Purity Results for p-Divisible Groups and Abelian Schemes over Regular Bases of Mixed Characteristic

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Abstract. Let p be a prime. Let $(R, \mathfrak{m})$ be a regular local ring of mixed characteristic $(0, p)$ and absolute index of ramification e. We provide general criteria of when each abelian scheme over $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ extends to an abelian scheme over $\operatorname{Spec} R$. We show that such extensions always exist if $e \leq p-1$, exist in most cases if $p \leq e \leq 2p-3$, and do not exist in general if $e \geq 2p-2$. The case $e \leq p-1$ implies the uniqueness of integral canonical models of Shimura varieties over a discrete valuation ring O of mixed characteristic $(0, p)$ and index of ramification at most $p-1$. This leads to large classes of examples of Néron models over O. If $p > 2$ and index $p-1$, the examples are new.

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To Michel Raynaud, for his 71th birthday

1 Introduction

Let p be a prime number. We recall the following *global purity* notion introduced in [V1], Definitions 3.2.1 2) and 9) and studied in [V1] and [V2].

Definition 1 *Let X be a regular scheme that is faithfully flat over* $\operatorname{Spec}\mathbb{Z}_{(p)}$*. We say X is healthy regular (resp. p-healthy regular), if for each open subscheme U of X which contains $X_{\mathbb{Q}}$ and all generic points of $X_{\mathbb{F}_p}$, every abelian scheme (resp. p-divisible group) over U extends uniquely to an abelian scheme (resp. a p-divisible group) over X.*

In (the proofs of) [FC], Chapter IV, Theorems 6.4, 6.4′, and 6.8 was claimed that every regular scheme which is faithfully flat over $\operatorname{Spec}\mathbb{Z}_{(p)}$ is healthy regular as well as p-healthy regular. This claim was disproved by an example of Raynaud–Gabber (see [Ga] and [dJO], Section 6): the regular scheme $\operatorname{Spec} W(k)[[T_1,T_2]]/(p-(T_1T_2)^{p-1})$ is neither p-healthy nor healthy regular. Here $W(k)$ is the ring of Witt vectors with coefficients in a perfect field k of characteristic p. The importance of healthy and p-healthy regular schemes stems from their applications to the study of integral models of *Shimura varieties*. We have a *local* version of Definition 1 as suggested by Grothendieck's work on the classical Nagata–Zariski purity theorem (see [Gr]).

Definition 2 *Let R be a local noetherian ring with maximal ideal $\mathfrak{m}$ such that* $\operatorname{depth} R \geq 2$*. We say that R is quasi-healthy (resp. p-quasi-healthy) if each abelian scheme (resp. p-divisible group) over* $\operatorname{Spec} R\setminus\{\mathfrak{m}\}$ *extends uniquely to an abelian scheme (resp. a p-divisible group) over* $\operatorname{Spec} R$.

If R is local, complete, regular of dimension 2 and mixed characteristic $(0,p)$, then the fact that R is p-quasi-healthy can be restated in terms of finite flat commutative group schemes annihilated by p over $\operatorname{Spec} R$ (cf. Lemma 20).
Our main result is the following theorem proved in Subsections 4.4 and 5.2.

Theorem 3 *Let R be a regular local ring of dimension $d \geq 2$ and of mixed characteristic $(0,p)$. We assume that there exists a faithfully flat local R-algebra $\hat{R}$ which is complete and regular of dimension d, which has an algebraically closed residue class field k, and which is equipped with an epimorphism $\hat{R} \twoheadrightarrow W(k)[[T_1,T_2]]/(p-h)$ where $h \in (T_1,T_2)W(k)[[T_1,T_2]]$ is a power series whose reduction modulo the ideal $(p,T_1^p,T_2^p,T_1^{p-1}T_2^{p-1})$ is non-zero. Then R is quasi-healthy. If moreover $d=2$, then R is also p-quasi-healthy.*

For instance, Theorem 3 applies if the strict completion of R is isomorphic to $W(k)[[T_1,\ldots,T_d]]/(p-T_1\cdot\ldots\cdot T_m)$ with $1\leq m\leq \min\{d,2p-3\}$ (cf. Subsection 5.3). The following consequence is also proved in Subsection 5.2.

Corollary 4 *Let R be a regular local ring of dimension $d \geq 2$ and of mixed characteristic $(0,p)$. Let $\mathfrak{m}$ be the maximal ideal of R. We assume that $p \notin \mathfrak{m}^p$. Then R is quasi-healthy. If moreover $d=2$, then R is also p-quasi-healthy.*

Directly from Theorem 3 and from very definitions we get:

Corollary 5 *Let X be a regular scheme that is faithfully flat over* $\operatorname{Spec}\mathbb{Z}_{(p)}$*. We assume that each local ring R of X of mixed characteristic $(0,p)$ and dimension at least 2 is such that the hypotheses of Theorem 3 hold for it (for* *instance, this holds if X is formally smooth over the spectrum of a discrete valuation ring O of mixed characteristic $(0,p)$ and index of ramification $e \leq p-1$). Then X is healthy regular. If moreover* $\dim X = 2$*, then X is also p-healthy regular.*

The importance of Corollary 5 stems from its applications to *Néron models* (see Section 6). Theorem 31 shows the existence of large classes of new types of Néron models that were not studied before in [N], [BLR], [V1], [V2], or [V3], Proposition 4.4.1. Corollary 5 encompasses (the correct parts of) [V1], Subsubsection 3.2.17 and [V2], Theorem 1.3. Theorem 28 (i) shows that if X is formally smooth over the spectrum of a discrete valuation ring O of mixed characteristic $(0,p)$ and index of ramification e at least p, then in general X is neither p-healthy nor healthy regular. From this and Raynaud–Gabber example we get that Theorem 3 and Corollary 4 are optimal. Even more, if $R = W(k)[[T_1, T_2]]/(p-h)$ with $h \in (T_1, T_2)$, then one would be inclined to expect that R is p-quasi-healthy if and only if h does not belong to the ideal $(p, T_1^p, T_2^p, T_1^{p-1}T_2^{p-1})$; this is supported by Theorems 3 and 28 and by Lemma 12. In particular, parts (ii) and (iii) of Theorem 28 present two generalizations of the Raynaud–Gabber example.

Our proofs are based on the classification of finite flat commutative group schemes of p power order over the spectrum of a local, complete, regular ring R of mixed characteristic $(0,p)$ and perfect residue class field. For $\dim R = 1$ this classification was a conjecture of Breuil [Br] proved by Kisin in [K1] and [K2] and reproved by us in [VZ], Theorem 1. Some cases with $\dim R \geq 2$ were also treated in [VZ]. The general case is proved by Lau in [L1], Theorems 1.2 and 10.7. Proposition 15 provides a new proof of Raynaud's result [R2], Corollary 3.3.6.

Subsection 5.1 disproves an additional claim of [FC], Chapter V, Section 6. It is the claim of [FC], top of p. 184 on torsors of liftings of p-divisible groups which was not previously disproved and which unfortunately was used in [V1] and [V2], Subsection 4.3. This explains why our results on p-healthy regular schemes and p-quasi-healthy regular local rings work only for dimension 2 (the difficulty is for the passage from dimension 2 to dimension 3). Implicitly, the p-healthy part of [V2], Theorem 1.3 is proved correctly in [V2] only for dimension 2.

The paper is structured as follows. Different preliminaries on *Breuil windows* and *modules* are introduced in Section 2. In Section 3 we study morphisms between Breuil modules. Our basic results on extending properties of finite flat group schemes, p-divisible groups, and abelian schemes are presented in Sections 4 and 5. Section 6 contains applications to integral models and Néron models.

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2 Preliminaries

In this paper the notions of *frame* and *window* are in a more general sense than in [Z1]. The new notions are suggested by the works [Br], [K1], [VZ], and [L1]. In all that follows we assume that a ring is unitary and commutative and that a finite flat group scheme is commutative and (locally) of p power order.

Definition 6 *A frame $\mathcal{F} = (R, S, J, \sigma, \dot{\sigma}, \theta)$ for a ring R consists of the following data:*

(a) A ring S and an ideal $J \subset S$.

(b) An isomorphism of rings $S/J \cong R$.

(c) A ring homomorphism $\sigma : S \to S$.

(d) A σ-linear map $\dot{\sigma} : J \to S$.

(e) An element $\theta \in S$.

We assume that $pS + J$ is in the radical of S, that σ induces the Frobenius endomorphism on S/pS, and that the following equation holds:

$$\sigma(\eta) = \theta\dot{\sigma}(\eta), \quad \textit{for all } \eta \in J. \tag{1}$$

Assume that

$$S\dot{\sigma}(J) = S. \tag{2}$$

In this case we find an equation $1 = \sum_i \xi_i \dot{\sigma}(\eta_i)$ with $\xi_i \in S$ and $\eta_i \in J$. From this and (1) we get that $\theta = \sum_i \xi_i \sigma(\eta_i)$. Therefore for each $\eta \in J$ we have

$$\sigma(\eta) = \sum_i \xi_i \sigma(\eta)\dot{\sigma}(\eta_i) = \sum_i \xi_i \dot{\sigma}(\eta\eta_i) = \sum_i \xi_i \sigma(\eta_i)\dot{\sigma}(\eta) = \theta\dot{\sigma}(\eta).$$

We conclude that the equation (2) implies the existence and uniqueness of an element θ such that the equation (1) is satisfied.
If M is an S-module we set $M^{(\sigma)} := S \otimes_{\sigma,S} M$. The linearization of a σ-linear map $\phi : M \to N$ is denoted by $\phi^\sharp : M^{(\sigma)} \to N$.

Definition 7 *A window with respect to $\mathcal{F}$ is a quadruple $(P, Q, F, \dot{F})$ where:*

(a) P is a finitely generated projective S-module.

(b) $Q \subset P$ is an S-submodule.

(c) $F : P \to P$ is a σ-linear map.

(d) $\dot{F} : Q \to P$ *is a* σ*-linear map.*

We assume that the following three conditions are satisfied:

(i) *There exists a decomposition* $P = T \oplus L$ *such that* $Q = JT \oplus L$.

(ii) $F(y) = \theta\dot{F}(y)$ *for* $y \in Q$ *and* $\dot{F}(\eta x) = \dot{\sigma}(\eta)F(x)$ *for* $x \in P$ *and* $\eta \in J$.

(iii) $F(P)$ *and* $\dot{F}(Q)$ *generate* P *as an* S*-module.*

If (2) holds, then from the second part of (ii) we get that for $x \in P$ we have $F(x) = 1F(x) = \sum_i \xi_i\dot{\sigma}(\eta_i)F(x) = \sum_i \xi_i\dot{F}(\eta_i x)$; if moreover $x \in Q$, then $F(x) = \sum_i \xi_i\sigma(\eta_i)\dot{F}(x) = \theta\dot{F}(x)$. Thus if (2) holds, then we have an inclusion $F(P) \subset S\dot{F}(Q)$ and the first condition of (ii) follows from the second condition of (ii). A decomposition as in (i) is called a *normal decomposition.*
We note that for each window $(P, Q, F, \dot{F})$ as above, the S-linear map

$$F^\sharp \oplus \dot{F}^\sharp : S \otimes_{\sigma,S} T \oplus S \otimes_{\sigma,S} L \to T \oplus L \tag{3}$$

is an isomorphism. Conversely an arbitrary isomorphism (3) defines uniquely a window with respect to $\mathcal{F}$ equipped with a given normal decomposition.
Let R be a regular local ring of mixed characteristic $(0, p)$. Let $\mathfrak{m}$ be the maximal ideal of R and let $k := R/\mathfrak{m}$.
A finite flat group scheme H over $\operatorname{Spec} R$ is called *residually connected* if its special fibre H_k over $\operatorname{Spec} k$ is connected (i.e., has a trivial étale part). If R is complete, then H is residually connected if and only if H as a scheme is connected and therefore in this case we will drop the word residually. We apply the same terminology to p-divisible groups over $\operatorname{Spec} R$.
In this section we will assume moreover that k is perfect and that R is complete (in the $\mathfrak{m}$-adic topology). Let $d = \dim R \geq 1$.
We choose regular parameters $t_1, \ldots, t_d$ of R. We denote by $W(\dagger)$ the ring of Witt vectors with coefficients in a ring $\dagger$. We set $\mathfrak{S} := W(k)[[T_1, \ldots, T_d]]$. We consider the epimorphism of rings

$$\mathfrak{S} \twoheadrightarrow R,$$

which maps each indeterminate T_i to t_i.
Let $h \in \mathfrak{S}$ be a power series without constant term such that we have

$$p = h(t_1, \ldots, t_d).$$

We set $E := p - h \in \mathfrak{S}$. Then we have a canonical isomorphism

$$\mathfrak{S}/E\mathfrak{S} \cong R.$$

As R is a regular local ring of mixed characteristic $(0, p)$, we have $E \notin p\mathfrak{S}$. We extend the Frobenius automorphism σ of $W(k)$ to $\mathfrak{S}$ by the rule

$$\sigma(T_i) = T_i^p.$$

Note that $\sigma : \mathfrak{S} \to \mathfrak{S}$ is flat. Let $\dot\sigma : E\mathfrak{S} \to \mathfrak{S}$ be the σ-linear map defined by the rule

$$\dot\sigma(Es) := \sigma(s).$$

Definition 8 *We refer to the sextuple $\mathcal{L} = (R, \mathfrak{S}, E\mathfrak{S}, \sigma, \dot\sigma, \sigma(E))$ as a standard frame for the ring R. A window for this frame will be called a Breuil window.*

As $\dot\sigma(E) = 1$, the condition (2) holds for $\mathcal{L}$. Let $\hat{W}(R) \subset W(R)$ be the subring defined in [Z2], Introduction and let σ_R be its Frobenius endomorphism. There exists a natural ring homomorphism

$$\varkappa : \mathfrak{S} \to \hat{W}(R)$$

which commutes with the Frobenius endomorphisms, i.e. for all $s \in \mathfrak{S}$ we have

$$\varkappa(\sigma(s)) = \sigma_R(\varkappa(s)).$$

The element $\varkappa(T_i)$ is the Teichmüller representative $[t_i] \in \hat{W}(R)$ of t_i.

Theorem 9 *(Lau [L1], Theorem 1.1) If $p \geq 3$, then the category of Breuil windows is equivalent to the category of p-divisible groups over* $\operatorname{Spec} R$.

This theorem was proved first in some cases in [VZ], Theorem 1. As in [VZ], the proof in [L1] shows first that the category of Breuil windows is equivalent (via $\varkappa$) to the category of Dieudonné displays over $\hat{W}(R)$. Theorem 9 follows from this and the fact (see [Z2], Theorem) that the category of Dieudonné displays over $\hat{W}(R)$ is equivalent to the category of p-divisible groups over $\operatorname{Spec} R$.
There exists a version of Theorem 9 for connected p-divisible groups over $\operatorname{Spec} R$ as described in the introductions of [VZ] and [L1]. This version holds as well for $p = 2$, cf. [L1], Corollary 10.6.
The categories of Theorem 9 have natural exact structures [Me2]. The equivalence of Theorem 9 (or of its version for $p \geq 2$) respects the exact structures.
For a prime ideal $\mathfrak{p}$ of R which contains p, let $\kappa(\mathfrak{p})^{\mathrm{perf}}$ be the perfect hull of the residue class field $\kappa(\mathfrak{p})$ of $\mathfrak{p}$. We deduce from $\varkappa$ a ring homomorphism $\varkappa_{\mathfrak{p}} : \mathfrak{S} \to W(\kappa(\mathfrak{p})^{\mathrm{perf}}) = \hat{W}(\kappa(\mathfrak{p})^{\mathrm{perf}})$.

Proposition 10 *Let G be a p-divisible group over R. If $p = 2$, we assume that G is connected. Let $(P, Q, F, \dot F)$ be the Breuil window of G. Then the classical Dieudonné module of $G_{\kappa(\mathfrak{p})^{\mathrm{perf}}}$ is canonically isomorphic (in a functorial way) to $W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\varkappa_{\mathfrak{p}}, \mathfrak{S}} P$ endowed with the $\sigma_{\kappa(\mathfrak{p})^{\mathrm{perf}}}$-linear map $\sigma_{\kappa(\mathfrak{p})^{\mathrm{perf}}} \otimes F$.*

Proof. If G is connected, we consider its display $(P', Q', F', \dot F')$ over $W(R)$. By [L1], Theorem 1.1 and Corollary 10.6 we have $P' = W(R) \otimes_{\varkappa, \mathfrak{S}} P$. Let us denote by $\mathbb{D}(G)$ the Grothendieck–Messing crystal associated to G, cf. [Me1]. By [Z1], Theorem 6 there is a canonical and functorial isomorphism $\mathbb{D}(G)_{W(R)} \cong P' \cong W(R) \otimes_{\varkappa, \mathfrak{S}} P$. The functor $\mathbb{D}$ commutes with base change. If we apply this

to the morphism of pd-extensions $W(R) \to W(\kappa(\mathfrak{p})^{\mathrm{perf}})$ of R and $\kappa(\mathfrak{p})^{\mathrm{perf}}$ (respectively), we get that $\mathbb{D}(G_{\kappa(\mathfrak{p})^{\mathrm{perf}}}) \cong \mathbb{D}(G)_{W(\kappa(\mathfrak{p})^{\mathrm{perf}})} \cong W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\varkappa_{\mathfrak{p}},\mathfrak{S}} P$. From this the proposition follows provided G is connected.
In the case $p \geq 3$ the same argument works if we replace $(P', Q', F', \dot{F}')$ by the Dieudonné display of G over $\hat{W}(R)$. But in this case the isomorphism $\mathbb{D}(G)_{\hat{W}(R)} \cong P'$ follows from [L2], Theorem 6.9. $\square$

All results in this paper are actually independent of the nonconnected part of the last Proposition. This is explained in the proof of Corollary 21.
In what follows we will not need to keep track of the $\sigma_{\kappa(\mathfrak{p})^{\mathrm{perf}}}$-linear maps $\sigma_{\kappa(\mathfrak{p})^{\mathrm{perf}}} \otimes F$ and thus we will simply call $W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\varkappa_{\mathfrak{p}},\mathfrak{S}} P$ the fibre of the Breuil window $(P, Q, F, \dot{F})$ over $\mathfrak{p}$.
We often write a Breuil window in the form (Q, ϕ) originally proposed by Breuil, where ϕ is the composite of the inclusion $Q \subset P$ with the inverse of the $\mathfrak{S}$-linear isomorphism $\dot{F}^{\sharp} : Q^{(\sigma)} \cong P$. In this notation P, F, and $\dot{F}$ are omitted as they are determined naturally by $\dot{F}^{\sharp}$ and thus by ϕ (see [VZ], Section 2). A Breuil window in this form is characterized as follows: Q is a finitely generated free $\mathfrak{S}$-module and $\phi : Q \to Q^{(\sigma)}$ is a $\mathfrak{S}$-linear map whose cokernel is annihilated by E. We note that this implies easily that there exists a unique $\mathfrak{S}$-linear map $\psi : Q^{(\sigma)} \to Q$ such that we have

$$\phi \circ \psi = E \,\mathrm{id}_{Q^{(\sigma)}}, \quad \psi \circ \phi = E \,\mathrm{id}_{Q}\,.$$

Clearly the datum (Q, ψ) is equivalent to the datum (Q, ϕ). In the notation (Q, ϕ), its fibre over $\mathfrak{p}$ is $W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\varkappa_{\mathfrak{p}},\mathfrak{S}} Q^{(\sigma)} = W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\sigma\varkappa_{\mathfrak{p}},\mathfrak{S}} Q$.
The dual of a Breuil window is defined as follows. Let M be a $\mathfrak{S}$-module. We set $\hat{M} := \mathrm{Hom}_{\mathfrak{S}}(M, \mathfrak{S})$. A $\mathfrak{S}$-linear map $M \to \mathfrak{S}$ defines a homomorphism $M^{(\sigma)} \to \mathfrak{S}^{(\sigma)} = \mathfrak{S}$. This defines a $\mathfrak{S}$–linear map:

$$\hat{M}^{(\sigma)} = \mathfrak{S} \otimes_{\sigma,\mathfrak{S}} \mathrm{Hom}_{\mathfrak{S}}(M, \mathfrak{S}) \to \mathrm{Hom}_{\mathfrak{S}}(M^{(\sigma)}, \mathfrak{S}) = \widehat{M^{(\sigma)}}.$$

It is clearly an isomorphism if M is a free $\mathfrak{S}$-module of finite rank and therefore also if M is a finitely generated $\mathfrak{S}$-module by a formal argument which uses the flatness of $\sigma : \mathfrak{S} \to \mathfrak{S}$.
If (Q, ϕ) is a Breuil window we obtain a $\mathfrak{S}$-linear map

$$\hat{\phi} : \hat{Q}^{(\sigma)} = \widehat{(Q^{(\sigma)})} \to \hat{Q}.$$

More symmetrically we can say that if (Q, ϕ, ψ) is a Breuil window then $(\hat{Q}, \hat{\psi}, \hat{\phi})$ is a Breuil window. We call $(\hat{Q}, \hat{\psi}, \hat{\phi})$ the *dual Breuil window* of (Q, ϕ, ψ).
Taking the fibre of a Breuil window (Q, ϕ) over $\mathfrak{p}$ is compatible with duals as we have:

$$W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\varkappa_{\mathfrak{p}},\mathfrak{S}} \hat{Q}^{(\sigma)} \cong \mathrm{Hom}_{W(\kappa(\mathfrak{p})^{\mathrm{perf}})}(W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\varkappa_{\mathfrak{p}},\mathfrak{S}} Q^{(\sigma)}, W(\kappa(\mathfrak{p})^{\mathrm{perf}})). \tag{4}$$

We recall from [VZ], Definition 2 and [L1] that a *Breuil module* (M,φ) is a pair, where M is a finitely generated $\mathfrak{S}$-module which is of projective dimension at most 1 and which is annihilated by a power of p and where $\varphi : M \to M^{(\sigma)}$ is a $\mathfrak{S}$-linear map whose cokernel is annihilated by E. We note that the map φ is always injective (the argument for this is the same as in [VZ], Proposition 2 (i)). It follows formally that there exists a unique $\mathfrak{S}$-linear map $\vartheta : M^{(\sigma)} \to M$ such that we have

$$\varphi \circ \vartheta = E\,\mathrm{id}_{M^{(\sigma)}}, \quad \vartheta \circ \varphi = E\,\mathrm{id}_M\,.$$

We define the *dual Breuil module* $(M^*, \vartheta^*, \varphi^*)$ of (M,φ,ϑ) by applying the functor $M^* = \mathrm{Ext}^1_{\mathfrak{S}}(M,\mathfrak{S})$ in the same manner we did for windows. It is easy to see that the $\mathfrak{S}$-module M^* has projective dimension at most 1. The fibre of (M,φ) (or of (M,φ,ϑ)) over $\mathfrak{p}$ is

$$W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\varkappa_{\mathfrak{p}},\mathfrak{S}} M^{(\sigma)} = W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\sigma\varkappa_{\mathfrak{p}},\mathfrak{S}} M. \tag{5}$$

The duals are again compatible with taking fibres as we have:

$$\begin{aligned} &W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\sigma\varkappa_{\mathfrak{p}},\mathfrak{S}} M^* \cong \\ &\cong \mathrm{Ext}^1_{W(\kappa(\mathfrak{p})^{\mathrm{perf}})}(W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\sigma\varkappa_{\mathfrak{p}},\mathfrak{S}} M, W(\kappa(\mathfrak{p})^{\mathrm{perf}})) \\ &= \mathrm{Hom}_{W(\kappa(\mathfrak{p})^{\mathrm{perf}})}(W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\sigma\varkappa_{\mathfrak{p}},\mathfrak{S}} M, W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\mathbb{Z}} \mathbb{Q}). \end{aligned} \tag{6}$$

Assume that p annihilates M, i.e. M is a module over $\bar{\mathfrak{S}} = \mathfrak{S}/p\mathfrak{S}$. As depth M is the same over either $\mathfrak{S}$ or $\bar{\mathfrak{S}}$ and it is d if $M \neq 0$, we easily get that M is a free $\bar{\mathfrak{S}}$-module. From this we get that $M^* = \mathrm{Hom}_{\bar{\mathfrak{S}}}(M, \bar{\mathfrak{S}})$ (to be compared with the last isomorphism of (6)). Thus in this case the duality works exactly as for windows.

If $p \geq 3$, it follows from Theorem 9 that the category of finite flat group schemes over $\operatorname{Spec} R$ is equivalent to the category of Breuil modules (the argument for this is the same as for [VZ], Theorem 2). We have a variant of this for $p = 2$ (cf. [L1], Theorem 10.7): the category of connected finite flat group schemes over $\operatorname{Spec} R$ is equivalent to the category of nilpotent Breuil modules (i.e., of Breuil modules (M,φ) that have the property that the reduction of φ modulo the maximal ideal of $\mathfrak{S}$ is nilpotent in the natural way).

We recall from [VZ], Subsection 6.1 that the Breuil module of a finite flat group scheme H over $\operatorname{Spec} R$ is obtained as follows. By a theorem of Raynaud (see [BBM], Theorem 3.1.1) we can represent H as the kernel

$$0 \to H \to G_1 \to G_2 \to 0$$

of an isogeny $G_1 \to G_2$ of p-divisible groups over $\operatorname{Spec} R$. If $p = 2$, then we assume that H and G_1 are connected. Let (Q_1, ϕ_1) and (Q_2, ϕ_2) be the Breuil windows of G_1 and G_2 (respectively). Then the Breuil module (M,φ) of H is the cokernel of the induced map $(Q_1,\phi_1) \to (Q_2,\phi_2)$ in a natural sense. From Proposition 10 we get that the classical covariant Dieudonné module of $H_{\kappa(\mathfrak{p})^{\mathrm{perf}}}$ is canonically given by (5).

3 Morphisms between Breuil modules

In this section, let R be a complete regular local ring of mixed characteristic $(0, p)$ with maximal ideal $\mathfrak{m}$ and perfect residue class field k. We write $R = \mathfrak{S}/E\mathfrak{S}$, where $d = \dim R \geq 1$, $\mathfrak{S} = W(k)[[T_1, \ldots, T_d]]$, and $E = p - h \in \mathfrak{S}$ are as in Section 2. We use the standard frame $\mathcal{L}$ of the Definition 8.
Let $e \in \mathbb{N}^*$ be such that $p \in \mathfrak{m}^e \setminus \mathfrak{m}^{e+1}$. It is the absolute ramification index of R. Let $\bar{\mathfrak{t}} := (T_1, \ldots, T_d) \subset \bar{\mathfrak{S}} := \mathfrak{S}/p\mathfrak{S} = k[[T_1, \ldots, T_d]]$. Let $\bar{h} \in \bar{\mathfrak{t}} \subset \bar{\mathfrak{S}}$ be the reduction modulo p of h. The surjective function $\mathrm{ord} : \bar{\mathfrak{S}} \twoheadrightarrow \mathbb{N} \cup \{\infty\}$ is such that $\mathrm{ord}(\bar{\mathfrak{t}}^i \setminus \bar{\mathfrak{t}}^{i+1}) = i$ for all $i \geq 0$ and $\mathrm{ord}(0) = \infty$.
Let $\mathrm{gr}\, R := \mathrm{gr}_{\mathfrak{m}} R$. The obvious isomorphism of graded rings

$$k[T_1, \ldots, T_d] \to \mathrm{gr}\, R \tag{7}$$

maps the initial form of $\bar{h}$ to the initial form of p. Thus e is the order of the power series $\bar{h}$.

Lemma 11 *We assume that R is such that $p \notin \mathfrak{m}^p$ (i.e., $e \leq p-1$). Let C be a $\mathfrak{S}$-module which is annihilated by a power of p. Let $\varphi : C \to C^{(\sigma)}$ be a $\mathfrak{S}$-linear map whose cokernel is annihilated by E. We assume that there exists a power series $f \in \mathfrak{S} \setminus p\mathfrak{S}$ which annihilates C. Then we have:*

(a) If $p \notin \mathfrak{m}^{p-1}$ (i.e., if $e \leq p-2$), then $C = 0$.

(b) If $e = p-1$, then either $C = 0$ or the initial form of p in $\mathrm{gr}\, R$ generates an ideal which is a $(p-1)$-th power.

Proof. It suffices to show that $C = 0$ provided either $e \leq p-2$ or $e = p-1$ and the initial form of p in $\mathrm{gr}\, R$ generates an ideal which is not a $(p-1)$-th power. By the lemma of Nakayama it suffices to show that $C/pC = 0$. It is clear that φ induces a $\mathfrak{S}$-linear map $C/pC \to (C/pC)^{(\sigma)}$. Therefore we can assume that C is annihilated by p.
Let u be the smallest non-negative integer with the following property: for each $c \in C$ there exists a power series $g_c \in \bar{\mathfrak{S}}$ such that $\mathrm{ord}(g_c) \leq u$ and g_c annihilates c.
From the existence of f in the annihilator of C we deduce that the number u exists. If $C \neq 0$, then we have $u > 0$. We will show that the assumption that $u > 0$ leads to a contradiction and therefore we have $C = 0$.
By the minimality of u there exists an element $x \in C$ such that for each power series a in the annihilator $\mathfrak{a} \subset \bar{\mathfrak{S}}$ of x we have $\mathrm{ord}(a) \geq u$. Consider the $\bar{\mathfrak{S}}$-linear injection

$$\bar{\mathfrak{S}}/\mathfrak{a} \hookrightarrow C \tag{8}$$

which maps 1 to x. Let $\mathfrak{a}^{(p)} \subset \bar{\mathfrak{S}}$ be the ideal generated by the p-th powers of elements in $\mathfrak{a}$. Each power series in $\mathfrak{a}^{(p)}$ has order $\geq pu$.
If we tensorize the injection (8) by $\sigma : \bar{\mathfrak{S}} \to \bar{\mathfrak{S}}$ we obtain a $\bar{\mathfrak{S}}$-linear injection

$$\bar{\mathfrak{S}}/\mathfrak{a}^{(p)} \cong \bar{\mathfrak{S}} \otimes_{\sigma, \bar{\mathfrak{S}}} \bar{\mathfrak{S}}/\mathfrak{a} \hookrightarrow C^{(\sigma)}.$$

Thus each power series in the annihilator of $1 \otimes x \in C^{(\sigma)}$ has at least order pu. On the other hand the cokernel of φ is by assumption annihilated by $\bar{h}$. Thus $\bar{h}(1 \otimes x)$ is in the image of φ. By the definition of u we find a power series $g \in \bar{\mathfrak{S}}$ with $\mathrm{ord}(g) \leq u$ which annihilates $\bar{h}(1 \otimes x)$. Thus $g\bar{h} \in \mathfrak{a}^{(p)}$. We get

$$u + \mathrm{ord}(\bar{h}) \geq \mathrm{ord}(g\bar{h}) \geq pu.$$

Therefore $e = \mathrm{ord}(\bar{h}) \geq (p-1)u \geq p-1$. In the case (a) we obtain a contradiction which shows that $C = 0$.
In the case $e = p-1$ we obtain a contradiction if $u > 1$. Assume that $u = 1$. As $g\bar{h} \in \mathfrak{a}^{(p)}$ and as $\mathrm{ord}(g\bar{h}) = p$, there exists a power series $\bar{f} \in \bar{\mathfrak{S}}$ of order 1 and a non-zero element $\xi \in k$ such that

$$g\bar{h} \equiv \xi \bar{f}^p \bmod \bar{\mathfrak{r}}^{p+1}.$$

This shows that the initial forms of g and $\bar{f}$ in the graded ring $\mathrm{gr}_{\bar{\mathfrak{r}}}\, \bar{\mathfrak{S}}$ differ by a constant in k. If we divide the last congruence by $\bar{f}$ we obtain

$$\bar{h} \equiv \xi \bar{f}^{p-1} \bmod \bar{\mathfrak{r}}^p.$$

By the isomorphism (7) this implies that initial form of p in $\mathrm{gr}\, R$ generates an ideal which is a $(p-1)$-th power. Contradiction. □

Lemma 12 *We assume that $d = 2$ and we consider the ring $\bar{\mathfrak{S}} = k[[T_1, T_2]]$. Let $\bar{h} \in \bar{\mathfrak{S}}$ be a power series of order $e \in \mathbb{N}^*$. Then the following two statements are equivalent:*

- *(a) The power series $\bar{h}$ does not belong to the ideal $\bar{\mathfrak{r}}^{(p)} + \bar{\mathfrak{r}}^{2(p-1)} = (T_1^p, T_2^p, T_1^{p-1}T_2^{p-1})$ of $\bar{\mathfrak{S}}$.*
- *(b) If C is a $\bar{\mathfrak{S}}$-module of finite length equipped with a $\bar{\mathfrak{S}}$-linear map $\varphi : C \to C^{(\sigma)}$ whose cokernel is annihilated by $\bar{h}$, then C is zero.*

Proof. We consider the $\bar{\mathfrak{S}}$-linear map

$$\tau : k = k[[T_1, T_2]]/(T_1, T_2) \to k^{(\sigma)} = k[[T_1, T_2]]/(T_1^p, T_2^p)$$

that maps 1 to $T_1^{p-1}T_2^{p-1}$ modulo (T_1^p, T_2^p). If $\bar{h} \in \bar{\mathfrak{r}}^{(p)} + \bar{\mathfrak{r}}^{2(p-1)}$, then the cokernel of τ is annihilated by $\bar{h}$. From this we get that (b) implies (a).
Before proving the other implication, we first make some general remarks. The Frobenius endomorphism $\sigma : \bar{\mathfrak{S}} \to \bar{\mathfrak{S}}$ is faithfully flat. If $\triangle$ is an ideal of $\bar{\mathfrak{S}}$, then with the same notations as before we have $\sigma(\triangle)\bar{\mathfrak{S}} = \triangle^{(p)}$.
Let C be a $\bar{\mathfrak{S}}$-module of finite type. Let $\mathfrak{a} \subset \bar{\mathfrak{S}}$ be the annihilator of C. Then $\mathfrak{a}^{(p)}$ is the annihilator of $C^{(\sigma)} = \bar{\mathfrak{S}} \otimes_{\sigma, \bar{\mathfrak{S}}} C$. This is clear for a monogenic module C. If we have more generators for C, then we can use the formula

$$\mathfrak{a}^{(p)} \cap \mathfrak{b}^{(p)} = (\mathfrak{a} \cap \mathfrak{b})^{(p)}$$

which holds for flat ring extensions in general.

We say that a power series $f \in \bar{\mathfrak{S}}$ of order u is normalized with respect to (T_1, T_2) if it contains T_1^u with a non-zero coefficient. This definition makes sense with respect to any regular system of parameters $\tilde{T}_1, \tilde{T}_2$ of $\bar{\mathfrak{S}}$.
We now assume that (a) holds and we show that (b) holds. Thus the $\bar{\mathfrak{S}}$-module C has finite length and the cokernel of $\varphi : C \to C^{(\sigma)}$ is annihilated by $\bar{h}$. Let k' be an infinite perfect field that contains k. Let $\bar{\mathfrak{S}}' := k'[[T_1, T_2]]$. To show that $C = 0$, it suffices to show that $\bar{\mathfrak{S}}' \otimes_{\bar{\mathfrak{S}}} C = 0$. Thus by replacing the role of k by the one of k', we can assume that k is infinite. This assumption implies that for almost all $\lambda \in k$, the power series $\bar{h}$ is normalized with respect to $(T_1, T_2 + \lambda T_1)$. By changing the regular system of parameters (T_1, T_2) in $\bar{\mathfrak{S}}$, we can assume that $\bar{h}$ is normalized with respect to (T_1, T_2). By the Weierstraß preparation theorem we can assume that $\bar{h}$ is a Weierstraß polynomial ([Bou], Chapter 7, Section 3, number 8). Thus we can write

$$\bar{h} = T_1^e + a_{e-1}(T_2)T_1^{e-1} + \ldots + a_1(T_2)T_1 + a_0(T_2),$$

where $a_0(T_2), \ldots, a_{e-1}(T_2) \in T_2 k[[T_2]]$.
We note that the assumption (a) implies that $e \leq 2p - 3$.
Let u be the minimal non-negative integer such that there exists a power series of the form $T_1^u + g \in \bar{\mathfrak{S}}$, with $g \in T_2\bar{\mathfrak{S}}$, for which we have $(T_1^u + g)C = 0$. By our assumptions, such a non-negative integer u exists.
We will show that the assumption that $C \neq 0$ leads to a contradiction. This assumption implies that $u \geq 1$. The annihilator $\mathfrak{a}$ of the module C has a set of generators of the following form:

$$T_1^{u_i} + g_i, (\text{with } i = 1, \ldots, l), \quad g_i \ (\text{with } i = l+1, \ldots, m),$$

where $u_i \geq u$ for $i = 1, \ldots, l$ and $g_i \in T_2\bar{\mathfrak{S}}$ for $i = 1, \ldots, m$.
As the cokernel of φ is annihilated by $\bar{h}$ we find that $(T_1^u + g)\bar{h}C^{(\sigma)} = 0$. As the annihilator of $C^{(\sigma)}$ is generated by the elements

$$T_1^{pu_i} + g_i^p \ (\text{with } i = 1, \ldots, l), \quad g_i^p \ (\text{with } i = l+1, \ldots, m), \tag{9}$$

we obtain the congruence

$$(T_1^u + g)\bar{h} \equiv 0 \mod (T_1^{up}, T_2^p). \tag{10}$$

We consider this congruence modulo T_2. We have $g \equiv 0 \mod (T_2)$ and $\bar{h} \equiv T_1^e \mod (T_2)$ because $\bar{h}$ is a Weierstraß polynomial. This proves that

$$T_1^u T_1^e \equiv 0 \mod (T_1^{up}).$$

But this implies that $u + e \geq up$. If $u \geq 2$, then $e \geq up - u \geq 2p - 2$ and this contradicts the inequality $e \leq 2p - 3$. Therefore we can assume that $u = 1$.
By replacing (T_1, T_2) with $(T_1 + g, T_2)$, without loss of generality we can assume that $T_1 C = 0$ and (cf. (10) and the equality $u = 1$) that

$$T_1\bar{h} \equiv 0 \mod \bar{\mathfrak{r}}^{(p)}.$$

This implies that up to a unit in $\bar{\mathfrak{S}}$ we can assume that $\bar{h}$ is of the form:

$$\bar{h} \equiv T_2^s T_1^{p-1} + \sum_{i=0}^{\infty} T_2^{i+p} \delta_i(T_1) \mod (T_1^p),$$

where $0 \leq s \leq p-2$ and where each $\delta_i \in k[T_1]$ has degree at most $p-2$.
Let now v be the smallest natural number such that $T_2^v C = 0$. Then the annihilator $\mathfrak{a}$ of C is generated by T_1, T_2^v and the annihilator $\mathfrak{a}^{(p)}$ of $C^{(\sigma)}$ is generated by T_1^p, T_2^{pv}. As the cokernel of φ is annihilated by $\bar{h}$, we find that $T_2^v \bar{h} C^{(\sigma)} = 0$. Thus we obtain the congruence

$$T_2^v (T_2^s T_1^{p-1} + \sum_{i=0}^{\infty} T_2^{i+p} \delta_i(T_1)) \equiv 0 \mod (T_1^p, T_2^{pv}).$$

But this implies $v + s \geq pv$. Thus $s \geq (p-1)v \geq p-1$ and (as $0 \leq s \leq p-2$) we reached a contradiction. Therefore $C = 0$ and thus (a) implies (b). □

Proposition 13 *We assume that $p \notin \mathfrak{m}^p$ (i.e., $e \leq p-1$). If $p \in \mathfrak{m}^{p-1}$, then we also assume that the ideal generated by the initial form of p in $\operatorname{gr} R$ is not a $(p-1)$-th power (thus $p > 2$).*
We consider a morphism $\alpha : (M_1, \varphi_1) \to (M_2, \varphi_2)$ of Breuil modules for the standard frame $\mathcal{L}$. Let $\mathfrak{p}$ be a prime ideal of R which contains p. We consider the $W(\kappa(\mathfrak{p})^{\mathrm{perf}})$-linear map obtained from α by base change

$$W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\sigma \varkappa_{\mathfrak{p}}, \mathfrak{S}} M_1 \to W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\sigma \varkappa_{\mathfrak{p}}, \mathfrak{S}} M_2. \tag{11}$$

Then the following two properties hold:

(a) If (11) is surjective, then the $\mathfrak{S}$-linear map $M_1 \to M_2$ is surjective.
(b) If (11) is injective, then the $\mathfrak{S}$-linear map $M_1 \to M_2$ is injective and its cokernel is a $\mathfrak{S}$-module of projective dimension at most 1.

Proof. We prove (a). We denote by $\tilde{\mathfrak{p}}$ the ideal of $\mathfrak{S}$ which corresponds to $\mathfrak{p}$ via the isomorphism $\mathfrak{S}/(E,p) \cong R/(p)$. It follows from the lemma of Nakayama that $(M_1)_{\tilde{\mathfrak{p}}} \to (M_2)_{\tilde{\mathfrak{p}}}$ is a surjection. Let us denote by (C, φ) the cokernel of α. We conclude that C is annihilated by an element $f \notin \tilde{\mathfrak{p}} \supset p\mathfrak{S}$. Therefore we conclude from Lemma 11 that $C = 0$. Thus (a) holds.
Part (b) follows from (a) by duality as it is compatible with taking fibres (6). Indeed for the last assertion it is enough to note that the kernel of the surjection $M_2^* \twoheadrightarrow M_1^*$ has clearly projective dimension at most 1. □

Remark. If $e = p-1$ and the ideal generated by the initial form of p in $\operatorname{gr} R$ is a $(p-1)$-th power, then the Proposition 13 is not true in general. This is so as there exist non-trivial homomorphisms $(\mathbb{Z}/p\mathbb{Z})_R \to \boldsymbol{\mu}_{p,R}$ for suitable such R's.

Proposition 14 *We assume that R has dimension $d = 2$ and that $h \in \mathfrak{S}$ does not belong to the ideal $(p, T_1^p, T_2^p, T_1^{p-1} T_2^{p-1})$ of $\mathfrak{S}$. Let $\alpha : (M_1, \varphi_1) \to (M_2, \varphi_2)$*

be a morphism of Breuil modules for the standard frame $\mathcal{L}$. We assume that for each prime ideal $\mathfrak{p}$ of R with $p \in \mathfrak{p} \neq \mathfrak{m}$, the $W(\kappa(\mathfrak{p})^{\mathrm{perf}})$-linear map obtained from α by base change

$$W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\sigma\varkappa_{\mathfrak{p}},\mathfrak{S}} M_1 \to W(\kappa(\mathfrak{p})^{\mathrm{perf}}) \otimes_{\sigma\varkappa_{\mathfrak{p}},\mathfrak{S}} M_2 \tag{12}$$

is surjective (resp. is injective).
Then the $\mathfrak{S}$-linear map $M_1 \to M_2$ is surjective (resp. is injective and its cokernel is a $\mathfrak{S}$-module of projective dimension at most 1).

PROOF. As in the last proof we only have to treat the case where the maps (12) are surjective. We consider the cokernel (C, φ) of α. As in the last proof we will argue that $C = 0$ but with the role of Lemma 11 being replaced by Lemma 12. It suffices to show that $\bar{C} := C/pC$ is zero. Let $\bar{\varphi} : \bar{C} \to \bar{C}^{(\sigma)}$ be the map induced naturally by φ.
Lemma 12 is applicable if we verify that $\bar{C}$ is a $\bar{\mathfrak{S}}$-module of finite length.
We denote by $\bar{\mathfrak{p}}$ the ideal of $\bar{\mathfrak{S}} = \mathfrak{S}/p\mathfrak{S}$ which corresponds to $\mathfrak{p}$ via the isomorphism $\bar{\mathfrak{S}}/(\bar{h}) \cong R/pR$. It follows from (12) by the lemma of Nakayama that for each prime ideal $\bar{\mathfrak{p}} \supset \bar{h}\bar{\mathfrak{S}}$ different from the maximal ideal of $\bar{\mathfrak{S}}$ the maps $\alpha_{\mathfrak{p}} : (M_1)_{\bar{\mathfrak{p}}} \to (M_2)_{\bar{\mathfrak{p}}}$ are surjective. Using this one constructs inductively a regular sequence $f_1, \ldots, f_{d-1}, \bar{h}$ in the ring $\bar{\mathfrak{S}}$ such that the elements $f_1, \ldots, f_{d-1}$ annihilate $\bar{C}$. As $\bar{C}$ is a finitely generated module over the 1-dimensional local ring $\bar{\mathfrak{S}}/(f_1, \ldots, f_{d-1})$, we get that $\bar{C}[1/\bar{h}]$ is a module of finite length over the regular ring $A = \bar{\mathfrak{S}}[1/\bar{h}]$. If we can show that $\bar{C}[1/\bar{h}] = 0$, then it follows that $\bar{C}$ is of finite length.
As we are in characteristic p, the Frobenius σ acts on the principal ideal domain $A = \bar{\mathfrak{S}}[1/\bar{h}]$. By the definition of a Breuil module the maps $\varphi_i[1/\bar{h}]$ for $i = 1, 2$ become isomorphisms. Therefore φ gives birth to an isomorphism:

$$\bar{\varphi}[1/\bar{h}] : \bar{C}[1/\bar{h}] \to (\bar{C}[1/\bar{h}])^{(\sigma)}. \tag{13}$$

As A is regular of dimension $d-1$, for each A-module $\ddagger$ of finite length we have

$$\text{length}\, \ddagger^{(\sigma)} = p^{d-1}\, \text{length}\, \ddagger.$$

We see that the isomorphism (13) is only possible if $\bar{C}[1/\bar{h}] = 0$. Thus $\bar{C}$ has finite length and therefore from Lemma 12 we get that $\bar{C} = 0$. This implies that $C = 0$. $\square$

4 EXTENDING EPIMORPHISMS AND MONOMORPHISMS

In this section let R be a regular local ring of mixed characteristic $(0, p)$ with maximal ideal $\mathfrak{m}$ and residue class field k. Let K be the field of fractions of R.

4.1 COMPLEMENTS ON RAYNAUD'S WORK

We first reprove Raynaud's result [R2], Corollary 3.3.6 by the methods of the previous sections. We state it in a slightly different form.

Proposition 15 *We assume that $p \notin \mathfrak{m}^{p-1}$ (thus $p > 2$). Let H_1 and H_2 be finite flat group schemes over* $\operatorname{Spec} R$. *Let $\beta : H_1 \to H_2$ be a homomorphism, which induces an epimorphism (resp. monomorphism) $H_{1,K} \to H_{2,K}$ between generic fibres. Then β is an epimorphism (resp. monomorphism).*

Proof. We prove only the statement about epimorphisms because the case of a monomorphism follows by Cartier duality. It is enough to show that the homomorphism $H_1 \to H_2$ is flat. By the fibre criterion of flatness it is enough to show that the homomorphism $\beta_k : H_{1,k} \to H_{2,k}$ between special fibres is an epimorphism. To see this we can assume that R is a complete local ring with algebraically closed residue class field k.
We write $R \cong \mathfrak{S}/(p - h)$, with $\mathfrak{S} = W(k)[[T_1, \ldots, T_d]]$. Then the reduction $\bar{h} \in k[[T_1, \ldots, T_d]]$ of h modulo p is a power series of order $e < p - 1$. By Noether normalization theorem we can assume that $\bar{h}$ contains the monom T_1^e. By replacing R with $R/(T_2, \ldots, T_d)$, we can assume that R is one-dimensional. We consider the morphism

$$(M_1, \varphi_1) \to (M_2, \varphi_2)$$

of Breuil modules associated to β. Let (C, φ) be its cokernel. Let $\bar{C} := C/C_0$, where C_0 is the $\mathfrak{S}$-submodule of C whose elements are annihilated by a power of the maximal ideal $\mathfrak{r}$ of $\mathfrak{S}$. The $\mathfrak{S}$-linear map φ factors as

$$\bar{\varphi} : \bar{C} \to \mathfrak{S} \otimes_{\sigma, \mathfrak{S}} \bar{C}$$

and the cokernel of $\bar{\varphi}$ is annihilated by $\bar{h}$. The maximal ideal $\mathfrak{r}$ of $\mathfrak{S}$ is not associated to $\mathfrak{S} \otimes_{\sigma, \mathfrak{S}} \bar{C}$. Thus either $\bar{C} = 0$ or $\operatorname{depth} \bar{C} \geq 1$. Therefore the $\mathfrak{S}$-module $\bar{C}$ is of projective dimension at most 1. As $\bar{C}$ is annihilated by a power of p, we conclude that $(\bar{C}, \bar{\varphi})$ is the Breuil module of a finite flat group scheme D over $\operatorname{Spec} R$. We have induced homomorphisms

$$H_1 \to H_2 \to D.$$

The composition of them is zero and the second homomorphism is an epimorphism because it is so after base change to k. As $H_{1,K} \twoheadrightarrow H_{2,K}$ is an epimorphism we conclude that $D_K = 0$. But then $D = 0$ and the Breuil module $(\bar{C}, \bar{\varphi})$ is zero as well. We conclude by Lemma 11 that $C = 0$. □

The next proposition is proved in [R2], Remark 3.3.5 in the case of biconnected finite flat group schemes H and D.

Proposition 16 *Let R be a discrete valuation ring of mixed characteristic $(0, p)$ and index of ramification $p - 1$; we have $K = R[1/p]$. Let $\beta : H \to D$ be a homomorphism of residually connected finite flat group schemes over* $\operatorname{Spec} R$ *which induces an isomorphism (resp. epimorphism) over* $\operatorname{Spec} K$.
Then β is an isomorphism (resp. epimorphism).

Proof. It is enough to show the statement about isomorphisms. Indeed, assume that β_K is an epimorphism. Consider the schematic closure H_1 of the kernel of β_K in H. Then $H/H_1 \to D$ is an isomorphism.
Therefore we can assume that β_K is an isomorphism. By extending R we can assume that R is complete and that k is algebraically closed. By considering the Cartier dual homomorphism $\beta^{\mathrm{t}} : D^{\mathrm{t}} \to H^{\mathrm{t}}$ one easily reduces the problem to the case when H is biconnected. As the case when D is also biconnected is known (cf. [R2], Remark 3.3.5), one can easily reduce to the case when D is of multiplicative type. Then D contains $\boldsymbol{\mu}_{p,R}$ as a closed subgroup scheme. We consider the schematic closure H_1 of $\boldsymbol{\mu}_{p,K}$ in H. Using an induction on the order of H it is enough to show that the natural homomorphism $\beta_1 : H_1 \to \boldsymbol{\mu}_{p,R}$ is an isomorphism. This follows from [R2], Proposition 3.3.2 3°. For the sake of completeness we reprove this in the spirit of the paper.
We write $R = \mathfrak{S}/(E)$, where $E \in \mathfrak{S} = W(k)[[T]]$ is an Eisenstein polynomial of degree $e = p-1$. The Breuil window of the p-divisible group $\boldsymbol{\mu}_{p^\infty,R}$ is given by

$$\begin{array}{ccc} \mathfrak{S} & \to & \mathfrak{S}^{(\sigma)} \cong \mathfrak{S} \\ f & \mapsto & Ef. \end{array}$$

The Breuil module (N,τ) of $\boldsymbol{\mu}_{p,R}$ is the kernel of the multiplication by p : $\boldsymbol{\mu}_{p^\infty,R} \to \boldsymbol{\mu}_{p^\infty,R}$ and therefore it can be identified with

$$\begin{array}{ccc} N = k[[T]] & \to & N^{(\sigma)} = k[[T]]^{(\sigma)} \cong k[[T]] \\ f & \mapsto & T^e f. \end{array}$$

Let (M,φ) be the Breuil module of H_1. As H_1 is of height 1 we can identify $M = k[[T]]$. Then $\varphi : M \to M^{(\sigma)} \cong M$ is the multiplication by a power series $g \in k[[T]]$ of order $\mathrm{ord}(g) \le e = p-1$. To the homomorphism $\beta_1 : H_1 \to \boldsymbol{\mu}_{p,R}$ corresponds a morphism $\alpha_1 : (M,\varphi) \to (N,\tau)$ that maps $1 \in M$ to some element $a \in N$. We get a commutative diagram:

$$\begin{tikzcd} k[[T]] \arrow[r, "a"] \arrow[d, "g"'] & k[[T]] \arrow[d, "T^e"] \\ k[[T]] \arrow[r, "a^p"] & k[[T]]. \end{tikzcd}$$

We obtain the equation $ga = a^pT^{p-1}$. As $a \neq 0$ this is only possible if

$$\mathrm{ord}(g) = (p-1)(\mathrm{ord}(a)+1).$$

As $\mathrm{ord}(g) \le p-1$, we get that $\mathrm{ord}(a) = 0$ and $\mathrm{ord}(g) = p-1$. As $\mathrm{ord}(a) = 0$, both α_1 and β_1 are isomorphisms. Thus $\beta : H \to D$ is an isomorphism. □

4.2 Basic extension properties

The next three results will be proved for $p = 2$ under certain residually connectivity assumptions. But in Subsection 4.3 below we will show how these results hold under no residually connectivity assumptions even if $p = 2$.

Proposition 17 *We assume that* $\dim R \geq 2$*, that* $p \notin \mathfrak{m}^p$*, and that the initial form of* p *in* $\operatorname{gr} R$ *generates an ideal which is not a* $(p-1)$*-th power. Let* $\beta : H_1 \to H_2$ *be a homomorphism of finite flat group schemes over* $\operatorname{Spec} R$ *which induces an epimorphism (resp. a monomorphism) over* $\operatorname{Spec} K$*. If* $p = 2$*, we assume as well that* H_1 *and* H_2 *(resp. that the Cartier duals of* H_1 *and* H_2*) are residually connected.*
Then β *is an epimorphism (resp. a monomorphism).*

Proof. As before we can assume that R is a complete regular local ring. We can also restrict our attention to the case of epimorphisms.
By Proposition 15 we can assume that $p \in \mathfrak{m}^{p-1}$. Let $\mathfrak{p}$ be a minimal prime ideal which contains p. We show that the assumption that $p \in \mathfrak{p}^{p-1}$ leads to a contradiction. Then we can write $p = uf^{p-1}$, where $u, f \in R$ and f is a generator of the prime ideal $\mathfrak{p}$. It follows from our assumptions that $u \notin \mathfrak{m}$ and that $f \in \mathfrak{m} \setminus \mathfrak{m}^2$. Therefore the initial form of p in $\operatorname{gr} R$ generates an ideal which is a $(p-1)$-th power. Contradiction.
We first consider the case when k is perfect. Let $\alpha : (M_1, \varphi_1) \to (M_2, \varphi_2)$ denote also the morphism of Breuil modules associated to $\beta : H_1 \to H_2$. As $p \notin \mathfrak{p}^{p-1}$, we can apply Proposition 15 to the ring $R_{\mathfrak{p}}$. It follows that β induces an epimorphism over the spectrum of the residue class field $\kappa(\mathfrak{p})$ of $\mathfrak{p}$ and thus also of its perfect hull $\kappa(\mathfrak{p})^{\mathrm{perf}}$. From this and the end of Section 2 we get that the hypotheses of Proposition 13 hold for α. We conclude that α is an epimorphism and thus β is also an epimorphism.
Let $R \to R'$ be a faithfully flat extension of noetherian local rings, such that $\mathfrak{m}R'$ is the maximal ideal of R' and the extension of residue class fields $k \hookrightarrow k'$ is radical. We consider the homomorphism of polynomial rings $\operatorname{gr} R \to \operatorname{gr} R'$. As $\operatorname{gr} R$ and $\operatorname{gr} R'$ are unique factorization domains, it is easy to see that the condition that the initial form of p is not a $(p-1)$-th power is stable by the extension $R \to R'$. But β is an epimorphism if and only if $\beta_{R'}$ is so. Therefore we can assume that the residue class field of R is perfect and this case was already proved. $\square$

Proposition 18 *We assume that* $\dim R = 2$*. Let* $U = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$*. We also assume that the following technical condition holds:*

(*) *there exists a faithfully flat local* R*-algebra* $\hat{R}$ *which is complete, which has an algebraically closed residue class field* k*, and which has a presentation* $\hat{R} = \mathfrak{S}/(p-h)$ *where* $\mathfrak{S} = W(k)[[T_1, T_2]]$ *and where* $h \in (T_1, T_2)$ *does not belong to the ideal* $(p, T_1^p, T_2^p, T_1^{p-1}T_2^{p-1})$*.*

Let $\beta : H_1 \to H_2$ *be a homomorphism of finite flat group schemes over* $\operatorname{Spec} R$*. We also assume that for each geometric point* $\operatorname{Spec} L \to U$ *such that* L *has characteristic* p *the homomorphism* β_L *is an epimorphism (resp. a monomorphism); thus* $\beta_U : H_{1,U} \to H_{2,U}$ *is an epimorphism (resp. a monomorphism). If* $p = 2$*, we assume that* H_1 *and* H_2 *(resp. that the Cartier duals of* H_1 *and* H_2*) are residually connected.*
Then β *is an epimorphism (resp. a monomorphism).*

Proof. We have $\dim \hat{R}/\mathfrak{m}\hat{R} = 0$ and thus $\mathrm{Spec}(\hat{R}) \setminus (\mathrm{Spec}(\hat{R}) \times_{\mathrm{Spec}\, R} U)$ is the closed point of $\mathrm{Spec}(\hat{R})$. Based on this we can assume that $R = \hat{R}$. Thus the proposition follows from Proposition 14 in the same way Proposition 17 followed from Propositions 15 and 13. $\square$

Corollary 19 *We assume that the assumptions on R of either Proposition 17 or Proposition 18 are satisfied (thus $\dim R \geq 2$). We consider a complex*

$$0 \to H_1 \to H_2 \to H_3 \to 0$$

of finite flat group schemes over $\mathrm{Spec}\, R$ whose restriction to $U = \mathrm{Spec}\, R \setminus \{\mathfrak{m}\}$ is a short exact sequence. If $p = 2$, then we assume that H_2 and H_3 are residually connected. Then $0 \to H_1 \to H_2 \to H_3 \to 0$ is a short exact sequence.

Proof. Propositions 17 and 18 imply that $H_2 \to H_3$ is an epimorphism. Its kernel is a finite flat group scheme isomorphic to H_1 over U and thus (as we have $\dim R \geq 2$) it is isomorphic to H_1. $\square$

4.3 Dévissage properties

In this subsection we explain how one can get the results of Subsection 4.2 under no residually connectivity assumptions for $p = 2$.
We assume that $\dim R = 2$ and that p is arbitrary. Let $U = \mathrm{Spec}\, R \setminus \{\mathfrak{m}\}$. Let $\mathcal{V}$ be a locally free $\mathcal{O}_U$-module of finite rank over U. The R-module $H^0(U, \mathcal{V})$ is free of finite rank. Using this it is easy to see that each finite flat group scheme over U extends uniquely to a finite flat group scheme over $\mathrm{Spec}\, R$. This is clearly an equivalence between the category of finite flat group schemes over U and the category of finite flat group schemes over $\mathrm{Spec}\, R$. The same holds if we restrict to finite flat group schemes annihilated by p.
Next we also assume that R is complete. Each finite flat group scheme H over $\mathrm{Spec}\, R$ is canonically an extension

$$0 \to H^\circ \to H \to H^{\text{ét}} \to 0,$$

where H° is connected and $H^{\text{ét}}$ is étale over $\mathrm{Spec}\, R$. In particular a homomorphism from a connected finite flat group scheme over $\mathrm{Spec}\, R$ to an étale finite flat group scheme over $\mathrm{Spec}\, R$ is zero. From this one easily checks that if $H_1 \to H_2$ is a homomorphism of finite flat group schemes over $\mathrm{Spec}\, R$ which is an epimorphism over U and if H_1 is connected, then H_2 is connected as well.

Lemma 20 *We assume that $\dim R = 2$ and that R is complete. Then the following four statements are equivalent:*

(a) *Each short exact sequence of finite flat group schemes over U extends uniquely to a short exact sequence of finite flat group schemes over* $\mathrm{Spec}\, R$.

(b) *Let H_1 and H_2 be connected finite flat group schemes over* $\operatorname{Spec} R$. *A homomorphism $H_1 \to H_2$ over* $\operatorname{Spec} R$ *is an epimorphism if its restriction to U is an epimorphism.*

(c) *Let H_1 and H_2 be connected finite flat group schemes over* $\operatorname{Spec} R$ *which are annihilated by p. A homomorphism $H_1 \to H_2$ over* $\operatorname{Spec} R$ *is an epimorphism if its restriction to U is an epimorphism.*

(d) *The regular ring R is p-quasi-healthy.*

Proof. It is clear that (a) implies (b). We show that (b) implies (a). Let

$$0 \to H_1 \to H_2 \to H_3 \to 0$$

be a complex of finite flat group schemes over $\operatorname{Spec} R$ whose restriction to U is a short exact sequence. It suffices to show that $\beta : H_1 \to H_2$ is a monomorphism. Indeed, in this case we can form the quotient group scheme H_2/H_1 and the homomorphism $H_2/H_1 \to H_3$ is an isomorphism as its restriction to U is so. We check that the homomorphism

$$\beta^\circ : H_1^\circ \to H_2^\circ$$

is a monomorphism. Let H_4 be the finite flat group scheme over R whose restriction to U is $H_{2,U}^\circ/H_{1,U}^\circ$. We have a complex $0 \to H_1^\circ \to H_2^\circ \to H_4 \to 0$ whose restriction to U is exact. We conclude that H_4 is connected. Thus we have an epimorphism $H_2^\circ \twoheadrightarrow H_4$ (as we are assuming that (b) holds) whose kernel is H_1°. Therefore (b) implies that β° is a monomorphism.
As β° is a monomorphism, it suffices to show that the induced homomorphism $\bar{\beta} : H_1/H_1^\circ \to H_2/H_1^\circ$ is a monomorphism. In other words, without loss of generality we can assume that $H_1 = H_1^{\text{ét}}$ is étale.
Let H_1'' be the kernel of $H_2^{\text{ét}} \to H_3^{\text{ét}}$; it is a finite étale group scheme over $\operatorname{Spec} R$. Let H_1' be the kernel of $H_1 \to H_1''$. It suffices to show that $H_1' \to H_2^\circ$ is a monomorphism. Therefore we can also assume that H_2 is connected. This implies that H_3 is connected. As we are assuming that (b) holds, $H_2 \to H_3$ is an epimorphism. Its kernel is H_1 and therefore $\beta : H_1 \to H_2$ is a monomorphism. Thus (b) implies (a).
We show that (c) implies (b). The last argument shows that a short exact sequence of finite flat group schemes over U annihilated by p extends to a short exact sequence of finite flat group schemes over $\operatorname{Spec} R$. We start with a homomorphism $H_2 \to H_3$ between connected finite flat group schemes over $\operatorname{Spec} R$ which induces an epimorphism over U. We extend the kernel of $H_{2,U} \to H_{3,U}$ to a finite flat group scheme H_1 over $\operatorname{Spec} R$. Then we find a complex

$$0 \to H_1 \to H_2 \to H_3 \to 0$$

whose restriction to U is a short exact sequence. To show that $H_2 \to H_3$ is an epimorphism it is equivalent to show that $H_1 \to H_2$ is a monomorphism.

As $H_{1,U}$ has a composition series whose factors are annihilated by p, we easily reduce to the case where H_1 is annihilated by p. We embed H_2 into a p-divisible group G over $\operatorname{Spec} R$. To check that $H_1 \to H_2$ is a monomorphism, it suffices to show that $H_1 \to G[p]$ is a monomorphism. But this follows from the second sentence of this paragraph.
It is clear that (a) implies (d). We are left to show that (d) implies (c). It suffices to show that a homomorphism $\beta : H_1 \to H_2$ of finite flat group schemes over $\operatorname{Spec} R$ annihilated by p is a monomorphism if its restriction $\beta_U : H_{1,U} \to H_{2,U}$ is a monomorphism. We embed H_2 into a p-divisible group G over $\operatorname{Spec} R$. The quotient $G_U/H_{1,U}$ is a p-divisible group over U which extends to a p-divisible group G' over $\operatorname{Spec} R$ (as we are assuming that (d) holds). The isogeny $G_U \to G'_U$ extends to an isogeny $G \to G'$. Its kernel is a finite flat group scheme and therefore isomorphic to H_1. We obtain a monomorphism $H_1 \to G$. Thus $\beta : H_1 \to H_2$ is a monomorphism, i.e. (d) implies (c). $\square$

Corollary 21 *Propositions 17 and 18 (and thus implicitly Corollary 19) hold without any residually connectivity assumption for $p = 2$.*

Proof. We can assume that R is complete. By considering an epimorphism $R \twoheadrightarrow R'$ with R' regular of dimension 2, we can also assume that $\dim R = 2$. Based on Lemma 20, it suffices to prove Propositions 17 and 18 in the case when connected finite flat group schemes are involved. But this case was already proved in Subsection 4.2. This also shows that for $p \geq 3$ we can restrict to the connected case and avoid to apply Proposition 10 for nonconnected p-divisible groups. $\square$

Corollary 22 *We assume that k is perfect and that $R = W(k)[[T_1, T_2]]/(p - h)$ with $h \in (T_1, T_2)$. Let $\bar{h} \in k[[T_1, T_2]]$ be the reduction of h modulo p. Then the fact that R is or is not p-quasi-healthy depends only on the orbit of the ideal $(\bar{h})$ of $k[[T_1, T_2]]$ under automorphisms of $k[[T_1, T_2]]$.*

Proof. The category of Breuil modules associated to connected finite flat group schemes over $\operatorname{Spec} R$ annihilated by p is equivalent to the category of pairs (M, φ), where M is a free $k[[T_1, T_2]]$-module of finite rank and where $\varphi : M \to M^{(\sigma)}$ is a $k[[T_1, T_2]]$-linear map whose cokernel is annihilated by $\bar{h}$ and whose reduction modulo the ideal (T_1, T_2) is nilpotent in the natural sense. The last category depends only on the orbit of $(\bar{h})$ under automorphisms of $k[[T_1, T_2]]$. The corollary follows from the last two sentences and the equivalence of (c) and (d) in Lemma 20. $\square$

4.4 The p-quasi-healthy part of Theorem 3

In this subsection we show that if R is as in Theorem 3 for $d = 2$, then R is p-quasi-healthy. It follows from the definition of a p-divisible group and the uniqueness part of the first paragraph of Subsection 4.3, that it is enough to show that a complex $0 \to H_1 \to H_2 \to H_3 \to 0$ of finite flat group schemes over $\operatorname{Spec} R$ is a short exact sequence if its restriction to $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ is a short exact sequence. This is a local statement in the faithfully flat topology of $\operatorname{Spec} R$ and thus to check it we can assume that $R = \hat{R} = W(k)[[T_1, T_2]/(p-h)$ with $h \in (T_1, T_2)$ but $h \notin (p, T_1^p, T_2^p, T_1^{p-1}T_2^{p-1})$. By Lemma 20 we can assume that H_2 and H_3 are connected. Thus $0 \to H_1 \to H_2 \to H_3 \to 0$ is a short exact sequence, cf. Corollary 19. □

5 Extending abelian schemes

Proposition 23 *Let R be a regular local ring of mixed characteristic $(0, p)$.*

(a) Let $U = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$, where $\mathfrak{m}$ is the maximal ideal of R. Let $\breve{A}$ be an abelian scheme over U. If the p-divisible group of $\breve{A}$ extends to a p-divisible group over $\operatorname{Spec} R$, then $\breve{A}$ extends uniquely to an abelian scheme A over $\operatorname{Spec} R$.

(b) If R is p-quasi-healthy, then R is quasi-healthy.

Proof. The uniqueness part of (a) is well known (cf. [R1], Chapter IX, Corollary 1.4). Part (a) is a particular case of either proof of [V2], Proposition 4.1 (see remark that starts the proof) or [V2], Remark 4.2. Part (b) follows from (a). □

Lemma 24 *Let $S \twoheadrightarrow R$ be a ring epimorphism between local noetherian rings whose kernel is an ideal $\mathfrak{a}$ with $\mathfrak{a}^2 = 0$ and $\operatorname{depth}_R \mathfrak{a} \geq 2$. We assume that $\operatorname{depth} R \geq 2$ and that R is quasi-healthy. Then S is quasi-healthy as well.*

Proof. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximal ideals of R and S (respectively). We set $U = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$ and $V = \operatorname{Spec} S \setminus \{\mathfrak{n}\}$.
Let $\breve{B}$ be an abelian scheme over V and let $\breve{A}$ be its reduction over U. Then $\breve{A}$ extends uniquely to an abelian scheme A over $\operatorname{Spec} R$. Let $\pi : A \twoheadrightarrow \operatorname{Spec} R$ be the projection. It is well-known that the set of liftings of A with respect to $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} S$ is a trivial torsor under the group $H^0(\operatorname{Spec} R, R^1\pi_*\underline{\operatorname{Hom}}(\Omega_{A/R}, \mathfrak{a}))$ and that the set of liftings of $\breve{A}$ with respect to $V \twoheadrightarrow U$ is a trivial torsor under the group $H^0(U, R^1\pi_*\underline{\operatorname{Hom}}(\Omega_{A/R}, \mathfrak{a}))$. As $\operatorname{depth}_R \mathfrak{a} \geq 2$, the last two groups are equal. Thus there exists a unique abelian scheme B over $\operatorname{Spec} S$ which lifts A and whose restriction to V is $\breve{B}$. □

Proposition 25 *Let S be a complete noetherian local ring of mixed characteristic $(0, p)$. Let $S \twoheadrightarrow R$ be a ring epimorphism with kernel $\mathfrak{a}$ (thus R is a complete noetherian local ring). We assume that there exists a sequence of ideals*

$$\mathfrak{a} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \dots$$

such that the intersection of these ideals is 0 and for all $i \geq 0$ we have $\mathfrak{a}_i^2 \subset \mathfrak{a}_{i+1}$ and $\operatorname{depth}_S \mathfrak{a}_i/\mathfrak{a}_{i+1} \geq 2$. We also assume that $\operatorname{depth} R \geq 2$ and that R is quasi-healthy. Let $\mathfrak{n}$ be the maximal ideal of S. Let $V := \operatorname{Spec} S \setminus \{\mathfrak{n}\}$.

(a) Then each abelian scheme over V that has a polarization extends to an abelian scheme over $\operatorname{Spec} S$.

(b) We assume that S is integral and geometrically unibranch (like S is normal). Then S is quasi-healthy.

Proof. By a well-known theorem we have $S = \lim\limits_{\leftarrow} S/\mathfrak{a}_i$.

For (b) (resp. (a)) we have to show that each abelian scheme $\breve{A}$ (resp. each abelian scheme $\breve{A}$ that has a polarization $\lambda_{\breve{A}}$) over V extends to an abelian scheme over $\operatorname{Spec} S$. We write $U_i = (\operatorname{Spec} S/\mathfrak{a}_i) \setminus \{\mathfrak{n}\}$. The topological space underlying U_i is independent of i and it will be denoted by U. It is easy to see that $\operatorname{depth} S/\mathfrak{a}_i \geq 2$. By Lemma 24 the ring $S/\mathfrak{a}_i$ is quasi-healthy. We denote by $\breve{A}_i$ the base change of $\breve{A}$ to U_i. Then $\breve{A}_i$ extends uniquely to an abelian scheme A_i over $\operatorname{Spec} S/\mathfrak{a}_i$. If S is integral and geometrically unibranch, then from [R1], Chapter XI, Theorem 1.4 we get that $\breve{A}$ is projective over V. Thus from now on we can assume that there exists a polarization $\lambda_{\breve{A}}$ of $\breve{A}$ and we will not anymore differentiate between parts (a) and (b).
From the uniqueness of A_i we easily get that the reduction of $\lambda_{\breve{A}}$ modulo $\mathfrak{a}_i$ extends uniquely to a polarization λ_{A_i} of A_i (this also follows from [R1], Chapter IX, Corollary 1.4). We get that the A_i's inherit a compatible system of polarizations. From this and the algebraization theorem of Grothendieck, we get that there exists an abelian scheme A over $\operatorname{Spec} S$ which lifts the A_i's.
Next we will prove that the p-divisible group G of A restricts over V to the p-divisible group $\breve{G}$ of $\breve{A}$. This implies that $\breve{A}$ extends to an abelian scheme over $\operatorname{Spec} S$ (cf. Proposition 23 (a)) which is then necessarily isomorphic to A.
Let $G_i[m]$ be the kernel of the multiplication by $p^m : A_i \to A_i$, and let $G[m]$ be the kernel of the multiplication by $p^m : A \to A$.
Let $C_i[m]$ be the $S/\mathfrak{a}_i$-algebra of global functions on $G_i[m]$. Then $B[m] = \lim\limits_{\leftarrow} C_i[m]$ is the S-algebra of global functions on $G[m]$.

We write $\underline{\operatorname{Spec}}\, \breve{\mathcal{C}}[m] = \breve{G}[m]$, where $\breve{\mathcal{C}}[m]$ is a finite $\mathcal{O}_{\breve{G}[m]}$-algebra. We have a natural homomorphism:

$$H^0(V, \breve{\mathcal{C}}[m]) \to H^0(U, (C_i[m])^\sim) = C_i[m].$$

Here $(C_i[m])^\sim$ is the restriction to U of the $\mathcal{O}_{\operatorname{Spec} S/\mathfrak{a}_i}$-algebra associated to $C_i[m]$. The last equality follows from the fact that $\operatorname{depth} S/\mathfrak{a}_i \geq 2$. This gives birth to an S-algebra homomorphism

$$H^0(V, \breve{\mathcal{C}}[m]) \to B[m].$$

If we restrict it to a homomorphism between $\mathcal{O}_V$-algebras we obtain a homomorphism of finite flat group schemes over V

$$G[m]_V \to \breve{G}[m]$$

and thus a homomorphism of corresponding p-divisible groups $G_V \to \breve{G}$ over V. By construction this is an isomorphism if we restrict it to U. As V is connected, from the following proposition we conclude that $G_V \to \breve{G}$ is an isomorphism. $\square$

Proposition 26 *Let $\beta : G \to G'$ be a homomorphism of p-divisible groups over a noetherian scheme X. Then the set Y of points $x \in X$ such that β_x is an isomorphism is open and closed in X. Moreover β_Y is an isomorphism.*

Proof. It is clear that β is an isomorphism if and only if $\beta[1] : G[1] \to G'[1]$ is an isomorphism. Therefore the subfunctor of X defined by the condition that β_Y is an isomorphism is representable by an open subscheme $Z \subset X$.
By the theorems of Tate and de Jong on extensions of homomorphisms between p-divisible groups, we get that the valuative criterion of properness holds for $Z \to X$. Thus Z is as well closed in X and therefore we can take $Y = Z$. $\square$

5.1 Counterexample for the p-quasi-healthy context

Lemma 24 does not hold for the p-quasi-healthy context even in the simplest cases. Here is an elementary counterexample. We take $R = W(k)[[T_1]]$. From Subsection 4.4 we get that R is p-quasi-healthy. We take $S = R[T_2]/(T_2^2)$. Let $V := \operatorname{Spec} S \setminus \{\mathfrak{n}\}$, where $\mathfrak{n}$ is the maximal ideal of S. Let $\mathcal{O} := R_{(T_1)}$; we have a natural identification $V = \operatorname{Spec} S[\frac{1}{T_1}] \cup \operatorname{Spec} \mathcal{O}[T_2]/(T_2^2)$.
Let $\mathcal{E}_k$ be an elliptic curve over $\operatorname{Spec} k$. We can identify the formal deformation space of $\mathcal{E}_k$ with $\operatorname{Spf} R$. Let $\mathcal{E}$ be the elliptic curve over $\operatorname{Spec} R$ which is the algebraization of the uni-versal elliptic curve over $\operatorname{Spf} R$. Let $\iota_1 : R \hookrightarrow S[\frac{1}{T_1}]$ be the $W(k)$-monomorphism that maps T_1 to $T_1 + T_2T_1^{-1}$; it lifts the natural $W(k)$-monomorphism $R \hookrightarrow R[\frac{1}{T_1}]$. The $W(k)$-monomorphism ι_1 gives birth to an elliptic curve $\mathcal{E}_1$ over $\operatorname{Spec} S[\frac{1}{T_1}]$ whose restriction to $\operatorname{Spec} S[\frac{1}{T_1}]/(T_2) = \operatorname{Spec} R[\frac{1}{T_1}]$ extends to the elliptic curve $\mathcal{E}$ over $\operatorname{Spec} R$.
We check that the assumption that $\mathcal{E}_1$ extends to an elliptic curve $\mathcal{E}_V$ over V leads to a contradiction. To $\mathcal{E}_V$ and $\mathcal{E}_U$ correspond morphisms $U \to V \to \mathbb{A}^1$, where $\mathbb{A}^1$ is the j-line over $\operatorname{Spec} W(k)$. As the topological spaces of U and V are equal, we get that we have a natural factorization $U \to V \to \operatorname{Spec} R \to \mathbb{A}^1$, where we identify $\operatorname{Spec} R$ with the completion of $\mathbb{A}^1$ at its k-valued point defined by $\mathcal{E}_k$. As $S = H^0(V, \mathcal{O}_V)$, the image of ι_1 is contained in S. Contradiction.
But the p-divisible group $\mathcal{G}_1$ of $\mathcal{E}_1$ extends to a p-divisible group $\mathcal{G}_V$ over V. This is so as each p-divisible group over $\operatorname{Spec} \mathcal{O}$ extends uniquely to an étale p-divisible group over $\operatorname{Spec} \mathcal{O}[T_2]/(T_2^2)$.
Finally we check that the assumption that $\mathcal{G}_V$ extends to a p-divisible group $\mathcal{G}_2$ over $\operatorname{Spec} S$ leads to a contradiction. Let $\mathcal{E}_2$ be the elliptic curve over $\operatorname{Spec} S$ which lifts $\mathcal{E}$ and whose p-divisible group is $\mathcal{G}_2$. Let $\iota_2 : R \to S$ be the $W(k)$-homomorphism that defines $\mathcal{E}_2$. We check that the resulting two $W(k)$-homomorphisms $\iota_1, \iota_2 : R \to S[\frac{1}{T_1}]$ are equal. It suffices to show that their composites $\iota_3, \iota_4 : R \to \widehat{S_{(p)}}$ with the natural $W(k)$-monomorphism $S[\frac{1}{T_1}] \hookrightarrow \widehat{S_{(p)}} = \widehat{R_{(p)}}[T_2]/(T_2^2)$ are equal (here $\widehat{\triangle}$ denotes the completion of the local ring $\triangle$). But this follows from Serre–Tate deformation theory and the fact that the

composites $\iota_5, \iota_6 : R \to \widehat{R_{(p)}}$ of ι_3, ι_4 with the $W(k)$-epimorphism $\widehat{S_{(p)}} \twoheadrightarrow \widehat{R_{(p)}}$, are equal. As the two $W(k)$-homomorphisms $\iota_1, \iota_2 : R \to S[\frac{1}{T_1}]$ are equal, the image of ι_1 is contained in S. Contradiction.
We conclude that S is not p-quasi-healthy. A similar argument shows that for all $n \geq 2$, the ring $R[T_2]/(T_2^n)$ is not p-quasi-healthy.
This counterexample disproves the claims on [FC], top of p. 184 on torsors of liftings of p-divisible groups.

5.2 Proofs of Theorem 3 and Corollary 4

Let $\mathfrak{m}$ be the maximal ideal of R. Let $U = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. Let $d = \dim R$.
We prove Theorem 3. We first assume that $d = 2$. It is enough to show that R is p-quasi-healthy, cf. Proposition 23 (b). But this follows from Subsection 4.4.
We next assume that $d \geq 3$. We have to show that each abelian scheme over U extends uniquely to an abelian scheme over $\operatorname{Spec} R$. This is a local statement in the faithfully flat topology of $\operatorname{Spec} R$ and thus to check it we can assume that $R = \hat{R}$ is complete with algebraically closed residue class field k. We have an epimorphism $R \twoheadrightarrow W(k)[[T_1, T_2]]/(p - h)$ where h is a power series in the maximal ideal of $W(k)[[T_1, T_2]]$ whose reduction modulo $(p, T_1^p, T_2^p, T_1^{p-1}T_2^{p-1})$ is non-zero. As $W(k)[[T_1, T_2]]/(p - h)$ is p-quasi-healthy (cf. the case $d = 2$), from Proposition 25 we get that R is quasi-healthy. This proves Theorem 3.
We prove Corollary 4. Thus $\dim R = d \geq 2$ and $p \notin \mathfrak{m}^p$. As in the previous paragraph we argue that we can assume that $R = \hat{R}$ is complete with algebraically closed residue class field k. We write $R = \mathfrak{S}/(p - h)$ where $h \in (T_1, \ldots, T_d)$ is such that its reduction $\bar{h} \in \bar{\mathfrak{S}} = \mathfrak{S}/p\mathfrak{S}$ modulo p is a power series of order $e = \operatorname{ord}(\bar{h}) \leq p - 1$. Due to Noether normalization theorem we can assume that $\bar{h}$ contains the monom T_1^e. We set $R' := \mathfrak{S}/(p - h, T_3, \ldots, T_d)$; if $d = 2$, then $R' = R$. From Proposition 25 applied to the epimorphism $R \twoheadrightarrow R'$, we get that it suffices to show that R' is quasi-healthy and p-quasi-healthy. Thus in order not to complicate the notations, we can assume that $d = 2$ (i.e., $R = R'$). As $e \leq p - 1$, the reduction of $\bar{h}$ modulo the ideal $(T_1^p, T_2^p, T_1^{p-1}T_2^{p-1})$ is non-zero. Thus Corollary 4 follows from Theorem 3. $\square$

5.3 Example

Let R be a regular local of mixed characteristic $(0, p)$ such that the strict completion of R is isomorphic to

$$C_k[[T_1, \ldots, T_d]]/(p - T_1^{e_1} \cdot \ldots \cdot T_m^{e_m})$$

where C_k is a Cohen ring of the field k, where $1 \leq m \leq d$, and where the m-tuple $(e_1, \ldots, e_m) \in \mathbb{N}^m$ has the property that there exists a disjoint union $\{1, \ldots, m\} = I_1 \bigsqcup I_2$ for which we have $m_1 := \sum_{i \in I_1} e_i \in \{1, \ldots, p - 1\}$ and $m_2 := \sum_{i \in I_2} e_i \in \{0, \ldots, p - 2\}$.
To check that R is quasi-healthy we can assume that the field k is algebraically closed and that $R = W(k)[[T_1, \ldots, T_d]]/(p - T_1^{e_1} \cdot \ldots \cdot T_m^{e_m})$ (thus $C_k = W(k)$).

We consider the ring epimorphism $R \twoheadrightarrow R' := W(k)[[T_1, T_2]]/(p - T_1^{m_1} T_2^{m_2})$ that maps T_i with $i \in I_1$ to T_1, that maps T_i with $i \in I_2$ to T_2, and that maps T_i with $i > m$ to 0. From Theorem 3 we get that R is quasi-healthy.
Concrete example: if $1 \leq m \leq \min\{d, 2p-3\}$ and if the strict completion of R is $C_k[[T_1, \ldots, T_d]]/(p - T_1 \cdot \ldots \cdot T_m)$, then R is quasi-healthy. From this and Corollary 5 we get that each étale scheme over $\operatorname{Spec} O[T_1, \ldots, T_d]/(p - T_1 \cdot \ldots \cdot T_m)$ is healthy regular, provided O is a discrete valuation ring of mixed characteristic $(0, p)$ and index of ramification 1.

5.4 Regular schemes which are not (p-) healthy

Let R be a local regular ring of dimension 2 and mixed characteristic $(0, p)$. Let $\mathfrak{m}$ be the maximal ideal of R. Let $U = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.
The ring R is (p-) quasi-healthy if and only if $\operatorname{Spec} R$ is (p-) healthy regular. The next lemma provides an easy criterion for when R is not (p-) quasi-healthy.

Lemma 27 *We assume that there exists a homomorphism $H \to D$ of finite flat group schemes over* $\operatorname{Spec} R$ *which is not an epimorphism and whose restriction to U is an epimorphism. Then R is neither quasi-healthy nor p-quasi-healthy.*

Proof. We embed the Cartier dual D^{t} of D into an abelian scheme A over $\operatorname{Spec} R$, cf. [BBM], Theorem 3.1.1. The Cartier dual homomorphism $H_U^{\mathrm{t}} \to D_U^{\mathrm{t}}$ is a closed immersion. The abelian scheme A_U/H_U^{t} over U does not extend to an abelian scheme over $\operatorname{Spec} R$. Based on [R1], Chapter IX, Corollary 1.4, the argument for this is similar to the one used to prove that (d) implies (c) in Lemma 20. Thus R is not quasi-healthy. From Proposition 23 (a) we get that the p-divisible group of A_U/H_U^{t} does not extend to $\operatorname{Spec} R$. Thus the fact that R is not p-quasi-healthy follows from either Lemma 20 or Proposition 23 (a). $\square$

Based on Lemma 27, the following theorem adds many examples to the classical example of Raynaud–Gabber.

Theorem 28 *We consider the ring $\mathfrak{S} = W(k)[[T_1, T_2]]$. Let $h \in (T_1, T_2) \setminus p\mathfrak{S}$. Let $R := \mathfrak{S}/(p - h)$. Let $\bar{\mathfrak{S}} := \mathfrak{S}/p\mathfrak{S}$ and let $\bar{h}$ be the reduction of h modulo p. We assume that one of the following three properties hold:*

(i) The element $\bar{h}$ is divisible by u^p, where u is a power series in the maximal ideal of $\bar{\mathfrak{S}}$ (the class of rings R for which (i) holds includes $O[[T]]$, where O is a totally ramified discrete valuation ring extension of $W(k)$ of index of ramification at least equal to p).

(ii) There exists a regular sequence u, v in $\bar{\mathfrak{S}}$ such that $u^{p-1}v^{p-1}$ divides $\bar{h}$.

(iii) We can write $\bar{h} = (aT_1^p + bT_2^p + cT_1^{p-1}T_2^{p-1})c$, where $a, b, c \in \bar{\mathfrak{S}}$.

Then R is neither p-quasi-healthy nor quasi-healthy.

Proof. It is enough to construct a homomorphism $\beta : H \to D$ of connected finite flat group schemes over $\operatorname{Spec} R$ which is not an epimorphism but whose restriction to U is an epimorphism, cf. Lemma 27.
We will construct H and D by specifying their Breuil modules (M,φ) and (N,τ) (respectively) associated to a standard frame for R. We first present with full details the case (i) and then we will only mention what are the changes required to be made for the other two cases (ii) and (iii).
We assume that the condition (i) holds. We set $M = \bar{\mathfrak{S}}^3$ and we identify $M^{(\sigma)}$ with $\bar{\mathfrak{S}}^3$. We choose an element $t \in \bar{\mathfrak{S}}$ such that t and u are a regular sequence in $\bar{\mathfrak{S}}$. We define the homomorphism φ by the following matrix:

$$\Gamma = \begin{pmatrix} 0 & 0 & u^p \\ t - t^p u^{p-1} & u & (u - t^{p-1})(t - t^p u^{p-1}) \\ u^{p-1} & 0 & u^{p-1}(u - t^{p-1}) \end{pmatrix}.$$

It is easy to see that there exists a matrix $\Delta \in M_{3\times 3}(\bar{\mathfrak{S}})$ such that

$$\Delta\Gamma = \Gamma\Delta = u^p I_3,$$

where I_3 is the unit matrix. It follows that the cokernel of φ is annihilated by u^p and thus also by $\bar{h}$. Moreover the image of φ is contained in $(t,u)M$. Therefore (M,φ) is the Breuil module H of a connected finite flat group scheme over $\operatorname{Spec} R$ annihilated by p.
We set $N = \bar{\mathfrak{S}}$ and $N^{(\sigma)} = \bar{\mathfrak{S}}$ and we define τ as the multiplication by u^p. This defines another connected finite flat group scheme D over $\operatorname{Spec} R$ annihilated by p.
One easily checks the following equation of matrices:

$$(t^p, u^p, (tu)^p)\Gamma = u^p(t, u, tu).$$

This equation shows that the $\mathfrak{S}$-linear map $M \to N$ defined by the matrix (t, u, tu) is a morphism of Breuil modules

$$\alpha : (M,\varphi) \to (N,\tau).$$

As α is not surjective, the homomorphism $\beta : H \to D$ associated to α is not an epimorphism. Let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal of R which contains p. The base change of α by $\kappa_{\mathfrak{p}} : \mathfrak{S} \to W(\kappa(\mathfrak{p})^{\mathrm{perf}})$ (of Section 2) is an epimorphism as the cokernel of α is k. This implies that $\beta_U : H_U \to D_U$ is an epimorphism.
We assume that the condition (ii) holds. The proof in this case is similar to the case (i) but with the definitions of M, Γ, τ, and $M \to N$ modified as follows. Let $M = \bar{\mathfrak{S}}^2$. Let

$$\Gamma = \begin{pmatrix} u^{p-1} & 0 \\ 0 & v^{p-1} \end{pmatrix}.$$

Let τ be defined by $(uv)^{p-1}$. Let $M \to N$ be defined by the matrix $(v\ u)$.

We assume that the condition (iii) holds. Let $M = \bar{\mathfrak{S}}^2$. Let

$$\Gamma = \begin{pmatrix} aT_1 + cT_2^{p-1} & aT_2 \\ bT_1 & bT_2 + cT_1^{p-1} \end{pmatrix}.$$

Let τ be defined by $aT_1^p + bT_2^p + cT_1^{p-1}T_2^{p-1}$. Let $M \to N$ be defined by the matrix $(T_1\ T_2)$. The determinant of Γ is $\bar{h}$. □

6 Integral models and Néron models

Let O be a discrete valuation ring of mixed characteristic $(0, p)$ and of index of ramification at most $p-1$. Let K be the field of fractions of O. A flat O-scheme $\bigstar$ is said to have the *extension property*, if for each O-scheme X which is a healthy regular scheme, every morphism $X_K \to \bigstar_K$ of K-schemes extends uniquely to a morphism $X \to \bigstar$ of O-schemes (cf. [V1], Definition 3.2.3 3)).

Lemma 29 *Let Z_K be a regular scheme which is formally smooth over* $\operatorname{Spec} K$*. Then there exists at most one regular scheme which is a formally smooth O-scheme, which has the extension property, and whose fibre over* $\operatorname{Spec} K$ *is Z_K.*

Proof. Let Z_1 and Z_2 be two regular schemes which are formally smooth over $\operatorname{Spec} O$, which satisfy the identity $Z_{1,K} = Z_{2,K} = Z_K$, and which have the extension property. Both Z_1 and Z_2 are healthy regular schemes, cf. Corollary 5. Thus the identity $Z_{1,K} = Z_{2,K}$ extends naturally to morphisms $Z_1 \to Z_2$ and $Z_2 \to Z_1$, cf. the fact that both Z_1 and Z_2 have the extension property. Due to the uniqueness part of the extension property, the composite morphisms $Z_1 \to Z_2 \to Z_1$ and $Z_2 \to Z_1 \to Z_2$ are identity automorphisms. Thus the identity $Z_{1,K} = Z_{2,K}$ extends uniquely to an isomorphism $Z_1 \to Z_2$. □

Corollary 30 *The integral canonical models of Shimura varieties defined in [V1], Definition 3.2.3 6) are unique, provided they are over the spectrum of a discrete valuation ring O as above.*

Let $d \geq 1$ and $n \geq 3$ be natural numbers. We assume that n is prime to p. Let $\mathcal{A}_{d,1,n}$ be the Mumford moduli scheme over $\operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$ that parameterizes principally polarized abelian scheme over $\operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$-schemes which are of relative dimension d and which have level-n symplectic similitude structures (cf. [MFK], Theorems 7.9 and 7.10). For Néron models over Dedekind domains we refer to [BLR], Chapter I, Subsection 1.2, Definition 1.

Theorem 31 *Let $\mathcal{D}$ be a Dedekind domain which is a flat $\mathbb{Z}[\frac{1}{n}]$-algebra. Let $\mathcal{K}$ be the field of fractions of $\mathcal{D}$. We assume that the following two things hold:*

(i) the only local ring of $\mathcal{D}$ whose residue class field has characteristic 0*, is $\mathcal{K}$;*

(ii) if v is a prime of $\mathcal{D}$ whose residue class field has a prime characteristic $p_v \in \mathbb{N}^*$, then the index of ramification of the local ring of v is at most $p_v - 1$.

Let $\mathcal{N}$ be a finite $\mathcal{A}_{d,1,n,\mathcal{D}}$-scheme which is a projective, smooth $\mathcal{D}$-scheme. Then $\mathcal{N}$ is the Néron model over $\mathcal{D}$ of its generic fibre $\mathcal{N}_{\mathcal{K}}$.

Proof. Let Y be a smooth $\mathcal{D}$-scheme. Let $\delta_{\mathcal{K}} : Y_{\mathcal{K}} \to \mathcal{N}_{\mathcal{K}}$ be a morphism of $\mathcal{K}$-scheme. Let V be an open subscheme of Y which contains $Y_{\mathcal{K}}$ and all generic points of fibres of Y in positive characteristic and for which $\delta_{\mathcal{K}}$ extends uniquely to a morphism $\delta_V : V \to \mathcal{N}$ (cf. the projectiveness of $\mathcal{N}$). Let $(\mathcal{B}_V, \lambda_V)$ be the pull back to V of the universal principally polarized abelian scheme over $\mathcal{A}_{d,1,n,\mathcal{D}}$ via the composite morphism $\nu_V : V \to \mathcal{N} \to \mathcal{A}_{d,1,n,\mathcal{D}}$. From Corollary 5 we get that $\mathcal{B}_V$ extends uniquely to an abelian scheme $\mathcal{B}$ over Y. From [R1], Chapter IX, Corollary 1.4 we get that λ_V extends (uniquely) to a polarization λ of $\mathcal{B}$. The level-n symplectic similitude structure of $(\mathcal{B}_V, \lambda_V)$ defined naturally by ν_V extends uniquely to a level-n symplectic similitude structure of $(\mathcal{B}, \lambda)$, cf. the classical Nagata–Zariski purity theorem. Thus ν_V extends uniquely to a morphism $\nu : Y \to \mathcal{A}_{d,1,n,\mathcal{D}}$. As Y is a normal scheme, as $\mathcal{N}$ is finite over $\mathcal{A}_{d,1,n,\mathcal{D}}$, and as ν restricted to V factors through $\mathcal{N}$, the morphism ν factors uniquely through a morphism $\delta : Y \to \mathcal{N}$ which extends δ_V and thus also $\delta_{\mathcal{K}}$. Hence the Theorem follows from the very definition of Néron models. □

Remarks. (a) From Theorem 31 and [V3], Remark 4.4.2 and Example 4.5 we get that there exist plenty of Néron models over O whose generic fibres are not finite schemes over torsors of smooth schemes over $\operatorname{Spec} K$.
We can take $\mathcal{N}$ to be the pull back to O of those Néron models of Theorem 31 whose generic fibres have the above property (cf. [V3], Remark 4.5). If $p > 2$ and $e = p - 1$, then these Néron models $\mathcal{N}$ are new (i.e., their existence does not follow from [N], [BLR], [V1], [V2], or [V3]).
(b) One can use Theorem 28 (i) and Artin's approximation theorem to show that Theorem 31 does not hold in general if there exists a prime v of $\mathcal{D}$ whose residue class field has a prime characteristic $p_v \in \mathbb{N}^*$ and whose index of ramification is at least p_v. Counterexamples can be obtained using integral models of projective Shimura varieties of PEL type, cf. [V3], Corollary 4.3.

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