Frames and Finite Group Schemes over Complete Regular Local Rings

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Abstract. Let p be an odd prime. We show that the classification of p-divisible groups by Breuil windows and the classification of commutative finite flat group schemes of p-power order by Breuil modules hold over every complete regular local ring with perfect residue field of characteristic p. We set up a formalism of frames and windows with an abstract deformation theory that applies to Breuil windows.

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1 Introduction

Let R be a complete regular local ring of dimension r with perfect residue field k of odd characteristic p. Let $W(k)$ be the ring of Witt vectors of k. One can write $R = \mathfrak{S}/E\mathfrak{S}$ with

$$\mathfrak{S} = W(k)[[x_1, \ldots, x_r]]$$

such that $E \in \mathfrak{S}$ is a power series with constant term p. Let σ be the endomorphism of $\mathfrak{S}$ that extends the Frobenius automorphism of $W(k)$ by $\sigma(x_i) = x_i^p$. Following Vasiu and Zink, a *Breuil window* relative to $\mathfrak{S} \to R$ is a pair (Q, ϕ) where Q is a free $\mathfrak{S}$-module of finite rank, and where

$$\phi : Q \to Q^{(\sigma)}$$

is an $\mathfrak{S}$-linear map with cokernel annihilated by E.

Theorem 1.1. *The category of p-divisible groups over R is equivalent to the category of Breuil windows relative to $\mathfrak{S} \to R$.*

If R has characteristic p, this follows from more general results of A. de Jong [dJ]; this case is included here only for completeness. If $r = 1$ and E is an Eisenstein polynomial, Theorem 1.1 was conjectured by Breuil [Br] and proved by Kisin [K1]. When E is a deformation of an Eisenstein polynomial the result is proved in [VZ1].

Like in these cases one can deduce a classification of commutative finite flat group schemes of p-power order over R: A *Breuil module* relative to $\mathfrak{S} \to R$ is a triple (M, φ, ψ) where M is a finitely generated $\mathfrak{S}$-module annihilated by a power of p and of projective dimension at most one, and where

$$\varphi : M \to M^{(\sigma)}, \qquad \psi : M^{(\sigma)} \to M$$

are $\mathfrak{S}$-linear maps with $\varphi\psi = E$ and $\psi\varphi = E$. If R has characteristic zero, such triples are equivalent to pairs (M, φ) such that the cokernel of φ is annihilated by E.

Theorem 1.2. *The category of commutative finite flat group schemes over R annihilated by a power of p is equivalent to the category of Breuil modules relative to $\mathfrak{S} \to R$.* [1]

This result is applied in [VZ2] to the question whether abelian schemes or p-divisible groups defined over the complement of the maximal ideal in $\operatorname{Spec} R$ extend to $\operatorname{Spec} R$.

Frames and windows

To prove Theorem 1.1 we show that Breuil windows are equivalent to Dieudonné displays over R, which are equivalent to p-divisible groups over R by [Z2]; the same route is followed in [VZ1]. So the main part of this article is purely module theoretic.

We introduce a notion of frames and windows (motivated by [Z3]) which allows to formulate a deformation theory that generalises the deformation theory of Dieudonné displays developed in [Z2] and that also applies to Breuil windows. Technically the main point is the formalism of σ_1 in Definition 2.1; the central result is the lifting of windows in Theorem 3.2.

This is applied as follows. Let $\mathfrak{m}_R$ be the maximal ideal of R. For each positive integer a we consider the rings $\mathfrak{S}_a = \mathfrak{S}/(x_1, \ldots, x_r)^a\mathfrak{S}$ and $R_a = R/\mathfrak{m}_R^a$. There is an obvious notion of Breuil windows relative to $\mathfrak{S}_a \to R_a$ and a functor

$$\varkappa_a : (\text{Breuil windows relative to } \mathfrak{S}_a \to R_a) \to (\text{Dieudonné displays over } R_a).$$

Here $\varkappa_1$ is trivially an equivalence because $\mathfrak{S}_1 = W(k)$ and $R_1 = k$. The deformation theory implies that on both sides lifts from a to $a+1$ are classified by lifts of the Hodge filtration in a compatible way. Thus $\varkappa_a$ is an equivalence for all a by induction, and Theorem 1.1 follows.

[1] Recently, Theorems 1.1 and 1.2 have been extended to the case $p = 2$. See: *A relation between Dieudonné displays and crystalline Dieudonné theory* (in preparation).

Complements

There is some freedom in the choice of the Frobenius lift on $\mathfrak{S}$. Namely, let σ be a ring endomorphism of $\mathfrak{S}$ which preserves the ideal $J = (x_1, \ldots, x_r)$ and which induces the Frobenius on $\mathfrak{S}/p\mathfrak{S}$. If the endomorphism σ/p of J/J^2 is nilpotent modulo p, Theorems 1.1 and 1.2 hold without change.
All of the above equivalences of categories are compatible with the natural duality operations on both sides.
If the residue field k is not perfect, there is an analogue of Theorems 1.1 and 1.2 for connected groups. Here $p = 2$ is allowed. The ring $W(k)$ is replaced by a Cohen ring of k, and the operators ϕ and φ must be nilpotent modulo the maximal ideal of $\mathfrak{S}$.
In the first version of this article [L3] the formalism of frames was introduced only to give an alternative proof of the results of Vasiu and Zink [VZ1]. In response, they pointed out that both their and this approach apply in greater generality, e.g. in the case where $E \in \mathfrak{S}$ takes the form $E = g + p\epsilon$ such that ϵ is a unit and g divides $\sigma(g)$ for a general Frobenius lift σ as above. However, the method of loc. cit. seems not to give Theorem 1.1 completely.

All rings in this article are commutative and have a unit. All finite flat group schemes are commutative.

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2 Frames and windows

Let p be a prime. The following notion of frames and windows differs from [Z3]. Some definitions and arguments could be simplified by assuming that the relevant rings are local, which is the case in our applications, but we work in greater generality until section 4.
If S is a ring equipped with a ring endomorphism σ, for an S-module M we write $M^{(\sigma)} = S \otimes_{\sigma,S} M$, and for a σ-linear map of S-modules $g : M \to N$ we denote by $g^\sharp : M^{(\sigma)} \to N$ its linearisation, $g^\sharp(s \otimes m) = sg(m)$. If $g^\sharp$ is invertible, g is called a σ-linear isomorphism.

Definition 2.1. A frame is a quintuple

$$\mathscr{F} = (S, I, R, \sigma, \sigma_1)$$

consisting of a ring S, an ideal I of S, the quotient ring $R = S/I$, a ring endomorphism $\sigma : S \to S$, and a σ-linear map of S-modules $\sigma_1 : I \to S$, such that the following conditions hold:

i. $I + pS \subseteq \mathrm{Rad}(S)$,

ii. $\sigma(a) \equiv a^p \bmod pS$ for $a \in S$,

iii. $\sigma_1(I)$ generates S as an S-module.

We do not assume here that R is the specific ring considered in the introduction. In our examples $\sigma_1(I)$ contains the element 1.

LEMMA 2.2. *For every frame $\mathscr{F}$ there is a unique element $\theta \in S$ such that $\sigma(a) = \theta\sigma_1(a)$ for $a \in I$.*

Proof. Condition iii means that the homomorphism $\sigma_1^\sharp : I^{(\sigma)} \to S$ is surjective. Let us choose $b \in I^{(\sigma)}$ such that $\sigma_1^\sharp(b) = 1$. Then necessarily $\theta = \sigma^\sharp(b)$. For $a \in I$ we compute $\sigma(a) = \sigma_1^\sharp(b)\sigma(a) = \sigma_1^\sharp(ba) = \sigma^\sharp(b)\sigma_1(a)$ as desired. □

DEFINITION 2.3. Let $\mathscr{F}$ be a frame. A window over $\mathscr{F}$, also called an $\mathscr{F}$-window, is a quadruple

$$\mathscr{P} = (P, Q, F, F_1)$$

where P is a finitely generated projective S-module, $Q \subseteq P$ is a submodule, $F : P \to P$ and $F_1 : Q \to P$ are σ-linear map of S-modules, such that the following conditions hold:

1. There is a decomposition $P = L \oplus T$ with $Q = L \oplus IT$,
2. $F_1(ax) = \sigma_1(a)F(x)$ for $a \in I$ and $x \in P$,
3. $F_1(Q)$ generates P as an S-module.

A decomposition as in 1 is called a normal decomposition of (P, Q) or of $\mathscr{P}$.

Remark 2.4. The operator F is determined by F_1. Indeed, if $b \in I^{(\sigma)}$ satisfies $\sigma_1^\sharp(b) = 1$, then condition 2 implies that $F(x) = F_1^\sharp(bx)$ for $x \in P$. In particular we have $F(x) = \theta F_1(x)$ when x lies in Q.

Remark 2.5. Condition 1 implies that

1′. P/Q is a projective R-module.

If finitely generated projective R-modules lift to projective S-modules, necessarily finitely generated because $I \subseteq \mathrm{Rad}(S)$, condition 1 is equivalent to 1′. In all our examples, this lifting property holds because S is either local or I-adic.

LEMMA 2.6. *Let $\mathscr{F}$ be a frame, let $P = L \oplus T$ be a finitely generated projective S-module, and let $Q = L \oplus IT$. The set of $\mathscr{F}$-window structures (P, Q, F, F_1) on these modules is mapped bijectively to the set of σ-linear isomorphisms*

$$\Psi : L \oplus T \to P$$

by the assignment $\Psi(l + t) = F_1(l) + F(t)$ for $l \in L$ and $t \in T$.

The triple (L, T, Ψ) is called a normal representation of (P, Q, F, F_1).

Proof. If (P, Q, F, F_1) is an $\mathscr{F}$-window, by conditions 2 and 3 of Definition 2.3 the linearisation of Ψ is surjective, thus bijective since P and $P^{(\sigma)}$ are projective S-modules of equal rank by conditions i and ii of Definition 2.1. Conversely, if Ψ is given, one gets an $\mathscr{F}$-window by $F(l+t) = \theta\Psi(l) + \Psi(t)$ and $F_1(l + at) = \Psi(l) + \sigma_1(a)\Psi(t)$ for $l \in L$, $t \in T$, and $a \in I$. □

Example. The Witt frame of a p-adic ring R is

$$\mathscr{W}_R = (W(R), I_R, R, f, f_1)$$

where $W(R)$ is the ring of p-typical Witt vectors of R, f is its Frobenius endomorphism, and $f_1 : I_R \to W(R)$ is the inverse of the Verschiebung homomorphism. Here $\theta = p$. We have $I_R \subseteq \mathrm{Rad}(W(R))$ because $W(R)$ is I_R-adic; see [Z1, Proposition 3]. Windows over $\mathscr{W}_R$ are 3n-displays over R in the sense of [Z1], called displays in [M2], which is the terminology we follow.

Functoriality

Let $\mathscr{F} = (S, I, R, \sigma, \sigma_1)$ and $\mathscr{F}' = (S', I', R', \sigma', \sigma_1')$ be frames.

Definition 2.7. A homomorphism of frames $\alpha : \mathscr{F} \to \mathscr{F}'$, also called a frame homomorphism, is a ring homomorphism $\alpha : S \to S'$ with $\alpha(I) \subseteq I'$ such that $\sigma'\alpha = \alpha\sigma$ and $\sigma_1'\alpha = u \cdot \alpha\sigma_1$ for a unit $u \in S'$. If $u = 1$, then α is called strict.

Remark 2.8. The unit u is unique because $\alpha\sigma_1(I)$ generates S' as an S'-module. We have $\alpha(\theta) = u\theta'$. If we want to specify u, we say that α is a u-homomorphism. There is a unique factorisation of α into frame homomorphisms

$$\mathscr{F} \xrightarrow{\alpha'} \mathscr{F}'' \xrightarrow{\omega} \mathscr{F}'$$

such that α' is strict and ω is an invertible u-homomorphism. Here $\mathscr{F}''$ is the u^{-1}-twist of $\mathscr{F}'$ defined as $\mathscr{F}'' = (S', I', R', \sigma', u^{-1}\sigma_1')$.

Let $\alpha : \mathscr{F} \to \mathscr{F}'$ be a u-homomorphism of frames.

Definition 2.9. Let $\mathscr{P}$ be an $\mathscr{F}$-window and let $\mathscr{P}'$ be an $\mathscr{F}'$-window. A homomorphism of windows $g : \mathscr{P} \to \mathscr{P}'$ over α, also called an α-homomorphism, is an S-linear map $g : P \to P'$ with $g(Q) \subseteq Q'$ such that $F'g = gF$ and $F_1'g = u \cdot gF_1$. A homomorphism of $\mathscr{F}$-windows is an $\mathrm{id}_{\mathscr{F}}$-homomorphism in the previous sense.

Lemma 2.10. *For each $\mathscr{F}$-window $\mathscr{P}$ there is a base change window $\alpha_*\mathscr{P}$ over $\mathscr{F}'$ together with an α-homomorphism of windows $\mathscr{P} \to \alpha_*\mathscr{P}$ that induces a bijection $\mathrm{Hom}_{\mathscr{F}'}(\alpha_*\mathscr{P}, \mathscr{P}') = \mathrm{Hom}_\alpha(\mathscr{P}, \mathscr{P}')$ for all $\mathscr{F}'$-windows $\mathscr{P}'$.*

Proof. Clearly this requirement determines $\alpha_*\mathscr{P}$ uniquely. It can be constructed explicitly as follows: If (L, T, Ψ) is a normal representation of $\mathscr{P}$, a normal representation of $\alpha_*\mathscr{P}$ is $(S' \otimes_S L, S' \otimes_S T, \Psi')$ where Ψ' is defined by $\Psi'(s' \otimes l) = u\sigma'(s') \otimes \Psi(l)$ and $\Psi'(s' \otimes t) = \sigma'(s') \otimes \Psi(t)$. □

If $\alpha_*\mathscr{P} = (P', Q', F', F_1')$, then $P' = S' \otimes_S P$, and $Q' \subseteq P'$ is the S'-submodule generated by $I'P'$ and by the image of Q.

Remark 2.11. As suggested in [VZ2], the above definitions of frames and windows can be generalised as follows. Instead of condition iii of Definition 2.1, the element θ given by Lemma 2.2 is taken as part of the data. For a u-homomorphism $\alpha : \mathscr{F} \to \mathscr{F}'$ of generalised frames in this sense it is necessary to require that $\alpha(\theta) = u\theta'$. For a window over a generalised frame the relation $F(x) = \theta F_1(x)$ of Remark 2.4 becomes part of the definition, and condition 3 of Definition 2.3 is replaced by the requirement that $F_1(Q) + F(P)$ generates P. Then the results of sections 2–4 hold for generalised frames and windows as well. Details are left to the reader.

Limits

Windows are compatible with projective limits of frames in the following sense. Assume that for each positive integer n we have a frame

$$\mathscr{F}_n = (S_n, I_n, R_n, \sigma_n, \sigma_{1n})$$

and a strict frame homomorphism $\pi_n : \mathscr{F}_{n+1} \to \mathscr{F}_n$ such that the involved maps $S_{n+1} \to S_n$ and $I_{n+1} \to I_n$ are surjective and $\operatorname{Ker}(\pi_n)$ is contained in $\operatorname{Rad}(S_{n+1})$. We obtain a frame $\varprojlim \mathscr{F}_n = (S, I, R, \sigma, \sigma_1)$ with $S = \varprojlim S_n$ etc. By definition, an $\mathscr{F}_*$-window is a system $\mathscr{P}_*$ of $\mathscr{F}_n$-windows $\mathscr{P}_n$ together with isomorphisms $\pi_{n*}\mathscr{F}_{n+1} \cong \mathscr{F}_n$.

Lemma 2.12. *The category of* $(\varprojlim \mathscr{F}_n)$*-windows is equivalent to the category of* $\mathscr{F}_*$*-windows.*

Proof. The obvious functor from $(\varprojlim \mathscr{F}_n)$-windows to $\mathscr{F}_*$-windows is fully faithful. We have to show that for an $\mathscr{F}_*$-window $\mathscr{P}_*$, the projective limit $\varprojlim \mathscr{P}_n = (P, Q, F, F_1)$ defined by $P = \varprojlim P_n$ etc. is a window over $\varprojlim \mathscr{F}_n$. The condition $\operatorname{Ker}(\pi_n) \subseteq \operatorname{Rad}(S_{n+1})$ implies that P is a finitely generated projective S-module and that P/Q is projective over R. In order that P has a normal decomposition it suffices to show that each normal decomposition of $\mathscr{P}_n$ lifts to a normal decomposition of $\mathscr{P}_{n+1}$. Assume that $P_n = L_n' \oplus T_n'$ and $P_{n+1} = L_{n+1} \oplus T_{n+1}$ are normal decompositions and let $P_n = L_n \oplus T_n$ be induced by the second. Since $T_n \otimes R_n \cong P_n/Q_n \cong T_n' \otimes R_n$ and $L_n \otimes R_n \cong Q_n/IP_n \cong L_n' \otimes R_n$, we have $T_n \cong T_n'$ and $L_n \cong L_n'$. Hence the two decompositions of P_n differ by an automorphism of $L_n \oplus T_n$ of the type $\omega = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c : L_n \to I_n T_n$. Now ω lifts to an endomorphism $\omega' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ of $L_{n+1} \oplus T_{n+1}$ with $c' : L_{n+1} \to I_{n+1}T_{n+1}$, and ω' is an automorphism since $\operatorname{Ker}(\pi_n) \subseteq \operatorname{Rad}(S_{n+1})$. The required lifting of normal decompositions follows. All remaining window axioms for $\varprojlim \mathscr{P}_n$ are easily checked. $\square$

Remark 2.13. Assume that S_1 is a local ring. Then all S_n and S are local too. Hence $\varprojlim \mathscr{F}_n$ satisfies the lifting property of Remark 2.5, so the normal decomposition of P in the preceding proof is automatic.

Duality

Let $\mathscr{P}$, $\mathscr{P}'$, and $\mathscr{P}''$ be windows over a frame $\mathscr{F}$. A bilinear form of $\mathscr{F}$-windows $\beta : \mathscr{P} \times \mathscr{P}' \to \mathscr{P}''$ is an S-bilinear map $\beta : P \times P' \to P''$ such that $\beta(Q \times Q') \subseteq Q''$ and

$$\beta(F_1(x), F_1'(x')) = F_1''(\beta(x, x')) \tag{2.1}$$

for $x \in Q$ and $x' \in Q'$. Let $\mathscr{F}$ also denote the $\mathscr{F}$-window (S, I, σ, σ_1). For every $\mathscr{F}$-window $\mathscr{P}$ there is a unique dual $\mathscr{F}$-window $\mathscr{P}^t$ together with a bilinear form $\mathscr{P} \times \mathscr{P}^t \to \mathscr{F}$ which induces for each $\mathscr{F}$-window $\mathscr{P}'$ an isomorphism $\operatorname{Hom}(\mathscr{P}', \mathscr{P}^t) \cong \operatorname{Bil}(\mathscr{P} \times \mathscr{P}', \mathscr{F})$. Explicitly we have $\mathscr{P}^t = (P^\vee, Q^t, F^t, F_1^t)$ where $P^\vee = \operatorname{Hom}_S(P, S)$ and

$$Q^t = \{x' \in P^\vee \mid x'(Q) \subseteq I\}.$$

The operators F_1^t and F^t are determined by (2.1) with σ_1 in place of F_1''. If (L, T, Ψ) is a normal representation for $\mathscr{P}$, a normal representation for $\mathscr{P}^t$ is given by $(T^\vee, L^\vee, \Psi^t)$ where $(\Psi^t)^\sharp$ is equal to $((\Psi^\sharp)^{-1})^\vee$. This shows that F_1^t and F^t are well-defined. There is a natural isomorphism $\mathscr{P}^{tt} \cong \mathscr{P}$.
For a more detailed exposition of the duality formalism in the case of (Diedonné) displays we refer to [Z1, Definition 19] or [L2, Section 3].

Lemma 2.14. *Let $\alpha : \mathscr{F} \to \mathscr{F}'$ be a u-homomorphism of frames and let $c \in S'$ be a unit such that $c^{-1}\sigma'(c) = u$. For all $\mathscr{F}$-windows $\mathscr{P}$ there is a natural isomorphism (depending on c)*

$$\alpha_*(\mathscr{P}^t) \cong (\alpha_*\mathscr{P})^t.$$

Proof. We consider the $\mathscr{F}'$-window $\mathscr{F}'_u = (S', I', u\sigma', u\sigma_1')$. The given bilinear form $\mathscr{P} \times \mathscr{P}^t \to \mathscr{F}$ induces a bilinear form $\alpha_*\mathscr{P} \times \alpha_*(\mathscr{P}^t) \to \mathscr{F}'_u$; this is easily verified using that under base change by α each of the operators F_1, F_1', and $F_1'' = \sigma_1$ accounts for one factor of u in (2.1). Multiplication by c is an isomorphism of $\mathscr{F}'$-windows $\mathscr{F}'_u \cong \mathscr{F}'$. The resulting bilinear form $\alpha_*\mathscr{P} \times \alpha_*(\mathscr{P}^t) \to \mathscr{F}'$ induces an isomorphism $\alpha_*(\mathscr{P}^t) \cong (\alpha_*\mathscr{P})^t$. □

3 Crystalline homomorphisms

Definition 3.1. A homomorphism of frames $\alpha : \mathscr{F} \to \mathscr{F}'$ is called crystalline if the functor $\alpha_* : (\mathscr{F}\text{-windows}) \to (\mathscr{F}'\text{-windows})$ is an equivalence of categories.

Theorem 3.2. *Let $\alpha : \mathscr{F} \to \mathscr{F}'$ be a strict frame homomorphism that induces an isomorphism $R \cong R'$ and a surjection $S \to S'$ with kernel $\mathfrak{a} \subset S$. We assume that there is a finite filtration $\mathfrak{a} = \mathfrak{a}_0 \supseteq \cdots \supseteq \mathfrak{a}_n = 0$ with $\sigma(\mathfrak{a}_i) \subseteq \mathfrak{a}_{i+1}$ and $\sigma_1(\mathfrak{a}_i) \subseteq \mathfrak{a}_i$ such that σ_1 is elementwise nilpotent on $\mathfrak{a}_i/\mathfrak{a}_{i+1}$. We assume that finitely generated projective S'-modules lift to projective S-modules. Then α is crystalline.*

In many applications the lifting property of projective modules holds because $\mathfrak{a}$ is nilpotent or S is local. The proof of Theorem 3.2 is a variation of the proofs of [Z1, Theorem 44] and [Z2, Theorem 3].

Proof. The homomorphism α factors into $\mathscr{F} \to \mathscr{F}'' \to \mathscr{F}'$ where the frame $\mathscr{F}''$ is determined by $S'' = S/\mathfrak{a}_1$, so by induction we may assume that $\sigma(\mathfrak{a}) = 0$. The functor α_* is essentially surjective because normal representations (L, T, Ψ) can be lifted from $\mathscr{F}'$ to $\mathscr{F}$. In order that α_* is fully faithful it suffices to show that α_* is fully faithful on automorphisms because a homomorphism $g : \mathscr{P} \to \mathscr{P}'$ can be encoded by the automorphism $\left(\begin{smallmatrix} 1 & 0 \\ g & 1 \end{smallmatrix}\right)$ of $\mathscr{P} \oplus \mathscr{P}'$. Since for a window $\mathscr{P}$ over $\mathscr{F}$ an automorphism of $\alpha_*\mathscr{P}$ can be lifted to an S-module automorphism of P, it suffices to prove the following assertion.
Assume that $\mathscr{P} = (P, Q, F, F_1)$ and $\mathscr{P}' = (P, Q, F', F_1')$ are two $\mathscr{F}$-windows such that $F \equiv F'$ and $F_1 \equiv F_1'$ modulo $\mathfrak{a}$. Then there is a unique $\mathscr{F}$-window isomorphism $g : \mathscr{P} \cong \mathscr{P}'$ with $g \equiv \mathrm{id}$ modulo $\mathfrak{a}$.
We write $F_1' = F_1 + \eta$ and $F' = F + \varepsilon$ and $g = 1 + \omega$, where the σ-linear maps $\eta : Q \to \mathfrak{a}P$ and $\varepsilon : P \to \mathfrak{a}P$ are given, and where $\omega : P \to \mathfrak{a}P$ is an arbitrary S-linear map. The induced g is an isomorphism of windows if and only if $gF_1 = F_1'g$ on Q, which translates into the identity

$$\eta = \omega F_1 - F_1'\omega. \tag{3.1}$$

We fix a normal decomposition $P = L \oplus T$, thus $Q = L \oplus IT$. For $l \in L$, $t \in T$, and $a \in I$ we have

$$\begin{aligned} \eta(l + at) &= \eta(l) + \sigma_1(a)\varepsilon(t), \\ \omega(F_1(l + at)) &= \omega(F_1(l)) + \sigma_1(a)\omega(F(t)), \\ F_1'(\omega(l + at)) &= F_1'(\omega(l)) + \sigma_1(a)F'(\omega(t)). \end{aligned}$$

Here $F'\omega = 0$ because for $a \in \mathfrak{a}$ and $x \in P$ we have $F'(ax) = \sigma(a)F'(x)$, and $\sigma(\mathfrak{a}) = 0$. As $\sigma_1(I)$ generates S we see that (3.1) is equivalent to:

$$\begin{cases} \varepsilon = \omega F & \text{on } T, \\ \eta = \omega F_1 - F_1'\omega & \text{on } L. \end{cases} \tag{3.2}$$

Since $\Psi : L \oplus T \xrightarrow{F_1 + F} P$ is a σ-linear isomorphism, to give ω is equivalent to give a pair of σ-linear maps

$$\omega_L = \omega F_1 : L \to \mathfrak{a}P, \qquad \omega_T = \omega F : T \to \mathfrak{a}P.$$

Let $\lambda : L \to L^{(\sigma)}$ be the composition $L \subseteq P \xrightarrow{(\Psi^\sharp)^{-1}} L^{(\sigma)} \oplus T^{(\sigma)} \xrightarrow{pr_1} L^{(\sigma)}$ and let $\tau : L \to T^{(\sigma)}$ be analogous with pr_2 in place of pr_1. Then the restriction $\omega|_L$ is equal to $\omega_L^\sharp\lambda + \omega_T^\sharp\tau$, and (3.2) becomes:

$$\begin{cases} \omega_T = \varepsilon|_T, \\ \omega_L - F_1'\omega_L^\sharp\lambda = \eta|_L + F_1'\omega_T^\sharp\tau. \end{cases} \tag{3.3}$$

Let $\mathscr{H}$ be the abelian group of σ-linear maps $L \to \mathfrak{a}P$. We claim that the endomorphism U of $\mathscr{H}$ given by $U(\omega_L) = F_1'\omega_L^\sharp\lambda$ is elementwise nilpotent, which implies that $1 - U$ is bijective, and (3.3) has a unique solution in (ω_L, ω_T) and thus in ω. The endomorphism F_1' of $\mathfrak{a}P$ is elementwise nilpotent because $F_1'(ax) = \sigma_1(a)F'(x)$ and because σ_1 is elementwise nilpotent on $\mathfrak{a}$ by assumption. Since L is finitely generated it follows that U is elementwise nilpotent. $\square$

Remark 3.3. The same argument applies if instead of σ_1 being elementwise nilpotent one demands that λ is (topologically) nilpotent, which is the original situation in [Z1, Theorem 44]; see section 10.

4 Abstract deformation theory

Definition 4.1. The Hodge filtration of a window $\mathscr{P}$ is the submodule

$$Q/IP \subseteq P/IP.$$

Lemma 4.2. *Let $\alpha : \mathscr{F} \to \mathscr{F}'$ be a strict homomorphism of frames such that $S = S'$; thus $R \to R'$ is surjective and we have $I \subseteq I'$. Then $\mathscr{F}$-windows $\mathscr{P}$ are equivalent to pairs consisting of an $\mathscr{F}'$-window $\mathscr{P}' = (P', Q', F', F_1')$ and a lift of its Hodge filtration to a direct summand $V \subseteq P'/IP'$.*

Proof. The equivalence is given by the functor $\mathscr{P} \mapsto (\alpha_*\mathscr{P}, Q/IP)$, which is easily seen to be fully faithful. We show that it is essentially surjective. Let an $\mathscr{F}'$-window $\mathscr{P}'$ and a lift of its Hodge filtration $V \subseteq P'/IP'$ be given and let $Q \subseteq P'$ be the inverse image of V; thus $Q \subseteq Q'$. We have to show that $\mathscr{P} = (P', Q, F', F_1'|_Q)$ is an $\mathscr{F}$-window. First we need a normal decomposition for $\mathscr{P}$; this is a decomposition $P' = L \oplus T$ such that $V = L/IL$. Since $\mathscr{P}'$ has a normal decomposition, $\mathscr{P}$ has one too for at least one choice of V. By modifying the isomorphism $P' \cong L \oplus T$ with an automorphism $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ of $L \oplus T$ for some homomorphism $c : L \to I'T$ one reaches every lift of the Hodge filtration. It remains to show that $F_1'(Q)$ generates P'. In terms of a normal decomposition $P' = L \oplus T$ for $\mathscr{P}$ this means that $F_1' + F' : L \oplus T \to P'$ is a σ-linear isomorphism, which holds because $\mathscr{P}'$ is an $\mathscr{F}'$-window. $\square$

Assume that a strict homomorphism of frames $\alpha : \mathscr{F} \to \mathscr{F}'$ is given such that $S \to S'$ is surjective with kernel $\mathfrak{a}$, and $I' = IS'$. We want to factor α into strict frame homomorphisms

$$(S, I, R, \sigma, \sigma_1) \xrightarrow{\alpha_1} (S, I'', R', \sigma, \sigma_1'') \xrightarrow{\alpha_2} (S', I', R', \sigma', \sigma_1') \tag{4.1}$$

such that α_2 satisfies the hypotheses of Theorem 3.2.
Necessarily $I'' = I + \mathfrak{a}$. The main point is to define $\sigma_1'' : I'' \to S$, which is equivalent to defining a σ-linear map $\sigma_1'' : \mathfrak{a} \to \mathfrak{a}$ that extends the restriction of σ_1 to $I \cap \mathfrak{a}$ and satisfies the hypotheses of Theorem 3.2. Once this is achieved, Theorem 3.2 and Lemma 4.2 will show that $\mathscr{F}$-windows are equivalent to $\mathscr{F}'$-windows $\mathscr{P}'$ plus a lift of the Hodge filtration of $\mathscr{P}'$ to a direct summand of P/IP, where $\mathscr{P}'' = (P, Q'', F, F_1'')$ is the unique lift of $\mathscr{P}'$ under α_2.

5 Dieudonné frames

Let R be a noetherian complete local ring with maximal ideal $\mathfrak{m}_R$ and with perfect residue field k of characteristic p. If $p = 2$, we assume that p annihilates R. Let $\hat{W}(\mathfrak{m}_R) \subset W(R)$ be the ideal of all Witt vectors whose coefficients lie in $\mathfrak{m}_R$ and converge to zero $\mathfrak{m}_R$-adically. There is a unique subring $\mathbb{W}(R)$ of $W(R)$ which is stable under the Frobenius f such that the projection $\mathbb{W}(R) \to W(k)$ is surjective with kernel $\hat{W}(\mathfrak{m}_R)$, and the ring $\mathbb{W}(R)$ is also stable under the Verschiebung v; see [Z2, Lemma 2]. Let $\mathbb{I}_R$ be the kernel of the projection to the first component $\mathbb{W}(R) \to R$. Then $v : \mathbb{W}(R) \to \mathbb{I}_R$ is bijective.

Definition 5.1. The Dieudonné frame associated to R is

$$\mathscr{D}_R = (\mathbb{W}(R), \mathbb{I}_R, R, f, f_1)$$

with $f_1 = v^{-1}$.

Here $\theta = p$. Windows over $\mathscr{D}_R$ are Dieudonné displays over R in the sense of [Z2]. We note that $\mathbb{W}(R)$ is a local ring, which guarantees the existence of normal decompositions; see Remark 2.5. The inclusion $\mathbb{W}(R) \to W(R)$ is a strict homomorphism of frames $\mathscr{D}_R \to \mathscr{W}_R$.
If R' has the same properties as R, a local ring homomorphism $R \to R'$ induces a strict frame homomorphism $\mathscr{D}_R \to \mathscr{D}_{R'}$.
Assume that $R' = R/\mathfrak{b}$ for an ideal $\mathfrak{b}$ which is equipped with elementwise nilpotent divided powers γ. Then $\mathbb{W}(R) \to \mathbb{W}(R')$ is surjective with kernel $\hat{W}(\mathfrak{b}) = W(\mathfrak{b}) \cap \hat{W}(\mathfrak{m}_R)$. In this situation, a factorisation (4.1) of the homomorphism $\mathscr{D}_R \to \mathscr{D}_{R'}$ can be defined as follows. We recall that the γ-divided Witt polynomials are defined as

$$w'_n(X_0, \ldots, X_n) = (p^n - 1)!\gamma_{p^n}(X_0) + (p^{n-1} - 1)!\gamma_{p^{n-1}}(X_1) + \cdots + X_n.$$

Thus $p^n w'_n$ is the usual Witt polynomial $w_n(X_0, \ldots, X_n) = X_0^{p^n} + \cdots + p^n X_n$. Let $\mathfrak{b}^{<\infty>}$ be the $W(R)$-module of all sequences $[b_0, b_1, \ldots]$ with elements $b_i \in \mathfrak{b}$ that converge to zero $\mathfrak{m}_R$-adically, such that $x \in W(R)$ acts on $\mathfrak{b}^{<\infty>}$ by $[b_0, b_1, \ldots] \mapsto [w_0(x)b_0, w_1(x)b_1, \ldots]$. We have an isomorphism of $W(R)$-modules

$$\log : \hat{W}(\mathfrak{b}) \cong \mathfrak{b}^{<\infty>}; \qquad b \mapsto (w'_0(b), w'_1(b), \ldots);$$

see the remark after [Z1, Cor. 82]. For $b \in \hat{W}(\mathfrak{b})$ we call $\log(b)$ the logarithmic coordinates of b. Let

$$\mathbb{I}_{R/R'} = \mathbb{I}_R + \hat{W}(\mathfrak{b}).$$

In logarithmic coordinates, the restriction of f_1 to $\mathbb{I}_R \cap \hat{W}(\mathfrak{b})$ is given by

$$f_1([0, b_1, b_2, \ldots]) = [b_1, b_2, \ldots].$$

Thus $f_1 : \mathbb{I}_R \to \mathbb{W}(R)$ extends uniquely to an f-linear map

$$\tilde{f}_1 : \mathbb{I}_{R/R'} \to \mathbb{W}(R)$$

with $\tilde{f}_1([b_0, b_1, \ldots]) = [b_1, b_2, \ldots]$ on $\hat{W}(\mathfrak{b})$, and we obtain a factorisation

$$\mathscr{D}_R \xrightarrow{\alpha_1} \mathscr{D}_{R/R'} = (\mathbb{W}(R), \mathbb{I}_{R/R'}, R', f, \tilde{f}_1) \xrightarrow{\alpha_2} \mathscr{D}_{R'}. \tag{5.1}$$

Proposition 5.2. *The frame homomorphism α_2 is crystalline.*

This is a reformulation of [Z2, Theorem 3] if $\mathfrak{m}_R$ is nilpotent, and the general case is an easy consequence. As explained in section 4, it follows that deformations of Dieudonné displays from R' to R are classified by lifts of the Hodge filtration; this is [Z2, Theorem 4].

Proof of Proposition 5.2. When $\mathfrak{m}_R$ is nilpotent, α_2 satisfies the hypotheses of Theorem 3.2; the required filtration of $\mathfrak{a} = \hat{W}(\mathfrak{b})$ is $\mathfrak{a}_i = p^i\mathfrak{a}$. In general, these hypotheses are not fulfilled because $f_1 : \mathfrak{a} \to \mathfrak{a}$ is only topologically nilpotent. However, one can find a sequence of ideals $R \supset I_1 \supset I_2 \cdots$ which define the $\mathfrak{m}_R$-adic topology such that each $\mathfrak{b} \cap I_n$ is stable under the divided powers of $\mathfrak{b}$. Indeed, for each n there is an l with $\mathfrak{m}_R^l \cap \mathfrak{b} \subseteq \mathfrak{m}_R^n\mathfrak{b}$; for $I_n = \mathfrak{m}_R^n\mathfrak{b} + \mathfrak{m}_R^l$ we have $\mathfrak{b} \cap I_n = \mathfrak{m}_R^n\mathfrak{b}$. The proposition holds for each R/I_n in place of R, and the general case follows by passing to the projective limit, using Lemma 2.12. $\square$

6 $\varkappa$-frames

The results in this section are essentially due to Th. Zink (private communication); see also [Z3, Section 1] and [VZ1, Section 3].

Definition 6.1. A $\varkappa$-frame is a frame $\mathscr{F} = (S, I, R, \sigma, \sigma_1)$ such that

iv. S has no p-torsion,

v. $W(R)$ has no p-torsion,

vi. $\sigma(\theta) - \theta^p = p \cdot$ unit in S.

The numbering extends i–iii of Definition 2.1. In the following we refer to conditions i–vi without explicitly mentioning Definitions 2.1 and 6.1.

Remark 6.2. If ii and iv hold, we have a (non-additive) map

$$\tau : S \to S, \qquad \tau(x) = \frac{\sigma(x) - x^p}{p},$$

and vi says that $\tau(\theta)$ is a unit. Condition v is satisfied if and only if the nilradical $\mathscr{N}(R)$ has no p-torsion, for example if R is reduced, or flat over $\mathbb{Z}_{(p)}$.

Proposition 6.3. *To each $\varkappa$-frame $\mathscr{F}$ one can associate a u-homomorphism of frames $\varkappa : \mathscr{F} \to \mathscr{W}_R$ lying over* id_R *for a well-defined unit u of $W(R)$. The homomorphism $\varkappa$ and the unit u are functorial in $\mathscr{F}$ with respect to strict frame homomorphisms.*

Proof. Conditions iv and ii imply that there is a well-defined ring homomorphism $\delta : S \to W(S)$ with $w_n\delta = \sigma^n$; see [Bou, IX.1, proposition 2]. We have $f\delta = \delta\sigma$. Let $\varkappa$ be the composite ring homomorphism

$$\varkappa : S \xrightarrow{\delta} W(S) \to W(R).$$

Then $f\varkappa = \varkappa\sigma$ and $\varkappa(I) \subseteq I_R$. Clearly $\varkappa$ is functorial in $\mathscr{F}$. To define u we write $1 = \sum y_i\sigma_1(x_i)$ in S with $x_i \in I$ and $y_i \in S$. This is possible by iii. Recall that $\theta = \sum y_i\sigma(x_i)$; see the proof of Lemma 2.2. Let

$$u = \sum \varkappa(y_i) f_1 \varkappa(x_i).$$

Then $pu = \varkappa(\theta)$ because $pf_1 = f$ and $f\varkappa = \varkappa\sigma$. We claim that $f_1\varkappa = u \cdot \varkappa\sigma_1$. By condition v this is equivalent to the relation $p \cdot f_1\varkappa = pu \cdot \varkappa\sigma_1$, which holds since $pf_1 = f$ and $pu = \varkappa(\theta)$ and $\theta\sigma_1 = \sigma$. It remains to show that u is a unit in $W(R)$. Let $pu = \varkappa(\theta) = (a_0, a_1, \ldots)$ as a Witt vector. By Lemma 6.4 below, u is a unit if and only if a_1 is a unit in R. In $W_2(S)$ we have $\delta(\theta) = (\theta, \tau(\theta))$ because (w_0, w_1) applied to both sides gives $(\theta, \sigma(\theta))$. Hence a_1 is a unit by vi. We conclude that $\varkappa : \mathscr{F} \to \mathscr{W}_R$ is a u-homomorphism of frames.
Finally, u is functorial in $\mathscr{F}$ by its uniqueness, see Remark 2.8. $\square$

Lemma 6.4. *Let R be a ring with $p \in \mathrm{Rad}(R)$ and let $u \in W(R)$ be given. For an integer $r \geq 0$ let $p^r u = (a_0, a_1, a_2, \ldots)$. The element u is a unit in $W(R)$ if and only if a_r is a unit in R.*

Proof. Assume first that $r = 0$. It suffices to show that an element $\bar{u}$ of $W_{n+1}(R)$ that maps to 1 in $W_n(R)$ is a unit. If $\bar{u} = 1 + v^n(x)$ with $x \in R$, then $\bar{u}^{-1} = 1 + v^n(y)$ where $y \in R$ is determined by the equation $x + y + p^n xy = 0$, which has a solution since $p \in \mathrm{Rad}(R)$. For general r, by the case $r = 0$ we may replace R by R/pR. Then we have $p(b_0, b_1, \ldots) = (0, b_0^p, b_1^p, \ldots)$ in $W(R)$, which reduces the assertion to the case $r = 0$. $\square$

Corollary 6.5. *Let $\mathscr{F}$ be a $\varkappa$-frame with $S = W(k)[[x_1, \ldots, x_r]]$ for a perfect field k of odd characteristic p. Assume that σ extends the Frobenius automorphism of $W(k)$ by $\sigma(x_i) = x_i^p$. Then u is a unit in $\mathbb{W}(R)$, and $\varkappa$ induces a u-homomorphism of frames $\varkappa : \mathscr{F} \to \mathscr{D}_R$.*

Proof. We claim that $\delta(S)$ lies in $\mathbb{W}(S)$. Indeed, $\delta(x_i) = [x_i]$ because w_n applied to both sides gives $x_i^{p^n}$. Thus for each multi-exponent $e = (e_1, \ldots, e_r)$ the element $\delta(x^e) = [x^e]$ lies in $\mathbb{W}(S)$. Let $\mathfrak{m}_S$ be the maximal ideal of S. Since $\mathbb{W}(S) = \varprojlim \mathbb{W}(S/\mathfrak{m}_S^n)$ and since for each n all but finitely many x^e lie in $\mathfrak{m}_S^n$, the claim follows. Hence the image of $\varkappa : S \to W(R)$ is contained in $\mathbb{W}(R)$. By its construction the unit u lies in $\mathbb{W}(R)$; it is invertible in $\mathbb{W}(R)$ because the inclusion $\mathbb{W}(R) \to W(R)$ is a local homomorphism of local rings. $\square$

7 The main frame

Let R be a complete regular local ring with perfect residue field k of characteristic $p \geq 3$. We choose a ring homomorphism

$$\mathfrak{S} = W(k)[[x_1, \ldots, x_r]] \xrightarrow{\pi} R$$

such that $x_1, \ldots, x_r$ map to a regular system of parameters of R. Since the graded ring of R is isomorphic to $k[x_1, \ldots, x_r]$, one can find a power series $E_0 \in \mathfrak{S}$ with constant term zero such that $\pi(E_0) = -p$. Let $E = E_0 + p$ and $I = E\mathfrak{S}$. Then $R = \mathfrak{S}/I$. Let $\sigma : \mathfrak{S} \to \mathfrak{S}$ be the ring endomorphism that extends the Frobenius automorphism of $W(k)$ by $\sigma(x_i) = x_i^p$. We have a frame

$$\mathscr{B} = (\mathfrak{S}, I, R, \sigma, \sigma_1)$$

where σ_1 is defined by $\sigma_1(Ey) = \sigma(y)$ for $y \in \mathfrak{S}$.

LEMMA 7.1. *The frame $\mathscr{B}$ is a $\varkappa$-frame.*

Proof. Let $\theta \in \mathfrak{S}$ be the element given by Lemma 2.2. The only condition to be checked is that $\tau(\theta)$ is a unit in $\mathfrak{S}$. Let $E_0' = \sigma(E_0)$. Since $\sigma_1(E) = 1$, we have $\theta = \sigma(E) = E_0' + p$. Hence

$$\tau(\theta) = \frac{\sigma(E_0') + p - (E_0' + p)^p}{p} \equiv 1 + \tau(E_0') \mod p.$$

Since the constant term of E_0 is zero, the same is true for $\tau(E_0')$, which implies that $\tau(\theta)$ is a unit as required. $\square$

Thus Proposition 6.3 and Corollary 6.5 give a ring homomorphism $\varkappa$ from $\mathfrak{S}$ to $\mathbb{W}(R)$, which is a u-homomorphism of frames

$$\varkappa : \mathscr{B} \to \mathscr{D}_R.$$

Here the unit $u \in \mathbb{W}(R)$ is determined by the identity $pu = \varkappa\sigma(E)$.

THEOREM 7.2. *The frame homomorphism $\varkappa$ is crystalline (Definition 3.1).*

To prove this we consider the following auxiliary frames. Let $J \subset \mathfrak{S}$ be the ideal $J = (x_1, \ldots, x_r)$, and let $\mathfrak{m}_R$ be the maximal ideal of R. For each positive integer a let $\mathfrak{S}_a = \mathfrak{S}/J^a\mathfrak{S}$ and $R_a = R/\mathfrak{m}_R^a$. Then $R_a = \mathfrak{S}_a/E\mathfrak{S}_a$, where E is not a zero divisor in $\mathfrak{S}_a$. There is a well-defined frame

$$\mathscr{B}_a = (\mathfrak{S}_a, I_a, R_a, \sigma_a, \sigma_{1a})$$

such that the projection $\mathfrak{S} \to \mathfrak{S}_a$ is a strict frame homomorphism $\mathscr{B} \to \mathscr{B}_a$. Indeed, σ induces an endomorphism σ_a of $\mathfrak{S}_a$ because $\sigma(J) \subseteq J$, and for $y \in \mathfrak{S}_a$ one can define $\sigma_{1a}(Ey) = \sigma_a(y)$.

For simplicity, the image of u in $\mathbb{W}(R_a)$ is denoted by u as well. The u-homomorphism $\varkappa$ induces a u-homomorphism

$$\varkappa_a : \mathscr{B}_a \to \mathscr{D}_{R_a}$$

because for $e \in \mathbb{N}^r$ we have $\varkappa(x^e) = [x^e]$, which maps to zero in $\mathbb{W}(R_a)$ when $e_1 + \cdots + e_r \geq a$. We note that $\mathscr{B}_a$ is again a $\varkappa$-frame, so the existence of $\varkappa_a$ can also be viewed as a consequence of Proposition 6.3.

Theorem 7.3. *For each positive integer a the homomorphism $\varkappa_a$ is crystalline.*

To prepare for the proof, for each $a \geq 1$ we will construct the following commutative diagram of frames, where vertical arrows are u-homomorphisms and where horizontal arrows are strict.

$$\begin{tikzcd} \mathscr{B}_{a+1} \arrow[r, "\iota"] \arrow[d, "\varkappa_{a+1}"] & \tilde{\mathscr{B}}_{a+1} \arrow[r, "\pi"] \arrow[d, "\tilde{\varkappa}_{a+1}"] & \mathscr{B}_a \arrow[d, "\varkappa_a"] \\ \mathscr{D}_{R_{a+1}} \arrow[r, "\iota'"] & \mathscr{D}_{R_{a+1}/R_a} \arrow[r, "\pi'"] & \mathscr{D}_{R_a} \end{tikzcd} \tag{7.1}$$

The upper line is a factorisation (4.1) of the projection $\mathscr{B}_{a+1} \to \mathscr{B}_a$. This means that the frame $\tilde{\mathscr{B}}_{a+1}$ necessarily takes the form

$$\tilde{\mathscr{B}}_{a+1} = (\mathfrak{S}_{a+1}, \tilde{I}_{a+1}, R_a, \sigma_{a+1}, \tilde{\sigma}_{1(a+1)})$$

with $\tilde{I}_{a+1} = E\mathfrak{S}_{a+1} + J^a/J^{a+1}$. We define $\tilde{\sigma}_{1(a+1)} : \tilde{I}_{a+1} \to \mathfrak{S}_{a+1}$ to be the extension of $\sigma_{1(a+1)} : E\mathfrak{S}_{a+1} \to \mathfrak{S}_{a+1}$ by zero on J^a/J^{a+1}. This is well-defined because

$$E\mathfrak{S}_{a+1} \cap J^a/J^{a+1} = E(J^a/J^{a+1})$$

and because for $x \in J^a/J^{a+1}$ we have $\sigma_{1(a+1)}(Ex) = \sigma_{a+1}(x)$, which is zero since $\sigma(J^a) \subseteq J^{ap}$.

The lower line of (7.1) is the factorisation (5.1) with respect to the trivial divided powers on the kernel $\mathfrak{m}_R^a/\mathfrak{m}_R^{a+1}$.

In order that the diagram commutes it is necessary and sufficient that $\tilde{\varkappa}_{a+1}$ is given by the ring homomorphism $\varkappa_{a+1}$.

It remains to show that $\tilde{\varkappa}_{a+1}$ is a u-homomorphism of frames. The only nontrivial condition is that $\tilde{f}_1\varkappa_{a+1} = u \cdot \varkappa_{a+1}\tilde{\sigma}_{1(a+1)}$ on $\tilde{I}_{a+1}$. This relation holds on $E\mathfrak{S}_{a+1}$ because $\varkappa_{a+1}$ is a u-homomorphism of frames. On J^a/J^{a+1} we have $\varkappa_{a+1}\tilde{\sigma}_{1(a+1)} = 0$ by definition. For $y \in \mathfrak{S}_{a+1}$ and $e \in \mathbb{N}^r$ with $e_1 + \cdots + e_r = a$ we compute

$$\tilde{f}_1(\varkappa_{a+1}(x^e y)) = \tilde{f}_1([x^e]\varkappa_{a+1}(y)) = \tilde{f}_1([x^e])f(\varkappa_{a+1}(y)) = 0$$

because $\log([x^e]) = [x^e, 0, 0, \ldots]$ and thus $\tilde{f}_1([x^e]) = 0$. As these x^e generate J^a, the required relation on $\tilde{I}_{a+1}$ follows. Thus the diagram is constructed.

Proof of Theorem 7.3. We use induction on a. The homomorphism $\varkappa_1$ is crystalline because it is invertible. Assume that $\varkappa_a$ is crystalline for some positive integer a and consider the diagram (7.1). The homomorphism π' is crystalline by Proposition 5.2, while π is crystalline by Theorem 3.2; the required filtration of J^a/J^{a+1} is trivial. Hence $\tilde{\varkappa}_{a+1}$ is crystalline. By Lemma 4.2, lifts of windows under ι or under ι' are classified by lifts of the Hodge filtration. Since $\varkappa_{a+1}$ lies over the identity of R_{a+1} and since $\tilde{\varkappa}_{a+1}$ lies over the identity of R_a, it follows that $\varkappa_{a+1}$ is crystalline too. $\square$

Proof of Theorem 7.2. The frame homomorphism $\varkappa : \mathscr{B} \to \mathscr{D}_R$ is the projective limit of the frame homomorphisms $\varkappa_a : \mathscr{B}_a \to \mathscr{D}_{R_a}$. By Lemma 2.12, $\mathscr{B}$-windows are equivalent to compatible systems of $\mathscr{B}_a$-windows for $a \geq 1$, and $\mathscr{D}_R$-windows are equivalent to compatible systems of $\mathscr{D}_{R_a}$-windows for $a \geq 1$. Thus Theorem 7.2 follows from Theorem 7.3. $\square$

8 Classification of group schemes

The following consequences of Theorem 7.2 are analogous to [VZ1]. Recall that we assume $p \geq 3$. Let $\mathscr{B} = (\mathfrak{S}, I, R, \sigma, \sigma_1)$ be the frame defined in section 7.

Definition 8.1. A Breuil window relative to $\mathfrak{S} \to R$ is a pair (Q, ϕ) where Q is a free $\mathfrak{S}$-module of finite rank and where $\phi : Q \to Q^{(\sigma)}$ is an $\mathfrak{S}$-linear map with cokernel annihilated by E.

Lemma 8.2. *Breuil windows relative to $\mathfrak{S} \to R$ are equivalent to $\mathscr{B}$-windows in the sense of Definition 2.3.*

Proof. This is similar to [VZ1, Lemma 1]. For a $\mathscr{B}$-window (P, Q, F, F_1) the module Q is free over $\mathfrak{S}$ because $I = E\mathfrak{S}$ is free. Hence $F_1^\sharp : Q^{(\sigma)} \to P$ is bijective, and we can define a Breuil window (Q, ϕ) where ϕ is the inclusion $Q \to P$ composed with the inverse of $F_1^\sharp$. Conversely, if (Q, ϕ) is a Breuil window, $\mathrm{Coker}(\phi)$ is a free R-module. Indeed, ϕ is injective because it becomes bijective over $\mathfrak{S}[E^{-1}]$, so $\mathrm{Coker}(\phi)$ has projective dimension at most one over $\mathfrak{S}$, which implies that it is free over R by using depth. Thus one can define a $\mathscr{B}$-window as follows: $P = Q^{(\sigma)}$, the inclusion $Q \to P$ is ϕ, $F_1 : Q \to Q^{(\sigma)}$ is given by $x \mapsto 1 \otimes x$, and $F(x) = F_1(Ex)$. The two constructions are mutually inverse. $\square$

By [Z2], p-divisible groups over R are equivalent to Dieudonné displays over R. Together with Theorem 7.2 and Lemma 8.2 this implies:

Corollary 8.3. *The category of p-divisible groups over R is equivalent to the category of Breuil windows relative to $\mathfrak{S} \to R$.* $\square$

Let us use the following abbreviation: An *admissible torsion $\mathfrak{S}$-module* is a finitely generated $\mathfrak{S}$-module annihilated by a power of p and of projective dimension at most one.

Definition 8.4. A Breuil module relative to $\mathfrak{S} \to R$ is a triple (M, φ, ψ) where M is an admissible torsion $\mathfrak{S}$-module together with $\mathfrak{S}$-linear maps $\varphi : M \to M^{(\sigma)}$ and $\psi : M^{(\sigma)} \to M$ such that $\varphi\psi = E$ and $\psi\varphi = E$.

When R has characteristic zero, each of the maps φ and ψ determines the other one; see Lemma 8.6 below.

Theorem 8.5. *The category of (commutative) finite flat group schemes over R annihilated by a power of p is equivalent to the category of Breuil modules relative to $\mathfrak{S} \to R$.*

This follows from Corollary 8.3 by the arguments of [K1] or [VZ1]. For completeness we give a detailed proof here.

Proof of Theorem 8.5. In this proof, all finite flat group schemes are of p-power order over R, and all Breuil modules or windows are relative to $\mathfrak{S} \to R$.
A homomorphism $g : (Q_0, \phi_0) \to (Q_1, \phi_1)$ of Breuil windows is called an isogeny if it becomes invertible over $\mathfrak{S}[1/p]$. Then g is injective, and its cokernel is naturally a Breuil module; the required ψ is induced by the $\mathfrak{S}$-linear map $E\phi_1^{-1} : Q_1^{(\sigma)} \to Q_1$. A homomorphism $\gamma : G_0 \to G_1$ of p-divisible groups is called an isogeny if it becomes invertible in $\mathrm{Hom}(G_0, G_1) \otimes \mathbb{Q}$. Then γ is a surjection of fppf sheaves, and its kernel is a finite flat group scheme.
We denote isogenies by $X_* = [X_0 \to X_1]$. A homomorphism of isogenies $q : X_* \to Y_*$ is called a quasi-isomorphism if its cone is a short exact sequence. In the case of p-divisible groups this means that q induces an isomorphism of finite flat group schemes on the kernels; in the case of Breuil windows this means that q induces an isomorphism of Breuil modules on the cokernels.
The equivalence between p-divisible groups and Breuil windows preserves isogenies and short exact sequences, and thus also quasi-isomorphisms of isogenies. We note the following two facts.

(a) Each finite flat group scheme over R of p-power order is the kernel of an isogeny of p-divisible groups over R. See [BBM, Théorème 3.1.1].

(b) Each Breuil module is the cokernel of an isogeny of Breuil windows. This is analogous to [VZ1, Proposition 2]; a proof is also given below.

Let us define an additive functor $H \mapsto M(H)$ from finite flat group schemes to Breuil modules. We write each H as the kernel of an isogeny of p-divisible groups $G_0 \to G_1$ and define $M(H)$ as the cokernel of the associated isogeny of Breuil windows. Assume that $h : H \to H'$ is a homomorphism of finite flat group schemes, and H' is written as the kernel of an isogeny of p-divisible groups $G_0' \to G_1'$. We embed H into $G_0'' = G_0 \oplus G_0'$ by $(1, h)$ and define $G_1'' = G_0''/H$. The coordinate projections $G_0 \leftarrow G_0'' \to G_0'$ induce homomorphisms of isogenies $G_* \leftarrow G_*'' \to G_*'$ such that the first map is a quasi-isomorphism, and the composition induces h on the kernels. Let $Q_* \leftarrow Q_*'' \to Q_*'$ be the associated homomorphisms of isogenies of Breuil windows. The first map is a quasi-isomorphism, and the composition induces a homomorphism $M(h) : M(H) \to M(H')$ on the cokernels.

One has to show that the construction is independent of the choice and defines an additive functor. This is an easy verification based on the following observation: If a homomorphism of isogenies of p-divisible groups $q : G_* \to G'_*$ induces zero on the kernels, then q is null-homotopic.

The construction of an additive functor $M \mapsto H(M)$ from Breuil modules to finite flat group schemes is analogous. Each M is written as the cokernel of an isogeny of Breuil windows $Q_0 \to Q_1$, and $H(M)$ is defined as the kernel of the associated isogeny of p-divisible groups. If $m : M \to M'$ is a homomorphism of Breuil modules and if M' is written as the cokernel of an isogeny of Breuil windows $Q'_0 \to Q'_1$, let Q''_0 be the kernel of the surjection $Q''_1 = Q_1 \oplus Q'_1 \to M'$ given by $(m, 1)$. The coordinate inclusions $Q_1 \to Q''_1 \leftarrow Q'_1$ induce homomorphisms of isogenies $Q_* \to Q''_* \leftarrow Q'_*$, where the second map is a quasi-isomorphism. The associated homomorphisms of isogenies of p-divisible groups induce a homomorphism of finite flat group schemes $H(m) : H(M) \to H(M')$ on the kernels.

Again, it is easy to verify that this construction is independent of the choice and defines an additive functor, using that a homomorphism of isogenies of Breuil windows is null-homotopic if and only if it induces zero on the cokernels. Clearly the two functors are mutually inverse.

Finally, let us prove (b). If (M, φ, ψ) is a Breuil module, one can find free $\mathfrak{S}$-modules P and Q together with surjective $\mathfrak{S}$-linear maps $\xi : Q \to M$ and $\xi' : P \to M^{(\sigma)}$ and $\mathfrak{S}$-linear maps $\tilde{\varphi} : Q \to P$ and $\tilde{\psi} : P \to Q$ which lift φ and ψ such that $\tilde{\varphi}\tilde{\psi} = E$ and $\tilde{\psi}\tilde{\varphi} = E$. Next one can choose an isomorphism $\alpha : P \cong Q^{(\sigma)}$ compatible with the projections ξ' and $\xi^{(\sigma)}$ to $M^{(\sigma)}$. Let $\phi = \alpha\tilde{\varphi}$. Then (Q, ϕ) is a Breuil window, and (M, φ, ψ) is the cokernel of the isogeny of Breuil windows $(\operatorname{Ker} \xi, \phi') \to (Q, \phi)$, where ϕ' is the restriction of ϕ. □

Lemma 8.6. *If R has characteristic zero, the category of Breuil modules relative to $\mathfrak{S} \to R$ is equivalent to the category of pairs (M, φ) where M is an admissible torsion $\mathfrak{S}$-module and where $\varphi : M \to M^{(\sigma)}$ is an $\mathfrak{S}$-linear map with cokernel annihilated by E.*

Proof. Cf. [VZ1, Proposition 2]. For a non-zero admissible torsion $\mathfrak{S}$-module M the set of zero divisors on M is equal to $\mathfrak{p} = p\mathfrak{S}$ because every associated prime of M has height one and contains p. In particular, $M \to M_{\mathfrak{p}}$ is injective. The hypothesis of the lemma means that $E \notin \mathfrak{p}$. For a given pair (M, φ) as in the lemma this implies that $\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to M_{\mathfrak{p}}^{(\sigma)}$ is surjective, thus bijective because both sides have the same finite length. It follows that φ is injective, and (M, φ) is extended uniquely to a Breuil module by $\psi(x) = \varphi^{-1}(Ex)$. □

Duality

The dual of a Breuil window (Q, ϕ) is the Breuil window $(Q, \phi)^t = (Q^\vee, \psi^\vee)$ where $Q^\vee = \operatorname{Hom}_{\mathfrak{S}}(Q, \mathfrak{S})$ and where $\psi : Q^{(\sigma)} \to Q$ is the unique $\mathfrak{S}$-linear map with $\psi\phi = E$. Here we identify $(Q^{(\sigma)})^\vee$ and $(Q^\vee)^{(\sigma)}$. For a p-divisible group

G over R let $G^\vee$ be the Serre dual of G, and let $\mathbb{M}(G)$ be the Breuil window associated to G by the equivalence of Corollary 8.3.

Proposition 8.7. *There is a functorial isomorphism* $\lambda_G : \mathbb{M}(G^\vee) \cong \mathbb{M}(G)^t$.

Proof. The equivalence between p-divisible groups over R and Dieudonné displays over R is compatible with duality by [L2, Theorem 3.4]. It is easy to see that the equivalence of Lemma 8.2 preserves duality, so it remains to show that the functor $\varkappa_*$ preserves duality as well. By Lemma 2.14 it suffices to find a unit $c \in \mathbb{W}(R)$ with $c^{-1}f(c) = u$. Since E has constant term p, u maps to 1 in $W(k)$ and thus lies in $1 + \hat{W}(\mathfrak{m}_R)$. Hence we can define c^{-1} by the infinite product $uf(u)f^2(u)\cdots$, which converges in $\mathbb{W}(R) = \varprojlim \mathbb{W}(R/\mathfrak{m}^n)$ in the sense that for each n, all but finitely many factors map to 1 in $\mathbb{W}(R/\mathfrak{m}^n)$. $\square$

The dual of a Breuil module $\mathbb{M} = (M, \varphi, \psi)$ is defined as the Breuil module $\mathbb{M}^t = (M^\star, \psi^\star, \varphi^\star)$ where $M^\star = \operatorname{Ext}^1_{\mathfrak{S}}(M, \mathfrak{S})$. Here we identify $(M^{(\sigma)})^\star$ and $(M^\star)^{(\sigma)}$ using that $(\ )^{(\sigma)}$ preserves projective resolutions as σ is flat. For a finite flat group scheme H over R of p-power order let $H^\vee$ be the Cartier dual of H and let $\mathbb{M}(H)$ be the Breuil module associated to H by the equivalence of Theorem 8.5.

Proposition 8.8. *There is a functorial isomorphism* $\lambda_H : \mathbb{M}(H^\vee) \cong \mathbb{M}(H)^t$.

Proof. Choose an isogeny of p-divisible groups $G_0 \to G_1$ with kernel H. Then $\mathbb{M}(H)$ is the cokernel of $\mathbb{M}(G_0) \to \mathbb{M}(G_1)$, which implies that $\mathbb{M}(H)^t$ is the cokernel of $\mathbb{M}(G_1)^t \to \mathbb{M}(G_0)^t$. On the other hand, $H^\vee$ is the kernel of $G_1^\vee \to G_0^\vee$, so $\mathbb{M}(H^\vee)$ is the cokernel of $\mathbb{M}(G_1^\vee) \to \mathbb{M}(G_0^\vee)$. The isomorphisms λ_{G_i} of Proposition 8.7 give an isomorphism $\lambda_H : \mathbb{M}(H^\vee) \cong \mathbb{M}(H)^t$. One easily checks that λ_H is independent of the choice of G_* and functorial in H. $\square$

9 Other lifts of Frobenius

One may ask how much freedom we have in the choice of σ for the frame $\mathscr{B}$. Let $R = \mathfrak{S}/E\mathfrak{S}$ be as in section 7; in particular we assume that $p \geq 3$. Let $J = (x_1, \ldots, x_r)$. To begin with, let $\sigma : \mathfrak{S} \to \mathfrak{S}$ be an arbitrary ring endomorphism such that $\sigma(J) \subset J$ and $\sigma(a) \equiv a^p$ modulo $p\mathfrak{S}$ for $a \in \mathfrak{S}$. We consider the frame

$$\mathscr{B} = (\mathfrak{S}, I, R, \sigma, \sigma_1)$$

with $\sigma_1(Ey) = \sigma(y)$. Again this is a $\varkappa$-frame; the proof of Lemma 7.1 uses only that σ preserves J. Thus Proposition 6.3 gives a homomorphism of frames

$$\varkappa : \mathscr{B} \to \mathscr{W}_R.$$

By the assumptions on σ we have $\sigma(J) \subseteq J^p + pJ$, which implies that the endomorphism $\sigma : J/J^2 \to J/J^2$ is divisible by p.

Proposition 9.1. *The image of* $\varkappa : \mathfrak{S} \to W(R)$ *lies in* $\mathbb{W}(R)$ *if and only if the endomorphism* σ/p *of* J/J^2 *is nilpotent modulo* p.

We have a non-additive map $\tau : J \to J$ given by $\tau(x) = (\sigma(x) - x^p)/p$. Let $\mathfrak{m}$ be the maximal ideal of $\mathfrak{S}$. We write $gr_n(J) = \mathfrak{m}^n J/\mathfrak{m}^{n+1} J$.

Lemma 9.2. *For $n \geq 0$ the map τ preserves $\mathfrak{m}^n J$ and induces a σ-linear endomorphism of k-modules $gr_n(\tau) : gr_n(J) \to gr_n(J)$. We have $gr_0(\tau) = \sigma/p$ as an endomorphism of $gr_0(J) = J/(J^2 + pJ)$. There is a commutative diagram of the following type with $\pi i = \mathrm{id}$*

$$\begin{tikzcd} gr_n(J) \arrow[r, "gr_n(\tau)"] \arrow[d, "\pi"'] & gr_n(J) \\ gr_0(J) \arrow[r, "gr_0(\tau)"] & gr_0(J). \arrow[u, "i"'] \end{tikzcd}$$

Proof. Let $J' = p^{-1}\mathfrak{m}J$ as an $\mathfrak{S}$-submodule of $J \otimes \mathbb{Q}$. Then $J \subset J'$, and $gr_n(J)$ is an $\mathfrak{S}$-submodule of $gr_n(J') = \mathfrak{m}^n J'/\mathfrak{m}^{n+1} J'$. The composition $J \xrightarrow{\tau} J \subset J'$ can be written as $\tau = \sigma/p - \varphi/p$, where $\varphi(x) = x^p$. One checks that $\varphi/p : \mathfrak{m}^n J \to \mathfrak{m}^{n+1} J'$ (which requires $p \geq 3$ when $n = 0$) and that $\sigma/p : \mathfrak{m}^n J \to \mathfrak{m}^n J'$. Hence σ/p and τ induce the same map $\mathfrak{m}^n J \to gr_n(J')$. This map is σ-linear and zero on $\mathfrak{m}^{n+1} J$ because this holds for σ/p, and its image lies in $gr_n(J)$ because this is true for τ.
We define $i : gr_0(J) \to gr_n(J)$ by $x \mapsto p^n x$. For $n \geq 1$ let K_n be the image of $\mathfrak{m}^{n-1} J^2 \to gr_n(J)$. Then i maps $gr_0(J)$ bijectively onto $gr_n(J)/K_n$, so there is a unique homomorphism $\pi : gr_n(J) \to gr_0(J)$ with kernel K_n such that $\pi i = \mathrm{id}$. Clearly i commutes with $gr(\tau)$. Thus, in order that the diagram commutes, it suffices that $gr_n(\tau)$ vanishes on K_n. We have $\sigma(J) \subseteq \mathfrak{m}J$, which implies that $(\sigma/p)(\mathfrak{m}^{n-1} J^2) \subseteq \mathfrak{m}^{n+1} J'$, and the assertion follows. □

Proof of Proposition 9.1. Recall that $\varkappa = \pi\delta$, where $\delta : \mathfrak{S} \to W(\mathfrak{S})$ is defined by $w_n \delta = \sigma^n$ for $n \geq 0$, and where $\pi : W(\mathfrak{S}) \to W(R)$ is the obvious projection. For $x \in J$ and $n \geq 1$ let

$$\tau_n(x) = (\sigma(x)^{p^{n-1}} - x^{p^n})/p^n,$$

thus $\tau_1 = \tau$. It is easy to see that

$$\tau_{n+1}(x) \in J \cdot \tau_n(x),$$

in particular we have $\tau_n : J \to J^n$. If $\delta(x) = (y_0, y_1, \ldots)$, the coefficients y_n are determined by $y_0 = x$ and $w_n(y) = \sigma w_{n-1}(y)$ for $n \geq 1$, which translates into the equations

$$y_n = \tau_n(y_0) + \tau_{n-1}(y_1) + \cdots + \tau_1(y_{n-1}).$$

Assume now that σ/p is nilpotent on J/J^2 modulo p. By Lemma 9.2 this implies that $gr_n(\tau)$ is nilpotent for every $n \geq 0$. We will show that for $x \in J$ the element $\delta(x)$ lies is $\mathbb{W}(\mathfrak{S})$, which means that the above sequence (y_n) converges to zero. Assume that for some $N \geq 0$ we have $y_n \in \mathfrak{m}^N J$ for all but finitely

many n. The last two displayed equations give that $y_n - \tau(y_{n-1}) \in \mathfrak{m}^{N+1}J$ for all but finitely many n. As $gr_N(\tau)$ is nilpotent it follows that $y_n \in \mathfrak{m}^{N+1}J$ for all but finitely many n. Thus $\delta(x) \in \mathbb{W}(\mathfrak{S})$ and in particular $\varkappa(x) \in \mathbb{W}(R)$.
Conversely, if σ/p is not nilpotent on J/J^2 modulo p, then $gr_0(\tau)$ is not nilpotent by Lemma 9.2, so there is an $x \in J$ such that $\tau^n(x) \notin \mathfrak{m}J$ for all $n \geq 0$. For $\delta(x) = (y_0, y_1, \ldots)$ we have $y_n \equiv \tau^n x$ modulo $\mathfrak{m}J$. The projection $\mathfrak{S} \to R$ induces an isomorphism $J/\mathfrak{m}J \cong \mathfrak{m}_R/\mathfrak{m}_R^2$. It follows that $\varkappa(x)$ lies in $W(\mathfrak{m}_R)$ but not in $\hat{W}(\mathfrak{m}_R)$, thus $\varkappa(x) \notin \mathbb{W}(R)$. □

Now we assume that σ/p is nilpotent on J/J^2 modulo p. Then we have a homomorphism of frames

$$\varkappa : \mathscr{B} \to \mathscr{D}_R.$$

As earlier let $\mathscr{B}_a = (\mathfrak{S}_a, I_a, R_a, \sigma_a, \sigma_{1a})$ with $\mathfrak{S}_a = \mathfrak{S}/J^a$ and $R_a = R/\mathfrak{m}_R^a$. The proof of Lemma 7.1 shows that $\mathscr{B}_a$ is a $\varkappa$-frame. Since $\mathbb{W}(R_a)$ is the image of $\mathbb{W}(R)$ in $W(R_a)$, we get a homomorphism of frames compatible with $\varkappa$:

$$\varkappa_a : \mathscr{B}_a \to \mathscr{D}_{R_a}.$$

Theorem 9.3. *The homomorphisms $\varkappa$ and $\varkappa_a$ are crystalline.*

Proof. The proof is similar to that of Theorems 7.2 and 7.3.
First we repeat the construction of the diagram (7.1). The restriction of $\sigma_{1(a+1)}$ to $E(J^a/J^{a+1}) = p(J^a/J^{a+1})$ is given by $\sigma_1 = \sigma/p = \tau$, which need not be zero in general, but still σ_1 extends uniquely to J^a/J^{a+1} by the formula $\sigma_1 = \sigma/p$. In order that $\tilde{\varkappa}_{a+1}$ is a u-homomorphism of frames we need that $\tilde{f}_1\varkappa_{a+1} = u \cdot \varkappa_{a+1}\tilde{\sigma}_{1(a+1)}$ on J^a/J^{a+1}. Here u acts on J^a/J^{a+1} as the identity. By the proof of Proposition 9.1, for $x \in J^a/J^{a+1}$ we have in $W(J^a/J^{a+1})$

$$\delta(x) = (x, \tau(x), \tau^2(x), \ldots).$$

Since $\tilde{\sigma}_{1(a+1)}(x) = \tau(x)$, the required relation follows easily.
To complete the proof we have to show that $\pi : \tilde{\mathscr{B}}_{a+1} \to \mathscr{B}_a$ is crystalline. Now σ/p is nilpotent modulo p on J^n/J^{n+1} for $n \geq 1$. Indeed, for $n = 1$ this is our assumption, and for $n \geq 2$ the endomorphism σ/p of J^n/J^{n+1} is divisible by p^{n-1} since $\sigma(J) \subseteq pJ + J^p$. In order to apply Theorem 3.2 we need another sequence of auxiliary frames: For $c \in \mathbb{N}$ let $\mathfrak{S}_{a+1,c} = \mathfrak{S}_{a+1}/p^cJ^a\mathfrak{S}_{a+1}$ and let $\tilde{\mathscr{B}}_{a+1,c} = (\mathfrak{S}_{a+1,c}, I_{a+1,c}, R_a, \ldots)$ be the obvious quotient frame of $\tilde{\mathscr{B}}_{a+1}$. Then $\mathscr{B}_a$ is isomorphic to $\tilde{\mathscr{B}}_{a+1,0}$, and $\tilde{\mathscr{B}}_{a+1}$ is the projective limit of $\tilde{\mathscr{B}}_{a+1,c}$ for $c \to \infty$. Theorem 3.2 shows that each projection $\tilde{\mathscr{B}}_{a+1,c+1} \to \tilde{\mathscr{B}}_{a+1,c}$ is crystalline, which implies that π is crystalline by Lemma 2.12. □

If σ/p is nilpotent on J/J^2 modulo p, then Corollary 8.3, Theorem 8.5, and the duality Propositions 8.7 and 8.8 follow as before.

10 Nilpotent windows

All results in this article have a nilpotent counterpart where only connected p-divisible groups and nilpotent windows are considered; in this case k need not be perfect and p need not be odd. The necessary modifications are standard, but for completeness we work out the details.

10.1 Nilpotence condition

Let $\mathscr{F} = (S, I, R, \sigma, \sigma_1)$ be a frame. For an $\mathscr{F}$-window $\mathscr{P} = (P, Q, F, F_1)$ there is a unique S-linear map

$$V^\sharp : P \to P^{(\sigma)}$$

with $V^\sharp(F_1(x)) = 1 \otimes x$ for $x \in Q$. In terms of a normal representation $\Psi : L \oplus T \to P$ of $\mathscr{P}$ we have $V^\sharp = (1 \oplus \theta)(\Psi^\sharp)^{-1}$ for θ as in Lemma 2.2. For simplicity, the composition

$$P \xrightarrow{V^\sharp} P^{(\sigma)} \xrightarrow{(V^\sharp)^{(\sigma)}} P^{(\sigma^2)} \to \cdots \to P^{(\sigma^n)}$$

is denoted $(V^\sharp)^n$. The nilpotence condition depends on the choice of an ideal $J \subset S$ such that $\sigma(J) + I + \theta S \subseteq J$, which we call an *ideal of definition* for $\mathscr{F}$.

Definition 10.1. Let $J \subset S$ be an ideal of definition for $\mathscr{F}$. An $\mathscr{F}$-window $\mathscr{P}$ is called nilpotent (with respect to J) if $(V^\sharp)^n \equiv 0$ modulo J for sufficiently large n.

Remark 10.2. For an $\mathscr{F}$-window $\mathscr{P}$ we consider the composition

$$\lambda : L \subseteq L \oplus T \xrightarrow{(\Psi^\sharp)^{-1}} L^{(\sigma)} \oplus T^{(\sigma)} \to L^{(\sigma)}.$$

Then $\mathscr{P}$ is nilpotent if and only if λ is nilpotent modulo J.

10.2 Nil-crystalline homomorphisms

If $\alpha : \mathscr{F} \to \mathscr{F}'$ is a homomorphism of frames and $J \subset S$ and $J' \subset S'$ are ideals of definition with $\alpha(J) \subseteq J'$, the functor α_* preserves nilpotent windows. We call α nil-crystalline if it induces an equivalence between nilpotent $\mathscr{F}$-windows and nilpotent $\mathscr{F}'$-windows. The following variant of Theorem 3.2 formalises [Z1, Theorem 44].

Theorem 10.3. *Let $\alpha : \mathscr{F} \to \mathscr{F}'$ be a homomorphism of frames that induces an isomorphism $R \cong R'$ and a surjection $S \to S'$ with kernel $\mathfrak{a} \subset S$. We assume that there is a finite filtration $\mathfrak{a} = \mathfrak{a}_0 \supseteq \cdots \supseteq \mathfrak{a}_n = 0$ such that $\sigma(\mathfrak{a}_i) \subseteq \mathfrak{a}_{i+1}$ and $\sigma_1(\mathfrak{a}_i) \subseteq \mathfrak{a}_i$. We assume that finitely generated projective S'-modules lift to projective S-modules. If $J \subset S$ is an ideal of definition for $\mathscr{F}$ such that $J^n \mathfrak{a} = 0$ for large n, then α is nil-crystalline with respect to $J \subset S$ and $J' = J/\mathfrak{a} \subset S'$.*

Proof. The assumptions imply that $\mathfrak{a} \subseteq I \subseteq J$, in particular J' is well-defined. An $\mathscr{F}$-window $\mathscr{P}$ is nilpotent if and only if $\alpha_*\mathscr{P}$ is nilpotent. Using this, the proof of Theorem 3.2 applies with the following modification in its final paragraph. We claim that the endomorphism U of $\mathscr{H}$ is nilpotent, which again implies that $1-U$ is bijective. Since $\mathscr{P}$ is nilpotent, λ is nilpotent modulo J, so λ is nilpotent modulo J^n for each $n \geq 1$ as J is stable under σ. Since $J^n\mathfrak{a} = 0$ by assumption, the claim follows from the definition of U. $\square$

10.3 Nilpotent displays

Let R be a ring which is complete and separated in the $\mathfrak{c}$-adic topology for an ideal $\mathfrak{c} \subset R$ containing p. We consider the Witt frame

$$\mathscr{W}_R = (W(R), I_R, R, f, f_1).$$

Here $I_R \subseteq \operatorname{Rad} R$ as required since $W(R) = \varprojlim W_n(R/\mathfrak{c}^n)$ and the successive kernels in this projective system are nilpotent. The inverse image of $\mathfrak{c}$ is an ideal of definition $J \subset W(R)$. Nilpotent windows over $\mathscr{W}_R$ with respect to J are displays over R which are nilpotent over $R/\mathfrak{c}$. By [Z1] and [L1] these are equivalent to p-divisible groups over R which are infinitesimal over $R/\mathfrak{c}$. (Here one uses that displays and p-divisible groups over R are equivalent to compatible systems of the same objects over $R/\mathfrak{c}^n$ for $n \geq 1$; cf. Lemma 2.12 above and [M1, Lemma 4.16].)
Assume that $R' = R/\mathfrak{b}$ for a closed ideal $\mathfrak{b} \subseteq \mathfrak{c}$ equipped with (not necessarily nilpotent) divided powers. One can define a factorisation

$$\mathscr{W}_R \xrightarrow{\alpha_1} \mathscr{W}_{R/R'} = (W(R), I_{R/R'}, R', f, \tilde{f}_1) \xrightarrow{\alpha_2} \mathscr{W}_{R'}$$

of the projection of frames $\mathscr{W}_R \to \mathscr{W}_{R'}$ as follows. Necessarily $I_{R/R'} = I_R + W(\mathfrak{b})$. The divided Witt polynomials define an isomorphism

$$\log : W(\mathfrak{b}) \cong \mathfrak{b}^\infty,$$

and $\tilde{f}_1 : I_{R/R'} \to W(R)$ extends f_1 such that $\tilde{f}_1([b_0, b_1, \ldots]) = [b_1, b_2, \ldots]$ in logarithmic coordinates on $W(\mathfrak{b})$. Let $J' \subset W(R')$ be the image of J. This is an ideal of definition for $\mathscr{W}_{R'}$, and J is an ideal of definition for $\mathscr{W}_{R/R'}$.
We assume that the $\mathfrak{c}$-adic topology of R can be defined by a sequence of ideals $R \supset I_1 \supset I_2 \cdots$ such that $\mathfrak{b} \cap I_n$ is stable under the divided powers of $\mathfrak{b}$ for each n. This is automatic when $\mathfrak{c}$ is nilpotent or when R is noetherian; see the proof of Proposition 5.2.

Proposition 10.4. *The homomorphism α_2 is nil-crystalline with respect to the ideals of definition J for $\mathscr{W}_{R/R'}$ and J' for $\mathscr{W}_{R'}$.*

This is essentially [Z1, Theorem 44].

Proof. By a limit argument the assertion is reduced to the case where $\mathfrak{c} \subset R$ is a nilpotent ideal; see Lemma 2.12. Then Theorem 10.3 applies: The required filtration of $\mathfrak{a} = W(\mathfrak{b})$ is $\mathfrak{a}_i = p^i\mathfrak{a}$. The condition $J^n\mathfrak{a} = 0$ for large n is satisfied because $J^n \subseteq I_R$ for some n and $I_R^{n+1} \subseteq p^n W(R)$ for all n, and $W(\mathfrak{b}) \cong \mathfrak{b}^\infty$ is annihilated by some power of p. $\square$

10.4 The main frame

Let now R be a complete regular local ring with arbitrary residue field k of characteristic p. Let C be a complete discrete valuation ring with maximal ideal pC and residue field k. We choose a surjective ring homomorphism

$$\mathfrak{S} = C[[x_1, \ldots, x_r]] \to R$$

that lifts the identity of k such that $x_1, \ldots, x_r$ map to a regular system of parameters for R. There is a power series $E \in \mathfrak{S}$ with constant term p such that $R = \mathfrak{S}/E\mathfrak{S}$. Let $\sigma : \mathfrak{S} \to \mathfrak{S}$ be a ring endomorphism which induces the Frobenius on $\mathfrak{S}/p\mathfrak{S}$ and preserves the ideal $(x_1, \ldots, x_r)$. Such σ exist because C has a Frobenius lift; see [Gr, Chap. 0, Théorème 19.8.6]. We consider the frame

$$\mathscr{B} = (\mathfrak{S}, I, R, \sigma, \sigma_1)$$

where $\sigma_1(Ey) = \sigma(y)$. Here $\theta = \sigma(E)$. The proof of Lemma 7.1 shows that $\mathscr{B}$ is again a $\varkappa$-frame, so we have a u-homomorphism of frames

$$\varkappa : \mathscr{B} \to \mathscr{W}_R.$$

Let $\mathfrak{m} \subset \mathfrak{S}$ and $\mathfrak{n} \subset W(R)$ be the maximal ideals.

Theorem 10.5. *The homomorphism $\varkappa$ is nil-crystalline with respect to the ideals of definition $\mathfrak{m}$ of $\mathscr{B}$ and $\mathfrak{n}$ of $\mathscr{W}_R$.*

Proof. The proof of Theorem 9.3 applies with the following modification: The initial case $a = 1$ is not trivial because C is not isomorphic to $W(k)$ if k is not perfect, but one can apply [Z3, Theorem 1.6]. In the diagram (7.1) the frame homomorphisms π' and π are only nil-crystalline in general; whether π is crystalline depends on the choice of σ. $\square$

10.5 Connected group schemes

One defines Breuil windows relative to $\mathfrak{S} \to R$ and Breuil modules relative to $\mathfrak{S} \to R$ as before. A Breuil window (Q, ϕ) or a Breuil module (M, φ, ψ) is called nilpotent if ϕ or φ is nilpotent modulo the maximal ideal of $\mathfrak{S}$. The proof of Lemma 8.2 shows that nilpotent Breuil windows are equivalent to nilpotent $\mathscr{B}$-windows. Hence Theorem 10.5 implies:

Corollary 10.6. *Connected p-divisible groups over R are equivalent to nilpotent Breuil windows relative to $\mathfrak{S} \to R$.* $\square$

Similarly we have:

Theorem 10.7. *Connected finite flat group schemes over R of p-power order are equivalent to nilpotent Breuil modules relative to $\mathfrak{S} \to R$.*

This is proved like Theorem 8.5, using two additional remarks:

Lemma 10.8. *Every connected finite flat group scheme H over R is the kernel of an isogeny of connected p-divisible groups.*

Proof. We know that H is the kernel of an isogeny of p-divisible groups $G \to G'$. There is a functorial exact sequence $0 \to G_0 \to G \to G_1 \to 0$ of p-divisible groups where G_0 is connected and G_1 is etale. Since $\mathrm{Hom}(H, G_1)$ is zero, H is the kernel of the isogeny $G_0 \to G_0'$. □

Lemma 10.9. *Every nilpotent Breuil module (M, φ, ψ) relative to $\mathfrak{S} \to R$ is the cokernel of an isogeny of nilpotent Breuil windows.*

Proof. See also [K2, Section 1.3]. As in the proof of Theorem 8.5 we see that (M, φ, ψ) is the cokernel of an isogeny of Breuil windows $(Q, \phi) \to (Q', \phi')$. There is a functorial exact sequence $0 \to Q_0 \to Q \to Q_1 \to 0$ of Breuil windows where Q_0 is nilpotent and where Q_1 is etale in the sense that $\phi : Q_1 \to Q_1^{(\sigma)}$ is bijective. Indeed, by [Z2, Lemma 10] it suffices to construct the sequence over k. Let $\phi_k : Q \otimes_{\mathfrak{S}} k \to Q^{(\sigma)} \otimes_{\mathfrak{S}} k$ be the special fibre of ϕ. Then $Q_0 \otimes_{\mathfrak{S}} k$ is the kernel of the obvious iterate $(\phi_k)^n : Q \otimes_{\mathfrak{S}} k \to Q^{(\sigma^n)} \otimes_{\mathfrak{S}} k$ for large n.
We claim that the free $\mathfrak{S}$-modules Q_1 and Q_1' have the same rank. Let us identify C with $\mathfrak{S}/(x_1, \ldots, x_r)$. Since $Q \to Q'$ becomes bijective over $\mathfrak{S}[1/p]$, the homomorphism $Q \otimes_{\mathfrak{S}} C \to Q' \otimes_{\mathfrak{S}} C$ becomes bijective over $C[1/p]$. Hence the etale parts $(Q \otimes_{\mathfrak{S}} C)_1$ and $(Q' \otimes_{\mathfrak{S}} C)_1$ have the same rank. The claim follows since $(Q \otimes_{\mathfrak{S}} C)_1 = Q_1 \otimes_{\mathfrak{S}} C$ and similarly for Q'.
Let us consider $\bar{M} = Q_1'/Q_1$. Here ϕ' induces a homomorphism $\bar{\varphi} : \bar{M} \to \bar{M}^{(\sigma)}$, which is surjective as Q_1' is etale. The natural surjection $\pi : M \to \bar{M}$ satisfies $\pi^{(\sigma)}\varphi = \bar{\varphi}\pi$. Since φ_k is nilpotent it follows that $\bar{\varphi}_k$ is nilpotent, thus $\bar{M} = 0$ by Nakayama's lemma. Hence $Q_1 \to Q_1'$ is bijective because both sides are free of the same rank, and consequently $M = Q_0'/Q_0$ as desired. □

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