_0\| < \infty$ *then* $\Omega^\beta_0 \in \mathrm{dom}(e^{-x(\mathcal{L}_0+Q)})$ *and*

$$\|e^{-x(\mathcal{L}_0+Q)}\Omega^\beta_0\| \leqslant \sup_{N\in\mathbb{N}} \|e^{-x(\mathcal{L}_0+Q_N)}\,\Omega^\beta_0\|$$

Proof. For $f \in \mathcal{C}^\infty_0(\mathbb{R})$ and $\phi \in \mathcal{K}$ we define $\psi_N \,:=\, f(\mathcal{L}_0 \,+\, Q_N)\,\phi$. Obviously, for $g(r) \,=\, e^{-x\,r}\, f(r) \in \mathcal{C}^\infty_0(\mathbb{R})$ we have $e^{-x\,(\mathcal{L}_0\,+\,Q_N)}\,\psi_N \,=\, g(\mathcal{L}_0 \,+\, Q_N)\,\phi$.

Since $\mathcal{L}_0 + Q_N$ tends to $\mathcal{L}_0 + Q$ in the strong resolvent sense as $N \to \infty$, we know from [16] that $\lim_{N\to\infty} \psi_N = f(\mathcal{L}_0 + Q)\,\phi =: \psi$ and

$$\lim_{N\to\infty} e^{-x\,(\mathcal{L}_0+Q_N)}\,\psi_N \;=\; \lim_{N\to\infty} g(\mathcal{L}_0+Q_N)\,\phi \;=\; g(\mathcal{L}_0+Q)\,\phi \;=\; e^{-x\,(\mathcal{L}_0+Q)}\,\psi.$$

Thus,

$$\begin{aligned} |\langle\, e^{-x(\mathcal{L}_0+Q)}\,\psi \,|\, \Omega_0^\beta\,\rangle| \;&=\; \lim_{N\to\infty} |\langle\, e^{-x(\mathcal{L}_0+Q_N)}\psi_N \,|\, \Omega_0^\beta\,\rangle| \\ &\leqslant\; \sup_{N\in\mathbb{N}} \|e^{-x\,(\mathcal{L}_0+Q_N)}\Omega_0^\beta\|\,\|\psi\|, \end{aligned}$$

Since $\{f(\mathcal{L}_0+Q)\,\phi\in\mathcal{K} \,:\, \phi\in\mathcal{K},\, f\in\mathcal{C}_0^\infty(\mathbb{R})\}$ is a core of $e^{-x(\mathcal{L}_0+Q)}$, we obtain $\Omega_0^\beta \in \mathrm{dom}(e^{-x(\mathcal{L}_0+Q)})$. ☐

Lemma 6.4. *For some $C>0$ we have*

$$\begin{aligned} &\int_{\Delta_\beta^n} \Big|\big\langle\, \Omega_0^\beta \,|\, Q_N(\underline{s}_n)\Omega_0^\beta\,\big\rangle\Big|\, d\underline{s}_n \\ &\qquad\leqslant \mathrm{const}\,(n+1)^2\,(1+\beta)^n \left(8\underline{\eta}_1 + \frac{(8C\underline{\eta}_2)^{1/2}}{(n+1)^{(1-2\gamma)/2}}\right)^n, \end{aligned}$$

where $\underline{\eta}_1$ and $\underline{\eta}_2$ are defined in (32).

Proof of 6.4. First recall the definition of Q_N and $Q_N(\underline{s}_n)$ in Equation (25) and Equation (31), respectively. Let

$$\int_{\Delta_\beta^n} \big|\big\langle\, \Omega_0^\beta \,|\, Q_N(\underline{s}_n)\,\Omega_0^\beta\,\big\rangle\big|\, d\underline{s}_n \;=:\; \int_{\Delta_1^n} \beta^n\, J_n(\beta,\underline{s})\, d\underline{s}_n,$$

The functions $J_n(\beta,\underline{s})$ clearly depends on N, but since we want to find an upper bound independent of N, we drop this index. Let $W_1 = \Phi(\vec{G})\,\Phi(\vec{H}) + \text{h.c.}$, $W_2 := \Phi(F)$ and $W := W_1 + W_2$. By definition of ω_0^β in (3.1), see also (13), we obtain

$$\begin{aligned} J_n(\beta,\,\underline{s}_n) \;&=\; \omega_0^\beta\Big(\big(e^{-\beta\, s_n\, H_0}\, W\, e^{\beta s_n H_0}\big)\cdots\big(e^{-\beta\, s_1\, H_0}\, W\, e^{\beta\, s_1\, H_0}\big)\Big) \\ &= (\mathcal{Z})^{-1} \sum_{\kappa\in\{1,2\}^n} \omega_f^\beta\Big(\,\mathrm{Tr}_{\mathcal{H}_{el}}\,\big\{e^{-\beta\, H_{el}}\big(e^{-\beta\, s_n\, H_0}\, W_{\kappa(n)}\, e^{\beta\, s_n\, H_0}\big)\cdots \\ &\qquad\qquad \cdots\big(e^{-\beta\, s_1\, H_0}\, W_{\kappa(1)}\, e^{\beta\, s_1\, H_0}\big)\big\}\Big) \end{aligned}$$

By definition of ω_f^β it suffices to consider expressions with an even number of field operators. In the next step we sum over all expression, where n_1 times W_1 occurs and $2n_2$ times W_2. The sum of n_1 and n_2 is denoted by m. For fixed n_1 and n_2 the remaining expressions are all expectations in ω_f^β of $2m$ field operators. In this case the expectations in ω_f^β can be expressed by an integral over $\mathbb{R}^{2m} \times \{\pm\}^{2m}$ with respect to ν, which is defined in Lemma A.4 below. To give a precise formula we define

$$M(m_1,\, m_2) \,=\, \{\kappa \in \{1,\, 2\}^n \,:\, \#\kappa^{-1}(\{i\}) \,=\, m_i, \quad i = 1,\, 2\}.$$

Thus we obtain

$$J_n(\beta,\, \underline{s}_n) \,=\, (\mathcal{Z})^{-1} \sum_{\substack{(n_1,\, n_2) \in \mathbb{N}^2 \\ n_1 + 2n_2 = n}} \quad \sum_{\substack{\kappa \in M(n_1,\, 2n_2) \\ m := n_1 + n_2}} \int \nu(d\underline{k}_{\,2m} \otimes d\underline{\tau}_{\,2m}) \tag{40}$$
$$\mathrm{Tr}_{\mathcal{H}_{el}} \left\{ e^{-(\beta \,-\, \beta(s_1 \,-\, s_{2m}))H_{el}} I_{2m} e^{-\beta\,(s_{2m-1} \,-\, s_{2m})\, H_{el}} \cdots e^{-\beta\,(s_1 \,-\, s_2)\, H_{el}}\, I_1 \right\},$$

Of course I_j depends on $\underline{k}_{\,2m} \times \underline{\tau}_{\,2m}$, namely for $\kappa(j) \,=\, 1,\, 2$ we have

$$I_j \quad = \quad \begin{cases} I_j(m,\, \tau,\, m',\, \tau'), & \kappa(j) \,=\, 1 \\ I_j(m,\, \tau), & \kappa(j) \,=\, 2, \end{cases}$$

where $(m,\, \tau)$, $(m',\, \tau') \in \{(k_j, \tau_j) \,:\, j = 1, \ldots, m\}$. For $\kappa(j) \,=\, 1$ we have that

$$\begin{aligned} I_j(m,\, +,\, m',\, -) \quad &= \quad \vec{G}^*(m)\, \vec{H}(m') \,+\, \vec{H}^*(m)\, \vec{G}(m') \\ I_j(m,\, -,\, m',\, +) \quad &= \quad \vec{G}(m)\, \vec{H}^*(m') \,+\, \vec{H}(m)\, \vec{G}^*(m') \\ I_j(m,\, +,\, m',\, +) \quad &= \quad \vec{G}^*(m)\, \vec{H}^*(m') \,+\, \vec{H}^*(m)\, \vec{G}^*(m') \\ I_j(m,\, -,\, m',\, -) \quad &= \quad \vec{G}(m)\, \vec{H}(m') \,+\, \vec{H}(m)\, \vec{G}(m') \end{aligned}$$

and for $\kappa(j) \,=\, 2$ we have that

$$\begin{aligned} I_j(m,\, +) \quad &= \quad F^*(m) \\ I_j(m,\, -) \quad &= \quad F(m). \end{aligned}$$

In the integral (40) we insert for $(m,\, \tau)$ and $(m',\, \tau')$ in the definition of I_j from left to right $k_{2m},\, \tau_{2m}, \ldots,\, k_1,\, \tau_1$.
For fixed $(\underline{k}_{\,2m}, \underline{\tau}_{\,2m})$ the integrand of (40) is a trace of a product of $4m$ operators in $\mathcal{H}_{el}$. We will apply Hölder's-inequality for the trace, i.e.,

$$|\,\mathrm{Tr}_{\mathcal{H}_{el}}\{A_{2m}\, B_{2m} \cdots A_1\, B_1\}| \leqslant \prod_{j=1}^{2m} \|B_j\|_{\mathcal{B}(\mathcal{H}_{el})} \cdot \prod_{j=1}^{2m} \mathrm{Tr}_{\mathcal{H}_{el}}\{A_i^{p_j}\}^{p_j^{-1}}.$$

In our case $p_i \,:=\, (s_{i-1} \,-\, s_i)^{-1}$ for $i = 2, \ldots, 2m$ and $p_1 \,:=\, (1 \,-\, s_1 \,+\, s_{2m})^{-1}$ and

$$(A_j,\, B_j) \,:=\, \begin{cases} \left(e^{-\beta\, p_j^{-1}\, H_{el}},\, I_j(m,\, \tau,\, m',\, \tau')\right), & \kappa(j) \,=\, 1 \\ \left(e^{-\beta\, p_j^{-1}\, H_{el}}\, H_{el,+}^{\gamma},\, H_{el,+}^{-\gamma}\, I_j(m, \tau,)\right), & \kappa(j) \,=\, 2 \end{cases}.$$

We define

$$\eta_1(k) = \max\left\{\|\vec{G}(k)\|_{\mathcal{B}(\mathcal{H}_{el})^r},\, \|\vec{H}(k)\|_{\mathcal{B}(\mathcal{H}_{el})^r}\right\}$$
$$\eta_2(k) = \max\left\{\|F(k)\, H_{el,+}^{-\gamma}\|_{\mathcal{B}(\mathcal{H}_{el})},\, \|F^*(k)\, H_{el,+}^{-\gamma}\|_{\mathcal{B}(\mathcal{H}_{el})}\right\}.$$

By definition of B_j we have

$$\|B_j\|_{\mathcal{B}(\mathcal{H}_{el})} \leqslant \begin{cases} \eta_1(m)\eta_1(m'), & \kappa(j) = 1 \\ \eta_2(m), & \kappa(j) = 2 \end{cases}. \tag{41}$$

Furthermore,

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}_{el}}\{A_i^{p_j}\}^{p_j^{-1}} &= \mathrm{Tr}_{\mathcal{H}_{el}}\left\{e^{-\beta H_{el}}\, H_{el,+}^{p_j\gamma}\right\}^{p_j^{-1}} \\ &\leqslant \|e^{-\epsilon H_{el}}\, H_{el,+}^{p_j\gamma}\|_{\mathcal{H}_{el}}^{p_j^{-1}}\, \mathrm{Tr}_{\mathcal{H}_{el}}\left\{e^{-(\beta-\epsilon)\, H_{el}}\right\}^{p_j^{-1}}, \quad k(j) = 2\end{aligned}$$

Let $E_{gs} := \inf \sigma(H_{el})$. The spectral theorem for self-adjoint operators implies

$$\|e^{-\epsilon\, H_{el}}\, H_{el,+}^{p_i\,\gamma}\|_{\mathcal{H}_{el}}^{p_i^{-1}} \leqslant \sup_{r \geqslant E_{gs}} e^{-\epsilon\, p_i^{-1}\, r}(r - E_{gs} + 1)^{\gamma} \leqslant \epsilon^{-\gamma}\, p_i^{\gamma}\, e^{-\epsilon\, p_i^{-1}\,(E_{gs}-1)}.$$

Inserting this estimates we get

$$\begin{aligned}&\mathrm{Tr}_{\mathcal{H}_{el}}\left\{e^{-(\beta-\beta(s_1-s_{2m}))H_{el}}I_{2m}e^{-\beta\,(s_{2m-1}-s_{2m})\,H_{el}}\cdots e^{-\beta\,(s_1-s_2)\,H_{el}}\,I_1\right\} \\ &\qquad \leqslant C_\kappa(\underline{s}_n)\prod_{j=1}^{2m}\|B_j\|_{\mathcal{B}(\mathcal{H}_{el})}\end{aligned}$$

where

$$C_\kappa(\underline{s}_n) := (1 - s_1 + s_n)^{-\alpha_1}\prod_{i=1}^{n-1}(s_i - s_{i+1})^{-\alpha_i} \tag{42}$$

and

$$\alpha_i = \begin{cases} 0, & \kappa(i) = 1 \\ 1/2, & \kappa(i) = 2 \end{cases} \tag{43}$$

Now, we recall the definition of ν. Roughly speaking, one picks a pair of variables (k_i, k_j) and integrates over $\delta_{k_i,k_j}\coth(\beta/2\alpha(k_i))\,dk_i dk_j$. Subsequently one picks the next pair and so on. At the end one sums up all $\frac{(2m)!}{2^m\, m!}$ pairings and all 4^m combinations of $\underline{\tau}_{2m}$. Inserting Estimate (41) and that

$$\int \eta_\nu(k)\eta_{\nu'}(k)\coth(\beta/2\alpha(k))\,dk \leqslant (1+\beta^{-1})\underline{\eta}_\nu^{1/2}\underline{\eta}_{\nu'}^{1/2},$$

we obtain

$$|J_n(\beta,\, \underline{s})| \leqslant \frac{(1+\beta^{-1})^n}{\mathcal{Z}}\sum_{\substack{(n_1,\,n_2)\in\mathbb{N}_0^2 \\ n_1+2n_2=n}}\ \sum_{\substack{\kappa\in M(n_1,\,2n_2) \\ m:=n_1+n_2}}(\,\underline{\eta}_1)^{n_1}\,(C\underline{\eta}_2)^{n_2}\,\frac{(2m)!2^m}{m!}C_\kappa(\underline{s})$$

By Lemma A.3 below and since $(2m)!/(m!)^2 \leqslant 4^m$ we have

$$\int_{\Delta_\beta^n} \big| \langle\, \Omega_0^\beta \,|\, Q_N(\underline{s}_n)\, \Omega_0^\beta \,\rangle \big|\, d\underline{s}_n$$

$$\leqslant \mathrm{const}(1+\beta)^n \sum_{\substack{(n_1,\, n_2) \in \mathbb{N}_0^2 \\ n_1 + 2n_2 = n}} \binom{n}{n_1} \frac{(8\underline{\eta}_1)^{n_1}\,(8C'\underline{\eta}_2)^{n_2}}{(n+1)^{(1-2\gamma)\, n_2 - 2}}$$

This completes the proof. □

7 The Harmonic Oscillator

Let $L^2(X,\, d\mu) = L^2(\mathbb{R})$ and $H_{el} =: H_{osc} := -\Delta_q + \Theta^2 q^2$ be the one dimensional harmonic oscillator and $\mathcal{H}_{ph} = L^2(\mathbb{R}^3)$. We define

$$H \,=\, H_{osc} \,+\, \Phi(F) \,+\, \check{H}, \qquad \check{H} \,:=\, d\Gamma(|k|), \tag{44}$$

where $\Phi(F) \,=\, q \,\cdot\, \Phi(f)$, with $\lambda\,(|k|^{-1/2} \,+\, |k|^{1/2})\, f \in L^2(\mathbb{R}^3)$.
H_{osc} is the harmonic oscillator, the form-factor F comes from the dipole approximation.
The Standard Liouvillean for this model is denoted by $\mathcal{L}_{osc}$. Now we prove Theorem 1.4.

Proof. We define the creation and annihilation operators for the electron.

$$A^* \,=\, \frac{\Theta^{1/2}\, q \,-\, i\,\Theta^{-1/2}\, p}{\sqrt{2}}, \quad A \,=\, \frac{\Theta^{1/2}\, q \,+\, i\,\Theta^{-1/2}\, p}{\sqrt{2}}, \quad p \,=\, -i\,\partial_x, \tag{45}$$

$$\Phi(c) \,=\, c_1\, q \,+\, c_2\, p, \quad \text{for } c \,=\, c_1 \,+\, i\, c_2 \in \mathbb{C},\ c_i \in \mathbb{R}. \tag{46}$$

These operators fulfill the CCR-relations and the harmonic- oscillator is the number-operator up to constants.

$$[A,\, A^*] \,=\, 1, \qquad [A^*,\, A^*] \,=\, [A,\, A] \,=\, 0, \qquad H_{osc} \,=\, \Theta A^*\, A \,+\, \Theta/2, \tag{47}$$

$$[H_{osc},\, A] \,=\, -\Theta\, A, \qquad [H_{osc},\, A^*] \,=\, \Theta\, A^*. \tag{48}$$

The vector $\Omega \,:=\, \big(\frac{\Theta}{\pi}\big)^{1/4}\, e^{-\Theta\, q^2\,/2}$ is called the vacuum vector. Note, that one can identify $\mathcal{F}_b[\mathbb{C}]$ with $L^2(\mathbb{R})$, since $\mathrm{LH}\{(A^*)^n\, \Omega \,|\, n \in \mathbb{N}^0\}$ is dense in $L^2(\mathbb{R})$. It follows, that ω_β^{osc} is quasi-free, as a state over $W(\mathbb{C})$ and

$$\omega_\beta^{osc}(W(c)) \,=\, (\mathcal{Z})^{-1}\, \mathrm{Tr}_{\mathcal{H}_{el}}\{e^{-\beta H_{el}}\, W(c)\} \,=\, \exp\big(-1/4\, \coth(\beta\,\Theta/2)\, |c|^2\big), \tag{49}$$

where $\mathcal{Z} \,=\, \mathrm{Tr}_{\mathcal{H}_{el}}\{e^{-\beta\, \mathcal{H}_{el}}\}$ is the partition function for $\mathcal{H}_{el}$.
First, we remark, that Equation (31) is defined for this model without regularization by $P_N \,:=\, \mathbf{1}[H_{el} \,\leqslant\, N]$. Moreover we obtain from Lemma 6.2,

that

$$\Big\| \int_{\Delta^n_{\beta/2}} Q(\underline{s}_n)\Omega_0^\beta \, d\underline{s}_{2n} \Big\|^2 \leqslant \int_{\Delta^{2n}_\beta} \big| \big\langle \, \Omega_0^\beta \, | \, Q(\underline{s}_{2n}) \, \Omega_0^\beta \, \big\rangle \big| \, d\underline{s}_{2n} =: h_{2n}(\beta, \lambda). \tag{50}$$

To show that $\Omega^\beta \in \mathrm{dom}(e^{-\beta/2 \, (\mathcal{L}_0 + Q)})$ is suffices to prove, that $\sum_{n=0}^\infty h_{2n}(\beta, \lambda)^{1/2} < \infty$. We have

$$\begin{aligned} h_{2n}(\beta, \lambda) = \frac{(-\beta\lambda)^{2n}}{\mathcal{Z}} \int_{\Delta_1^{2n}} & \omega_\beta^{osc}\big(\big(e^{-\beta\, s_{2n}\, H_{el}} \, q \, e^{\beta\, s_{2n}\, H_{el}}\big) \\ & \cdots \big(e^{-\beta\, s_1\, H_{el}} \, q \, e^{\beta\, s_1\, H_{el}}\big)\big) \\ \cdot \omega_f^\beta \big((e^{-\beta\, s_{2n}\, \check{H}} \, \Phi(f) \, e^{\beta\, s_{2n}\, \check{H}}) & \cdots (e^{-\beta\, s_1\, \check{H}} \, \Phi(f) \, e^{\beta\, s_1\, \check{H}})\big) \, d\underline{s}_{2n}. \end{aligned} \tag{51}$$

Moreover, we have

$$\begin{aligned} e^{-\beta\, s_i\, H_{el}} \, q \, e^{\beta\, s_i\, H_{el}} &= (2\Theta)^{-1/2}\big(e^{-\beta\,\Theta\, s_i} \, A^* + e^{\beta\,\Theta\, s_i} \, A\big) \\ e^{-\beta\, s_i\, \check{H}} \, \Phi(f) \, e^{\beta\, s_i\, \check{H}} &= 2^{-1/2}\Big(a^*(e^{-\beta\, s_i\, |k|} \, f) + a(e^{\beta\, s_i\, |k|} \, f)\Big). \end{aligned} \tag{52}$$

Inserting the identities of Equation (52) in Equation (51) and applying Wick's theorem [5, p. 40] yields

$$\begin{aligned} h_{2n}(\beta, \lambda) = (\beta\,\lambda)^{2n} \int_{\Delta_1^{2n}} & \sum_{P \in \mathcal{Z}_2} \prod_{\{i,\, j\} \in P} K_{osc}(|s_i - s_j|, \beta) \\ & \cdot \sum_{P' \in \mathcal{Z}_2} \prod_{\{k,\, l\} \in P'} K_f(|s_k - s_l|, \beta) \, d\underline{s}_{2n} \\ = \frac{(\beta\,\lambda)^{2n}}{(2n)!} \int_{[0,\,1]^{2n}} & \sum_{P,\, P' \in \mathcal{Z}_2} \prod_{\substack{\{i,\, j\} \in P \\ \{k,\, l\} \in P'}} K_{osc}(|s_i - s_j|, \beta) \, K_f(|s_k - s_l|, \beta) \, d\underline{s}_{2n}, \end{aligned} \tag{53}$$

where for $k < l$ and $i < j$, such as

$$\begin{aligned} K_f(|s_k - s_l|, \beta) &:= \omega_f^\beta((e^{-\beta\, s_k\, \check{H}} \, \Phi(f) \, e^{\beta\, s_k\, \check{H}}) \, (e^{-\beta\, s_l\, \check{H}} \, \Phi(f) \, e^{\beta\, s_l\, \check{H}})) \\ K_{osc}(|s_i - s_j|, \beta) &:= \omega_\beta^{osc}(e^{-\beta\, s_i\, H_{el}} \, q \, e^{\beta\, s_i\, H_{el}} \, e^{-\beta\, s_j\, H_{el}} \, q \, e^{\beta\, s_j\, H_{el}}). \end{aligned}$$

The last equality in (53) holds, since the integrand is invariant with respect to a change of the axis of coordinates.

We interpret two pairings P and $P' \in \mathcal{Z}_2$ as an indirected graph $G = G(P, P')$, where $M_{2n} = \{1, \ldots, 2n\}$ is the set of points. Any graph in G has two kinds of lines, namely lines in $L_{osc}(G)$, which belong to elements of P and lines in $L_f(G)$, which belong to elements of P'.

Let $\mathcal{G}(A)$ be the set of undirected graphs with points in $A \subset M_{2n}$, such that for each point "i" in A, there is exact one line in $L_f(G)$, which begins in "i", and

exact one line in $L_{osc}(G)$, which begins with "i". $\mathcal{G}_c(A)$ is the set of connected graphs. We do not distinguish, if points are connected by lines in $L_f(G)$ or by lines in $L_{osc}(G)$.
Let

$$\mathcal{P}_k := \Big\{ P : P = \{A_1, \dots, A_k\},\ \emptyset \neq A_i \subset M_{2n}, \\ A_i \cap A_j = \emptyset \text{ for } i \neq j,\ \bigcup_{i=1}^{k} A_i = M_{2n} \Big\}$$

be the family of decompositions of M_{2n} in k disjoint set. It follows

$$\begin{aligned} h_{2n}(\beta, \lambda) &= \frac{(\beta\,\lambda)^{2n}}{(2n)!} \sum_{G \in \mathcal{G}(M_{2n})} \int_{M_{2n}} \prod_{\substack{\{i,j\} \in L_{osc}(G) \\ \{k,l\} \in L_f(G)}} K_{osc}(|s_i - s_j|, \beta) \\ &\quad K_f(|s_k - s_l|, \beta)\, d\underline{s}_n \\ &= \frac{(\beta\lambda)^{2n}}{(2n)!} \sum_{k=1}^{2n} \sum_{\{A_1, \dots, A_k\} \in \mathcal{P}_k} \sum_{\substack{(G_1, \dots, G_k) \\ G_a \in \mathcal{G}_c(A_a)}} \prod_{a=1}^{k} J(G_a, A_a, \beta) \\ &= \frac{(\beta\lambda)^{2n}}{(2n)!} \sum_{k=1}^{2n} \frac{1}{k!} \sum_{\substack{A_1, \dots, A_k \subset M_{2n}, \\ \{A_1, \dots, A_k\} \in \mathcal{P}_k}} \sum_{\substack{(G_1, \dots, G_k) \\ G_a \in \mathcal{G}_c(A_a)}} \prod_{a=1}^{k} J(G_a, A_a, \beta), \end{aligned} \tag{54}$$

where

$$J(G_a, A_a, \beta) := \int_{A_a} \prod_{\substack{\{i,j\} \in L_{osc}(G_a) \\ \{k,l\} \in L_f(G_a)}} K_{osc}(|s_i - s_j|, \beta) K_f(|s_k - s_l|, \beta)\, d\underline{s}. \tag{55}$$

$\int_{A_a} d\underline{s}$ means, $\int_{-1}^{1} ds_{j_1} \int_{-1}^{1} ds_{j_2} \dots \int_{-1}^{1} ds_{j_m}$, where $A_a = \{j_1, \dots, j_m\}$ and $\# A_a = m$.
From the first to the second line we summarize terms with graphs, having connected components containing the same set of points. From the second to the third line the order of the components is respected, hence the correction factor $\frac{1}{k!}$ is introduced. Due to Lemma 7.2 the integral depends only on the number of points in the connected graph, i. e. $J(G, A, \beta) = J(\# A, \beta)$. Moreover, Lemma 7.2 states that $\beta^{\# A} \cdot J(\# A, \beta) \leqslant (2 \| |k|^{-1/2} f \|_2 (\Theta\,\beta)^{-1})^{\# A} (C\,\beta + 1)$. To ensure that $\mathcal{G}_c(A_a)$ is not empty, $\# A_a$ must be even. For $(m_1, \dots, m_k) \in \mathbb{N}^k$ with $m_1 + \dots + m_k = n$ we obtain

$$\sum_{\substack{A_1, \dots, A_k \subset M_{2n}, \# A_i = 2m_i \\ \{A_1, \dots, A_k\} \in \mathcal{P}_k}} 1 = \frac{(2n)!}{(2m_1)! \cdots (2m_k)!}. \tag{56}$$

Let now be $A_a \subset M_{2n}$ with $\#A_a = 2m_a > 2$ fixed. In G_a are $\#A_a$ lines in $L_{osc}(G_a)$, since such lines have no points in common, we have $\frac{(2m_a)!}{m_a!\,2^{m_a}}$ choices. Let now be the lines in $L_{osc}(G_a)$ fixed. We have now $\big((2m_a-2)(2m_a-4)\cdots 1\big)$ choices for m_a lines in $L_f(G_a)$, which yield a connected graph. Thus

$$\sum_{G_a \in \mathcal{G}_c(A_a),} 1 = \frac{(2m_a)!}{m_a!\,2^{m_a}}\big((2m_a - 2)(2m_a - 4)\cdots 1\big) = \frac{(2m_a)!}{2m_a}. \tag{57}$$

For $\#A_a = 2$ exists only one connected graph. We obtain for h_{2n}

$$\begin{aligned} h_{2n}(\beta,\lambda) &= (\lambda)^{2n}\sum_{k=1}^{2n}\frac{1}{k!}\sum_{\substack{(m_1,\dots,m_k)\in\mathbb{N}^k\\ m_1+\dots+m_k=n}}\prod_{a=1}^{k}\frac{J(2m_a,\beta)(\beta^2)^{m_a}}{2m_a} \qquad (58)\\ &\leqslant (2\Theta^{-1}\,\||k|^{-1/2}f\|\,\lambda)^{2n}\sum_{k=1}^{2n}\frac{1}{k!}\sum_{\substack{(m_1,\dots,m_k)\in\mathbb{N}^k\\ m_1+\dots+m_k=n}}\prod_{a=1}^{k}\frac{(C\beta+1)}{2m_a}\\ &\leqslant (2\Theta^{-1}\,\||k|^{-1/2}f\|\,\lambda)^{2n}\sum_{k=1}^{2n}\frac{\big((C\beta+1)/2\sum_{m=1}^{n}\frac{1}{m}\big)^k}{k!}. \end{aligned}$$

Since the $\sum_{m=1}^{n}\frac{1}{m}$ can be considered as a lower Riemann sum for the integral $\int_1^{m+1} r^{-1}\,dr$, we have $\sum_{m=1}^{n}\frac{1}{m} \leqslant \ln(n+1)$. Thus,

$$\begin{aligned} h_{2n}(\beta,\lambda) &\leqslant (2\Theta^{-1}\,\||k|^{-1/2}f\|\,\lambda)^{2n}\sum_{k=1}^{2n}\frac{\big((C\beta+1)/2\,\ln(n+1)\big)^k}{k!} \qquad (59)\\ &\leqslant (2\Theta^{-1}\,\||k|^{-1/2}f\|\,\lambda)^{2n}(n+1)^{(C\beta+1)/2}. \end{aligned}$$

Since $2|\lambda|\,\||k|^{1/2}f\| < \Theta$ the series $\sum_{n=0}^{\infty} h_{2n}(\beta,\lambda)^{1/2}$ converges absolutely for all $\beta > 0$. It follows, that

$$e^{-\beta/2\,(\mathcal{L}_0+Q)}\,\Omega_0^\beta = \Omega_0^\beta + \sum_{n=1}^{\infty}\int_{\Delta^n_{\beta/2}} Q(\underline{s}_n)\,\Omega_0^\beta\,d\underline{s}_n$$

exists. □

Conversely, Equation (58) and Lemma 7.2 imply

$$h_{2n}(\beta,\lambda) \geqslant \lambda^{2n}\frac{J(2n,\beta)\,\beta^{2n}}{2n} = \frac{\Big(\Theta^{-1}\int\frac{\beta^2\,\lambda^2\,|f(k)|^2}{\sinh(|k|\,\beta/2)\sinh(\beta\,\Theta/2)}\,dk\Big)^n}{2n}. \tag{60}$$

Hence for every $\beta > 0$ exists a $\lambda \in \mathbb{R}$, such that $h_{2n}(\beta,\lambda) \geqslant \frac{1}{2n}$. Thus $\sum_{n=1}^{\infty} h_{2n}(\beta,\lambda)^{1/2} = \infty$

REMARK 7.1. *We can therefore not extended Theorem 1.4 to an existence proof for all $\lambda > 0$.*

Lemma 7.2. *Following statements are true.*

$$J(G,\,A,\,\beta) = J(\#A,\beta),\;\; G\in\mathcal{G}_c(A)$$
$$J(\#A,\,\beta) \leqslant (2\||k|^{-1/2}\,f\|_2\,(\Theta\,\beta)^{-1})^{\#A}\cdot(C\,\beta\,+\,1)$$
$$J(\#A,\,\beta) \geqslant \Big(\Theta^{-1}\int\frac{|f(k)|^2}{\sinh(|k|\,\beta/2)\sinh(\Theta\,\beta/2)}\,dk\Big)^{\#A/2},$$

where $\#A\,=\,2m$ *and* $C\,=\,(1/2)\,\frac{\|f\|^2}{\||k|^{1/2}\,f\|^2}$.

Proof of 7.2. A relabeling of the integration variables yields

$$J(G,\,A,\,\beta) \leqslant \overline{K}_f\int_{[0,1]^{2m}} K_{osc}(|t_1-t_2|,\,\beta)\,K_f(|t_2\,-\,t_3|,\,\beta)\cdots$$
$$\cdots K_{osc}(|t_{2m-1}\,-\,t_{2m}|,\,\beta)\,d\underline{t}$$

for $\overline{K}_f\;:=\;\sup_{s\in[0,1]}K_f(s,\,\beta)$. We transform due to $s_i\;:=\;t_i\,-\,t_{i+1},\;i\;\leqslant\;2m-1$ and $s_{2m}\;=\;t_{2m}$, hence $-1\;\leqslant\;s_i\;\leqslant\;1,\;i\;=\;1,\ldots,\,2m$, since integrating a positive function we obtain

$$J(G,\,A,\,\beta) \leqslant \Big(\int_{-1}^{1}K_{osc}(|s|,\,\beta)\,ds\Big)^m\Big(\int_{-1}^{1}K_f(|s|,\,\beta)\,ds\Big)^{m-1}$$
$$\cdot\sup_{s\in[0,1]}K_f(s,\,\beta).$$

We recall that

$$\int_{-1}^{1}K_{osc}(|s|,\,\beta)\,ds\;=\;(2\Theta)^{-1}\int_{-1}^{1}\frac{\cosh(\beta\,\Theta\,|s|\;-\;\Theta\,\beta/2)}{\sinh(\Theta\,\beta/2)}\,ds\;=\;2(\Theta^2\,\beta)^{-1}$$

and

$$\int_{-1}^{1}K_f(|s|,\,\beta)\,ds\;=\;\int_{-1}^{1}\int\frac{\cosh(\beta\,|s|\,|k|-\beta|k|/2)\,|f(k)|^2}{2\sinh(\beta\,|k|/2)}\,dk\,ds$$
$$=\;2\int\frac{|f(k)|^2}{\beta\,|k|}\,dk.$$

Using $\coth(x)\leqslant 1+1/x$ and using convexity of cosh, we obtain

$$\sup_{s\in[0,\,1]}K_f(s,\,\beta)\leqslant(1/2)\int|f(k)|^2\,dk\,+\,\frac{1}{\beta}\int\frac{|f(k)|^2}{|k|}\,dk.$$

Due to the fact, that $t\,\mapsto\,K_f(t,\,\beta)$ and $t\,\mapsto\,K_{osc}(t,\,\beta)$ attain their minima at $t\,=\,1/2$, we obtain the lower bound for $J(\#A,\,\beta)$. ☐

Remark 7.3. *In the literature there is one criterion for* $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2\,(\mathcal{L}_0+Q)})$*, to our knowledge, that can be applied in this situation [6]. One has to show that* $\|e^{-\beta/2\,Q}\,\Omega_0^\beta\| < \infty$*. If we consider the case, where the criterion holds for* $\pm\lambda$*, then the expansion in* λ *converges,*

$$\begin{aligned}
\|e^{-\beta/2\,Q}\,\Omega_0^\beta\|^2 &= \sum_{n=0}^{\infty} \frac{(\lambda\,\beta)^{2n}}{(2n)!}\,\omega_{el}^\beta(q^{2n})\,\omega_f^\beta(\Phi(f)^{2n}) \\
&= \sum_{n=0}^{\infty} \frac{(\lambda\,\beta)^{2n}}{(2n)!}\Big(\frac{(2n)!}{n!\,2^n}\Big)^2 K_{osc}(0,\,\beta)^n\,K_f(0,\,\beta)^n \\
&= \sum_{n=0}^{\infty}(\lambda\,\beta)^{2n}\,\Theta^{-n}\binom{2n}{n}2^{-2n}\Big(\coth(\Theta\,\beta/2)\int|f(k)|^2\,\coth(\beta\,|k|/2)\,dk\Big)^n \\
&\geqslant \sum_{n=0}^{\infty}(\lambda\,\beta)^{2n}(4\,\Theta)^{-n}\Big(\int|f(k)|^2\,dk\Big)^n.
\end{aligned}$$

Obviously, for any value of $\lambda \neq 0$*, there is a* $\beta > 0$*, for which* $\|e^{-\beta/2\,Q}\,\Omega_0^\beta\| < \infty$ *is not fulfilled.*

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A

Lemma A.1. *Let* $f,\,g\,:\,\{z\in\mathbb{C}\,:\,0\leqslant Re(z)\leqslant\alpha\}\to\mathbb{C}$ *continuous and analytic in the interior. Moreover, assume that* $f(t) = g(t)$ *for* $t\in\mathbb{R}$*. Then* $f = g$*.*

Proof of A.1. Let $h\,:\,\{z\in\mathbb{C}\,:\,|\,\mathrm{Im}\,(z) < \alpha\}\to\mathbb{C}$ defined by

$$h(z) := \begin{cases} f(\,z\,)\,-\,g(\,z\,), & \text{on } \{z\in\mathbb{C}\,:\,0\leqslant \mathrm{Im}\,(z) < \alpha\} \\ \overline{f(\,\overline{z}\,)} - \overline{g(\,\overline{z}\,)}, & \text{on } \{z\in\mathbb{C}\,:\,-\alpha < \mathrm{Im}\,(z) < 0\} \end{cases} \tag{61}$$

Thanks to the Schwarz reflection principle h is analytic. Since $h(t) = 0$ for all $t\in\mathbb{R}$, we get $h = 0$. Hence $f = g$ on $\{z\in\mathbb{C}\,:\,0\leqslant \mathrm{Re}\,(z) < \alpha\}$. Since both f and g are continuous, we infer that $f = g$ on the whole domain. ☐

Lemma A.2. *Let* H *be some self-adjoint operator in* $\mathcal{H}$*,* $\alpha > 0$ *and* $\phi \in \mathrm{dom}(e^{\alpha\,H})$*. Then* $\phi\in\mathrm{dom}(e^{z\,H})$ *for* $z\in\{z\in\mathbb{C}\,:\,0\leqslant Re(z)\leqslant\alpha\}$*.* $z\mapsto e^{z\,H}\phi$ *is continuous on* $\{z\in\mathbb{C}\,:\,0\leqslant Re(z)\leqslant\alpha\}$ *and analytic in the interior.*

Proof of A.2. Due to the spectral calculus we have

$$\int e^{2\mathrm{Re}\, z\, s} d\langle\phi\,|\,\mathbb{E}_s\,\phi\rangle \leqslant \int (1+e^{2\alpha\, s}) d\langle\phi\,|\,\mathbb{E}_s\,\phi\rangle =: C_1^2 < \infty.$$

Thus $\phi \in \mathrm{dom}(e^{z\,H})$. Let $\psi \in \mathcal{H}$ and $f(z) = \langle\psi\,|\,e^{z\,H}\,\phi\rangle$. There is a sequence $\{\psi_n\}$ with $\psi_n \in \bigcup_{m\in\mathbb{N}} \mathrm{ran}\ \mathbb{1}[|H| \leqslant m]$ and $\lim_{n\to\infty}\psi_n = \psi$. We set $f_n(z) = \langle\psi_n\,|\,e^{z\,H}\,\phi\rangle$. It is not hard to see that f_n is analytic, since ψ_n is an analytic vector for H, and that $|f_n(z)| \leqslant C_1\,\|\psi_n\|$ and $\lim_{n\to\infty} f_n(z) = f(z)$. Thus f is analytic and hence $z \mapsto e^{z\,H}\phi$ is analytic. Thanks to the dominated convergence theorem the right side of

$$\|e^{z_n H}\phi - e^{zH}\phi\|^2 \leqslant \int (e^{2\mathrm{Re}\, z_n s} + e^{2\mathrm{Re}\, zs} - e^{\bar{z_n}s+zs} - e^{\bar{z}s+z_n s}) d\langle\phi\,|\,\mathbb{E}_s\,\phi\rangle \quad (62)$$

tends to zero for $\lim_{n\to\infty} z_n = z$. This implies the continuity of $z \mapsto e^{z\,H}\phi$. □

Lemma A.3. *We have for $n_1 + n_2 \geqslant 1$*

$$\int_{\Delta_1^n} C_\kappa(\underline{s})\, d\underline{s}_n \leqslant \frac{\mathrm{const}\ C^{n_2}}{(n_1+n_2)!\,(n+1)^{(1-2\gamma)\,n_2-2}} \quad (63)$$

Proof of A.3. We turn now to the integral

$$\int_{\Delta_1^n} C_\kappa(\underline{s})\, d\underline{s}_n = \int_{\Delta_1^n} (1 - s_1 + s_n)^{-\alpha_1} \prod_{i=1}^{n-1} (s_i - s_{i+1})^{-\alpha_i}\, d\underline{s}_n. \quad (64)$$

We define for $k = 1, \ldots, 2n$, a change of coordinates by $s_k = r_1 - \sum_{j=2}^{k} r_j$, the integral transforms to

$$\begin{aligned} &\int_{S^n} (1 - (r_2 + \cdots + r_n))^{-\alpha_1} \prod_{i=2}^{n} r_i^{-\alpha_i}\, d\underline{r}_n \quad (65) \\ &= \int_{T^{n-1}} (1 - (r_2 + \cdots + r_n))^{1-\alpha_1} \prod_{i=2}^{n} r_i^{-\alpha_i}\, d\underline{r}_{n-1} \\ &= \frac{\Gamma(1-\alpha_1)^{-1}\Gamma\big(1-\gamma\big)^{2n_2}}{\Gamma\big(n_1 + 2n_2\,(1-\gamma)\big)} \end{aligned}$$

where $S^{2n} := \{\underline{r} \in \mathbb{R}^{2n} : 0 \leqslant r_i \leqslant 1,\ r_2 + \cdots + r_{2n} \leqslant r_1\}$ and $T^{2n-1} := \{\underline{r} \in \mathbb{R}^{2n-1} : 0 \leqslant r_i \leqslant 1,\ r_2 + \cdots + r_{2n} \leqslant 1\}$. From the first to the second formula we integrate over dr_1. The last equality follows from [11, Formula 4.635 (4)], here Γ denotes the Gamma-function.
From Stirling's formula we obtain

$$(2\pi)^{1/2}\, x^{x-1/2}\, e^{-x} \leqslant \Gamma(x) \leqslant (2\pi)^{1/2}\, x^{x-1/2}\, e^{-x+1}, \quad x \geqslant 1. \quad (66)$$

Since $n_1 + n_2 \geqslant 1$ get

$$\frac{\Gamma(n_1+n_2+1)}{\Gamma(n_1+2(1-\gamma)\,n_2)} \leqslant (n+1)^2\Big(\frac{n_1+2(1-\gamma)n_2}{e}\Big)^{-(1-2\gamma)n_2}. \tag{67}$$

Note that $\Gamma(n_1+n_2+1) = (n_1+n_2)!$. □

Lemma A.4. *Let* $(1+\alpha(k)^{-1/2})\,f_1,\ldots,(1+\alpha(k)^{-1/2})\,f_{2m} \in \mathcal{H}_{ph}$ *and* $\sigma \in \{+,-\}^{2m}$. *Let* $a^+ = a^*$ *and* $a^- = a$

$$\begin{aligned}&\omega_f^\beta\big(a^{\sigma_{2m}}(e^{-\sigma_{2m}\,s_{2m}\,\alpha(k)}\,f_{2m})\cdots a^{\sigma_1}(e^{-\sigma_1\,s_1\,\alpha(k)}\,f_1)\big)\\ &\quad= \int f_{2m}^{\sigma_{2m}}(k_{2m},\,\tau_{2m})\cdots f_1^{\sigma_1}(k_1,\,\tau_1)\,\nu(d\underline{k}_{2m}\otimes d\underline{\tau}_{2m}),\end{aligned}$$

where $\nu(d\underline{k}_{2m}\otimes d\underline{\tau}_{2m})$ *is a measure on* $(\mathbb{R}^3)^{2m}\times\{+,\,-\}^{2m}$ *for phonons, respectively on* $(\mathbb{R}^3\times\{\pm\})^{2m}\times\{+,\,-\}^{2m}$ *for photons, and*

$$\nu(d\underline{k}_{2m}\otimes d\underline{\tau}_{2m}) \leqslant \sum_{P\in\mathcal{Z}_{2m}}\ \sum_{\underline{\tau}\in\{+,\,-\}^{2m}}\ \prod_{\{i>j\}\in P}\Big(\delta_{k_i,\,k_j}\coth(\beta\,\alpha(k_i)/2)\Big)\,d\underline{k}_{2m}. \tag{68}$$

for $f^+(k,\,\tau) := f(k)\,\mathbf{1}[\tau = +]$ *and* $f^+(k,\,\tau) := \overline{f(k)}\,\mathbf{1}[\tau = -]$.

Proof of A.4. Since ω_f^β is quasi-free, we obtain with $a^+ := a^*$ and $a^- := a$

$$\begin{aligned}&\omega_f^\beta\big(a^{\sigma_{2m}}(e^{-\sigma_{2m}\,s_{2m}\,\alpha(k)}\,f_{2m})\cdots a^{\sigma_1}(e^{-\sigma_1\,s_1\,\alpha(k)}\,f_1)\big)\\ &\quad= \sum_{P\in\mathcal{Z}_2}\ \prod_{\substack{\{i,\,j\}\in P\\ i>j}}\omega_f^\beta\big(a^{\sigma_i}(e^{-\sigma_i\,s_i\,\alpha(k)}\,f_i)\,a^{\sigma_j}(e^{-\sigma_j\,s_j\,\alpha(k)}\,f_j)\big),\end{aligned}$$

see Equation (12). For the expectation of the so called two point functions we obtain:

$$\omega_f^\beta\big(a^+(e^{s_i\,\alpha(k)}\,f_i)\,a^+(e^{s_j\,\alpha(k)}\,f_j)\big) = 0 = \omega_f^\beta\big(a(e^{-s_i\,\alpha(k)}\,f_i)\,a(e^{-s_j\,\alpha(k)}\,f_j)\big),$$

such as

$$\begin{aligned}\omega_f^\beta\big(a^+(e^{xs_i\alpha(k)}f_i)a^-(e^{-x\,s_j\alpha(k)}f_j)\big) &= \int f_i(k)\,\overline{f_j(k)}\ \frac{e^{x\,(s_i\,-\,s_j)\alpha(k)}}{e^{\beta\,\alpha(k)}-1}\,dk\\ \omega_f^\beta\big(a^-(e^{xs_i\alpha(k)}f_i)\,a^+(e^{-xs_j\alpha(k)}f_j)\big) &= \int f_j(k)\,\overline{f_i(k)}\ \frac{e^{(\beta+xs_j-xs_i)\alpha(k)}}{e^{\beta\alpha(k)}-1}\,dk\end{aligned}$$

Hence it follows

$$\begin{aligned}&\omega_f^\beta\big(a^{\sigma_{2m}}(e^{-\sigma_{2m}\,s_{2m}\,\alpha(k)}\,f_{2m})\cdots a^{\sigma_1}(e^{-\sigma_1\,s_1\,\alpha(k)}\,f_1)\big)\\ &\quad= \int f_{2m}^{\sigma_{2m}}(k_{2m},\,\tau_{2m})\cdots f_1^{\sigma_1}(k_1,\,\tau_1)\,\nu(d\underline{k}_{2m}\otimes d\underline{\tau}_{2m}),\end{aligned}$$

where $f^+(k,\tau) := f(k)\,\mathbf{1}[\tau=+]$ and $f^-(k,\tau) := \overline{f(k)}\,\mathbf{1}[\tau=-]$. $\nu(d^{3(2m)}k \otimes d^{2m}\tau)$ is a measure on $(\mathbb{R}^3)^{2m} \times \{+,-\}^{2m}$, which is defined by

$$\sum_{P\in\mathcal{Z}_{2m}} \sum_{\underline{\tau}\in\{+,-\}^{2m}} \prod_{\{i>j\}\in P} \delta_{\tau,-\tau}\,\delta_{k_i,k_j} \left(\delta_{\tau,+}\frac{e^{x\,(s_i-s_j)\,\alpha(k_i)}}{e^{\beta\,\alpha(k_i)}-1} + \delta_{\tau,-}\frac{e^{(\beta-x\,(s_i-s_j))\,\alpha(k_i)}}{e^{\beta\,\alpha(k_i)}-1}\right) d\underline{k}_{2m}. \tag{69}$$

□

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