Log-Growth Filtration and Frobenius Slope Filtration of F-Isocrystals at the Generic and Special Points

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Abstract. We study, locally on a curve of characteristic $p > 0$, the relation between the log-growth filtration and the Frobenius slope filtration for F-isocrystals, which we will indicate as φ-∇-modules, both at the generic point and at the special point. We prove that a bounded φ-∇-module at the generic point is a direct sum of pure φ-∇-modules. By this splitting of Frobenius slope filtration for bounded modules we will introduce a filtration for φ-∇-modules (PBQ filtration). We solve our conjectures of comparison of the log-growth filtration and the Frobenius slope filtration at the special point for particular φ-∇-modules (HPBQ modules). Moreover we prove the analogous comparison conjecture for PBQ modules at the generic point. These comparison conjectures were stated in our previous work [CT09]. Using PBQ filtrations for φ-∇-modules, we conclude that our conjecture of comparison of the log-growth filtration and the Frobenius slope filtration at the special point implies Dwork's conjecture, that is, the special log-growth polygon is above the generic log-growth polygon including the coincidence of both end points.

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Contents

1 Introduction

The local behavior of p-adic linear differential equations is, in one sense, very easy. If the equation has a geometric origin (i.e., if it is furnished with a Frobenius structure), then the radius of convergence of solutions at any nonsingular point is at least 1. In general, the p-adic norm of the coefficients a_n in the Taylor series of a solution is an increasing function on n. However, one knows that some solutions are p-adically integral power series. B.Dwork discovered these mysterious phenomena and introduced a measure, called logarithmic growth (or log-growth, for simplicity), for power series in order to investigate this delicate difference systematically (see [Dw73] and [Ka73, Section 7]). He studied the log-growth of solutions of p-adic linear differential equations both at the generic point and at special points (see [Ro75], [Ch83]), and asked whether the behaviors are similar to those of Frobenius slopes or not. He conjectured that the Newton polygon of log-growth of solutions at a special point is above the Newton polygon of log-growth of solutions at the generic point [Dw73, Conjecture 2]. We refer to it as Conjecture $\mathbf{LG}_{\mathrm{Dw}}$ when there are not Frobenius structures, and as Conjecture $\mathbf{LGF}_{\mathrm{Dw}}$ where there are Frobenius structures (see Conjecture 2.7). He also proved that the Newton polygon of log-growth of solutions at the generic (resp. special) point coincides with the Newton polygon of Frobenius slopes in the case of hypergeometric Frobenius-differential systems if the systems are nonconstant, thus establishing the conjecture in these cases [Dw82, 9.6, 9.7, 16.9].

On the other hand P.Robba studied the generic log-growth of differential modules defined over the completion of $\mathbb{Q}(x)$ under the p-adic Gauss norm by introducing a filtration on them via p-adic functional analysis [Ro75] (see Theorem 2.2). His theory works on more general p-adically complete fields, for example our field $\mathcal{E}$.

Let k be a field of characteristic $p > 0$, let $\mathcal{V}$ be a discrete valuation ring with residue field k, and let K be the field of fractions of $\mathcal{V}$ such that the characteristic of K is 0. In [CT09] we studied Dwork's problem on the log-growth for φ-∇-modules over $\mathcal{E}$ or $K[\![x]\!]_0$ which should be seen as localizations of F-isocrystals on a curve over k with coefficients in K. Here $K[\![x]\!]_0$ is the ring of bounded functions on the unit disk around $x = 0$, $\mathcal{E}$ is the p-adically complete field which is the field of fractions of $K[\![x]\!]_0$, and φ (resp. ∇) indicates the Frobenius structure (resp. the connection) (See the notation and terminology introduced in Section 2). We gave careful attention to Dwork's result on the comparison between the log-growth and the Frobenius slopes of hypergeometric Frobenius-differential equations. We compared the log-growth and the Frobenius slopes at the level of filtrations.

Let M be a φ-∇-module over $K[\![x]\!]_0$. Let $M_\eta = M \otimes_{K[\![x]\!]_0} \mathcal{E}$ be a φ-∇-module over $\mathcal{E}$ which is the generic fiber of M and let $V(M)$ be the φ-module over K consisting of horizontal sections on the open unit disk. Denote by M_η^λ the log-growth filtration on M_η at the generic point indexed by $\lambda \in \mathbb{R}$, and by $V(M)^\lambda$ be the log-growth filtration with real indices on the φ-module $V(M)$. Furthermore, let $S_\lambda(\,\cdot\,)$ be the Frobenius slope filtration such that $S_\lambda(\,\cdot\,)/S_{\lambda-}(\,\cdot\,)$ is pure of slope λ.

We proved that the log-growth filtration is included in the orthogonal part of the Frobenius slope filtration of the dual module under the natural perfect pairing $M_\eta \otimes_{\mathcal{E}} M_\eta^\vee \to \mathcal{E}$ (resp. $V(M) \otimes_K V(M^\vee) \to K$) at the generic point (resp. the special point) [CT09, Theorem 6.17] (see the precise form in Theorem 2.3):

$$M_\eta^\lambda \subset (S_{\lambda-\lambda_{\max}}(M_\eta^\vee))^\perp \qquad (\text{resp.}\, V(M)^\lambda \subset (S_{\lambda-\lambda_{\max}}(V(M^\vee)))^\perp)$$

for any $\lambda \in \mathbb{R}$ if $\lambda_{\max}$ is the highest Frobenius slope of M_η. We then conjectured: (a) the rationality of log-breaks λ (both at the generic and special fibers) and (b) if the bounded quotient M_η/M_η^0 is pure as a φ-module then the inclusion relation becomes equality both at the generic and special points [CT09, Conjectures 6.8, 6.9]. The hypothesis of (b) will be called the condition of being "pure of bounded quotient" (PBQ) in Definition 5.1. Note that there are examples with irrational breaks, and that both $M^{\lambda-} \supsetneq M^\lambda$ and $M^\lambda \supsetneq M^{\lambda+}$ can indeed occur for log-growth filtrations in absence of Frobenius structures [CT09, Examples 5.3, 5.4]. We state the precise forms of our conjectures in Conjecture 2.4 on $\mathcal{E}$ and Conjecture 2.5 on $K[\![x]\!]_0$, and denote the conjectures by $\mathbf{LGF}_{\mathcal{E}}$ and $\mathbf{LGF}_{K[\![x]\!]_0}$, respectively. We have proved our conjectures $\mathbf{LGF}_{\mathcal{E}}$ and $\mathbf{LGF}_{K[\![x]\!]_0}$ if M is of rank ≤ 2 [CT09, Theorem 7.1, Corollary 7.2], and then we established Dwork's conjecture $\mathbf{LGF}_{\mathrm{Dw}}$ if M is of rank ≤ 2 [CT09, Corollary 7.3].

Let us now explain the results in the present paper. First we characterize bounded φ-∇-modules over $\mathcal{E}$ by using Frobenius structures (Theorem 4.1):

(1) A bounded φ-∇-module M over $\mathcal{E}$ (i.e., $M^0 = 0$, which means that all the solutions on the generic disk are bounded) is isomorphic to a direct sum of several pure φ-∇-modules if the residue field k of $\mathcal{V}$ is perfect.

Note that the assertion corresponding to (1) is trivial for a φ-∇-module M over $K[\![x]\!]_0$ such that M_η is bounded by Christol's transfer theorem (see [CT09, Proposition 4.3]). This characterization implies the existence of a unique increasing filtration $\{P_i(M)\}$ of φ-∇-modules M over $\mathcal{E}$ such that $P_i(M)/P_{i-1}(M)$ is the maximally PBQ submodule of $M/P_{i-1}(M)$ (Corollary 5.5). This filtration is called the PBQ filtration. When we start with a φ-∇-module M over $K[\![x]\!]_0$, we can introduce a similar PBQ filtration for M, i.e., a filtration consisting of φ-∇-submodules over $K[\![x]\!]_0$ whose generic fibers will induce the PBQ filtration of the generic fiber M_η (Corollary 5.10). To this end we use an argument of A.J. de Jong in [dJ98] establishing the full faithfulness of the forgetful functor from the category of φ-∇-modules over $K[\![x]\!]_0$ to the category of φ-∇-modules over $\mathcal{E}$.

The need to study horizontality behavior for the PBQ condition with respect to the special and generic points leads us to introduce a new condition for φ-∇-modules over $K[\![x]\!]_0$, namely, the property of being "horizontally pure of bounded quotient " (which, for simplicity, we abbreviate as HPBQ, cf. Definition 6.1). Then in Theorem 6.5 we prove that

(2) our conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (see 2.5) on the comparison between the log-growth filtration and the Frobenius slope filtration at the special point holds for a HPBQ module.

A HPBQ module should be understood as a φ-∇-module for which the bounded quotient is horizontal and pure with respect to the Frobenius. Our method of proof will lead us to introduce the related definition of equislope φ-∇-modules over $K[\![x]\!]_0$ (Definition 6.7): they admit a filtration as φ-∇-modules over $K[\![x]\!]_0$ which induces the Frobenius slope filtration at the generic point. Note that a PBQ equislope object is HPBQ. Using this result, we prove in Theorem 7.1 that

(3) our conjecture $\mathbf{LGF}_{\mathcal{E}}$ (see 2.4) on comparison between the log-growth filtration and the Frobenius slope filtration at the generic point holds for PBQ modules over $\mathcal{E}$.

Indeed, for a φ-∇-module M over $\mathcal{E}$, the induced φ-∇-module $M_\tau = M \otimes_{\mathcal{E}} \mathcal{E}_t[\![X-t]\!]_0$ (where $\mathcal{E}_t[\![X-t]\!]_0$ is the ring of bounded functions on the open unit disk at generic point t) is equislope. For the proof of comparison for HPBQ modules, we use an explicit calculation of log-growth for solutions of certain Frobenius equations (Lemma 4.8) and a technical induction argument.
For a submodule L of a φ-∇-module M over $\mathcal{E}$ with $N = M/L$, the induced right exact sequence

$$L/L^\lambda \to M/M^\lambda \to N/N^\lambda \to 0$$

is also left exact for any λ if L is a maximally PBQ submodule of M by Proposition 2.6. Since there do exist PBQ filtrations, the comparison between the log-growth filtrations and the Frobenius slope filtrations for PBQ modules both at the generic point and at the special point implies the rationality of breaks (Theorem 7.2 and Proposition 7.3) as well as Dwork's conjecture (Theorem 8.1) that the special log-growth polygon lies above the generic log-growth polygon (including the coincidence of both end points):

(4) Our conjecture of comparison between the log-growth filtration and the Frobenius slope filtration at the special point (Conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$, 2.5) implies Dwork's conjecture (Conjecture $\mathbf{LGF}_{\mathrm{Dw}}$, 2.7).

As an application, we have the following theorem (Theorem 8.8) without any assumptions.

(5) The coincidence of both log-growth polygons at the generic and special points is equivalent to the coincidence of both Frobenius slope polygons at the generic and special points.

Let us also mention some recent work on log-growth. Y.André ([An08]) proved the conjecture $\mathbf{LG}_{\mathrm{Dw}}$ of Dwork without Frobenius structures, that is, the log-growth polygon at the special point is above the log-growth filtration at the generic point for ∇-modules, but without coincidence of both end points. (Note that his convention on the Newton polygon is different from ours, see Remark 2.8). He used semi-continuity of log-growth on Berkovich spaces. K.Kedlaya

defined the log-growth at the special point for regular singular connections and studied the properties of log-growth [Ke09, Chapter 18].
This paper is organized in the following manner. In Section 2 we recall our notation and results from [CT09]. In Section 3 we establish the independence of the category of φ-∇-modules over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) of the choices of Frobenius on $\mathcal{E}$ (resp. $K[\![x]\!]_0$). In Section 4 we study when the Frobenius slope filtration of φ-∇-modules over $\mathcal{E}$ is split and prove (1) above. In Section 5 we introduce the notion of PBQ and prove the existence of PBQ filtrations. In Section 6 we study the log-growth filtration for HPBQ φ-∇-modules over $K[\![x]\!]_0$ and prove the comparison (2) between the log-growth filtration and the Frobenius slope filtration. This comparison implies the comparison (3) for PBQ φ-∇-modules over $\mathcal{E}$ in Section 7. In Section 8 we show that (4) our conjecture of comparison at the special point implies Dwork's conjecture.

2 Preliminaries

We fix notation and recall the terminology in [CT09]. We also review Dwork's conjecture and our conjectures.

2.1 Notation

Let us fix the basic notation which follows from [CT09].

p : a prime number.

K : a complete discrete valuation field of mixed characteristic $(0, p)$.

$\mathcal{V}$: the ring of integers of K.

k : the residue field of $\mathcal{V}$.

$\mathfrak{m}$: the maximal ideal of $\mathcal{V}$.

$|\ |$: a p-adically absolute value on K and its extension as a valuation field, which is normalized by $|p| = p^{-1}$.

q : a positive power of p.

σ : (q-)Frobenius on K, i.e., a continuous lift of q-Frobenius endomorphism ($a \mapsto a^q$ on k). We suppose the existence of Frobenius on K. We also denote by σ a K-algebra endomorphism on $\mathcal{A}_K(0, 1^-)$, which is an extension of Frobenius on K, such that $\sigma(x)$ is bounded and $|\sigma(x) - x^q|_0 < 1$. Then $K[\![x]\!]_\lambda$ is stable under σ. We also denote by σ the unique extension of σ on $\mathcal{E}$, which is a Frobenius on $\mathcal{E}$. In the case we only discuss φ-∇-modules over $\mathcal{E}$, one can take a Frobenius σ on K such that $\sigma(x) \in \mathcal{E}$ with $|\sigma(x) - x^q|_0 < 1$.

$\widehat{K^{\mathrm{perf}}}$: the p-adic completion of the inductive limit K^{perf} of $K \xrightarrow{\sigma} K \xrightarrow{\sigma} \cdots$. Then $\widehat{K^{\mathrm{perf}}}$ is a complete discrete valuation field such that the residue field of the ring of integers of $\widehat{K^{\mathrm{perf}}}$ is the perfection of k and that the value group of $\widehat{K^{\mathrm{perf}}}$ coincides with the value group of K. The Frobenius σ uniquely extend to $\widehat{K^{\mathrm{perf}}}$. Moreover, taking the p-adic completion $\widehat{K^{\mathrm{al}}}$ of the maximally unramified extension K^{al} of K^{perf}, we have a canonical extension of K as a discrete valuation field with the same value group such that the residue field of the ring of integers is algebraically closed and the Frobenius extends on it. We use the same symbol σ for Frobenius on the extension.

q^λ : an element of K with $\log_q|q^\lambda| = -\lambda$ for a rational number λ such that $\sigma(q^\lambda) = q^\lambda$. Such a q^λ always exists if the residue field k is algebraically closed and $\lambda \in \log_q|K^\times|$. In particular, if k is algebraically closed, then there exists an extension L of K as a discrete valuation field with an extension of Frobenius such that q^λ is contained in L for a fix λ. In this paper we freely extend K as above.

$\mathcal{A}_K(c, r^-)$: the K-algebra of analytic functions on the open disk of radius r at the center c, i.e.,

$$\mathcal{A}_K(c, r^-) = \left\{ \sum_{n=0}^{\infty} a_n(x-c)^n \in K[\![x-c]\!] \;\middle|\; \begin{array}{l} |a_n|\gamma^n \to 0 \,\mathrm{as}\, n \to \infty \\ \mathrm{for\ any}\, 0 < \gamma < r \end{array} \right\}.$$

$K[\![x]\!]_0$: the ring of bounded power series over K, i.e.,

$$K[\![x]\!]_0 = \left\{ \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}_K(0, 1^-) \;\middle|\; \sup_n |a_n| < \infty \right\}.$$

An element of $K[\![x]\!]_0$ is said to be a bounded function.

$K[\![x]\!]_\lambda$: the Banach K-module of power series of log-growth λ in $\mathcal{A}_K(0, 1^-)$ for a nonnegative real number $\lambda \in \mathbb{R}_{\geq 0}$, i.e.,

$$K[\![x]\!]_\lambda = \left\{ \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}_K(0, 1^-) \;\middle|\; \sup_n |a_n|/(n+1)^\lambda < \infty \right\},$$

with a norm $|\sum_{n=0}^\infty a_n x^n|_\lambda = \sup_n |a_n|/(n+1)^\lambda$. $K[\![x]\!]_\lambda$ is a $K[\![x]\!]_0$-modules. $K[\![x]\!]_\lambda = 0$ for $\lambda < 0$ for the convenient. An element $f \in K[\![x]\!]_\lambda$ which is not contained in $K[\![x]\!]_\gamma$ for $\gamma < \lambda$ is said to be exactly of log-growth λ.

$\mathcal{E}$: the p-adic completion of the field of fractions of $K[\![x]\!]_0$ under the Gauss norm $|\ |_0$, i.e.,

$$\mathcal{E} = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \;\middle|\; a_n \in K, \sup_n |a_n| < \infty, \, |a_n| \to 0 \,(\mathrm{as}\, n \to -\infty) \right\}.$$

$\mathcal{E}$ is a complete discrete valuation field under the Gauss norm $|\ |_0$ in fact K is discrete valuated. The residue field of the ring $\mathcal{O}_{\mathcal{E}}$ of integers of $\mathcal{E}$ is $k(\!(x)\!)$.

t : a generic point of radius 1.

$\mathcal{E}_t$: the valuation field corresponding to the generic point t, i.e., the same field as $\mathcal{E}$ in which x is replaced by t: we emphasize t in the notation with the respect to [CT09]. We regard the Frobenius σ as a Frobenius on $\mathcal{E}_t$.

$\mathcal{E}_t[\![X-t]\!]_0$: the ring of bounded functions in $\mathcal{A}_{\mathcal{E}_t}(t, 1^-)$. Then

$$\tau : \mathcal{E} \to \mathcal{E}_t[\![X-t]\!]_0 \qquad \tau(f) = \sum_{n=0}^{\infty} \frac{1}{n!}(\frac{d^n}{dx^n} f)|_{x=t}(X-t)^n$$

is a K-algebra homomorphism which is equivariant under the derivations $\frac{d}{dx}$ and $\frac{d}{dX}$. The Frobenius σ on $\mathcal{E}_t[\![X-t]\!]_0$ is defined by σ on $\mathcal{E}_t$ and $\sigma(X-t) = \tau(\sigma(x)) - \sigma(x)|_{x=t}$. τ is again σ-equivariant.

For a function f on R and for a matrix $A = (a_{ij})$ with entries in R, we define $f(A) = (f(a_{ij}))$. In case where f is a norm $|\ |$, then $|A| = \sup_{i,j} |a_{ij}|$. We use 1 (resp. 1_r) to denote the unit matrix of suitable degree (resp. of degree r). For a decreasing filtration $\{V^\lambda\}$ indexed by the set $\mathbb{R}$ of real numbers, we put

$$V^{\lambda-} = \cap_{\mu<\lambda} V^\mu, \qquad V^{\lambda+} = \cup_{\mu>\lambda} V^\mu.$$

We denote by $W_{\lambda-} = \cup_{\mu<\lambda} W_\mu$ and $W_{\lambda+} = \cap_{\mu>\lambda} W_\mu$ the analogous objects for an increasing filtration $\{W_\lambda\}_\lambda$, respectively.

2.2 Terminology

We recall some terminology and results from [CT09].
Let R be either K (K might be $\mathcal{E}$) or $K[\![x]\!]_0$. A φ-module over R consists of a free R-module M of finite rank and an R-linear isomorphism $\varphi : \sigma^* M \to M$. For a φ-module over K, there is an increasing filtration $\{S_\lambda(M)\}_{\lambda\in\mathbb{R}}$ which is called the Frobenius slope filtration. Then there is a sequence $\lambda_1 < \cdots < \lambda_r$ of real numbers, called the Frobenius slopes of M, such that $S_{\lambda_i}(M)/S_{\lambda_i-}(M)$ is pure of slope λ_i and $M \otimes \widehat{K^{\mathrm{al}}} \cong \oplus_i S_{\lambda_i}(M) \otimes_K \widehat{K^{\mathrm{al}}}/S_{\lambda_i-}(M) \otimes_K \widehat{K^{\mathrm{al}}}$ is the Dieudonné-Manin decomposition as φ-modules over $\widehat{K^{\mathrm{al}}}$. We call λ_1 the first Frobenius slope and λ_r the highest Frobenius slope, respectively.
Let R be either $\mathcal{E}$ or $K[\![x]\!]_0$. A φ-∇-module over R consists of a φ-module (M, φ) over R and a K-connection $\nabla : M \to M \otimes_R \Omega_R$, where $\Omega_R = Rdx$, such that $\varphi \circ \sigma^*(\nabla) = \nabla \circ \varphi$. For a basis $(e_1, \cdots, e_r)$, the matrices A and G with entries R,

$$\varphi(1\otimes e_1, \cdots, 1\otimes e_r) = (e_1, \cdots, e_r)A, \qquad \nabla(e_1, \cdots, e_r) = (e_1, \cdots, e_r)G dx$$

are called the Frobenius matrix and the connection matrix of R, respectively. Then one has

$$\frac{d}{dx}A + GA = (\frac{d}{dx}\sigma(x))A\sigma(G) \tag{FC}$$

by the horizontality of φ. We denote the dual of M by $M^\vee$.
Let M be a φ-∇-module over $K[\![x]\!]_0$. We define the K-space

$$V(M) = \{s \in M \otimes_{K[\![x]\!]_0} \mathcal{A}_K(0,1^-) \mid \nabla(s) = 0\}$$

of horizontal sections and the K-space of solutions,

$$\mathrm{Sol}(M) = \mathrm{Hom}_{K[\![x]\!]_0[\nabla]}(M, \mathcal{A}_K(0,1^-)),$$

on the unit disk. Both $\dim_K V(M)$ and $\dim_K \mathrm{Sol}(M)$ equal to $\mathrm{rank}_{K[\![x]\!]_0} M$ by the solvability. If one fixes a basis of M, the solution Y of the equations

$$\begin{cases} A(0)\sigma(Y) = YA \\ \frac{d}{dx}Y = YG \\ Y(0) = 1 \end{cases}$$

in $\mathcal{A}_K(0,1^-)$ is a solution matrix of M, where $A(0)$ and $Y(0)$ are the constant terms of A and Y, respectively. The log-growth filtration $\{V(M)^\lambda\}_{\lambda\in\mathbb{R}}$ is defined by the orthogonal space of the K-space $\mathrm{Sol}_\lambda(M) = \mathrm{Hom}_{K[\![x]\!]_0[\nabla]}(M, K[\![x]\!]_\lambda)$ under the natural bilinear perfect pairing

$$V(M) \times \mathrm{Sol}(M) \to K.$$

Then $V(M)^\lambda = 0$ for $\lambda >> 0$ by the solvability of M and the log-growth filtration is a decreasing filtration of $V(M)$ as φ-modules over K. The following proposition allows one to change the coefficient field K to a suitable extension K'.

Proposition 2.1 ([CT09, Proposition 1.10]) *Let M be a φ-module over $K[\![x]\!]_0$. For any extension K' over K as a complete discrete valuation field with an extension of Frobenius, there is a canonical isomorphism $V(M \otimes_{K[\![x]\!]_0} K'[\![x]\!]_0) \cong V(M) \otimes_K K'$ as log-growth filtered φ-modules.*

The induced φ-∇-module $M_\eta = M \otimes_{K[\![x]\!]_0} \mathcal{E}$ over $\mathcal{E}$ is said to be the generic fiber of M, and the K-module $V(M)$ is called the special fiber of M.

Let M be a φ-∇-module over $\mathcal{E}$. We denote by M_τ the induced φ-∇-module $M \otimes_{\mathcal{E}} \mathcal{E}_t[\![X-t]\!]_0$ over $\mathcal{E}_t[\![X-t]\!]_0$. Applying the theory of Robba [Ro75], we have a decreasing filtration $\{M^\lambda\}_{\lambda\in\mathbb{R}}$ of M as φ-∇-modules over $\mathcal{E}$ which is characterized by the following universal property.

Theorem 2.2 [Ro75, 2.6, 3.5] (See [CT09, Theorem 3.2].) *For any real number λ, M/M^λ is the maximum quotient of M such that all solutions of log-growth λ of M_τ on the generic unit disk come from the solutions of $(M/M^\lambda)_\tau$.*

The filtration $\{M^\lambda\}$ is called the log-growth filtration of M. Note that $M^\lambda = M$ for $\lambda < 0$ by definition and $M^\lambda = 0$ for $\lambda >> 0$ by the solvability. The quotient module M/M^0 is called the bounded quotient, and, in particular, if $M^0 = 0$, then M is called bounded.

Our main theorem in [CT09] is the following:

Theorem 2.3 ([CT09, Theorem 6.17])

(1) *Let M be a φ-∇-module over $\mathcal{E}$. If $\lambda_{\max}$ is the highest Frobenius slope of M, then $M^\lambda \subset (S_{\lambda-\lambda_{\max}}(M^\vee))^\perp$.*

(2) *Let M be a φ-∇-module over $K[\![x]\!]_0$. If $\lambda_{\max}$ is the highest Frobenius slope of M_η, then $V(M)^\lambda \subset (S_{\lambda-\lambda_{\max}}(V(M^\vee)))^\perp$.*

Here $S^\perp$ denotes the orthogonal space of S under the natural bilinear perfect pairing

$$M \otimes_{\mathcal{E}} M^\vee \to \mathcal{E} \text{ or } V(M) \otimes_K V(M^\vee) \to K.$$

We conjectured that equalities hold in Theorem 2.3 if M is PBQ (Definition 5.1) in [CT09], and proved them if M is of rank ≤ 2 [CT09, Theorem 7.1, Corollary 7.2].

Conjecture 2.4 ([CT09, Conjectures 6.8]) *Let M be a φ-∇-module over $\mathcal{E}$.*

(1) *All breaks of log-growth filtration of M are rational and $M^\lambda = M^{\lambda+}$ for any λ.*

(2) *Let $\lambda_{\max}$ be the highest Frobenius slope of M. If M/M^0 is pure as φ-module (PBQ in Definition 5.1 (1)), then $M^\lambda = (S_{\lambda-\lambda_{\max}}(M^\vee))^\perp$.*

We denote Conjecture 2.4 above by $\mathbf{LGF}_{\mathcal{E}}$.

Conjecture 2.5 ([CT09, Conjectures 6.9]) *Let M be a φ-∇-module over $K[\![x]\!]_0$.*

(1) *All breaks of log-growth filtration of $V(M)$ are rational and $V(M)^\lambda = V(M)^{\lambda+}$ for any λ.*

(2) *Let $\lambda_{\max}$ be the highest Frobenius slope of M_η. If M_η/M_η^0 is pure as φ-module (PBQ in Definition 5.1 (2)), then $V(M)^\lambda = (S_{\lambda-\lambda_{\max}}(V(M)^\vee))^\perp$.*

We denote Conjecture 2.5 above by $\mathbf{LGF}_{K[\![x]\!]_0}$.

Note that we formulate the theorem and the conjecture in the case where $\lambda_{\max} = 0$ in [CT09]. However, the theorem holds for an arbitrary $\lambda_{\max}$ by Proposition 2.1 (and the conjecture should also hold). Moreover, it suffices to establish the conjecture when the residue field k of $\mathcal{V}$ is algebraically closed.

In section 7 we will reduce the conjecture $\mathbf{LGF}_{\mathcal{E}}\,(1)$ (resp. $\mathbf{LGF}_{K[\![x]\!]_0}\,(1)$) to the conjecture $\mathbf{LGF}_{\mathcal{E}}\,(2)$ (resp. $\mathbf{LGF}_{K[\![x]\!]_0}\,(2)$) by applying the proposition below to the PBQ filtration which is introduced in section 5. The following proposition is useful for attacking log-growth questions by induction.

Proposition 2.6 *Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-∇-modules over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) and let $\lambda_{\max}$ be the highest Frobenius slope of M and L (resp. M_η and L_η).*

(1) *Suppose that $L^\lambda = (S_{\lambda-\lambda_{\max}}(L^\vee))^\perp$ for λ. Then the induced sequence*

$$0 \to L/L^\lambda \to M/M^\lambda \to N/N^\lambda \to 0$$

is exact.

(2) *Suppose that $V(L)^\lambda = (S_{\lambda-\lambda_{\max}}(V(L)^\vee))^\perp$ for λ. Then the induced sequence*

$$0 \to V(L)/V(L)^\lambda \to V(M)/V(M)^\lambda \to V(N)/V(N)^\lambda \to 0$$

is exact.

Proof. (1) Since

$$L/L^\lambda \to M/M^\lambda \to N/N^\lambda \to 0$$

is right exact by [CT09, Proposition 3.6], we have only to prove the injectivity of the first morphism. There is an inclusion relation

$$M^\lambda \subset (S_{\lambda-\lambda_{\max}}(M^\vee))^\perp = S_{(\lambda_{\max}-\lambda)-}(M)$$

by Theorem 2.3 and the equality

$$L^\lambda = (S_{\lambda-\lambda_{\max}}(L^\vee))^\perp = S_{(\lambda_{\max}-\lambda)-}(L).$$

holds by our hypothesis on L. Since the Frobenius slope filtrations are strict for any morphism, the bottom horizontal morphism in the natural commutative diagram

$$\begin{array}{ccc} L/L^\lambda & \longrightarrow & M/M^\lambda \\ =\downarrow & & \downarrow \\ L/S_{(\lambda_{\max}-\lambda)-}(L) & \longrightarrow & M/S_{(\lambda_{\max}-\lambda)-}(M) \end{array}$$

is injective. Hence we have the desired injectivity.
(2) The proof here is similar to that of (1) on replacing [CT09, Proposition 3.6] by [CT09, Proposition 1.8]. $\square$

2.3 Dwork's conjecture

We recall Dwork's conjecture. We have proved it in the case where M is of rank ≤ 2 [CT09, Corollary 7.3].

Conjecture 2.7 ([Dw73, Conjecture 2], [CT09, Conjecture 4.9]) *Let M be a φ-∇-module over $K[\![x]\!]_0$. Then the special log-growth is above the generic log-growth polygon (with coincidence at both endpoints).*

We denote Conjecture 2.7 above by $\mathbf{LGF}_{\mathrm{Dw}}$. We will prove that the conjecture $\mathbf{LGF}_{\mathrm{Dw}}$ follows from the conjectures $\mathbf{LGF}_{\mathcal{E}}$ and $\mathbf{LGF}_{K[\![x]\!]_0}$ in section 8. There is also a version of Dwork's conjecture without Frobenius structures, we denote it by $\mathbf{LG}_{\mathrm{Dw}}$.
Let us recall the definition of the log-growth polygon: the generic log-growth polygon is the piecewise linear curve defined by the vertices

$$(0,0), (\dim_{\mathcal{E}} \tfrac{M_\eta}{M_\eta^{\lambda_1+}}, \lambda_1 \dim_{\mathcal{E}} \tfrac{M_\eta^{\lambda_1-}}{M_\eta^{\lambda_1+}}), \cdots, (\dim_{\mathcal{E}} \tfrac{M_\eta}{M_\eta^{\lambda_i+}}, \textstyle\sum_{j=1}^{i} \lambda_j \dim_{\mathcal{E}} \tfrac{M_\eta^{\lambda_j-}}{M_\eta^{\lambda_j+}}),$$
$$\cdots, (\dim_{\mathcal{E}} M_\eta, \textstyle\sum_{j=1}^{r} \lambda_j \dim_{\mathcal{E}} \tfrac{M_\eta^{\lambda_j-}}{M_\eta^{\lambda_j+}}),$$

where $0 = \lambda_1 < \cdots < \lambda_r$ are breaks (i.e., $M^{\lambda-} \neq M^{\lambda+}$) of the log-growth filtration of M_η. The special log-growth polygon is defined in the same way using the log-growth filtration of $V(M)$.

Remark 2.8 (1) *The convention of André's polygon of log-growth [An08] is different from ours. His polygon at the generic fiber is $\sum_{j=1}^{r} \lambda_j \dim_{\mathcal{E}} \frac{M^{\lambda_j-}}{M^{\lambda_j+}}$ below our polygon in the direction of the vertical axis and the starting point of the polygon is $(\dim_{\mathcal{E}} M, 0)$, and the same at the special fiber. André proved the conjecture $\mathbf{LG}_{\mathrm{Dw}}$ except the coincidence of both endpoints in [An08].*

(2) *If the special log-growth polygon lies above the generic log-growth polygon in both conventions of André's and ours, then both endpoints coincide with each other. However even if this is the case, we cannot prove $M_\eta^{\lambda} = M_\eta^{\lambda+}$ (resp. $V(M)^{\lambda} = V(M)^{\lambda+}$) for a break λ.*

3 Choices of Frobenius

Let us recall the precise form of equivalence between categories of φ-∇-modules with respect to different choices of Frobenius on $\mathcal{E}$ (resp. $K[\![x]\!]_0$) (see [Ts98a, Section 3.4] for example). We will use it in the next section.

3.1 Comparison morphism $\vartheta_{\sigma_1,\sigma_2}$

Let σ_1 and σ_2 be Frobenius maps on $\mathcal{E}$ (resp. $K[\![x]\!]_0$) such that the restriction of each σ_i to K is the given Frobenius on K. Let M be a φ-∇-module. We define an $\mathcal{E}$-linear (resp. $K[\![x]\!]_0$-linear) morphism

$$\vartheta_{\sigma_1,\sigma_2} : \sigma_1^* M \to \sigma_2^* M$$

by

$$\vartheta_{\sigma_1,\sigma_2}(a \otimes m) = a \sum_{n=0}^{\infty} (\sigma_2(x) - \sigma_1(x))^n \otimes \frac{1}{n!} \nabla(\frac{d^n}{dx^n})(m).$$

Since M is solvable and $|\sigma_2(x) - \sigma_1(x)| < 1$, the right hand side converges in $\sigma_2^* M$. As a matrix representation, the transformation matrix is

$$H = \sum_{n=0}^{\infty} \sigma_2(G_n) \frac{(\sigma_2(x) - \sigma_1(x))^n}{n!}$$

for the induced basis $1 \otimes e_1, \cdots, 1 \otimes e_r$, where G is the matrix of connection, $G_0 = 1$ and $G_{n+1} = GG_n + \frac{d}{dx} G_n$ for $n \geq 0$.

Proposition 3.1 *Let $\sigma_1, \sigma_2, \sigma_3, \sigma$ be Frobenius maps of $\mathcal{E}$ (resp. $K[\![x]\!]_0$) as above. Then we have the cocycle conditions:*

(1) $\vartheta_{\sigma_2,\sigma_3} \circ \vartheta_{\sigma_1,\sigma_2} = \vartheta_{\sigma_1,\sigma_3}$.

(2) $\vartheta_{\sigma,\sigma} = \mathrm{id}_{\sigma^* M}$.

Proposition 3.2 *Let M be a φ-∇-module pure of slope λ over $\mathcal{E}$ and let A be the Frobenius matrix of M with respect to a basis. Suppose that $|A - q^{\lambda} 1|_0 \leq q^{-\mu}$ for $\mu \geq \lambda$. Then the representation matrix H of the comparison morphism $\vartheta_{\sigma_1,\sigma_2}$ with respect to the bases which are the pull-backs by σ_1 and σ_2 respectively, satisfies $|H - 1|_0 < q^{\lambda - \mu}$.*

Proof. By replacing the Frobenius φ by $q^{-\lambda}\varphi$, we may assume that $\lambda = 0$. The assertion then follows from the fact that under these assumptions the solution matrix Y at the generic point satisfies $Y \equiv 1 \,(\mathrm{mod}\, (X-t)\mathfrak{m}^n \mathcal{O}_{\mathcal{E}_t}[\![X-t]\!])$. Here n is the least integer such that $|\mathfrak{m}^n| \leq q^{-\mu}$. □

3.2 Equivalence of categories

Let R be either $\mathcal{E}$ or $K[\![x]\!]_0$ and let σ_1 and σ_2 be Frobenius maps on R as in the previous subsection. We define a functor

$$\vartheta^*_{\sigma_1,\sigma_2} : (\varphi\text{-}\nabla\text{-modules over } (R, \sigma_2)) \to (\varphi\text{-}\nabla\text{-modules over } (R, \sigma_1))$$

by $(M, \nabla, \varphi) \mapsto (M, \nabla, \varphi \circ \vartheta_{\sigma_1,\sigma_2})$. Here $\vartheta_{\sigma_1,\sigma_2}$ is defined as in the previous section. The propositions of the previous subsection then give

Theorem 3.3 *$\vartheta^*_{\sigma_1,\sigma_2}$ is an equivalence of categories which preserves tensor products and duals. Moreover, $\vartheta^*_{\sigma_1,\sigma_2}$ preserves the Frobenius slope filtration and the log-growth filtration of M (resp. $V(M)$) for a φ-∇-module M over $\mathcal{E}$ (resp. $K[\![x]\!]_0$).*

4 Boundedness and splitting of the Frobenius slope filtration

4.1 Splitting theorem

Theorem 4.1 *Suppose that the residue field k of $\mathcal{V}$ is perfect. A φ-∇-module M over $\mathcal{E}$ is bounded if and only if M is a direct sum of pure φ-∇-modules, that is,*

$$M \cong \oplus_{i=1}^{r} S_{\lambda_i}(M)/S_{\lambda_i -}(M)$$

as φ-∇-modules, where $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ are Frobenius slopes of M.

Since any pure φ-∇-module over $\mathcal{E}$ is bounded by [CT09, Corollary 6.5]. Hence, Theorem 4.1 above follows from the next proposition.

Proposition 4.2 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-∇-modules over $\mathcal{E}$ such that both L and N are pure of Frobenius slope λ and ν, respectively. If one of the conditions*

(1) $\nu - \lambda < 0$*;*

(2) $\nu - \lambda > 1$*;*

(3) *M is bounded and* $0 < \nu - \lambda \leq 1$,

holds, then the exact sequence is split, that is, $M \cong L \oplus N$ as φ-∇-modules.

In the case (1) the assertion easily follows from the fact that, for $a \in \mathcal{E}$ with $|a|_0 < 1$, $a\sigma$ is a contractive operator on the p-adic complete field $\mathcal{E}$. The rest of this section will be dedicated to proving the assertion in cases (2) and (3).

4.2 Descent of splittings

Proposition 4.3 *Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-modules over $\mathcal{E}$ such that L and N are pure and the two slopes are different. Let $\mathcal{E}'$ be one of the following:*

(i) *$\mathcal{E}'$ is a p-adic completion of an unramified extension of $\mathcal{E}$;*

(ii) *$\mathcal{E}'$ is the p-adic completion of $\mathcal{E} \otimes_K K'$ for some extension K' of K as a complete discrete valuation field with an extension σ' of σ such that, if G is the group of continuous automorphisms of K' over K, then the invariant subfield of K' by the action of G is K.*

If the exact sequence is split over $\mathcal{E}'$, then it is split over $\mathcal{E}$. The same holds for φ-∇-modules over $\mathcal{E}$.

Proof. In each case we may assume that $\mathcal{E}$ is the invariant subfield of $\mathcal{E}'$ by the action of continuous automorphism group G. Let $e_1, \cdots, e_r, e_{r+1}, \cdots, e_{r+s}$ be a basis of M over $\mathcal{E}$ such that $e_1, \cdots, e_r$ is a basis of L. Put

$$\varphi(e_1, \cdots, e_r, e_{r+1}, \cdots, e_{r+s}) = (e_1, \cdots, e_r, e_{r+1}, \cdots, e_{r+s}) \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where A_{11} is of degree r and A_{22} is of degree s, respectively, and all entries of A_{11}, A_{12} and A_{22} are contained in $\mathcal{E}$. By the hypothesis of splitting over $\mathcal{E}'$ there exists a matrix Y with entries in $\mathcal{E}'$ such that

$$A_{11}\sigma(Y) - YA_{22} + A_{12} = 0.$$

For any $\rho \in G$, $\rho(Y)$ also gives a splitting. Hence $A_{11}\sigma(Y - \rho(Y)) = (Y - \rho(Y))A_{22}$. By the assumption on slopes, $\rho(Y) = Y$. Therefore, all entries of Y are contained in $\mathcal{E}$ and the exact sequence is split over $\mathcal{E}$. $\square$

Definition 4.4 *An extension $\mathcal{E}'$ (resp. K') of $\mathcal{E}$ (resp. K) is allowable if $\mathcal{E}'$ is a finitely successive extension of $\mathcal{E}$ (resp. K) of type in (i) or (ii) (resp. (ii)) of Proposition 4.3.*

4.3 Preparations

In this subsection we assume that the residue field k of $\mathcal{V}$ is algebraically closed. Moreover we assume that the Frobenius on $\mathcal{E}$ (resp. $K[\![x]\!]_0$) is defined by $\sigma(x) = x^q$. For an element $a = \sum_n a_n x^n$ in $\mathcal{E}$ (resp. $K[\![x]\!]$) we define the subseries $a^{(q)}$ by $\sum_n a_{qn} x^{qn}$.

Lemma 4.5 *Let $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ be an invertible matrix of degree $r + s$ over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) with A_{11} of degree r and A_{22} of degree s such that the matrix satisfies the conditions:*

(i) *$A_{11} = A_{11}^{(q)}$ and $A_{11} = P^{-1}$ for a matrix P over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) with $|P|_0 < 1$,*

(ii) *$A_{22} = A_{22}^{(q)}$ and $|A_{22} - 1_s|_0 < 1$.*

Suppose that $A_{12}^{(q)} \neq 0$. Then there exists an $r \times s$ matrix Y over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) with $|Y|_0 < |A_{12}^{(q)}|_0$ such that, if one puts $B = A_{11}\sigma(Y) - YA_{22} + A_{12}$, then $|B^{(q)}|_0 < |A_{12}^{(q)}|_0$. Moreover, there exists an $r \times s$ matrix Y over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) such that if one defines B_{12} by

$$\begin{pmatrix} A_{11} & B_{12} \\ 0 & A_{22} \end{pmatrix} = \begin{pmatrix} 1_r & -Y \\ 0 & 1_s \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} 1_r & \sigma(Y) \\ 0 & 1_s \end{pmatrix},$$

then $B_{12}^{(q)} = 0$.

PROOF. Take a matrix Y such that $\sigma(Y) = -PA_{12}^{(q)}$. Such a Y exists since the residue field k of $\mathcal{V}$ is perfect. Then $|Y|_0 < |A_{12}^{(q)}|_0$ and $B = A_{11}\sigma(Y) - YA_{22} + A_{12} = A_{11}PA_{12}^{(q)} - YA_{22} + A_{12} = A_{12} - A_{12}^{(q)} - YA_{22}$. Hence $|B^{(q)}|_0 = |YA_{22}^{(q)}|_0 < |A_{12}^{(q)}|_0$ and we have the first assertion. Applying the first assertion inductively on the value $|A_{12}^{(q)}|_0$, we have a desired matrix Y of the second assertion since $\mathcal{E}$ (resp. $K[\![x]\!]_0$) is complete under the norm $|\ |_0$. $\square$

We give a corollary of the preceding lemma for φ-∇-modules over $\mathcal{E}$.

PROPOSITION 4.6 *Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-∇-modules over $\mathcal{E}$. Suppose that N is pure of Frobenius slope ν and all Frobenius slopes of L are less than ν. Then there exist an allowable extension $\mathcal{E}'$ of $\mathcal{E}$ and a basis $e_1, \cdots, e_r, e_{r+1}, \cdots, e_{r+s}$ of $M \otimes_{\mathcal{E}} \mathcal{E}'$ with respect to the exact sequence such that, if one fixes an element x' in the ring $\mathcal{O}_{\mathcal{E}'}$ of integers of $\mathcal{E}'$ whose image gives a uniformizer of the residue field of $\mathcal{O}_{\mathcal{E}'}$ and a Frobenius σ' on $\mathcal{E}'$ with $\sigma'(x') = x'^q$, then the Frobenius matrix $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ of $M \otimes_{\mathcal{E}} \mathcal{E}'$ with respect to σ' (here we use Theorem 3.3) has the following form:*

(i) $A_{11} = A_{11}^{(q)}$ *and* $A_{11} = P^{-1}$ *for a matrix* $|P|_0 < q^{\nu}$,

(ii) $A_{22} = A_{22}^{(q)}$ *and* $|A_{22} - q^{\nu}1_s|_0 < q^{-\nu}$,

(iii) $A_{12}^{(q)} = 0$,

where $a^{(q)}$ is defined by using the parameter x'. Moreover, one can replaces the inequality $|A_{22} - q^{\nu}1_s|_0 < q^{-\nu}$ in (ii) by the inequality $|A_{22} - q^{\nu}1_s|_0 < q^{-\nu}\eta$ for a given $0 < \eta \leq 1$ (the extension $\mathcal{E}'$ depends on η).

PROOF. Since k is algebraically closed, there is a uniformizer π of K such that $\sigma(\pi) = \pi$. Let K_m be a Galois extension $K(\pi^{1/m}, \zeta_m)$ of K for a positive integer m, where ζ_m denotes a primitive m-th root of unity. Then σ on K extends on K_m. If we choose a positive integer m such that $m/\log_q|\pi|$ is a common multiple of denominators of ν and the highest Frobenius slope of L, then ν and the highest Frobenius slope of L are contained in $\log_q|K_m^{\times}|$. Hence we may assume that $\nu = 0$ and all Frobenius slopes of the twist $\pi\varphi_L$ of the Frobenius φ_L of L are less than or equal to 0.

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ be a Frobenius matrix of M with respect to the given exact sequence. Since any φ-module over $\mathcal{E}$ has a cyclic vector [Ts96, Proposition 3.2.1], we may assume that $A_{22} \in \mathrm{GL}_s(\mathcal{O}_{\mathcal{E}})$ by $\nu = 0$. Then there is a matrix $X \in \mathrm{GL}_s(\mathcal{O}_{\mathcal{E}'})$ such that $X^{-1}A_{22}\sigma(X) \equiv 1_s \pmod{\mathfrak{m}\mathcal{O}_{\mathcal{E}'}}$ for some finite unramified extension $\mathcal{E}'$ over $\mathcal{E}$ by [Ts98b, Lemma 5.2.2]. By applying the existence of a cyclic vector again, we may assume that the all entries of Frobenius matrix of $L^{\vee}$ are contained in $\mathfrak{m}\mathcal{O}_{\mathcal{E}}$ by the hypothesis on Frobenius slopes of L.

Now we fix a parameter x' of $\mathcal{E}'$ and change a Frobenius σ' on $\mathcal{E}'$ such that $\sigma'(x') = x'^q$. The the hypothesis of the matrices A_{11} and A_{12} are stable by Theorem 3.3. If one replaces the basis $(e_1, \cdots, e_{r+s})$ by $(e_1, \cdots, e_{r+s})A$, then the Frobenius matrix becomes $\sigma'(A)$. Since the hypothesis in Lemma 4.5 hold in our Frobenius matrix A, we have the assertion. □

Now a variant of Proposition 4.6 for φ-∇-modules over $K[\![x]\!]_0$, which we use it in section 6, is given.

Proposition 4.7 *Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-∇-modules over $K[\![x]\!]_0$. Suppose that N_η is pure of Frobenius slope ν and all Frobenius slopes of L_η are less than ν. Then there exist an allowable extension K' of K with an extension of Frobenius σ' and a basis $e_1, \cdots, e_r, e_{r+1}, \cdots, e_{r+s}$ of $M \otimes_{K[\![x]\!]_0} K'[\![x]\!]_0$ with respect to the exact sequence such that the Frobenius matrix $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ of $M \otimes_{K[\![x]\!]_0} K'[\![x]\!]_0$ with respect to σ' has the following form:*

(i) *$A_{11} = A_{11}^{(q)}$ and $A_{11} = P^{-1}$ for a matrix $|P|_0 < q^\nu$,*

(ii) *$A_{22} = q^\nu 1_s$,*

(iii) *$A_{12}^{(q)} = 0$*

Proof. We may assume $\mu = 0$ and the highest Frobenius slope of L_η is contained in $\log_q |K_m^\times|$ as in the proof of Proposition 4.6. Then N is a direct sum of copies of the unit object $(K[\![x]\!]_0, d, \sigma)$'s since k is algebraically closed. In order to find the matrix P, we apply the isogeny theorem [Ka79, Theorem 2.6.1] and the existence of a free lattice over $\mathcal{V}[\![x]\!]$ in [dJ98, Lemma 6.1] for $L^\vee$. The rest is again same as the proof of Proposition 4.6. □

Lemma 4.8 *Let ν be a nonnegative rational number. Suppose that $y \in xK[\![x]\!]$ satisfies a Frobenius equation*

$$y - q^{-\nu} a\sigma(y) = f.$$

for $a \in K$ with $|a| = 1$ and for $f = \sum_n f_n x^n \in xK[\![x]\!]$.

(1) *Suppose that $f^{(q)} = 0$. If $f \in K[\![x]\!]_\nu \setminus \{0\}$, then $y \in K[\![x]\!]_\nu \setminus K[\![x]\!]_{\nu-}$, and if $f \in K[\![x]\!]_\lambda \setminus K[\![x]\!]_{\lambda-}$ for $\lambda > \nu$, then $y \in K[\![x]\!]_\lambda \setminus K[\![x]\!]_{\lambda-}$.*

(2) *Let l be a nonnegative integer with $q \not| \, l$. If $f \in K[\![x]\!]_0$ and $|f_l| > |q^\nu f|_0 = q^{-\nu}|f|_0 \neq 0$, then $y \in K[\![x]\!]_\nu \setminus K[\![x]\!]_{\nu-}$.*

Proof. Since the residue field k of $\mathcal{V}$ is algebraically closed, we may assume that $a = 1$. Formally in $K[\![x]\!]$,

$$y = \sum_n \sum_{m=0}^{\infty} (q^{-\nu})^m \sigma^m(f_n) x^{q^m n}$$

is a solution of the equation.
(1) If $q^m n = q^{m'} n'$, then $m = m'$ and $n = n'$ because $q \not| n, n'$. Hence, $y \neq 0$. By considering a subseries $\sum_{m=0}^{\infty} (q^{-\nu})^m \sigma^m(f_n) x^{q^m n}$ for $f_n \neq 0$, y is of log-growth equal to or greater than ν. Moreover, we have

$$|(q^{-\nu})^m \sigma^m(f_n)|/(q^m n + 1)^\nu = |f_n|/(n + 1/q^m)^\nu$$

Hence, if $f \in K[\![x]\!]_\nu$, then y is exactly of log-growth ν. Suppose $f \in K[\![x]\!]_\lambda \setminus K[\![x]\!]_{\lambda-}$. Since for each m, n

$$|(q^{-\nu})^m \sigma^m(f_n)|/(q^m n + 1)^\lambda = |f_n|/(q^{m(1-\nu/\lambda)} n + 1/q^{m\nu/\lambda})^\lambda,$$

the log-growth of y is exactly λ.
(2) There exists $z \in xK[\![x]\!]_0$ with $|z|_0 \leq |q^\nu f|_0 = q^{-\nu}|f|_0$ such that, if $g = f - z + q^{-\nu}\sigma(z) = \sum_n g_n x^n$, then $g^{(q)} = 0$ and $g_l \neq 0$ by the same construction of the proof of Lemma 4.5. The assertion now follows from (1). □

4.4 PROOF OF PROPOSITION 4.2

Replacing K by an extension, we may assume that k is algebraically closed and that $\lambda = 0$, $\nu > 0$ and $\nu \in \log_q|K^\times|$ by Proposition 4.3 (see the beginning of proof of Proposition 4.6). We may also assume $\sigma(x) = x^q$ by Theorem 3.3. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ be a Frobenius matrix of M with respect to the basis which is compatible with the given extension (i.e., the $(1,1)$-part (resp. $(2,2)$-part) corresponds to L (resp. N)) and let $G = \begin{pmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{pmatrix}$ be the matrix of the connection, respectively. The commutativity of Frobenius and connection (the relation (FC) in section 2.2) gives the relation

$$1^\circ \qquad \frac{d}{dx}A_{12} + G_{11}A_{12} + G_{12}A_{22} = qx^{q-1}(A_{11}\sigma(G_{12}) + A_{12}\sigma(G_{22}))$$

of the $(1,2)$-part of the matrix. We may assume that

2° $A_{11} = A_{11}^{(q)}$, $|A_{11} - 1_r|_0 \leq q^{-1}$ and hence $|G_{11}|_0 < q^{-1}$ (r is rank of L);

3° $A_{22} = A_{22}^{(q)}$, $|A_{22} - q^\nu 1_s|_0 \leq q^{-\nu-1}$ and $|G_{22}|_0 < q^{-1}$ (s is rank of N);

4° $A_{12}^{(q)} = 0$.

by Proposition 4.6 Note that both inequalities $|G_{11}|_0 < q^{-1}$ and $|G_{11}|_0 < q^{-1}$ above follow from the relation (FC) in section 2.2 for L and N, respectively. When $\nu \neq 1$, we will first prove $A_{12} = 0$ and then prove $G_{12} = 0$. When $\nu = 1$, we will first prove $G_{12} = 0$ and then prove $A_{12} = 0$. Hence, we will have a splitting in all cases.

4.4.1 The case where $\nu > 1$

Suppose $\nu > 1$ (and $\lambda = 0$). Assume that $A_{12} \neq 0$. By 4° we have $|\frac{d}{dx}A_{12}|_0 > |qA_{12}|_0 = q^{-1}|A_{12}|_0$. Then $|G_{11}A_{12}|_0 < q^{-1}|A_{12}|_0 < |\frac{d}{dx}A_{12}|_0$ and $|qx^{q-1}A_{12}\sigma(G_{22})|_0 < q^{-1}|A_{12}|_0 < |\frac{d}{dx}A_{12}|_0$. On the other hand, $|G_{12}A_{22}|_0 < |qx^{q-1}A_{11}\sigma(G_{12})|_0$ by $\nu > 1$ since A_{11} (resp. A_{22}) is a unit matrix (resp. a unit matrix times q^{ν}) modulo $\mathfrak{m}\mathcal{O}_{\mathcal{E}}$ (resp. $q^{\nu}\mathfrak{m}\mathcal{O}_{\mathcal{E}}$) by 2° (resp. 3°). So we have

$$\frac{d}{dx}A_{12} \equiv qx^{q-1}A_{11}\sigma(G_{12}) \ (\mathrm{mod}\, q^{-\log_q|\frac{d}{dx}A_{12}|_0}\mathfrak{m}\mathcal{O}_{\mathcal{E}})$$

But, on comparing the x-adic order of both sides, this is seen to be impossible by 2°, 3° and 4°. Hence $A_{12} = 0$. Now the commutativity of Frobenius and connection (the relation 1°) is just

$$G_{12}A_{22} = qx^{q-1}A_{11}\sigma(G_{12}).$$

Since any morphism between pure φ-modules with different Frobenius slopes are 0, we have $G_{12} = 0$ by $\nu > 1$.

4.4.2 The case where $0 < \nu < 1$

Suppose $0 < \nu < 1$ (and $\lambda = 0$). Assuming that $A_{12} \neq 0$, we will show the existence of unbounded solutions on the generic disk by applying Lemma 4.8 (2). This is a contradiction to our hypothesis of boundedness of M, and thus we must have $A_{12} = 0$. Since $\nu \neq 1$, we again have $G_{12} = 0$ by the slope reason. Therefore, the extension is split.

Assume that $A_{12} = \sum_n A_{12,n}x^n \neq 0$. Since $|G_{12}A_{22}|_0 = q^{-\nu}|G_{12}|_0$, $|qx^{q-1}A_{11}\sigma(G_{12})|_0 = q^{-1}|G_{12}|_0$, and $|\frac{d}{dx}A_{12}|_0 > q^{-1}|A_{12}|_0$ by 3°, 2° and our hypothesis, respectively, the formula 4° gives estimates

5° $\quad q^{-1}|A_{12}|_0 < q^{-\nu}|G_{12}|_0 = |G_{12}A_{22}|_0 = |\frac{d}{dx}A_{12}|_0 \leq |A_{12}|_0.$

We also claim that

6° $\quad$ there is a positive integer m with $q \not| m$ such that $|\frac{1}{m!}\frac{d^m}{dx^m}A_{12}|_0 = |A_{12}|_0$

by 1°. Indeed, let l be an integer such that $|A_{12,l}| = |A_{12}|_0$. When $l > 0$, we put $m = l$. Then the coefficient of $\frac{1}{m!}\frac{d^l}{dx^l}A_{12}$ in the 0-th term x^0 is $A_{12,l}$ and we have $|\frac{1}{l!}\frac{d^l}{dx^l}A_{12}|_0 = |A_{12,l}| = |A_{12}|_0$. When $l < 0$, we put $m = q^{-l} + l$ (remark that any sufficient large power of q can be replaced by q^{-l}). Then the coefficient of $\frac{1}{m!}\frac{d^m}{dx^m}A_{12}$ in the $l-m(=-q^{-l})$-th term x^{l-m} is $(-1)^m\begin{pmatrix} m-l-1 \\ m \end{pmatrix}A_{12,l}$ and we have $|\frac{1}{m!}\frac{d^m}{dx^m}A_{12}|_0 = |A_{12,l}| = |A_{12}|_0$ since $(-1)^m\begin{pmatrix} m-l-1 \\ m \end{pmatrix}$ is a p-adic unit.

In proving the assertion, we will consider the following two cases for A_{12}:

(i) $|\frac{d}{dx}A_{12}|_0 > q^{-\nu}|A_{12}|_0$.

(ii) $|\frac{d}{dx}A_{12}|_0 \leq q^{-\nu}|A_{12}|_0$. (Hence we have $|G_{12}|_0 \leq |A_{12}|_0$ by 5°)

In order to prove the existence of unbounded solutions above, let us reorganize the matrix representation by using changes of basis of M, a change of Frobenius and an extension of scalar field. Let us consider the induced φ-∇-module $M_\tau = M \otimes_{\mathcal{E}} \mathcal{E}_t[\![X-t]\!]_0$ over the bounded functions $\mathcal{E}_t[\![X-t]\!]_0$ at the generic disk. Since L_τ and N_τ are pure, we have bounded solution matrices Y_{11} of L and Y_{22} of N, that is,

$$7^\circ \quad \begin{array}{l} L: \left\{ \begin{array}{l} A_{11}(t)\sigma(Y_{11}) = Y_{11}\tau(A_{11}) \\ \dfrac{d}{dX}Y_{11} = Y_{11}\tau(G_{11}) \\ Y_{11} \in 1_r + q(X-t)\mathrm{Mat}_r(\mathcal{O}_{\mathcal{E}_t}[\![X-t]\!]) \end{array} \right. \\ N: \left\{ \begin{array}{l} A_{22}(t)\sigma(Y_{22}) = Y_{22}\tau(A_{22}) \\ \dfrac{d}{dX}Y_{22} = Y_{22}\tau(G_{22}) \\ Y_{22} \in 1_s + q(X-t)\mathrm{Mat}_s(\mathcal{O}_{\mathcal{E}_t}[\![X-t]\!]) \end{array} \right. \end{array}$$

by 2° and 3°. Note that $\tau(f) = \sum_n \frac{1}{n!}(\frac{d^n}{dx^n}f)(t)(X-t)^n$ for $f \in \mathcal{E}$ and it is an isometry. Consider a change of basis of M_τ by the matrix $Y^{-1} = \begin{pmatrix} Y_{11}^{-1} & 0 \\ 0 & Y_{22}^{-1} \end{pmatrix}$. Then the new Frobenius matrix and the new connection matrix are as follows:

$$\begin{aligned} A^\tau &= YA\sigma(Y)^{-1} = \begin{pmatrix} A_{11}(t) & Y_{11}\tau(A_{12})\sigma(Y_{22})^{-1} \\ 0 & A_{22}(t) \end{pmatrix} \\ G^\tau &= Y\tfrac{d}{dX}Y^{-1} + YGY^{-1} = \begin{pmatrix} 0 & Y_{11}\tau(G_{12})Y_{22}^{-1} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Let us put $A^\tau_{12} = \sum_n A^\tau_{12,n}(X-t)^n$ (resp. G^τ_{12}) to be the $(1,2)$-part of the Frobenius matrix A^τ (resp. G^τ), and define $B^\tau_{12} = \sum_{n>0} A^\tau_{12,n}(X-t)^n$ by the subseries of positive powers. Then we have

$$8^\circ \qquad |B^\tau_{12}|_0 = |A_{12}|_0$$

$$9^\circ \qquad |G^\tau_{12}|_0 = |\tau(G_{12})|_0 = |G_{12}|_0.$$

by 6° and $Y \equiv 1_{r+s} \,(\mathrm{mod}\, q(X-t)\mathcal{O}_{\mathcal{E}_t}[\![X-t]\!])$.

Now we consider a change of Frobenius. At first our Frobenius on $\mathcal{E}$ is given by $\sigma(x) = x^q$. Hence the induced Frobenius on the generic disk is given by $\sigma(X-t) = ((X-t)+t)^q - t^q$. Let us replace σ by the Frobenius $\widetilde{\sigma}$ defined by $\widetilde{\sigma}(X-t) = (X-t)^q$. Note that

$$10^\circ \qquad \sigma(X-t) - \widetilde{\sigma}(X-t) \equiv qt^{q-1}(X-t) \;(\mathrm{mod}\, p(X-t)^2\mathcal{O}_{\mathcal{E}_t}[\![X-t]\!]).$$

Since $|\frac{1}{n!}\frac{d^{n-1}}{dX^{n-1}}G^\tau_{12}|_0 \leq |n|^{-1}|G_{12}|_0$ and $|p^n/n| \leq |p|$ for all $n \geq 1$, the matrix H of comparison transform $\vartheta^*_{\widetilde{\sigma},\sigma}(M_\tau)$ in section 3.1 satisfies the congruence relation

$$
\begin{array}{lcl}
H & = & 1_{r+s} + \displaystyle\sum_{n=1}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & \sigma(\frac{d^{n-1}}{dX^{n-1}} G_{12}^{\tau}) \\ 0 & 0 \end{pmatrix} (\sigma(X-t) - \widetilde{\sigma}(X-t))^n \\
11^\circ & \equiv & 1_{r+s} + qt^{q-1} \begin{pmatrix} 0 & \sigma(G_{12}(t)) \\ 0 & 0 \end{pmatrix} (X-t) \\
& & \qquad (\mathrm{mod}\, pq^{-\log_q |G_{12}|_0} (X-t)^2 \mathcal{O}_{\mathcal{E}_t} [\![X-t]\!])
\end{array}
$$

by 9° and 10°. Our Frobenius matrix of M_τ with respect to the Frobenius $\widetilde{\sigma}$ is

$$
\widetilde{A} = A^\tau H = \begin{pmatrix} A_{11}(t) & A_{12}^\tau + A_{11}(t) H_{12} \\ 0 & A_{22}(t) \end{pmatrix}
$$

by the definition of the equivalence (Theorem 3.3), where $H_{12} = \sum_n H_{12,n}(X-t)^n$ is the $(1,2)$-part of H. If we put $\widetilde{A}_{12} = \sum_n \widetilde{A}_{12,n}(X-t)^n$ to be the $(1,2)$-part of $\widetilde{A}$ and put $\widetilde{B}_{12} = \sum_{n>0} \widetilde{A}_{12,n}(X-t)^n$, then

12° there is a positive integer m with $q \not| \, m$ such that $|\widetilde{A}_{12,m}|_0 > q^{-\nu} |\widetilde{B}_{12}|_0$.

Indeed, in the case (i) for A_{12}, since $\widetilde{A}_{12,1} = A_{12,1}^\tau + A_{11}(t) H_{12,1}$ and $|H_{12,1}|_0 \leq q^{-1}|G_{12}|_0$, we have $|\widetilde{A}_{12,1}|_0 = |A_{12,1}^\tau|_0 = |\frac{d}{dx} A_{12}|_0$ by 5° and 11°. On the other hand, $|\widetilde{B}_{12}|_0 \leq \max\{|B_{12}^\tau|_0, |H_{12}|_0\} \leq \max\{|A_{12}|_0, |p||G_{12}|_0\} < q^\nu |\frac{d}{dx} A_{12}|_0$ by 5°, 8° and 11° because of our hypothesis (i), $|\frac{d}{dx} A_{12}|_0 > q^{-\nu}|A_{12}|_0$. Hence we can take $m = 1$. In the case (ii), we take a positive integer m such as 6°. Since $|G_{12}|_0 \leq |A_{12}|_0$ by the hypothesis (ii), we have $|\widetilde{B}_{12}|_0 \leq \max\{|B_{12}^\tau|_0, |H_{12}|_0\} = |A_{12}|_0$ by 8° and 11°.

By Proposition 2.1 we may replace $\mathcal{E}_t$ by the p-adic completion $\widehat{\mathcal{E}_t^{\mathrm{ur}}}$ of the maximally unramified extension of $\mathcal{E}_t$. Then we may assume that $\widetilde{A}_{11} = 1_r$ and $\widetilde{A}_{22} = q^\nu 1_s$ since the solutions of both $(1,1)$-part and $(2,2)$-part is 1 modulo q by 2° and 3°. The solution matrix of $M_\tau \otimes_{\mathcal{E}_t} \widehat{\mathcal{E}_t^{\mathrm{ur}}}$ has a form $Z = \begin{pmatrix} 1_r & Z_{12} \\ 0 & 1_s \end{pmatrix}$ satisfying $\widetilde{A}|_{X=t} \widetilde{\sigma}(Z) = Z \widetilde{A}$ and $Z_{12}|_{X=t} = 0$. In particular, Z_{12} satisfies the relation

$$
\widetilde{\sigma}(Z_{12}) = q^\nu Z_{12} + \widetilde{B}_{12}.
$$

On applying Lemma 4.8 (2) to Z_{12}, one sees that one of entries of Z_{12} must be exactly of log-growth ν by 12°. Hence the non-vanishing of A_{12} implies that M is unbounded.

This completes the proof for the case $0 < \nu < 1$.

4.4.3 The case where $\nu = 1$

Suppose that $\nu = 1$. Suppose that $G_{12} \neq 0$. Let us develop $G_{12} = \sum_n G_{12,n} x^n$ and let m be the least integer such that $|G_{12,m}| = |G_{12}|_0$. If $A_{12} \neq 0$, we have $|\frac{d}{dx} A_{12}|_0 > q^{-1}|A_{12}|_0$ by 4°. So the relation 1° induces a congruence

$$13^\circ \qquad \frac{d}{dx}A_{12} + qG_{12} \equiv qx^{q-1}\sigma(G_{12}) \pmod{q^{1+\log_q|G_{12}|_0}\mathfrak{m}\mathcal{O}_{\mathcal{E}}}$$

by 2° and 3°. This congruence 13° also holds when $A_{12} = 0$.
Suppose that $m < -1$. The least power of x which should appear in the right hand side of the congruence 13° above is $qm + q - 1$. Since $qm + q - 1 < m$, this is precluded by 4°.
Suppose that $m = -1$. Then

$$14^\circ \qquad \begin{aligned} \tau(G_{12}) &= \sum_{n=0}^{\infty} \frac{1}{n!}(\frac{d^n}{dx^n}G_{12})(t)(X-t)^n \\ &= \sum_{n=0}^{\infty}(G_{12,-1}\frac{(-1)^n}{t^n} + q^{-\log_q|G_{12}|_0}\mathrm{Mat}(t^{1-n}\mathcal{V}[\![t]\!]_0 + \mathfrak{m}\mathcal{O}_{\mathcal{E}}))(X-t)^n. \end{aligned}$$

Let us calculate the solution matrix of M_τ by using 7° as in the previous case. By changing a basis of M_τ by the invertible matrix $Y = \begin{pmatrix} Y_{11}^{-1} & 0 \\ 0 & Y_{22}^{-1} \end{pmatrix}$ as before, our differential equation becomes

$$\frac{d}{dX}\begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & Y_{11}\tau(G_{12})Y_{22}^{-1} \\ 0 & 0 \end{pmatrix}.$$

in Z, that is, $\frac{d}{dX}Z = Y_{11}\tau(G_{12})Y_{22}^{-1}$. Since all the coefficients of all the power series which appear on the entries of $Y_{11}\tau(G_{12})Y_{22}^{-1}$ do not vanish modulo $q^{-\log_q|G_{12}|_0}\mathfrak{m}\mathcal{O}_{\mathcal{E}}$ by 7° and 14°, at least one of entries of Z is exactly of log-growth 1. This contradicts to our hypothesis of boundness of M. Hence, $m \neq -1$.
Suppose that $m > 0$. Then we have

$$G_{12} \equiv -q^{-1}x^{-1}\left(x\tfrac{d}{dx}A_{12} + \sigma(x\tfrac{d}{dx}A_{12}) + \sigma^2(x\tfrac{d}{dx}A_{12}) + \cdots\right) \pmod{q^{-\log_q|G_{12}|_0}\mathfrak{m}\mathcal{O}_{\mathcal{E}}}$$

by 4° and 13°. The case where $A_{12} = 0$ is impossible since $G_{12} \neq 0$. If $A_{12} \neq 0$, then we have a solution exactly of log-growth 1 on the generic disk by the similar construction in the case $m = -1$. This contradicts our hypothesis. Therefore, we have $G_{12} = 0$ in any case.
Now we prove $A_{12} = 0$. Suppose that $A_{12} \neq 0$. Then the relation 1° is

$$\frac{d}{dx}A_{12} + G_{11}A_{12} = qx^{q-1}A_{12}\sigma(G_{22}).$$

This is impossible by 2°, 3° and 4°. Hence, $A_{12} = 0$.
This completes the proof of Proposition 4.2. □

Remark 4.9 *There is another proof of Proposition 4.2: one can reduce Proposition 4.2 to the case where $q = p$, that is, the Frobenius σ is a p-Frobenius. Then, in the proof of the case $0 < \nu < 1$, it is enough to discuss only in the case $|\frac{d}{dx}A_{12}|_0 = |A_{12}|_0$.*

5 PBQ φ-∇-modules

5.1 Definition of PBQ φ-∇-modules

Definition 5.1 *(Definition of "PBQ" φ-∇-modules)*

(1) *A φ-∇-module M over $\mathcal{E}$ is said to be pure of bounded quotient (called PBQ for simplicity) if M/M^0 is pure as a φ-module.*

(2) *A φ-∇-module M over $K[\![x]\!]_0$ is said to be pure of bounded quotient (called PBQ for simplicity) if the generic fiber M_η of M is PBQ as a φ-∇-module over $\mathcal{E}$.*

The notion "PBQ" depends only on the Frobenius slopes of the bounded quotient of the generic fiber of φ-∇-modules. As we saw in Theorem 4.1, the bounded quotient of the generic fiber always admits a splitting by Frobenius slopes when it has different slopes.

Example 5.2 (1) *A bounded φ-∇-module M over $\mathcal{E}$ is PBQ if and only if M is pure as a φ-module. In particular, any φ-∇-module M over $\mathcal{E}$ of rank 1 is PBQ.*

(2) *Any φ-∇-module M over $\mathcal{E}$ of rank 2 which is not bounded is PBQ [CT09, Theorem 7.1].*

(3) *Let us fix a Frobenius on σ with $\sigma(x) = x^q$. Let M be a φ-∇-module over $K[\![x]\!]_0$ with a basis (e_1, e_2, e_3) such that the Frobenius matrix A and the connection matrix G are defined by*

$$A = \begin{pmatrix} 1 & -q^{1/2}x & -qx \\ 0 & q^{1/2} & 0 \\ 0 & 0 & q \end{pmatrix}, \ G = \begin{pmatrix} 0 & \sum_{n=0}^{\infty} q^{n/2}x^{q^n-1} & \sum_{n=0}^{\infty} x^{q^n-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then M_η is not bounded and M is not PBQ. Indeed, the $K[\![x]\!]_0$-submodule L generated by e_1 is a φ-∇-submodule of M such that the quotient $(M/L)_\eta$ is bounded and $(M/L)_\eta$ is not pure. On the other hand the dual $M^\vee$ of M is PBQ.

Proposition 5.3 *Any quotient of PBQ φ-∇-modules over $\mathcal{E}$ (resp. $K[\![x]\!]_0$) is PBQ.*

Proof. Let M be a PBQ φ-∇-module over $\mathcal{E}$ and let M' be a quotient of M. The assertion follows from that the natural morphism $M/M^0 \to M'/(M')^0$ is surjective by [CT09, Corollary 3.5]. □

5.2 Existence of the maximally PBQ φ-∇-submodules over $\mathcal{E}$

Proposition 5.4 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let M be a φ-∇-module over $\mathcal{E}$ with highest Frobenius slope $\lambda_{\max}$ and let N' be a φ-∇-submodule of $M/S_{\lambda_{\max}-}(M)$. Then there is a unique φ-∇-submodule N of M such that N is PBQ and the natural morphism $N/N^0 \to M/S_{\lambda_{\max}-}(M)$ gives an isomorphism between N/N^0 and N'. When $N' = M/S_{\lambda_{\max}-}(M)$, we call the corresponding N the maximally PBQ submodule of M.*

Proof. First we prove the uniqueness of N. Let N_1 and N_2 be a PBQ submodule of M such that both natural morphisms $N_1/N_1^0 \to M/S_{\lambda_{\max}-}(M) \leftarrow N_2/N_2^0$ give isomorphisms with N'. Let N be the image of $N_1 \oplus N_2 \to M$ $(a, b) \mapsto a+b$. Then N is PBQ by Proposition 5.3. Since $N_1/N_1^0 \oplus N_2/N_2^0 \to N/N^0$ is surjective by [CT09, Proposition 3.6], the natural morphism $N/N^0 \to M/S_{\lambda_{\max}-}(M)$ gives an isomorphism with N'. If N_1 (resp. N_2) is not N, then the quotient N/N_1 (resp. N/N_2) has a bounded solution at the generic disk whose Frobenius slope is different from $\lambda_{\max}$. But this is impossible because N is PBQ. Hence $N = N_1 = N_2$.
Now we prove the existence of N. We use the induction on the dimension of M. Let $f : M \to M/M^0$ be a natural surjection. Since M/M^0 is bounded, $M/S_{\lambda_{\max}-}(M)$ is a direct summand of M/M^0 by the maximality of slopes by Theorem 4.1. Put $L = f^{-1}(N')$. If L is PBQ, then one can put $N = L$. If L is not PBQ, then L is a proper submodule of M and there is a PBQ submodule L' of L such that $L'/(L')^0 \cong L/S_{\lambda_{\max}-}(L) = N'$ by the induction hypothesis. □

Corollary 5.5 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let M be a φ-∇-module over $\mathcal{E}$. Then there is a unique filtration $0 = P_0(M) \subsetneq P_1(M) \subsetneq \cdots \subsetneq P_r(M) = M$ of φ-∇-modules over $\mathcal{E}$ such that $P_i(M)/P_{i-1}(M)$ is the maximally PBQ submodule of $M/P_{i-1}(M)$ for any $i = 1, \cdots, r$. We call $\{P_i(M)\}$ the PBQ filtration of M.*

5.3 Existence of the maximally PBQ φ-∇-submodules over $K[\![x]\!]_0$

Theorem 5.6 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let M be a φ-∇-module over $K[\![x]\!]_0$. Then there is a unique φ-∇-submodule N of M over $K[\![x]\!]_0$ such that the generic fiber N_η of N is the maximally PBQ submodule of the generic fiber M_η of M. We call N the maximally PBQ submodule of M.*

Proof. The proof of uniqueness of the maximally PBQ submodules is same to the proof of Proposition 5.4.
We prove the existence of the maximally PBQ submodules by induction on the rank of M. If M is of rank 1, then the assertion is trivial. For general M, if M is PBQ, then there is nothing to prove. Suppose that M is not PBQ. Then there is a direct summand L_η of M_η/M_η^0 such that L_η is pure with the Frobenius slope which is less than the highest slope $\lambda_{\max}$ of M by Theorem

4.1. Consider the composite of natural morphisms $M \to M_\eta/M_\eta^0 \to L_\eta$. It is not injective by Lemma 5.7 below. Put M' to be the kernel. Then M' is a φ-∇-submodule of M such that $M'_\eta/S_{\lambda_{\max}}(M'_\eta) \cong M_\eta/S_{\lambda_{\max}}(M_\eta)$. By the induction hypothesis there is a maximally PBQ submodule N of M' which becomes the maximally PBQ submodule N of M. □

Lemma 5.7 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let M be a φ-module over $K[\![x]\!]_0$ such that the highest Frobenius slope of the generic fiber M_η of M is $\lambda_{\max}$. Suppose that there exists an injective $K[\![x]\!]_0$-homomorphism $f : M \to L_\eta$ which is φ-equivariant, i.e., $\varphi \circ f = f \circ \varphi$ for a pure φ-module L_η over $\mathcal{E}$. Then the Frobenius slope of L_η is $\lambda_{\max}$.*

Proof. In [dJ98, Corollary 8.2] A.J. de Jong proved this assertion when L_η is a generic fiber of a rank 1 pure φ-∇ module L over $K[\![x]\!]_0$. (Indeed, he proved a stronger assertion.) We give a sketch of the proof which is due to [dJ98, Propositions 5.5, 6.4 and 8.1]. Our $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$, resp. $\widetilde{\mathcal{E}}$, resp. $\widetilde{\mathcal{E}}^\dagger$ introduced below) corresponds to Γ (resp. Γ_c, resp. Γ_2, resp. $\Gamma_{2,c}$) in [dJ98]. We also remark that $\widetilde{\mathcal{E}}^\dagger$ is the extended bounded Robba ring $\widetilde{\mathcal{R}}^{\mathrm{bd}}$ in [Ke08, 2.2].

We may assume that the residue field k of $\mathcal{V}$ is algebraically closed and all slopes of M are contained in the value group of $\log_q|K^\times|$. We may also assume that $\sigma(x) = x^q$ by Theorem 3.3. Let us define K-algebras

$$\widetilde{\mathcal{E}} = \left\{ \sum_{n\in\mathbb{Q}} a_n x^n \;\middle|\; \begin{array}{l} a_n \in K,\ \sup_n |a_n| < \infty,\ |a_n| \to -\infty\ (n \to -\infty), \\ \{n \mid |a_n| \geq \alpha\} \text{ is a well-ordered set with respect to} \\ \text{the order } \leq \text{ for any } \alpha \in \mathbb{R}. \end{array} \right\}$$

$$\widetilde{\mathcal{E}}^\dagger = \left\{ \sum_{n\in\mathbb{Q}} a_n x^n \in \widetilde{\mathcal{E}} \;\middle|\; |a_n|\eta^n \to 0\ (n \to -\infty) \text{ for some } 0 < \eta < 1. \right\}.$$

Both $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{E}}^\dagger$ are discrete valuation fields such that both ring of integers have a same residue field

$$k((x^{\mathbb{Q}})) = \left\{ \sum_{n\in\mathbb{Q}} a_n x^n \;\middle|\; \begin{array}{l} a_n \in k,\ \{n \mid a_n \neq 0\} \text{ is a well-ordered set} \\ \text{with respect to the order } \leq . \end{array} \right\},$$

which includes an algebraic closure of $k((x))$ [Ke01], and that the p-adic completion of $\widetilde{\mathcal{E}}^\dagger$ is $\widetilde{\mathcal{E}}$. $\widetilde{\mathcal{E}}$ is naturally an $\mathcal{E}$-algebra and σ naturally extends to $\widetilde{\mathcal{E}}$ by $\sigma(\sum_n a_n x^n) = \sum_n \sigma(a_n) x^{qn}$. Put

$$\mathcal{E}^\dagger = \widetilde{\mathcal{E}}^\dagger \cap \mathcal{E}.$$

Then $\mathcal{E}^\dagger$ is stable under σ and the K-derivation d/dx. We also denote by $\mathcal{O}_{\widetilde{\mathcal{E}}^\dagger}$ the ring of integer of $\widetilde{\mathcal{E}}^\dagger$.

By explicit calculations we have the following sublemmas.

Sublemma 5.8 *For* $0 < \eta < 1$ *and for* $\sum_{n \in \mathbb{Q}} a_n x^n \in \widetilde{\mathcal{E}}^\dagger$*, let us consider a condition:*

$$(*)_\eta : \quad \sup_n |a_n| \max\{\eta^n, 1\} \leq 1.$$

If f *and* g *in* $\widetilde{\mathcal{E}}^\dagger$ *satisfy the condition* $(*)_\eta$*, then so are* $f+g$ *and* fg*. Moreover, if* $f = \sum_n a_n x^n$ *satisfies the condition* $(*)_\eta$ *and* $|a_0| = 1$*, then so is* f^{-1}*.*

Note that, if $\eta < \mu$, then the condition $(*)_\eta$ implies the condition $(*)_\mu$.

Sublemma 5.9 (1) *Let* $A = 1 + B$ *be a square matrix such that* 1 *is the unit matrix and all entries of* B *contained in* $\mathfrak{m}^n \mathcal{O}_{\widetilde{\mathcal{E}}^\dagger}$ *for a positive integer* n*. Suppose that all entries of* A *satisfy the condition* $(*)_\eta$ *in Sublemma 5.8. Then there is a matrix* $Y = 1 + Z$ *with* $A\sigma(Y) = Y$ *such that all entries of* Z *are contained in* $\mathfrak{m}^n \mathcal{O}_{\widetilde{\mathcal{E}}^\dagger}$ *and satisfy the condition* $(*)_{\eta^q}$*.*

(2) *Let* C *be a matrix such that all entries are contained in* $\mathfrak{m}^n \mathcal{O}_{\widetilde{\mathcal{E}}^\dagger}$ *for a nonnegative integer* n *and satisfy the condition* $(*)_\eta$*. Then there is a matrix* Z *satisfying* $\sigma(Z) - Z = C$ *such that all entries of* Z *are contained in* $\mathfrak{m}^n \mathcal{O}_{\widetilde{\mathcal{E}}^\dagger}$ *and satisfy the condition* $(*)_{\eta^q}$*.*

Proof. (1) follows from (2) by considering a congruence equation $A\sigma(Y) \equiv Y \,(\mathrm{mod}\, \mathfrak{m}^l \mathcal{O}_{\widetilde{\mathcal{E}}^\dagger})$ inductively on l.
(2) Since the residue field k of $\mathcal{V}$ is perfect, σ is bijective. Put $C = \sum_n C_n x^n = C_- + C_0 + C_+$, where they are subseries of negative powers, a constant term, and subseries of positive powers, respectively. The series $Z_- = \sum_{n<0} \sum_{i=1}^\infty \sigma^{-i}(C_n) x^{n/q^i}$ converges and all entries of Z_- satisfies the condition $(*)_{\eta^q}$, and the equation $\sigma(Z_-) - Z_- = C_-$ holds. Since k is algebraically closed, there is a matrix Z_0 over $\mathcal{V}$ with $|Z_0| \leq |C_0|$ such that $\sigma(Z_0) - Z_0 = C_0$. The series $Z_+ = -\sum_{i=0}^\infty \sigma(C_+)$ converges and satisfies $\sigma(Z_+) - Z_+ = C_+$. Hence, $Z = Z_- + Z_0 + Z_+$ is the desired solution. □

If $N^\dagger$ is a φ-∇-submodule of $M \otimes_{K[\![x]\!]_0} \mathcal{E}^\dagger$ over $\mathcal{E}^\dagger$, then there is a φ-∇-submodule N of M over $K[\![x]\!]_0$ with $N \otimes_{K[\![x]\!]_0} \mathcal{E}^\dagger \cong N$ by [dJ98, Proposition 6.4]. Hence, the induced morphism $M \otimes_{K[\![x]\!]_0} \mathcal{E}^\dagger \to L_\eta$ is also injective. Moreover, since $\widetilde{\mathcal{E}}^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E} \to \widetilde{\mathcal{E}}$ is injective (the similar proof of [dJ98, Proposition 8.1] works), the induced morphism $M \otimes_{K[\![x]\!]_0} \widetilde{\mathcal{E}}^\dagger \to L_\eta \otimes_{\mathcal{E}} \widetilde{\mathcal{E}}$ is again injective. Let $\lambda_1 < \cdots < \lambda_r (= \lambda_{\max})$ be Frobenius slopes of M_η. One can prove that there exists an increasing filtration $0 = \widetilde{M}_0 \subsetneq \widetilde{M}_1 \subsetneq \cdots \subsetneq \widetilde{M}_r = M \otimes_{K[\![x]\!]_0} \widetilde{\mathcal{E}}^\dagger$ of φ-modules over $\widetilde{\mathcal{E}}^\dagger$ such that $(\widetilde{M}_i / \widetilde{M}_{i-1}) \otimes_{\widetilde{\mathcal{E}}^\dagger} \widetilde{\mathcal{E}}$ is pure of slope λ_{r-i+1}. This existence of filtration of opposite direction corresponds to Proposition 5.5 in [dJ98]. Indeed, since the residue field $k(\!(x^{\mathbb{Q}})\!)$ includes an algebraic closure of $k(\!(x)\!)$, there is a basis of $M \otimes_{K[\![x]\!]_0} \widetilde{\mathcal{E}}^\dagger$ such that the Frobenius matrix of

$M \otimes_{K[\![x]\!]_0} \widetilde{\mathcal{E}}^\dagger$ has a form

$$\begin{pmatrix} q^{\lambda_1}1 & & \\ & \ddots & \\ & & q^{\lambda_r}1 \end{pmatrix} + (\text{a square matrix with entries in } \mathfrak{m}^n\mathcal{O}_{\widetilde{\mathcal{E}}})$$

by Dieudonné-Manin classification of φ-modules and the density of $\widetilde{\mathcal{E}}^\dagger$ in $\widetilde{\mathcal{E}}$. Here q^λ is a element of K with $\log_q|q^\lambda| = -\lambda$, 1 is the unit matrix with a certain size (the first matrix is a diagonal matrix), and n is sufficiently large. One can find a basis of $M \otimes_{K[\![x]\!]_0} \widetilde{\mathcal{E}}^\dagger$ such that the Frobenius matrix of $M \otimes_{K[\![x]\!]_0} \widetilde{\mathcal{E}}^\dagger$ is a lower triangle matrix

$$\begin{pmatrix} q^{\lambda_1}1 & & 0 \\ & \ddots & \\ * & & q^{\lambda_r}1 \end{pmatrix}$$

by Sublemmas 5.8 and 5.9. Hence, one has a filtration of opposite direction. Since $\widetilde{M_1}$ is pure of slope $\lambda_r = \lambda_{\max}$ and the inclusion $\widetilde{M_1} \subset L_\eta \otimes_{\mathcal{E}} \widetilde{\mathcal{E}}$ is φ-equivariant, the slope of L_η must be $\lambda_{\max}$. □

Corollary 5.10 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let M be a φ-∇-module over $K[\![x]\!]_0$. Then there is a unique filtration $0 = P_0(M) \subsetneq P_1(M) \subsetneq \cdots \subsetneq P_r(M) = M$ as φ-∇-modules over $K[\![x]\!]_0$ such that $P_i(M)/P_{i-1}(M)$ is the maximally PBQ submodule of $M/P_{i-1}(M)$ for any $i = 1, \cdots, r$. We call $\{P_i(M)\}$ the PBQ filtration of M.*

Example 5.11 *Let M be a φ-∇-module over $K[\![x]\!]_0$ which is introduced in Example 5.2 (3). If $P_1(M)$ is a φ-∇-submodule of M over $K[\![x]\!]_0$ generated by e_1 and e_3, the sequence $0 = P_0(M) \subsetneq P_1(M) \subsetneq P_2(M) = M$ is the PBQ filtration of M.*

6 Log-growth and Frobenius slope for HPBQ φ-∇-modules over $K[\![x]\!]_0$

6.1 Log-growth for HPBQ φ-∇-modules

Definition 6.1 (1) *A φ-∇-module M over $K[\![x]\!]_0$ is horizontal of bounded quotient (HBQ for simplicity) if there is a quotient N of M as a φ-∇-module over $K[\![x]\!]_0$ such that the canonical surjection induces an isomorphism $M_\eta/M_\eta^0 \cong N_\eta$ at the generic fiber.*

(2) *A φ-∇-module M over $K[\![x]\!]_0$ is horizontally pure of bounded quotient (HPBQ for simplicity) if M is PBQ and HBQ.*

Example 6.2 (1) *A bounded φ-∇-module M over $K[\![x]\!]_0$ is HBQ. A bounded φ-∇-module M over $K[\![x]\!]_0$ is HPBQ if and only if M_η is pure as a φ-module.*

(2) *Let M be a φ-∇-module M over $K[\![x]\!]_0$ of rank 2 which arises from the first crystalline cohomology of a projective smooth family E of elliptic curves over* $\operatorname{Spec} k[\![x]\!]$. *Then M is HBQ if and only if either (i) E is a non-isotrivial family over* $\operatorname{Spec} k[\![x]\!]$ *and the special fiber E_s of E is ordinary or (ii) E is an isotrivial family over* $\operatorname{Spec} k[\![x]\!]$. *In the case (i) M is HPBQ, but in the case (ii) M is HPBQ if and only if E is an isotrivial family of supersingular elliptic curves.*

(3) *Let M be a φ-∇-module over $K[\![x]\!]_0$ which is introduced in Example 5.2 (3). Then M is HBQ but is not HPBQ. The dual $M^\vee$ of M is HPBQ.*

Proposition 6.3 *Let M be a φ-∇-module over $K[\![x]\!]_0$. Then M is HBQ if and only if*

$$\dim_K V(M)/V(M)^0 = \dim_{\mathcal{E}} M_\eta/M_\eta^0.$$

Moreover, when M is HBQ, the natural pairing $M \otimes_K \mathrm{Sol}_0(M) \to K[\![x]\!]_0$ induces an isomorphism

$$M_\eta/M_\eta^0 \cong V(M)/V(M)^0 \otimes_K \mathcal{E}$$

as φ-∇-modules.

Proof. Suppose that M is HBQ. Let N be the quotient as in Definition 6.1 (1). Since N_η is bounded, we have $V(N)^0 = 0$ by Christol's transfer theorem (see [CT09, Proposition 4.3]) and $\dim_K V(M)/V(M)^0 \geq \dim_K V(N)/V(N)^0 = \mathrm{rank}_{K[\![x]\!]_0} N = \dim_{\mathcal{E}} M_\eta/M_\eta^0$. On the other hand, one knows an inequality $\dim_K V(M)/V(M)^0 \leq \dim_{\mathcal{E}} M_\eta/M_\eta^0$ by [CT09, Proposition 4.10]. Hence, we have an equality $\dim_K V(M)/V(M)^0 = \dim_{\mathcal{E}} M_\eta/M_\eta^0$. Now we prove the inverse. The natural pairing $M \otimes_K \mathrm{Sol}_0(M) \to K[\![x]\!]_0$ induces the surjection $M \to V(M)/V(M)^0 \otimes_K K[\![x]\!]_0$. If $\dim_K V(M)/V(M)^0 = \dim_{\mathcal{E}} M_\eta/M_\eta^0$, we have an isomorphism $M_\eta/M_\eta^0 \cong V(M)/V(M)^0 \otimes_K \mathcal{E}$ since $V(M)/V(M)^0 \otimes_K \mathcal{E}$ is bounded. □

Since any quotient of bounded φ-∇-modules over $\mathcal{E}$ is again bounded, the proposition below follows from the chase of commutative diagrams.

Proposition 6.4 *Any quotient of HBQ φ-∇-modules over $K[\![x]\!]_0$ is HBQ. In particular, any quotient of HPBQ modules is HPBQ.*

Proof. We may assume that the residue field of $\mathcal{V}$ is algebraically closed and $q^{\lambda_{\max}} \in K$. Since M is HBQ, there is a surjection $M \to V(M)/V(M)^0 \otimes_K K[\![x]\!]_0$ by Propoition 6.3 whose kernel is denoted by L. Then $M_\eta^0 = L_\eta$. If $f : M \to N$ be the given surjection, $N/f(L)$ is a quotient of $V(M)/V(M)^0 \otimes_K K[\![x]\!]_0$ and hence a direct sum of copies of $(K[\![x]\!]_0, q^\lambda\sigma, d)$ for some λ. Since f gives a surjection from M_η^0 to N_η^0 by [CT09, Proposition 3.6], we have

$$\dim_K V(N)/V(N)^0 \geq \mathrm{rank}_{K[\![x]\!]_0} N/f(L) = N_\eta/N_\eta^0.$$

On the other hand, $\dim_K V(N)/V(N)^0 \leq \dim_{\mathcal{E}} N_\eta/N_\eta^0$ by [CT09, Proposition 4.10]. Hence $\dim_K V(N)/V(N)^0 = \dim_{\mathcal{E}} N_\eta/N_\eta^0$. The rest follows from Proposition 5.3. □

Note that the notion PBQ is determined only by the generic fiber. On the other hand, for "HPBQ", the bounded quotient is horizontal.

Theorem 6.5 *Let M be a φ-∇-module M over $K[\![x]\!]_0$ which is HPBQ. Then the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (see 2.5) holds for M.*

Proof. We have only to prove the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}(2)$ for M. Then the property of Frobenius slopes implies the conjecture the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}(1)$ for M. We may assume that the residue field of $\mathcal{V}$ is algebraically closed and all Frobenius slopes of $V(M)$ are contained in the valued group $\log_q|K^\times|$ by Proposition 2.1. We may also assume that our Frobenius σ is defined by $\sigma(x) = x^q$ by Theorem 3.3. Let us denote by $\lambda_{\max}$ the highest Frobenius slope of M_η (= the highest Frobenius slope of $V(M)$). Let $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ be a filtration of M as φ-∇-modules over $K[\![x]\!]_0$ such that $M_i/M_{i-1}\,(i = 1, \cdots, r)$ is irreducible (i.e., it has no nontrivial φ-∇-submodule over $K[\![x]\!]_0$). We will prove the induction on r. If $r = 1$, then $M \cong (K[\![x]\!]_0, q^{\lambda_{\max}}\sigma, d)$ and the assertion is trivial.
Now suppose $r > 1$. We may also assume $\dim_K V(M)/V(M)^0 = 1$, hence $M_r/M_{r-1} \cong (K[\![x]\!]_0, q^{\lambda_{\max}}\sigma, d)$. Indeed, suppose that $s = \dim_K V(M)/V(M)^0 > 1$. By our assumption, there is a φ-∇-submodule L' over $K[\![x]\!]_0$ such that the highest Frobenius slope of L' is $\lambda_{\max}$ with multiplicity 1 (note that L' is M_{r-s+1}). Take the maximally PBQ submodule L of L'. Then L is HPBQ such that the highest Frobenius slope is $\lambda_{\max}$ with multiplicity 1. Since both highest Frobenius slopes of L and M/L are $\lambda_{\max}$, the assertion follows from the induction hypothesis by Propositions 2.6 and 6.4.
Since all Frobenius slopes of $(M_{r-1})_\eta$ are less than $\lambda_{\max}$, one can take a basis $e_1, \cdots, e_s$ of M such that the Frobenius matrix $A = \begin{pmatrix} A_1 & B \\ 0 & q^{\lambda_{\max}} \end{pmatrix}$ (A_1 is the Frobenius matrix of M_{r-1}) satisfies (i) all entries of A_1 are contained in $K[\![x]\!]_0 \cap x^q K[\![x^q]\!]$ and (ii) all entries of B are contained in $xK[\![x]\!]_0 \backslash x^q K[\![x^q]\!] \cup \{0\}$ by Proposition 4.7. Moreover $B \neq 0$ by Lemma 6.6 below since M is PBQ. Let G be the matrix of connection of M. Then the identification

$$\mathrm{Sol}(M) = \left\{ y \in \mathcal{A}_K(0, 1^-) \;\middle|\; \frac{d}{dx} y = yG \right\}$$

is given by $f \mapsto (f(e_1), \cdots, f(e_s))$. The inclusion relation in Theorem 2.3 for the solution space is

$$\mathrm{Sol}_\lambda(M) \supset S_{\lambda-\lambda_{\max}}(\mathrm{Sol}(M)).$$

Then it is sufficient to prove the inclusion is equal for all λ. The φ-module is a direct sum of 1-dimensional φ-spaces, on which φ acts by $q^\delta\sigma$ for some rational number δ such that $\lambda_{\max} - \delta$ is a Frobenius slope of M, by our assumption of K. Let $f \in \mathrm{Sol}_\lambda(M)$ with $\varphi(f) = q^\delta f$. Then the restriction of f on M_{r-1} gives a $(\varphi, \frac{d}{dx})$-equivariant morphism

$$M_{r-1} \to (\mathcal{A}_K(0,1^-), q^{-\delta}\sigma, d).$$

The kernel L of f is a φ-∇-module over $K[\![x]\!]_0$ and f is a solution of M/L of log-growth λ.
Suppose that $L \neq 0$. Then the length of M/L is smaller than M and M/L is HPBQ by Proposition 6.4. Considering f as a solution of M/L, we have $\delta \leq \lambda - \lambda_{\max}$ by the hypothesis of induction.
Suppose that $L = 0$. The Frobenius relation $\varphi(f) = q^\delta f$ is equivalent to

$$q^{-\delta}\sigma(f(e_1), \cdots, f(e_s)) = (f(e_1), \cdots, f(e_s))A.$$

By the assumption of A_1 we have $f(e_i) \in \mathcal{A}_K(0,1^-) \cap x^q K[\![x^q]\!]$. Let us focus on the s-th entry, then it is

$$q^{-\delta}\sigma(f(e_s)) = q^{\lambda_{\max}} f(e_s) + (f(e_1), \cdots, f(e_{s-1}))B.$$

Since the highest Frobenius slope of M_{r-1} is less than $\lambda_{\max}$, the log-growth of the restriction of f on M_{r-1} is of log-growth less than $\lambda_{\max} + \delta$, and so is $(f(e_1), \cdots, f(e_{s-1}))B$. Since f is injective, $(f(e_1), \cdots, f(e_{s-1}))B \in \mathcal{A}_K(0,1^-) \setminus x^q K[\![x^q]\!]$ is not 0. Hence, $f(e_s)$ is exactly of log-growth $\lambda_{\max} + \delta$ by Lemma 4.8 (1). This provides an inequality $\lambda_{\max} + \delta \leq \lambda$, and we have $\delta \leq \lambda - \lambda_{\max}$.
Therefore, $f \in S_{\lambda - \lambda_{\max}}(\mathrm{Sol}(M))$. This completes the proof of Theorem 6.5. □

Lemma 6.6 *Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-∇-modules over $K[\![x]\!]_0$. If the exact sequence is split as φ-modules, then it is split as φ-∇-modules.*

Proof. Let $A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$ and $G = \begin{pmatrix} G_1 & H \\ 0 & G_2 \end{pmatrix}$ be the matrices of Frobenius and connection, respectively. We should prove that $B = 0$ implies $H = 0$. It is sufficient to prove the assertion above as $\mathcal{A}_K(0,1^-)$-modules with Frobenius and connection. Solving the differential modules L and N, we may assume that A_1 and A_2 are constant regular matrices and $G_1 = G_2 = 0$. Then the horizontality of Frobenius structure means he relation

$$HA_2 = qx^{q-1}A_1\sigma(H).$$

Then we have $H = 0$ by comparing the x-adic order of both sides. □

6.2 Equislope φ-∇-modules over $K[\![x]\!]_0$

Definition 6.7 *A φ-∇-module M over $K[\![x]\!]_0$ is equislope if there is an increasing filtration $\{S_\lambda(M)\}_{\lambda\in\mathbb{R}}$ of φ-∇-module over $K[\![x]\!]_0$ such that $S_\lambda(M)\otimes\mathcal{E}$ gives the Frobenius slope filtration of the generic fiber M_η of M. We also call $\{S_\lambda(M)\}_{\lambda\in\mathbb{R}}$ the Frobenius slope filtration of M.*

By [Ka79, 2.6.2] (see [CT09, Theorem 6.21]) we have

Proposition 6.8 *A φ-∇-module M over $K[\![x]\!]_0$ is equislope if and only if both the special polygon and generic polygon of Frobenius slopes of M coincides with each other.*

Corollary 6.9 *Any subquotients, direct sums, extensions, tensor products, duals of equislope φ-∇-modules over $K[\![x]\!]_0$ are equislope.*

Proposition 6.10 *Let M be an equislope φ-∇-module over $K[\![x]\!]_0$.*

(1) *M is HBQ. In particular, if M is PBQ, then M is HPBQ.*

(2) *If $V(M)/V(M)^0$ is pure as a φ-module, then M is HPBQ.*

Proof. (1) We may assume that the residue field of $\mathcal{V}$ is algebraically closed and all slopes of M_η is contained in the value group $\log_q|K^\times|$ of $K^\times$ by Proposition 2.1. Let us take a φ-∇-submodule L such that its generic fiber L_η is M_η^0. Such an L exists by Lemma 6.11 below. Since $(M/L)_\eta\cong M_\eta/L_\eta$ is bounded, M is HBQ by definition.
(2) The assertion follows from (1) and Proposition 6.3. □

Lemma 6.11 *Let M be an equislope φ-∇-module over $K[\![x]\!]_0$. Suppose that the residue field of $\mathcal{V}$ is algebraically closed and all slopes of M_η are contained in the valued group $\log_q|K^\times|$. The map taking generic fibers gives a bijection from the set of φ-∇-submodules of M to the set of φ-∇-submodules of M_η.*

Proof. Since the functor from the category φ-∇-module over $K[\![x]\!]_0$ to the category φ-∇-module over $\mathcal{E}$ is fully faithful, it is sufficient to prove the surjectivity [dJ98, Theorem 1.1].
We may assume that $\sigma(x)=x^q$ by Theorem 3.3. We use the induction on the number of Frobenius slopes of M in order to prove the existence of a submodule N over $K[\![x]\!]_0$ for a given submodule N_η over $\mathcal{E}$. Suppose that M is pure of slope λ. There are a basis $e_1,\cdots,e_r$ of M such that the Frobenius matrix is $q^\lambda 1_r$ since M is bounded. Let N_η be a φ-∇-submodule of M_η over $\mathcal{E}$ which is generated by $(e_1,\cdots,e_r)P$ for $P\in\mathrm{Mat}_{rs}(\mathcal{E})$ with $s=\dim_{\mathcal{E}} N_\eta$. Since N_η is a φ-submodule, there is a $B\in\mathrm{GL}_s(\mathcal{E})$ such that $q^\lambda\sigma(P)=PB$. Since $\mathrm{rank}(P)=s$, there is a regular minor Q of P of degree s such that $q^\lambda\sigma(Q)=QB$. If one puts $R=PQ^{-1}\in\mathrm{Mat}_{rs}(\mathcal{E})$, then $\sigma(R)=R$. Hence, $R\in\mathrm{Mat}_{rs}(K)$. Since $(e_1,\cdots,e_r)R$ is a basis of N_η such that $(e_1,\cdots,e_r)R$

are included in M, the submodule N is given by the $K[\![x]\!]_0$-submodule of M generated by $(e_1, \cdots, e_r)R$.
Let λ_1 be the first slopes of M_η. By the induction hypothesis there are a φ-∇-submodule N_1 of $S_{\lambda_1}(M)$ such that the generic fiber $(N_1)_\eta$ of N_1 is $N_\eta \cap S_{\lambda_1}(M_\eta)$ and a φ-∇-submodule N_2 of $M/S_{\lambda_1}(M)$ such that the generic fiber of N_2 is $N_\eta/(S_{\lambda_1}(M_\eta) \cap N_\eta) = N_\eta/(N_1)_\eta$. Let N_3 be the inverse image of N_2 by the surjection $M/N_1 \to M/S_{\lambda_1}(M)$. Since the intersection of $N_\eta/(N_1)_\eta$ and $S_{\lambda_1}(M_\eta)/(N_1)_\eta$ is 0 in $M_\eta/(N_1)_\eta$, $(N_3)_\eta$ is a direct sum of $N_\eta/(N_1)_\eta$ and $S_\lambda(M_\eta)/(N_1)_\eta$. By applying the fully faithfulness of the functor from the category of φ-∇-modules over $K[\![x]\!]_0$ to the category of φ-∇-modules over $\mathcal{E}$ [dJ98, Theorem 1.1], there is a direct summand N_4 of N_3 as φ-∇-module over $K[\![x]\!]_0$ such that the generic fiber of N_4 is $N_\eta/(N_1)_\eta$. Then the inverse image N of N_4 by the surjection $M \to M/N_1$ is our desired one. □

THEOREM 6.12 *The conjecture* $\mathbf{LGF}_{K[\![x]\!]_0}$ *(see 2.5) holds for any equislope and PBQ φ-∇-module over $K[\![x]\!]_0$.*

PROOF. The assertion follows from Theorem 6.5 and Proposition 6.10 (1). □

7 LOG-GROWTH FILTRATION AND FROBENIUS FILTRATION AT THE GENERIC POINT

7.1 THE LOG-GROWTH OF PBQ φ-∇-MODULES OVER $\mathcal{E}$

THEOREM 7.1 *The conjecture* $\mathbf{LGF}_{\mathcal{E}}$ *(see 2.4) holds for any PBQ φ-∇-module over $\mathcal{E}$.*

PROOF. Let M be a PBQ φ-∇-module over $\mathcal{E}$ such that $\lambda_{\max}$ is the highest Frobenius slope of M, and let us consider a φ-∇-module $M_\tau = M \otimes_{\mathcal{E}} \mathcal{E}_t[\![X-t]\!]_0$ over the $\mathcal{E}_t$-algebra $\mathcal{E}_t[\![X-t]\!]_0$ of bounded functions on the generic disk. Then M_τ is equislope since $\{(S_\lambda(M))_\tau\}$ gives a Frobenius slope filtration of M_τ. Moreover, since M is PBQ, $\mathrm{Sol}_0(M, \mathcal{A}_{\mathcal{E}_t}(t, 1^-))$ is a pure φ-module. Hence $V(M_\tau)/V(M_\tau)^0$ is pure, and M_τ is HPBQ by Proposition 6.10 (2). Applying Theorem 6.5 to M_τ, we have

$$\begin{aligned}
\dim_{\mathcal{E}} M/M^\lambda &= \dim_{\mathcal{E}_t} \mathrm{Sol}_\lambda(M, \mathcal{A}_{\mathcal{E}_t}(t, 1^-)) = \dim_{\mathcal{E}_t} V(M_\tau)/V(M_\tau)^\lambda \\
&= \dim_{\mathcal{E}_t} V(M_\tau^\vee) - \dim_{\mathcal{E}_t} (S_{\lambda-\lambda_{\max}}(V(M_\tau^\vee)))^\perp \\
&= \dim_{\mathcal{E}} M^\vee - \dim_{\mathcal{E}} (S_{\lambda-\lambda_{\max}}(M^\vee))^\perp \\
&= \dim_{\mathcal{E}} M^\vee/(S_{\lambda-\lambda_{\max}}(M^\vee))^\perp
\end{aligned}$$

for any λ. Hence, $M^\lambda = (S_{\lambda-\lambda_{\max}}(M^\vee))^\perp$ by Theorem 2.3. Therefore, the conjecture $\mathbf{LGF}_{\mathcal{E}}$ holds for M. □

7.2 Rationality of breaks of log-growth filtrations

Theorem 7.2 *Let M be a φ-∇-module over $\mathcal{E}$ and let λ be a break of log-growth filtration of M, i.e., $M^{\lambda-} \supsetneq M^{\lambda+}$. Then λ is rational and $M^{\lambda} = M^{\lambda+}$. In other words, the conjecture $\mathbf{LGF}_{\mathcal{E}}$ (1) (see 2.4) holds for any φ-∇-modules over $\mathcal{E}$.*

Proof. We may assume that the residue field k of $\mathcal{V}$ is perfect by Proposition 2.1. Suppose that $\lambda_{\max}$ be the maximal Frobenius slope of M. If M is PBQ, then $M^{\lambda} = (S_{\lambda-\lambda_{\max}}(M^{\vee}))^{\perp} = S_{(\lambda_{\max}-\lambda)-}(M)$ for any λ by Theorem 7.1. Then we have

$$M^{\lambda+} = \cup_{\mu>\lambda} S_{(\lambda_{\max}-\mu)-}(M) = \cup_{\mu>\lambda} S_{(\lambda_{\max}-\mu)}(M) = S_{(\lambda_{\max}-\lambda)-}(M) = M^{\lambda}.$$

If λ is a break of log-growth filtration, then

$$S_{\lambda_{\max}-\lambda}(M) = S_{(\lambda_{\max}-\lambda)+}(M) = M^{\lambda-} \supsetneq M^{\lambda} = S_{(\lambda_{\max}-\lambda)-}(M)$$

and λ is also a Frobenius slope filtration. Hence λ is rational.
For a general M, we use the induction on the length of the PBQ filtration of M. Let L be the maximally PBQ submodule of M and suppose $N = M/L$. Then we have the assertion by Proposition 2.6 (1), the PBQ case and the induction hypothesis on L and N. □

Proposition 7.3 *Suppose that the residue field of $\mathcal{V}$ is perfect. Let M be a φ-∇-module over $K[\![x]\!]_0$ and let λ be a break of log-growth filtration of $V(M)$, i.e., $V(M)^{\lambda-} \supsetneq V(M)^{\lambda+}$, and let $\{P_i(M)\}$ be the PBQ filtration of M. Suppose that the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (2) (see 2.5) holds for all $P_i(M)/P_{i-1}(M)$. Then λ is rational and $V(M)^{\lambda} = V(M)^{\lambda+}$. In particular, the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (2) implies the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (1) for any φ-∇-modules over $K[\![x]\!]_0$.*

Proof. The proof is similar to that of Theorem 7.2 by replacing Proposition 2.6 (1) by Proposition 2.6 (2). □

8 Toward Dwork's conjecture $\mathbf{LGF}_{\mathrm{Dw}}$

8.1 The comparison at the special point and Dwork's conjecture $\mathbf{LGF}_{\mathrm{Dw}}$

Theorem 8.1 *The conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (2) (see 2.5) implies the conjecture $\mathbf{LGF}_{\mathrm{Dw}}$ (see 2.7), that is, the special log-growth polygon lies above the generic log-growth polygon (and they have the same endpoints).*

The theorem above follows from the proposition below by Proposition 2.1.

Proposition 8.2 *Suppose that the residue field k of $\mathcal{V}$ is perfect. Let M be a φ-∇-module over $K[\![x]\!]_0$ and let $\{P_i(M)\}$ be the PBQ filtration of M. Suppose that the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (2) (see 2.5) holds for all $P_i(M)/P_{i-1}(M)$. Then the special log-growth polygon of M lies above the generic log-growth polygon of M (and they have the same endpoints).*

Proof. For the PBQ φ-∇-modules arising from the PBQ filtration of M, the log-growth polygons at the generic (resp. special) fiber coincides with the Newton polygon of Frobenius slopes of the dual at the generic (resp. special) fiber under the suitable shifts of Frobenius actions by Theorem 7.1 (resp. our hypothesis). The assertion follows from Proposition 2.6, Lemma 8.3 below and the fact that the special Newton polygon of Frobenius slopes is above the generic Newton polygon of Frobenius slopes and they have the same endpoints. □

Lemma 8.3 *Let $0 \to L \to M \to N \to 0$ be an exact sequence of φ-∇-modules over $K[\![x]\!]_0$ such that the induced sequences*

$$\begin{array}{ccccccccc} 0 & \to & L_\eta/L_\eta^\lambda & \to & M_\eta/M_\eta^\lambda & \to & N_\eta/N_\eta^\lambda & \to & 0 \\ 0 & \to & V(L)/V(L)^\lambda & \to & V(M)/V(M)^\lambda & \to & V(N)/V(N)^\lambda & \to & 0 \end{array}$$

on both the generic fiber and the special fiber are exact for any λ.

(1) *If the special log-growth polygon lies above the generic log-growth polygon (the endpoints might be different) for both L and N, then the same holds for M.*

(2) *If the special log-growth polygon and the generic log-growth polygon have the same endpoints for both L and N, then the same holds for M.*

(3) *Suppose that the special log-growth polygon lies above the generic log-growth polygon for both L and N. Then both the special and the generic log-growth polygons coincide with each other for M if and only if the same hold for L and N.*

Proof. Let r be the rank of M. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$ be breaks of log-growth filtration of M_η with multiplicities, and put $b_0(M_\eta) = 0$ and

$$b_j(M_\eta) = \lambda_1 + \cdots + \lambda_j$$

for $1 \leq j \leq r$. Then the generic log-growth polygon of M is a polygon which connects points $(0, b_0(M_\eta)), (1, b_1(M_\eta)), \cdots, (r, b_r(M_\eta))$ by lines. We also define $b_j(V(M))$ for the special log-growth of M. Then the exactness for any λ implies the equality

$$b_j(M_\eta) = \min \left\{ b_i(L_\eta) + b_k(N_\eta) \;\middle|\; \begin{array}{c} 0 \leq i \leq \operatorname{rank} L,\ 0 \leq k \leq \operatorname{rank} N, \\ i + k = j \end{array} \right\}$$

for all $0 \le j \le r$, and the same holds for the special log-growth. The special log-growth polygon lies above the generic log-growth polygon for M if and only if $b_j(M_\eta) \le b_j(V(M))$ for all j, the special log-growth polygon and the generic log-growth polygon have the same endpoints for M if and only if $b_r(M_\eta) = b_r(V(M))$, and both the special and the generic log-growth polygons coincide with each other for M if and only if $b_j(M_\eta) = b_j(V(M))$ for all j. Hence we have the assertions. □

Remark 8.4 *If L is supposed to be HPBQ in the short exact sequence of the previous lemma, then the induced sequences are automatically exact for all λ: in fact one has Theorems 7.1 and 6.5 and can apply Proposition 2.6.*

Remark 8.5 *If one assumes that the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (2) (see 2.5) for any PBQ φ-∇-module over $K[\![x]\!]_0$ of rank $\le r$, then the proofs of Proposition 7.3 and Theorem 8.1 works for any φ-∇-module over $K[\![x]\!]_0$ of rank $\le r$.*

8.2 Dwork's conjecture in the HBQ cases

Lemma 8.6 *Let M be a HBQ φ-∇-module over $K[\![x]\!]_0$ and let N be a φ-∇-submodule of M over $K[\![x]\!]_0$ which is PBQ. Then N is HPBQ. In particular, suppose that the residue field of $\mathcal{V}$ is perfect and let $\{P_i(M)\}$ be the PBQ filtration of M, then $P_i(M)/P_{i-1}(M)$ is HPBQ for all i.*

Proof. We have $\dim_K V(M)/V(M)^0 = \dim_{\mathcal{E}} M_\eta/M_\eta^0$ and $\dim_K V(M/N)/V(M/N)^0 = \dim_{\mathcal{E}} (M/N)_\eta/(M/N)_\eta^0$ by Proposition 6.3 since the quotient M/N is HBQ by Proposition 6.4. Comparing the induced exact sequence $0 \to N_\eta/N_\eta^0 \to M_\eta/M_\eta^0 \to (M/N)_\eta/(M/N)_\eta^0 \to 0$ at the generic point by Theorem 7.1 and Proposition 2.6 (1) to the corresponding right exact sequence at the special point, we have an inequality $\dim_K V(N)/V(N)^0 \ge \dim_{\mathcal{E}} N_\eta/N_\eta^0$. On the contrary, we know the inequality $\dim_K V(N)/V(N)^0 \le \dim_{\mathcal{E}} N_\eta/N_\eta^0$ by [CT09, Proposition 4.10]. Hence, $\dim_K V(N)/V(N)^0 = \dim_{\mathcal{E}} N_\eta/N_\eta^0$ and N is HPBQ.
The rest follows from the first part and Proposition 6.4. □

Theorem 8.7 *Let M be a HBQ φ-∇-module over $K[\![x]\!]_0$. Then the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ (1) (see 2.5) and the conjecture $\mathbf{LGF}_{\mathrm{Dw}}$ (see 2.7) hold for M.*

Proof. The assertions follows from the similar arguments of Theorems 7.2 and 8.1, respectively, by using Theorem 6.5 and Lemma 8.6. □

8.3 When do the generic and special log-growth polygons coincide?

Theorem 8.8 *Let M be a φ-∇-module over $K[\![x]\!]_0$. The special log-growth polygon and the generic log-growth polygon coincide with each other if and only if M is equislope.*

PROOF. We may assume that the residue field of $\mathcal{V}$ is algebraically closed by Proposition 2.1. Let $\{P_i(M)\}$ be the PBQ filtration of M (Theorem 5.6). Each condition (i) the coincidence of special and generic log-growth polygons or (ii) equislope implies that $P_i(M)/P_{i-1}(M)$ is HPBQ and $M/P_i(M)$ is HBQ for all i by Propositions 6.3, 6.4, and Lemma 8.6 for (i) and by Corollary 6.9 and Proposition 6.10 (1) for (ii). Then we can apply Lemma 8.3 (3) inductively on i by Remark 8.4 and Theorem 8.7. Hence it is sufficient to prove the assertion when M is HPBQ by Corollary 6.9. Then the coincidence of the log-growth filtration and the Frobenius slope filtration both at the special point (Theorem 8.7) and at the generic point (Theorem 7.1) implies our desired equivalence. □

EXAMPLE 8.9 (1) *Let M be a φ-∇-module over $K[\![x]\!]_0$ such that M_η is bounded. Then there is a φ-module L over K such that $M \cong L \otimes_K K[\![x]\!]_0$ by Christol's transfer theorem (see [CT09, Proposition 4.3]). Hence, M is equislope.*

(2) *Let M be a φ-∇-module over $K[\![x]\!]_0$ of rank 2 such that M_η is not bounded. Then we have identities $M^\lambda = (S_{\lambda-\lambda_{\max}}(M^\vee))^\perp$ and $V(M)^\lambda = (S_{\lambda-\lambda_{\max}}(V(M^\vee)))^\perp$ for any λ [CT09, Theorem 7.1], where $\lambda_{\max}$ is the highest Frobenius slope of M_η. Hence the special log-growth polygon and the generic log-growth polygon coincide with each other if and only if M is equislope.*

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