Interface and mixed boundary value problems on n-dimensional polyhedral domains

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Abstract. Let $\mu \in \mathbb{Z}_+$ be arbitrary. We prove a well-posedness result for mixed boundary value/interface problems of second-order, positive, strongly elliptic operators in weighted Sobolev spaces $\mathcal{K}^\mu_a(\Omega)$ on a bounded, curvilinear polyhedral domain Ω in a manifold M of dimension n. The typical weight η that we consider is the (smoothed) distance to the set of singular boundary points of $\partial\Omega$. Our model problem is $Pu := -\operatorname{div}(A\nabla u) = f$, in Ω, $u = 0$ on $\partial_D\Omega$, and $D^P_\nu u = 0$ on $\partial_\nu\Omega$, where the function $A \geq \epsilon > 0$ is piece-wise smooth on the polyhedral decomposition $\bar{\Omega} = \cup_j \bar{\Omega}_j$, and $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$ is a decomposition of the boundary into polyhedral subsets corresponding, respectively, to Dirichlet and Neumann boundary conditions. If there are no interfaces and no adjacent faces with Neumann boundary conditions, our main result gives an isomorphism $P : \mathcal{K}^{\mu+1}_{a+1}(\Omega) \cap \{u = 0 \text{ on } \partial_D\Omega,\ D^P_\nu u = 0 \text{ on } \partial_N\Omega\} \to \mathcal{K}^{\mu-1}_{a-1}(\Omega)$ for $\mu \geq 0$ and $|a| < \eta$, for some $\eta > 0$ that depends on Ω and P but not on μ. If interfaces are present, then we only obtain regularity on each subdomain Ω_j. Unlike in the case of the usual Sobolev spaces, μ can be arbitrarily large, which is useful in certain applications. An important step in our proof is a *regularity* result, which holds for general strongly elliptic operators that are not necessarily positive. The regularity result is based, in turn, on a study of the geometry of our polyhedral domain when endowed with the metric $(dx/\eta)^2$, where η is the weight (the smoothed distance to the singular set). The well-posedness result applies to positive operators, provided the interfaces are smooth and there are no adjacent faces with Neumann boundary conditions.

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Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set. Consider the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g, & \text{on } \Omega, \end{cases} \tag{1}$$

where Δ is the Laplace operator. For Ω smooth, this boundary value problem has a unique solution $u \in H^{s+2}(\Omega)$ depending continuously on $f \in H^s(\Omega)$ and $g \in H^{s+3/2}(\partial\Omega)$, $s \geq 0$. See the books of Evans [25], Lions and Magenes [49], or Taylor [72] for proofs of this basic and well known result.

It is also well known that this result does not extend to non-smooth domains Ω. For instance, Jerison and Kenig prove in [35] that if $g = 0$ and $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is an open, bounded set such that $\partial\Omega$ is Lipschitz, then Equation (1) has a unique solution in $W^{s,p}(\Omega)$ depending continuously on $f \in W^{s-2,p}(\Omega)$ if, and only if, $(1/p, s)$ belongs to a certain explicit hexagon. They also prove a similar result if $\Omega \subset \mathbb{R}^2$. A consequence of this result is that the smoothness of the solution u (measured by the order s of the Sobolev space $W^{s,p}(\Omega)$ containing it) will not exceed, in general, a certain bound that depends on the domain Ω and p, even if f is smooth.

In addition to the Jerison and Kenig paper mentioned above, a deep analysis of the difficulties that arise for $\partial\Omega$ Lipschitz is contained in the papers of Babuška [4], Baouendi and Sjöstrand [9], Băcuţă, Bramble, and Xu [14], Babuška and Guo [31, 30], Brown and Ott [13], Jerison and Kenig [33, 34], Kenig [38], Kenig and Toro [39], Koskela, Koskela and Zhong [43, 44], Mitrea and Taylor [58, 60, 61], Verchota [73], and others (see the references in the aforementioned papers). Results more specific to curvilinear polyhedral domains are contained in the papers of Costabel [17], Dauge [19], Elschner [20, 21], Kondratiev [41, 42], Mazya and Rossmann [54], Rossmann [63] and others. Excellent references are also the monographs of Grisvard [27, 28] as well as the recent books [45, 46, 52, 53, 62], where more references can be found.

In this paper, we consider the boundary value problem (1) when Ω is a *bounded curvilinear polyhedral domain* in $\mathbb{R}^n$, or, more generally, in a manifold M of dimension n and, Poisson's equation $\Delta u = f$ is replaced by a positive, strongly elliptic scalar equation. We define curvilinear polyhedral domains inductively in Section 2. We allow polyhedral domains to be disconnected for technical reasons, more precisely, for the purpose of defining them inductively. Our results, however, are formulated for connected polyhedral domains. Many polyhedral domains are Lipschitz domains, but not all. This fact is discussed in detail by Vogel and Verchota in [74], where they also prove that the harmonic measure is absolutely continuous with respect to the Lebesgue measure on the boundary as well as the solvability of Equation (1) if $f = 0$ and $g \in L^{2-\epsilon}(\partial\Omega)$, thus generalizing several earlier, classical results. See also the excellent book [50]. The generalized polyhedra we considered are of combinatorial type if no cracks are present. (For a discussion of more general domains, see the references [68, 74, 75].)

Instead of working with the usual Sobolev spaces, as in several of the papers mentioned above, we shall work in some weighted analogues of these papers. Let $\Omega^{(n-2)} \subset \partial\Omega$ be the set of singular (or non-smooth) boundary points of Ω, that is, the set of points $p \in \partial\Omega$ such $\partial\Omega$ is not smooth in a neighborhood of p. We shall denote by $\eta_{n-2}(x)$ the distance from a point $x \in \Omega$ to the set $\Omega^{(n-2)}$. We agree to take $\eta_{n-2} = 1$ if there are no such points, that is, if $\partial\Omega$ is smooth. We then consider the weighted Sobolev spaces

$$\mathcal{K}_a^\mu(\Omega) = \{u \in L^2_{\mathrm{loc}}(\Omega),\, \eta_{n-2}^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ \text{for all } |\alpha| \le \mu\}, \quad \mu \in \mathbb{Z}_+, \quad (2)$$

which we endow with the induced Hilbert space norm. A similar definition yields the weighted Sobolev spaces $\mathcal{K}_a^s(\partial\Omega)$, $s \in \mathbb{R}_+$. By including an extra weight h in the above spaces we obtain the spaces $h\mathcal{K}_a^\mu(\Omega)$ and $h\mathcal{K}_a^s(\partial\Omega)$ (where h is required to be an admissible weight, see Definition 3.8 and Subsection 5.1). These spaces are closely related to the weighted Sobolev spaces on non-compact manifolds considered, for example in the references [41, 42, 46, 54, 62, 63] mentioned above, as well as in the works of Erkip and Schrohe [22], Grubb [29], Schrohe [65], as well as the sequence of papers of Schrohe and Schulze [66, 67] concerning related results on boundary value problems on non-compact manifolds and, more generally, on the analysis on non-compact manifolds.

The main result of this article, Theorem 1.2 applies to operators with *piece-wise smooth coefficients*, such as $\operatorname{div} a\nabla u = f$, where a is allowed to have only jumps across the interface. A simplified version of that result, when formulated for the Laplace operator Δ on $\mathbb{R}^n$ with Dirichlet boundary conditions, reads as follows. In this theorem and throughout this paper, Ω will always denote an open set.

Theorem 0.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, curvilinear polyhedral domain and $\mu \in \mathbb{Z}_+$. Then there exists $\eta > 0$ such that the boundary value problem (1) has a unique solution $u \in \mathcal{K}_{a+1}^{\mu+1}(\Omega)$ for any $f \in \mathcal{K}_{a-1}^{\mu-1}(\Omega)$, any $g \in \mathcal{K}_{a+1/2}^{\mu+1/2}(\partial\Omega)$, and any $|a| < \eta$. This solution depends continuously on f and g. If $a = \mu = 0$, this solution is the solution of the associated variational problem.*

The case $n = 2$ of the above theorem is Theorem 6.6.1 in the excellent monograph [45]. Results in higher dimensions related to the ones in our paper can be found, for instance, in [19, 45, 51, 54, 62]. These works also use the framework of the $\mathcal{K}_a^\mu(\Omega)$ spaces. The spaces $h\mathcal{K}_a^\mu(\Omega)$, with h an admissible weight are somewhat more general (see Definition 3.8 for a definition of admissible weights). We also take the dimension n of the ambient Euclidean space $\mathbb{R}^n \supset \Omega$ to be arbitrary. Furthermore, we impose mixed Dirichlet/Neumann boundary conditions and allow the boundary conditions to change along $(n-2)$-dimensional, piecewise smooth hypersurfaces in each hyperface of Ω. To handle this situation, we include in the singular set of Ω all points where the boundary conditions change, giving rise to a polyhedral structure on Ω which is not entirely determined by geometry, but also takes into account the specifics of the boundary value problem. However, we consider only second order, strongly elliptic systems. For $n = 3$, mixed boundary value problems for such systems in polyhedral domains were studied in weighted L^p spaces by Mazya and Rossmann [54] using point estimates for the Green's function [55]. Since we work in L^2-based spaces, we use instead coercive estimates, which directly generalize to arbitrary dimension and to transmission problems. We use manifolds in order to be able to prove estimates inductively. The method of layer potentials seems to give more precise results, but is less elementary (see for example [38, 59, 60, 75]). Solvability of mixed boundary value problems from the point of view of parametrices and pseudodifferential calculus can be found in the papers by Eskin [23, 24], Vishik and Eskin [76, 77, 78], and Boutet de Monvel [10, 11] among others.

As we have already pointed out, it is not possible to obtain full regularity in the usual Sobolev spaces, when the smoothness of the solution as measured by $\mu + 1$ in Theorem 0.1 is too large. On the other hand, the weighted Sobolev spaces have proved themselves to be as convenient as the usual Sobolev spaces in applications. Possible applications are to partial differential equations, algebraic geometry, representation theory, and other areas of pure and applied mathematics, as well as to other areas of science, such as continuum mechanics, quantum mechanics, and financial mathematics. See for example [7, 8, 48], where optimal rates of convergence were obtained for the Finite element method on 3D polyhedral domains and for 2D transmission problems using weighted Sobolev spaces.

The paper is organized as follows. In Section 1, we introduce the mixed boundary value/interface problem that we study, namely Equation (6), and state the main results of the paper, Theorem 1.1 on the regularity of (6) in weighted spaces of arbitrarily high Sobolev index, and Theorem 1.2 on the solvability of (6) under additional conditions on the operator (positivity) and on the domain (smooth interfaces and no two adjacent faces with Neumann boundary conditions). In Section 2, we give a notion of curvilinear, polyhedral domain in arbitrary dimension using induction, then we specialize to the familiar case of polygonal domains in $\mathbb{R}^2$ and polyhedral domains in $\mathbb{R}^3$, and describe the desingularization $\Sigma(\Omega)$ of the domain Ω in these familiar settings. Before discussing the desingularization in higher dimension, we recall briefly needed notions from the theory of Lie manifolds with boundary in Section 3. Then, in Section 4 we show that $\Sigma(\Omega)$, also defined by induction on the dimension, naturally carries a structure of Lie manifold with boundary. We also discuss the construction of the canonical weight function r_Ω, which extends smoothly to $\Sigma(\Omega)$ and is comparable to the distance to the singular set. In turn, the Lie manifold structure on $\Sigma(\Omega)$ allows to identify the spaces $\mathcal{K}^{\mu}_{a}(\Omega)$, $\mu \in \mathbb{Z}_+$, with standard Sobolev spaces on $\Sigma(\Omega)$, and hence lead to a definition of the weighted Sobolev spaces on the boundary $\mathcal{K}^{s}_{a}(\partial\Omega)$, $s \in \mathbb{R}$. Lastly Section 6 contains the proofs of the main results and most of lemmas of the paper; in particular, it contains a proof of the weighted Hardy-Poincaré inequality used to establish positivity or strict coercivity for the problem of Equation (6). An earlier version of this paper was first circulated as an IMA Preprint in August 2004.

We conclude this Introduction with a word on notation. By Ω we always mean an open set in $\mathbb{R}^n$. By a diffeomorphisms, we mean a C^∞ invertible map with C^∞ inverse. By C we shall denote a generic constant that may change from line to line. We also denote $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$.

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1 The problem and statement of the main results

We begin by introducing the class of differential operators and the associated mixed Dirichlet-Neumann boundary value/interface problem that will be the object of study. For simplicity, we consider primarily the scalar case, although our results extend to systems as well. Then, we state the main results of this article, namely the regularity and the well-posedness of the mixed boundary value/interface problem (6) in weighted Sobolev spaces for n-dimensional, curvilinear polyhedral domains $\Omega \subset \mathbb{R}^n$. These are stated in Theorems 1.2 and 1.1.

Our analysis is general enough to extend to a bounded subdomain $\Omega \subset M$ of a compact Riemannian manifold M. Initially the reader may assume the polyhedron is straight, that is, informally, that every j-dimensional component of the boundary, $j = 1, \ldots, n-1$ is a subset of an affine space. A complete definition of a curvilinear polyhedral domain is given in Section 2.

1.1 The differential operator P and the associated problem

Let us denote by $\Omega \subset \mathbb{R}^n$ a bounded, curvilinear stratified polyhedral domain (see Definition 2.1). The domain Ω need not be connected, nor convex. We assume that we are given a decomposition

$$\overline{\Omega} = \cup_{j=1}^{N} \overline{\Omega}_j, \tag{3}$$

where Ω_j are disjoint polyhedral subdomains. In particular, every face of Ω is also a face of one of the domains Ω_j. This is possible since the faces of Ω are not determined only by the geometry of Ω. As discussed in Section 4, a face of codimension 1 of Ω is called a hyperface. For well-posedness results, we shall assume that

$$\Gamma = \cup_{j=1}^{N} \partial\Omega_j \setminus \partial\Omega, \tag{4}$$

is a finite collection of disjoint, smooth $(n-1)$-hypersurfaces. We observe that, since each Ω_j is a polyhedron, each component of Γ intersects $\partial\Omega$ transversely. We refer to Γ as the *interface*.

We also assume that the boundary of Ω is partition into two *disjoint* subsets

$$\partial\Omega = \partial_D\Omega \cup \partial_N\Omega, \tag{5}$$

with $\partial_N\Omega$ consisting of a union of open faces of Ω. For well-posedness results, we shall assume that $\partial_N\Omega$ does not contain adjacent faces of $\partial\Omega$.

We are interested in studying the following mixed boundary value/interface problem for a certain class of elliptic, scalar operators P described below:

$$\begin{cases} \ Pu = f & \text{on } \Omega, \\ u|_{\partial_D\Omega} = g_D & \text{on } \partial_D\Omega, \\ D^P_\nu u|_{\partial_N\Omega} = g_N & \text{on } \partial_N\Omega, \\ u^+ = u^-, \ D^{P+}_\nu u = D^{P-}_\nu u & \text{on } \Gamma. \end{cases} \tag{6}$$

Above, ν is the unit outer normal to $\partial\Omega$, which is defined almost everywhere, D^P_ν is the conormal derivative associated to the operator P (see (10)), and $\pm$ refers to one-sided, non-tangential limits at the interface Γ. We observe that $D^{P\pm}_\nu$ is well-defined a.e. on each side of the interface Γ, since each smooth component of Γ is the boundary of exactly two adjacent polyhedral domains Ω_j, by (4). The coefficients of P will have in general jump discontinuities along Γ.

We next introduce the class of differential operators that we consider. At first, the reader may assume $P = -\Delta$, the Laplace operator. We shall write $Re(z) := \frac{1}{2}(z + \overline{z})$, or simply $Re\, z$ for the real part of a complex number z.

Let $u \in H^2_{\text{loc}}(\Omega)$. We shall study the following scalar, differential operator in divergence form

$$Pu(x) = -\sum_{j,k=1}^{n} \partial_j \left[A_{jk}(x) \partial_k u(x) \right] + \sum_{j=1}^{n} B_j(x) \partial_j u(x) + C(x) u(x). \tag{7}$$

The coefficients A_{jk}, B_j, C are real valued with only jump discontinuities on the interface Γ, the operator P is required to be uniformly strongly elliptic and to satisfy another positivity condition. More precisely, the coefficients of P are assumed to satisfy:

$$A_{jk},\, B_j,\, C \in \oplus_{j=1}^{N}\mathcal{C}^\infty(\overline{\Omega_j}) \cap L^\infty(\overline{\Omega}) \tag{8a}$$

$$Re\Big(\sum_{j,k=1}^{n} [A_{jk}(x)]\xi_j\overline{\xi}_k\Big) \geq \epsilon \sum_{j=1}^{n} |\xi_j|^2, \quad \forall \xi_j \in \mathbb{C},\ \forall x \in \Omega, \quad \text{and} \tag{8b}$$

$$2C(x) - \sum_{j=1}^{n} \partial_j B_j(x) \geq 0, \tag{8c}$$

for some $\epsilon > 0$.
For scalar equations, one may weaken the uniform strong ellipticity condition (8b), but this is not needed for our purposes. Our results extend to systems satisfying the *strong Legendre–Hadamard* condition, namely

$$Re\Big(\sum_{j,k=1}^{n}\sum_{p,q=1}^{m} [A_{jk}(x)]_{pq}\xi_{jp}\overline{\xi}_{kq}\Big) \geq \epsilon \sum_{j=1}^{n}\sum_{p=1}^{m} |\xi_{jp}|^2, \quad \forall \xi_{jp} \in \mathbb{C}, \tag{9}$$

and a condition on the lower-order terms equivalent to (8c). This condition is not satisfied however by the system of anisotropic elasticity in $\mathbb{R}^3$, for which nevertheless the well-posedness result holds if the elasticity tensor is positive definite on symmetric matrices [56].
In (8a), the "regularity condition on the coefficients of P" means that the coefficients and their derivatives of all orders have well-defined limits from each side of Γ, but as equivalence classes in L^∞ they may have jump discontinuities along the interface. This condition can be relaxed, but it allows us to state a regularity result of arbitrary order in each subdomains for the solution to the problem (6). The conormal derivative associated to the operator P is formally defined by:

$$D_P^\nu u(x) = \sum_{i,j=1}^{n} \nu_i\, A_{ij}\partial_j u(x), \tag{10}$$

where $\nu = (\nu^i)$ is the unit outer normal vector to the boundary of Ω. We give meaning to (10) in the sense of trace at the boundary. In particular, for u regular enough $D_\nu^P u$ is defined almost everywhere on the boundary as a non-tangential limit.
The problem (6) with $g_D = 0$ is interpreted in a weak (or variational) sense, using the bilinear form $B(u, v)$ defined by:

$$B(u,v) := \sum_{j,k=1}^{n} \big(A_{jk}\partial_k u, \partial_j v\big) + \sum_{j=1}^{n} \big(B_j \partial_j u, v\big) + \big(Cu, v\big), \tag{11}$$

which is well-defined for any $u, v \in H^1(\Omega)$. Then, (6) is weakly equivalent to

$$B(u,v) = (f,v)_{L^2(\Omega)} + (g_N, v)_{\partial_N\Omega}, \tag{12}$$

where the second parenthesis denotes the pairing between a distribution and a (suitable) function. The jump or *transmission* conditions, $u^+ = u^-$, $D^P_\nu u^+ = D^P_\nu u^-$ at the interface Γ follow from the weak formulation and the H^1-regularity of weak solutions, and hence justify passing from the strong formulation (6) to the weak one (12). Otherwise, in general, the difference $D^{P+}_\nu u - D^{P-}_\nu u$ may be non-zero and may be included as a distributional term in f.

Condition (8c) implies the Hardy-Poincaré type inequality

$$ReB(u,u) > C(\eta_{n-2}u, \eta_{n-2}u)_{L^2}, \tag{13}$$

if there are no adjacent faces with Neumann boundary conditions and the interface is smooth. In fact, it is enough to assume that the latter is satisfied instead of (8c). For applications, however, it is more convenient to have the concrete condition (8c).

Problems of the form (6) arise in many applications. An important example is given by (linear) elastostatics. In this case, $[Pu]^i = -\sum_{jkl=1}^3 \partial_j\, C^{ijkl}\, \partial_k u^l$, $i = 1, 2, 3$, where C is the fourth-order elasticity tensor, modelling the response of an elastic body under small deformations. Dirichlet or displacement boundary conditions correspond to clamping (parts of) the boundary, while Neumann or traction boundary conditions correspond to loading mechanically (parts of) the boundary. Interfaces arise due to the use of different materials. A careful analysis of mixed Dirichlet/Neumann boundary value problems for linear elastostatics in 3-dimensional curvilinear, polyhedral domains, was carried out by two of the authors in [56]. There, the concept of a "domain with polyhedral structure" is more general than in this paper and includes cracks. In [48], they studied mixed boundary value/interface problems and the implementation of the Finite Element Method on "domains with polygonal structure" with non-smooth interfaces (see also [15]). The results of this paper can be extended to include domains with cracks, as in [56] and [48], but the topological machinery used there, including the notion of an "unfolded boundary" [19] in arbitrary dimensions is significantly more complex. (See [12] for related results.)

1.1.1 Operators on manifolds

We turn to consider the assumptions on P when the domain Ω is a curvilinear, polyhedral domain in a manifold M of the same dimension. Let then E be a vector bundle on M endowed with a hermitian metric. A coordinate free expression of the conditions in Equations (8a)–(8c) is obtained as follows. We assume that there exist a metric preserving connection $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$, a smooth endomorphism $A \in \mathrm{End}(E \otimes T^*M)$, and a first order differential operator $P_2 : \Gamma(E) \to \Gamma(E)$ with smooth coefficients such that

$$A + A^* \geq 2\epsilon I \text{ for some } \epsilon > 0. \tag{14}$$

Then we define $P_1 = \nabla^* A \nabla$ and $P = P_1 + P_2$. In particular, the operator P will satisfy the strong Lagrange–Hadamard condition in a neighborhood of Ω in M.

Note that if $\Omega \subset \mathbb{R}^n$ and the vector bundle E is trivial, then the condition of (14) reduces to the conditions of (8), by taking ∇ to be the trivial connection. We can allow A to have jump discontinuities as well along polyhedral interfaces.

1.2 The main results

We are ready to state the principal results of this paper. *We continue to assume hypotheses* (3)–(5) *on the domain* Ω *and its decomposition into disjoint subdomains* Ω_j *separated by the interface* Γ.

We begin with a regularity results for solutions to the problem (6) in weighted Sobolev spaces $h\mathcal{K}_a^\mu$, $\mu \in \mathbb{Z}_+$, $a \in \mathbb{R}$, where

$$\mathcal{K}_a^\mu(\Omega) := \{u \in L^2_{\mathrm{loc}}(\Omega),\, \eta_{n-2}^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ \text{for all } |\alpha| \leq \mu\}\,, \quad \mu \in \mathbb{Z}_+,$$

and

$$h\mathcal{K}_a^\mu(\Omega) := \{hu,\, u \in \mathcal{K}_a^\mu(\Omega)\}.$$

(See Section 5 for a detailed discussion and main properties of these spaces.) Above, η_{n-2} is the distance to the singular set in Ω given in Definition 2.5, while h is a so-called admissible weight described in Definition 3.8. Initially, the reader may assume that $h = r_\Omega^b$, $b \in \mathbb{R}$, where r_Ω is a function comparable to the distance function η_{n-2} close to the singular set, but with better regularity than η_{n-2} away from the singular set. (We refer again to Subsection 5.1 for more details). The weight h is important in the applications of the theory developed here for numerical methods, where appropriate choices of h yield quasi-optimal rates of convergence for the Finite Element approximation to the weak solution of the problem (6) (see [6, 7, 8, 48]).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, curvilinear polyhedral domain of dimension n. Assume that the operator P satisfies conditions* (8a) *and* (8b). *Let $\mu \in \mathbb{Z}_+$, $a \in \mathbb{R}$, and $u \in h\mathcal{K}^1_{a+1}(\Omega)$ be such that $Pu \in h\mathcal{K}^{\mu-1}_{a-1}(\Omega_j)$, for all j, $u|_{\partial_D\Omega} \in h\mathcal{K}^{\mu+1/2}_{a+1/2}(\partial_D\Omega)$, $D^P_\nu u|_{\partial_N\Omega} \in h\mathcal{K}^{\mu-1/2}_{a-1/2}(\partial_N\Omega)$. If h is an admissible weight, then $u \in h\mathcal{K}^{\mu+1}_{a+1}(\Omega_j)$, for all $j = 1, \dots, N$, and*

$$\begin{aligned}\|u\|_{h\mathcal{K}^{\mu+1}_{a+1}(\Omega_j)} \leq C\Big(\sum_{k=1}^N \|Pu\|_{h\mathcal{K}^{\mu-1}_{a-1}(\Omega_k)} + \|u\|_{h\mathcal{K}^0_{a+1}(\Omega)} +\\ \|u|_{\partial_D\Omega}\|_{h\mathcal{K}^{\mu+1/2}_{a+1/2}(\partial_D\Omega)} + \|u|_{\partial_N\Omega}\|_{h\mathcal{K}^{\mu-1/2}_{a-1/2}(\partial_N\Omega)}\Big)\end{aligned} \tag{15}$$

for a constant $C = C(\Omega, P, \mu, a, h) > 0$, independent of u.

The proof of the regularity theorem exploits Lie manifolds and their structure to reduce to the classical case of bounded, smooth domains. The proof can be found in Section 6. Note that in this theorem we do not require the interfaces to be smooth and we allow for adjacent faces with Neumann boundary conditions. Under additional conditions on the set Ω and its boundary ensuring strict coercivity of the bilinear form B of equation (11), we obtain a well-posedness result for problem (6). In [48], two of the authors obtained a well-posedness result in an augmented space on polygonal domains with "Neumann-Neumann vertices," *i.e.,* vertices for which both sides joining at the vertex are given Neumann boundary conditions, and for which the interface Γ is not smooth. Such result is based on specific spectral properties of operator pencils near the vertices and is not easily extendable to higher dimension. Note that $\mathcal{K}_a^\mu(\Omega) = h\mathcal{K}_0^\mu(\Omega)$ for a suitable admissible weight h and hence there is no loss of generality to assume $a = 0$ in Theorem 1.1. We will use the same reasoning to simplify the statements of the following results.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected curvilinear polyhedral domain of dimension n. Assume that $\partial\Omega_N$ does not contain any two adjacent hyperfaces, that $\partial_D\Omega$ is not empty, and that the interface Γ is smooth. In addition, assume that the operator P satisfies conditions* (8). *Let $\mathcal{W}_\mu(\Omega)$, $\mu \in \mathbb{Z}_+$, be the set of admissible weights h such that the map $\tilde{P}(u) := (Pu, u|_{\partial_D\Omega}, D_\nu^P u|_{\partial_N\Omega})$ establishes an isomorphism*

$$\tilde{P} : \{u \in \bigoplus_{j=1}^{N} h\mathcal{K}_1^{\mu+1}(\Omega_j) \cap h\mathcal{K}_1^1(\Omega),\ D_\nu^P u^+ = D_\nu^P u^- \text{ on } \Gamma\} \\ \to \bigoplus_{j=1}^{N} h\mathcal{K}_{-1}^{\mu-1}(\Omega_j) \oplus h\mathcal{K}_{1/2}^{\mu+1/2}(\partial_D\Omega) \oplus h\mathcal{K}_{-1/2}^{\mu-1/2}(\partial_N\Omega).$$

Then the set $\mathcal{W}_\mu(\Omega)$ is an open set containing 1.

Theorem 1.2 reduces to a well-known, classical result when Ω is a smooth bounded domain. (See Remark 6.11 for a result on smooth domains that is not classical.) The same is true for the following result, Theorem 1.3, which works for general domains on manifolds. Note however that for manifolds it is more difficult to express the coercive property, so for more complete results we restrict to the case of operators of Laplace type.

Theorem 1.3. *Let $\Omega \subset M$ be a bounded, connected curvilinear polyhedral domain of dimension n. Assume that every connected component of Ω has a non-empty boundary and that the operator P satisfies condition* (14). *Assume additionally that no two adjacent hyperfaces of $\partial\Omega$ are endowed with Neumann boundary conditions and that the interface Γ is smooth. Let $c \in \mathbb{C}$ and $\mathcal{W}'_\mu(\Omega)$ be the set of admissible weights h such that the map $\tilde{P}_c(u) := (Pu + cu, u|_{\partial\Omega}, D_\nu^P u|_{\partial\Omega})$ establishes an isomorphism*

$$\tilde{P}_c : \{u \in \bigoplus_{j=1}^{N} h\mathcal{K}_1^{\mu+1}(\Omega_j) \cap h\mathcal{K}_1^1(\Omega),\ u^+ = u^-,\ D_\nu^P u^+ = D_\nu^P u^- \text{ on } \Gamma\} \\ \to \bigoplus_{j=1}^{N} h\mathcal{K}_{-1}^{\mu-1}(\Omega_j) \oplus h\mathcal{K}_{1/2}^{\mu+1/2}(\partial_D\Omega) \oplus h\mathcal{K}_{-1/2}^{\mu-1/2}(\partial_N\Omega).$$

Then the set $\mathcal{W}'_\mu(\Omega)$ is an open set, which contains 1 *if the real part of c is large or if $P = \nabla^* A \nabla$ with A satisfying* (14).

For the rest of this section, Ω and P will be as in Theorem 1.2. We discuss some immediate consequences of Theorem 1.2. Analogous results can be obtained from Theorem 1.3, but we will not state them explicitly. The continuity of the inverse of $\tilde{P}$ is made explicit in the following corollary.

Corollary 1.4. *Let A satisfy* (14). *There exists a constant $C = C(\Omega, P, \mu, a, h) > 0$, independent of f, g_D, and g_N, such that*

$$\|u\|_{h\mathcal{K}^1_1(\Omega)} + \|u\|_{h\mathcal{K}^{\mu+1}_1(\Omega_j)} \leq C\,\Big(\sum_{j=1}^N \|Pu\|_{h\mathcal{K}^{\mu-1}_{-1}(\Omega_j)} + \|u|_{\partial_D\Omega}\|_{h\mathcal{K}^{\mu+1/2}_{1/2}(\partial_D\Omega)} + \|D^P_\nu u|_{\partial_N\Omega}\|_{h\mathcal{K}^{\mu+1/2}_{1/2}(\partial_N\Omega)} \Big),$$

for any $u \in h\mathcal{K}^1_1(\Omega)$ and any j.

From the fact that η_{n-2} is equivalent to r_Ω by Proposition 4.9 and Corollary 4.11, we obtain the following corollary.

Corollary 1.5. *Let A satisfy* (14). *There exists $\eta > 0$ such that*

$$(P, D^P_\nu) : \{u \in \bigoplus_{j=1}^N \mathcal{K}^{\mu+1}_{a+1}(\Omega_j) \cap \mathcal{K}^1_{a+1}(\Omega),\ u|_{\partial_D\Omega} = 0, \quad D^P_\nu u^+ = D^P_\nu u^- \ \text{on } \Gamma\} \to \bigoplus_{j=1}^N \mathcal{K}^{\mu-1}_{a-1}(\Omega_j) \oplus \mathcal{K}^{\mu-1/2}_{a-1/2}(\partial_N\Omega)$$

is an isomorphism for all $\mu \in \mathbb{Z}_+$ and all $|a| < \eta$.

Note above and in what follows that the interface matching condition $u^+ = u^-$ follows from $u \in \mathcal{K}^1_{a+1}(\Omega)$.

Proof. From the results in Sections 5 and 5.1, $\mathcal{K}^{\mu+1}_{a+1} = r^a_\Omega \mathcal{K}^{\mu+1}_1$ and r^a_Ω is an admissible weight for any $a \in \mathbb{R}$. The result then follows from the fact that $\mathcal{W}_\mu(\Omega)$ is an open set containing the weight 1 by Theorem 1.2. $\square$

The following corollary gives a characterization of the set $\mathcal{W}_\mu(\Omega)$ in the spirit of [15]. There, similar arguments give that for $n = 2$ the constant η in the previous corollary is $\eta = \pi/\alpha_M$, where α_M is the largest angle of Ω. See also [42].

Corollary 1.6. *Let $h = r^a_\Omega$ and A satisfy* (14). *Assume that for all $\lambda \in [0, 1]$ the map*

$$(P, D^P_\nu) : \{u \in h^\lambda \mathcal{K}^1_1(\Omega),\ u|_{\partial_D\Omega} = 0, \quad D^P_\nu u|_{\partial_N\Omega} = 0\} \to h^\lambda \mathcal{K}^{-1}_{-1}(\Omega)$$

is Fredholm. Then $h \in \mathcal{W}_\mu(\Omega)$.

The corollary holds for more general weights $h = \prod_H x_H^{a_H}$, where x_H is the distance to an hyperface H at infinity (see Section 5.1), as long as all $a_H \geq 0$ or all $a_H \leq 0$.

Proof. We proceed as in [15]. The family $P_\lambda := h^{-\lambda} P h^\lambda$ is continuous for $\lambda \in [0,1]$, consists of Fredholm operators by hypothesis, and is invertible for $\lambda = 0$ by Theorem 1.2. It follows that the family P_λ consists of Fredholm operators of index zero. To prove that these operators are isomorphisms, it is hence enough to prove that they are either injective or surjective. Assume first that $a \geq 0$ in the definition of h. Then $\mathcal{K}_1^{1+\lambda}(\Omega) = h^\lambda \mathcal{K}_1^1(\Omega) \subset \mathcal{K}_1^1(\Omega)$. Therefore P is injective on

$$h^\lambda \mathcal{K}_1^1(\Omega) \cap \{u|_{\partial_D \Omega} = 0,\ D_P^\nu u|_{\partial_N \Omega} = 0\}.$$

This, in turn, gives that P_λ is injective.
Assume that $a \leq 0$ and consider

$$P_\lambda : h^\lambda \mathcal{K}_1^1(\Omega) \cap \{u|_{\partial \Omega} = 0,\ D_P^\nu u|_{\partial_N \Omega} = 0\} \to h^\lambda \mathcal{K}_{-1}^{-1}(\Omega). \tag{16}$$

We have $(P_\lambda)^* = (P^*)_{-\lambda}$. The same argument as above shows that P_λ^* is injective, and hence that it is an isomorphism, for all $0 \leq \lambda \leq 1$. Hence P_λ is an isomorphism for all $0 \leq \lambda \leq 1$. □

2 Polyhedral domains

In this section we introduce the class of domains to which the results of the previous sections apply. We then specialize to domains in 2 and 3 dimensions and provide ample examples. The reader may at first concentrate on this case. We describe how to desingularize the domain in arbitrary dimension later in the paper, using the theory of Lie manifolds, which we recall in the next section.
Let Ω be a proper open set in $\mathbb{R}^n$ or more generally in a smooth manifold M of dimension n. Our main focus is the analysis of partial differential equations on Ω, specifically the mixed boundary value/interface problem (6). For this reason, we give Ω a structure that is not entirely determined by geometry, rather it takes into account the boundary and interface conditions for the operator P in problem (6).
We assume that Ω is given together with a smooth stratification:

$$\Omega^{(0)} \subset \Omega^{(1)} \subset \ldots \subset \Omega^{(n-2)} \subset \Omega^{(n-1)} := \partial\Omega \subset \Omega^{(n)} := \bar{\Omega}. \tag{17}$$

We recall that a *smooth stratification* $S_0 \subset S_1 \subset \ldots \subset X$ of a topological space X is an increasing sequence of closed sets $S_j = S_j(X)$ such that each point of X has a neighborhood that meets only finitely many of the sets S_j, S_0 is a discrete subset, $S_{j+1} \smallsetminus S_j$, $j \geq 0$, is a disjoint union of smooth manifolds of dimension $j+1$, and $X = \cup S_j$. Some of the sets S_j may be empty for $0 \leq j \leq j_0 < \dim(X)$.

We will always assume that the stratification $\{\Omega^{(j)}\}$ satisfies the condition that $\Omega^{(j)} \smallsetminus \Omega^{(j-1)}$ has finitely many connected components, for all j. This assumption is automatically satisfied if Ω is bounded, and it is not crucial, but simplifies some of the later constructions.

We proceed by induction on the dimension to define a polyhedral structure on Ω. Our definition is very closely related to that of Whitney stratified spaces [79]. We first agree that a curvilinear polyhedral domain of dimension $n = 0$ is simply a finite set of points. Then, we assume that we have defined curvilinear polyhedral domains in dimension $\leq n-1$, $n \geq 1$, and define a curvilinear polyhedral domain in a manifold M of dimension n next. We shall denote by B^l the open unit ball in $\mathbb{R}^l$ and by $S^{l-1} := \partial B^l$ its boundary. In particular, we identify $B^0 = \{1\}$, $B^1 = (-1, 1)$, and $S^0 = \{-1, 1\}$.

Definition 2.1. Let M be a smooth manifold of dimension $n \geq 1$. Let $\Omega \subset M$ be an open subset endowed with the stratification (17). Then $\Omega \subset M$ is a *stratified, curvilinear polyhedral domain* if for every point $p \in \partial\Omega$, there exist a neighborhood V_p in M such that:

(i) if $p \in \Omega^{(l)} \setminus \Omega^{(l-1)}$, $l = 1, \ldots, n-1$, there is a curvilinear polyhedral domain $\omega_p \subset S^{n-l-1}$, $\overline{\omega_p} \neq S^{n-l-1}$, and

(ii) a diffeomorphism $\phi_p : V_p \to B^{n-l} \times B^l$ such that $\phi_p(p) = 0$ and

$$\phi_p(\Omega \cap V_p) = \{rx',\ 0 < r < 1,\ x' \in \omega_p\} \times B^l, \tag{18}$$

inducing a homeomorphism of stratified spaces.

Given any $p \in \partial\Omega$, let $0 \leq \ell(p) \leq n-1$ be the smallest integer such that $p \in \Omega^{(\ell(p))}$, but $p \notin \Omega^{(\ell(p)-1)}$ (by convention we set $\Omega^{(l)} = \emptyset$ if $l < 0$). By construction, $\ell(p)$ is unique given p. Then, the domain $\omega_p \subset S^{n-\ell(p)-1}$ in the definition above will be called the *link of* Ω *at* p. We identify the "ball" $B^0 = \{1\}$ and the "sphere" $S^0 = \partial B^1 = \{-1, 1\}$. In particular if $\ell(p) = n-1$, then ω_p is a point.

The notion of a stratified polyhedron is well known in the literature (see for example the monograph [71]). However, our definition is more general, and well suited for applications to partial differential equations. See the papers of Babuška and Guo [5], Mazya and Rossmann [54], and Verchota and Vogel [74, 75] for related definitions. We remark that, according to the above definition, Ω does not need to be bounded, nor connected, nor convex. *For applications to the analysis of boundary value/interface problems, however, we will always assume* Ω *is connected.* The boundary $\partial\Omega$ need not be connected either, but it does have finitely many connected components. We also stress that polyhedral domains will always be open subsets.

The condition $\overline{\omega_p} \neq S^{n-l-1}$ can be relaxed to $\omega_p \neq S^{n-l-1}$, thus allowing for cracks and slits, but not punctured domains of the form $M \smallsetminus \{p\}$. We will not pursue this generality in the paper, given also that submanifolds of codimension greater than 2 consists of irregular boundary points for elliptic equations and

may lead to ill-posedness in boundary value problems. We refer to the articles [48, 56] for a detailed analysis of polyhedral domains with cracks in 2 and 3 dimensions.
We continue with some comments on Definition 2.1 before providing several concrete examples in dimension $n = 1, 2, 3$. We denote by tB^l the ball of radius t in $\mathbb{R}^l$, $l \in \mathbb{N}$, centered at the origin. We also let tB^0 to be a point independent of t. Sometimes it is convenient to replace Condition (18) with the equivalent condition that there exist $t > 0$ such that

$$\phi_p(\Omega \cap V_p) = \{rx',\, 0 < r < t, x' \in \omega_p\} \times tB^l. \tag{19}$$

We shall interchange conditions (18) and (19) at will from now on. For a cone or an infinite wedge, $t = +\infty$, so cones and wedges are particular examples of polyhedral domain.
We have the following simple result that is an immediate consequence of the definitions.

Proposition 2.2. *Let $\psi : M \to M'$ be a diffeomorphism and let $\Omega \subset M$ be a curvilinear polyhedral domain. Then $\psi(\Omega)$ is also a curvilinear polyhedral domain.*

Next, we introduce the *singular set* of Ω, $\Omega_{\text{sing}} := \Omega^{(n-2)}$. A point $p \in \Omega^{(n-2)}$ will be called a *singular point* for Ω. We recall that a point $x \in \partial\Omega$ is called a *smooth boundary point* of Ω if the intersection of $\partial\Omega$ with a small neighborhood of p is a smooth manifold of dimension $n - 1$. In view of Definition 2.1, the point p is smooth if ϕ_p satisfies

$$\phi_p(\Omega \cap V_p) = (0, t) \times B^{n-1}. \tag{20}$$

This observation is consistent with ω_p being a point in this case, since it is a polyhedral domain of dimension 0.
Any point $p \in \partial\Omega$ that is not a smooth boundary point in this sense is a singular point. But the singular set may include other points as well, in particular the points where the boundary conditions change, *i.e.,* the points of the boundary of $\partial_D\Omega$ in $\partial\Omega$, and the points where the interface Γ meets $\partial\Omega$. It is known [36, 37] that the solution to the problem (6) near such points behaves in a similar way as in the neighborhood of non-smooth boundary points. We call the non-smooth points in $\partial\Omega$ the *true* or *geometric* singular points, while we call all the other singular points *artificial* singular points.
The true singular points can be characterized by the condition that the domain ω_p of Definition 2.1 be an "irreducible" subset of the sphere S^{n-l-1}, in the sense of the following definition.

Definition 2.3. A subset $\omega \subset S^{n-1} := \partial B^n$, the unit sphere in $\mathbb{R}^n$ will be called *irreducible* if $\mathbb{R}_+\omega := \{rx',\, r > 0, x' \in \omega\}$ cannot be written as $V + V'$ for a linear subspace $V \subset \mathbb{R}^k$ of dimension ≥ 1 and V' an arbitrary subset of $\mathbb{R}^{n-k}$. (The sum does not have to be a direct sum and, in fact, V' is not assumed to be an affine subspace.)

For example, $(0,\alpha) \subset S^1$ is irreducible if, and only if, $\alpha \neq \pi$. A subset $\omega \subset S^{n-1}$ strictly contained in an open half-space is irreducible, but the intersection of S^{n-1}, $n \geq 2$, with an open half-space is not irreducible.
If $p \in \Omega^{(0)}$, then we shall call p a *vertex* of Ω and we shall interpret the condition (18) as saying that ϕ_p defines a diffeomorphism such that

$$\phi_p(\Omega \cap V_p) = \{rx',\, 0 < r < t, x' \in \omega_p\}. \tag{21}$$

This interpretation is consistent with our convention that the set B^0 (the zero dimensional unit ball) consists of a single point. We shall call any open, connected component of $\Omega^{(1)} \smallsetminus \Omega^{(0)}$ an (open) *edge* of Ω, necessarily a smooth curve in M. Similarly, any open, connected component of $\Omega^{(j)} \smallsetminus \Omega^{(j-1)}$ shall be called a (open) j*–face* if $2 \leq j \leq n-1$. A $n-1$-face will be called a *hyperface.* A j–face H is a smooth manifold of dimension j, but in general it is not a curvilinear polyhedral domain (except if $n = 2$), because there might not exist a j-manifold containing the closure of H in $\partial\Omega$. This point will be addressed in terms of the desingularization $\Sigma(\Omega)$ of Ω constructed in Section 4.

Notations 2.4. *From now on, Ω will denote a curvilinear polyhedral domain in a manifold M of dimension n with given stratification $\Omega^{(0)} \subset \Omega^{(1)} \subset \ldots \subset \Omega^{(n)} := \overline{\Omega}$.*

Some or all of the sets $\Omega^{(j)}$, $j = 0, \ldots, n-2$, in the stratification of Ω may be empty. In fact, $\Omega^{(n-2)}$ is empty if, and only if, $\overline{\Omega}$ is a smooth manifold, possibly with boundary, a particular case of a curvilinear, stratified polyhedron. Finally we introduce the notion of distance to the singular set $\Omega^{(n-2)}$ of Ω (if not empty) on which the constructions of the Sobolev spaces $\mathcal{K}^\mu_a(\Omega)$ given in Section 5 is based. If $\Omega^{(n-2)} = \emptyset$, we let $\eta_{n-2} \equiv 1$.

Definition 2.5. Let Ω be a curvilinear, stratified polyhedral domain of dimension n. The distance function $\eta_{n-2}(x)$ from x to the singular set $\Omega^{(n-2)}$ is

$$\eta_{n-2}(x) := \inf_\gamma \ell(\gamma), \tag{22}$$

where $\ell(\gamma)$ is the length of the curve γ, and γ ranges through all smooth curves $\gamma : [0,1] \to \overline{\Omega}$, $\gamma(0) = x$, $p := \gamma(1) \in \Omega^{(n-2)}$.

If Ω is not bounded, for example Ω is an infinite cone, then we modify the definition of the distance function as follows:

$$\eta_{n-2}(x) := \chi(\inf_\gamma \ell(\gamma)), \qquad \text{where}$$
$$\chi \in C^\infty([0,+\infty)), \quad \chi(s) = \begin{cases} s, & 0 \leq s \leq 1 \\ \geq 1, & s \geq 1 \\ 2, & s \geq 3, \end{cases} \tag{23}$$

which has the effect of making η_{n-2} a bounded function.

2.1 Curvilinear polyhedral domains in 1, 2, and 3 dimensions

In this subsection we give some examples of curvilinear polyhedral domains Ω in $\mathbb{R}^2$, in S^2, or in $\mathbb{R}^3$. These examples are crucial in understanding Definition 2.1, which we specialize here for $n = 2$, $n = 3$. The desingularization $\Sigma(\Omega)$ and the function r_Ω will be introduced in the next subsection in these special cases. We have already defined a polyhedron in dimension 0 as a finite collection of points. Accordingly, a subset $\Omega \subset \mathbb{R}$ or $\Omega \subset S^1$ is a *curvilinear polyhedral domain* if, and only if, it is a finite union of open intervals.
Let M be a smooth 2-manifold or $\mathbb{R}^2$. Definition 2.1 can be more explicitly stated as follows.

Definition 2.6. A subset $\Omega \subset M$ together with smooth stratification $\Omega^{(0)} \subset \Omega^{(1)} \equiv \partial\Omega \subset \Omega^{(2)} \equiv \Omega$ will be called *a curvilinear, stratified polygonal domain* if, for every point of the boundary $p \in \partial\Omega$, there exists a neighborhood $V_p \subset M$ of p and a diffeomorphism $\phi_p : V_p \to B^2$, $\phi_p(p) = 0$, such that:

(a) $\phi_p(V_p \cap \Omega) = \{\, (r\cos\theta, r\sin\theta),\ 0 < r < 1,\ \theta \in \omega_p \,\}$, where ω_p is a union of open intervals of the unit circle such that $\overline{\omega_p} \neq S^1$;

(b) if $p \in \Omega^{(1)} \smallsetminus \Omega^{(0)}$, then ω_p is exactly an interval of length π.

Any point $p \in \Omega^{(0)}$ is a *vertex* of Ω, and p is a true vertex precisely when ω_p is not an interval of length π. The open, connected components of $\partial\Omega \smallsetminus \Omega^{(0)}$ are the (open) *sides* of Ω. In view of condition (b) above, sides are smooth curves $\gamma_j : [0, 1] \to M$, $j = 1, \ldots, N$, with no common interior points. Recall that by hypothesis, there are finitely many vertices and sides. The condition that $\overline{\omega_p} \neq S^1$ implies that either a side γ_j has a vertex in common with another side γ_k or γ_j is a closed smooth curve or an unbounded smooth curve. In the special case $\Omega^{(1)} \smallsetminus \Omega^{(0)} = \emptyset$, Ω has only isolated conical points (see Example 2.11 in the next subsection), while if $\Omega^{(0)} = \emptyset$, Ω has smooth boundary.

Notations 2.7. *Any curvilinear, stratified polygon in $\mathbb{R}^2$ will be denoted by $\mathbb{P}$ and its stratification by* $\mathbb{P}^{(0)} \subset \mathbb{P}^{(1)} = \partial\mathbb{P} \subset \mathbb{P}^{(2)} = \overline{\mathbb{P}}$.

Let now M be a smooth 3-manifold or $\mathbb{R}^3$. Definition 2.1 can also be stated more explicitly.

Definition 2.8. A subset $\Omega \subset M$ together with a smooth stratification $\Omega^{(0)} \subset \Omega^{(1)} \subset \Omega^{(2)} \equiv \partial\Omega \subset \Omega^{(3)} \equiv \overline{\Omega}$ will be called *a curvilinear, stratified polyhedral domain* if, for every point of the boundary $p \in \partial\Omega$, there exists a neighborhood $V_p \subset M$ of p and a diffeomorphism $\phi_p : V_p \to B^l \times B^{3-l}$, $\phi_p(p) = 0$, such that:

(a) $\phi_p(V_p \cap \Omega) = \{\, (y, rx'),\ y \in B^l,\ 0 < r < t,\ x' \in \omega_p \,\}$, where $t \in (0, +\infty]$ and $\omega_p \subset S^{2-l}$ is such that $\overline{\omega_p} \neq S^{2-l}$;

(b) if $l = 0$ (*i.e.,* if $p \in \Omega^{(0)}$), then $\omega_p \subset S^2$ is a stratified, curvilinear polygonal domain;

(c) if $l = 1$ (*i.e.,* if $p \in \Omega^{(1)} \smallsetminus \Omega^{(0)}$), then ω_p is a finite, disjoint union of finitely many open intervals in S^1 of total length less than 2π.

(d) if $l = 2$ then p is a smooth boundary point;

(e) ϕ_p preserves the stratifications.

Each point $p \in \Omega^{(0)}$ is a *vertex* of Ω and p is a true vertex precisely when ω_p is an irreducible subset of S^2 (according to Definition 2.3). The open, connected components of $\Omega^{(1)} \smallsetminus \Omega^{(0)}$ are the *edges* of Ω, smooth curves with no interior common points by condition (c) above. The open, connected components of $\Omega^{(2)} \smallsetminus \Omega^{(1)}$, smooth surfaces with no common interior points, are the *faces* of Ω. Recall that by hypothesis, there are only finitely many vertices, edges, and faces in Ω. The condition that $\overline{\omega_p}$ be not the whole sphere S^{2-l} ($l = 1, 0$) implies that either an edge γ_j has a vertex in common with another edge γ_k or γ_j is a closed smooth curve or an unbounded smooth curve (such as in a wedge), and similarly for faces. Again, in the the case $\Omega^{(1)} = \Omega^{(0)}$, Ω has only isolated conical points, in the case $\Omega^{(0)} = \emptyset$, Ω has only edge singularities, and in the case $\Omega^{(1)} = \Omega^{(0)} = \emptyset$, Ω is smooth.
The following subsection contains several examples.

2.2 Definition of $\Sigma(\Omega)$ and of r_Ω if $n = 2$ or $n = 3$

We now introduce the desingularization $\Sigma(\Omega)$ for some of the typical examples of curvilinear polyhedral domains in $n = 2$ or $n = 3$. The desingularization of a domain $\Omega \subset M$ depends in general on M, but we do not explicitly show this dependence in the notation, and generally ignore it in order to streamline the presentation, given that the manifold M will be mostly implicit. Associated to the singularization is the function r_Ω, which is comparable with the distance to the singular set η_{n-2} but is more regular. We also frame these definitions as examples. The general case (of which the examples considered here are particular cases) is in Section 4. The reader can skip this part at first reading. The case $n = 2$ of a polygonal domain $\mathbb{P}$ in $\mathbb{R}^2$ is particularly simple. We use the notation in Definition 2.6.

Example 2.9. The desingularization $\Sigma(\mathbb{P})$ of $\mathbb{P}$ will replace each of the vertices A_j, $j = 1, \dots, k$, of $\mathbb{P}$ with a segment of length $\alpha_j > 0$, where α_j is the magnitude of the angle at A_j (if A_j is an artificial vertex, then $\alpha_j = \pi$). We can realize $\Sigma(\mathbb{P})$ in three dimensions as follows. Let θ_j be the angle in a polar coordinates system (r_j, θ_j) centered at A_j. Let ϕ_j be a smooth function on $\mathbb{P}$ that is equal to 1 on $\{r_j < \epsilon\}$ and vanishes outside $V_j := \{r_j < 2\epsilon\}$. By choosing $\epsilon > 0$ small enough, we can arrange that the sets V_j do not intersect. We define then

$$\Phi : \overline{\mathbb{P}} \smallsetminus \{A_1, A_2, \dots, A_k\} \to \mathbb{P} \times \mathbb{R} \subset \mathbb{R}^3$$

by $\Phi(p) = (p, \sum \phi_j(p)\theta_j(p))$. Then $\Sigma(\mathbb{P})$ is (up to a diffeomorphism) the closure in $\mathbb{R}^3$ of $\Phi(\mathbb{P})$. The desingularization map is $\kappa(p, z) = p$. The structural Lie algebra of vector fields $\mathcal{V}(\mathbb{P})$ on $\Sigma(\mathbb{P})$ is given by (the lifts of) the smooth vector fields X on $\overline{\mathbb{P}} \smallsetminus \{A_1, A_2, \ldots, A_k\}$ that on $V_j = \{r_j < 2\epsilon\}$ can be written as

$$X = a_r(r_j, \theta_j) r_j \partial_{r_j} + a_\theta(r_j, \theta_j)\partial_{\theta_j}, \tag{24}$$

with a_r and a_θ smooth functions of (r_j, θ_j) on $[0, 2\epsilon) \times [0, \alpha_j]$. We can take $r_\Omega(x) := \psi(x) \prod_{j=1}^k r_j(x)$, where ψ is a smooth, nowhere vanishing function on $\Sigma(\Omega)$. (Such a factor ψ can always be introduced, and the function r_Ω is determined only up to this factor. We shall omit this factor in the examples below.)

The examples of a domain with a single edge or of a domain with a single vertex are among of the most instructive.

Example 2.10. Let first Ω be the wedge

$$\mathbb{W} := \{(r\cos\theta, r\sin\theta, z),\, 0 < r, 0 < \theta < \alpha, z \in \mathbb{R}\}, \tag{25}$$

where $0 < \alpha < 2\pi$, and $x = r\cos\theta$ and $y = r\sin\theta$ define the usual cylindrical coordinates (r, θ, z), with $(r, \theta, z) \in [0, \infty) \times [0, 2\pi) \times \mathbb{R}$. Then the manifold of generalized cylindrical coordinates is, in this case, just the domain of the cylindrical coordinates on $\overline{\mathbb{W}}$:

$$\Sigma(\mathbb{W}) = [0, \infty) \times [0, \alpha] \times \mathbb{R}.$$

The desingularization map is $\kappa(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$ and the structural Lie algebra of vector fields of $\Sigma(\mathbb{W})$ is

$$a_r(r, \theta, z) r\partial_r + a_\theta(r, \theta, z)\partial_\theta + a_z(r, \theta, z) r\partial_z,$$

where a_r, a_z, and a_θ are smooth functions on $\Sigma(\mathbb{W})$. Note that the vector fields in $\mathcal{V}(\mathbb{W})$ may not extend to the closure $\overline{\mathbb{W}}$. We can take $r_\Omega = r$, the distance to the Oz-axis.

At this stage, we can describe a domain with one conical point and its desingularization in any dimension.

Example 2.11. Let next Ω be *a domain with one conical point*, that is, Ω is a curvilinear, stratified polyhedron in $\mathbb{R}^n$ such that $\Omega^{(j)} = \Omega^{(0)}$ for all $1 \leq j \leq n-2$. We assume Ω is bounded for simplicity. Let $p \in \Omega^{(0)}$ denote the single vertex of Ω. There exists a neighborhood V_p of p such that, up to a local change of coordinates,

$$V_p \cap \Omega = \{rx',\, 0 \leq r < \epsilon, x' \in \omega\}, \tag{26}$$

for some smooth, connected domain $\omega \subset S^{n-1} := \partial B^n$. Then we can realize $\Sigma(\Omega)$ in $\mathbb{R}^{2n}$ as follows. Assume $p = 0$, the origin, for simplicity. We define $\Phi(x) = (x, |x|^{-1}x)$ for $x \neq p$, where $|x|$ is the distance from x to the origin (*i.e.,* to p). The set $\Sigma(\Omega)$ is defined to be the closure of the range of Φ. The map κ is the projection onto the first n components. The map κ is one-to-one, except that $\kappa^{-1}(p) = \{p\} \times \omega$. We can take $r_\Omega(x) = |x|$. The Lie algebra of vector fields $\mathcal{V}(\Omega)$ consists of the vector fields on $\Sigma(\Omega)$ that are tangent to $\kappa^{-1}(p)$. This example is due to Melrose [57].

Example 2.12. Let $\Omega \subset \mathbb{R}^3$ be a convex polyhedral domain, such that all edges are straight segments. To construct $\Sigma(\Omega)$, we combine the ideas used in the previous examples. First, for each edge e we define (r_e, θ_e, z_e) to be a coordinate system aligned to that edge and such that $\theta_e \in (0, \alpha_e)$, as in Example 2.10. Let $v_1, v_2, \ldots, v_b$ be the set of vertices of Ω and $e_1, \ldots, e_a$ be the set of edges. Then, for x not on any edge of Ω, we define $\Phi(x) \in \mathbb{R}^{3+a+b}$ by

$$\Phi(x) = \big(x, \theta_{e_1}, \theta_{e_2}, \ldots, \theta_{e_a}, |x - v_1|^{-1}(x - v_1), \ldots, |x - v_b|^{-1}(x - v_b)\big)\,.$$

The desingularization $\Sigma(\Omega) \subset \mathbb{R}^{3+a+b}$ is defined as the closure of the range of Φ. The resulting set will be a manifold with corners with several different types of hyperfaces. Namely, the manifold $\Sigma(\Omega)$ will have a hyperface for each face of Ω, a hyperface for each edge of Ω, and, finally, a hyperface for each vertex of Ω. The last two types of hyperfaces are the so-called *hyperfaces at infinity* of $\Sigma(\Omega)$. Let x_H be the distance to the hyperface H. We can take then $r_\Omega = \prod_H x_H$, where H ranges through the hyperfaces at infinity of $\Sigma(\Omega)$.
We can imagine $\Sigma(\Omega)$ as follows. Let $\epsilon > 0$. Remove the sets $\{x \in \Omega, |x - v_j| \leq \epsilon\}$ and $\{x \in \Omega, |x - e_k| \leq \epsilon^2\}$. Call the resulting set Ω_ϵ. Then, for ϵ small enough, the closure of Ω_ϵ is diffeomorphic to $\Sigma(\Omega)$.

The example above can be generalized to a curvilinear, stratified polyhedron, using local change of coordinates as in Example 2.9 in 2 dimensions. A detailed construction will be given in Section 4.
A nonstandard example of a curvilinear polyhedral domain is given below.

Example 2.13. We start with a connected polygonal domain $\mathbb{P}$ with connected boundary and we deform it, within the class of connected polygonal domains, until one, and exactly one of the vertices, say A, touches the interior of another edge, say $[B, C]$. (It is clear that such a deformation exists since we allow each side to have arbitrary finite curvature and length.) Let Ω be the resulting connected open set. Then Ω will be a curvilinear polyhedral domain. We define the set $\Sigma(\Omega)$ as for the polygonal domain $\mathbb{P}$, but by introducing polar coordinates in the whole neighborhood of the point A.
If we deform $\mathbb{P}$ to Ω, $\Sigma(\mathbb{P})$ will deform continuously to a space $\Sigma'(\Omega)$, different from $\Sigma(\Omega)$. For certain purposes, the desingularization $\Sigma'(\Omega)$ is better suited than $\Sigma(\Omega)$.

3 Lie manifolds with boundary

The construction of the desingularization $\Sigma(\Omega)$ of a general, curvilinear, stratified polyhedron Ω in n dimensions will be discussed in Section 4. $\Sigma(\Omega)$ will be used both in the definition of weighted Sobolev spaces on the boundary and the proof of a weighted Hardy-Poincaré inequality in Subsection 6.2, which in turn is crucial in establishing coercive estimates for the mixed boundary value/interface problem (6). Since the construction of the desingularization $\Sigma(\Omega)$ relies on properties of manifolds with a Lie structure at infinity, we now recall the definition of a Lie manifold from [2] and of a Lie manifold with boundary from [1], in order to make this paper as self-contained as possible. We also recall a few other needed definitions and results from those papers.

3.1 Definition

We recall that a topological space $\mathfrak{M}$ is, by definition, *a manifold with corners* if every point $p \in \mathfrak{M}$ has a coordinate neighborhood diffeomorphic to $[0,1)^k \times (-1,1)^{n-k}$, $k = 0, 1, \ldots, n$, such that the transition functions are smooth (including at the boundary). Given $p \in \mathfrak{M}$, the least integer k with the above property is called the *depth* of p. Since the transition functions are smooth, it therefore makes sense to talk about smooth functions on $\mathfrak{M}$, these being the functions that correspond to smooth functions on $[0,1)^k \times (-1,1)^{n-k}$. We denote by $\mathcal{C}^\infty(\mathfrak{M})$ the set of smooth functions on a manifold with corners $\mathfrak{M}$.

Throughout this paper, $\mathfrak{M}$ will denote a manifold with corners, not necessarily compact. We shall denote by $\mathfrak{M}_0$ the interior of $\mathfrak{M}$ and by $\partial\mathfrak{M} = \mathfrak{M} \smallsetminus \mathfrak{M}_0$ the boundary of $\mathfrak{M}$. The set $\mathfrak{M}_0$ consists of the set of points of depth zero of $\mathfrak{M}$. It is usually no loss of generality to assume that $\mathfrak{M}_0$ is connected. Let $\mathfrak{M}_k$ denote the set of points of $\mathfrak{M}$ of depth k and F_0 be a connected component of $\mathfrak{M}_k$. We shall call F_0 an *open face of codimension* k of $\mathfrak{M}$ and $F := \overline{F}_0$ a *face of codimension* k of $\mathfrak{M}$. A face of codimension 1 will be called a *hyperface* of $\mathfrak{M}$, so that $\partial\mathfrak{M}$ is the union of all hyperfaces of $\mathfrak{M}$. In general, a face of $\mathfrak{M}$ need not be a smooth manifold (with or without corners). A face $F \subset \mathfrak{M}$ which is a submanifold with corners of $\mathfrak{M}$ will be called an *embedded face*.

Anticipating, a Lie manifold with boundary $\mathfrak{M}_0$ is the interior of a manifold with corners $\mathfrak{M}$ together with a Lie algebra of vector fields $\mathcal{V}$ on $\mathfrak{M}$ satisfying certain conditions. To state these conditions, it will be convenient first to introduce a few other concepts.

Definition 3.1. Let $\mathfrak{M}$ be a manifold with corners and $\mathcal{V}$ be a space of vector fields on $\mathfrak{M}$. Let $U \subset \mathfrak{M}$ be an open set and $Y_1, Y_2, \ldots, Y_k$ be vector fields on $U \cap \mathfrak{M}_0$. We shall say that $Y_1, Y_2, \ldots, Y_k$ form a *local basis of* $\mathcal{V}$ *on* U if the following three conditions are satisfied:

(i) there exist vector fields $X_1, X_2, \ldots, X_k \in \mathcal{V}$, $Y_j = X_j$ on $U \cap \mathfrak{M}_0$;

(ii) $\mathcal{V}$ is closed under products with smooth functions in $\mathcal{C}^\infty(\mathfrak{M})$ (*i.e.,* $\mathcal{V} = \mathcal{C}^\infty(\mathfrak{M})\mathcal{V}$) and for any $X \in \mathcal{V}$, there exist smooth functions $\phi_1, \phi_2, \ldots, \phi_k \in \mathcal{C}^\infty(\mathfrak{M}_0)$ such that

$$X = \phi_1 X_1 + \phi_2 X_2 + \ldots + \phi_k X_k \quad \text{on } U \cap \mathfrak{M}_0\,; \tag{27}$$

and

(iii) if $\phi_1, \phi_2, \ldots, \phi_k \in \mathcal{C}^\infty(\mathfrak{M})$ and $\phi_1 X_1 + \phi_2 X_2 + \ldots + \phi_k X_k = 0$ on $U \cap \mathfrak{M}_0$, then $\phi_1 = \phi_2 = \ldots = \phi_k = 0$ on U.

We now recall structural Lie algebras of vector fields from [2].

Definition 3.2. A subspace $\mathcal{V} \subseteq \Gamma(\mathfrak{M}, T\mathfrak{M})$ of the Lie algebra of all smooth vector fields on $\mathfrak{M}$ is said to be a *structural Lie algebra of vector fields on* $\mathfrak{M}$ provided that the following conditions are satisfied:

(i) $\mathcal{V}$ is closed under the Lie bracket of vector fields;

(ii) every vector field $X \in \mathcal{V}$ is tangent to all hyperfaces of $\mathfrak{M}$;

(iii) $\mathcal{C}^\infty(\mathfrak{M})\mathcal{V} = \mathcal{V}$; and

(iv) for each point $p \in \mathfrak{M}$ there exist a neighborhood U_p of p in $\mathfrak{M}$ and a local basis of $\mathcal{V}$ on U_p.

The concept of Lie structure at infinity, defined next, is also taken from [2].

Definition 3.3. A *Lie structure at infinity* on a smooth manifold $\mathfrak{M}_0$ is a pair $(\mathfrak{M}, \mathcal{V})$, where $\mathfrak{M}$ is a compact manifold, possibly with corners, and $\mathcal{V} \subset \Gamma(\mathfrak{M}, T\mathfrak{M})$ is a structural Lie algebra of vector fields on $\mathfrak{M}$ with the following properties:

(i) $\mathfrak{M}_0 = \mathfrak{M} \smallsetminus \partial\mathfrak{M}$, the interior of $\mathfrak{M}$, and

(ii) If $p \in \mathfrak{M}_0$, then any local basis of $\mathcal{V}$ in a neighborhood of p is also a local basis of the tangent space to $\mathfrak{M}_0$. (In particular, the constant k of Equation (27) equals n, the dimension of $\mathfrak{M}_0$.)

A *manifold with a Lie structure at infinity* (or, simply, a *Lie manifold*) is a manifold $\mathfrak{M}_0$ together with a Lie structure at infinity $(\mathfrak{M}, \mathcal{V})$ on $\mathfrak{M}_0$. We shall sometimes denote a Lie manifold as above by $(\mathfrak{M}_0, \mathfrak{M}, \mathcal{V})$, or, simply, by $(\mathfrak{M}, \mathcal{V})$, because $\mathfrak{M}_0$ is determined as the interior of $\mathfrak{M}$.
Let $\mathcal{V}_b$ be the set of vector fields on $\mathfrak{M}$ that are tangent to all faces of $\mathfrak{M}$. Then $(\mathfrak{M}, \mathcal{V}_b)$ is a Lie manifold [57]. See [1, 2, 47] for more examples.

3.2 Riemannian metric

Let $(\mathfrak{M}, \mathcal{V})$ be a Lie manifold and g a Riemannian metric on $\mathfrak{M}_0 := \mathfrak{M} \smallsetminus \partial\mathfrak{M}$. We shall say that *$g$ is compatible* (with the Lie structure at infinity $(\mathfrak{M}, \mathcal{V})$) if, for any $p \in \mathfrak{M}$, there exist a neighborhood U_p of p in $\mathfrak{M}$ and a local basis $X_1, X_2, \ldots, X_n$ of $\mathcal{V}$ on U_p that is orthonormal with respect to g on U_p.
It was proved in [2] that $(\mathfrak{M}_0, g_0)$ is necessarily of infinite volume and complete. Moreover, all the covariant derivatives of the Riemannian curvature tensor of g are bounded.
We also know that the injectivity radius is bounded from below by a positive constant, *i.e.,* $(\mathfrak{M}_0, g_0)$ is of bounded geometry [18]. (A *manifold with bounded geometry* is a Riemannian manifold with positive injectivity radius and with bounded covariant derivatives of the curvature tensor, see for example [16] or [69] and references therein).

3.3 $\mathcal{V}$-differential operators

We are especially interested in the analysis of the differential operators generated using only derivatives in $\mathcal{V}$. Let $\mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M})$ be the algebra of differential operators on $\mathfrak{M}$ generated by multiplication with functions in $\mathcal{C}^\infty(\mathfrak{M})$ and by differentiation with vector fields $X \in \mathcal{V}$. The space of order m differential operators in $\mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M})$ will be denoted $\mathrm{Diff}^m_{\mathcal{V}}(\mathfrak{M})$. A differential operator in $\mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M})$ will be called a $\mathcal{V}$-differential operator. We define the set $\mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M}; E, F)$ of $\mathcal{V}$-differential operators acting between sections of smooth vector bundles $E, F \to \mathfrak{M}$ in the usual way [1, 2].
A simple but useful property of the differential operator in $\mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M})$ is that

$$x^s P x^{-s} \in \mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M}) \tag{28}$$

for any $P \in \mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M})$ and any defining function x of some hyperface of $\mathfrak{M}$ [3]. This property is easily proved using the fact that X is tangent to the hyperface defined by x, for any $X \in \mathcal{V}$ (a proof of a slightly more general fact is included in Corollary 6.3).

3.4 Lie manifolds with boundary

A subset $\mathfrak{N} \subset \mathfrak{M}$ is called a *submanifold with corners of* $\mathfrak{M}$ if $\mathfrak{N}$ is a closed submanifold of $\mathfrak{M}$ such that $\mathfrak{N}$ is transverse to all faces of $\mathfrak{M}$ and any face of $\mathfrak{N}$ is a component of $\mathfrak{N} \cap F$ for some face F of $\mathfrak{M}$.
The following definition is a reformulation of a definition of [1].

Definition 3.4. Let $(\mathfrak{N}, \mathcal{W})$ and $(\mathfrak{M}, \mathcal{V})$ be Lie manifolds, where $\mathfrak{N} \subset \mathfrak{M}$ is a submanifold with corners and

$$\mathcal{W} = \{X|_{\mathfrak{N}},\ X \in \mathcal{V}, X|_{\mathfrak{N}} \text{ tangent to } \mathfrak{N}\}.$$

We shall say that $(\mathfrak{N}, \mathcal{W})$ is a *tame* submanifold of $(\mathfrak{M}, \mathcal{V})$ if, for any $p \in \partial\mathfrak{N}$ and any $X \in T_p\mathfrak{M}$, there exist $Y \in \mathcal{V}$ and $Z \in T_p\mathfrak{N}$ such that $X = Y(p) + Z$.

Let $\mathfrak{N} \subset \mathfrak{M}$ be a submanifold with corners. We assume that $\mathfrak{M}$ and $\mathfrak{N}$ are endowed with the Lie structures $(\mathfrak{N}, \mathcal{W})$ and $(\mathfrak{M}, \mathcal{V})$. We shall say that $\mathfrak{N}$ is a *regular* submanifold of $(\mathfrak{M}, \mathcal{V})$ if we can choose a tubular neighborhood V of $\mathfrak{N}_0 := \mathfrak{N} \smallsetminus \partial\mathfrak{N} = \mathfrak{N} \cap \mathfrak{M}_0$ in $\mathfrak{M}_0$, a compatible metric g_1 on $\mathfrak{N}_0$, a product-type metric g_1 on V that reduces to g_1 on $\mathfrak{N}_0$, and a compatible metric on $\mathfrak{M}_0$ that restricts to g_1 on V. Theorem 5.8 of [2] states that every tame submanifold is regular. The point of this result is that it is much easier to check that a submanifold is tame than to check that it is regular.
In the case when $\mathfrak{N}$ is of codimension one in $\mathfrak{M}$, the condition that $\mathfrak{N}$ be tame is equivalent to the fact that there exists a vector field $X \in \mathcal{V}$ that restricts to a normal vector of $\mathfrak{N}$ in $\mathfrak{M}$. The neighborhood V will then be of the form $V \simeq (\partial\mathfrak{N}_0) \times (-\varepsilon_0, \varepsilon_0)$. Moreover, there will exist a compatible metric on $\mathfrak{M}_0$ that restricts to the product metric $g_1 + dt^2$ on V, where g_1 is a compatible metric on $\mathfrak{N}_0$.

Let A be a subset of $\mathfrak{M}$. We denote by $\partial_{\mathfrak{M}} A := \overline{A} \smallsetminus A$ the boundary of A computed within $\mathfrak{M}$, which should not be confused with $\partial A = A \smallsetminus A_0$, where A_0 is the interior of A, as a manifold with corners. Let $\mathbb{D} \subset \mathfrak{M}$ be an open subset. We say that $\mathbb{D}$ is a *Lie domain* in $\mathfrak{M}$ if, and only if,

$$\partial_{\mathfrak{M}} \mathbb{D} = \partial_{\mathfrak{M}} \overline{\mathbb{D}} \tag{29}$$

and $\partial_{\mathfrak{M}} \mathbb{D}$ is a regular submanifold of $\mathfrak{M}$. The condition (29) is included in order to make sure that $\mathbb{D}$ is on only one side of its boundary. A typical example of a Lie domain $\mathbb{D} \subset \mathfrak{M}$ is obtained by considering a regular submanifold with corners $\mathfrak{N} \subset \mathfrak{M}$ of codimension one with the property that $\mathfrak{M} \smallsetminus \mathfrak{N}$ consists of two connected components. Any of these two components will be a Lie domain.

Definition 3.5. A *Lie manifold with boundary* is a triple $(\mathfrak{O}_0, \mathfrak{O}, \mathcal{V}')$, where $\mathfrak{O}_0$ is a smooth manifold with boundary, $\mathfrak{O}$ is a compact manifold with corners containing $\mathfrak{O}_0$ as an open subset, and $\mathcal{V}'$ is a Lie algebra of vector fields on $\mathfrak{O}$ with the property that there exists a Lie manifold $(\mathfrak{M}_0, \mathfrak{M}, \mathcal{V})$, a Lie domain $\mathbb{D}$ in $\mathfrak{M}$ and a diffeomorphism $\phi : \mathfrak{O} \to \overline{\mathbb{D}}$ such that $\phi(\mathfrak{O}_0) = \overline{\mathbb{D}} \cap \mathfrak{M}_0$ and $D\phi(\mathcal{V}|_{\mathbb{D}}) = \mathcal{V}'$.

We continue with some simple observations. First note that if $(\mathfrak{O}_0, \mathfrak{O}, \mathcal{V})$ is a Lie manifold with boundary, then $\mathfrak{O}_0$ is determined by $(\mathfrak{O}, \mathcal{V})$. Indeed, if we remove from $\mathfrak{O}$ the hyperfaces H with the property that $\mathcal{V}$ consists only of vectors tangent to H, then the resulting set is $\mathfrak{O}_0$. Therefore, we can denote the Lie manifold with boundary $(\mathfrak{O}_0, \mathfrak{O}, \mathcal{V})$ simply by $(\mathfrak{O}, \mathcal{V})$.
Another observation is that $\partial\mathfrak{O}_0$, the boundary of $\mathfrak{O}_0$ (as a smooth manifold with boundary), has a canonical structure of Lie manifold $(\partial\mathfrak{O}_0, D = \partial_{\mathfrak{M}}\mathbb{D}, \mathcal{W})$, where $\mathcal{W} = \{X|_D,\, X \in \mathcal{V}, X|_D \text{ is tangent to } D\}$. The compactification D is the closure of $\partial\mathfrak{O}_0$ in $\mathfrak{O}$.

3.5 Sobolev spaces

The main reason for considering Lie manifolds (with or without boundary) in our setting is that they carry some naturally defined Sobolev spaces and these Sobolev spaces behave almost exactly like the Sobolev spaces on a compact manifold with a smooth boundary. Let us recall one of the equivalent definitions in [1]. See also [16, 32, 57, 64, 70] for results on Sobolev spaces on non-compact manifolds.

Definition 3.6. Fix a Lie manifold $(\mathfrak{M}, \mathcal{V})$. The spaces $L^2(\mathfrak{M}_0) = L^2(\mathfrak{M}_0)$ are defined using the natural volume form on $\mathfrak{M}_0$ given by an arbitrary compatible metric g on $\mathfrak{M}_0$ (*i.e.,* compatible with the Lie structure at infinity). All such volume forms are known to define the same space $L^2(\mathfrak{M})$, but with possibly different norms. Let $k \in \mathbb{Z}_+$. Choose a finite set of vector fields $\mathcal{X} \subset \mathcal{V}$ such that $\mathcal{C}^\infty(\mathfrak{M})\mathcal{X} = \mathcal{V}$. The system $\mathcal{X}$ gives rise to the norm

$$\|u\|^2_{\mathcal{X},\Omega} := \sum \|X_1 X_2 \dots X_\mu u\|^2_{L^2(\Omega)}\,, \quad 1 \le p < \infty, \tag{30}$$

the sum being over all possible choices of $0 \leq l \leq k$ and all possible choices of vector fields $X_1, X_2, \ldots, X_l \in \mathcal{X}$, not necessarily distinct. We then set

$$H^k(\mathfrak{M}_0) = H^k(\mathfrak{M}) := \{u \in L^2(\mathfrak{M}),\ \|u\|_{\mathcal{X},\mathfrak{M}} < \infty\}.$$

The spaces $H^s(\mathfrak{M}_0) = H^s(\mathfrak{M})$ are defined by duality (with pivot $L^2(\mathfrak{M}_0)$) when $-s \in \mathbb{Z}_+$, and then by interpolation, as above.
Let $(\mathfrak{O}_0, \mathfrak{O}, \mathcal{V})$ be a Lie manifold with boundary. We shall assume that $\mathfrak{O}$ is the closure of a Lie domain $\mathbb{D}$ of the Lie manifold $\mathfrak{M}$. The Sobolev spaces $H^k(\mathfrak{O}_0)$ are defined as the set of *restrictions* to $\mathfrak{O}_0$ of distributions $u \in H^k(\mathfrak{M}_0)$, using the notation of Definition 3.5, $k \in \mathbb{Z}$. In particular, we obtain the following description of $H^k(\mathfrak{O}_0)$ from [1].

Lemma 3.7. *We have, for $k \geq 0$,*

$$H^k(\mathfrak{O}_0) = \{u \in L^2(\mathfrak{O}_0),\ \|u\|_{\mathcal{X}} < \infty\},$$

and

$$H^{-k}(\mathfrak{O}_0) = H^k_0(\mathfrak{O}_0)^*,$$

where $H^k_0(\mathfrak{O}_0)$ is the closure of $\mathcal{C}^\infty_c(\mathfrak{O}_0)$ in $H^k(\mathfrak{M}_0)$.

Definition 3.8. The hyperfaces of $\mathfrak{O}$ that do not intersect the boundary $\partial\mathfrak{O}_0$ of the manifold with boundary $\mathfrak{O}_0$ will be called *hyperfaces at infinity.* Let x_H be a defining function of the hyperface H of $\mathfrak{O}$. Any function of the form $h = \prod x_H^{a_H}$, where H ranges through the set of hyperfaces at infinity of $\mathfrak{O}$ and $a_H \in \mathbb{R}$, will be called an *admissible weight.* If h is an admissible weight, we set

$$hH^\mu(\mathfrak{O}_0) = \{h\,u;\ u \in H^\mu(\mathfrak{O}_0)\}$$

with the induced norm.

Later in the paper, we will identify the weighted Sobolev spaces $\mathcal{K}^s_a(\Omega)$ with suitable spaces $hH^s(\mathfrak{O}_0)$ in Proposition 5.7 and utilize the spaces $hH^s(\partial\mathfrak{O}_0)$ to define the spaces $\mathcal{K}^s_a(\partial\Omega)$ on the boundary in Definition 5.8, for Ω a curvilinear, stratified polyhedral domain in dimension n. The following proposition, which summarizes the relevant results from Theorem 3.4 and 3.7 from [1], will then imply Theorem 5.9.

Proposition 3.9. *The restriction to the boundary extends to a continuous, surjective map $hH^\mu(\mathfrak{O}_0) \to hH^{\mu-1/2}(\partial\mathfrak{O}_0)$, for any $\mu \geq 1$ and any admissible weight h. The kernel of this map, for $\mu = 1$, consists of the closure of $\mathcal{C}^\infty_c(\mathfrak{O}_0)$ in $hH^1(\mathfrak{O}_0)$.*

For $\mathbb{D}$, $\mathfrak{O}$, $\mathfrak{O}_0$ as in the proposition above, $hH^s(\mathbb{D})$, $hH^s(\mathfrak{O})$, and $hH^s(\mathfrak{O}_0)$ will all denote the same space.

4 Desingularization of polyhedra

In this section, we introduce a desingularization procedure that we shall use for studying curvilinear polyhedral domains. The desingularization will carry a natural structure of Lie manifold with boundary. This construction will allow us to study curvilinear polyhedral domains using Lie manifolds with boundary. The desingularization of a domain $\Omega \subset M$ depends on M in general (since $\overline{\Omega}$ depends on M), but we do not include the dependence on M in the notation, and generally ignore this issue in order to streamline the presentation, since the manifold M will be clear from the context in most cases.
As before, $\Omega \subset M$ denotes a curvilinear, stratified polyhedral domain in an n-dimensional manifold M. We shall construct by induction on n a *canonical* manifold with corners $\Sigma(\Omega)$ and a differentiable map $\kappa : \Sigma(\Omega) \to \overline{\Omega}$ that is a diffeomorphism from the interior of $\Sigma(\Omega)$ to Ω. In particular, the map κ allows us to identify Ω with a subset of $\Sigma(\Omega)$. We shall also construct a canonical Lie algebra of vector fields $\mathcal{V}(\Omega)$ on $\Sigma(\Omega)$. The manifold $\Sigma(\Omega)$ will be called the *desingularization of* Ω, the map κ will be called the *desingularization* map, and the Lie algebra of vector fields will be called the *structural Lie algebra of vector fields of* $\Sigma(\Omega)$. We shall also introduce in this section a smooth weight function r_Ω equivalent to η_{n-2}.
The space $\Sigma(\Omega)$ that we construct is not optimal if the links ω_p are not connected. A better desingularization would be obtained if one considers a diffeomorphism ϕ_{pC} for each connected component C of $V_p \cap \Omega$ that maps C to a conic set of the form $\omega_{p,C} \times B^\lambda$, with λ largest possible. The difference between these two constructions is seen by looking at the Example 2.13.

Notations 4.1. *From now on V_p and $\phi_p : V_p \to tB^{n-l} \times tB^l$, $l = \ell(p)$, will denote a neighborhood of $p \in \partial\Omega$ in $M \supset \Omega$ and ϕ_p will be a diffeomorphism satisfying the conditions of Definition* (2.1). *In addition, $\omega_p \subset S^{n-l-1}$ will be the curvilinear, stratified polyhedron such that*

$$\phi_p(V_p \cap \Omega) = \{(rx', x'')\},\ r \in (0,t), x' \in \omega_p \text{ and } x'' \in tB^l\},$$

i.e., *ω_p is the link of Ω at p. This notation will remain fixed throughout the paper.*

Recall that $0 \le \ell(p) \le n-1$ is defined to be the smallest integer such that $p \in \Omega^{(\ell(p))}$, but $p \notin \Omega^{(\ell(p)-1)}$. If $\ell(p) = 0$, then B^l is reduced to a point, and we just drop x'' from the notation above. We will assume that ϕ_p extends to the closure of V_p, if necessary.

4.1 The desingularization $\Sigma(\Omega)$

We now define the canonical desingularization of a curvilinear polyhedral domain $\Omega \subset M$, M an n-dimensional smooth manifold. For $n = 0$, Ω consists of finitely many points. Then we define $\Sigma(\Omega) = \Omega$ and $\kappa = id$. To define $\Sigma(\Omega)$ for general Ω, we shall proceed by induction.

We need first to make the important observation that the set ω_p, $p \in \partial\Omega$, of Definition 2.1 is determined up to a linear isomorphism of $\mathbb{R}^{n-l-1}$. Indeed, let $S_p \subset \partial\Omega$ be the maximal connected manifold of dimension $l = \ell(p)$ passing through p that is, the connected component of $\Omega^{(l)} \smallsetminus \Omega^{(l-1)}$ containing p. Let $(T_pS_p)^\perp = T_pM/T_pS_p$. The differential $D\phi_p : T_pM \to \mathbb{R}^n = T_0\mathbb{R}^n$ of the map ϕ_p at p has then the property that $D\phi_p(T_pS_p) = T_0\mathbb{R}^l$, $D\phi_p\big((T_pS_p)^\perp\big) = T_0\mathbb{R}^n/T_0\mathbb{R}^l = \mathbb{R}^{n-l}$. We will define a canonical set $\mathcal{N}_p \subset (T_pS_p)^\perp$ such that

$$D\phi_p(\mathcal{N}_p) \simeq \mathbb{R}_+\omega_p.$$

Since the definition of $\mathcal{N}_p$, which we give next, is independent of any choices used in the definition of a polyhedral domain, it follows that ω_p is unique, up to a linear isomorphism of $\mathbb{R}^{n-l-1}$. It remains to define the set $\mathcal{N}_p$ with the desired independence property. It is enough to define the *complement* of $\mathcal{N}_p$. This complement is the projection onto $(T_pS_p)^\perp = T_pM/T_pS_p$ of the set $\gamma'(0) \in T_pM$, where γ ranges through the set of smooth curves $\gamma : [0,1] \to M$, with $\gamma(t) \notin \Omega$ for $t > 0$, and $\gamma(0) = p$.
We let then $\sigma_p := \mathcal{N}_p/\mathbb{R}_+$, the set of rays in $\mathcal{N}_p$, for $p \in \partial\Omega$. Any choice of a metric on $T_pM/T_pS_p \supset \mathcal{N}_p$ will identify σ_p with a subset of the unit sphere of T_pM/T_pS_p, which depends however on the metric. In particular, $D\phi_p : \sigma_p \to \omega_p$ is a diffeomorphism. If p is not in the singular set $\Omega^{(n-2)}$ of Ω, then σ_p consists exactly of one point. The map κ is the projection onto the second component and is one-to-one above Ω and above $\Omega^{(n-1)} \smallsetminus \Omega^{(n-2)} \subset \partial\Omega$.
We now proceed with the induction step. Assume $\Sigma(\omega)$ and $\kappa : \Sigma(\omega) \to \overline{\omega}$ have been constructed for all curvilinear, stratified polyhedral domains ω of dimension at most $n-1$. If $p \in \Omega$, we then set $\sigma_p = \{0\} = \Sigma(\sigma_p)$. Let Ω be an arbitrary curvilinear, stratified polyhedral domain of dimension n. We define

$$\Sigma(\Omega) := \bigcup_{p\in\Omega} \{p\} \times \Sigma(\sigma_p) = \Omega \cup \bigcup_{p\in\partial\Omega} \{p\} \times \Sigma(\sigma_p). \tag{31}$$

In particular, if Ω is a bounded domain with smooth boundary, then $\Sigma(\Omega) \simeq \Omega$. This definition is consistent as ω_p is a curvilinear polyhedral domain of dimension at most $n-1$. Below, an open embedding will mean a diffeomorphism onto an open subset of the codomain.

Proposition 4.2. *Let $\Omega \subset M$ and $\Omega' \subset M'$ be curvilinear, stratified polyhedral domains and $\Phi : M \to M'$ be an open embedding such that $\Phi(\Omega)$ is a union of connected components of $\Omega' \cap \Phi(M)$. Then the embedding Φ defines a canonical map $\Sigma(\Phi) : \Sigma(\Omega) \to \Sigma(\Omega')$ such that*

$$\Sigma(\Phi \circ \Phi') = \Sigma(\Phi) \circ \Sigma(\Phi'),$$

for all open embeddings Φ and Φ' for which $\Sigma(\Phi \circ \Phi')$, $\Sigma(\Phi) \circ \Sigma(\Phi')$ are well-defined.

Proof. The proof is by induction. There is nothing to prove for $n = 0$. Let $p \in \overline{\Omega}$. We have that $\Phi(\overline{\Omega}) \subset \overline{\Omega'}$, and hence $\Phi(p) \in \overline{\Omega'}$, as well. Let V_p' be an

open neighborhood of $\Phi(p)$ in M' such that there exists a diffeomorphism $\phi'_p : V'_p \to B^{n-l} \times B^l$ satisfying the condition (18) of the definition of a polyhedral domain (*i.e.,* $\phi'_p(\Omega' \cap V'_p)$ is $\mathbb{R}_+\omega'_p \times B^l$, for some curvilinear polyhedral domain $\omega'_p \subset S^{n-l-1}$). By decreasing V'_p, if necessary, we can assume that $V'_p \subset \Phi(M)$. Then $V'_p \cap \Phi(\Omega)$ is a union of connected components of $V'_p \cap \Omega'$. Therefore ω'_p is a union of connected components of $\Phi(\omega_p)$, where $\omega_p \subset S^{n-l-1}$ is associated to $p \in \overline{\Omega}$ in the same way as ω'_p was associated to $\Phi(p) \in \overline{\Omega'}$. The induction hypothesis then gives rise to a canonical, injective map $\Sigma(\omega_p) \to \Sigma(\omega'_p)$. The map $\Sigma(\Phi)$ is obtained by combining these different maps.
The functoriality (*i.e.,* the relation $\Sigma(\Phi \circ \Phi') = \Sigma(\Phi) \circ \Sigma(\Phi')$) is proved similarly by induction. □

Here is a corollary of the above proof.

COROLLARY 4.3. *If* $\Omega = \Omega' \cup \Omega''$ *is the disjoint union of two open sets, then the inclusions* $\Sigma(\Omega') \subset \Sigma(\Omega)$ *and* $\Sigma(\Omega') \subset \Sigma(\Omega)$ *defined in Proposition 4.2 realize* $\Sigma(\Omega) = \Sigma(\Omega') \cup \Sigma(\Omega'')$, *where the union is a disjoint union.*

Proof. We use the same argument as in the proof of Proposition 4.2. □

The desingularization has a simple behavior with respect to products.

LEMMA 4.4. *We have a canonical identification*

$$\Sigma(M' \times \Omega) = M' \times \Sigma(\Omega),$$

for any smooth manifolds M *and* M' *and any curvilinear polyhedral domain* $\Omega \subset M$.

Proof. Since M' is smooth, we can choose the structural local diffeomorphism $\phi_{(p,q)}$ in $M' \times \Omega$ to be given by $\phi_p \times \psi_q$, where ψ_p is a local coordinate chart defined in a neighborhood of $p \in M'$ and ϕ_q is the local diffeomorphism of a neighborhood of q in Ω. Indeed, then

$$\Sigma(M' \times \Omega) := \cup_{p,q}\{(p,q)\} \times \Sigma(\sigma_{(p,q)}) = \cup_{p,q}\{(p,q)\} \times \Sigma(\sigma_q) = M' \times \Sigma(\Omega), \quad (32)$$

where $q \in \overline{\Omega}$ and $p \in M'$. Consequently, there is a canonical bijection $\sigma_{(p,q)} \simeq \sigma_q$ for any $q \in \overline{\Omega}$ and any $p \in M'$ (so (p,q) is in the closure of $M' \times \Omega$ in $M' \times M$). □

It remains to define the topology and differentiable structure on $\Sigma(\Omega)$. These definitions will again be canonical if we require that the map of the above lemma, as well as the maps κ and $\Sigma(\phi)$, be differentiable, for any open embedding ϕ.
Let $V_p \subset M$ and ϕ_p be as in Equation (19). By Proposition 4.2, we may assume that ϕ_p is the identity, so that $p = 0$, $V_p = B^{n-l} \times B^l$, and $V_p \cap \Omega = I\omega_p \times B^l$, with $I = (0,1)$. Let $(\Sigma(\omega_p), \kappa'_p)$ be the canonical desingularization of ω_p in S^{n-l-1}. We shall need the following lemma.

Lemma 4.5. *We have a canonical identification*

$$\Sigma(V_p \cap \Omega) = [0,1) \times \Sigma(\omega_p) \times B^l$$

such that the desingularization map

$$\kappa_p : [0,1) \times \Sigma(\omega_p) \times B^l \to \overline{V_p \cap \Omega} \subset B^{n-l} \times B^l$$

is given by $\kappa_p(r, x', y) = (r\kappa_p'(x'), y)$.

Proof. We may assume $p = 0$. Let $I = (0,1)$. The closure of $V_0 \cap \Omega$ in $V_0 = V_p$ is the disjoint union $\{0\} \times B^l \cup I\overline{\omega}_0 \times B^l$. Accordingly, we decompose $\Sigma(V_0 \cap \Omega, M)$ into two disjoint sets, corresponding to this splitting of the closure of $V_0 \cap \Omega$. Recall that by definition $\Sigma(V_0 \cap \Omega)$ is the union $\cup_{p \in \overline{V_0 \cap \Omega}} \{p\} \times \Sigma(\sigma_p)$. Using also Lemma 4.4, we then obtain

$$\begin{aligned}
\Sigma(V_0 \cap \Omega) = \Sigma(V_0 \cap \Omega) \cup \bigcup_{q \in B^l} \{(0,q)\} \times \Sigma(\omega_0) \\
= \Sigma((0,1) \times \omega_0 \times B^l) \cup \bigcup_{q \in B^l} \{(0,q)\} \times \Sigma(\omega_0) \\
= (0,1) \times \Sigma(\omega_0) \times B^l \cup \{0\} \times \Sigma(\omega_0) \times B^l = [0,1) \times \Sigma(\omega_0) \times B^l.
\end{aligned}$$

The formula for κ_0 follows from the definition. $\square$

Since $\Sigma(\Omega)$ is the union of all the sets $\Sigma(V_p \cap \Omega)$, with V_p in the covering above, we can define the topology and smooth structure on $\Sigma(\Omega)$ by induction as follows (there is nothing to define in the case Ω has dimension zero, since then $\Sigma(\Omega) = \Omega$).

Definition 4.6. Let $\phi_p : V_p \to B^{n-l} \times B^l$ and ω_p be as in Definition 2.1. The topology and smooth structure on $\Sigma(\Omega)$ are such that the induced structure on $\Sigma(V_p \cap \Omega)$ is the same as the one obtained from the canonical identification $\Sigma(V_p \cap \Omega) = [0,1) \times \Sigma(\omega_p) \times B^l$ of Lemma 4.5.

The smooth structure on $\Sigma(\Omega)$ is therefore defined using a covering with sets of the form $\Sigma(V_p \cap \Omega)$ (this desingularization is with respect to V_p and not $M \subset \Omega$!). We need to prove that the transition functions are smooth. This property follows from the fact that the maps ϕ_p are diffeomorphisms and from Lemma 4.5.

We have therefore completed the definition of the desingularization $\Sigma(\Omega)$ and of its smooth structure.

4.2 The distance to singular boundary points

We continue with a study of the geometric and, especially, metric properties of $\Sigma(\Omega)$. We first argue that $\Sigma(\Omega)$ has embedded faces and hence that every hyperface of $\Sigma(\Omega)$ has a defining function.

Let F_0 be an open hyperface of a manifold with corners $\mathfrak{M}$. Then F_0 is a manifold of dimension $n-1$. Its closure F, in general, will not necessarily be a manifold, because it may have self-intersections. (A typical example is the boundary of a curvilinear polygonal domain with only one vertex, the "tear drop domain.") By induction, however, it follows that $F_0 \cap V_p$ will be a manifold with corners, for any p. In particular, we obtain that all (closed) faces of $\Sigma(\Omega)$ are embedded submanifolds of $\Sigma(\Omega)$. Let H be a hyperface of $\Sigma(\Omega)$, since H is an embedded submanifold of codimension 1, there will exist a function $x_H > 0$ on Ω, $H = \{x_H = 0\}$, and $dx_H \neq 0$ on H. A function x_H with this property is called a *defining function of H* [57].

One of the main reasons for introducing the desingularization space $\Sigma(\Omega)$ is the following result.

Proposition 4.7. *Let Ω be a bounded, curvilinear, stratified polyhedral domain and g_1 and g_2 be two smooth Riemannian metrics on M. Let us fix k and assume $\Omega^{(k)} \neq \emptyset$. Let $f_j(x)$ be the modified distance from $x \in \overline{\Omega}$ to the set $\Omega^{(k)}$ in the metric g_j, computed within $\overline{\Omega}$. Then the quotient f_2/f_1 extends to a continuous function on $\Sigma(\Omega)$.*

Proof. It is enough to prove the given property in the neighborhood of every point $p \in \overline{\Omega}$. So let us fix $p \in \overline{\Omega}$. By replacing V_p with a smaller neighborhood of p, if necessary, we can also assume that $g_2(\xi) \leq Cg_1(\xi)$, which implies that $f_2 \leq Cf_1$, and hence that f_2/f_1 is bounded.

We shall prove the statement by induction on n. In the case $n = 1$, the only possibility is that $k = 0$, or otherwise $\Omega^{(k)} = \emptyset$. Then $f(x)$ is the distance to the vertices of Ω. Recall that Ω is a disjoint union of open intervals in this case, so that we can reduce to consider a single interval. If say $\Omega = [a, b]$, then close to a, $f_j(x) = a_j(x)(x-a)$, with a_j smooth near a and $a_j(a) \neq 0$. The same situation holds at b. This proves our result in the case $n = 1$. We now proceed with the induction step.

The function f_1/f_2 is clearly continuous on the open set Ω. Fix $p \in \partial\Omega$. We shall construct an open neighborhood U_p of p in $\overline{\Omega}$ such that f_1/f_2 extends to a continuous function on $\kappa^{-1}(U_p)$. Let V_p be as in the definition of polyhedral domains (Definition 2.1). We shall identify $V_p \cap \Omega$ with $I\omega_p \times B^l$ using the diffeomorphism ϕ_p of Equation (19). If $l > k$, that is, $p \in \Omega^{(l)} \smallsetminus \Omega^{(k)}$, then both f_1 and f_2 extend to continuous, non-vanishing functions on $V_p \cap \overline{\Omega}$, which can be lifted to continuous, non-vanishing functions on $\kappa^{-1}(V_p \cap \Omega)$. We shall assume hence that $k \geq l$.

On a smaller neighborhood $V' \subset V_p$, if necessary, we can arrange that the function f_1 gives the distance to $V_p^{(k)}$, that is, that the point of $\Omega^{(k)}$ closest to $x \in V' \cap \Omega$ is, in fact, in V_p. By decreasing V' even further, we can further arrange that the same holds for f_2. Then we shall take $U_p := V'$.

To prove that f_2/f_1 extends to a continuous function on $\kappa^{-1}(U_p)$, it is enough to do that in the case $\Omega = V_p \cap \Omega$, because the quotient f_2/f_1 does not change on $U_p \cap \Omega$ if we replace Ω with $V_p \cap \Omega$, as explained in the paragraph above. We can also assume that g_2 is the standard Euclidean metric, but then we have to

prove that f_1/f_2 extends to a *nowhere vanishing* continuous function on $\Sigma(\Omega)$. (Using also Proposition 4.2, we have reduced to the case $\Omega = I\omega_p \times B^l \subset \mathbb{R}^n$, $I = (0,t)$.)
The scaling property of the Euclidean metric and our assumption that $k \geq l$ imply that

$$f_2(rx', x'') = rf_2(x', x''),$$

for any $r \in [0,1]$. Let g_0 be a constant metric on $\mathbb{R}^n$ that coincides with g_1 at the origin.
Let f_0 be associated to g_0 in the same way as f_j is associated to g_j, for $j = 1, 2$, *i.e.,* $f_0(x) = \mathrm{dist}(x, \Omega^{(k)})$ using the metric g_0. We then have similarly $f_0(rx', x'') = rf_0(x', x'')$, so that the quotient $f_0(rx', x'')/f_1(rx', x'')$ does not depend on r. We can therefore fix $r = 1$. Consequently, we can work with the lower dimensional polyhedral domain $\omega := \omega_p \times B^l$ instead of $\Omega = I\omega_p \times B^l$, and prove that f_0/f_1 extends by continuity to $\Sigma(\omega)$. It remains to see that we can use induction to prove the existence of this extension. Since ω is by construction a stratified polyhedron, we denote by $\omega^{(k)} = \omega^{(k)} \times B^l$ $k < n$, its associated stratification, where we set $\omega_p^{(k-l-1)} = \emptyset$ if $k - l - 1 < 0$ as before. Let f_1' be the distance function to $\omega^{(k-1)}$ on ω (*i.e.,* computed *within* $\overline{\omega}$, with respect to the metric induced by g_1, as in the statement of Proposition 4.7). We let $f_1' = 1$ if $\omega_p^{(k-l-1)} = \emptyset$.
We define f_0' similarly. The inductive hypothesis guarantees that f_0'/f_1' extends to a continuous function on $\Sigma(\omega) = \Sigma(\omega_p) \times B^l$. On the other hand, it is easy to see that both f_1/f_1' and f_1'/f_1 extend to continuous functions on $\overline{\omega}$ if we set them to be equal to 1 on $\omega^{(k-1)}$. The same is true of f_0/f_0' and f_0'/f_0. Putting all these estimates together, it follows that

$$f_0/f_1 = (f_0/f_0')(f_0'/f_1')(f_1'/f_1)$$

extends to a continuous, nowhere vanishing function on $\Sigma(\omega)$.
Let us tackle now the case g_2 arbitrary. Let f_0 be defined as before. We then have that $f_2(rx', x'') = rf_0(x', x'') + r^2 h(rx', x'')$, with h a continuous function on $\Sigma(V_p \times \Omega)$ that vanishes on $\Omega^{(k)}$. Then

$$\frac{f_2}{f_1} = \frac{f_0}{f_1} + r\frac{h(rx', x'')}{f_1(x', x'')}.$$

The function f_0/f_1 was already shown to extend by continuity to $\Sigma(\Omega)$. The same argument as above shows that h/f_1 extends by continuity to a nowhere vanishing function on

$$[\epsilon, 1) \times \Sigma(\omega_p) \times B^l \subset [0,1) \times \Sigma(\omega_p) \times B^l =: \Sigma(\Omega).$$

The continuity of f_2/f_1 then follows from the boundedness of f_2/f_1.
The resulting function does not vanish at $r = 0$, because it is equal to f_0/f_1 there. It was already proved that it does not vanish for $\epsilon > 0$. The proof is complete. $\square$

We shall need also the following corollary of the above proof.

Corollary 4.8. *Identify* $V_p \cap \Omega = \Omega$ *with* $I\omega_p \times B^l$, $I = (0, a)$, $l = \ell(p)$, *using the diffeomorphism* ϕ_p *given in Definition 2.1. Let* g *be a smooth metric on* V_p, *and let* $f(x)$ *be the distance from* x *to* $\Omega^{(k)}$, $k \geq l$, $f'(x', x'')$ *be the distance from* $(x', x'') \in \omega := \omega_p \times B^l$ *to* $\omega^{(k-1)}$ *(within* $\overline{\omega}$, *as in Proposition 4.7) if* $\omega^{(k-1)} \neq \emptyset$, *and* $f'(x', x'') = 1$ *otherwise. Assume* ω_p *is connected. Then*

$$f(rx', x'')/rf'(x', x'')$$

extends to a continuous, nowhere vanishing function on $\Sigma(\Omega) = [0, a) \times \Sigma(\omega_p) \times B^l$.

Proof. Assume first that $\omega^{(k-1)} \neq \emptyset$, where $\omega^{(k)}$ is defined as in Proposition 4.7. Let f_0 and f_0' be defined in the same way f and f' were defined, but replacing g with a constant metric g_0. Then the proof of Proposition 4.7 gives that $f_0(rx', x'')/rf_0'(x', x'')$ is independent of r. Hence $f_0(rx', x'')/rf_0'(x', x'')$ extends to a continuous, nowhere vanishing function on $\Sigma(\Omega)$, as it was shown in the proof of Proposition 4.7. Then

$$\frac{f(rx', x'')}{rf'(x', x'')} = \frac{f(rx', x'')}{f_0(rx', x'')} \times \frac{f_0(rx', x'')}{rf_0'(rx', x'')} \times \frac{f_0'(x', x'')}{f'(x', x'')}.$$

We have just argued that the middle quotient in the above product extends to a continuous function on $\Sigma(\Omega)$. The other two quotients also extend to continuous functions on $\Sigma(\Omega)$, by Proposition 4.7 applied to Ω and ω.
Let us assume now that $\omega^{(k-1)} = \emptyset$. Then the same proof applies, given that $f_0'/f' = 1$ clearly extends to a continuous function on $\Sigma(\Omega)$. ☐

4.3 The weight function r_Ω

Recall that $\eta_{n-2}(x)$, given in Definition 2.5, denotes the distance from $x \in \overline{\Omega}$ to the singular set $\Omega^{(n-2)}$.
The main goal of this subsection is to define on any curvilinear polyhedral domain Ω a function

$$r_\Omega : \overline{\Omega} \to [0, \infty)$$

that lifts to a *smooth* function on $\Sigma(\Omega)$ and is *equivalent* to η_{n-2}. (Additional properties of r_Ω will be established later on.) This will lead to a definition of the Sobolev spaces $\mathcal{K}_a^m(\Omega)$ as weighted Sobolev spaces on Lie manifolds with boundary, Proposition 5.7. We again proceed by induction on n.
We define $r_\Omega = 1$ if $n \leq 1$ (recall $\Omega^{(n-2)} = \emptyset$ if $n < 2$) or if $\Omega^{(n-2)} = \emptyset$, that is, Ω is a smooth manifold, possibly with boundary.
Assume now that a function r_ω was defined for all curvilinear polyhedral domains ω of dimension $< n$ and let us define it for a given bounded n-dimensional curvilinear polyhedral domain Ω.
We localize first to a neighborhood of a generic point $p \in \partial\Omega$ and then use a partition of unity argument. We recall that by definition there exists a

neighborhood V_p of p in M, a diffeomorphism $\phi_p : V_p \to B^{n-l} \times B^l$, for some $0 \leq l = \ell(p) \leq n-1$, and a polyhedral domain $\omega_p \subset S^{n-l-1}$ such that $\phi_p(V_p \cap \Omega) = I\,\omega_p \times B^l$, $I = (0, \epsilon)$, see Condition (18)). Therefore, we can assume that ϕ_p is the identity map and replace in what follows V_p with $\phi_p^{-1}\big(\frac{1}{2}B^{n-l} \times \frac{1}{2}B^l\big)$. Since r_Ω is already defined equal to 1 if $p \in \Omega^{(n-1)} \smallsetminus \Omega^{(n-2)}$, we restrict $n - l - 1 \geq 1$ above. Let r_{ω_p} be the function associated to the curvilinear polyhedral domain ω_p. Then we define

$$r_{V_P}(rx', x'') := r r_{\omega_p}(x'), \quad (rx', x'') \in \Omega \subset V_p, \tag{33}$$

if $x' \in \omega_p$, $x'' \in B^l$, and $1 \leq l = \ell(p) \leq n-2$. Following our usual procedures, we set $r_{V_p}(rx') = r r_{\omega_p}(x')$ if $l = 0$.

We consider next a locally finite covering of $\overline{\Omega}$ with open sets U_α of one of the three following forms

(i) $U_\alpha \subset \overline{U_\alpha} \subset \Omega$ with ∂U_α smooth;

(ii) $U_\alpha = V_p$ with $\ell(p) = n-1$ (*i.e.,* p is not in the singular set of $\overline{\Omega}$); or

(iii) such that for any $x \in U_\alpha \cap \Omega$, the point of $\Omega^{(n-2)}$ closest to x is in V_p with $\ell(p) \leq n-2$, and

$$p \in U_\alpha \subset \overline{U}_\alpha \subset V_p. \tag{34}$$

A condition similar to (iii) above was already used in the proof of Proposition 4.7. The conditions (i) and (ii) above correspond exactly to the case when $(\partial U_\alpha \cap \partial\Omega)$ is smooth (this includes the case when $(\partial U_\alpha \cap \partial\Omega)$ is empty). We then set

$$r_\alpha = \begin{cases} 1 & \text{if } (\partial U_\alpha \cap \partial\Omega) \text{ is smooth} \\ r_{V_p} & \text{if } U_\alpha \text{ is as in (34).} \end{cases} \tag{35}$$

and define

$$r_\Omega = \sum_\alpha \varphi_\alpha r_\alpha, \tag{36}$$

where φ_α is a smooth partition of unity subordinated to U_α. If Ω is not bounded, we define instead:

$$r_\Omega = \chi\big(\sum_\alpha \varphi_\alpha r_\alpha\big), \tag{37}$$

where χ is defined as in (23). We notice that the definition of r_Ω is not canonical, because it depends on a choice of a covering $\{U_\alpha\}$ of $\overline{\Omega}$ as above and a choice of a subordinated partition of unity.

Proposition 4.9. *Let Ω be a curvilinear, stratified polyhedral domain of dimension $n \geq 2$. Then r_Ω defined in Equation* (36) *(or* (37)*) is continuous on $\overline{\Omega}$ and $r_\Omega \circ \kappa$ is smooth on $\Sigma(\Omega)$. Moreover, η_{n-2}/r_Ω extends to a continuous, nowhere vanishing function on $\Sigma(\Omega)$ and r_α/r_Ω extends to a smooth function on $\Sigma(V_p \cap \Omega)$.*

Proof. Let $\eta_{-1} := 1$ for the inductive step. We shall prove the statement on η_{n-2}/r_Ω by induction on $n \geq 1$. Since $r_\Omega = 1$ for polyhedral domains of dimension $n = 0$, the result is obviously true for $n = 1$. We now proceed with the inductive step.
We shall use the above results, in particular, Proposition 4.7, for $k = n-2 \geq 0$. Thus $f = \eta_{n-2}$ in the notation of Proposition 4.7. Let $f_\alpha(x)$ be the distance from $x \in V_p$ to $V_p \cap \Omega^{(n-2)}$, if $U_\alpha \subset V_p$ is as in Equation (34) (so $\ell(p) \leq n-2$ in this case). Thus $f_\alpha = f$ on $U_\alpha \cap \Omega$, by the construction of U_α. We identify once again $V_p \cap \Omega$ with $(0, \epsilon)\, \omega_p \times B^l$, $l = \ell(p)$, using the diffeomorphism ϕ_p, and set again $\omega := \omega_p \times B^l$. Also, for any $x \in \omega$, let $f'_\alpha(x)$ be the distance from x to the singular set $\omega^{(n-l-2)}$ of ω if it is not empty, $f'_\alpha(x) = 1$ otherwise. Let r_α be as in the definition of r_Ω, Equation (36). Then

$$\frac{f_\alpha(rx', x'')}{r_\alpha(rx', x'')} = \frac{f_\alpha(rx', x'')}{rf'_\alpha(x', x'')}\, \frac{f'_\alpha(x', x'')}{r_{\omega_p}(x', x'')}, \qquad \text{for } (rx', x'') \in V_p \cap \Omega.$$

The quotient $f'_\alpha(rx', x'')/rf'_\alpha(x', x'')$ extends to a continuous, nowhere vanishing function on $\Sigma(V_p \cap \Omega)$, by Corollary 4.8. By the induction hypothesis, the quotient $f'_\alpha(x', x'')/r_{\omega_p}(x', x'')$ also extends to a continuous, nowhere vanishing function on $\Sigma(\omega) = \Sigma(\omega_p) \times B^l$. Since this quotient is independent of r, it also extends to a continuous, nowhere vanishing function on $\Sigma(V_p \cap \Omega)$. Hence f_α/r_α extends to a continuous, nowhere vanishing function on $\Sigma(V_p)$. Therefore

$$r/f = \sum_\alpha \varphi_\alpha r_\alpha / f = \sum_\alpha \varphi_\alpha r_\alpha / f_\alpha$$

extends to a continuous function on $\Sigma(\Omega)$.
The quotient r/f is immediately seen to be non-zero everywhere, from the definition. Hence f/r also extends to a continuous function on $\Sigma(\Omega)$.
We have already noticed that r_α/f extends to a continuous, nowhere vanishing function on $\Sigma(V_p)$. Hence $r_\alpha/r_\Omega = (r_\alpha/f)(f/r_\Omega)$ extends to a continuous, nowhere vanishing function on $\Sigma(V_p \cap \Omega)$. Since both r_α and r_Ω are smooth on $\Sigma(V_p \cap \Omega)$ and the set of zeroes of r_Ω is the union of transversal manifolds on which r_Ω has simple zeroes, it follows that r_α/r_Ω extends to a smooth function on $\Sigma(V_p)$. Since $\overline{U}_\alpha \subset V_p$ is compact, it follows from a compactness argument that r_α and r are equivalent on U_α. The proof is complete. ∎

We can now prove the following result, which will be used in the proof of Theorem 6.4.

Proposition 4.10. *Let Ω be a bounded, curvilinear, stratified polyhedral domain. Suppose r_Ω, r'_Ω are two functions on $\overline{\Omega}$ defined by formula* (36) *(or* (37)*) with possibly different choices of open covering $\{U_\alpha\}$, subordinate partition $\{\varphi_\alpha\}$, and diffeomorphisms ϕ_p. Then r'_Ω/r_Ω extends to a smooth, nowhere vanishing function on $\Sigma(\Omega)$.*

Proof. We know from Proposition 4.9, that f/r'_Ω and f/r_Ω extend to continuous, nowhere vanishing functions on $\Sigma(\Omega)$. Hence r'_Ω/r_Ω extends to a continuous, nowhere vanishing function on $\Sigma(\Omega)$. Since both r'_Ω and r_Ω are smooth functions on $\Sigma(\Omega)$ and the set of zeroes of r_Ω is a union of transverse manifolds, each a set of simple zeroes of r_Ω, it follows that the quotient r'_Ω/r_Ω is smooth on $\Sigma(\Omega)$. $\square$

We obtain the following corollary. Let $H \subset \Sigma(\Omega)$ be a hyperface (*i.e.,* face of maximal dimension) of $\Sigma(\Omega)$. Recall that *a defining function of H* is a smooth function $x_H \geq 0$ defined on $\Sigma(\Omega)$, such that $H = \{x = 0\}$ and $dx_H \neq 0$ on H. All the faces of $\Sigma(\Omega)$ are closed subsets of $\Sigma(\Omega)$, by definition. We have already noticed that any face of $\Sigma(\Omega)$ has a defining function. We then have the following corollary.

Corollary 4.11. *Let $\eta = \prod_H x_H$, where H ranges through the set of hyperfaces of $\Sigma(\Omega)$ that do not intersect $\partial\Omega \smallsetminus \Omega^{(n-2)}$. Then η/r_Ω extends to a smooth, nowhere vanishing function on $\Sigma(\Omega)$.*

Proof. This is a local statement that can be checked by induction, as in the previous proofs. $\square$

In particular, since the function r_Ω is anyway determined only up to a factor of $h \in \mathcal{C}^\infty(\Sigma(\Omega))$, $h \neq 0$, we obtain that we could take $r_\Omega = \prod_H x_H$, where H ranges through the set of hyperfaces of $\Sigma(\Omega)$ that do not intersect $\partial\Omega \smallsetminus \Omega^{(n-2)}$. The function r_Ω, for various versions of the set Ω, will play an important role in the inductive definition of the structural Lie algebra of vector fields $\mathcal{V}(\Omega)$ on $\Sigma(\Omega)$, which we address next. The faces considered in the above corollary are the hyperfaces at infinity of $\Sigma(\Omega)$. See Definition 3.8.

4.4 The structural Lie algebra of vector fields

We now proceed to define by induction a canonical Lie algebra of vector fields $\mathcal{V}(\Omega)$ on $\Sigma(\Omega)$, for Ω a curvilinear, stratified polyhedral domain of dimension $n \geq 1$. In view of Corollary 4.3, we can assume that Ω is connected. We denote by

$$\mathcal{X}(M) := \Gamma\big(M; TM\big)$$

the space of vector fields on a manifold M. We let

$$\mathcal{V}(\Omega) = \mathcal{X}(\overline{\Omega}) = \mathcal{X}(\Sigma(\Omega)), \quad \text{if } n = 1. \tag{38}$$

In other words, there is no restriction on the vector fields $X \in \mathcal{V}(\Omega)$, if Ω has dimension one.

Assume now that the Lie algebra of vector fields $\mathcal{V}(\omega)$ has been defined on $\Sigma(\omega)$ for all curvilinear polyhedral domains ω of dimension $1 \leq k \leq n-1$ and let us define $\mathcal{V}(\Omega)$ for a curvilinear polyhedral domain of dimension n. We fix $p \in \partial\Omega$ and let V_p and ϕ_p be as in Definition 2.1, as usual. We identify $V_p \cap \Omega$ with $(0,1)\omega_p \times B^l$ using ϕ_p. Assume $1 \leq \ell(p) \leq n-2$, so that in particular ω_p is a curvilinear polyhedral domain of dimension ≥ 1. Let $M_1 := [0,1) \times \Sigma(\omega_p) \times B^l$. We notice that

$$TM_1 = T([0,1) \times \Sigma(\omega_p) \times B^l) = T([0,1)) \times T\Sigma(\omega_p) \times TB^l$$

and hence

$$\begin{aligned}\mathcal{X}\big(M_1\big) = \Gamma\big(M_1;\, T[0,1)\big) \times_{M_1} \Gamma\big(M_1;\, T\Sigma(\omega_p)\big) \times_{M_1} \Gamma\big(M_1;\, TB^l\big)\\ \subset \Gamma\big(M_1;\, T[0,1)\big) \times \Gamma\big(M_1;\, T\Sigma(\omega_p)\big) \times \Gamma\big(M_1;\, TB^l\big).\end{aligned}$$

Then we define

$$\begin{aligned}\mathcal{V}(V_p \cap \Omega) = \{X = (X_1, X_2, X_3) \in \mathcal{X}\big(M_1\big)\\ X_1 \in \Gamma\big(M_1;\, T[0,1)\big),\quad X_2 \in \Gamma\big(M_1;\, T\Sigma(\omega_p)\big),\quad X_3 \in \Gamma\big(M_1;\, TB^l\big)\\ Y_1 := r_\Omega^{-1} X_1 \text{ and } Y_3 := r_\Omega^{-1} X_3 \text{ are smooth, and}\\ X_2(t, x', x'') \in \mathcal{V}\big(\{t\} \times \omega_p \times \{x''\}\big) = \mathcal{V}(\omega_p), \quad \text{for any fixed } t, x''\}. \quad (39)\end{aligned}$$

In Equation (39) above, "smooth" means, "smooth *including at* $r = 0$." If $\ell(p) = 0$, then we just drop the component X_3, but keep the same conditions on X_1 and X_2. By Proposition 4.10, the definition of $\mathcal{V}(V_p \cap \Omega)$ is independent of the choice of r_Ω. All vector fields are assumed to be smooth.
Finally, we define $\mathcal{V}(\Omega)$ to consist of the vector fields $X \in \mathcal{X}(\Sigma(\Omega))$ such that $X|_{V_p \cap \Omega} \in \mathcal{V}(V_p \cap \Omega)$ for all $p \in \Omega^{(n-2)}$. In particular, only the smoothness condition is imposed on our vector fields at the smooth points of $\partial\Omega$. Note that the vector fields in $\mathcal{V}(\Omega)$ may not extend to the closure $\overline{\Omega}$, in general. This was seen in Example 2.10.

4.5 Lie manifolds with boundary

We now proceed to show that the pair $(\Sigma(\Omega), \mathcal{V}(\Omega))$ defines a Lie manifold with boundary, introduced in [1], and the construction of which was recalled in Definition 3.5.
We first establish some lemmata.

Lemma 4.12. *Let $X \in \mathcal{X}(\Sigma(\Omega))$ be such that $X = 0$ in a neighborhood of the boundary of $\Sigma(\Omega)$. Then $X \in \mathcal{V}(\Omega)$.*

Proof. The result follows immediately by induction from the definition of $\mathcal{V}(\Omega)$. □

We also get the following simple fact.

Lemma 4.13. *If $f : \Sigma(\Omega) \to \mathbb{C}$ is a smooth function and $X \in \mathcal{V}(\Omega)$, then $X(f)$ is a smooth function on $\Sigma(\Omega)$ and $fX \in \mathcal{V}$.*

Proof. The vector field X is smooth on $\Sigma(\Omega)$, hence $X(f)$ is smooth on $\Sigma(\Omega)$. The second statement is local, so it is enough to check it on Ω and on each V_p, on which it is as a direct consequence of the definition and induction. □

Lemma 4.14. *For any $X \in \mathcal{V}(\Omega)$ and any continuous function $f : \overline{\Omega} \to \mathbb{C}$ such that $f \circ \kappa$ is smooth on $\Sigma(\Omega)$, we have*

$$X(f) = \tilde{f}\, r_\Omega,$$

where $\tilde{f}$ is a smooth function on $\Sigma(\Omega)$. In particular, $X(r_\Omega) = f_X r_\Omega$, where f_X is a smooth function on $\Sigma(\Omega)$.

Proof. This is a local statement that can be checked by induction in any neighborhood V_p, using the definition, as follows. Let us use the notation of Equation (39) and write

$$X = (X_1, 0, 0) + (0, X_2, 0) + (0, 0, X_3).$$

We shall write, with abuse of notation, $X_1 = (X_1, 0, 0)$. Define X_2 and X_3 similarly. It is enough to check that $X_j f(rx', x'')$ is of the indicated form, for $j = 1, 2, 3$. We have $X_1 = r_\Omega Y_1$ and $X_3 = r_\Omega Y_3$, where Y_1 and Y_3 are smooth (in appropriate spaces), by Equation (39). This observation proves our lemma if $X = X_1$ or $X = X_3$. If $X = X_2$, then we have

$$(Xf)(r, x', x'') = X_2(f(rx', x'')) = r_{\omega_p} f_1(r, x', x''), \tag{40}$$

with f_1 a smooth function on $\Sigma(V_p \cap \Omega) = [0, \epsilon) \times \Sigma(\omega_p) \times \mathbb{R}^l$, by the induction hypothesis. Moreover, given that by assumption (39) $\kappa_* X$ is a vector field tangent to the sphere S^{n-l-1}, we see that $Xf(0, x', x'') = 0$. Therefore $Xf = r r_{\omega_p} \tilde{f}$, for some smooth function $\tilde{f}$ on $\Sigma(V_p \cap \Omega)$. Let us denote $r_\alpha = r r_{\omega_p}$, as in Equation (35) and in Proposition 4.9. Proposition 4.9 gives that r_α / r_Ω is smooth on its domain of definition. Hence $Xf = r_\alpha f_1 = r_\Omega (r_\alpha / r_\Omega) f_1 = r_\Omega \tilde{f}$, with $\tilde{f}$ smooth on each $\Sigma(V_p \cap \Omega)$. Hence $\tilde{f}$ is smooth on $\Sigma(\Omega)$. □

We next characterize which vector fields on Ω are restrictions of vector fields on $\mathcal{V}(\Omega)$. We begin by showing that the restriction property is local.

Lemma 4.15. *Let Y be a vector field on Ω with the property that every point $p \in \overline{\Omega}$ has a neighborhood U_p in M such that $Y = X_U$ on $U \cap \Omega$, for some $X_U \in \mathcal{V}(\Omega)$. Then there exists $X \in \mathcal{V}(\Omega)$ such that Y is the restriction of X to Ω.*

Proof. Let us cover $\overline{\Omega}$ with a locally finite family of sets U_p, $p \in B \subset \overline{\Omega}$. Let ψ_p, $p \in B$, be a subordinated partition of unity.
We claim that $X = \sum_{p \in B} \psi_p X_{U_p} \in \mathcal{V}(\Omega)$ (by Lemma 4.13) satisfies $X(x) = Y(x)$, $x \in \Omega$. Indeed, $X(x) = \sum_{p \in B} \psi_p(x) X_{U_p}(x) = \big(\sum \psi_p(x)\big) Y(x) = Y(x)$. □

We can now prove the following lemma.

Lemma 4.16. *Let Y be a smooth vector field on $\overline{\Omega}$. Then $r_\Omega Y$ is the restriction to $\Omega \subset \Sigma(\Omega)$ of a vector field X in $\mathcal{V}(\Omega)$.*

Proof. By Lemma 4.15, it is enough to check this statement on a neighborhood V_p of some $p \in \partial\Omega$. We shall proceed by induction. Since the desingularization and the definition of r_Ω are covariant with respect to diffeomorphism (that respect the stratification of Ω), we can assume that $V_p = B^{n-l} \times B^l$ and that $V_p \cap \Omega \simeq (0,1)\omega_p \times B^l$. Assume first that $Y = \partial_j$ is a constant vector field on V_p. Let $\alpha_t(x', x'') = (tx', x'')$. Then $D\alpha_t(\partial_j) = t\partial_j$. Therefore,

$$D\alpha_t(X) = X, \tag{41}$$

where $X = r_\Omega \partial_j$, where r_Ω can be taken, on V_p, to be given by rr_{ω_p}. Let us decompose $\partial_j = (Y_1, Y_2, Y_3)$ on V_p using the notation of Equation (39). Then Y_3 is constant. In fact, either $Y_3 = \partial_j$ or $Y_3 = 0$. In any instance, if we write $X = (X_1, X_2, X_3)$, then $X_3 = r_\Omega Y_3$ satisfies the condition of Equation (39). The relation (41) gives that $Y_1(r, x', x'') = a_1(x')\partial_r$ and $Y_2(r, x', x'') = r^{-1}Z(x')$, with a_1 a smooth function and Z a smooth vector field on $\overline{\omega}_p$. Clearly $X_1 = r_\Omega Y_1$ will satisfy the conditions of Equation (39). The induction hypothesis then gives that $X_2(r, x', x'') = r_\Omega Y_2(r, x', x'') = r_{\omega_p}(x')Z(x')$ is the restriction to $V_p \cap \Omega$ of a smooth vector field in $\mathcal{V}(V_p \cap \Omega)$. (This vector field depends only on the second factor in $\Sigma(V_p \cap \Omega) = [0,1) \times \omega_p \times B^l$.) □

We now identify a canonical metric on the vector fields $\mathcal{V}$. Recall that the concept of local basis of a space of vector fields was defined in Definition 3.1.

Proposition 4.17. *Let us fix a metric h on $M \supset \Omega$. Let $q \in \Sigma(\Omega)$ be arbitrary. Then there exists a neighborhood U of q in $\Sigma(\Omega)$ and $X_1, X_2, \ldots, X_n \in \mathcal{V}(\Omega)$ that form a local basis of $\mathcal{V}(\Omega)$ on U and satisfy*

$$h(X_j, X_k) = r_\Omega^2 \delta_{jk}.$$

In other words, the vectors $X_1, X_2, \ldots, X_n$ form an orthonormal system on $\Omega \cap U$ for the metric $r_\Omega^{-2}h$. A local basis $X_1, X_2, \ldots, X_n$ with this property will be called *a local orthonormal basis of $\mathcal{V}(\Omega)$ over U.*

Proof. If $q \in \Omega \subset \Sigma(\Omega)$, the result follows from Lemma 4.12. Let $p = \kappa(q)$. We shall hence assume that $p \in \partial\Omega$. This is again a local statement in $p \in \partial\Omega$. We can therefore proceed by induction. If the dimension n of Ω is 1, then there is nothing to prove because $r_\Omega = 1$ in this case.
Once again, we let $\phi_p : V_p \to B^{n-l} \times B^l$ and ω_p be as in Definition 2.1. We can assume that ϕ_p is the identity map. If we can prove the result for the function $r = r_\Omega$, then we can prove it for the function $r' = f'r$, where $f', 1/f' \in \mathcal{C}^\infty(\Sigma(\Omega))$, simply by replacing X_j with $f'X_j$. By Proposition 4.9, we can therefore assume that $r_\Omega = rr_{\omega_p}$ on $V_p \cap \Omega$. Let $q = (0, x', x'') \in [0,1) \times \Sigma(\omega_p) \times B^l$.
Let h_0 be the standard metric on V_p. For the induction hypothesis, we shall need that the metric h_0 is given by

$$h_0(r, x', x'') = (dr)^2 + r^2(dx')^2 + (dx'')^2 \tag{42}$$

on $\Omega \cap V_p = (0,1)\omega_p \times B^l$. Here $(dx')^2$ denotes the metric on ω_p induced by the Euclidean metric on the sphere S^{n-l-1}. In other words, if $X = (X_1, X_2, X_3)$ is a vector field on $V_p \cap \Omega$, written using the product decomposition explained above (or as in the Equation (39)), then

$$h_0(X) = \|X_1\|^2 + r^2\|X_2\|^2 + \|X_3\|^2$$

where the norms come from the standard metrics, respectively, on $T[0,1)$, on $TS^{n-l-1} \supset T\omega_p$, and on $T\mathbb{R}^l$.
Let us assume first that $h = h_0$, the standard metric on $\mathbb{R}^n$. By the induction hypothesis, we can construct $Y_2, \ldots, Y_{n-l} \in \mathcal{V}(\omega_p)$ forming a local orthonormal basis of $\mathcal{V}$ over some small neighborhood U' of x' in $\Sigma(\omega_p)$ (*i.e.,* $\{Y_2, \ldots, Y_{n-l}\} \subset \mathcal{V}(\omega_p)$ is orthonormal with respect to the metric $r_{\omega_p}^{-2}(dx')^2$). Here $(dx')^2$ denotes the metric on ω_p induced by the Euclidean metric on the sphere S^{n-l-1}, as above. Let $Y_1 = r_\Omega \partial_r$ and $Y_j = r_\Omega \partial_j$, $j = n-l+1, \ldots, n$, where ∂_j forms the standard basis of $\mathbb{R}^{l-1}$. Then we claim that we can take $U = [0,1) \times U' \times B^l$ and

$$\{X_1, X_2, \ldots, X_n\} = \{Y_1\} \cup \{Y_2, \ldots, Y_{n-l}\} \cup \{Y_{n-l+1}, \ldots, Y_n\}. \tag{43}$$

(If $n-l=1$, then the second set in the above union is empty. If $l=0$, then the third set in the above union is empty.) Indeed, $\{X_1, \ldots, X_n\}$ is a local basis by construction and by the local definition of $\mathcal{V}(\Omega)$ in Equation (39). Let us check that this is an orthonormal local basis. To this end, we shall use the form of the standard metric h_0 given in Equation (42), to obtain

$$h_0(X_1) = r_\Omega^2\|\partial_r\|^2 = r_\Omega^2\,, \quad h_0(X_{n-l+1}) = \ldots = h_0(X_n) = r_\Omega^2$$
$$\text{and } \ h_0(X_2) = \ldots = h_0(X_{n-l}) = r^2\|X_2\|^2 = r^2 r_{\omega_p}^2 = r_\Omega^2\,.$$

It is also clear that $\{X_1, X_2, \ldots, X_n\}$ is an orthogonal system. This completes the induction step if $h = h_0$, the standard metric on $\mathbb{R}^n$.
If h is not the standard metric on V_l, we can nevertheless chose a matrix valued function T defined on a neighborhood of q in U such that $h(T\xi, T\eta) = h_0(\xi, \eta)$. We then let $X_j = TY_j$ and replace U with this smaller neighborhood. □

This lemma gives the following corollary.

Corollary 4.18. *Let $X, Y \in \mathcal{V}(\Omega)$ and h be a fixed metric on M. Then the function $r_\Omega^{-2}h(X,Y)$, defined first on Ω, extends to a smooth function on $\Sigma(\Omega)$.*

Proof. This is a local statement in the neighborhood of each point $q \in \Sigma(\Omega)$. Let $X_1, X_2, \ldots, X_n$ be a local basis of $\mathcal{V}$ on a neighborhood U of q in $\Sigma(\Omega)$ satisfying the conditions of Proposition 4.17 (*i.e.,* orthogonal with respect to $r_\Omega^{-2}h$). Let $X = \sum \phi_j X_j$ and $Y = \sum \psi_j X_j$ on $U \cap \Omega$, where ϕ_j, ψ_j are smooth functions on $\Sigma(\Omega)$. Then $r_\Omega^{-2}h(X,Y) = \sum \phi_j \overline{\psi_j}$ is smooth on U. □

Lemma 4.19. *Let $p \in \partial\Omega$ and $X_1, X_2, \ldots, X_n$ be vector fields on $\overline{\Omega}$ that define a local basis of TM on $\overline{U}$, for some neighborhood U of p. Then $r_\Omega X_1, r_\Omega X_2, \ldots, r_\Omega X_n$ is a local basis of $\mathcal{V}(\Omega)$ on U, that is, for any $Y \in \mathcal{V}(\Omega)$, there exist unique smooth function $\phi_1, \phi_2, \ldots, \phi_n$ on $\Sigma(\Omega)$ satisfying*

$$Y = \phi_1 r_\Omega X_1 + \phi_2 r_\Omega X_2 + \ldots + \phi_n r_\Omega X_n \quad \text{on } U \cap \Omega \subset \Sigma(\Omega). \tag{44}$$

Conversely, if a vector field Y on Ω satisfies Condition (44) for any p and any local basis $X_1, \ldots, X_n$ of TM at p, then Y is the restriction to Ω of a vector field in $\mathcal{V}(\Omega)$.

Proof. The converse part is easier, so we prove it first. Let Y be a vector field on Ω that satisfies Condition (44) for any p and any local basis $X_1, \ldots, X_n$ of TM at p. Fix an arbitrary $p \in \Omega$. Lemmata 4.13 and 4.16 give that $\phi_j r_\Omega X_j$ is the restriction to Ω of a vector field in $\mathcal{V}(\Omega)$. Hence on each $U \cap \Omega$, Y is the restriction of a vector field $Y_U \in \mathcal{V}(\Omega)$. Lemma 4.15 then gives the converse part of our lemma.
We now prove the direct part of the lemma. We can assume that the vector fields $X_1, \ldots, X_n$ form an orthonormal system on U with respect to some fixed metric h on M. We know from Lemma 4.16 that $r_\Omega X_j \in \mathcal{V}(\Omega)$.
Let then $Y \in \mathcal{V}(\Omega)$ and note that $\phi_j = r_\Omega^{-1} h(Y, X_j) = r_\Omega^{-2} h(Y, r_\Omega X_j) \in \mathcal{C}^\infty(\Sigma(\Omega))$, by Corollary 4.18. Then $Y = \sum_{j=1}^n \phi_j r_\Omega X_j$ on $U \cap \Omega$. The local uniqueness of the functions ϕ_j follows from the fact that $r_\Omega X_1, r_\Omega X_2, \ldots, r_\Omega X_n$ also form a local basis of $T\Omega$ on $U \cap \Omega$. ◻

We are now ready to prove the following characterizations of $\mathcal{V}(\Omega)$. We notice that the restriction map $\mathcal{V}(\Omega) \ni X \to X|_\Omega$ is injective, so we may identify $\mathcal{V}(\Omega)$ with a subspace of the space $\Gamma(\Omega, TM)$ of vector fields on Ω.

Proposition 4.20. *Let $\Omega \subset M$ be a curvilinear, stratified polyhedral domain of dimension n and let X be a smooth vector field on Ω. Fix an arbitrary metric h on M. Then $X \in \mathcal{V}(\Omega)$ if, and only if, $r_\Omega^{-1} h(X, Y)$ extends to a smooth function on $\Sigma(\Omega)$ for any smooth vector field Y on $\overline{\Omega}$.*

Proof. In one direction the result follows from Lemma 4.16 and Corollary 4.18. Indeed, let $X \in \mathcal{V}(\Omega)$ and Y be a smooth vector field on $\overline{\Omega}$. Then $r_\Omega Y \in \mathcal{V}(\Omega)$ by Lemma 4.16 and hence $r_\Omega^{-1} h(X, Y) = r_\Omega^{-2} h(X, r_\Omega Y)$ extends to a smooth function on $\Sigma(\Omega)$ by Corollary 4.18. (We have already used this argument in the proof of the previous lemma.)
Conversely, assume that $r_\Omega^{-1} h(X, Y)$ extends to a smooth function on $\Sigma(\Omega)$ for any smooth vector field on $\overline{\Omega}$. The statement that $X \in \mathcal{V}(\Omega)$ is a local statement, by Lemma 4.15. So let $p \in \overline{\Omega}$ and let U be an arbitrary neighborhood of p. Choose smooth vector fields defined in a neighborhood of $\overline{\Omega}$ in M such that $X_1, X_2, \ldots, X_n$ is a local orthonormal basis on U (orthonormal with respect to h). Let

$$\phi_j = r_\Omega^{-1} h(Y, X_j),$$

by assumption $\phi_j \in \mathcal{C}^\infty(\Sigma(\Omega))$. Then $Y = \sum_{j=1}^n \phi_j X_j$ on $U \cap \Omega$ and $\sum_{j=1}^n \phi_j X_j \in \mathcal{V}(\Omega)$. Lemma 4.15 then shows that $X \in \mathcal{V}(\Omega)$. ◻

We now prove the main characterization of the structural Lie algebra of vector fields $\mathcal{V}(\Omega)$.

Theorem 4.21. *Let $\Omega \subset M$ be a bounded curvilinear, stratified polyhedral domain of dimension n. Then $\mathcal{V}(\Omega)$ is generated as a vector space by the vector fields of the form $\phi\, r_\Omega X$, where $\phi \in \mathcal{C}^\infty(\Sigma(\Omega))$ and X is a smooth vector field on $\overline{\Omega}$.*

Proof. We know that $\phi r_\Omega X \in \mathcal{V}(\Omega)$ whenever X is a smooth vector field on $\overline{\Omega}$, by Lemmata 4.13 and 4.16. This remark shows that the linear span of vectors of the form $\phi\, r_\Omega X$, where $\phi \in \mathcal{C}^\infty(\Sigma(\Omega))$ and X is a smooth vector field in a neighborhood of Σ, is contained in $\mathcal{V}(\Omega)$.
Conversely, let $Y \in \mathcal{V}(\Omega)$. Then Lemma 4.19 shows that we can find, in the neighborhood U_p of any point $p \in \overline{\Omega}$ vector fields $X_{1p}, X_{2p}, \ldots, X_{np}$ and smooth functions ϕ_{jp} such that $Y = \sum \phi_{jp} r_\Omega X_{jp}$ on U_p. The result then follows using a finite partition of unity on $\Sigma(\Omega)$ subordinated to the covering U_p. □

If we drop the condition that Ω be bounded, we obtain the following result, which was established in the first half of the above proof.

Proposition 4.22. *Let $\Omega \subset M$ be a curvilinear polyhedral domain of dimension n. Then $\mathcal{V}(\Omega)$ consists of the set of vector fields that locally can be written as linear combinations of vector fields of the form $\phi r_\Omega X$, where $\phi \in \mathcal{C}^\infty(\Sigma(\Omega))$ and X is a smooth vector field on $\overline{\Omega}$.*

We are finally in the position to endow $\Sigma(\Omega)$ with a structure of Lie manifold, which we will exploit in the following sections to study the mixed boundary value/interface problem (6). We set $\partial''\Sigma(\Omega)$ to be the union of all hyperfaces (*i.e.,* faces of maximal dimension) H of $\Sigma(\Omega)$ such that $\kappa(H) \subset \overline{\Omega}$ lies in the singular set $\Omega^{(n-2)}$, and let $\partial'\Sigma(\Omega) = \partial\Sigma(\Omega) \smallsetminus \partial''\Sigma(\Omega)$. In particular, $\partial''\Sigma(\Omega)$ is the union of the hyperfaces at infinity of $\Sigma(\Omega)$, see Definition 3.8. The next theorem is the main result of this subsection.

Theorem 4.23. *Let Ω be a bounded curvilinear, stratified polyhedral domain and let*

$$\mathfrak{O}_0 := \Sigma(\Omega) \smallsetminus \partial''\Sigma(\Omega) = \Omega \cup \partial'\Sigma(\Omega) = \kappa^{-1}(\overline{\Omega} \smallsetminus \Omega^{(n-2)}).$$

Then $(\mathfrak{O}_0, \Sigma(\Omega), \mathcal{V}(\Omega))$ is a Lie manifold with boundary $\partial'\Sigma(\Omega)$. The projection map $\kappa : \mathfrak{O}_0 \to \overline{\Omega} \smallsetminus \Omega^{(n-2)}$ is such that $\kappa^{-1}(p)$ consists of exactly one point if $p \in \overline{\Omega} \smallsetminus \Omega^{(n-2)}$.

Proof. The last statement (on the number of elements in $\kappa^{-1}(p)$, $p \in \overline{\Omega} \smallsetminus \Omega^{(n-2)}$) follows from the definition. Therefore, to prove the proposition, we need, using the notation of Definition 3.5, to construct a compactification $\mathfrak{O}$ of $\mathfrak{O}_0$ that identifies with the closure of a Lie domain in a Lie manifold $\mathfrak{M}$.
We shall choose then $\mathfrak{O} = \Sigma(\Omega)$. Then we shall let $\mathfrak{M}$ be the "double" of $\Sigma(\Omega)$, also denoted ${}^d\Sigma(\Omega)$. More precisely, $\mathfrak{M}$ is obtained from the disjoint union of

two copies of $\Sigma(\Omega)$ by identifying the hyperfaces that are not at infinity. We let $\mathcal{V}$ to be the set of smooth vector fields on $\mathfrak{M}$ such that the restriction to either copy of $\Sigma(\Omega)$ is in $\mathcal{V}(\Omega)$.

Let $\mathbb{D}$ be obtained from the closure of Ω in $\mathfrak{M}$ by removing the closure of $\partial'\Sigma(\Omega)$. Then $\mathbb{D}$ is an open subset of $\mathfrak{M}$ whose closure is $\Sigma(\Omega)$. Moreover, $\partial_{\mathfrak{M}}\mathbb{D}$ (the boundary of $\mathbb{D}$ regarded as a subset of $\mathfrak{M}$) is the closure of $\partial'\Sigma(\Omega)$.

To prove our theorem, we shall check that $\mathfrak{M}$ is a manifold with corners, that $(\mathfrak{M}, \mathcal{V})$ is a Lie manifold, and that $\partial\mathbb{D}$ is a regular submanifold of $\mathfrak{M}$. Each of these properties is local, so it can be checked in the neighborhood of a point of $\mathfrak{M}$.

Fix $V_p = (0, \epsilon) \times \omega_p \times B^l$. Then the union of the two copies of $\Sigma(V_p)$ is the double ${}^d\Sigma(V_p)$ of $\Sigma(V_p)$. Denote by ${}^d\omega_p$ the double of ω_p. Then

$$ {}^d\Sigma(V_p) = [0, \epsilon) \times {}^d\omega_p \times B^l. $$

An inductive argument then shows that ${}^d\Sigma(\Omega)$ is a manifold with corners and that $\partial\mathbb{D}$ is a regular submanifold of $\mathfrak{M}$.

Let us check that $\mathcal{V}$ satisfies the conditions defining a Lie manifold structure on $\mathfrak{M}$. It follows from Theorem 4.21 that $\mathcal{V}$ is a $\mathcal{C}^\infty(\mathfrak{M})$–module (this checks condition (iii) of Definition 3.2). Theorem 4.21 and Lemmata 4.14, 4.13 show that $\mathcal{V}$ is closed under Lie brackets (this checks condition (i) of Definition 3.2). Condition (ii) of that definition follows from the definition of $\mathcal{V}(\Omega)$. Condition (iv) of Definition 3.2 as well as Condition (ii) of Definition 3.3 were proved in Lemma 4.19. This shows that $(\mathfrak{M}, \mathcal{V})$ is a Lie manifold. ☐

An immediate consequence of the above Proposition is that the boundary $\partial\mathfrak{O}_0 = \partial'\Sigma(\Omega)$ of $\mathfrak{O}_0 = \Sigma(\Omega) \smallsetminus \partial''\Sigma(\Omega)$ will acquire the structure of a Lie manifold, as explained after the definition of a Lie manifold with boundary, Definition 3.5. Let D be the closure of $\partial\mathfrak{O}_0$ in $\mathfrak{O}$. Then the Lie structure at infinity is $(\partial\mathfrak{O}_0, D, \mathcal{W})$, where

$$ \mathcal{W} = \{X|_D,\, X \in \mathcal{V}, X|_D \text{ is tangent to } D\}. \tag{45} $$

As always, $X \in \mathcal{W}$ is completely determined by its restriction to $\mathfrak{O}_0$.

5 Weighted Sobolev spaces

One of the main goals of this work, as mentioned already, is the study of mixed boundary value/interface problems for second-order elliptic operators on n-dimensional curvilinear, stratified polyhedral domains Ω in the framework of certain weighted Sobolev spaces. This framework is adapted to the singular geometry of polyhedral domains and allows to obtain optimal regularity, which does not hold in general in the standard (unweighted) spaces.

We begin by giving a rigorous definition of the weighted spaces. Let f be a continuous function on Ω, $f > 0$ on the interior of Ω. We define the μ-th

Sobolev space with weight f and index a by

$$\mathcal{K}^{\mu}_{a,f}(\Omega) = \{u \in L^2_{\mathrm{loc}}(\Omega),\ f^{|\alpha|-a}\partial^{\alpha}u \in L^2(\Omega),\ \text{for all } |\alpha| \le \mu\}\,, \quad \mu \in \mathbb{Z}_+. \tag{46}$$

The norm on $\mathcal{K}^{\mu}_{a,f}(\Omega)$ is given by

$$\|u\|^2_{\mathcal{K}^{\mu}_{a,f}(\Omega)} := \sum_{|\alpha|\le\mu} \|f^{|\alpha|-a}\partial^{\alpha}u\|^2_{L^2(\Omega)}\,. \tag{47}$$

Definition 5.1. Let f, g be two continuous, non-negative functions on Ω. We shall say that f and g are *equivalent* (written $f \sim g$) if there exists a constant $C > 0$ such that

$$C^{-1}f(x) \le g(x) \le Cf(x),$$

for all $x \in \Omega$.

Clearly, if $f \sim g$, then the norms $\|u\|_{\mathcal{K}^{\mu}_{a,f}(\Omega)}$ and $\|u\|_{\mathcal{K}^{\mu}_{a,g}(\Omega)}$ are equivalent, and hence we have $\mathcal{K}^{\mu}_{a,f}(\Omega) = \mathcal{K}^{\mu}_{a,g}(\Omega)$ as Banach spaces.

Definition 5.2. Let $f = \eta_{n-2}$ be the distance to $\Omega^{(n-2)}$, as before. Then we define $\mathcal{K}^{\mu}_{a}(\Omega) = \mathcal{K}^{\mu}_{a,f}(\Omega)$ and $\|u\|_{\mathcal{K}^{\mu}_{a}(\Omega)} = \|u\|_{\mathcal{K}^{\mu}_{a,f}(\Omega)}$.

For example, $\mathcal{K}^0_0(\Omega) = L^2(\Omega)$. For Ω a polygon in the plane, $\eta_{n-2}(x) = \eta_0(x)$ is the distance from x to the vertices of Ω and the resulting spaces $\mathcal{K}^{\mu}_{a}(\Omega)$ are the spaces considered in Kondratiev's paper [41]. Above in Definition 5.2, we can and will replace η_{n-2} with the equivalent function r_Ω by Proposition 4.9. If $\mu \in \mathbb{N} = \mathbb{Z}_+ \smallsetminus \{0\}$, we define $\mathcal{K}^{-\mu}_{-a}(\Omega)$ to be the dual of

$$\overset{\circ}{\mathcal{K}}{}^{\mu}_{a}(\Omega) := \mathcal{K}^{\mu}_{a}(\Omega) \cap \{\partial^j_\nu u|_{\partial\Omega} = 0,\ j = 0, 1, \ldots, \mu - 1\} \tag{48}$$

with pivot $\mathcal{K}^0_0(\Omega)$. Later in this Section, we will identify $\mathcal{K}^{m}_{a}(\Omega)$ with a suitable space $hH^{\mu}(\Sigma(\Omega))$ using the Lie structure on $\Sigma(\Omega)$. Then, Theorem 3.4 of [1] (see also Lemma 3.7 and Proposition 3.9) gives that $\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ is dense in $\overset{\circ}{\mathcal{K}}{}^{\mu}_{a}(\Omega)$ and consequently $\mathcal{K}^{-\mu}_{-a}(\Omega)$ is the completion of the space of smooth functions u on Ω satisfying

$$\|u\|_{\mathcal{K}^{-\mu}_{-a}(\Omega)} = \sup_{0\ne v\in\mathcal{C}^\infty_{\mathrm{c}}(\Omega)} \frac{|(u,v)|}{\|v\|_{\mathcal{K}^{\mu}_{a}(\Omega)}} < +\infty. \tag{49}$$

In order to make the identification $\mathcal{K}^{\mu}_{a}(\Omega) \approx hH^{\mu}(\Sigma(\Omega))$, for suitable h, we introduce next a class of "admissible weights" h.

5.1 The set of weights

If $h > 0$ on Ω, we denote

$$h\mathcal{K}^{\mu}_{a}(\Omega) := \{hu,\ u \in \mathcal{K}^{\mu}_{a}(\Omega)\}, \tag{50}$$

with induced norm, that is $\|hu\|_{h\mathcal{K}^\mu_a(\Omega)} = \|u\|_{\mathcal{K}^\mu_a(\Omega)}$.
A weight h on Ω will be called *admissible* if it is admissible on $\Sigma(\Omega)$. One of the main examples of an admissible weight is η^a_{n-2}, for $a \in \mathbb{R}$. We recall that an admissible weight on $\Sigma(\Omega)$ is a function h of the form $h = \prod_H x_H^{a_H}$, where H ranges through the set of hyperfaces at infinity of $\Sigma(\Omega)$ and $a_H \in \mathbb{R}$. The topology is induced from the topology on the set $\{(a_H)\}$ of exponents.
As discussed after Corollary 4.11, we can always assume $r_\Omega := \prod_H x_H$. In particular, r^a_Ω, $a \in \mathbb{R}$, is the most important example of an admissible weight. It is more suitable to use this weight, which is intimately related to the structure of Lie manifold on $\Sigma(\Omega)$ ($\smallsetminus\partial$"$\Sigma(\Omega)$) described in Theorem 4.23, instead of η^a_{n-2}.
We also have that

$$r^t_\Omega \mathcal{K}^\mu_a(\Omega) = \mathcal{K}^\mu_{a+t}(\Omega), \tag{51}$$

so in a statement about the spaces $h\mathcal{K}^\mu_a(\Omega)$, where h is an admissible weight, we can usually assume that $a = 0$, without loss of generality. These spaces are *weighted Sobolev spaces* in the sense of the following definition. (These spaces are sometimes called *Babuška–Kondratiev spaces.*)

Definition 5.3. Let h be an admissible weight on Ω. The *weighted Sobolev space* of order $\mu \in \mathbb{Z}$ and weight h on Ω is the space $h\mathcal{K}^\mu_0(\Omega)$.

5.2 Sobolev spaces and Lie manifolds

We now identify the weighted Sobolev space $\mathcal{K}^\mu_a(\Omega)$ with $hH^\mu(\Sigma(\Omega))$, for a suitable admissible weight h; more precisely, $h = r_\Omega^{a-n/2}$. The following description of $\mathcal{V}(\Omega)$ for Ω a curvilinear polyhedral domain in $\mathbb{R}^n$ will be useful. It follows readily from Theorem 4.21.

Corollary 5.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded curvilinear, polyhedral domain. Then*

$$\mathcal{V}(\Omega) = \{\phi_1 r_\Omega \partial_1 + \phi_2 r_\Omega \partial_2 + \ldots + \phi_n r_\Omega \partial_n, \ \textit{where } \phi_j \in \mathcal{C}^\infty(\Sigma(\Omega))\}.$$

We shall denote by

$$\mathrm{Diff}^k_\Omega := \mathrm{Diff}_{\mathcal{V}(\Omega)}(\Sigma(\Omega)) \tag{52}$$

the space of differential operators with coefficients in $\mathcal{C}^\infty(\Sigma(\Omega))$ of order $\le k$ on $\Sigma(\Omega)$ generated by $\mathcal{V}(\Omega)$. The algebra of differential operators $\mathrm{Diff}^\infty_\Omega$ is an example of the algebra of differential operators considered in 3.3. From the last corollary, we obtain directly the following lemma.

Lemma 5.5. *Let $X_1, X_2, \ldots, X_k$ be smooth vector fields on M. Then*

$$P := r^k_\Omega X_1 X_2 \ldots X_k \in \mathrm{Diff}^k_\Omega.$$

Moreover, Diff^k_Ω *is generated linearly by ϕP, with P as above and $\phi \in \mathcal{C}^\infty(\Sigma(\Omega))$.*

Proof. For $k = 1$, this follows from Lemma 4.16. Next, we have

$$r_\Omega^{k+1} X_0 X_1 \dots X_k = r_\Omega X_0 r_\Omega^k X_1 \dots X_k - k X_0(r_\Omega) r_\Omega^k X_1 \dots X_k.$$

The fact that $P \in \mathrm{Diff}_\Omega^k$ then follows by induction, since $X_0(r_\Omega) \in \mathcal{C}^\infty(\Sigma(\Omega))$, by Lemma 4.14.
Conversely, we can similarly check by induction (using the same identity above) that the product $r_\Omega X_1 r_\Omega X_2 \dots r_\Omega X_k$ can be written as a linearly combination of differential operators of the form ϕP, with $\phi \in \mathcal{C}^\infty(\Sigma(\Omega))$ and P as above. Since $r_\Omega X$ generate $\mathcal{V}(\Omega)$ as a $\mathcal{C}^\infty(\Sigma(\Omega))$–module (see Corollary 5.4 or the second part of Theorem 4.21), the result follows. □

We next provide a different description of the weighted Sobolev spaces $\mathcal{K}_a^\mu(\Omega)$, $\mu \in \mathbb{Z}_+$. For a multiindex α, we denote

$$(r_\Omega \partial)^\alpha := (r_\Omega \partial_1)^{\alpha_1} (r_\Omega \partial_2)^{\alpha_2} \dots (r_\Omega \partial_n)^{\alpha_n}. \tag{53}$$

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded curvilinear, stratified polyhedral domain and*

$$\|u\|_{\mu,a}^2 := \sum_{|\alpha| \le \mu} \|r_\Omega^{-a} (r_\Omega \partial)^\alpha\, u\|_{L^2(\Omega)}^2.$$

Then $\|u\|_{\mu,a}$ is equivalent to $\|u\|_{\mathcal{K}_a^\mu(\Omega)}$ of Definition 5.2. In particular,

$$\mathcal{K}_a^\mu(\Omega) = \{u,\, \|u\|_{\mu,a} < \infty\}.$$

Proof. We have that

$$\begin{aligned} u \in \mathcal{K}_a^\mu(\Omega) \ &\Leftrightarrow r_\Omega^{|\alpha|-a} \partial^\alpha u \in L^2(\Omega) \quad \text{for all } |\alpha| \le \mu \quad \text{ by Proposition 4.9} \\ &\Leftrightarrow r_\Omega^{-a} (r_\Omega \partial)^\alpha u \in L^2(\Omega) \quad \text{for all } |\alpha| \le \mu \quad \text{ by Proposition 5.5.} \end{aligned}$$

Above the corresponding square integrability conditions define the topology on the indicated spaces. Therefore $\Leftrightarrow$ also means that the topologies are the same. □

We are in position to identify the spaces $\mathcal{K}_a^\mu$ with Sobolev spaces on Lie manifolds. If Ω is a bounded curvilinear, stratified polyhedral domain, we let

$$\mathfrak{O}_0 := \Sigma(\Omega) \smallsetminus \partial''\Sigma(\Omega) = \Omega \cup \partial'\Sigma(\Omega) = \kappa^{-1}(\overline{\Omega} \smallsetminus \Omega^{(n-2)}),$$

as in Theorem 4.23. Since $(\mathfrak{O}_0, \mathfrak{O} := \Sigma(\Omega), \mathcal{V}(\Omega))$ is a Lie manifold with boundary by the same theorem, the definitions of Sobolev spaces on Lie manifolds (with or without boundary) of Subsection 3.5 provide us with natural spaces $H^s(\Sigma(\Omega)) = H^s(\mathfrak{O}) = H^s(\mathfrak{O}_0)$ and $H^s(\partial'\Sigma(\Omega)) = H^s(\partial\mathfrak{O}_0)$. For the last equality we used that the boundary of $\mathfrak{O}_0$ is $\partial'\Sigma(\Omega)$.

Proposition 5.7. *Let Ω be an n-dimensional, bounded curvilinear, stratified polyhedral domain and let h be an admissible weight on Ω. We have an equality*

$$h\mathcal{K}_a^\mu(\Omega) = hr_\Omega^{a-n/2}H^\mu(\Sigma(\Omega)), \quad \mu \in \mathbb{Z}.$$

Proof. This is again a local statement. We can therefore assume that $\Omega \subset \mathbb{R}^n$. Furthermore, it is enough to prove the statement in the case $h = 1$, since the weight h does not enter into the condition on derivatives in the definition 50 of weighted spaces. Equation (51) and Proposition 4.9 show that we can also assume $a = 0$. Recall from Lemma 3.7 that the spaces $H^k(\Sigma(\Omega))$ are defined using $L^2(\Sigma(\Omega))$. In turn, $L^2(\Sigma(\Omega))$ is defined using the volume element of a compatible metric. A typical compatible metric is $r_\Omega^{-2}g_e$, where g_e is the Euclidean metric. Therefore the volume element on $\Sigma(\Omega)$ is $r_\Omega^{-n}dx$, where dx is the Euclidean volume element. In particular, $v \in L^2(\Omega) \Leftrightarrow v \in r_\Omega^{-n/2}L^2(\Sigma(\Omega))$. We notice next that $r_\Omega^{-t}(r_\Omega\partial)^\alpha r_\Omega^t - (r_\Omega\partial)^\alpha$ is a linear combination with $\mathcal{C}^\infty(\Sigma(\Omega))$–coefficients of monomials $(r_\Omega\partial)^\beta$, with $|\beta| < |\alpha|$, by the second part of Lemma 4.14. From this observation we obtain

$$\begin{aligned} u \in \mathcal{K}_0^\mu(\Omega) &\Leftrightarrow (r_\Omega\partial)^\alpha u \in L^2(\Omega) \quad \text{for all } |\alpha| \le \mu \quad \text{by Theorem 5.6}\\ &\Leftrightarrow (r_\Omega\partial)^\alpha u \in r_\Omega^{-n/2}L^2(\Sigma(\Omega)) \quad \text{for all } |\alpha| \le \mu\\ &\Leftrightarrow (r_\Omega\partial)^\alpha r_\Omega^{n/2} u \in L^2(\Sigma(\Omega)) \quad \text{for all } |\alpha| \le \mu\\ &\Leftrightarrow u \in r_\Omega^{-n/2}H^\mu(\Sigma(\Omega)). \end{aligned}$$

This proves that $\mathcal{K}_a^\mu(\Omega) = r_\Omega^{a-n/2}H^\mu(\Sigma(\Omega))$ for $\mu \in \mathbb{Z}_+$. For $\mu \in \mathbb{Z}_-$, we observe that, for $(\mathfrak{D}, \mathfrak{D}_0, \mathcal{V})$ a Lie manifold with boundary in a manifold with corner $\mathfrak{M}$, the set of restrictions of distributions $u \in H^{-\mu}(\mathfrak{M})$ to $\mathfrak{D}_0$ is the dual of the closure of $\mathcal{C}_c^\infty(\mathfrak{D}_0)$ in $H^{-\mu}(\mathfrak{M})$. Hence

$$\mathcal{K}_0^{-\mu}(\Omega) := \overset{\circ}{\mathcal{K}_a^\mu}(\Omega)^* = \big(r_\Omega^{-n/2}\overset{\circ}{H}{}^\mu(\Sigma(\Omega))\big)^* = r_\Omega^{-n/2}H^{-\mu}(\Sigma(\Omega)).$$

The proof is concluded. ☐

The identification given in Proposition 5.7 above allows to define weighted spaces on the boundary $h\mathcal{K}_a^m(\partial\Omega)$. We recall that the closure of a hyperface of a curvilinear, stratified polyhedral domain Ω need not be contained in any smooth $n-1$ manifold. Consequently, we utilize the desingularization $\Sigma(\Omega)$. In the special case that $\Omega \subset \mathbb{R}^n$ is a (bounded) convex, stratified polyhedron that in addition has straight faces (*i.e.,* each connected component $D_j^{(l)}$ of $\Omega^{(l)} \smallsetminus \Omega^{(l-1)}$, $l = 1, \dots, n-1$ is contained in an affine space $V_j^{(l)}$ of dimension l), for example an n-simplex, we can more simply define the spaces on the boundary as follows:

$$\mathcal{K}_a^\mu(D_j^{(n-1)}) = \{u \in L^2_{\text{loc}}(D_j^{(n-1)}),\ r_\Omega^{k-a}X_1\dots X_k u \in L^2(D_j^{(n-1)}),\ 0 \le k \le l\}\,,$$

for all choices of vector fields X_j in a basis of the linear space containing $D_j^{(n-1)}$. Then for any admissible weight h,

$$h\mathcal{K}_a^\mu(\partial\Omega) = \{hu,\ u \in L^2_{\text{loc}}(\partial\Omega),\ u|_{D_j^{(n-1)}} \in \mathcal{K}_a^\mu(D_j^{(n-1)}),\ \text{ for all } j\}. \tag{54}$$

In the general case of a curvilinear, stratified polyhedral domain, we exploit the structure of Lie manifold on $\Sigma(\Omega)$, following the notation of Proposition 5.7.

Definition 5.8. Let Ω be a bounded, curvilinear, stratified polyhedral domain. Then we define

$$h\mathcal{K}_a^\mu(\partial\Omega) := hr_\Omega^{a-(n-1)/2} H^\mu(\partial'\Sigma(\Omega)),$$

for any admissible weight h.

Note that on each hyperface, the natural weight is the distance to the boundary of that face, not the distance to the set of singular boundary points of that face. The spaces $\mathcal{K}_{-a}^{-\mu}(\partial\Omega)$ are defined to be the duals of $\mathcal{K}_a^\mu(\partial\Omega)$ with pivot $L^2(\partial\Omega)$. For reasons that will be explained later, we do not have to restrict to functions with vanishing trace when studying weighted Sobolev spaces on the boundary. In particular, the usual difficulties that appear in the treatment of Sobolev spaces of fractional order on smooth, bounded domains [49], do not arise when studying the weighted Sobolev spaces on $\partial\Omega$, and we can define the spaces $\mathcal{K}_a^s(\partial\Omega)$, with $s \notin \mathbb{Z}$, by complex interpolation. A similar attempt at defining $\mathcal{K}_a^s(\Omega)$, with $s \in \mathbb{Z} + 1/2$, would lead to the usual difficulties encountered in the case of smooth domain [49].

We next prove a trace theorem, generalizing the corresponding well-known result for smooth domains. Let $\mathcal{C}_c^\infty(\Omega)$ be the space of compactly supported functions on the open set Ω.

Theorem 5.9. *The restriction* $\mathcal{C}_c^\infty(\overline{\Omega}\smallsetminus\Omega^{(n-2)}) \ni u \to u|_{\partial\Omega} \in \mathcal{C}_c^\infty(\partial\Omega\smallsetminus\Omega^{(n-2)})$ *extends to a continuous, surjective map*

$$Tr:\ \mathcal{K}_a^\mu(\Omega) \to \mathcal{K}_{a-1/2}^{\mu-1/2}(\partial\Omega), \quad \mu \geq 1.$$

Moreover, $\mathcal{C}_c^\infty(\Omega)$ *is dense in the kernel of this map if* $\mu = 1$*. Similarly, the normal derivative* ∂_ν *extends to a continuous, surjective map*

$$\partial_\nu:\ \mathcal{K}_a^\mu(\Omega) \to \mathcal{K}_{a-3/2}^{\mu-3/2}(\partial\Omega), \quad \mu \geq 1.$$

The result is a consequence of similar results for Lie manifolds contained in Theorems 3.4 and 3.7 of [1] recalled here in Proposition 3.9. For the normal derivative, we also use the fact that the rescaled normal vector $r_\Omega\nu$ extends to a smooth vector field, first on the boundary of $\Sigma(\Omega)$, and then on the whole of $\Sigma(\Omega)$. The rescaled normal vector is then a unit normal vector for the boundary of $\Sigma(\Omega)$.

Proof. The map $H^{\mu}(\Sigma(\Omega)) \to H^{\mu-1/2}(\partial'\Sigma(\Omega))$, where we follow the notation of Proposition 5.7, is well defined, continuous, and surjective by Proposition 3.9. Proposition 5.7 then shows that the map

$$h\mathcal{K}_a^{\mu}(\Omega) = hr_{\Omega}^{a-n/2}H^{\mu}(\Sigma(\Omega)) \to hr_{\Omega}^{a-n/2}H^{\mu-1/2}(\partial\Sigma(\Omega)) = h\mathcal{K}_{a-1/2}^{\mu-1/2}(\partial\Omega)$$

is also well defined, continuous, and surjective.
The density of $\mathcal{C}_c^{\infty}(\Omega)$ in the subspace of elements in $h\mathcal{K}_a^1(\Omega)$ with trace zero also follows from Proposition 3.9 and Proposition 5.7. ☐

6 Proofs

In this section, we establish the main results of the paper, Theorems 1.1, 1.2, 1.3, using material from previous sections. We first discuss some results on the behavior of differential operators on the spaces $h\mathcal{K}_a^m(\Omega)$.

6.1 Differential operators

We recall that the algebra $\mathrm{Diff}_{\Omega}^{\infty}$ is the natural algebra of differential operators on Ω associated to the Lie algebra of vector fields $\mathcal{V}(\Omega)$, namely, it is generated as an algebra by $X \in \mathcal{V}(\Omega)$ and $\phi \in \mathcal{C}^{\infty}(\Sigma(\Omega))$. (This algebra was used also in Equation (52) and in Subsection 3.3.)

Proposition 6.1. *Let P be a differential operator of order m on a manifold M with smooth coefficients. Let $\Omega \subset M$ be a curvilinear, stratified polyhedral domain. Then P maps $h\mathcal{K}_a^{\mu}(\Omega)$ to $h\mathcal{K}_{a-m}^{\mu-m}(\Omega)$ continuously, for any admissible weight h and any $\mu \in \mathbb{Z}$. Moreover, the resulting family $h^{-\lambda}Ph^{\lambda} : \mathcal{K}_a^{\mu}(\Omega) \to \mathcal{K}_{a-m}^{\mu-m}(\Omega)$ of bounded operators depends continuously on λ.*

Before proceeding with the proof, we discuss a corollary, which will be relevant in showing that Theorems 1.2 and 1.3 hold. Following the notation of those theorems, below $\mathcal{W}_{\mu}(\Omega)$ represents the set of admissible weights h such that the map $\tilde{P}(u) := (Pu, u|_{\partial_D\Omega}, D_{\nu}^P u|_{\partial_N\Omega})$ is an isomorphism $\{\bigoplus_{j=1}^{N} h\mathcal{K}_1^{\mu+1}(\Omega_j) \cap h\mathcal{K}_1^1(\Omega);\ u^+ = u^-,\ D_{\nu}^P u^+ = D_{\nu}^P u^- \text{ on } \Gamma\} \simeq \bigoplus_{j=1}^{N} h\mathcal{K}_{-1}^{\mu-1}(\Omega_j) \oplus h\mathcal{K}_{1/2}^{\mu+1/2}(\partial_D\Omega) \oplus h\mathcal{K}_{-1/2}^{\mu-1/2}(\partial_N\Omega)$.

Proposition 6.2. *The set $\mathcal{W}_{\mu}(\Omega)$ is open.*

Proof. This follows directly from Proposition 6.1. Indeed, the family $P : \bigoplus_{j=1}^{N} h\mathcal{K}_1^{\mu+1}(\Omega_j) \cap h\mathcal{K}_1^1(\Omega) \to h\mathcal{K}_{-1}^{\mu-1}(\Omega)$ is unitarily equivalent to $h^{-1}Ph : \bigoplus_{j=1}^{N} \mathcal{K}_1^{\mu+1}(\Omega_j) \cap \mathcal{K}_1^1(\Omega) \to \mathcal{K}_{-1}^{\mu-1}(\Omega)$. The result then follows since the set of invertible operators is open. ☐

For the proof of Proposition 6.1, we observe that if $\Omega \subset \mathbb{R}^n$, the principal symbol of $(r_\Omega \partial)^\alpha$ is $(\imath\xi)^\alpha$. This result follows from the definition of the principal symbol in [2, 1] and from Corollary 5.4. (The reader can just assume $\sigma\big((r_\Omega\partial)^\alpha\big) = (\imath\xi)^\alpha$ by definition.)

Corollary 6.3. *Let P be a differential operator of order m on M with smooth coefficients. Then*

(i) $P_0 := r_\Omega^m P \in \mathrm{Diff}_\Omega$;

(ii) *P is uniformly strongly elliptic if, and only if, $r_\Omega^m P$ is uniformly strongly elliptic in* Diff_Ω^m;

(iii) *$h^\lambda P h^{-\lambda}$ depends continuously on λ;*

(iv) *P maps $h\mathcal{K}_a^\mu(\Omega) \to h\mathcal{K}_{a-m}^{\mu-m}(\Omega)$ continuously.*

Proof. The relation $r_\Omega^m P \in \mathrm{Diff}_\Omega^m$ was proved as part of Lemma 5.5. Strong ellipticity is a local property, so we can assume $\Omega \subset \mathbb{R}^n$. The proof of Lemma 5.5 shows that P and $r_\Omega^m P$ have the same principal symbol. Therefore they are elliptic (or strongly elliptic) at the same time.

For any $X \in \mathcal{V}$ and any defining function x of some hyperface at infinity of $\Sigma(\Omega)$, we have that $x^\lambda X x^{-\lambda} = X - \lambda x^{-1} X(x)$. Since $x^{-1}X(x)$ is a smooth function (as X is tangent to the face defined by x), we see that $x^\lambda X x^{-\lambda} \in \mathrm{Diff}_\Omega^1$ and depends continuously on λ, establishing (iii). It also shows, in particular, that Diff_Ω^k is conjugation invariant with respect to defining functions of hyperfaces at infinity (Equation (28)). We can therefore assume that $h = 1$.

Since $(\Sigma(\Omega), \mathcal{V}(\Omega))$ is a Lie manifold with boundary (Theorem 4.23) any $P_0 \in \mathrm{Diff}_\Omega^k$ maps $H^\mu(\Sigma(\Omega)) \to H^{\mu-k}(\Sigma(\Omega))$ continuously. (This simple property, proved in [1], is an immediate consequence of the definitions.) The continuity of $P = r_\Omega^{-m} P_0 : \mathcal{K}_a^\mu(\Omega) \to \mathcal{K}_{a-m}^{\mu-m}(\Omega)$ then follows using also the fact that multiplication by r_Ω^{-m} defines an isometry $\mathcal{K}_a^{\mu-m}(\Omega) \simeq \mathcal{K}_{a-m}^{\mu-m}(\Omega)$. □

6.2 A weighted Hardy–Poincaré's inequality

The stepping stones in the proof of our main result on the solvability of the mixed boundary value/interface problem (6), Theorem 1.2, consist of

(i) a Hardy–Poincaré type inequality (Theorem 6.4);

(ii) the regularity result for polyhedra (Theorem 1.1).

We address the Hardy-Poincaré inequality first and turn to the proof of the regularity result, which is more general and of independent interest in the next subsection. Let $dx = dx_1 dx_2 \ldots dx_n$ denote the standard volume element in $\mathbb{R}^n$. We continue to denote by Ω a curvilinear, stratified polyhedral domain satisfying hypotheses (3)–(5).

Theorem 6.4. *Let Ω be a bounded, connected, curvilinear, stratified polyhedral domain $\Omega \subset M$. Assume that $\partial_D\Omega \neq \emptyset$ and $\partial_N\Omega$ does not contain any two adjacent hyperfaces. Then there exists a constant $\kappa_\Omega > 0$, depending only on the polyhedral structure of Ω, such that*

$$\|u\|^2_{\mathcal{K}^0_1(\Omega)} := \int_\Omega \frac{|u(x)|^2}{\eta_{n-2}(x)^2}\, dx \leq \kappa_\Omega \int_\Omega |\nabla u(x)|^2\, dx, \tag{55}$$

for any function $u \in H^1_{\mathrm{loc}}(\Omega)$ such that $u|_{\partial_D\Omega} = 0$.

Above, if u/η_{n-2} is not square integrable, the statement of the theorem is understood to mean that ∇u is not square integrable either. By Propositions 4.9 and 4.10, we can replace the distance to the singular set η_{n-2} with the more regular weight r_Ω.
The proof proceeds by induction on the dimension n. We discuss first the case $n = 2, 3$.
The case $n = 2$. In view of the local nature of the definition of a curvilinear, stratified polygonal domain, Definition 2.6, it will be sufficient to have the Hardy-Poincaré inequality in a sector. By abuse of notation, we shall write $u(r,\theta) := u(r\cos\theta, r\sin\theta)$ for a function $u(x_1,x_2)$ expressed in polar coordinates. The proof of the following elementary lemma can be found in e.g. [62][Subsection 2.3.1]. See also [40].

Lemma 6.5. *Let $\mathcal{C} = \mathcal{C}_R(\alpha,\beta) := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2, 0 < r < R, \beta < \theta < \alpha\}$, $0 < \alpha - \beta < 2\pi$. Then*

$$\int_{\mathcal{C}} \frac{|u|^2}{r^2}\, dx \leq \frac{\pi^2}{(\alpha-\beta)^2} \int_{\mathcal{C}} |\nabla u|^2 dx$$

for any $u \in H^1_{\mathrm{loc}}(\mathcal{C})$ satisfying $u(r,\theta) = 0$ if $\theta = \beta$ or $\theta = \alpha$. The same result holds if $\mathcal{C}$ is the disjoint union of domains $\mathcal{C}_R(\alpha,\beta)$, for different values of R, α, and β.

From the Lemma above, we obtain the case $n = 2$ in Theorem 6.4, the first step in our induction proof. A detailed proof can be found e.g. in the papers [15, 56].

Lemma 6.6. *Let Ω be a connected, curvilinear, stratified polygonal domain in a two dimensional manifold M. Assume that $\partial_D\Omega \neq \emptyset$ and $\partial_N\Omega$ does not contain any two adjacent sides of Ω. Fix an arbitrary metric g on M and let $\eta_0(z)$ be the distance from z to the vertices of Ω. Then there exists a constant $\kappa_\Omega > 0$ such that*

$$\int_\Omega \frac{|u(w)|^2}{\eta_0(w)^2}\, dz \leq \kappa_\Omega \int_\Omega |\nabla u(w)|^2 dz$$

for any $u \in H^1_{\mathrm{loc}}(\Omega)$ satisfying $u = 0$ on $\partial_D\Omega$.

The case $n = 3$. The proof of Theorem 6.4 for $n = 3$ combines the methods used in the previous two Lemmata and the inequality for the case $n = 2$. We give a self-contained proof again, especially because the induction step in the general case is very similar. The general case $n > 3$ will be completed using Proposition 4.10.

Proof. Let us fix, for any $p \in \partial\Omega$, a neighborhood V_p of p in M and a diffeomorphism $\phi_p : V_p \to U = B^{3-l} \times B^l$ as in Definition 2.8, where $l = \ell(p)$. We denote $\mathcal{C} := \phi_p(V_p \cap \Omega)$. We shall use the notation ω_p introduced in that definition. By decreasing V_p, if necessary, we may assume that ϕ_p extends to a diffeomorphism defined in a neighborhood of $\overline{V}_p$ in $\mathbb{R}^3$.
Since $\eta_{n-2} = \eta_1$ is the distance to the singular set $\Omega^{(1)}$ of Ω, we need only discuss two cases:

(a) $l = \ell(p) = 0$, *i.e.,* p is a true or artificial vertex;

(b) $l = \ell(p) = 1$, *i.e.,* p belongs to a true or artificial edge.

If $l = 0$, we denote by $\psi_0(x')$ the distance from a point $x' \in \omega_p \subset S^2$ to the vertices of ω_p and let $r_p(w) = \rho\psi_0(x')$, if $\phi_p(w) = \rho x'$, where $0 < \rho$ and $x' \in \omega_p$. If $l = 1$, we let $r_p(w) = r$ if $\phi_p(w) = (r\cos\theta, r\sin\theta, z)$, where $0 < r$, $0 < \theta < \alpha$, and $z \in \mathbb{R}$. (These definitions agree with the general definition of r_Ω given in (36) with $r_p = r_\alpha$ given in (35).) As before, the function $\eta_1(x)/r_p(x)$ is bounded for any p, provided that we choose the neighborhoods V_p small enough, uniformly in p. Below, we shall write $u(x)$ instead of $u(\phi_p^{-1}(x))$, by abuse of notation.
If $l = 1$, $\mathcal{C} = \mathcal{C}' \times (-1, 1)$, so that we exploit the Hardy-Poincaré inequality in a sector of Lemma 6.5. In fact

$$\int_{V_p\cap\Omega} \frac{|u(w)|^2}{\eta_1(w)^2}\, dw = C \int_{\Omega\cap V_p} \frac{|u(x)|^2}{r^2} \left|\frac{\partial w}{\partial x}\right| dx \leq C \int_{\mathcal{C}} \frac{|u(x)|^2}{r^2}\, dx. \tag{56}$$

so that we obtain

$$\begin{aligned} \int_{\mathcal{C}} \frac{|u(x)|^2}{r^2}\, dx &= \int_{-1}^{1} \left(\int_{\mathcal{C}'} \frac{|u(x)|^2}{r^2}\, dx_1 dx_2 \right) dx_3 \\ \leq \int_{-1}^{1} \left(\int_{\mathcal{C}'} |\nabla_{(x_1,x_2)} u(x)|^2\, dx_1 dx_2 \right) dx_3 &\leq \int_{-1}^{1} \left(\int_{\mathcal{C}'} |\nabla u(x)|^2\, dx_1 dx_2 \right) dx_3 \\ &\leq C \int_{V_p\cap\Omega} |\nabla u(w)|^2\, dw. \end{aligned} \tag{57}$$

We perform a similar calculation on $V_p \cap \Omega$ when $l = 0$, using spherical coordinates instead. Recall that $\mathcal{C} = \phi_p(V_p \cap \Omega) = \{\rho x',\, 0 < \rho < 1,\, x' \in \omega_p\}$, hence following (56) and using that $C\eta_1(x) \geq \rho\psi_0(x)$ The inequality

$$\int_{V_p\cap\Omega} \frac{|u(w)|^2}{\eta_1(w)^2}\, dw \leq C \int_{\mathcal{C}} \frac{|u(x)|^2}{\rho^2\psi_0(x')^2}\, dx, \quad x = \rho x',\ |x'| = 1, \tag{58}$$

Next, we observe that $\nabla u(\rho x') = \rho^{-1}\nabla' u(\rho x') + \partial_\rho u(\rho x')$, with ∇' the gradient of a function defined on ω_p, so that $|\nabla' u(\rho x')|^2 \leq \rho^2 |\nabla u(\rho x')|$, which gives

$$\int_{\mathcal{C}} \frac{|u(x)|^2}{\rho^2 \psi_0(x)^2}\, dx = \int_0^1 \left(\int_{\omega_p} \frac{|u(\rho x')|^2}{\psi_0^2} dx' \right) d\rho$$
$$\leq C \int_0^1 \left(\int_{\omega_p} |\nabla' u(\rho x')|^2 dx' \right) d\rho \leq C \int_0^1 \left(\int_{\omega_p} \rho^2 |\nabla u(\rho x')|^2 dx' \right) d\rho$$
$$\leq C \int_{V_p \cap \Omega} |\nabla u(w)|^2\, dw. \quad (59)$$

We can rewrite the above inequalities simply as

$$\int_{V_p \cap \Omega} \frac{|u(w)|^2}{\eta_1(w)^2}\, dz \leq C_p \int_{V_p \cap \Omega} |\nabla u(w)|^2\, dz \leq C_p \int_{\Omega} |\nabla u(w)|^2\, dw, \quad (60)$$

where the constant C_p depends on the point $p \in \Omega^{(1)}$ but not on u.

To conclude the proof, as before we cover the singular set $\Omega^{(1)}$ with finitely many sets $V_p = V_{p_k}$. Let $C_0 > \eta_1^{-2}$ *outside* the union of the sets V_{p_k}. Let $\kappa_\Omega = C_0 C_\Omega + \sum C_{p_k}$, where C_Ω is the standard Poincaré inequality constant for the domain Ω. We add all inequalities (60) for $p = p_k$ and combine it with the Poincaré inequality to get

$$\int_{\Omega} \frac{|u(w)|^2}{\eta_1^2(w)} dw \leq \kappa_\Omega \int_{\Omega} |\nabla u(w)|^2 dw. \quad (61)$$

The proof of Theorem 6.4 is now complete for $n = 3$. □

The general case $n > 3$. To conclude the proof of theorem 6.4, we need only establish the induction step. The induction step follows very closely the proof of the case $n = 3$. The only delicate point is showing that the ratio $\eta_{n-2}(x)/r_\alpha(x)$ is bounded on $\overline{\Omega}$, where η_{n-2} is the distance to the singular set $\Omega^{(n-2)}$ of Ω and r_α is as in Equation 35. This fact was established in Proposition 4.9.

We conclude with an immediate corollary of Theorem 6.4, which will be used in the proof of Theorem 1.2

Corollary 6.7. *There exists a constant $\kappa'_\Omega > 0$, depending only on Ω, such that*

$$\frac{1}{\kappa'_\Omega} \|u\|^2_{\mathcal{K}^1_1(\Omega)} \leq \int_{\Omega} |\nabla u(x)|^2\, dx\,,$$

for any function $u \in H^1_{\text{loc}}(\Omega)$ such that $u|_{\partial_D \Omega} = 0$, if $\partial_D \Omega \neq \emptyset$ and $\partial_N \Omega$ does not contain any two adjacent hyperfaces.

6.3 Proofs of the main results

In this subsection, we finally tackle the proofs of the main results of the paper. We first show how the proof of the regularity property for the mixed boundary value/interface problem (6), Theorem 1.1 can be obtained from the results of [1] and the theory developed in Section 4. The following result was proved in [1].

Theorem 6.8. *Let $(\mathfrak{M}, \mathcal{V})$ be a Lie manifold with boundary and $P_0 := r_\Omega^2 P \in \mathrm{Diff}^2(\mathfrak{M})$ be a second order, uniformly strongly elliptic operator. Let h be an admissible weight and $u \in hH^1(\mathfrak{M})$ be such that $Pu \in hH^{\mu-1}(\mathfrak{M})$ and $u|_{\partial\mathfrak{M}} \in hH^{\mu+1/2}(\partial\mathfrak{M})$, $\mu \in \mathbb{Z}_+$. Then $u \in hH^{\mu+1}(\mathfrak{M})$ and*

$$\|u\|_{hH^{\mu+1}(\mathfrak{M})} \leq C\big(\|Pu\|_{hH^{\mu-1}(\mathfrak{M})} + \|u\|_{hH^0(\mathfrak{M})} + \|u|_{\partial\Omega}\|_{hH^{\mu+1/2}(\partial\mathfrak{M})}\big). \tag{62}$$

For mixed boundary value/interface problems we need the following extension of this theorem, which is proved exactly in the same way.

Theorem 6.9. *Let $(\mathfrak{M}, \mathcal{V})$ be a Lie manifold with boundary and $P_0 = r_\Omega^2 P \in \mathrm{Diff}^2(\mathfrak{M})$ be a second order, uniformly strongly elliptic operator with jump discontinuities on sub Lie manifolds of $\mathfrak{M}$ that partition it into subsets $\mathfrak{M}_j$. Assume that $\partial\mathfrak{M} = \partial_D\mathfrak{M} \cup \partial_N\mathfrak{M}$ is a disjoint decomposition into* open, *disjoint subsets. Let h be an admissible weight and $u \in hH^1(\mathfrak{M})$ be such that $Pu \in hH^{\mu-1}(\mathfrak{M}_j)$ and $u|_{\partial\mathfrak{M}} \in hH^{\mu+1/2}(\partial\mathfrak{M})$, $\mu \in \mathbb{Z}_+$. Then $u \in hH^{\mu+1}(\mathfrak{M}_j)$ and*

$$\|u\|_{hH^{\mu+1}(\mathfrak{M}_j)} + \|u\|_{hH^1(\mathfrak{M})} \leq C\big(\sum_k \|Pu\|_{hH^{\mu-1}(\mathfrak{M}_k)} + \|u\|_{hH^0(\mathfrak{M})} \\ + \|u|_{\partial\Omega}\|_{hH^{\mu+1/2}(\partial_D\mathfrak{M})} + \|D_\nu^P u|_{\partial\Omega}\|_{hH^{\mu-1/2}(\partial_N\mathfrak{M})}\big).$$

Theorem 1.1 then follows by applying the above theorem to $P_0 := r_\Omega^2 P$, which is strongly elliptic by Corollary 6.3(ii), and using the identifications of Proposition 5.7 and Definition 5.8.

We now prove Theorem 1.2 assuming the results stated in the previous subsection. The proof of Theorem 1.3 is completely similar.

Remark 6.10. In the statement of Theorems 1.2 and 1.3, the spaces $\mathcal{K}_1^{\mu+1}(\Omega_j)$ are defined intrinsically, without reference to Ω. However, the interface Γ is assumed smooth for well-posedness in this paper (more general conditions on Γ were for example considered in [48]) and the points where Γ intersects $\partial\Omega$, necessarily transversely, are included in the singular sets $\Omega^{(n-2)}$ of Ω; consequently, r_Ω is equivalent to r_{Ω_j} in each Ω_j.

Proof. We first notice that Theorem 5.9 allows us to reduce the proof to the case $g_D = 0$.

We continue to denote with $\mathcal{W}_\mu(\Omega)$ the set of weights such that the operator $\tilde{P}$, defined below, is an isomorphism

$$\begin{aligned}
&\tilde{P} := (Pu, u|_{\partial_D\Omega}, D^P_\nu u|_{\partial_N\Omega}) : \\
&\{u \in \bigoplus_{j=1}^N h\mathcal{K}^{\mu+1}_1(\Omega_j) \cap h\mathcal{K}^1_1(\Omega); u|_{\partial_D\Omega} = 0, u^+ = u^-, D^P_\nu u^+ = D^P_\nu u^- \text{ on } \Gamma\} \\
&\qquad\qquad \to \bigoplus_{j=1}^N h\mathcal{K}^{\mu-1}_{-1}(\Omega_j) \oplus h\mathcal{K}^{\mu-1/2}_{-1/2}(\partial_N\Omega),
\end{aligned} \tag{63}$$

which is an open set by Proposition 6.2. Therefore, it is enough to show that $1 \in \mathcal{W}_\mu(\Omega)$ to complete the proof.

For solvability, we consider the case $\mu = 0$. For $\mu = 0$, the problem (6) is interpreted in the weak sense (11), using that $\mathcal{K}^1_1(\Omega) \subset H^1(\Omega)$. More precisely, we let

$$\mathcal{H} := \{u \in \mathcal{K}^1_1(\Omega),\ u = 0 \text{ on } \partial_D\Omega\}, \tag{64}$$

and we define the weak solution u of Equation (11) with $g_D = 0$ as the unique $u \in \mathcal{K}^1_1(\Omega)$ satisfying $u = 0$ on $\partial_D\Omega$ in trace sense and

$$B_P(u, v) = \Phi(v) \quad \text{for all } \ v \in \mathcal{H}, \tag{65}$$

where the element $\Phi \in \mathcal{H}^*$ is defined by $\Phi(u) = \int_\Omega fu\, dx + \int_{\partial_N\Omega} g_N u\, dS(x)$, this last integral being the pairing between $\mathcal{K}^{1/2}_{1/2}(\partial\Omega)$ and $\mathcal{K}^{-1/2}_{-1/2}(\partial\Omega)$. Here, we have employed the trace property, Theorem 5.9. We will establish the existence and uniqueness of u by using the Lax-Milgram Lemma and coercive estimates for P in weighted Sobolev spaces, which in turn follow from the (uniform) strong ellipticity of P and the Hardy-Poincaré inequality of Theorem 6.4. This result gives the first step of the proof, that is, $1 \in \mathcal{W}_0(\Omega)$. We refer to [26] for the version of the Lax–Milgram lemma needed in this proof, where P contains lower-order terms.

Indeed, the sesquilinear form B is continuous on $\mathcal{H} \times \mathcal{H}$ by Proposition 6.1. Furthermore, assumptions 8 on the coefficients A_{jk}, B_j, and C of the operator P, together with Corollary 6.7 imply the following inequality for the real part of $B(u, v)$:

$$\begin{aligned}
Re\big(Pu, u\big) &= \int_\Omega \Big(Re \sum_{j,k=1}^n A_{jk}\partial_k u\overline{\partial_j u}\Big)\, dx + \big((2C - \sum_j \partial_j B_j)u, u\big)/2 \\
&\geq \epsilon \sum_{j=1}^n \|\partial_j u\|^2 \geq \epsilon \|u\|^2_{\mathcal{K}^1_1(\Omega)} =: \epsilon\|u\|^2_{\mathcal{K}^1_1(\Omega)},
\end{aligned} \tag{66}$$

which shows that B is strictly coercive on $\mathcal{H}$.

The assumptions of the Lax-Milgram lemma are therefore satisfied, Hence $P : \mathcal{H} \to \mathcal{H}^*$ is an isomorphism (*i.e.,* P is continuous with continuous inverse), proving that $1 \in \mathcal{W}_0(\Omega)$.

We next consider the case $\mu \geq 1$ and prove that $\mathcal{W}_0(\Omega) \subset \mathcal{W}_\mu(\Omega)$ for any $\mu \in \mathbb{Z}_+$, so that, in particular, $1 \in \mathcal{W}_\mu(\Omega)$. We pick $h \in \mathcal{W}_0(\Omega)$ and observe that by the regularity theorem, Theorem 1.1, the map $\tilde{P}$ of Equation 63 above is surjective. Since this map is also continuous (Proposition 6.1) and injective (because $h \in \mathcal{W}_0(\Omega)$), it is an isomorphism by the open mapping theorem. This observation shows that $\mathcal{W}_0(\Omega) \subset \mathcal{W}_\mu(\Omega)$, for any $\mu \in \mathbb{Z}_+$.
Since we have already proved that $\mathcal{W}_\mu(\Omega)$ is open, the proof is complete. □

Remark 6.11. It seems that it would be more natural to work in the framework of stratified spaces than in the framework of polyhedral domains. For example, if we consider a smooth, bounded domain $\Omega \subset \mathbb{R}^n$ and a submanifold $X \subset \partial\Omega$ of lower dimension, then we can consider $\eta_{n-2}(x)$ to be the distance from x to X. Then Theorem 1.2 remains true, with essentially the same proof, by taking $\Omega^{(n-2)} := X$ in this case.

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