Almost Proper GIT-Stacks and Discriminant Avoidance

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Abstract. We prove that the classifying stack of an reductive group scheme over a field is very close to being proper. Using this we prove a result about isotrivial families of varieties. Fix a polarized variety with reductive automorphism group. To prove that every isotrivial family with this fibre has a rational section it suffices to prove this when the base is projective, i.e., the discriminant of the family is empty.

1. Introduction

Consider an algebraic stack of the form $[\mathrm{Spec}(k)/G]$ where G is a geometrically reductive group scheme over a field k. It turns out that such a stack is nearly proper, see Proposition 2.5.1. Our proof of this uses ideas very similar to those used by Totaro and Edidin-Graham in their work on equivariant Chow theory. It seems the application of these ideas here is novel.

Next, consider a pair $(V, \mathcal{L})$ consisting of a projective variety V over k and an invertible sheaf. Also, fix an integer $d \geq 1$. We would like to know if every d-dimensional family of polarized varieties $X \to S$, $\mathcal{N} \in \mathrm{Pic}(X)$, all of whose fibres are isomorphic to $(V, \mathcal{L})$, has a rational section. For example this is true if V is a nodal plane cubic.

Theorem 1.0.1. *(See Theorem 2.2.3 which is slightly more general.) Assume $G = Aut(V, \mathcal{L})$ is geometrically reductive. If $X \to S$ has a rational section whenever S is a projective variety of dimension d then there is a rational section whenever S is a quasi-projective variety, provided* $\dim S \leq d$.

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Loosely speaking this means that if you prove the existence of rational sections whenever the discriminant is empty then you prove it in general. For example, it implies that if you are trying to find rational sections of families of polarized homogenous varieties over surfaces then it suffices to do so in the case of families of homogenous varieties over projective nonsingular surfaces. Our proof of this theorem depends on the result on GIT-stacks mentioned above.

In a forthcoming article, joint with Xuhua He, we use this to prove that certain families of polarized homogeneous varieties over surfaces always have rational sections. This is the crucial step in resolving Serre's "Conjecture II" over function fields of surfaces, cf. [Ser02, p. 137], and also gives a proof of the first author's *Period-Index Theorem*, cf. [dJ04], valid in arbitrary characteristic (another proof valid in arbitrary characteristic was proved independently by Max Lieblich, who also proved some beautiful extensions, cf. [Lie08]).

2. Isotrivial families

The title of this section is a little misleading as usually one thinks of an isotrivial family as having finite monodromy. As the reader will see such families are certainly examples to which our discussion applies, but we also allow for a positive dimensional structure group. The families will be isotrivial in the sense that the fibres over a Zariski open will be all isomorphic to a fixed variety V.

2.1. Generalities on Isom. Let U be a base scheme. Let $f : X \to U$ and $g : Y \to U$ be proper, flat morphisms. Let $\mathcal{N}$ be an f-ample invertible sheaf on X, and let $\mathcal{L}$ be a g-ample invertible sheaf on Y. Consider the functor that associates to a scheme $T \to U$ over U the set of pairs (ϕ, α), where $\phi : X_T \to Y_T$ is an isomorphism over T and $\alpha : \phi^*\mathcal{L}_T \to \mathcal{N}_T$ is an isomorphism of invertible sheaves. This functor is representable, see [Gro62, No. 221-19, §4.c], [Gro63, Corollaire 7.7.8], and [LMB00, Théorème 4.6.2.1]. We will call the representing U-scheme $\mathrm{Isom}_U((X,\mathcal{N}),(Y,\mathcal{L}))$.

In fact this U-scheme is affine over U. To see this, it is first convenient to change $\mathcal{L}$ and $\mathcal{N}$. For every integer $N > 0$ there is an obvious morphism

$$\mathrm{Isom}_U((X,\mathcal{N}),(Y,\mathcal{L})) \to \mathrm{Isom}_U((X,\mathcal{N}^N),(Y,\mathcal{L}^N)).$$

It is straightforward to verify that this morphism is finite. Therefore we can reduce to the case that $\mathcal{L}$ and $\mathcal{N}$ are relatively *very ample* and also have vanishing higher direct images. Then the natural map

$$\mathrm{Isom}_U((X,\mathcal{N}),(Y,\mathcal{L})) \to \mathrm{Isom}_U(f_*\mathcal{N}, g_*\mathcal{L})$$

is a closed immersion whose target is clearly affine over U, cf. [Gro63, 7.7.8, 7.7.9].

2.2. Statement of the result. Let k be an algebraically closed field of any characteristic. We assume given a projective scheme V over k and an ample invertible sheaf $\mathcal{L}$ over V. We let $m = \dim V$. We introduce another integer $d \geq 1$ which will be an upper bound for the dimension of the base of our families. We are going to ask the following question: Is it true that for any polarized family of schemes over a $\leq d$-dimensional base whose general fibre is V, there is a rational point on the generic fibre? We make this more precise as follows.

Situation 2.2.1. Here we are given a triple $(K/k, X \to S, \mathcal{N})$, with the following properties: (a) The field K is an algebraically closed field extension of k. (b) The map $X \to S$ is a proper morphism to a projective variety S over K. (c) The dimension of S is at most d. (d) We are given an invertible sheaf $\mathcal{N}$ on X. (e) For a general point $s \in S(K)$ we have $(X_s, \mathcal{N}_s) \cong (V_K, \mathcal{L}_K)$.

The notation $(V_K, \mathcal{L}_K)$ refers to the base change of the pair $(V, \mathcal{L})$ to Spec K. Part (e) means that there exists a nonempty Zariski open $U \subset S$ over which the morphism is flat and such that $(X_s, \mathcal{N}_s) \cong (V_K, \mathcal{L}_K)$ as pairs over K for all $s \in U$. We will see in Lemma 2.3.2 that this implies $\mathrm{Isom}_U((X, \mathcal{N}), (V_U, \mathcal{L}_U))$ is a torsor, hence all geometric fibres of $X \to S$ over U are isomorphic to a suitable base change of V.

Question 2.2.2. Suppose we are in Situation 2.2.1. Is there a rational point on the generic fibre of $X \to S$? In other words: Is $X(K(S))$ not empty?

A natural problem that arises when studying this question is the possibility of bad fibres in the family $X \to S$. Let us define the discriminant Δ of a family $(K/k, X \to S, \mathcal{N})$ as in Situation 2.2.1 as the Zariski closure of the set of points $s \in S(K)$ such that $(X_s, \mathcal{L}_s)$ is not isomorphic to $(V_K, \mathcal{L}_K)$. A priori the codimension of (the closure of) Δ is assumed ≥ 1, and typically it will be 1. In this section we show that it often suffices to answer Question 2.2.2 in cases where the codimension of Δ is bigger, at least as long as we are answering the question for all families.

It is not surprising that the automorphism group G of the pair $(V, \mathcal{L})$ is an important invariant of the situation. The group scheme G has T-valued points which are pairs (ϕ, α), where $\phi : V_T \to V_T$ is an automorphism of schemes over T, and $\alpha : \phi^*\mathcal{L}_T \to \mathcal{L}_T$ is an isomorphism of invertible sheaves. It is representable by Subsection 2.1. The group law is given by $(\phi, \alpha) \cdot (\psi, \beta) = (\phi \circ \psi, \beta \circ \psi^*(\alpha))$. And G is an affine group scheme over k. In the following theorem G°_{red} denotes the reduction of the identity component of G. Note that G°_{red} is a smooth affine group scheme (since k is algebraically closed, and hence perfect).

Theorem 2.2.3. *Fix $(V, \mathcal{L})$ and d as above. Assume that G°_{red} is reductive. If the answer to Question 2.2.2 is yes whenever $\Delta = \emptyset$, then the answer to Question 2.2.2 is yes in all cases.*

The proof has 2 parts: deformation and specialization. The deformation argument proves the following: For every triple $(K/k, X \to S, \mathcal{N})$, there is a dense open subset $U \subset S$ and a deformation of $(X_U \to U, \mathcal{N}|_{X_U})$ to a triple $(K'/k, X' \to S', \mathcal{N}')$ with trivial discriminant. The specialization argument proves the following: Every rational point of the generic fiber of $X' \to S'$ specializes to a rational point of the generic fiber of $X \to S$. Thus Question 2.2.2

has a positive answer for $(K/k, X \to S, \mathcal{N})$ if it has a positive answer for $(K'/k, X' \to S', \mathcal{N}')$.

2.3. A bijective correspondence. To deform the pair $(X_U \to U, \mathcal{N}|_{X_U})$, it is convenient to first convert the pair into a G-torsor over U, deform the torsor, and then convert this back into a triple. This subsection describes how to convert between pairs and G-torsors. As in subsection 2.2, denote by G the automorphism group scheme of $(V, \mathcal{L})$. Following is a quick proof of a well-known result about homogeneous spaces.

Lemma 2.3.1. *Let Γ be a finite type group scheme over k acting on a nonempty, reduced, finite type k-scheme X. If the induced morphism*

$$\Psi : \Gamma \times_{Spec\ k} X \to X \times_{Spec\ k} X, \quad (g, x) \mapsto (g \cdot x, x)$$

is surjective on geometric points, then it is flat so that X is a homogeneous space under Γ. If, moreover, Γ is smooth over k, then also X is smooth over k.

Proof. By [Gro67, Théorème 11.1.1], the set U of points in $\Gamma \times_{\mathrm{Spec}\ k} X$ at which Ψ is flat is open. The morphism Ψ is equivariant for the $\Gamma \times_{\mathrm{Spec}\ k} \Gamma$-actions,

$$(\Gamma \times_{\mathrm{Spec}\ k} \Gamma) \times_{\mathrm{Spec}\ k} (\Gamma \times_{\mathrm{Spec}\ k} X) \to \Gamma \times_{\mathrm{Spec}\ k} X, \ \ (\gamma', \gamma) \cdot (g, x) := (\gamma' g \gamma^{-1}, \gamma \cdot x),$$

$$(\Gamma \times_{\mathrm{Spec}\ k} \Gamma) \times_{\mathrm{Spec}\ k} (X \times_{\mathrm{Spec}\ k} X) \to X \times_{\mathrm{Spec}\ k} X, \ \ (\gamma', \gamma) \cdot (x', x) := (\gamma' \cdot x', \gamma \cdot x).$$

Therefore U is $(\Gamma \times_{\mathrm{Spec}\ k} \Gamma)$-invariant. Every invariant subset of $\Gamma \times_{\mathrm{Spec}\ k} X$ is of the form $\Gamma \times_{\mathrm{Spec}\ k} V$ for a Γ-invariant subset V of X. Since $X \times_{\mathrm{Spec}\ k} X$ is reduced, Ψ is flat at every point of $\Gamma \times_{\mathrm{Spec}\ k} X$ mapping to a generic point of $X \times_{\mathrm{Spec}\ k} X$. And such points exist by the hypothesis that Ψ is surjective. Therefore U is nonempty, i.e., V is nonempty. Finally, by the hypothesis that Ψ is surjective, the only nonempty, Γ-invariant open subset V of X is $V = X$. Therefore U equals $\Gamma \times_{\mathrm{Spec}\ k} X$, i.e., Ψ is flat.

Finally, assume that Γ is smooth over k. For any k-point x of X (which exists since X is nonempty), the induced morphism

$$\Psi_x : \Gamma \to X, \quad g \mapsto g \cdot x$$

is flat, since it is the base change of Ψ by the morphism

$$X \mapsto X \times_{\mathrm{Spec}\ k} X, \ x' \mapsto (x', x).$$

Therefore, by [Gro67, Proposition 17.7.7], X is smooth over k. $\square$

Lemma 2.3.2. *Let U be a k-scheme. Let $(X \to U, \mathcal{N})$ be a pair where $X \to U$ is a flat proper morphism and $\mathcal{N}$ is an invertible sheaf on X. Assume that the geometric fiber of $(X, \mathcal{N})$ over U is isomorphic to the base change of $(V, \mathcal{L})$ for a dense set of geometric points of U. Also assume that U is reduced. Then the scheme $\mathcal{T} := Isom_U((X, \mathcal{N}), (V, \mathcal{L}))$, with its natural G-action, is a G-torsor over U.*

Proof. It suffices to prove that $(X, \mathcal{N})$ is locally in the fppf topology of U isomorphic to the constant family $(V, \mathcal{L}) \times U$. To prove this we need some notation.

Take N so large that $\mathcal{L}^N$ is very ample on V and has vanishing higher cohomology groups. Let $n = \dim \Gamma(V, \mathcal{L}^N)$. A choice of basis of $\Gamma(V, \mathcal{L}^N)$ determines a closed immersion $i : V \to \mathrm{P}^{n-1}$. This determines a point $[i]$ of the Hilbert scheme $\mathrm{Hilb} = \mathrm{Hilb}_{\mathrm{P}^{n-1}/k}$. The smooth algebraic group PGL_n acts on Hilb, and we denote by Z the orbit of $[i]$, which is a locally closed subscheme of Hilb. By Lemma 2.3.1, Z is a smooth scheme and the morphism $\mathrm{PGL}_n \to Z$ associated to any k-point of Z is flat. By construction the pullback of the universal family over Z to PGL_n is canonically isomorphic to $V \times \mathrm{PGL}_n$, and the invertible sheaf $\mathcal{O}(1)$ pulls back to $\mathcal{L}^N \boxtimes \mathcal{O}$.

The question is local on U so we may assume that U is affine. By our choice of N above, the invertible sheaf $\mathcal{N}^N$ is very ample on every fibre of X over U with vanishing higher cohomology groups. Hence after possibly shrinking U we can find a closed immersion $X \to \mathrm{P}_U^{n-1}$ which restricts to the embedding given by the full linear series of $\mathcal{N}^N$ on every geometric fibre. Consider the associated moduli map $m : U \to \mathrm{Hilb}$. Since U is reduced, and since each pair $(X_s, \mathcal{N}_s)$ for a dense set of geometric points s is isomorphic to a base change of $(V, \mathcal{L})$, we see that $m(U) \subset Z$.

This implies there is some surjective flat morphism $U' \to U$ and an U'-isomorphism $X' \cong V \times U'$ with the property that $\mathcal{N}^N$ pulls back to $\mathcal{L}^N$. The fiber product $U' = U \times_Z \mathrm{PGL}_n$ parameterizes points of U together with an autmorphism of $\mathbb{P}^n$ transforming the fiber of X isomorphically to $i(V)$. Since $\mathrm{PGL}_n \to Z$ is surjective and flat, $U' \to U$ is also surjective and flat. To finish, do the same thing for $N+1$ to get some $U'' \to U$. Then over $U''' := U' \times_U U''$ there is an isomorphism of the pullback of $(X, \mathcal{N})$ and the base change of $(V, \mathcal{L})$. This proves the result. $\square$

Conversely, given a left G-torsor $\mathcal{T}$ over U we will construct a flat proper family of varieties $X \to U$ and an invertible sheaf $\mathcal{N}$ on X such that $\mathrm{Isom}_U((X, \mathcal{N}), (V_U, \mathcal{L}_U))$ is isomorphic to $\mathcal{T}$. Of course it will turn out that X equals $(V \times \mathcal{T})/G$ (as an fppf sheaf), but we need to prove this is a scheme.

The structure morphism $\pi : \mathcal{T} \to U$ is a flat surjective morphism of finite type. We are going to descend the constant family $V \times \mathcal{T}$ to U using a descent datum

$$\phi : V \times \mathcal{T} \times_U \mathcal{T} \to V \times \mathcal{T} \times_U \mathcal{T}.$$

Before we describe the descent datum, we recall that the map

$$\Psi : G \times \mathcal{T} \to \mathcal{T} \times_U \mathcal{T}, \quad (g, t) \mapsto (g \cdot t, t)$$

is an isomorphism. Also, let us denote $m : V \times G \to V$ the map $(v, g) \mapsto gv$, where gv denote the natural action of $g \in G$ on $v \in V$. Finally, we take

$$\phi = \mathrm{Id}_V \times \Psi \circ m \times \mathrm{Id}_{\mathcal{T}} \circ (\mathrm{Id}_V \times \Psi)^{-1}.$$

To verify the cocycle condition on $\mathcal{T} \times_U \mathcal{T} \times_U \mathcal{T}$, we can think of ϕ as the map $(v, gt, t) \mapsto (g^{-1}v, gt, t)$. If on $V \times \mathcal{T} \times_U \mathcal{T} \times_U \mathcal{T}$ we have a point (v, g_1g_2t, g_2t, t) then $\mathrm{pr}_{23}^*(\phi)(v, g_1g_2t, g_2t, t) = (g_2v, g_1g_2t, g_2t, t)$ and $\mathrm{pr}_{12}^*(\phi) \circ \mathrm{pr}_{23}^*(\phi)(v, g_1g_2t, g_2t, t) = (g_1g_2v, g_1g_2t, g_2t, t)$ and $\mathrm{pr}_{13}^*(\phi)(v, g_1g_2t, g_2t, t) = ((g_1g_2)v, g_1g_2t, g_2t, t)$. Thus $\mathrm{pr}_{13}^*(\phi) = \mathrm{pr}_{12}^*(\phi) \circ \mathrm{pr}_{23}^*(\phi)$ as desired.

Because all the maps in question lift canonically to the invertible ample sheaf $\mathcal{L}$ this actually defines a descent datum on the pair $(V, \mathcal{L})$ for $\mathcal{T} \to U$. As $\mathcal{L}$ is ample, this descent datum is effective, cf. [Gro62, No. 190, §B.1]. Thus there exists a pair $(X \to U, \mathcal{N})$ over U and an isomorphism $\delta : \mathcal{T} \times_U (X, \mathcal{N}) \to \mathcal{T} \times (V, \mathcal{L})$ such that ϕ equals $\mathrm{pr}_1^*\delta \circ \mathrm{pr}_2^*\delta^{-1}$.

Conclusion 2.3.3. The above constructions give a bijective correspondence between pairs $(X \to U, \mathcal{N})$ and left G-torsors over U in case U is a reduced scheme over k.

Remark 2.3.4. The construction of the family $(X, \mathcal{N})/U$ starting from the torsor $\mathcal{T}$ works more generally when k is a ring as long as: (1) V is a flat projective scheme of finite presentation over k, (2) $\mathcal{L}$ is ample, and (3) the automorphism group scheme $G = \mathrm{Aut}(V, \mathcal{L})$ is flat over k.

2.4. Deforming torsors over a Henselian DVR.

Before proving Theorem 2.2.3, it is useful to say what is known without the hypothesis that G is reductive. We thank Ofer Gabber, Jean-Louis Colliot-Thélène and Max Lieblich for explaining the following proposition.

Proposition 2.4.1. *Let R be a Henselian DVR with residue field k, and let G be a flat separated group scheme of finite type over Spec R. Every torsor for the closed fiber G_k over Spec k is the closed fiber of a torsor for G over Spec R.*

Proof. We first give a proof when G is affine which is all we will use in this paper. The usual proof that every affine group scheme over a field is linear extends to affine, flat group schemes over a DVR, see [ABD$^+$65, Exposé $\mathrm{VI_B}$, Remarque 11.11.1]. Choose a closed immersion $G \to \mathrm{GL}_{n,R}$. The quotient fppf sheaf $X = \mathrm{GL}_{n,R}/G$ is an algebraic space over R, cf. [Art74, Corollary 6.3]. In fact, by [Ana73, Proposition 3.4.2], there exists an fpqc cover Spec $R' \to$ Spec R such that the pullback Spec $R' \times_{\mathrm{Spec}\ R} X$ is a scheme. After base change to R', by [ABD$^+$65, Exposé $\mathrm{VI_A}$, Proposition 9.2] the quotient morphism

$$\mathrm{GL}_{n,R'} \to \mathrm{Spec}\ R' \times_{\mathrm{Spec}\ R} X$$

is faithfully flat, in fact is a G-torsor, and Spec $R' \times_{\mathrm{Spec}\ R} X$ is smooth over R' . But each of these statements (in the category of algebraic spaces) can be checked after faithfully flat base change. Thus also $\mathrm{GL}_{n,R} \to X$ is faithfully flat, in fact a G-torsor, and X is smooth over R. Since $H^1(k, \mathrm{GL}_{n,k}) = \{1\}$, any torsor for G_k is the fibre of the map $\mathrm{GL}_{n,k} \to X_k$ over a k-point of X. Since R is Henselian and since X is smooth, the map $X(R) \to X(k)$ is surjective, and hence every G_k-torsor lifts.

In the general case, i.e., when G is not necessarily affine, we argue as follows. By [LMB00, Prop. 10.13.1], which relies upon Artin's criterion for algebraicity of a stack, the classifying stack BG is an algebraic stack over Spec R. By [LMB00, Thm. 6.3], for each G_k-torsor there exists an affine R-scheme X, a smooth morphism $\phi : X \to BG$, and a k-point x of X such that $\phi(x)$ corresponds to the given G_k-torsor. Denote by t : Spec $R \to BG$ the 1-morphism associated to the trivial G-torsor. Since ϕ is smooth, the base-change pr_R : Spec $R \times_{t,BG,\phi} X \to$ Spec R is smooth. Since t is a surjective flat morphism, the base-change, pr_X : Spec $R \times_{t,BG,\phi} X \to X$ is surjective and flat. By [Gro67, §6.5], it follows that X is smooth over Spec R. Since R is Henselian and X is smooth over Spec R, $X(R) \to X(k)$ is surjective; in particular there is an R-morphism Spec $R \to X$ extending the given k-point of X. The composition of this morphism with ϕ determines a G-torsor over Spec R whose closed fiber is isomorphic to the given G_k-torsor over Spec k. □

Corollary 2.4.2. *Let R be a DVR with residue field k, and let G be a separated, finite type, flat group scheme over Spec R. Let U be a finite type, integral k-scheme, and let $\mathcal{T}_U \to U$ be a G_k-torsor. There exists an integral, flat, quasi-projective R-scheme Y, with nonempty special fibre Y_k, a G-torsor $\mathcal{T} \to Y$, and an open immersion $j : Y_k \to U$ such that $j^*\mathcal{T}_U$ is isomorphic to $\mathcal{T}_k$ as G_k-torsors over Y_k.*

$$\begin{tikzcd} \mathcal{T} \arrow[d] & \mathcal{T}_k \arrow[l] \arrow[r, "j"] \arrow[d] & \mathcal{T}_U \arrow[d] \\ Y & Y_k \arrow[l] \arrow[r, "j"] & U \end{tikzcd}$$

Proof. First we show there exists an integral, flat, quasi-projective R-scheme Z and an open immersion $j : Z_k \to U$. It suffices to prove this after replacing U by a dense open subset. Thus first replace U by a dense open affine. And then replace U by the regular locus $\mathrm{Reg}(U)$ which is open by [Gro67, Corollaire 6.12.5] and which is dense since it contains the generic point of U (the stalk being a field since U is integral). In particular this implies that $U \to$ Spec k is a local complete intersection morphism, see [Gro67, Proposition 19.3.2]. So after shrinking U some more we may assume that $U =$ Spec $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a complete intersection, i.e., $\dim U = n - c$. At this point we simply put $Z =$ Spec $R[x_1, \ldots, x_n]/(F_1, \ldots, F_c)$, where $F_i \in R[\underline{x}]$ lifts f_i.

Define R' to be the local ring of Z at the generic point of Z_k. Then R' is a Noetherian 1-dimensional local ring. Denote by π a uniformizer of R. Clearly, π maps into $\mathfrak{m}_{R'}$ and $R'/\pi R'$ is the function field of Z_k, i.e., the function field of U. Because R' is R-flat, π is a nonzerodivisor. Thus R' is a DVR with residue field $K = k(U)$.

By Proposition 2.4.1, the G_k torsor over $R'/\pi R'$ lifts to a G-torsor $\mathcal{T}^h$ over the Henselization of R'. By a standard limit argument, this lift exists over an étale extension $R' \to R''$ contained in the Henselization of R'. Note that the residue field $R''/\pi R''$ of R'' is still the function field of U. By a standard limit argument, there is an étale morphism $Y \to Z$ such that $Y_k \to Z_k$ is an open immersion and such that R'' is the local ring of Y at the generic point of Y_k. After replacing Y by an open subscheme, there is a G-torsor $\mathcal{T}$ over Y that pulls back to $\mathcal{T}^h$ over R''. We leave it to the reader to see that, after possible shrinking Y again, this torsor satisfies the conditions of the corollary. □

Corollary 2.4.2 above is a weak version of the deformation principle that we will establish later on. The remaining issue is whether there exists a datum $(Y \to \text{Spec } R, \mathcal{T} \to Y, j : Y_K \to U)$ such that the generic fiber of $Y \to \text{Spec } R$ is projective. Presumably this is not always possible, but in case G is reductive we will show that it is.

Corollary 2.4.2 can be used to lift problems in char $p > 0$ to characteristic 0. Suppose that R is a complete discrete valuation ring with algebraically closed residue field k. Let Ω be an algebraic closure of the fraction field of R. We have in mind the case where $\text{char}(k) = p > \text{char}(\Omega) = 0$. Suppose that V_R is a flat projective R scheme, and that $\mathcal{L}_R$ is an ample invertible sheaf over V_R. We assume that V_Ω and V_k are varieties. Let G_R denote the automorphism group scheme of $(V_R, \mathcal{L}_R)$ over R.

Corollary 2.4.3. *Notations and assumptions as above. Fix $d \in \mathbb{N}$. Assume that G_R is flat over R. If the answer to Question 2.2.2 is always "yes" for the pair $(V_\Omega, \mathcal{L}_\Omega)$ then the answer is always "yes" for the pair $(V_k, \mathcal{L}_k)$.*

Proof. Let $(K/k, X \to S, \mathcal{N})$ be a triple as in Situation 2.2.1 for the pair $(V_k, \mathcal{L}_k)$. Let U be the open subscheme of S over which all geometric fibres of $(X, \mathcal{N})$ are isomorophic to the base change of $(V_k, \mathcal{L}_k)$. The construction in Subsection 2.3 gives a corresponding G_K-torsor $\mathcal{T}_U$ over U.

There exists an extension of complete discrete valuation rings $R \subset R'$ such that the induced extension of residue fields is K/k, see [Gro63, Chapitre 0, 10.3.1]. We apply Corollary 2.4.2 to obtain $Y \to \text{Spec } R'$, $\mathcal{T} \to Y$ and $j : Y_K \to U$. According to Conclusion 2.3.3 and Remark 2.3.4 there exists a pair $(X' \to Y, \mathcal{N}')$ over Y whose restriction to Y_K is isomorphic to $(j^*X|_U, j^*\mathcal{N}|_U)$.

Let Ω' be an algebraic closure of the field of fractions $Q(R')$ of R'. Since $R \subset R'$ we may and do assume that $\Omega \subset \Omega'$. Note that we do not know that the geometric fibre $Y_{\Omega'}$ is irreducible. However, our assumptions imply that X' has a $\Omega'(Y')$-valued point for every irreducible component Y' of $Y_{\Omega'}$. To conclude we apply the lemma below. □

Lemma 2.4.4. *Suppose that R is a DVR with algebraically closed residue field K. Let Ω be an algebraic closure of $Q(R)$. Let $Y \to \text{Spec } R$ be a flat, finite type morphism, $X \to Y$ a projective morphism and let $\xi \in Y_K$. Assume in addition that (a) ξ is the generic point of an irreducible component C of the scheme Y_K, (b) the scheme Y_K is reduced at ξ, and (c) for every irreducible component Y' of Y_Ω there exists a $\Omega(Y')$-valued point of X. Then X has a $K(C)$-valued point.*

Proof. Note that right from the start we may replace R by its completion, and hence we may assume that R is excellent, cf. [Gro67, Scholie 7.8.3(iii)]. This will guarantee that the integral closure of R in a finite extension of $Q(R)$ is finite over R, cf. [Gro67, Scholie 7.8.3(vi)]. (In fact this is not necessary if the fraction field of R has characteristic 0, cf. [Mat89, Lemma 1, p. 262].)

By hypothesis, and a standard limit argument, there is a section of $X_\Omega \to Y_\Omega$ over a dense open $V \subset Y_\Omega$, say $s : V \to X_\Omega$. By a standard limit argument, there is a finite extension $Q(R) \subset L$ such that V and s are defined over L. Let R' be the integral closure of R in L. Since R is excellent the extension $R \subset R'$ is a finite extension of DVRs. The residue field of R' is isomorphic to K as K is algebraically closed.

By construction the scheme $Y_{R'} = Y \times_R R'$ has special fibre equal to Y_K. The local ring $\mathcal{O}$ of $Y_{R'}$ at ξ is a DVR. This follows from flatness of $Y_{R'}/R'$ and property (b), see proof of 2.4.2. Thus the image of Spec $Q(\mathcal{O}) \to Y_L$ is one of the generic points of Y_L and hence contained in V. Since $X_{R'} \to Y_{R'}$ is proper, we see that $s|_{\mathrm{Spec}\ Q(\mathcal{O})}$ extends to a $\mathcal{O}$-valued point of $X_{R'}$, and in particular we obtain a $\kappa(\xi) = K(C)$-valued point of $(X_K)_{\kappa(\xi)} = X_{K(C)}$ as desired. □

For example this corollary always applies to the case where $(V, \mathcal{L})$ is the pair consisting of a Grassmanian and its ample generator.

2.5. Deforming torsors for a reductive group. Under the additional hypothesis that G is a geometrically reductive linear algebraic group we can prove a stronger version of Corollary 2.4.2. First we prove that BG is proper over k in some approximate sense.

Proposition 2.5.1. *Let G be a geometrically reductive group scheme over the field k. For each integer c, there exists a smooth k-scheme X, a smooth morphism $\phi : X \to BG$, and an open immersion $j : X \to \overline{X}$ such that*

(i) *$\overline{X}$ is a projective k-scheme,*
(ii) *for every infinite field K and every morphism Spec $K \to BG$, there exists a lift Spec $K \to X$ under ϕ.*
(iii) *$\overline{X} - X$ has codimension $\geq c$,*

The proof uses geometric invariant theory to construct $X \subset \bar{X}$. With more care it may be possible to remove the assumption that K is infinite from (ii).

Proof. Step 1. A "nice" projective representation. By definition G is a linear group scheme. Let V be a finite dimensional k-vector space, and let $\rho' : G \to \mathrm{GL}(V)$ be a closed immersion of group schemes. Consider $\rho : G \to \mathrm{SL}(V \oplus k \oplus k)$ defined by $\rho(g) = \mathrm{diag}(\rho'(g), \det(\rho'(g))^{-1}, 1)$ (diagonal blocks). Observe that the intersection of Image(ρ) and $\mathrm{G}_m\mathrm{Id}$ is the trivial group scheme. Thus, without loss of generality, assume ρ is a closed embedding of G into $\mathrm{SL}(V)$ such that Image(ρ) $\cap\ \mathrm{G}_m\mathrm{Id}$ is the trivial group scheme. In other words, the induced morphism of group schemes $\mathbb{P}\rho : G \to \mathrm{PGL}(V)$ is a closed immmersion.

Step 2. Making the "non-free" locus have codimension $\geq c$. Denote the dimension of V by $n > 1$. Let W be a finite-dimensional k-vector space of dimension c. Denote by H the finite-dimensional k-vector space $\mathrm{Hom}(W, \mathrm{Hom}(V, V))$. There is a linear action $\sigma : \mathrm{GL}(V) \times H \to H$, where an element $g \in \mathrm{GL}(V)$ acts on a linear map $h : W \to \mathrm{Hom}(V, V)$ by $\sigma(g, h)(w) = g \circ h(w)$. This restricts to a linear action of G on H.

Step 3. The GIT quotient. The linear action of G on H determines an action of G on the projective space $\mathbb{P}H$ of lines in H. It comes with a natural linearization of the invertible sheaf $L := \mathcal{O}_{\mathbb{P}H}(1)$ so that the action of G on $H^0(\mathbb{P}H, \mathcal{O}(1)) = \mathrm{Hom}(H, k)$ is the dual of ρ. Denote by $\mathbb{P}H^{\mathrm{ss}}$, resp. $\mathbb{P}H^{\mathrm{s}}_{(0)}$, the semistable, resp. properly stable, locus for the action of G on the pair $(\mathbb{P}H, L)$. Denote by $\overline{X}$ the uniform categorical quotient $\mathbb{P}H^{\mathrm{ss}} /\!\!/ G$ and denote by $p : \mathbb{P}H^{\mathrm{ss}} \to \overline{X}$ the quotient morphism. These exist by [MFK94, Thm. 1.10, App. 1.A, App. 1.C]. By the remark on [MFK94, p. 40], $\overline{X}$ is projective. Also, some power of L is the pullback under p of an ample invertible sheaf on $\overline{X}$. Thus (i) is satisfied for $\overline{X}$.

Step 4. A large open subset of $\mathbb{P}H^{\mathrm{ss}}_{(0)}$ which is a G-torsor. For every element $w \in W - \{0\}$, define F_w to be the homogeneous, degree n polynomial on H defined by $F_w(h) = \det(h(w))$. For every $g \in \mathrm{SL}(V)$,

$$\begin{aligned} F_w(\sigma(g, h)) &= \det(\sigma(g, h)(w)) = \det(gh(w)) \\ &= \det(g)\det(h(w)) = \det(h(w)) = F_w(h). \end{aligned}$$

Thus F_w is invariant for the action of $\mathrm{SL}(V)$. Thinking of F_w as an element of $\Gamma(\mathbb{P}H, \mathcal{O}(n))$ it is invariant for the action of G. Denote by $H_w \subset H$, resp. $\mathbb{P}H_w \subset \mathbb{P}H$, the open complement of the zero locus of F_w. By what was said above, $\mathbb{P}H_w$ is contained in $\mathbb{P}H^{\mathrm{ss}}$. The next step is to prove that $\mathbb{P}H_w$ is contained in $\mathbb{P}H^{\mathrm{s}}_{(0)}$, and, in fact, the geometric quotient $\mathbb{P}H_w \to \mathbb{P}H_w/G$ is a G-torsor.

Let W' be a subspace of W such that $W = \mathrm{span}(w) \oplus W'$. Denote by $H' \subset H$ the subspace $H' = \mathrm{Hom}(W', \mathrm{Hom}(V, V))$. There is a morphism

$$q_w : H_w \to \mathrm{GL}(V) \times H', \quad h \mapsto (h(w), h(w)^{-1} h|_{W'}).$$

The morphism q_w is $\mathrm{GL}(V)$-equivariant if we act on $\mathrm{GL}(V) \times H'$ on the first factor only. There is an inverse morphism

$$r_w : \mathrm{GL}(V) \times H' \to H_w$$

sending a pair (g, h') to the unique linear map $W \to \mathrm{Hom}(V, V)$ such that $w \mapsto g$ and $w' \mapsto gh'(w')$ for every $w' \in W'$. Thus, as a scheme with a left $\mathrm{GL}(V)$-action, H_w is isomorphic to $\mathrm{GL}(V) \times H'$. For the same reason, as a scheme with a $\mathrm{PGL}(V)$-action, $\mathbb{P}H_w$ is isomorphic to $\mathrm{PGL}(V) \times H'$. Thus the categorical quotient of $\mathbb{P}H_w$ by the action of G is the induced morphism $\mathbb{P}H_w \to (\mathrm{PGL}(V)/G) \times H'$. Now the categorical quotient $\mathrm{PGL}(V) \to \mathrm{PGL}(V)/G$, which is also a geometric quotient, is a G-torsor, see [ABD+65, Exposé $\mathrm{VI_A}$, Théorème 3.2] or [MFK94, Proposition 0.9]. Thus also the categorical quotient

$\mathbb{P}H_w \to (\mathrm{PGL}(V)/G) \times H'$ is a G-torsor. In particular, the action of G on $\mathbb{P}H_w$ is proper and free so that $\mathbb{P}H_w$ is contained in $\mathbb{P}H^{\mathrm{ss}}_{(0)}$.

Denote $U = \bigcup \mathbb{P}H_w$, where the union is over all $w \in W - \{0\}$. This is a G-invariant open subscheme of $\mathbb{P}H^{\mathrm{s}}_{(0)}$. Therefore there exists a unique open subscheme $X \subset \overline{X}$ such that $p^{-1}(X) = U$. By the last paragraph, $p : \mathcal{T} \to X$ is a G-torsor. Since U is smooth and p is flat, by [Gro67, §6.5] also X is smooth.

Step 5. Lifting K-valued points of BG to X, K infinite. Associated to the G-torsor U over X, there is a 1-morphism $\phi : X \to BG$. There are also morphisms of stacks $[H/G] \to BG$ and $[\mathbb{P}H/G] \to BG$ because $BG = [\mathrm{Spec}\ k/G]$. By construction, X is 2-equivalent to an open substack of $[\mathbb{P}H/G]$ as a stack over BG. The morphism $[\mathbb{P}H/G] \to BG$ is smooth, since $\mathbb{P}H$ is smooth. Hence $X \to BG$ is smooth. For every field K and 1-morphism $\mathrm{Spec}\ K \to BG$, the 2-fibered product $\mathrm{Spec}\ K \times_{BG} [H/G]$ is a K-vector space, and $\mathrm{Spec}\ K \times_{BG} [\mathbb{P}H/G]$ is the associated projective space. Thus $\mathrm{Spec}\ K \times_{BG} [\mathbb{P}H/G] \cong \mathbb{P}^{cn^2-1}$. Finally, $\mathrm{Spec}\ K \times_{BG} X$ is a nonempty open subscheme of $\mathrm{Spec}\ K \times_{BG} [\mathbb{P}H/G]$. Since K is infinite every dense open subset of $\mathbb{P}^{dn^2-1}_K$ contains a K-point. This proves (ii).

Step 6. The codimension of $\overline{X} - X$ is large. Finally, the codimension of $\overline{X} - X$ is at least as large as the codimension of $\mathbb{P}H - U$. Choosing a basis $(w_1, \dots, w_d)$ for W, $\mathbb{P}H - U$ is contained in the common zero locus of $F_{w_1}, \dots, F_{w_c}$, which clearly has codimension c. Therefore $\overline{X} - X$ has codimension at least c in $\overline{X}$. This proves (iii). □

Corollary 2.5.2. *Let the field k and the group scheme G be as in Proposition 2.5.1. Let R be a DVR containing k with residue field K. Let U be a finite type, integral K-scheme, and let $\mathcal{T}_U \to U$ be a G-torsor. There exists a triple $(Y \to \mathrm{Spec}\ R, \mathcal{T} \to Y, j : Y_K \to U)$ as in Corollary 2.4.2 with the additional property that the generic fiber of Y is projective.*

Proof. We may assume that $\dim U > 0$. Let c be an integer larger than $\dim(U)$. Let $(\phi : X \to BG, X \subset \overline{X})$ be as in Proposition 2.5.1. The torsor $\mathcal{T}_U$ corresponds to a 1-morphism $U \to BG$. By condition (ii), the base-change morphism $\mathrm{Spec}\ K(U) \to BG$ lifts to a morphism $\mathrm{Spec}\ K(U) \to X$. (Note that $K(U)$ is infinite since $\dim U > 0$.) After replacing U by a dense open subscheme, this comes from a morphism $f : U \to X$ lifting $U \to BG$. Also, replace U by an open subscheme that is quasi-projective, say a nonempty open affine. Then for some positive integer N, there is a locally closed immersion of K-schemes, $f' : U \to (X \times \mathbb{P}^N_k)_K$ such $\mathrm{pr}_X \circ f'$ equals f. Denote by m the codimension of $f'(U)$ in $(X \times \mathbb{P}^N_k)_K$.

The scheme $(\overline{X} \times \mathbb{P}^N_k)_R$ is flat and projective over $\mathrm{Spec}\ R$. Choose a closed immersion in $\mathbb{P}^M_R$ for some positive integer M. As in the proof of 2.4.2 we will use that the scheme U is a local complete intersection at a general point, and we will use that X is smooth over k. This implies that $f'(U)$ is dense in a component of a complete intersection of $(\overline{X} \times \mathbb{P}^N_k)_K$ in $\mathbb{P}^M_K$. More precisely, for some positive integer e, there exist homogoneous, degree e polynomials $F_1, \dots, F_m$ on $\mathbb{P}^M_K$ such that the scheme $\overline{Y}_K := \mathrm{V}(F_1, \dots, F_m) \cap (\overline{X} \times \mathbb{P}^N_k)_K$ has pure dimension $\dim(U)$ and contains a nonempty open subscheme U' that is an open subscheme of $f'(U)$. Let $\widetilde{F}_1, \dots, \widetilde{F}_c$ be homogeneous, degree e polynomials on $\mathbb{P}^M_R$ such that for every $i = 1, \dots, m$,

$$(*) \qquad \widetilde{F}_i \equiv F_i \pmod{\mathfrak{m}_R}.$$

Denote by $\overline{Y}$ the zero scheme $\mathrm{V}(\widetilde{F}_1, \dots, \widetilde{F}_m) \cap (\overline{X} \times \mathbb{P}^N_k)_R$. Then $\overline{Y}$ is flat over Spec R by Grothendieck's lemma, see [Mat89, Corollary, p. 179]. The closed fiber of $\overline{Y}$ equals $\overline{Y}_K$. Moreover,

$$\dim((\overline{X} - X) \times \mathbb{P}^N_k) - m \leq \dim X - c + N - m = \dim f'(U) - c < 0.$$

It is easy to see that the set of all possible choices of $\widetilde{F}_i$ satisfying $(*)$ forms a Zariski dense set of points in the relevant vector space of degree e polynomials over the field of fractions $Q(R)$ of R. Thus the dimension count shows there exists a choice of $\widetilde{F}_1, \dots, \widetilde{F}_c$ such that $\overline{Y}_{Q(R)}$ does not intersect $\big((\overline{X} - X) \times \mathbb{P}^N_k\big)_{Q(R)}$. In other words, the generic fiber of $\overline{Y} \to \mathrm{Spec}\ R$ is contained in $(X \times \mathbb{P}^N_k)_{Q(R)}$.

Let η be a generic point of $\overline{Y}$ that specializes to the generic point of U'. Replace $\overline{Y}$ by the closure of η, so that now $\overline{Y}$ is integral. (Presumably, a suitable application of Bertini's theorem could be used to replace this step.) Then $\overline{Y}$ is an integral, flat, projective R-scheme, the closed fiber contains U' as an open subscheme, and the generic fiber is contained in Spec $R \times_{\mathrm{Spec}\ k} (X \times \mathbb{P}^N_k)$. Define

$$Y = \overline{Y} - \big(\overline{Y} \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ K - U'\big).$$

This is an integral, flat, quasi-projective R-scheme whose generic fiber is projective. Moreover, Y_K equals U', which admits a dense, open immersion in S. Finally, the projection $\mathrm{pr}_X : Y \to X$, and the 1-morphism $\phi \circ \mathrm{pr}_X : Y \to BG$ determine a G-torsor $\mathcal{T}$ over Y. By construction, the restriction of this G-torsor to U' is isomorphic to the pullback of $\mathcal{T}_U$ by the open immersion, as desired. $\square$

Remark 2.5.3. We remark that we did not claim that the generic fibre of $Y \to$ Spec (R) is geometrically irreducible. Since X is smooth and geometrically irreducible over k, it seems that with a careful choice of the $\widetilde{F}_i$ and some additional arguments one can obtain this property as well having $Y_{Q(R)}$ smooth over $Q(R)$.

Next we deduce a corollary to help prove Theorem 2.2.3. Let k be an algebraically closed field, and let $(V, \mathcal{L})$ be a pair of a projective k-scheme and an ample invertible sheaf. Denote by G/k the group scheme $G = \mathrm{Aut}(V, \mathcal{L})$. Let $(K/k, X \to S, \mathcal{N})$ be as in Situation 2.2.1. Denote by G°_{red} the reduced, connected component of the identity of G.

Corollary 2.5.4. *Notations as above. Let R be a DVR containing k and with residue field K. If G°_{red} is reductive, there exists an integral, flat, quasi-projective R-scheme Y, a projective, flat morphism $f : \widetilde{X} \to Y$, an invertible sheaf $\widetilde{\mathcal{N}}$ on X, and an open immersion $j : Y_K \to S$ such that:*

(i) *every geometric fiber of $(\widetilde{X}, \widetilde{\mathcal{N}})$ over Y equals the base-change of $(V, \mathcal{L})$,*
(ii) *the restriction of $(\widetilde{X}, \widetilde{\mathcal{N}})$ to Y_K is isomorphic to the pullback of $j^*(X, \mathcal{N})$, and*
(iii) *the generic fiber of $Y \to$ Spec R is projective.*

In particular, let S' be an irreducible component of the geometric generic fibre of $Y \to$ Spec R. Then $(\widetilde{X} \to S', \widetilde{\mathcal{N}})$ over R is a triple $(K'/k, X' \to S', \mathcal{N}')$ with empty discriminant.

Proof. The hypothesis that G°_{red} is reductive implies that it is a geometrically reductive group scheme over k by a result of Haboush, see [Hab75] and [MFK94, Appendix 1.A, p. 191]. Note that G°_{red} is a closed normal subgroup scheme of G and that the quotient G/G°_{red} is a finite group scheme. A finite group scheme over k is geometrically reductive, and an extension of geometrically reductive group schemes is reductive, see [Fog69, Exercise, p. 189 and Lemma 5.57, p. 193]. Hence G is geometrically reductive. Thus the result of this Corollary follows from Corollary 2.5.2 above by applying the bijective correspondence of Conclusion 2.3.3. □

Proof of Theorem 2.2.3. Let us start with an arbitrary triple $(K/k, X \to S, \mathcal{N})$. Let $R = K[[t]]$. So R is Henselian, contains k and has residue field K. Let $\widetilde{X} \to Y \to$ Spec R and $\widetilde{\mathcal{N}}$ be as in Corollary 2.5.4. Denote by Ω/k an algebraic closure of the field of fractions $Q(R)$ of R. Let S' be any irreducible component of Y_Ω and let $X' = \widetilde{X}|_{S'}$, $\mathcal{N}' = \widetilde{\mathcal{N}}|_{S'}$. Thus $(\Omega/k, X' \to S', \mathcal{N}')$ is a triple as in Situation 2.2.1. By construction, this has empty discriminant. By hypothesis, the generic fiber of $X' \to S'$ has a $K'(S')$-point. At this point we apply Lemma 2.4.4 to conclude. □

3. Simple applications

As mentioned in the introduction, our main application of these results is to homogeneous spaces over fraction fields, which will appear in a forthcoming article. But in this section we want to indicate some simple applications of Theorem 2.2.3.

3.1. Fermat Hypersurfaces. As a first case we take V a Fermat hypersurface of degree d in $\mathbb{P}^{d^2-1}$

$$V : T_0^d + T_1^d + \dots T_{d^2-1}^d = 0,$$

with $\mathcal{L} = \mathcal{O}_V(1)$, say over the complex numbers $\mathbf{C}$. In this case the group scheme G is an extension of a finite group by $\mathbf{G}_m$ so certainly reductive. Consider the following family with general fibre $(V, \mathcal{L})$ over $\mathbb{P}^2$:

$$(*) \qquad \sum_{0 \leq i,j \leq d-1} X^i Y^j Z^{2d-2-i-j} T_{i+dj}^d = 0,$$

We learned about this family in personal communication with Tom Graber. This family does not have a rational point over $k(\mathbb{P}^2)$. The reader may enjoy finding an elementary proof of this by looking at what it means to have a polynomial solution to the above. We conclude from Theorem 3.1 that there is a smooth projective family over a projective surface with *every* fibre isomorphic to $(V, \mathcal{L})$, without a rational section. We like this example because it is not immediately obvious how to write one down explicitly.

There is another reason why the family given by $(*)$ is interesting. Tsen's theorem asserts that, if $n \geq d^2$ then any degree d hypersurface $X \subset \mathbb{P}^n_F$, where F is the function field of a surface has a rational point. The authors of this paper wonder what the obstruction to the existence of a rational point is in the boundary case, namely degree d in $\mathbb{P}^{d^2-1}$. One guess is that it is a Brauer class, i.e., an element α in the Brauer group of F such that for finite extensions F'/F one has: $X(F') \neq \emptyset \Leftrightarrow \alpha|_{F'} = 0$. However, the example above shows that this is not the case.

Namely, in our example $F = \mathrm{C}(x, y)$ where $x = X/Z$ and $y = Y/Z$. Anand Depokar pointed out that $(*)$ obtains a rational point over $F(\xi)$ where ξ is a dth root of a nonzero polynomial of the form

$$f(x, y) = - \sum_{0 \leq i,j \leq d-1, (i,j) \neq (0,0)} a_{i,j} x^i y^j.$$

(Just take $T_0 = \xi$ and $T_{i+jd} = a_{i,j}^{1/d}$.) Let $C \subset \mathbb{P}^2$ be an irreducible curve, not the line at infinity $Z = 0$. Suppose that α ramifies along C. The ramification data gives a cyclic extension $\mathrm{C}(C) \subset \mathrm{C}(C)[g^{1/d'}]$ of degree d', where $1 < d'|d$. There is a choice of of $a_{i,j}$ such that the rational function $f(x, y)$ restricts to a rational function on C such that both $f|_C$ and $g^{-1}f|_C$ are not d'th powers. (Left to the reader.) Thus the pullback of α to F' is still ramified along the pullback of C to the surface whose function field is $\mathrm{C}(x, y)(\xi)$. Contradiction. Hence C does not exist. However, the only Brauer class on $\mathbb{P}^2$ ramified along a single line is 0.

3.2. Projective spaces. Another case is where we take the pair $(V, \mathcal{L})$ to be $(\mathbf{P}^n, \mathcal{O}(n+1))$. Note that $\mathcal{O}(n+1) = \omega_{\mathbf{P}^n}^{-1}$ so the families in question are canonically polarized, and we are just talking about the problem of having nontrivial families of Brauer-Severi varieties. In particular, our theorem reduces the problem of proving the nullity of the Brauer group of a curve to the problem of proving the nonexistence of Brauer-Severi varieties having no rational sections over projective nonsingular curves. As far as we know this is not really helpful, since the proof of Tsen's theorem is pretty straigthforward anyway. However, it illustrates the idea!

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