Hankel Operators and the Dixmier Trace on Strictly Pseudoconvex Domains

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Abstract. Generalizing earlier results for the disc and the ball, we give a formula for the Dixmier trace of the product of $2n$ Hankel operators on Bergman spaces of strictly pseudoconvex domains in $\mathbf{C}^n$. The answer turns out to involve the dual Levi form evaluated on boundary derivatives of the symbols. Our main tool is the theory of generalized Toeplitz operators due to Boutet de Monvel and Guillemin.

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1. Introduction

Let Ω be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary, and $L^2_{\text{hol}}(\Omega)$ the Bergman space of all holomorphic functions in $L^2(\Omega)$. For a bounded measurable function f on Ω, the Toeplitz and the Hankel operator with symbol f are the operators $\mathbf{T}_f : L^2_{\text{hol}}(\Omega) \to L^2_{\text{hol}}(\Omega)$ and $H_f : L^2_{\text{hol}}(\Omega) \to L^2(\Omega) \ominus L^2_{\text{hol}}(\Omega)$, respectively, defined by

$$\mathbf{T}_f g := \mathbf{\Pi}(fg), \quad H_f g := (I - \mathbf{\Pi})(fg), \tag{1}$$

where $\mathbf{\Pi} : L^2(\Omega) \to L^2_{\text{hol}}(\Omega)$ is the orthogonal projection. It has been known for some time that for f holomorphic and $n > 1$, the Hankel operator $H_{\overline{f}}$

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belongs to the Schatten ideal $\mathcal{S}^p$ if and only if f is in the diagonal Besov space $B^p(\Omega)$ and $p > 2n$, or f is constant (so $H_f = 0$) and $p \le 2n$; see Arazy, Fisher and Peetre [1] for $\Omega = \mathbf{B}^n$, the unit ball of $\mathbf{C}^n$, and Li and Luecking [21] for general smoothly bounded strictly pseudoconvex domains Ω. This phenomenon is called a *cutoff* at $p = 2n$. In dimension $n = 1$, the situation is slightly different, in that the cutoff occurs not at $p = 2$ but at $p = 1$. One can rephrase the above results also in terms of membership in the Schatten classes of the commutators $[\mathbf{T}_{\overline{f}}, \mathbf{T}_g] := \mathbf{T}_{\overline{f}}\mathbf{T}_g - \mathbf{T}_g\mathbf{T}_{\overline{f}}$ of Toeplitz operators. In fact, it is immediate from (1) that

$$\mathbf{T}_{\overline{f}g} - \mathbf{T}_g\mathbf{T}_{\overline{f}} = H^*_{\overline{g}} H_{\overline{f}},$$

and also that $\mathbf{T}_{\overline{f}}\mathbf{T}_g = \mathbf{T}_{\overline{f}g}$ if f or g is holomorphic; thus for holomorphic f and g

$$[\mathbf{T}_{\overline{f}}, \mathbf{T}_g] = H^*_{\overline{g}} H_{\overline{f}}.$$

In any case, it follows that there are no nonzero trace-class Hankel operators $H_{\overline{f}}$, with f holomorphic, if $n = 1$, and similarly the product $H^*_{\overline{f}_1} H_{\overline{f}_2} \dots H^*_{\overline{f}_{2n-1}} H_{\overline{f}_{2n}} = [\mathbf{T}_{\overline{f}_2}, \mathbf{T}_{f_1}] \dots [\mathbf{T}_{\overline{f}_{2n}}, \mathbf{T}_{f_{2n-1}}]$, with $f_1, \dots, f_{2n}$ holomorphic, is never trace-class if $n > 1$. In particular, there is no hope for $n > 1$ of having an analogue of the well-known formula for the unit disc,

$$\operatorname{tr}[\mathbf{T}_{\overline{f}}, \mathbf{T}_f] = \int_{\mathbf{D}} |f'(z)|^2 \, dm(z) \tag{2}$$

expressing the trace of the commutator $[\mathbf{T}_{\overline{f}}, \mathbf{T}_f]$ as the square of the Dirichlet norm of the holomorphic function f, which is one of the best known Moebius invariant integrals. (This formula actually holds for Toeplitz operators on any Bergman space of a bounded planar domain, if the Lebesgue area measure $dm(z)$ is replaced by an appropriate measure associated to the domain, see [2].) A remarkable substitute for (2) on the unit ball $\mathbf{B}^n$ is the result of Helton and Howe [19], who showed that for smooth functions $f_1, \dots, f_{2n}$ on the closed ball, the complete anti-symmetrization $[\mathbf{T}_{f_1}, \mathbf{T}_{f_2}, \dots, \mathbf{T}_{f_{2n}}]$ of the $2n$ operators $\mathbf{T}_{f_1}, \dots, \mathbf{T}_{f_{2n}}$ is trace-class and

$$\operatorname{tr}[\mathbf{T}_{f_1}, \mathbf{T}_{f_2}, \dots, \mathbf{T}_{f_{2n}}] = \int_{\mathbf{B}^n} df_1 \wedge df_2 \wedge \dots \wedge df_{2n}.$$

There is, however, a generalization of (2) to the unit ball $\mathbf{B}^n$, $n > 1$, in a different direction — using the Dixmier trace. This may be notable especially in view of the prominent applications of the Dixmier trace in noncommutative differential geometry [9].

Namely, it was shown by the present authors and Guo [12] that for $f_1, \dots, f_n$ and $g_1, \dots, g_n$ smooth on the closed ball, the product $[\mathbf{T}_{f_1}, \mathbf{T}_{g_1}] \dots [\mathbf{T}_{f_n}, \mathbf{T}_{g_n}]$ belongs to the Dixmier class $\mathcal{S}^{\mathrm{Dixm}}$ and has Dixmier trace equal to

$$\operatorname{Tr}_\omega([\mathbf{T}_{f_1}, \mathbf{T}_{g_1}] \dots [\mathbf{T}_{f_n}, \mathbf{T}_{g_n}]) = \frac{1}{n!} \int_{\partial \mathbf{B}^n} \prod_{j=1}^n \{f_j, g_j\}_b \, d\sigma, \tag{3}$$

where $d\sigma$ is the normalized surface measure on $\partial\mathbf{B}^n$ and $\{f,g\}_b$ is the "boundary Poisson bracket" given by

$$\{f,g\}_b := \sum_{j=1}^n \Big(\frac{\partial f}{\partial \overline{z}_j}\frac{\partial g}{\partial z_j} - \frac{\partial f}{\partial z_j}\frac{\partial g}{\partial \overline{z}_j}\Big) - (\overline{R}fRg - Rf\overline{R}g),$$

with $\overline{R} := \sum_{j=1}^n \overline{z}_j \frac{\partial}{\partial \overline{z}_j}$ and $R := \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$ the anti-holomorphic and the holomorphic part of the radial derivative, respectively. In particular, for f holomorphic on $\mathbf{B}^n$ and smooth on the closed ball, $(H^*_{\overline{f}}H_{\overline{f}})^n = [\mathbf{T}_{\overline{f}},\mathbf{T}_f]^n \in \mathcal{S}^{\mathrm{Dixm}}$ and

$$\mathrm{Tr}_\omega((H^*_{\overline{f}}H_{\overline{f}})^n) = \frac{1}{n!}\int_{\partial\mathbf{B}^n}\Big(\sum_{j=1}^n\Big|\frac{\partial f}{\partial z_j}\Big|^2 - |Rf|^2\Big)^n\, d\sigma.$$

Note that for $n=1$ the right-hand side vanishes, in accordance with the fact that in dimension 1 the cutoff occurs at $p=1$ instead of $p=2n=2$; in fact, it was shown by Rochberg and the first author [13] that for $n=1$ actually $|H_f| = (H_f^*H_f)^{1/2}$, rather than $H_f^*H_f$, is in the Dixmier class for any $f\in C^\infty(\overline{\mathbf{D}})$, and

$$\mathrm{Tr}_\omega(|H_f|) = \int_{\partial\mathbf{D}} |\overline{\partial}f|\, d\sigma,$$

so, in particular,

$$\mathrm{Tr}_\omega(|H_{\overline{f}}|) = \int_{\partial\mathbf{D}} |f'|\, d\sigma = \|f'\|_{H_1}$$

for $f\in C^\infty(\overline{\mathbf{D}})$ holomorphic on $\mathbf{D}$, where H^1 denotes the Hardy 1-space on the unit circle.

In this paper, we generalize the result of [12] to arbitrary bounded strictly pseudoconvex domains Ω with smooth boundary. Our result is that for any $2n$ functions $f_1,g_1,\dots,f_n,g_n\in C^\infty(\overline{\Omega})$,

$$\text{(4)}\qquad \mathrm{Tr}_\omega(H^*_{f_1}H_{g_1}\dots H^*_{f_n}H_{g_n}) = \frac{1}{n!(2\pi)^n}\int_{\partial\Omega}\prod_{j=1}^n \mathcal{L}(\overline{\partial}_b g_j,\overline{\partial}_b f_j)\ \eta\wedge(d\eta)^{n-1},$$

where $\overline{\partial}_b$ stands for the boundary $\overline{\partial}$-operator [14], $\eta\wedge(d\eta)^{n-1}$ is a certain measure on $\partial\Omega$, and $\mathcal{L}$ stands for the dual of the Levi form on the anti-holomorphic tangent bundle; see §§ 2 and 4 below for the details.

In contrast to [12], where we were using the so-called pseudo-Toeplitz operators of Howe [18], our proof here relies on Boutet de Monvel's and Guillemin's theory of Toeplitz operators on the Hardy space $H^2(\partial\Omega)$ with pseudodifferential symbols. (This is also the approach used in [13], however the situation $\Omega=\mathbf{D}$ treated there is much more manageable.)

In fact, it turns out that for any classical pseudodifferential operator Q on $\partial\Omega$ of order $-n$, the corresponding Hardy-Toeplitz operator T_Q belongs to the Dixmier class and

$$\operatorname{Tr}_\omega(T_Q) = \frac{1}{n!(2\pi)^n} \int_{\partial\Omega} \sigma_{-n}(Q)(x,\eta(x))\, \eta(x) \wedge (d\eta(x))^{n-1}, \tag{5}$$

where $\sigma_{-n}(Q)$ is the principal symbol of Q, and η is a certain 1-form on $\partial\Omega$; see again §2 below for the details. In particular, in view of the results of Guillemin [16] [17], this means that on Toeplitz operators T_Q of order $\leq -n$, the Dixmier trace $\operatorname{Tr}_\omega T_Q$ coincides with the residual trace $\operatorname{Tr}_{\mathrm{Res}} T_Q$, a quantity constructed using the meromorphic continuation of the ζ function of T_Q (Wodzicki [24], Boutet de Monvel [7], Ponge [23], Lesch [20], Connes [9]).

We recall the necessary prerequisites on the Dixmier trace, Hankel operators and the Boutet de Monvel-Guillemin theory in Section 2. The proofs of (5) and (4) appear in Sections 3 and 4, respectively. Some concluding comments are assembled in the final Section 5.

Throughout the paper, we will denote Bergman-space Toeplitz operators by $\mathbf{T}_f$, in order to distinguish them from the Hardy-space Toeplitz operators T_f and T_Q. Since Hankel operators on the Hardy space never appear in this paper, Hankel operators on the Bergman space are denoted simply by H_f.

2. Background

2.1 Generalized Toeplitz operators. Let r be a defining function for Ω, that is, $r \in C^\infty(\overline{\Omega})$, $r < 0$ on Ω, and $r = 0$, $\|\partial r\| > 0$ on $\partial\Omega$. Denote by η the restriction to $\partial\Omega$ of the 1-form $\operatorname{Im}(\partial r) = (\partial r - \overline{\partial} r)/2i$. The strict pseudoconvexity of Ω guarantees that η is a contact form, i.e. the half-line bundle

$$\Sigma := \{(x,\xi) \in \mathcal{T}^*(\partial\Omega) :\ \xi = t\eta_x,\ t > 0\}$$

is a symplectic submanifold of $\mathcal{T}^*(\partial\Omega)$. Equip $\partial\Omega$ with a measure with smooth positive density, and let $L^2(\partial\Omega)$ be the Lebesgue space with respect to this measure. The Hardy space $H^2(\partial\Omega)$ is the subspace in $L^2(\partial\Omega)$ of functions whose Poisson extension is holomorphic in Ω; or, equivalently, the closure in $L^2(\partial\Omega)$ of $C^\infty_{\mathrm{hol}}(\partial\Omega)$, the space of boundary values of all the functions in $C^\infty(\overline{\Omega})$ that are holomorphic on Ω. (In dimensions greater than 1, $H^2(\partial\Omega)$ can also be characterized as the null-space of the $\overline{\partial}_b$-operator, which will appear in Section 4 further on.) We will also denote by $W^s(\partial\Omega)$, $s \in \mathbf{R}$, the Sobolev spaces on $\partial\Omega$, and by $W^s_{\mathrm{hol}}(\partial\Omega)$ the corresponding subspaces of nontangential boundary values of functions holomorphic in Ω. (Thus $W^0(\partial\Omega) = L^2(\partial\Omega)$ and $W^0_{\mathrm{hol}}(\partial\Omega) = H^2(\partial\Omega)$.)

Unless otherwise specified, by a pseudodifferential operator or Fourier integral operator (ΨDO or FIO for short) on $\partial\Omega$ we will always mean an operator which is "classical", i.e. whose total symbol (or amplitude) in any local coordinate system has an asymptotic expansion

$$p(x,\xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x,\xi),$$

where p_{m-j} is C^∞ in x,ξ, and is positive homogeneous of degree $m-j$ in ξ for $|\xi|>1$. Here j runs through nonnegative integers, while m can be any integer; and the symbol "$\sim$" means that the difference between p and $\sum_{j=0}^{k-1} p_{m-j}$ should belong to the Hörmander class S^{m-k}, for each $k=0,1,2,\dots$. The set of all classical ΨDOs on $\partial\Omega$ as above (i.e. of order m) will be denoted by $\Psi^m_{\rm cl}$; and we set, as usual, $\Psi_{\rm cl} := \bigcup_{m\in\mathbf{Z}} \Psi^m_{\rm cl}$ and $\Psi^{-\infty} := \bigcap_{m\in\mathbf{Z}} \Psi^m_{\rm cl}$. The operators in $\Psi^{-\infty}$ are precisely the *smoothing* operators, i.e. those given by a C^∞ Schwartz kernel; and for any $P,Q\in\Psi_{\rm cl}$, we will write $P\sim Q$ if $P-Q$ is smoothing. Note that if $P\in\Psi^m_{\rm cl}$, then P is continuous from $W^s(\partial\Omega)$ into $W^{s-m}(\partial\Omega)$, for any $s\in\mathbf{R}$.

For $Q\in\Psi^m_{\rm cl}$, the *generalized Toeplitz operator* $T_Q : W^m_{\rm hol}(\partial\Omega)\to H^2(\partial\Omega)$ is defined as

$$T_Q = \Pi Q,$$

where $\Pi : L^2(\partial\Omega)\to H^2(\partial\Omega)$ is the orthogonal projection (the Szegö projection). Alternatively, one may view T_Q as the operator

$$T_Q = \Pi Q \Pi$$

on all of $W^m(\partial\Omega)$. Actually, T_Q maps continuously $W^s(\partial\Omega)$ into $W^{s-m}_{\rm hol}(\partial\Omega)$, for each $s\in\mathbf{R}$, because Π is bounded on $W^s(\partial\Omega)$ for any $s\in\mathbf{R}$ (see [6]).

It is known that the generalized Toeplitz operators T_P, $P\in\Psi_{\rm cl}$, have the following properties.

- (P1) They form an algebra which is, modulo smoothing operators, locally isomorphic to the algebra of classical ΨDOs on $\mathbf{R}^n$.
- (P2) In fact, for any T_Q there exists a ΨDO P of the same order such that $T_Q = T_P$ and $P\Pi = \Pi P$.
- (P3) If P,Q are of the same order and $T_P = T_Q$, then the principal symbols $\sigma(P)$ and $\sigma(Q)$ coincide on Σ. One can thus define unambiguously the *order* of a generalized Toeplitz operator as $\mathrm{ord}(T_Q) := \min\{\mathrm{ord}(P) : T_P = T_Q\}$, and its *principal symbol* (or just "symbol") as $\sigma(T_Q) := \sigma(Q)|_\Sigma$ if $\mathrm{ord}(Q) = \mathrm{ord}(T_Q)$. (The symbol is undefined if $\mathrm{ord}(T_Q) = -\infty$.)
- (P4) The order and the symbol are multiplicative: $\mathrm{ord}(T_PT_Q) = \mathrm{ord}(T_P) + \mathrm{ord}(T_Q)$ and $\sigma(T_PT_Q) = \sigma(T_P)\sigma(T_Q)$.
- (P5) If $\mathrm{ord}(T_Q)\le 0$, then T_Q is a bounded operator on $L^2(\partial\Omega)$; if $\mathrm{ord}(T_Q) < 0$, then it is even compact.
- (P6) If $Q\in\Psi^m_{\rm cl}$ and $\sigma(T_Q)=0$, then there exists $P\in\Psi^{m-1}_{\rm cl}$ with $T_P = T_Q$. In particular, if $T_Q\sim 0$, then there exists a ΨDO $P\sim 0$ such that $T_Q = T_P$.

(P7) We will say that a generalized Toeplitz operator T_Q of order m is *elliptic* if $\sigma(T_Q)$ does not vanish. Then T_Q has a parametrix, i.e. there exists a Toeplitz operator T_P of order $-m$, with $\sigma(T_P) = \sigma(T_Q)^{-1}$, such that $T_Q T_P \sim I_{H^2(\partial\Omega)} \sim T_P T_Q$.

We refer to the book [5], especially its Appendix, and to the paper [4] (which we have loosely followed in this section) for the proofs and additional information on generalized Toeplitz operators.

2.2 The Poisson operator. Let $\mathbf{K}$ denote the Poisson extension operator on Ω, i.e. $\mathbf{K}$ solves the Dirichlet problem

$$\Delta \mathbf{K} u = 0 \quad \text{on } \Omega, \qquad \mathbf{K}u|_{\partial\Omega} = u. \tag{6}$$

(Thus $\mathbf{K}$ acts from functions on $\partial\Omega$ into functions on Ω. Here Δ is the ordinary Laplace operator.) By the standard elliptic regularity theory (see e.g. [22]), $\mathbf{K}$ acts continuously from $W^s(\partial\Omega)$ onto the subspace $W^{s+1/2}_{\text{harm}}(\Omega)$ of all harmonic functions in $W^{s+1/2}(\Omega)$. In particular, it is continuous from $L^2(\partial\Omega)$ into $L^2(\Omega)$, and thus has a continuous Hilbert space adjoint $\mathbf{K}^* : L^2(\Omega) \to L^2(\partial\Omega)$. The composition

$$\mathbf{K}^*\mathbf{K} =: \Lambda$$

is known to be an elliptic positive ΨDO on $\partial\Omega$ of order -1. We have

$$\Lambda^{-1}\mathbf{K}^*\mathbf{K} = I_{L^2(\partial\Omega)}, \tag{7}$$

while

$$\mathbf{K}\Lambda^{-1}\mathbf{K}^* = \mathbf{\Pi}_{\text{harm}},$$

the orthogonal projection in $L^2(\Omega)$ onto the subspace $L^2_{\text{harm}}(\Omega)$ of all harmonic functions. (Indeed, from (7) it is immediate that the left-hand side acts as the identity on the range of $\mathbf{K}$, while it trivially vanishes on $\operatorname{Ker}\mathbf{K}^* = (\operatorname{Ran}\mathbf{K})^\perp$.) Comparing (7) with (6), we also see that the restriction

$$\gamma := \Lambda^{-1}\mathbf{K}^*|_{L^2_{\text{harm}}(\Omega)}$$

is the operator of "taking the boundary values" of a harmonic function. Again, by elliptic regularity, γ extends to a continuous operator from $W^s_{\text{harm}}(\Omega)$ onto $W^{s-1/2}(\partial\Omega)$, for any $s \in \mathbf{R}$, which is the inverse of $\mathbf{K}$.

The operators

$$\Lambda_w := \mathbf{K}^* w \mathbf{K},$$

with w a smooth function on $\overline{\Omega}$, are governed by a calculus developed by Boutet de Monvel [3]. It was shown there that for w of the form

$$w = r^m g, \qquad m = 0, 1, 2, \dots, \; g \in C^\infty(\overline{\Omega}), \tag{8}$$

Λ_w is a ΨDO on $\partial\Omega$ of order $-m-1$, with symbol

$$\sigma(\Lambda_w)(x,\xi) = \frac{(-1)^m m!}{2|\xi|^{m+1}}\, g(x)\,\|\eta_x\|^m. \tag{9}$$

(In particular, $\sigma(\Lambda)(x,\xi) = 1/2|\xi|$.)
By abstract Hilbert space theory, $\mathbf{K}$ has, as an operator from $L^2(\partial\Omega)$ into $L^2(\Omega)$, the polar decomposition

$$\mathbf{K} = U(\mathbf{K}^*\mathbf{K})^{1/2} = U\Lambda^{1/2}, \tag{10}$$

where U is a partial isometry with initial space $\overline{\operatorname{Ran}\mathbf{K}^*} = (\operatorname{Ker}\mathbf{K})^\perp$ and final space $\overline{\operatorname{Ran}\mathbf{K}}$; that is, U is a unitary operator from $L^2(\partial\Omega)$ onto $L^2_{\text{harm}}(\Omega)$.
The operators γ, $\mathbf{K}$ and $U = \mathbf{K}\Lambda^{-1/2}$ can be used to "transfer" operators on $L^2_{\text{harm}}(\Omega) \subset L^2(\Omega)$ into operators on $L^2(\partial\Omega)$. The following proposition appears as Proposition 8 in [11]; we reproduce its (short) proof here for completeness.

Proposition 1. $\gamma\mathbf{\Pi}\mathbf{K} = T_\Lambda^{-1}\Pi\Lambda$.

Proof. Set $\mathbf{\Pi}_\Lambda := \mathbf{K}T_\Lambda^{-1}\Pi\Lambda\gamma$, an operator on $L^2_{\text{harm}}(\Omega)$; we need to show that $\mathbf{\Pi}_\Lambda = \mathbf{\Pi}|_{L^2_{\text{harm}}}$. Since $T_\Lambda^{-1}\Pi\Lambda$ acts as the identity on the range of Π, it is immediate that $\mathbf{\Pi}_\Lambda^2 = \mathbf{\Pi}_\Lambda$; furthermore, $\mathbf{\Pi}_\Lambda = \mathbf{K}T_\Lambda^{-1}\Pi\mathbf{K}^* = \mathbf{K}\Pi T_\Lambda^{-1}\Pi\mathbf{K}^*$ is evidently self-adjoint. Thus $\mathbf{\Pi}_\Lambda$ is the orthogonal projection in $L^2_{\text{harm}}(\Omega)$ onto $\operatorname{Ran}\mathbf{\Pi}_\Lambda$. But

$$\begin{aligned}\operatorname{Ran}\mathbf{\Pi}_\Lambda &= (\operatorname{Ker}\mathbf{\Pi}_\Lambda)^\perp = (\operatorname{Ker}\mathbf{K}\Pi T_\Lambda^{-1}\Pi\mathbf{K}^*)^\perp = (\operatorname{Ker}T_\Lambda^{-1/2}\Pi\mathbf{K}^*)^\perp \\ &= (\operatorname{Ker}\Pi\mathbf{K}^*)^\perp = \overline{\operatorname{Ran}\mathbf{K}\Pi} = \overline{\mathbf{K}H^2(\partial\Omega)} \\ &= \overline{W^{1/2}_{\text{hol}}(\Omega)} = L^2_{\text{hol}}(\Omega).\end{aligned}$$

So, indeed, $\mathbf{\Pi}_\Lambda = \mathbf{\Pi}$. $\square$

Similarly to (10), the bounded (in fact — since Λ is of order < 0 — even compact) operator $\Lambda^{1/2}\Pi$ on $L^2(\partial\Omega)$ has polar decomposition

$$\Lambda^{1/2}\Pi = W(\Pi\Lambda\Pi)^{1/2} = WT_\Lambda^{1/2},$$

where W is a partial isometry with initial space $\overline{\operatorname{Ran}\Pi\Lambda^{1/2}} = H^2(\partial\Omega)$ and final space $\overline{\operatorname{Ran}\Lambda^{1/2}\Pi} = \overline{\Lambda^{1/2}H^2(\partial\Omega)}$; in particular,

$$W^*W = I \quad \text{on } H^2(\partial\Omega). \tag{11}$$

The following proposition is analogous to Corollary 9 of [11].

Proposition 2. *Let $w \in C^\infty(\overline{\Omega})$ be of the form* (8). *Then*

$$U^*\mathbf{T}_w U = WT_\Lambda^{-1/2} T_{\Lambda_w} T_\Lambda^{-1/2} W^* = WT_{Q_w} W^*,$$

where Q_w is a ΨDO on $\partial\Omega$ of order $-m$ with

$$\sigma(Q_w)(x,\xi)|_\Sigma = \frac{(-1)^m m!}{|\xi|^m}\, g(x)\, \|\eta_x\|^m.$$

Proof. By Proposition 1, $\mathbf{\Pi K} = \mathbf{K} T_\Lambda^{-1} \Pi\Lambda = \mathbf{K}\Pi T_\Lambda^{-1}\Pi\Lambda$; hence

$$\begin{aligned} U^*\mathbf{T}_w U &= \Lambda^{-1/2}\mathbf{K}^*\mathbf{\Pi} w \mathbf{\Pi K}\Lambda^{-1/2} \\ &= \Lambda^{1/2}\Pi T_\Lambda^{-1}\Pi\mathbf{K}^* w\mathbf{K}\Pi T_\Lambda^{-1}\Pi\Lambda^{1/2} \\ &= \Lambda^{1/2}\Pi T_\Lambda^{-1}\Pi\Lambda_w\Pi T_\Lambda^{-1}\Pi\Lambda^{1/2} \\ &= \Lambda^{1/2}\Pi T_\Lambda^{-1} T_{\Lambda_w} T_\Lambda^{-1}\Pi\Lambda^{1/2} \\ &= WT_\Lambda^{-1/2} T_{\Lambda_w} T_\Lambda^{-1/2} W^*, \end{aligned}$$

proving the first equality. The second equality follows from (9) and the properties (P1) and (P4). □

2.3 The Dixmier trace. Recall that if A is a compact operator acting on a Hilbert space then its sequence of singular values $\{s_j(A)\}_{j=1}^\infty$ is the sequence of eigenvalues of $|A| = (A^*A)^{1/2}$ arranged in nonincreasing order. In particular if $A \gg 0$ this will also be the sequence of eigenvalues of A in nonincreasing order. For $0 < p < \infty$ we say that A is in the Schatten ideal $\mathcal{S}_p$ if $\{s_j(A)\} \in l^p(\mathbf{Z}_{>0})$. If $A \gg 0$ is in $\mathcal{S}_1$, the trace class, then A has a finite trace and, in fact, $\operatorname{tr}(A) = \sum_j s_j(A)$. If however we only know that

$$\begin{aligned} &s_j(A) = O(j^{-1}) \text{ or that} \\ &S_k(A) := \sum_{j=1}^k s_j(A) = O(\log(1+k)) \end{aligned}$$

then A may have infinite trace. However in this case we may still try to compute its *Dixmier trace*, $\operatorname{Tr}_\omega(A)$. Informally $\operatorname{Tr}_\omega(A) = \lim_k \frac{1}{\log k} S_k(A)$ and this will actually be true in the cases of interest to us. We begin with the definition. Select a continuous positive linear functional ω on $l^\infty(\mathbf{Z}_{>0})$ and denote its value on $a = (a_1, a_2, ...)$ by $\operatorname{Lim}_\omega(a_k)$. We require of this choice that $\operatorname{Lim}_\omega(a_k) = \lim a_k$ if the latter exists. We require further that ω be *scale invariant*; a technical requirement that is fundamental for the theory but will not be of further concern to us.

Let $\mathcal{S}^{\mathrm{Dixm}}$ be the class of all compact operators A which satisfy

$$\left(\frac{S_k(A)}{\log(1+k)}\right) \in l^\infty. \tag{12}$$

With the norm defined as the l^∞-norm of the left-hand side of (12), $\mathcal{S}^{\mathrm{Dixm}}$ becomes a Banach space [15]. For a positive operator $A \in \mathcal{S}^{\mathrm{Dixm}}$, we define the Dixmier trace of A, $\mathrm{Tr}_\omega(A)$, as $\mathrm{Tr}_\omega(A) = \mathrm{Lim}_\omega(\frac{S_k(A)}{\log(1+k)})$. $\mathrm{Tr}_\omega(\cdot)$ is then extended by linearity to all of $\mathcal{S}^{\mathrm{Dixm}}$. Although this definition does depend on ω the operators A we consider are *measurable*, that is, the value of $\mathrm{Tr}_\omega(A)$ is independent of the particular choice of ω. We refer to [9] for details and for discussion of the role of these functionals.
It is a result of Connes [8] that if Q is a ΨDO on a compact manifold M of real dimension n and $\mathrm{ord}(Q) = -n$, then $Q \in \mathcal{S}^{\mathrm{Dixm}}$ and

$$\mathrm{Tr}_\omega(Q) = \frac{1}{n!(2\pi)^n} \int_{(\mathcal{T}^*M)_1} \sigma(Q). \tag{13}$$

(Here $(\mathcal{T}^*M)_1$ denotes the unit sphere bundle in the cotangent bundle $\mathcal{T}^*M$, and the integral is taken with respect to a measure induced by any Riemannian metric on M; since $\sigma(Q)$ is homogeneous of degree $-n$, the value of the integral is independent of the choice of such metric.) In the next section, we will see that for Toeplitz operators T_Q on $\partial\Omega$, $\Omega \subset \mathbf{C}^n$, the "right" order for T_Q to belong to $\mathcal{S}^{\mathrm{Dixm}}$ is not $-\dim_{\mathbf{R}} \partial\Omega = -(2n-1)$, but $-\dim_{\mathbf{C}} \Omega = -n$.

3. Dixmier trace of generalized Toeplitz operators

Let T be a positive self-adjoint generalized Toeplitz operator on $\partial\Omega$ of order 1 with $\sigma(T) > 0$. Let $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$ be the points of its spectrum (counting multiplicities) and let $N(\lambda)$ denote the number of λ_j's less than λ. It was shown in Theorem 13.1 in [5] that as $\lambda \to +\infty$,

$$N(\lambda) = \frac{\mathrm{vol}(\Sigma_T)}{(2\pi)^n} \lambda^n + O(\lambda^{n-1}), \tag{14}$$

where Σ_T is the subset of Σ where $\sigma(T) \le 1$, and $\mathrm{vol}(\Sigma_T)$ is its symplectic volume.
Using properties of generalized Toeplitz operators, it is easy to derive from here the formula for the Dixmier trace.

Theorem 3. *Let T be a generalized Toeplitz operator on $H^2(\partial\Omega)$ of order $-n$. Then $T \in \mathcal{S}^{\mathrm{Dixm}}$, and*

$$\mathrm{Tr}_\omega(T) = \frac{1}{n!(2\pi)^n} \int_{\partial\Omega} \sigma(T)(x, \eta_x)\, \eta \wedge (d\eta)^{n-1}.$$

In particular, T is measurable.

Proof. As the Dixmier trace is defined first on positive operators and then extended to all of $\mathcal{S}^{\mathrm{Dixm}}$ by linearity, while T may be split into its real and imaginary parts each of which can be expressed as a difference of two positive generalized Toeplitz operators of the same order, it is enough to prove the assertion when T is positive self-adjoint with $\sigma(T)>0$. Then T is elliptic, and it follows from Seeley's theorem on complex powers of ΨDO's and from the property (P2) that $T^{-1/n}$ is also a generalized Toeplitz operator, with symbol $\sigma(T)^{-1/n}$ and of order 1 (see [10], Proposition 16, for the detailed argument). Thus the eigenvalues $\lambda_1 \le \lambda_2 \le \dots$ of $T^{-1/n}$ satisfy (14). Consequently,

$$\begin{aligned} S_k(T) = \sum_{j=1}^k s_j(T) = \sum_{j=1}^k \lambda_j^{-n} &= \int_{[\lambda_1,\lambda_k]} \lambda^{-n}\, dN(\lambda) \\ &= \int_{[\lambda_1,\lambda_k]} \Big(\frac{c}{N(\lambda)} + O\big(N(\lambda)^{-1-\frac1n}\big)\Big)\, dN(\lambda) \\ &= \int_1^k \Big(\frac{c}{N} + O(N^{-1-\frac1n})\Big)\, dN \\ &= c\log k + O(1). \end{aligned}$$

Here we have temporarily denoted $c := (2\pi)^{-n}\operatorname{vol}(\Sigma_{T^{-1/n}})$. Dividing by $\log(k+1)$ and letting k tend to infinity, it follows that $T\in\mathcal{S}^{\mathrm{Dixm}}$ and

$$\operatorname{Tr}_\omega(T) = \lim_{k\to\infty} \frac{S_k(T)}{\log(k+1)} = c. \tag{15}$$

Let us parameterize Σ as $(x, t\eta_x)$ with $x\in\partial\Omega$, $t>0$. The subset $\Sigma_{T^{-1/n}}$ is then characterized by

$$\sigma(T)(x,t\eta_x)^{-1/n} \le 1, \quad \text{or} \quad t\le \sigma(T)(x,\eta_x)^{1/n}.$$

A routine computation, which we postpone to the next lemma, shows that the symplectic volume on Σ with respect to the above parameterization is given by $\frac{t^{n-1}}{(n-1)!}\, dt\wedge\eta(x)\wedge(d\eta(x))^{n-1}$. Consequently,

$$\begin{aligned} \operatorname{vol}(\Sigma_{T^{-1/n}}) &= \int_{\partial\Omega}\int_0^{\sigma(T)(x,\eta_x)^{1/n}} \frac{t^{n-1}}{(n-1)!}\, dt\wedge\eta\wedge(d\eta)^{n-1} \\ &= \frac{1}{n!}\int_{\partial\Omega} \sigma(T)(x,\eta_x)\, \eta\wedge(d\eta)^{n-1}. \end{aligned}$$

Combining this with (15) and the definition of c, the assertion follows. □

Remark 4. Observe that, in analogy with (13), the last integral is independent of the choice of the defining function. Indeed, if r is replaced by gr, with $g>0$ on $\partial\Omega$, then $\eta=\operatorname{Im}(\partial r)$ is replaced by $g\eta$ (since $\partial(gr)=g\partial r$ on the set where $r=0$), and $\eta\wedge(d\eta)^{n-1}$ by $g\eta\wedge(g\,d\eta+dg\wedge\eta)^{n-1}=g^n\,\eta\wedge(d\eta)^{n-1}$ (because $\eta\wedge\eta=0$); as $\sigma(T)(x,\xi)$ is homogeneous of degree $-n$ in ξ, the integrand remains unchanged. □

Lemma 5. *With respect to the parameterization* $\Sigma = \{(x, t\eta_x) : x \in \partial\Omega, t > 0\}$, *the symplectic form on* Σ *is given by*

$$\omega = t\,d\eta + dt \wedge \eta = d(t\eta).$$

Consequently, the symplectic volume in the (x,t) *coordinates is given by*

$$\frac{\omega^n}{n!} = \frac{t^{n-1}}{(n-1)!}\,dt \wedge \eta \wedge (d\eta)^{n-1}.$$

We are supplying a proof of this simple fact below, since we were unable to locate it in the literature (though we expect that it must be at least implicitly contained e.g. somewhere in [5]).

Proof. Recall that if $(x_1, x_2, \dots, x_{2n-1})$ is a real coordinate chart on $\partial\Omega$ and (x,ξ) the corresponding local coordinates for a point $(x; \xi_1 dx_1 + \dots + \xi_{2n-1} dx_{2n-1})$ in $\mathcal{T}^*\partial\Omega$, then the form $\alpha = \xi_1 dx_1 + \dots + \xi_{2n-1} dx_{2n-1}$ is globally defined and the symplectic form is given by $\omega = d\alpha = d\xi_1 \wedge dx_1 + \dots + d\xi_{2n-1} \wedge dx_{2n-1}$. Since exterior differentiation commutes with restriction (or, more precisely, with the pullback j^* under the inclusion map $j : \Sigma \to \mathcal{T}^*\partial\Omega$), it follows that the symplectic form $\omega_\Sigma = j^*\omega$ on Σ is given by $\omega_\Sigma = d(j^*\alpha)$. As in our case $j^*\alpha = t\eta$, the first formula follows. (We will drop the subscript Σ from now on.) The second formula is immediate from the first since $\eta \wedge \eta = 0$ and $(d\eta)^n = 0$. $\square$

The following corollary is immediate upon combining Theorem 3 and Proposition 2.

Corollary 6. *Assume that* $f \in C^\infty(\overline{\Omega})$ *vanishes at* $\partial\Omega$ *to order* n. *Then* $\mathbf{T}_f$ *belongs to the Dixmier class, is measurable, and*

$$\operatorname{Tr}_\omega(\mathbf{T}_f) = \frac{1}{n!(4\pi)^n} \int_{\partial\Omega} \mathfrak{N}^n f \,\frac{\eta \wedge (d\eta)^{n-1}}{\|\eta\|^n},$$

where $\mathfrak{N}$ *denotes the interior unit normal derivative.*

4. Dixmier trace for products of Hankel operators

It is known [5] that the symbol of the commutator of two generalized Toeplitz operators is given by the Poisson bracket (with respect to the symplectic structure of Σ) of their symbols:

$$\sigma([T_P, T_Q]) = \tfrac{1}{i}\{\sigma(T_P), \sigma(T_Q)\}_\Sigma.$$

We need an analogous formula for the semi-commutator $T_{PQ} - T_P T_Q$ of two generalized Toeplitz operators. Not surprisingly, it turns out to be given (at least in the cases of interest to us) by an appropriate "half" of the Poisson bracket.

Let us denote by $\mathcal{T}'' \subset \mathcal{T}\partial\Omega \otimes \mathbf{C}$ the anti-holomorphic complex tangent space to $\partial\Omega$, i.e. the elements of $\mathcal{T}''_x$, $x \in \partial\Omega$, are the vectors $\sum_{j=1}^n a_j \frac{\partial}{\partial \overline{z}_j}$, $a_j \in \mathbf{C}$, such that $\sum_j a_j \frac{\partial r}{\partial \overline{z}_j}(x) = 0$. (This notation follows [6], p. 141.) On the open subset $\mathcal{U}_m$ of $\partial\Omega$ where $\frac{\partial r}{\partial \overline{z}_m} \neq 0$ (as m ranges from 1 to n, these subsets cover all of $\partial\Omega$), $\mathcal{T}''$ is spanned by the $n-1$ vector fields

$$\overline{R}_j := \frac{\partial}{\partial \overline{z}_j} - \frac{\partial r/\partial \overline{z}_j}{\partial r/\partial \overline{z}_m} \frac{\partial}{\partial \overline{z}_m}, \qquad j \neq m.$$

(Thus $\overline{R}_j$ depends also on m, although this is not reflected by the notation.) The (similarly defined) holomorphic complex tangent space $\mathcal{T}'$ is, analogously, spanned on $\mathcal{U}_m$ by the $n-1$ vector fields

$$R_j := \frac{\partial}{\partial z_j} - \frac{\partial r/\partial z_j}{\partial r/\partial z_m} \frac{\partial}{\partial z_m}, \qquad j \neq m,$$

while the whole complex tangent space $\mathcal{T}\partial\Omega \otimes \mathbf{C}$ is spanned there by the R_j, $\overline{R}_j$ and

$$E := \sum_{j=1}^n \frac{\partial r}{\partial z_j} \frac{\partial}{\partial \overline{z}_j} - \frac{\partial r}{\partial \overline{z}_j} \frac{\partial}{\partial z_j}$$

(the "complex normal" direction).
The boundary d-bar operator $\overline{\partial}_b : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega, \mathcal{T}''^*)$ is defined as the restriction

$$\overline{\partial}_b f := df|_{\mathcal{T}''},$$

or, more precisely, $\overline{\partial}_b f = d\tilde{f}|_{\mathcal{T}''}$ for any smooth extension $\tilde{f}$ of f to a neighbourhood of $\partial\Omega$ in $\mathbf{C}^n$ (the right-hand side is independent of the choice of such extension). On $\mathcal{U}_m$, $\mathcal{T}''^*$ admits $d\overline{z}_j|_{\mathcal{T}''}$, $j \neq m$, as a basis and

$$\overline{\partial}_b f = \sum_j \overline{R}_j f \, d\overline{z}_j|_{\mathcal{T}''}.$$

Under our parameterization of Σ by $(x,t) \in \partial\Omega \times \mathbf{R}_+$, the tangent bundle $\mathcal{T}\Sigma$ is identified with $\mathcal{T}\partial\Omega \times \mathbf{R}$, being spanned at each $(x, t\eta_x) \in \Sigma$ by $\overline{R}_j$, R_j, E and the extra vector $T := \frac{\partial}{\partial t}$. Recall that the Levi form L' is the Hermitian form on $\mathcal{T}'$ defined by

$$L'(X,Y) := \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} X_j \overline{Y}_k \qquad \text{if } X = \sum_j X_j \frac{\partial}{\partial z_j},\ Y = \sum_k Y_k \frac{\partial}{\partial z_k}.$$

The strong pseudoconvexity of Ω means that L' is positive definite. Similarly, one has the positive-definite Levi form L'' on $\mathcal{T}''$ defined by

$$L''(X,Y) := \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_k \partial \overline{z}_j} X_j \overline{Y}_k \qquad \text{if } X = \sum_j X_j \frac{\partial}{\partial \overline{z}_j},\ Y = \sum_k Y_k \frac{\partial}{\partial \overline{z}_k}.$$

In terms of the complex conjugation $X \mapsto \overline{X}$ given by $\overline{X_j \frac{\partial}{\partial z_j}} = \overline{X}_j \frac{\partial}{\partial \overline{z}_j}$, mapping $\mathcal{T}'$ onto $\mathcal{T}''$ and vice versa, the two forms are related by

$$L''(X,Y) = L'(\overline{Y},\overline{X}) \qquad \forall X,Y \in \mathcal{T}''. \tag{16}$$

By the usual formalism, L'' induces a positive definite Hermitian form[1] on the dual space $\mathcal{T}''^*$ of $\mathcal{T}''$; we denote it by $\mathcal{L}$. Namely, if L'' is given by a matrix $\mathbb{L}$ with respect to some basis $\{e_j\}$, then $\mathcal{L}$ is given by the inverse matrix $\mathbb{L}^{-1}$ with respect to the dual basis $\{\hat{e}_k\}$ satisfying $\hat{e}_k(e_j) = \delta_{jk}$. An alternative description is the following. For any $\alpha \in \mathcal{T}''^*$, let $Z''_\alpha \in \mathcal{T}''$ be defined by

$$L''(X, Z''_\alpha) = \alpha(X) \qquad \forall X \in \mathcal{T}''.$$

(This is possible, and Z''_α is unique, owing to the non-degeneracy of L''. Note that $\alpha \mapsto Z''_\alpha$ is conjugate-linear.) Then

$$\mathcal{L}(\alpha,\beta) = L''(Z''_\beta, Z''_\alpha) = \alpha(Z''_\beta) = \overline{\beta(Z''_\alpha)}.$$

Let, in particular, $Z''_f := Z''_{\overline{\partial}_b f}$, so that

$$L''(X, Z''_f) = \overline{\partial}_b f(X) \qquad \forall X \in \mathcal{T}'',$$

and denote by $Z'_f \in \mathcal{T}'$ the similarly defined holomorphic vector field satisfying

$$L'(Y, Z'_f) = \partial_b f(Y) \qquad \forall Y \in \mathcal{T}',$$

where $\partial_b f := df|_{\mathcal{T}'}$. Set

$$Z_f := i(\overline{Z''_f} - \overline{Z'_f}) \in \mathcal{T}' + \mathcal{T}''.$$

These objects are related to the symplectic structure of Σ as follows. Note that

$$d\eta = i\partial\overline{\partial} r = i \sum_{k,l=1}^{n} \frac{\partial^2 r}{\partial z_k \partial \overline{z}_l}\, dz_k \wedge d\overline{z}_l,$$

hence

$$d\eta(X' + X'', Y' + Y'') = iL'(X', \overline{Y''}) - iL'(Y', \overline{X''})$$

for all $X', Y' \in \mathcal{T}'$ and $X'', Y'' \in \mathcal{T}''$. It follows that $d\eta$ is a non-degenerate skew-symmetric bilinear form on $\mathcal{T}' + \mathcal{T}''$, and

$$d\eta(X, Z_f) = Xf \qquad \forall X \in \mathcal{T}' + \mathcal{T}''. \tag{17}$$

[1] or, perhaps more appropriately, a positive definite Hermitian *bivector*

Indeed,

$$\begin{aligned} d\eta(X'+X'',Z_f) &= iL'(X',\overline{-i\overline{Z'_f}}) - iL'(i\overline{Z''_f},\overline{X''}) \\ &= L'(X',Z'_f) + L''(X'',Z''_f) \\ &= \partial_b f(X') + \overline{\partial}_b f(X'') = df(X'+X''). \end{aligned}$$

Let us define $E_{\mathcal{T}} \in \mathcal{T}' + \mathcal{T}''$ by

$$d\eta(X,E_{\mathcal{T}}) = d\eta(X,E) \qquad \forall X \in \mathcal{T}' + \mathcal{T}'' \tag{18}$$

(again, this is possible and unambiguous by virtue of the non-degeneracy of $d\eta$ on $\mathcal{T}'+\mathcal{T}''$), and set

$$E_\perp := \frac{E-E_{\mathcal{T}}}{\eta(E)} = \frac{E-E_{\mathcal{T}}}{i\|\eta\|^2}.$$

The vector field $E_\perp$ is usually called the Reeb vector field, and is defined by the conditions $\eta(E_\perp)=1$, $i_{E_\perp}d\eta=0$.

Proposition 7. *Let $f,g\in C^\infty(\partial\Omega)$, and let F,G be the functions on $\Sigma\cong\partial\Omega\times\mathbf{R}_+$ given by*

$$F(x,t)=t^{-k}f(x), \quad G(x,t)=t^{-m}g(x).$$

Then the Poisson bracket of F and G is given by

$$\{F,G\}_\Sigma = t^{-k-m-1}\Big(Z_f g + mgE_\perp f - kfE_\perp g\Big).$$

Proof. Recall that the Hamiltonian vector field H_F of F is the pre-dual of dF with respect to the symplectic form $\omega_\Sigma\equiv\omega$ on Σ, namely

$$\omega(X,H_F) = dF(X) = XF, \qquad \forall X\in\mathcal{T}\Sigma.$$

Since $F=t^{-k}f(x)$, we have $dF=t^{-k}df - kt^{-k-1}f\,dt$, so

$$H_F = t^{-k}H_f - kt^{-k-1}fH_t. \tag{19}$$

We claim that

$$H_t = E_\perp, \qquad H_f = \frac{1}{t}Z_f - E_\perp f\,T. \tag{20}$$

We check the formula for H_t, i.e.

$$\omega(X,H_t) = dt(X) \qquad \forall X\in\mathcal{T}\Sigma.$$

For $X = T$,

$$\begin{aligned}\omega(T, E_\perp) &= \frac{1}{\eta(E)}(td\eta + dt\wedge\eta)(T, E - E_{\mathcal{T}})\\ &= \frac{1}{\eta(E)}\,dt\wedge\eta(T, E - E_{\mathcal{T}}) = \frac{\eta(E) - \eta(E_{\mathcal{T}})}{\eta(E)}\\ &= 1 = dt(T),\end{aligned}$$

since η vanishes on $\mathcal{T}' + \mathcal{T}'' \ni E_{\mathcal{T}}$. Similarly, for $X = X' \in \mathcal{T}'$,

$$\omega(X', E_\perp) = \frac{1}{\eta(E)}\,t\,d\eta(X', E_\perp)$$

vanishes by the definition (18) of $E_{\mathcal{T}}$, and so does $dt(E_\perp)$ since $E_\perp$ contains no t-differentiations. Analogously for $X = X'' \in \mathcal{T}''$. Finally, for $X = E$ we have

$$\begin{aligned}\omega(E, E_\perp) &= -\frac{1}{\eta(E)}\omega(E, E_{\mathcal{T}}) = -\frac{1}{\eta(E)}\,t\,d\eta(E, E_{\mathcal{T}})\\ &= -\frac{1}{\eta(E)}\,t\,d\eta(E_{\mathcal{T}}, E_{\mathcal{T}}) = 0 = dt(E),\end{aligned}$$

where in the third equality we have used (18) for $X = E_{\mathcal{T}}$.
Next we check the formula for H_f. For $X = T$, both $\omega(X, H_f)$ and $df(X)$ are zero. For $X \in \mathcal{T}' + \mathcal{T}''$, we have $\omega(X, T) = dt\wedge\eta(X, T) = -\eta(X) = 0$ and the equality follows by (17). Finally for $X = E$

$$\begin{aligned}\omega(E, H_f) &= t\,d\eta(E, \tfrac{1}{t}Z_f) + dt\wedge\eta(E, -E_\perp f\,T)\\ &= d\eta(E_{\mathcal{T}}, Z_f) + \eta(E)E_\perp f\\ &= E_{\mathcal{T}}f + \eta(E)E_\perp f \qquad \text{by (17)}\\ &= E_{\mathcal{T}}f + (E - E_{\mathcal{T}})f,\end{aligned}$$

which indeed coincides with $df(E) = Ef$.
By (20) and (19), we thus get

$$H_F = t^{-k-1}Z_f - t^{-k}E_\perp f\,T - kt^{-k-1}fE_\perp.$$

Consequently,

$$\begin{aligned}\{F, G\}_\Sigma &= \omega(H_F, H_G) = H_F G\\ &= t^{-k-m-1}Z_f g + mt^{-k-m-1}gE_\perp f - kt^{-k-m-1}fE_\perp g,\end{aligned}$$

and the assertion follows. $\square$

Corollary 8. *Let $f, g \in C^\infty(\partial\Omega)$, and denote by f, g also the corresponding functions on $\Sigma \cong \partial\Omega \times \mathbf{R}_+$ constant on each fiber. Then*

$$\{f, g\}_\Sigma = \frac{1}{t} Z_f g = i\, \frac{\mathcal{L}(\overline{\partial}_b f, \overline{\partial}_b \overline{g}) - \mathcal{L}(\overline{\partial}_b g, \overline{\partial}_b \overline{f})}{t}.$$

Proof. Immediate upon taking $m = k = 0$ in the last proposition, and observing that

$$\begin{aligned} \frac{1}{i} Z_f g &= \overline{Z''_f} g - \overline{Z'_f} g = dg(\overline{Z''_f}) - dg(\overline{Z'_f}) \\ &= \partial_b g(\overline{Z''_f}) - \overline{\partial}_b g(\overline{Z'_f}) = \overline{\overline{\partial}_b \overline{g}(Z''_f)} - \overline{\partial}_b g(\overline{Z'_f}) \\ &= \overline{\mathcal{L}(\overline{\partial}_b \overline{g}, \overline{\partial}_b f)} - \mathcal{L}(\overline{\partial}_b g, \overline{\partial}_b \overline{f}), \end{aligned}$$

since $\overline{Z'_f} = Z''_{\overline{f}}$ by virtue of (16). $\square$

We are now ready to state the main result of this section and, in some sense, of this paper.

Theorem 9. *Let U, W have the same meaning as in Proposition 2. Then for $f, g \in C^\infty(\overline{\Omega})$,*

$$U^*(\mathbf{T}_{f\overline{g}} - \mathbf{T}_{\overline{g}} \mathbf{T}_f) U = W T_Q W^*,$$

where T_Q is a generalized Toeplitz operator on $\partial\Omega$ of order -1 with principal symbol

$$\sigma(T_Q)(x, t\eta_x) = \frac{1}{t}\, \mathcal{L}(\overline{\partial}_b f, \overline{\partial}_b g)(x). \tag{21}$$

Proof. By Proposition 2,

$$U^*(\mathbf{T}_{f\overline{g}} - \mathbf{T}_{\overline{g}} \mathbf{T}_f) U = W (T_{Q_{f\overline{g}}} - T_{Q_{\overline{g}}} T_{Q_f}) W^*,$$

where $T_{Q_f} = T_\Lambda^{-1/2} T_{\Lambda_f} T_\Lambda^{-1/2}$ is a generalized Toeplitz operator of order 0 with symbol $\sigma(T_{Q_f})(x, \xi) = f(x)$. By (P1) and (P4), the expression $T_{Q_{f\overline{g}}} - T_{Q_{\overline{g}}} T_{Q_f}$ is thus a generalized Toeplitz operator T_Q of order 0 with symbol $\sigma(T_Q) = \sigma(T_{Q_{f\overline{g}}}) - \sigma(T_{Q_{\overline{g}}}) \sigma(T_{Q_f}) = f\overline{g} - \overline{g} f = 0$; thus by (P6), it is indeed, in fact, a generalized Toeplitz operator of order -1. It remains to show that its symbol, which we denote by $\rho(f, g)$, is given by (21).

By the general theory, $\rho(f, g)$ is given by a local expression, i.e. one involving only finitely many derivatives of f and g at the given point, and linear in f and $\overline{g}$. (Indeed, the proof of Proposition 2.5 in [5] shows that the construction, for a given ΨDO Q, of the ΨDO P from property (P2), i.e. such that $T_Q = T_P$ and $[P, \Pi] = 0$, is completely local in nature, so the total symbol of the P corresponding to $Q = \Lambda_f$ is given by local expressions in terms of the total symbol of Λ_f, hence, by local expressions in terms of f; the claim thus follows from the product formula for the symbol of ΨDOs.) It is therefore enough to show that

$$\rho(f,g)=\frac{1}{t}\mathcal{L}(\overline{\partial}_b f,\overline{\partial}_b g) \tag{22}$$

for functions f,g of the form $u\overline{v}$, with u,v holomorphic on $\overline{\Omega}$.[2] Next, if u and v are holomorphic on $\overline{\Omega}$, then $\mathbf{T}_{\overline{v}}\mathbf{T}_f=\mathbf{T}_{\overline{v}f}$ and $\mathbf{T}_f\mathbf{T}_u=\mathbf{T}_{fu}$ for any f; consequently, using Proposition 2 and (11),

$$\begin{aligned}
U^*(\mathbf{T}_{uf\overline{vg}}-\mathbf{T}_{\overline{vg}}\mathbf{T}_{uf})U &= U^*\mathbf{T}_{\overline{v}}(\mathbf{T}_{f\overline{g}}-\mathbf{T}_{\overline{g}}\mathbf{T}_f)\mathbf{T}_uU \\
&= U^*\mathbf{T}_{\overline{v}}UU^*(\mathbf{T}_{f\overline{g}}-\mathbf{T}_{\overline{g}}\mathbf{T}_f)UU^*\mathbf{T}_uU \\
&= WT_{Q_{\overline{v}}}W^*W(T_{Q_{f\overline{g}}}-T_{Q_{\overline{g}}}T_{Q_f})W^*WT_{Q_u}W^* \\
&= WT_{Q_{\overline{v}}}(T_{Q_{f\overline{g}}}-T_{Q_{\overline{g}}}T_{Q_f})T_{Q_u}W^*.
\end{aligned}$$

By (P4) we see that

$$\rho(uf,vg)=u\,\rho(f,g)\,\overline{v}.$$

Since also

$$\mathcal{L}(\overline{\partial}_b uf,\overline{\partial}_b vg)=u\,\mathcal{L}(\overline{\partial}_b f,\overline{\partial}_b g)\,\overline{v}$$

(because $\overline{\partial}_b(uf)=u\,\overline{\partial}_b f$ for holomorphic u), it in fact suffices to prove (22) when f,g are both conjugate-holomorphic, i.e. $\overline{\partial}_b\overline{f}=\overline{\partial}_b\overline{g}=0$. However, in that case $\mathbf{T}_{f\overline{g}}=\mathbf{T}_f\mathbf{T}_{\overline{g}}$, so, using again Proposition 2 and (11),

$$\begin{aligned}
U^*(\mathbf{T}_{f\overline{g}}-\mathbf{T}_{\overline{g}}\mathbf{T}_f)U &= U^*[\mathbf{T}_f,\mathbf{T}_{\overline{g}}]U=[U^*\mathbf{T}_fU,U^*\mathbf{T}_{\overline{g}}U] \\
&= [WT_{Q_f}W^*,WT_{Q_{\overline{g}}}W^*]=W[T_{Q_f},T_{Q_{\overline{g}}}]W^*,
\end{aligned}$$

implying that

$$\begin{aligned}
\rho(f,g) &= \sigma([T_{Q_f},T_{Q_{\overline{g}}}]) \\
&= \tfrac{1}{i}\{\sigma(T_{Q_f}),\sigma(T_{Q_{\overline{g}}})\}_\Sigma \\
&= \tfrac{1}{i}\{f,\overline{g}\}_\Sigma \\
&= \frac{\mathcal{L}(\overline{\partial}_b f,\overline{\partial}_b g)-\mathcal{L}(\overline{\partial}_b\overline{g},\overline{\partial}_b\overline{f})}{t} \qquad \text{by Corollary 8} \\
&= \tfrac{1}{t}\,\mathcal{L}(\overline{\partial}_b f,\overline{\partial}_b g),
\end{aligned}$$

completing the proof. □

Remark 10. It seems much more difficult to obtain a formula for the symbol of $T_{PQ}-T_PT_Q$ for general ΨDOs P and Q. □

We are now ready to prove the main result on Dixmier traces.

[2] In fact, even holomorphic polynomials u,v would do.

Theorem 11. *Let $f_1, g_1, \dots, f_n, g_n \in C^\infty(\overline{\Omega})$. Then the operator*

$$H = H_{g_1}^* H_{f_1} H_{g_2}^* H_{f_2} \dots H_{g_n}^* H_{f_n}$$

on $L^2_{\mathrm{hol}}(\Omega)$ belongs to the Dixmier class, and

$$\operatorname{Tr}_\omega(H) = \frac{1}{n!(2\pi)^n} \int_{\partial\Omega} \mathcal{L}(\overline{\partial}_b f_1, \overline{\partial}_b g_1) \dots \mathcal{L}(\overline{\partial}_b f_n, \overline{\partial}_b g_n)\, \eta \wedge (d\eta)^{n-1}. \tag{23}$$

In particular, H is measurable.

Proof. Denote, for brevity, $V_j := T_\Lambda^{-1/2}(T_{\Lambda_{f_j \overline{g}_j}} - T_{\Lambda_{\overline{g}_j}} T_\Lambda^{-1} T_{\Lambda_{f_j}}) T_\Lambda^{-1/2}$. We have seen in the last theorem that $H_{g_j}^* H_{f_j} = \mathbf{T}_{\overline{g}_j f_j} - \mathbf{T}_{\overline{g}_j} \mathbf{T}_{f_j}$ satisfies

$$U^* H_{g_j}^* H_{f_j} U = W V_j W^*$$

and that V_j is a generalized Toeplitz operator of order -1 with symbol given by $\sigma(V_j)(x, t\eta_x) = \frac{1}{t}\mathcal{L}(\overline{\partial}_b f_j, \overline{\partial}_b g_j)$. By iteration and using (11), it follows that

$$U^* H_{g_1}^* H_{f_1} H_{g_2}^* H_{f_2} \dots H_{g_n}^* H_{f_n} U = W V_1 V_2 \dots V_n W^* = W V W^*,$$

where $V := V_1 V_2 \dots V_n$ is a generalized Toeplitz operator of order $-n$ with symbol $\sigma(V)(x, t\eta_x) = t^{-n} \prod_{j=1}^n \mathcal{L}(\overline{\partial}_b f_j, \overline{\partial}_b g_j)$. An application of Theorem 3 completes the proof. $\square$

Corollary 12. *Let f be holomorphic on Ω and C^∞ on $\overline{\Omega}$. Then $|H_{\overline{f}}|^{2n}$ is in the Dixmier class, measurable, and*

$$\operatorname{Tr}_\omega(|H_{\overline{f}}|^{2n}) = \frac{1}{n!(2\pi)^n} \int_{\partial\Omega} \mathcal{L}(\overline{\partial}_b \overline{f}, \overline{\partial}_b \overline{f})^n\, \eta \wedge (d\eta)^{n-1}.$$

By standard matrix algebra, one has[3]

$$\mathcal{L}(\overline{\partial}_b f, \overline{\partial}_b g) = \begin{bmatrix} \overline{\partial}\tilde{g} \\ 0 \end{bmatrix}^* \begin{bmatrix} \partial\overline{\partial} r & \overline{\partial} r \\ \partial r & 0 \end{bmatrix}^{-1} \begin{bmatrix} \overline{\partial}\tilde{f} \\ 0 \end{bmatrix},$$

[3] Let, quite generally, X be an operator on $\mathbf{C}^n$, $u \in \mathbf{C}^n$, and denote by A the compression of X to the orthogonal complement $u^\perp$ of u, i.e. $A = PX|_{\operatorname{Ran} P}$ where $P: \mathbf{C}^n \to u^\perp$ is the orthogonal projection. Assume that A is invertible. Then the block matrix $\begin{bmatrix} X & u \\ u^* & 0 \end{bmatrix} \in \mathbf{C}^{(n+1)\times(n+1)}$ is invertible, and for any $v, w \in \mathbf{C}^n$,

$$\langle A^{-1} P v, P w \rangle = \begin{bmatrix} w \\ 0 \end{bmatrix}^* \begin{bmatrix} X & u \\ u^* & 0 \end{bmatrix}^{-1} \begin{bmatrix} v \\ 0 \end{bmatrix}.$$

Indeed, switching to a convenient basis we may assume that $u = [0, \dots, 0, 1]^t$. Write $X = \begin{bmatrix} A & b \\ c^* & d \end{bmatrix}$, with $b, c \in \mathbf{C}^n$, $d \in \mathbf{C}$. Then

$$\begin{bmatrix} X & u \\ u^* & 0 \end{bmatrix} = \begin{bmatrix} A & b & 0 \\ c^* & d & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c^* & 0 & 1 \end{bmatrix},$$

where $\tilde f, \tilde g$ are any smooth extensions of $f, g \in C^\infty(\partial\Omega)$ to a neighbourhood of $\partial\Omega$.
In particular, for $\Omega = \mathbf{B}^d$, the unit ball, with the defining function $r(z) = |z|^2 - 1$, we obtain

$$\mathcal{L}(\overline{\partial}_b f, \overline{\partial}_b g) = \sum_{j=1}^n \frac{\partial \tilde f}{\partial \overline{z}_j} \overline{\frac{\partial \tilde g}{\partial \overline{z}_j}} - \overline{R}\tilde f \overline{\overline{R}\tilde g}, \tag{24}$$

where $\overline{R} := \sum_{j=1}^n \overline{z}_j \frac{\partial}{\partial \overline{z}_j}$ is the anti-holomorphic radial derivative. One also easily checks that $\eta \wedge (d\eta)^{n-1} = (2\pi)^n \, d\sigma$ where $d\sigma$ is the normalized surface measure on $\partial\mathbf{B}^n$. The last two theorems thus recover, as they should, the results from [12] (Theorem 4.4 — which is the formula (3) above — and Corollary 4.5 there).
Note also that for $n = 1$, the expression (24) vanishes; in this case $U^* H_g^* H_f U$ is thus in fact of order not -1 but -2 (so that $|H_g^* H_f|^{1/2}$ is in the Dixmier class rather than $H_g^* H_f$), and some additional work is needed to compute the symbol (and, from it, the Dixmier trace); see [13].
Finally, we pause to remark that the value of the integral (23) remains unchanged under biholomorphic mappings, as well as changes of the defining function. Indeed, if r is replaced by gr, with $g > 0$ on $\partial\Omega$, then $\mathcal{T}''$ and $\overline{\partial}_b$ are unaffected, while the Levi form L on $\mathcal{T}''$ gets multiplied by g. Hence its dual $\mathcal{L}$ gets multiplied by g^{-1}, and as $\eta \wedge (d\eta)^{n-1}$ transforms into $g^n \eta \wedge (d\eta)^{n-1}$ (cf. Remark 4), the integrand in (23) does not change. Similarly, if $\phi : \Omega_1 \to \Omega_2$ is a biholomorphic map and r is a defining function for Ω_2, one can choose $\phi \circ r$ as the defining function for Ω_1; then it is immediate, in turn, that ϕ sends $\mathcal{T}'$ into $\mathcal{T}'$ and $\mathcal{T}''$ into $\mathcal{T}''$, and that it transforms each of η, $\eta \wedge (d\eta)^{n-1}$, $\overline{\partial}_b$, ∂_b, L and $\mathcal{L}$ into the corresponding object on the other domain. Hence $\mathcal{L}(\overline{\partial}_b f, \overline{\partial}_b g) = (\phi_* \mathcal{L})(\phi^* \overline{\partial}_b f, \phi^* \overline{\partial}_b g) = \mathcal{L}(\overline{\partial}_b (f \circ \phi), \overline{\partial}_b (g \circ \phi))$ and, finally, $\phi^*(\prod_j \mathcal{L}(\overline{\partial}_b f_j, \overline{\partial}_b g_j) \, \eta \wedge (d\eta)^{n-1}) = \prod_j \mathcal{L}(\overline{\partial}_b (f_j \circ \phi), \overline{\partial}_b (g_j \circ \phi)) \, \eta \wedge (d\eta)^{n-1}$, proving the claim. Note that e.g. even in the formula (3) for $\Omega = \mathbf{B}^n$, the invariance of the value of the integral under biholomorphic self-maps of the ball is definitely not apparent.

5. Concluding remarks

5.1 Manifolds. The results in this paper should all be generalizable to arbitrary strictly pseudoconvex manifolds.

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whence

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c^* & 0 & 1 \end{bmatrix} \begin{bmatrix} X & u \\ u^* & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and the claim follows.
The formula for $\mathcal{L}(\overline{\partial}_b f, \overline{\partial}_b g)$ is obtained upon taking $X = \mathbb{L}$, $u = \overline{\partial} r$, $v = \overline{\partial} f$ and $w = \overline{\partial} g$.

5.2 Residual trace. Comparing Theorem 3 with the results of Guillemin [16] [17], we see that the Dixmier trace for generalized Toeplitz operators coincides (possibly up to different normalization) with the residual trace of Wodzicki, Guillemin, Manin and Adler. This is completely analogous to the situation for ΨDOs, cf. Connes [8], Theorem 1.

5.3 Nonsmooth symbols. For the unit disc $\mathbf{D}$ in $\mathbf{C}$, the analogue of Corollary 12 is

$$\operatorname{Tr}_\omega(|H_{\overline{f}}|) = \int_{\partial\mathbf{D}} |f'(e^{i\theta})| \, \frac{d\theta}{2\pi}$$

for f holomorphic on $\mathbf{D}$ and smooth on $\overline{\mathbf{D}}$; see [13]. It was shown in [13] that the smoothness assumption can be dispensed with: namely, for f holomorphic on $\mathbf{D}$, $|H_{\overline{f}}| \in \mathcal{S}^{\mathrm{Dixm}}$ if and only if f' belongs to the Hardy 1-space $H^1(\partial\mathbf{D})$, and then

$$\operatorname{Tr}_\omega(|H_{\overline{f}}|) = \|f'\|_{H^1}.$$

We expect that the same situation prevails also for general domains Ω of the kind studied in this paper, in the following sense. For f holomorphic on Ω, denote

$$\mathcal{L}_f(z) := \begin{bmatrix} \partial f \\ 0 \end{bmatrix}^* \begin{bmatrix} \partial\overline{\partial} r & \partial r \\ \overline{\partial} r & 0 \end{bmatrix}^{-1} \begin{bmatrix} \partial f \\ 0 \end{bmatrix} (z).$$

This is a smooth function defined in some neighbourhood of $\partial\Omega$ in $\overline{\Omega}$, whose boundary values coincide with $\mathcal{L}(\overline{\partial}_b \overline{f}, \overline{\partial}_b \overline{f})$ if f is smooth up to the boundary.

Conjecture. *Let f be holomorphic on Ω. Then $|H_{\overline{f}}|^{2n} \in \mathcal{S}^{\mathrm{Dixm}}$ if and only if*

$$\|f\|_{\mathcal{L}} := \limsup_{\epsilon\searrow 0} \Big(\frac{1}{n!(2\pi)^n} \int_{r=-\epsilon} |\mathcal{L}_f|^n \, |\eta \wedge (d\eta)^{n-1}| \Big)^{1/2n}$$

is finite, and then

$$\operatorname{Tr}_\omega(|H_{\overline{f}}|^{2n}) = \|f\|_{\mathcal{L}}^{2n}.$$

The proof for the disc went by showing first that $\|f'\|_{H_1}$ actually dominates the $\mathcal{S}^{\mathrm{Dixm}}$ norm of $|H_{\overline{f}}|$; the result then followed from the one for $f \in C^\infty(\overline{\mathbf{D}})$ by a straightforward approximation argument. This approach might also work for general domains Ω (with $\|f\|_{\mathcal{L}}$ and $|H_{\overline{f}}|^{2n}$ replacing $\|f'\|_{H^1}$ and $|H_{\overline{f}}|$), but the techniques for doing so (estimates for the oscillation of f' on Carleson-type rectangles, etc.) are outside the scope of this paper.

5.4 Higher type. The generalized Toeplitz operators on $H^2(\partial\Omega)$ of *higher type m*, $m = 1, 2, \dots$, are defined as $T_Q^{(m)} = \Pi_m Q \Pi_m$, where Q is a ΨDO on $\partial\Omega$ as before and Π_m is the orthogonal projection in $L^2(\partial\Omega)$ onto the subspace $H^2_{(m)}(\partial\Omega)$ of functions annihilated by the m-th symmetric power of $\overline{\partial}_b$; in other words,

$$H^2_{(m)}(\partial\Omega) = \text{closure of } \{f \in C^\infty(\partial\Omega) : \overline{R}_{j_1} \overline{R}_{j_2} \dots \overline{R}_{j_m} f = 0 \ \forall j_1, j_2, \dots, j_m\}.$$

For $m = 1$, this recovers the ordinary Szegö projector Π and the generalized Toeplitz operators discussed so far. As shown in §15.3 of [5], the projectors Π_m have almost the same microlocal description as Π, so it is conceivable that our results could also be extended to these higher type Toeplitz operators.

5.5 Weighted spaces. Our methods also work, with only minimal modifications, for $L^2_{\rm hol}(\Omega)$ replaced by the weighted Bergman spaces $L^2_{\rm hol}(\Omega, |r|^\nu) \subset L^2(\Omega, |r|^\nu)$, with any $\nu > -1$. The formulas in Theorems 9 and 11, and in Corollary 12, remain unchanged (i.e. they do not depend on ν).
Finally, it is immediate from Theorem 3, the property (P4) and the proof of Theorem 9 that the formulas in Theorem 11 and Corollary 12 also remain valid for Hankel operators on the Hardy space $H^2(\partial\Omega)$.

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