Locally Well Generated Homotopy Categories of Complexes

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Abstract. We show that the homotopy category of complexes $\mathbf{K}(\mathcal{B})$ over any finitely accessible additive category $\mathcal{B}$ is locally well generated. That is, any localizing subcategory $\mathcal{L}$ in $\mathbf{K}(\mathcal{B})$ which is generated by a set is well generated in the sense of Neeman. We also show that $\mathbf{K}(\mathcal{B})$ itself being well generated is equivalent to $\mathcal{B}$ being pure semisimple, a concept which naturally generalizes right pure semisimplicity of a ring R for $\mathcal{B} = \text{Mod-}R$.

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Introduction

The main motivation for this paper is to study when the homotopy category of complexes $\mathbf{K}(\mathcal{B})$ over an additive category $\mathcal{B}$ is compactly generated or, more generally, well generated.

In the last few decades, the theory of compactly generated triangulated categories has become an important tool unifying concepts from various fields of mathematics. Standard examples are the unbounded derived category of a ring or the stable homotopy category of spectra. The key property of such a category $\mathcal{T}$ is the Brown Representability Theorem, cf. [30, 25], originally due to Brown [9]:

> Any contravariant cohomological functor $F : \mathcal{T} \to \mathrm{Ab}$ which sends coproducts to products is representable.

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This theorem is an important tool and has been used in several places. We mention Neeman's proof of the Grothendieck Duality Theorem [30], Krause's work on the Telescope Conjecture [28, 24], or Keller's representation theorem for algebraic compactly generated triangulated categories [23].

Recently, there has been a growing interest in giving criteria for certain homotopy categories $\mathbf{K}(\mathcal{B})$ to be compactly generated, [15, 20, 29, 31]. Here, $\mathcal{B}$ typically was a suitable subcategory of a module category. The main reason for studying such homotopy categories were results concerning the Grothendieck Duality Theorem [17, 31] and relative homological algebra [19]. There is, however, a conceptual reason, too. Namely, every algebraic triangulated category is triangle equivalent to a full subcategory of some homotopy category, [25, §7.5].

It turned out when studying the homotopy category of complexes of projective modules over a ring R in [31] that it is useful to consider well generated triangulated categories in this context. More precisely, $\mathbf{K}(\text{Proj-}R)$ is always well generated, but may not be compactly generated. Well generated categories have been defined by Neeman [32] in a natural attempt to extend results such as the Brown Representability from compactly generated triangulated categories to a wider class of triangulated categories.

Although one has already known for some time that there exist rather natural triangulated categories, such as the homotopy category of complexes of abelian groups, which are not even well generated, one has typically viewed those as rare and exceptional cases.

We will give some arguments to show that this interpretation is not very accurate. First, the categories $\mathbf{K}(\text{Mod-}R)$ for a ring R are rarely well generated. It happens if and only if R is right pure semisimple, which establishes the converse of [15, §4 (3), p. 17]. Moreover, we generalize this result to the homotopy categories $\mathbf{K}(\mathcal{B})$ with $\mathcal{B}$ additive finitely accessible. This way, we obtain a fairly complete answer regarding when $\mathbf{K}(\text{Flat-}R)$ is compactly or well generated, see [15, Question 4.2].

We also give a partial remedy for the typical failure of $\mathbf{K}(\mathcal{B})$ to be well generated. Roughly speaking, the main problem with $\mathbf{K}(\mathcal{B})$, where $\mathcal{B}$ is finitely accessible, is that it may not have any set of generators at all. But if we take a localizing subcategory $\mathcal{L}$ generated by any set of objects, it will automatically be well generated. We will call a triangulated category with this property locally well generated.

We will also give basic properties of locally well generated categories and see that some of the usual results regarding localization hold in the new setting. For example, any localizing subcategory generated by a set of objects is realized as the kernel of a localization endofunctor. This version of a Bousfield localization theorem generalizes [26, §7.2] and [2, 5.7]. However, one has to be more careful. The Brown Representability theorem as stated above does not work for locally well generated categories in general, and there are localizing subcategories which are not associated to any localization endofunctor. We illustrate this in Example 3.7.

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1. PRELIMINARIES

Let $\mathcal{T}$ be a triangulated category. A triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ is called *thick* if, whenever $X \amalg Y \in \mathcal{S}$, then also $X \in \mathcal{S}$. From now on, we will assume that $\mathcal{T}$ has arbitrary (set-indexed) coproducts. A full triangulated subcategory $\mathcal{L} \subseteq \mathcal{T}$ is called *localizing* if it is closed under forming coproducts. Note that by [32, 1.6.8], $\mathcal{T}$ has splitting idempotents and any localizing subcategory $\mathcal{L} \subseteq \mathcal{T}$ is thick.

If $\mathcal{S}$ is any class of objects of $\mathcal{T}$, we denote by $\operatorname{Loc} \mathcal{S}$ the smallest localizing subcategory of $\mathcal{T}$ which contains $\mathcal{S}$. In other words, $\operatorname{Loc} \mathcal{S}$ is the closure of $\mathcal{S}$ under shifts, coproducts and triangle completions.

Given $\mathcal{T}$ and a localizing subcategory $\mathcal{L} \subseteq \mathcal{T}$, one can construct the so-called *Verdier quotient* $\mathcal{T}/\mathcal{L}$ by formally inverting in $\mathcal{T}$ all morphisms in the class $\Sigma(\mathcal{L})$ defined as

$$\Sigma(\mathcal{L}) = \{f \mid \exists \text{ triangle } X \xrightarrow{f} Y \to Z \to X[1] \text{ in } \mathcal{T} \text{ such that } Z \in \mathcal{L}\}.$$

It is a well known fact that the Verdier quotient always has coproducts, admits a natural triangulated structure, and the canonical localization functor $Q : \mathcal{T} \to \mathcal{T}/\mathcal{L}$ is exact and preserves coproducts, [32, Chapter 2]. However, one has to be careful, since $\mathcal{T}/\mathcal{L}$ might not be a usual category in the sense that the homomorphism spaces might be proper classes rather than sets. This fact, although often inessential and neglected, as $\mathcal{T}/\mathcal{L}$ has a very straightforward and constructive description, may nevertheless have important consequences in some cases; see eg. [6].

Let $L : \mathcal{T} \to \mathcal{T}$ be an exact endofunctor of $\mathcal{T}$. Then L is called a *localization functor* if there exists a natural transformation $\eta : \mathrm{Id}_{\mathcal{T}} \to L$ such that $L\eta_X = \eta_{LX}$ and $\eta_{LX} : LX \to L^2X$ is an isomorphism for each $X \in \mathcal{T}$.

It is easy to check that the full subcategory $\operatorname{Ker} L$ of $\mathcal{T}$ given by

$$\operatorname{Ker} L = \{X \in \mathcal{T} \mid LX = 0\}$$

is always localizing [2, 1.2]. Moreover, there is a canonical triangle equivalence between $\mathcal{T}/\operatorname{Ker} L$ and $\operatorname{Im} L$, the essential image of L; see [32, 9.1.16] or [26, 4.9.1]. This among other things implies that all morphism spaces in $\mathcal{T}/\operatorname{Ker} L$ are sets. Note that although $\operatorname{Im} L$ has coproducts as a category, it might *not* be closed under coproducts in $\mathcal{T}$. This type of localization, coming from a localization functor, is often referred to as *Bousfield localization.* However, not every localizing subcategory $\mathcal{L}$ is realized as the kernel of a localization functor, [6, 1.3]. Namely, $\mathcal{L}$ is of the form $\operatorname{Ker} L$ for some localization functor if and only if the inclusion $\mathcal{L} \to \mathcal{T}$ has a right adjoint, [2, 1.6].

A central concept in this paper is that of a well generated triangulated category. Let κ be a regular cardinal number. An object Y in a category with arbitrary coproducts is called κ–*small* provided that every morphism of the form

$$Y \longrightarrow \coprod_{i \in I} X_i$$

factorizes through a subcoproduct $\coprod_{i \in J} X_i$ with $|J| < \kappa$.

Definition 1.1. Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts and κ be a regular cardinal. Then $\mathcal{T}$ is called κ–*well generated* provided there is a set $\mathcal{S}$ of objects of $\mathcal{T}$ satisfying the following conditions:

(1) If $X \in \mathcal{T}$ such that $\mathcal{T}(Y, X) = 0$ for each $Y \in \mathcal{S}$, then $X = 0$;
(2) Each object $Y \in \mathcal{S}$ is κ–small;
(3) For any morphism in $\mathcal{T}$ of the form $f : Y \to \coprod_{i \in I} X_i$ with $Y \in \mathcal{S}$, there exists a family of morphisms $f_i : Y_i \to X_i$ such that $Y_i \in \mathcal{S}$ for each $i \in I$ and f factorizes as

$$Y \longrightarrow \coprod_{i \in I} Y_i \xrightarrow{\coprod f_i} \coprod_{i \in I} X_i.$$

The category $\mathcal{T}$ is called *well generated* if it is κ–well generated for some regular cardinal κ.

This definition differs to some extent from Neeman's original definition in [32, 8.1.7]. The equivalence between the two follows from [27, Theorem A] and [27, Lemmas 4 and 5]. Note that if $\kappa = \aleph_0$, then condition (3) is vacuous and $\aleph_0$–well generated triangulated categories are precisely the *compactly generated* triangulated categories in the usual sense.
The key property of well generated categories is that the Brown Representability Theorem holds:

Proposition 1.2. [32, 8.3.3] *Let $\mathcal{T}$ be a well generated triangulated category. Then:*

(1) *Any contravariant cohomological functor $F : \mathcal{T} \to \mathrm{Ab}$ which takes coproducts to products is, up to isomorphism, of the form $\mathcal{T}(-, X)$ for some $X \in \mathcal{T}$.*
(2) *If $\mathcal{S}$ is a set of objects of $\mathcal{T}$ which meets assumptions* (1), (2) *and* (3) *of Definition 1.1 for some cardinal κ, then $\mathcal{T} = \operatorname{Loc} \mathcal{S}$.*

Next we turn our attention to categories of complexes. Let $\mathcal{B}$ be an additive category. Using a standard notation, we denote by $\mathbf{C}(\mathcal{B})$ the category of chain complexes

$$X : \qquad \cdots \to X^{n-1} \stackrel{d^{n-1}}{\to} X^n \stackrel{d^n}{\to} X^{n+1} \to \dots,$$

of objects of $\mathcal{B}$. By $\mathbf{K}(\mathcal{B})$, we denote the factor-category of $\mathbf{C}(\mathcal{B})$ modulo the ideal of null-homotopic chain complex morphisms. It is well known that $\mathbf{K}(\mathcal{B})$ has a triangulated structure where triangle completions are constructed using mapping cones (see for example [14, Chapter I]). Moreover, if $\mathcal{B}$ has arbitrary coproducts, so have them both $\mathbf{C}(\mathcal{B})$ and $\mathbf{K}(\mathcal{B})$, and the canonical functor $\mathbf{C}(\mathcal{B}) \to \mathbf{K}(\mathcal{B})$ preserves coproducts.

We will often take for $\mathcal{B}$ module categories or their subcategories. In this case, R will denote an associative unital ring and Mod-R the category of all (unital) right R–modules. By Proj-R and Flat-R we denote, respectively, the full subcategories of projective and flat R–modules.

In fact, our considerations will usually work in a more general setting. Let $\mathcal{A}$ be a skeletally small additive category and Mod-$\mathcal{A}$ be the category of all contravariant additive functors $\mathcal{A} \to \mathrm{Ab}$. We will call such functors *right $\mathcal{A}$–modules*. Then Mod-$\mathcal{A}$ shares many formal properties with usual module categories. We refer to [18, Appendix B] for more details. Correspondingly, we denote by Proj-$\mathcal{A}$ the full subcategory of projective functors and by Flat-$\mathcal{A}$ the category of flat functors. We discuss the categories of the form Flat-$\mathcal{A}$ more in detail in Section 4 since those are, up to equivalence, precisely the so called additive finitely accessible categories. Many natural abelian categories are of this form.

Finally, we spend a few words on set-theoretic considerations. All our proofs work in ZFC with an extra technical assumption: the axiom of choice for proper classes. The latter assumption has no algebraic significance, it is only used to keep arguments simple in the following case:

Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant additive functor. If we know, for example by the Brown Representability Theorem, that the composition of functors

$$\mathcal{C} \xrightarrow{\quad F \quad} \mathcal{D} \xrightarrow{\mathcal{D}(-,X)} \mathrm{Ab}$$

is representable for each $X \in \mathcal{D}$, we would like to conclude that F has a right adjoint $G : \mathcal{D} \to \mathcal{C}$. In order to do that, we must for each $Y \in \mathcal{C}$ *choose* one particular value for GY from a class of mutually isomorphic candidates.

2. Pure semisiplicity

A relatively straightforward but crucial obstacle causing a homotopy category of complexes $\mathbf{K}(\mathcal{B})$ not to be well generated is that the additive base category $\mathcal{B}$ is not pure semisimple. Here, we use the following very general definition:

Definition 2.1. An additive category $\mathcal{B}$ with arbitrary coproducts is called *pure semisimple* if it has an additive generator. That is, there is an object $X \in \mathcal{B}$ such that $\mathcal{B} = \mathrm{Add}X$, where $\mathrm{Add}X$ stands for the full subcategory formed by all objects which are summands in (possibly infinite) coproducts of copies of X.

The term is inspired by the case $\mathcal{B} = $ Mod-R, where we have the following proposition:

Proposition 2.2. *A ring R is right pure semisimple (that is, each pure monomorphism between right R–modules splits) if and only if* Mod-R *is pure semisimple in the sense of Definition 2.1.*

Proof. If every pure monomorphism in Mod-R splits, then also every pure epimorphism splits. That is, every module is pure projective, or equivalently a summand in a direct sum of finitely presented modules. By a theorem of

Kaplansky, [21, Theorem 1], it follows that every module is a direct sum of countably generated modules. Hence, Mod-R is pure semisimple according to our definition. In fact, one can show more in this case: Every module is even a direct sum of finitely presented modules; see for example [16] or [18, App. B]. Let us conversely assume that Mod-R is a pure semisimple additive category. Using [3, Theorem 26.1], which is a variation of [21, Theorem 1] for higher cardinalities, we see that if Mod-$R = \mathrm{Add} X$ for some κ–generated module X, then each module in Mod-R is a direct sum of λ–generated modules where $\lambda = \max(\kappa, \aleph_0)$. This fact implies that every module is Σ–pure injective, [12]. In particular, each pure monomorphism in Mod-R splits and R is right pure semisimple. □

If R is an artin algebra, then the conditions of Proposition 2.2 are well-known to be further equivalent to R being of finite representation type; see [4, Theorem A]. For more details and references on this topic, we also refer to [16]. It turns out that the pure semisimplicity condition has a nice interpretation for finitely accessible additive categories as well. We will discuss this more in detail in Section 4.

For giving a connection between pure semisimplicity of $\mathcal{B}$ and properties of $\mathbf{K}(\mathcal{B})$, we recall a structure result for the so-called contractible complexes in $\mathbf{C}(\mathcal{B})$. A complex $Y \in \mathbf{C}(\mathcal{B})$ is *contractible* if it is mapped to a zero object under $\mathbf{C}(\mathcal{B}) \to \mathbf{K}(\mathcal{B})$. It is clear that the complexes of the form

$$I_{X,n}: \qquad \cdots \to 0 \to 0 \to X = X \to 0 \to 0 \to \dots,$$

such that the first X is in degree n, are contractible. Moreover, all other contractible complexes are obtained in the following way:

Lemma 2.3. *Let $\mathcal{B}$ be an additive category with splitting idempotents and $Y \in \mathbf{C}(\mathcal{B})$. Then the following are equivalent:*

(1) *Y is contractible;*
(2) *Y is isomorphic in $\mathbf{C}(\mathcal{B})$ to a complex of the form $\coprod_{n\in\mathbb{Z}} I_{X_n,n}$.*

Proof. (2) $\implies$ (1). This is trivial given the fact that the functor $\mathbf{C}(\mathcal{B}) \to \mathbf{K}(\mathcal{B})$ preserves those componentwise coproducts of complexes which exist in $\mathbf{C}(\mathcal{B})$.
(1) $\implies$ (2). Let us fix a contractible complex in $\mathbf{K}(\mathcal{B})$:

$$Y: \qquad \dots \xrightarrow{d^{n-2}} Y^{n-1} \xrightarrow{d^{n-1}} Y^n \xrightarrow{d^n} Y^{n+1} \xrightarrow{d^{n+1}} \dots.$$

By definition, the identity morphism of Y is homotopy equivalent to the zero morphism in $\mathbf{C}(\mathcal{B})$, so there are morphisms $s^n : Y^n \to Y^{n-1}$ in $\mathcal{B}$ such that

$$1_{Y^n} = d^{n-1}s^n + s^{n+1}d^n.$$

When composing with d^n, we get $d^n = d^n s^{n+1} d^n$, so $s^{n+1}d^n : Y^n \to Y^n$ is idempotent in $\mathcal{B}$ for each $n \in \mathbb{Z}$. Hence there are morphisms $p^n : Y^n \to X_n$ and $j^n : X_n \to Y^n$ in $\mathcal{B}$ such that $p^n j^n = 1_{X_n}$ and $j^n p^n = s^{n+1}d^n$. Let us denote by $f^n : X_{n-1} \amalg X_n \to Y^n$ and $g^n : Y^n \to X_{n-1} \amalg X_n$ the morphisms defined as follows:

$$f^n = (d^{n-1}j^{n-1}, j^n), \quad \text{and} \quad g^n = \begin{pmatrix} p^{n-1}s^n \\ p^n \end{pmatrix}.$$

Using the identities above, it is easy to check that $f^n g^n = 1_{Y^n}$ and $g^n f^n$ is an isomorphism in $\mathcal{B}$ for each n. Therefore, both f^n and g^n are isomorphisms and $g^n f^n$ is the identity morphism. Finally, it is straightforward to check that the family of morphisms $(f_n \mid n \in \mathbb{Z})$ induces an (iso)morphism $f : \coprod_{n\in Z} I_{X_n,n} \to Y$ in $\mathbf{C}(\mathcal{B})$. $\square$

It is not difficult to see that the condition of $\mathcal{B}$ having splitting idempotents is really necessary in Lemma 2.3. However, there is a standard construction which allows us to amend $\mathcal{B}$ with the missing summands if $\mathcal{B}$ does not have splitting idempotents.

DEFINITION 2.4. Let $\mathcal{B}$ be an additive category. Then an additive category $\bar{\mathcal{B}}$ is called an *idempotent completion* of $\mathcal{B}$ if

(1) $\bar{\mathcal{B}}$ has splitting idempotents;
(2) $\mathcal{B}$ is a full subcategory of $\bar{\mathcal{B}}$;
(3) Every object in $\bar{\mathcal{B}}$ is a direct summand of an object in $\mathcal{B}$.

It is a classical result that idempotent completions always exist. We refer for example to [5, §1] for a particular construction. Moreover, it is well-known that if $\mathcal{B}$ has arbitrary coproducts, then also $\bar{\mathcal{B}}$ has them and they are compatible with coproducts in $\mathcal{B}$.
Now we can state the main result of the section showing that for $\mathbf{K}(\mathcal{B})$ being generated by a set (and, in particular, for $\mathbf{K}(\mathcal{B})$ being well generated), the category $\mathcal{B}$ is necessarily pure semisimple.

THEOREM 2.5. *Let $\mathcal{B}$ be an additive category with arbitrary coproducts and assume that there is a set of objects $\mathcal{S} \subseteq \mathbf{K}(\mathcal{B})$ such that $\mathbf{K}(\mathcal{B}) = \operatorname{Loc} \mathcal{S}$. Then $\mathcal{B}$ is pure semisimple.*

Proof. Note that we can replace $\mathcal{S}$ by a singleton $\{Y\}$; take for instance $Y = \coprod_{Z\in\mathcal{S}} Z$. Let us denote by $X \in \mathcal{B}$ the coproduct $\coprod_{n\in\mathbb{Z}} Y^n$ of all components of Y. We will show that $\mathcal{B} = \operatorname{Add} X$. First, we claim that $\mathbf{K}(\operatorname{Add} X)$ is a dense subcategory of $\mathbf{K}(\mathcal{B})$, that is, each object in $\mathbf{K}(\mathcal{B})$ is isomorphic to one in $\mathbf{K}(\operatorname{Add} X)$. Indeed, $Y \in \mathbf{K}(\operatorname{Add} X)$ and one easily checks that the closure of $\mathbf{K}(\operatorname{Add} X)$ under taking isomorphic objects in $\mathbf{K}(\mathcal{B})$ is a localizing subcategory. Hence $\mathbf{K}(\operatorname{Add} X)$ is dense in $\mathbf{K}(\mathcal{B})$ and the claim is proved.
Suppose for the moment that $\mathcal{B}$ has splitting idempotents. If we identify $\mathcal{B}$ with the full subcategory of $\mathbf{K}(\mathcal{B})$ formed by complexes concentrated in degree zero, we have proved that each object $Z \in \mathcal{B}$ is isomorphic to a complex $Q \in \mathbf{K}(\operatorname{Add} X)$. That is, there is a chain complex homomorphism $f : Z \to Q$ such that $Q \in \mathbf{C}(\operatorname{Add} X)$ and f becomes an isomorphism in $\mathbf{K}(\mathcal{B})$. In particular,

the mapping cone C_f of f is contractible:

$$C_f: \qquad \dots \longrightarrow Q^{-3} \xrightarrow{d^{-3}} Q^{-2} \xrightarrow{\binom{d^{-2}}{0}} Q^{-1} \amalg Z \xrightarrow{(d^{-1}, f^0)} Q^0 \xrightarrow{d^0} Q^1 \longrightarrow \dots$$

Here, f^0 is the degree 0 component of f. Consequently, Lemma 2.3 yields the following commutative diagram in $\mathcal{B}$ with isomorphisms in columns:

$$\begin{tikzcd}
Q^{-2} \arrow[r, "\binom{d^{-2}}{0}"] \arrow[d, "\cong"'] & Q^{-1} \amalg Z \arrow[r, "(d^{-1}, f^0)"] \arrow[d, "\cong"'] & Q^0 \arrow[d, "\cong"'] \\
U \amalg V \arrow[r, "\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] & V \amalg W \arrow[r, "\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] & W \amalg Z
\end{tikzcd}$$

It follows that V, W and also $Q^{-1} \amalg Z$ and Z are in $\operatorname{Add} X$. Hence $\mathcal{B} = \operatorname{Add} X$. Finally, let $\mathcal{B}$ be a general additive category with coproducts and $\bar{\mathcal{B}}$ be its idempotent completion. From the fact that $\mathbf{K}(\mathcal{B})$ has splitting idempotents, [32, 1.6.8], one easily sees that the full embedding $\mathbf{K}(\mathcal{B}) \to \mathbf{K}(\bar{\mathcal{B}})$ is dense. We already know that if $\mathbf{K}(\mathcal{B}) = \operatorname{Loc} \mathcal{S}$ for a set $\mathcal{S}$, then $\bar{\mathcal{B}} = \operatorname{Add} X$ for some $X \in \bar{\mathcal{B}}$. In fact, we can take $X \in \mathcal{B}$ by the above construction. But then clearly $\mathcal{B} = \operatorname{Add} X$ when the additive closure is taken in $\mathcal{B}$. Hence $\mathcal{B}$ is pure semisimple. $\square$

Remark. When studying well generated triangulated categories, an important role is played by so-called κ–localizing subcategories, see [32, 26]. We recall that given a cardinal number κ, a *κ–coproduct* is a coproduct with fewer than κ summands. If $\mathcal{T}$ is a triangulated category with arbitrary κ–coproducts, a thick subcategory $\mathcal{L} \subseteq \mathcal{T}$ is called *κ–localizing* if it is closed under taking κ–coproducts. In this context, one can state the following "bounded" version of Theorem 2.5:
Let κ be an uncountable regular cardinal and $\mathcal{B}$ be an additive category with κ–coproducts. If $\mathbf{K}(\mathcal{B})$ is generated as a κ–localizing subcategory by a set $\mathcal{S}$ of fewer than κ objects, then there is $X \in \mathcal{B}$ such that every object of $\mathcal{B}$ is a summand in a κ–coproduct of copies of X.

Note that Theorem 2.5 gives immediately a wide range of examples of categories which are not well generated. For instance, $\mathbf{K}(\text{Mod-}R)$ is not well generated for any ring R which is not right pure semisimple. One can take $R = \mathbb{Z}$ or $R = k(\cdot \rightrightarrows \cdot)$, the Kronecker algebra over a field k. The fact that $\mathbf{K}(\text{Ab})$ is not well generated was first observed by Neeman, [32, E.3.2], using different arguments. In fact, we can state the following proposition, which we later generalize in Section 5:

Proposition 2.6. *Let R be a ring. Then the following are equivalent:*

(1) $\mathbf{K}(\text{Mod-}R)$ *is well generated;*
(2) $\mathbf{K}(\text{Mod-}R)$ *is compactly generated;*
(3) R *is right pure semisimple.*

If R is an artin algebra, the conditions are further equivalent to:

(4) R *is of finite representation type.*

Proof. (2) $\implies$ (1) is clear, as compactly generated is the same as $\aleph_0$–well generated. (1) $\implies$ (3) follows by Theorem 2.5 and Proposition 2.2. (3) $\implies$ (2) has been proved by Holm and Jørgensen, [15, §4 (3), p. 17]. Finally, the equivalence between (3) and (4) is due to Auslander, [4, Theorem A]. □

3. Locally well generated triangulated categories

We have seen in the last section that a triangulated category of the form $\mathbf{K}(\text{Mod-}R)$ is often not well generated. One might get an impression that handling such categories is hopeless, but the main problem here actually is that the category is very big in the sense that it is not generated by any set. Otherwise, it has a very reasonable structure. We shall see that it is locally well generated in the following sense:

Definition 3.1. A triangulated category $\mathcal{T}$ with arbitrary coproducts is called *locally well generated* if $\operatorname{Loc}\mathcal{S}$ is well generated for any set $\mathcal{S}$ of objects of $\mathcal{T}$.

In fact, we prove that $\mathbf{K}(\text{Mod-}\mathcal{A})$ is locally well generated for any skeletally small additive category $\mathcal{A}$. To this end, we first need to be able to measure the size of modules and complexes.

Definition 3.2. Let $\mathcal{A}$ be a skeletally small additive category and $M \in \text{Mod-}\mathcal{A}$. Recall that M is a contravariant additive functor $\mathcal{A} \to \text{Ab}$ by definition. Then the *cardinality of* M, denoted by $|M|$, is defined as

$$|M| = \sum_{A \in \mathcal{S}} |M(A)|,$$

where $|M(A)|$ is just the usual cardinality of the group $M(A)$ and $\mathcal{S}$ is a fixed representative set for isomorphism classes of objects from $\mathcal{A}$. The *cardinality of a complex* $Y = (Y^n, d^n) \in \mathbf{K}(\text{Mod-}\mathcal{A})$ is defined as

$$|Y| = \sum_{n \in \mathbb{Z}} |Y^n|.$$

It is not so difficult to see that the category of all complexes whose cardinalities are bounded by a given regular cardinal always gives rise to a well-generated subcategory of $\mathbf{K}(\text{Mod-}\mathcal{A})$:

Lemma 3.3. *Let $\mathcal{A}$ be a skeletally small additive category and κ be an infinite cardinal. Then the full subcategory $\mathcal{S}_\kappa$ formed by all complexes of cardinality less than κ meets conditions* (2) *and* (3) *of Definition 1.1.*
In particular, $\mathcal{T}_\kappa = \operatorname{Loc}\mathcal{S}_\kappa$ is a κ–well generated subcategory of $\mathbf{K}(\text{Mod-}\mathcal{A})$ *for any regular cardinal κ.*

Proof. Let $Y \in \mathbf{K}(\text{Mod-}\mathcal{A})$ such that $|Y| < \kappa$. If $(Z_i \mid i \in I)$ is an arbitrary family of complexes in $\mathbf{K}(\text{Mod-}\mathcal{A})$, we can construct their coproduct as a componentwise coproduct in $\mathbf{C}(\text{Mod-}\mathcal{A})$. Then whenever $f : Y \to \coprod_{i \in I} Z_i$ is a morphism in $\mathbf{C}(\text{Mod-}\mathcal{A})$, it is straightforward to see that f factorizes through $\coprod_{i \in J} Z_i$ for some $J \subseteq I$ of cardinality less than κ. Hence Y is κ–small in $\mathbf{K}(\text{Mod-}\mathcal{A})$.

Regarding part (3) of Definition 1.1, consider a morphism $f : Y \to \coprod_{i\in I} Z_i$. We have the following factorization in the abelian category of complexes $\mathbf{C}(\text{Mod-}\mathcal{A})$:

$$Y \xrightarrow{(f_i)} \coprod_{i\in I} \operatorname{Im} f_i \xrightarrow{j} \coprod_{i\in I} Z_i.$$

Here, $f_i : Y \to Z_i$ are the compositions of f with the canonical projections $\pi_i : \coprod_{i'\in I} Z_{i'} \to Z_i$, and j stands for the obvious inclusion. It is easy to see that $|\operatorname{Im} f_i| < \kappa$ for each $i \in I$ and that the morphism j is a coproduct of the inclusions $\operatorname{Im} f_i \to Z_i$. Hence (3) is satisfied.

For the second part, let κ be regular and $\mathcal{T}_\kappa = \operatorname{Loc} \mathcal{S}_\kappa$. Let us denote by $\mathcal{S}'$ a representative set of objects in $\mathcal{S}_\kappa$. It only remains to prove that $\mathcal{S}'$ satisfies condition (1) of Definition 1.1, which is rather easy. Namely, let $X \in \mathcal{T}_\kappa$ such that $\mathcal{T}_\kappa(Y, X) = 0$ for each $Y \in \mathcal{S}'$. Then $\mathcal{T}' = \{Y \in \mathcal{T}_\kappa \mid \mathcal{T}_\kappa(Y, X) = 0\}$ defines a localizing subcategory of $\mathcal{T}_\kappa$ containing $\mathcal{S}_\kappa$. Hence, $\mathcal{T}' = \mathcal{T}_\kappa$ and $X = 0$. □

We will also need (a simplified version of) an important result, which is essentially contained already in [32]. It says that the property of being well generated is preserved when passing to any localizing subcategory generated by a set. In particular, every well generated category is locally well generated.

Proposition 3.4. [26, Theorem 7.2.1] *Let $\mathcal{T}$ be a well generated triangulated category and $\mathcal{S} \subseteq \mathcal{T}$ be a set of objects. Then $\operatorname{Loc} \mathcal{S}$ is a well generated triangulated category, too.*

Now, we are in a position to state a theorem which gives us a major source of examples of locally well generated triangulated categories.

Theorem 3.5. *Let $\mathcal{A}$ be a skeletally small additive category. Then the triangulated category $\mathbf{K}(\text{Mod-}\mathcal{A})$ is locally well generated.*

Proof. As in Lemma 3.3, we denote by $\mathcal{S}_\kappa$ the full subcategory of $\mathbf{K}(\text{Mod-}\mathcal{A})$ formed by complexes of cardinality less than κ and put $\mathcal{T}_\kappa = \operatorname{Loc} \mathcal{S}_\kappa$, the localizing class generated by $\mathcal{S}_\kappa$ in $\mathbf{K}(\text{Mod-}\mathcal{A})$. Then $\mathcal{T}_\kappa$ is (κ–)well generated for each regular cardinal κ by Lemma 3.3 and clearly

$$\mathbf{K}(\text{Mod-}\mathcal{A}) = \bigcup_{\kappa \text{ regular}} \mathcal{S}_\kappa = \bigcup_{\kappa \text{ regular}} \mathcal{T}_\kappa.$$

Now, if $\mathcal{S} \subseteq \mathbf{K}(\text{Mod-}\mathcal{A})$ is a set of objects, then $\mathcal{S} \subseteq \mathcal{T}_\kappa$ for some κ. Hence also $\operatorname{Loc} \mathcal{S} \subseteq \mathcal{T}_\kappa$ and $\operatorname{Loc} \mathcal{S}$ is well generated by Proposition 3.4. It follows that $\mathbf{K}(\text{Mod-}\mathcal{A})$ is locally well generated. □

Having obtained a large class of examples of locally well generated triangulated categories, one might ask for some basic properties of such categories. We will prove a version of the so-called Bousfield Localization Theorem here:

Proposition 3.6. *Let $\mathcal{T}$ be a locally well generated triangulated category and $\mathcal{S} \subseteq \mathcal{T}$ be a set of objects. Then $\mathcal{T}/\operatorname{Loc} \mathcal{S}$ is a Bousfield localization; that* *is, there is a localization functor* $L : \mathcal{T} \to \mathcal{T}$ *such that* $\operatorname{Ker} L = \operatorname{Loc} \mathcal{S}$. *In particular, we have*

$$\operatorname{Im} L = \{X \in \mathcal{T} \mid \mathcal{T}(Y, X) = 0 \text{ for each } Y \in \mathcal{S}\},$$

there is a canonical triangle equivalence between $\mathcal{T}/\operatorname{Loc}\mathcal{S}$ *and* $\operatorname{Im} L$ *given by the composition*

$$\operatorname{Im} L \xrightarrow{\subseteq} \mathcal{T} \xrightarrow{Q} \mathcal{T}/\operatorname{Loc}\mathcal{S},$$

and all morphism spaces in $\mathcal{T}/\operatorname{Loc}\mathcal{S}$ *are sets.*

Proof. The proof is rather standard. $\operatorname{Loc}\mathcal{S}$ is well generated, so it satisfies the Brown Representability Theorem (see Proposition 1.2). Hence the inclusion $\mathbf{i} : \operatorname{Loc}\mathcal{S} \to \mathcal{T}$ has a right adjoint by [32, 8.4.4]. The composition of this right adjoint with $\mathbf{i}$ gives a so-called colocalization functor $\Gamma : \mathcal{T} \to \mathcal{T}$ whose essential image is equal to $\operatorname{Loc}\mathcal{S}$. The definition of a colocalization functor is formally dual to the one of a localization functor; see [26, §4.12] for details. A well-known construction then yields a localization functor $L : \mathcal{T} \to \mathcal{T}$ such that $\operatorname{Ker} L = \operatorname{Loc}\mathcal{S}$. We refer to [32, 9.1.14] or [26, 4.12.1] for details. The rest follows from [32, 9.1.16] or [26, 4.9.1]. □

Remark. Proposition 3.6 has been proved before for well generated triangulated categories. This is implicitly contained for example in [26, §7.2]. It also generalizes more classical results, such as a corresponding statement for the derived category $\mathbf{D}(\mathcal{B})$ of a Grothendieck category $\mathcal{B}$, [2, 5.7]. To see this, one only needs to observe that $\mathbf{D}(\mathcal{B})$ is well generated, see [26, Example 7.7].

An obvious question is whether the Brown Representability Theorem also holds for locally well generated categories, as this was the crucial feature of well generated categories. Unfortunately, this is not the case in general, as the following example suggested by Henning Krause shows.

Example 3.7. According to [10, Exercise 1, p. 131], one can construct an abelian category $\mathcal{B}$ with some Ext-spaces being proper classes. Namely, let U be the class of all cardinals, and let $\mathcal{B} = \text{Mod-}\mathbb{Z}\langle U\rangle$, the category of all "modules over the free ring on the proper class of generators U." That is, an object X of $\mathcal{B}$ is an abelian group such that each $\kappa \in U$ has a $\mathbb{Z}$-linear action on X and this action is trivial for all but a set of cardinals. Such a category admits a valid set-theoretical description in ZFC. If we denote by $\mathbb{Z}$ the object of $\mathcal{B}$ whose underlying group is free of rank 1 and $\kappa \cdot \mathbb{Z} = 0$ for each $\kappa \in U$, then $\operatorname{Ext}^1_{\mathcal{B}}(\mathbb{Z}, \mathbb{Z})$ is a proper class (see also [26, 4.15] or [6, 1.1]).
Given the above description of objects of $\mathcal{B}$, one can easily adjust the proof of Theorem 3.5 to see that $\mathbf{K}(\mathcal{B})$ is locally well generated. Let $\mathbf{K}_{\mathrm{ac}}(\mathcal{B})$ stand for the full subcategory of all acyclic complexes in $\mathbf{K}(\mathcal{B})$. Then $\mathbf{K}_{\mathrm{ac}}(\mathcal{B})$ is clearly a localizing subcategory of $\mathbf{K}(\mathcal{B})$, hence locally well-generated.
It has been shown in [6] that $\mathbf{K}_{\mathrm{ac}}(\mathcal{B})$ does not satisfy the Brown Representability Theorem. In fact, one proved even more: $\mathbf{K}_{\mathrm{ac}}(\mathcal{B})$ is localizing in $\mathbf{K}(\mathcal{B})$, but it is not a kernel of any localization functor $L : \mathbf{K}(\mathcal{B}) \to \mathbf{K}(\mathcal{B})$. More

specifically, the composition of functors, the second of which is contravariant,

$$\mathbf{K}_{\mathrm{ac}}(\mathcal{B}) \xrightarrow{\subseteq} \mathbf{K}(\mathcal{B}) \xrightarrow{\mathbf{K}(\mathcal{B})(-,\mathbb{Z})} \mathrm{Ab}$$

is not representable by any object of $\mathbf{K}_{\mathrm{ac}}(\mathcal{B})$.

Yet another natural question is what other triangulated categories are locally well generated. A deeper analysis of this problem is left for future research, but we will see in Section 4 that $\mathbf{K}(\mathcal{B})$ is locally well generated for any finitely accessible additive category $\mathcal{B}$. For now, we will prove that the class of locally well generated triangulated categories is closed under some natural constructions. Let us start with a general lemma, which holds even if morphism spaces in the quotient $\mathcal{T}/\mathcal{L}$ are proper classes:

LEMMA 3.8. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{L} \subseteq \mathcal{L}'$ be two localizing subcategories of $\mathcal{T}$. Then $\mathcal{L}'/\mathcal{L}$ is a localizing subcategory of $\mathcal{T}/\mathcal{L}$.*

Proof. It is easy to see that $\mathcal{L}'/\mathcal{L}$ is a full subcategory of $\mathcal{T}/\mathcal{L}$ which is closed under taking isomorphic objects, see [33, Théorème 4-2] or [22, Proposition 1.6.5]. The rest follows directly from the construction of $\mathcal{T}/\mathcal{L}$. □

Now we can show that taking localizing subcategories and localizing with respect to a set of objects preserves the locally well generated property.

PROPOSITION 3.9. *Let $\mathcal{T}$ be a locally well generated triangulated category.*

1. *Any localizing subcategory $\mathcal{L}$ of $\mathcal{T}$ is itself locally well generated.*
2. *The Verdier quotient $\mathcal{T}/\mathrm{Loc}\,\mathcal{S}$ is locally well generated for any set $\mathcal{S}$ of objects in $\mathcal{T}$.*

Proof. (1) is trivial. For (2), put $\mathcal{L} = \mathrm{Loc}\,\mathcal{S}$ and consider a set $\mathcal{C}$ of objects in $\mathcal{T}/\mathcal{L}$. We have to prove that the localizing subcategory generated by $\mathcal{C}$ in $\mathcal{T}/\mathcal{L}$ is well generated. Since the objects of $\mathcal{T}$ and $\mathcal{T}/\mathcal{L}$ coincide by definition, we can consider a localizing subcategory $\mathcal{L}' \subseteq \mathcal{T}$ defined by $\mathcal{L}' = \mathrm{Loc}\,(\mathcal{S} \cup \mathcal{C})$. One easily sees using Lemma 3.8 that $\mathcal{L}'/\mathcal{L} = \mathrm{Loc}\,\mathcal{C}$ in $\mathcal{T}/\mathcal{L}$. Since both $\mathcal{L}$ and $\mathcal{L}'$ are well generated by definition, so is $\mathcal{L}'/\mathcal{L}$ by [26, 7.2.1]. Hence $\mathcal{T}/\mathcal{L}$ is locally well generated. □

We conclude this section with an immediate consequence of Theorem 3.5 and Proposition 3.9, which will be useful in the next section:

COROLLARY 3.10. *Let $\mathcal{A}$ be a small additive category and $\mathcal{B}$ be a full subcategory of* Mod-$\mathcal{A}$ *which is closed under arbitrary coproducts. Then $\mathbf{K}(\mathcal{B})$ is locally well generated.*

4. FINITELY ACCESSIBLE ADDITIVE CATEGORIES

There is a natural generalization of module categories, namely the additive version of finitely accessible categories in the terminology of [1]. As we have seen, there is quite a lot of freedom to choose $\mathcal{B}$ in the above Corollary 3.10. We will use this fact and a standard trick to (seemingly) generalize Theorem 3.5

from module categories to finitely accessible additive categories. We start with a definition.

Definition 4.1. Let $\mathcal{B}$ be an additive category which admits arbitrary filtered colimits. Then:

- An object $X \in \mathcal{B}$ is called *finitely presentable* if the representable functor $\mathcal{B}(X, -) : \mathcal{B} \to \mathrm{Ab}$ preserves filtered colimits.
- The category $\mathcal{B}$ is called *finitely accessible* if there is a set $\mathcal{A}$ of finitely presentable objects from $\mathcal{B}$ such that every object in $\mathcal{B}$ is a filtered colimit of objects from $\mathcal{A}$.

Note that if $\mathcal{B}$ is finitely accessible, the full subcategory $\mathrm{fp}(\mathcal{B})$ of $\mathcal{B}$ formed by all finitely presentable objects in $\mathcal{B}$ is skeletally small, [1, 2.2]. Several other general properties of finitely accessible categories will follow from Proposition 4.2.
Finitely accessible categories occur at many occasions. The simplest and most natural example is the module category Mod-R over an associative unital ring. It is well-known that finitely presentable objects in Mod-R coincide with finitely presented R–modules in the usual sense. The same holds for Mod-$\mathcal{A}$, the category of modules over a small additive category $\mathcal{A}$. Motivated by representation theory, finitely accessible categories were studied by Crawley-Boevey [8] under the name locally finitely presented categories; see [8, §5] for further examples. The term from [8], however, may cause some confusion in the light of other definitions. Namely, Gabriel and Ulmer [11] have defined the concept of a *locally finitely presentable* category which is, in our terminology, a cocomplete finitely accessible category. As the latter concept has been used quite substantially in one of our main references, [26], we stick to the terminology of [1].
The crucial fact about finitely accessible additive categories is the following representation theorem:

Proposition 4.2. *The assignments*

$$\mathcal{A} \mapsto \text{Flat-}\mathcal{A} \quad \textit{and} \quad \mathcal{B} \mapsto \mathrm{fp}(\mathcal{B})$$

form a bijective correspondence between

(1) *equivalence classes of skeletally small additive categories $\mathcal{A}$ with splitting idempotents, and*
(2) *equivalence classes of additive finitely accessible categories $\mathcal{B}$.*

Proof. See [8, §1.4]. □

Remark. The correspondence from Proposition 4.2 restricts, using [8, §2.2], to a bijection between equivalence classes of skeletally small additive categories with finite colimits (equivalently, with cokernels) and equivalence classes of locally finitely presentable categories in the sense of Gabriel and Ulmer [11].

One of the main results of this paper has now become a mere corollary of preceding results:

Theorem 4.3. *Let $\mathcal{B}$ be a finitely accessible additive category. Then $\mathbf{K}(\mathcal{B})$ is locally well generated.*

Proof. Let us put $\mathcal{A} = \mathrm{fp}(\mathcal{B})$, the full subcategory of $\mathcal{B}$ formed by all finitely presentable objects. Using Proposition 4.2, we see that $\mathcal{B}$ is equivalent to the category Flat-$\mathcal{A}$. The category $\mathbf{K}(\text{Flat-}\mathcal{A})$ is locally well generated by Corollary 3.10, and so must be $\mathbf{K}(\mathcal{B})$. $\square$

The remaining question when $\mathbf{K}(\mathcal{B})$ is κ–well generated and which cardinals κ can occur will be answered in the next section. For now, we know by Theorem 2.5 that a necessary condition is that $\mathcal{B}$ be pure semisimple. In fact, we will show that this is also sufficient, but at the moment we will only give a better description of pure semisimple finitely accessible additive categories.

Proposition 4.4. *Let $\mathcal{B}$ be a finitely accessible additive category. Then the following are equivalent:*

1. *$\mathcal{B}$ is pure semisimple in the sense of Definition 2.1;*
2. *Each object in $\mathcal{B}$ is a coproduct of (indecomposable) finitely presentable objects;*
3. *Each flat right $\mathcal{A}$–module is projective, where $\mathcal{A} = \mathrm{fp}(\mathcal{B})$.*

Proof. For the whole argument, we put $\mathcal{A} = \mathrm{fp}(\mathcal{B})$ and without loss of generality assume that $\mathcal{B} = \text{Flat-}\mathcal{A}$.
(1) $\implies$ (3). Assume that Flat-$\mathcal{A}$ is pure semisimple. As in the proof for Proposition 2.2, we can use a generalization [3, Theorem 26.1] of Kaplansky's theorem, to deduce that there is a cardinal number λ such that each flat $\mathcal{A}$–module is a direct sum of at most λ–generated flat $\mathcal{A}$–modules. The key step is then contained in [13, Corollary 3.6] which says that under the latter condition $\mathcal{A}$ is a right perfect category. That is, it satisfies the equivalent conditions of Bass' theorem [18, B.12] (or more precisely, its version for contravariant functors $\mathcal{A} \to \mathrm{Ab}$). One of the equivalent conditions is condition (3).
(3) $\implies$ (2). This is a consequence of Bass' theorem; see [18, B.13].
(2) $\implies$ (1). Trivial, $\mathcal{B} = \mathrm{Add} X$ where $X = \bigoplus_{Y \in \mathcal{A}} Y$. $\square$

For further reference, we mention one more condition which one might impose on a finitely accessible additive category. Namely, it is well known that for a ring R, the category Flat-R is closed under products if and only if R is left coherent. This generalizes in a natural way for finitely accessible additive categories. Let us recall that an additive category $\mathcal{A}$ is said to have *weak cokernels* if for each morphism $X \to Y$ there is a morphism $Y \to Z$ such that $\mathcal{A}(Z, W) \to \mathcal{A}(Y, W) \to \mathcal{A}(X, W)$ is exact for all $W \in \mathcal{A}$.

Lemma 4.5. *Let $\mathcal{B}$ be a finitely accessible additive category and $\mathcal{A} = \mathrm{fp}(\mathcal{B})$. Then the following are equivalent:*

1. *$\mathcal{B}$ has products.*
2. Flat-$\mathcal{A}$ *is closed under products in* Mod-$\mathcal{A}$.
3. *$\mathcal{A}$ has weak cokernels.*

Proof. See [8, §2.1]. $\square$

Remark. If $\mathcal{B}$ has products, one can give a more classical proof for Proposition 4.4. Namely, one can then replace the argument by Guil Asensio, Izurdiaga and Torrecillas [13] by an older and simpler argument by Chase [7, Theorem 3.1].

5. When is the homotopy category well generated?

In this final section, we have developed enough tools to answer the question when exactly is the homotopy category of complexes $\mathbf{K}(\mathcal{B})$ well generated if $\mathcal{B}$ is a finitely accessible additive category. This way, we will generalize Proposition 2.6 and also give a rather complete answer to [15, Question 4.2] asked by Holm and Jørgensen. Finally, we will give another criterion for a triangulated category to be (or not to be) well generated and this way construct other classes of examples of categories which are not well generated.
First, we recall a crucial result due to Neeman:

Lemma 5.1. *Let $\mathcal{A}$ be a skeletally small additive category. Then the homotopy category $\mathbf{K}(\text{Proj-}\mathcal{A})$ is $\aleph_1$–well generated. If, moreover, $\mathcal{A}$ has weak cokernels, then $\mathbf{K}(\text{Proj-}\mathcal{A})$ is compactly generated.*

Proof. Neeman has proved in [31, Theorem 1.1] that, given a ring R, the category $\mathbf{K}(\text{Proj-}R)$ is $\aleph_1$–well generated, and if R is left coherent then $\mathbf{K}(\text{Proj-}R)$ is even compactly generated. The actual arguments, contained in [31, §§4–7], immediately generalize to the setting of projective modules over small categories. The role of finitely generated free modules over R is taken by representable functors, and instead of the duality between the categories of left and right projective finitely generated modules we consider the duality between the idempotent completions of the categories of covariant and contravariant representable functors. □

We already know that $\mathbf{K}(\mathcal{B})$ is always locally well generated. When employing Lemma 5.1, we can show the following statement, which is one of the main results of this paper:

Theorem 5.2. *Let $\mathcal{B}$ be a finitely accessible additive category. Then the following are equivalent:*

(1) $\mathbf{K}(\mathcal{B})$ *is well generated;*
(2) $\mathbf{K}(\mathcal{B})$ *is $\aleph_1$–well generated;*
(3) $\mathcal{B}$ *is pure semisimple.*

If, moreover, $\mathcal{B}$ has products, then the conditions are further equivalent to

(4) $\mathbf{K}(\mathcal{B})$ *is compactly generated.*

Proof. (1) $\implies$ (3). If $\mathbf{K}(\mathcal{B})$ is well generated, it is in particular generated by a set of objects as a localizing subcategory of itself; see Proposition 1.2. Hence $\mathcal{B}$ is pure semisimple by Theorem 2.5.
(3) $\implies$ (2) and (4). If $\mathcal{B}$ is pure semisimple and $\mathcal{A} = \text{fp}(\mathcal{B})$, then $\mathcal{B}$ is equivalent to Flat-$\mathcal{A}$ by Proposition 4.2, and Flat-$\mathcal{A}$ = Proj-$\mathcal{A}$ by Proposition 4.4. The conclusion follows by Lemmas 5.1 and 4.5.

(2) or (4) $\implies$ (1). This is obvious. □

Remark. (1) Neeman proved in [31] more than stated in Lemma 5.1. He described a particular set of generators for $\mathbf{K}(\text{Proj-}\mathcal{A})$ satisfying conditions of Definition 1.1. Namely, $\mathbf{K}(\text{Proj-}\mathcal{A})$ is always $\aleph_1$–well generated by a representative set of bounded below complexes of finitely generated projectives. Moreover, he gave an explicit description of compact objects in $\mathbf{K}(\text{Proj-}\mathcal{A})$ in [31, 7.12].
(2) An exact characterization of when $\mathbf{K}(\mathcal{B})$ is compactly generated and thereby a complete answer to [15, Question 4.2] does not seem to be known. We have shown that this reduces to the problem when $\mathbf{K}(\text{Proj-}\mathcal{A})$ is compactly generated. A sufficient condition is given in Lemma 5.1, but it is probably not necessary. On the other hand, if $R = k[x_1, x_2, x_3, \dots]/(x_i x_j;\ i, j \in \mathbb{N})$ where k is a field, then $\mathbf{K}(\text{Flat-}R)$ coincides with $\mathbf{K}(\text{Proj-}R)$, but the latter is not a compactly generated triangulated category; see [31, 7.16] for details.

Example 5.3. The above theorem adds other locally well generated but not well generated triangulated categories to our repertoire. For example $\mathbf{K}(\mathcal{TF})$, where $\mathcal{TF}$ stands for the category of all torsion-free abelian groups, has this property.

We finish the paper with some examples of triangulated categories where the fact that they are not generated by a set is less obvious. For this purpose, we will use the following criterion:

Proposition 5.4. *Let $\mathcal{T}$ be a locally well generated triangulated category and $\mathcal{L}$ be a localizing subcategory. Consider the diagram*

$$\mathcal{L} \xrightarrow{\ \subseteq\ } \mathcal{T} \xrightarrow{\ Q\ } \mathcal{T}/\mathcal{L}.$$

If two of the categories $\mathcal{L}$, $\mathcal{T}$ and $\mathcal{T}/\mathcal{L}$ are well generated, so is the third.

Proof. If $\mathcal{L} = \operatorname{Loc}\mathcal{S}$ and $\mathcal{T}/\mathcal{L} = \operatorname{Loc}\mathcal{C}$ for some sets $\mathcal{S}, \mathcal{C}$, let $\mathcal{L}'$ be the localizing subcategory of $\mathcal{T}$ generated by the set of objects $\mathcal{S} \cup \mathcal{C}$. Lemma 3.8 yields the equality $\mathcal{T}/\mathcal{L} = \mathcal{L}'/\mathcal{L}$. Hence also $\mathcal{T} = \mathcal{L}'$, so $\mathcal{T}$ is generated by a set, and consequently $\mathcal{T}$ is well generated.
If $\mathcal{L}$ and $\mathcal{T}$ are well generated, so is $\mathcal{T}/\mathcal{L}$ by [26, 7.2.1]. Finally, one knows that $X \in \mathcal{T}$ belongs to $\mathcal{L}$ if and only if $QX = 0$; see [32, 2.1.33 and 1.6.8]. Therefore, if $\mathcal{T}$ and $\mathcal{T}/\mathcal{L}$ are well generated, so is $\mathcal{L}$ by [26, 7.4.1]. □

Remark. We stress here that by saying that $\mathcal{T}/\mathcal{L}$ is well generated, we in particular mean that $\mathcal{T}/\mathcal{L}$ is a usual category in the sense that all morphism spaces are sets and *not* proper classes.

Now we can conclude by showing that some homotopy categories of acyclic complexes are not well generated.

Example 5.5. Let R be a ring, $\mathbf{K}_{\mathrm{ac}}(\text{Mod-}R)$ be the full subcategory of $\mathbf{K}(\text{Mod-}R)$ formed by all acyclic complexes, and $\mathcal{L} = \operatorname{Loc}\{R\}$. It is well-known but also an easy consequence of Proposition 3.6 that the composition

$$\mathbf{K}_{\mathrm{ac}}(\text{Mod-}R) \xrightarrow{\subseteq} \mathbf{K}(\text{Mod-}R) \xrightarrow{Q} \mathbf{K}(\text{Mod-}R)/\mathcal{L}$$

is a triangle equivalence between $\mathbf{K}(\text{Mod-}R)/\mathcal{L}$ and $\mathbf{K}_{\text{ac}}(\text{Mod-}R)$.
By Proposition 2.6, $\mathbf{K}(\text{Mod-}R)$ is well generated if and only if R is right pure semisimple. Therefore, $\mathbf{K}_{\text{ac}}(\text{Mod-}R)$ is well generated if and only if R is right pure semisimple by Proposition 5.4. In fact, $\mathbf{K}_{\text{ac}}(\text{Mod-}R)$ is not generated by any set of objects if R is not right pure semisimple. As particular examples, we may take $R = \mathbb{Z}$ or $R = k(\cdot \rightrightarrows \cdot)$ for any field k.

Example 5.6. Let $\mathcal{B}$ be a finitely accessible category. Recall that $\mathcal{B}$ is equivalent to Flat-$\mathcal{A}$ for $\mathcal{A} = \text{fp}(\mathcal{B})$. Then the natural exact structure on Flat-$\mathcal{A}$ coming from Mod-$\mathcal{A}$ is nothing else than the well-known exact structure given by pure exact short sequences in $\mathcal{B}$ (see eg. [8]).
We denote by $\mathbf{K}_{\text{pac}}(\text{Flat-}\mathcal{A})$ the full subcategory of $\mathbf{K}(\text{Flat-}\mathcal{A})$ formed by all complexes exact with respect to this exact structure, and call such complexes *pure acyclic*. More explicitly, $X \in \mathbf{K}(\text{Flat-}\mathcal{A})$ is pure acyclic if and only if X is acyclic in Mod-$\mathcal{A}$ and all the cycles $Z^i(X)$ are flat. Note that $\mathbf{K}_{\text{pac}}(\text{Flat-}\mathcal{A})$ is closed under taking coproducts in $\mathbf{K}(\text{Flat-}\mathcal{A})$.
Neeman proved in [31, Theorem 8.6] that $X \in \mathbf{K}(\text{Flat-}\mathcal{A})$ is pure acyclic if and only if there are no non-zero homomorphisms from any $Y \in \mathbf{K}(\text{Proj-}\mathcal{A})$ to X. Then either by combining Proposition 3.6 with Lemma 5.1 or by using [31, 8.1 and 8.2], one shows that the composition

$$\mathbf{K}_{\text{pac}}(\text{Flat-}\mathcal{A}) \xrightarrow{\subseteq} \mathbf{K}(\text{Flat-}\mathcal{A}) \xrightarrow{Q} \mathbf{K}(\text{Flat-}\mathcal{A})/\mathbf{K}(\text{Proj-}\mathcal{A})$$

is a triangle equivalence. Now again, Proposition 5.4 implies that $\mathbf{K}_{\text{pac}}(\text{Flat-}\mathcal{A})$ is well generated if and only if $\mathcal{B}$ is pure semisimple. If $\mathcal{B}$ is of the form Flat-R for a ring R, this precisely means that R is right perfect.
As a particular example, $\mathbf{K}_{\text{pac}}(\mathcal{TF})$ is locally well generated but not well generated, where $\mathcal{TF}$ stands for the class of all torsion-free abelian groups.

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