The Critical Values of Generalizations of the Hurwitz Zeta Function

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Abstract. We investigate a few types of generalizations of the Hurwitz zeta function, written $Z(s, a)$ in this abstract, where s is a complex variable and a is a parameter in the domain that depends on the type. In the easiest case we take $a \in \mathbf{R}$, and one of our main results is that $Z(-m, a)$ is a constant times $E_m(a)$ for $0 \le m \in \mathbf{Z}$, where E_m is the generalized Euler polynomial of degree n. In another case, a is a positive definite real symmetric matrix of size n, and $Z(-m, a)$ for $0 \le m \in \mathbf{Z}$ is a polynomial function of the entries of a of degree $\le mn$. We will also define Z with a totally real number field as the base field, and will show that $Z(-m, a) \in \mathbf{Q}$ in a typical case.

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Introduction

This paper is divided into four parts. In the first part we consider a generalization of Hurwitz zeta function given by

$$\zeta(s;\, a,\, \gamma) = \sum_{n=0}^{\infty} \gamma^n (n+a)^{-s}, \tag{0.1}$$

where $s \in \mathbf{C}$, $0 < a \in \mathbf{R}$, and $\gamma \in \mathbf{C}$, $0 < |\gamma| \le 1$. Clearly the infinite series is convergent for $\mathrm{Re}(s) > 1$. For $\gamma = 1$ this becomes $\sum_{n=0}^{\infty}(n+a)^{-s}$, which is the classical Hurwitz zeta function usually denoted by $\zeta(s,\, a)$. This generalization is not new. It was considered by Lerch in [Le], a work five years after the paper [Hu] of Hurwitz in 1882. Its analytic properties can be summarized as follows.

Theorem 0.1. *For a and γ as above the product $(e^{2\pi i s}-1)\Gamma(s)\zeta(s;\, a,\, \gamma)$ can be continued to an entire function in s. In addition, there exists a holomorphic function in $(s,\, a,\, \gamma)\in\mathbf{C}^3$, defined for $\mathrm{Re}(a)>0$ and $\gamma \notin \{x\in\mathbf{R} \mid x\ge 1\}$ with no condition on s, that coincides with the product when $\mathrm{Re}(s) > 1$, $0 < a \in \mathbf{R}$, and $0 < |\gamma| \le 1$.*

The proof will be given in §1.1.

To state a more interesting fact, we first put

$$\mathbf{e}(z) = \exp(2\pi i z) \qquad (z \in \mathbf{C}), \tag{0.2}$$

and define a function $E_{c,n}(t)$ in t for $c \in \mathbf{C}$ and $0 < n \in \mathbf{Z}$, that is called the *nth generalized Euler polynomial*, by

$$\frac{(1+c)e^{tz}}{e^z + c} = \sum_{n=0}^{\infty} \frac{E_{c,n}(t)}{n!} z^n. \tag{0.3}$$

We assume $c = -\mathbf{e}(\alpha)$ with $\alpha \in \mathbf{R}, \notin \mathbf{Z}$. The function $E_{c,n}(t)$ was introduced in [S07]. If $c = 1$, $E_{1,n}(t)$ is the classical Euler polynomial of degree n. In [S07] we showed that $E_{c,n}(t)$ is a polynomial in t of degree n; it is also a polynomial in $(1+c)^{-1}$. Its properties are listed in [S07, pp. 25–26]. We mention here only

$$E_{c,n}(1-t) = (-1)^n E_{c^{-1},n}(t), \tag{0.3a}$$

(see [S07, (4.3f)]), which will become necessary later. Now we have

Theorem 0.2. *For $0 < k \in \mathbf{Z}$, $\mathrm{Re}(a) > 0$, and $\gamma \notin \{x \in \mathbf{R} \mid x \geq 1\}$ the value $\zeta(1-k,\, a;\, \gamma)$ is a polynomial function of a and $(\gamma - 1)^{-1}$. More precisely, we have*

$$\zeta(1-k;\, a,\, \gamma) = E_{c,k-1}(a)/(1 + c^{-1}) \tag{0.4}$$

for such k, a, and γ, where $c = -\gamma^{-1}$.

This will be proven in §1.2.

As for the original Hurwitz function, there is a well known relation

$$\zeta(1-k,\, a) = -B_k(a)/k \quad \text{for} \quad 0 < k \in \mathbf{Z}, \tag{0.5}$$

where B_k is the k-th Bernoulli polynomial. This is essentially due to Hurwitz; see [Hu, p. 92]; cf. also [E, p.27, (11)] and [WW, p. 267, 13.14].

In [S07] and [S08] we investigated the critical values of the L-function $L(s,\, \chi)$ with a Dirichlet character χ, and proved especially (see [S07, Theorem 4.14] and [S08, Theorem 1.4])

Theorem 0.3. *Let χ be a nontrivial primitive Dirichlet character modulo a positive integer d, and let k be a positive integer such that $\chi(-1) = (-1)^k$.*

(i) *If $d = 2q + 1$ with $0 < q \in \mathbf{Z}$, then*

$$L(1-k,\, \chi) = \frac{d^{k-1}}{2^k \chi(2) - 1} \sum_{b=1}^{q} (-1)^b \chi(b) E_{1,k-1}(b/d). \tag{0.6}$$

(ii) *If $d = 4d_0$ with $1 < d_0 \in \mathbf{Z}$, then*

$$L(1-k,\, \chi) = (2d_0)^{k-1} \sum_{a=1}^{d_0 - 1} \chi(a) E_{1,k-1}(2a/d). \tag{0.7}$$

In §1.4 we will give a shorter proof for these formulas by means of (0.4), and in Section 2 we will prove a functional equation for $\zeta(s;\, a, \gamma)$ by producing an expression for $\zeta(1-s;\, a,\, \gamma)$.

The second part of the paper concerns the analogue of (0.1) defined when the base field is a totally real algebraic number field F. If $F \neq \mathbf{Q}$, there are nontrivial units, which cause considerable difficulties, and for this reason we cannot give a full generalization of which (0.1) is a special case. However, taking such an F as the base field, we will present a function of a complex variable s and two parameters a and p in F, that includes as a special case at least $\zeta(s;\, a,\, \gamma)$ with $a \in \mathbf{Q}$ and γ a root of unity. We then prove in Theorem 3.4 a rationality result on its critical values.

The third part is a kind of interlude. Observing that the formula for $L(k,\, \chi)$ (not $L(1-k,\, \chi)$) involves the Gauss sum $G(\chi)$ of χ, we will give in Section 4 a formula for $G(\chi\lambda)/G(\chi)$ for certain Dirichlet characters χ and λ.

The fourth and final part of the paper, which has a potential of future development, concerns the analogue of (0.1) defined for a complex variable s, with nonnegative and positive definite symmetric matrices of size n in place of n and a. We will show in Section 5 that it is an entire function of s and also that its value at $s = -m$ for $0 \le m \in \mathbf{Z}$ is a polynomial function of the variable symmetric matrix of degree $\le mn$.

1. Proof of Theorems 0.1, 0.2, and 0.3

1.1. To prove Theorem 0.1, assuming that $0 < a \in \mathbf{R}$ and $0 < |\gamma| \le 1$, we start from an easy equality $\Gamma(s)(n+a)^{-s} = \int_0^\infty x^{s-1}e^{-(n+a)x}dx$. Therefore

$$\begin{aligned}\Gamma(s)\zeta(s;\, a,\, \gamma) &= \sum_{n=0}^{\infty}\Gamma(s)\gamma^n(n+a)^{-s} = \sum_{n=0}^{\infty}\int_0^\infty x^{s-1}\gamma^n e^{-(n+a)x}dx \\ &= \int_0^\infty \sum_{n=0}^{\infty} x^{s-1}\gamma^n e^{-(n+a)x}dx = \int_0^\infty \frac{x^{s-1}e^{-ax}}{1-\gamma e^{-x}}dx.\end{aligned}$$

Our calculation is justified for $\sigma = \mathrm{Re}(s) > 1$, since

$$\sum_{n=0}^{\infty}\int_0^\infty \left|x^{s-1}\gamma^n e^{-(n+a)x}\right|dx \le \sum_{n=0}^{\infty}\Gamma(\sigma)(n+a)^{-\sigma} < \infty.$$

Thus we obtain

$$\Gamma(s)\zeta(s;\, a,\, \gamma) = \int_0^\infty \frac{x^{s-1}e^{x(1-a)}}{e^x-\gamma}dx \quad \text{for} \quad \mathrm{Re}(s) > 1. \tag{1.1}$$

We now consider

$$\int_\infty^{0+} \frac{z^{s-1}e^{z(1-a)}}{e^z-\gamma}dz$$

with the standard symbol $\int_\infty^{0+}$ of contour integration. The integral is the sum of three integrals: $\int_\infty^\delta$, $\oint$ on the circle $|z| = \delta$, and $\int_\delta^\infty$, where $0 < \delta \in \mathbf{R}$; we naturally take the limit as δ tends to 0. We take $z^{s-1} = \exp\bigl((s-1)\log z\bigr)$ for the first integral $\int_\infty^\delta$ with $\log z \in \mathbf{R}$ for $0 < z \in \mathbf{R}$; for the evaluation of the other integrals we continue z^{s-1} analytically without passing through the positive real axis. Then the first and third integrals produce

$$(\mathbf{e}(s)-1)\int_\delta^\infty \frac{x^{s-1}e^{x(1-a)}}{e^x-\gamma}dx,$$

which is meaningful for every $s \in \mathbf{C}$ and every $(a,\, \gamma) \in \mathbf{C}^2$ such that

$$\text{(1.1a)} \qquad \mathrm{Re}(a) > 0 \quad \text{and} \quad \gamma \notin \{x \in \mathbf{R} \,|\, x > 1\}.$$

As for $\oint$, we first observe that given $\gamma \in \mathbf{C}$, we can find a small $\delta_0 \in \mathbf{R}, > 0$, such that $e^z \neq \gamma$ for $0 < |z| \leq \delta_0$, since the map $z \mapsto w = e^z$ sends the punctured disc $0 < |z| \leq \delta_0$ into a punctured disc $0 < |w - 1| < \varepsilon$ that does not contain γ. (This is clearly so even for $\gamma = 1$.) Therefore $\oint$ is meaningful for every $s \in \mathbf{C}$ and sufficiently small δ in both cases $\gamma \neq 1$ and $\gamma = 1$; the integral is independent of δ because of Cauchy's theorem. Now put $z = \delta e^{i\theta}$ with δ such that $0 < \delta < \delta_0$ and $0 \leq \theta < 2\pi$. Then $z^{s-1} = \exp\{(s-1)(\log \delta + i\theta)\}$, and so for $s = \sigma + i\tau$ with real σ and τ, we have $|z^{s-1}| = \delta^{\sigma-1}|e^{-\theta\tau}| \leq \delta^{\sigma-1}e^{2\pi|\tau|}$. If $\gamma \neq 1$, we see that $\mathrm{Min}_{|z|\leq\delta_0}|e^z - \gamma| > 0$, and so $|e^{z(1-a)}/(e^z - \gamma)|$ is bounded for $|z| \leq \delta_0$. If $\gamma = 1$, the function $e^{z(1-a)}/(e^z - \gamma)$ is $1/z$ plus a holomorphic function at $z = 0$. Thus for $0 < \delta \leq \delta_0$ we see that $\left|\oint\right| \leq M\delta^\sigma$ if $\gamma \neq 1$ and $\left|\oint\right| \leq M\delta^{\sigma-1}$ if $\gamma = 1$ with a constant M that depends on a, γ, and δ_0, and so $\oint$ tends to 0 as $\delta \to 0$ if $\mathrm{Re}(s) > 1$, and we obtain

$$\text{(1.2)} \qquad (\mathbf{e}(s) - 1)\Gamma(s)\zeta(s;\, a,\, \gamma) = \int_\infty^{0+} \frac{z^{s-1}e^{z(1-a)}}{e^z - \gamma}dz$$

for $0 < |\gamma| \leq 1$, $0 < a \in \mathbf{R}$, and $\mathrm{Re}(s) > 1$. (If $\gamma \neq 1$, the condition $\mathrm{Re}(s) > 0$ instead of $\mathrm{Re}(s) > 1$ is sufficient.) Now the right-hand side of (1.2) is meaningful for every $(s,\, a,\, \gamma) \in \mathbf{C}^3$ under condition (1.1a) and so it establishes the product on the left-hand side as an entire function of s. If $\gamma \neq 1$, the contour integral can define the product for the variables s, a, γ as described in Theorem 0.1. The integrand as a function of $(z,\, s,\, a,\, \gamma)$ is not finite in a domain including $\gamma = 1$, and so the result for $\gamma = 1$ must be stated separately.

Remark. If $0 < |\gamma| \leq 1$ and $\gamma \neq 1$, then the series of (0.1) is convergent for $\mathrm{Re}(s) > 0$ and defines a holomorphic function of s there. This is clear if $|\gamma| < 1$. For $|\gamma| = 1$ and $\gamma \neq 1$, this follows from [S07, Lemma 4.3].

We are interested in the value of $\zeta(s;\, a,\, \gamma)$ at $s = -m$ with $0 \leq m \in \mathbf{Z}$. We first note

$$\{(\mathbf{e}(s) - 1)\Gamma(s)\}_{s=-m} = 2\pi i(-1)^m/m!.$$

For $s = -m$ the function z^{s-1} in the integrand is a one-valued function, and so $\int_\infty^\delta + \int_\delta^\infty = 0$. Therefore

$$\text{(1.3)} \qquad \int_\infty^{0+} = \oint = 2\pi i \cdot \mathrm{Res}_{z=0} \frac{z^{-m-1}e^{(1-a)z}}{e^z - \gamma}.$$

By (0.3), for $\gamma \neq 1$ we have

$$\frac{z^{-m-1}e^{(1-a)z}}{e^z - \gamma} = \frac{1}{1-\gamma}\sum_{n=0}^\infty \frac{E_{-\gamma,n}(1-a)}{n!}z^{n-m-1},$$

and so the residue in question is $(1-\gamma)^{-1}E_{-\gamma,m}(1-a)/m!$. Combining this with (0.3a) and (1.2), we obtain

$$\text{(1.4)} \qquad \zeta(-m;\, a,\, \gamma) = E_{c,m}(a)/(1 + c^{-1}) \quad \text{for} \quad 0 \leq m \in \mathbf{Z},$$

where $c = -\gamma^{-1}$. This proves (0.4). In [S07, p. 26] we showed that $E_{c,\,n}(t)$ is a polynomial in t and $(1+c)^{-1}$. For $c = -\gamma^{-1}$ we have $(1+c)^{-1} = 1+(\gamma-1)^{-1}$, and so we obtain Theorem 0.2.

For the reader's convenience, we give a proof of (0.5) here. If $\gamma = 1$, instead of (0.2) we use $ze^{tz}/(e^z - 1) = \sum_{n=0}^{\infty} B_n(t)z^n/n!$. Then we obtain (0.5) in the same manner as in the case $\gamma \neq 1$.

1.3. Let us insert here a historical remark. Fixing a positive integer m and an integer a such that $0 \le a \le m$, Hurwitz considered in [Hu] an infinite series

$$f(s,\,a) = \sum_{n=0}^{\infty}(mn+a)^{-s}. \tag{1.5}$$

Since this depends on m, he also denoted it by $f(s,\,a|m)$. He proved analytic continuation of these functions and stated a functional equation for $f(1-s,\,a)$, basically following Riemann's methods for the investigation of $\zeta(s)$ in [R]. At that time not much was known about Dirichlet's L-function beyond his formulas for the class number of a binary quadratic form and his theorem about prime numbers in an arithmetic progression. Employing the results on $f(s,\,a)$, Hurwitz was able to prove that the L-function for a quadratic character has analytic continuation and satisfies a functional equation. Using the standard notation $\zeta(s,\,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$ employed at present, we have $f(s,\,a|m) = m^{-s}\zeta(s,\,a/m)$, and so he considered $\zeta(s,\,a)$ only for $a \in \mathbf{Q}$. It is noticeable however that he proved essentially (0.5) as we already said.

As noted at the beginning of the paper, Lerch investigated the series of (0.1); one can also find an exposition of this topic in [E, p. 27, §1.11]. The paper [Li] of Lipschitz may be mentioned in this connection.

1.4. Let the symbols be as in Theorem 0.3(i). Put

$$\Lambda(s) = \sum_{n=1}^{\infty}(-1)^n\chi(n)n^{-s}.$$

Then $\Lambda(s) + L(s,\,\chi) = 2\sum_{n=1}^{\infty}\chi(2n)(2n)^{-s} = \chi(2)2^{1-s}L(s,\,\chi)$, and so

$$\Lambda(s) = L(s,\chi)\bigl\{\chi(2)2^{1-s} - 1\bigr\}. \tag{1.6}$$

Since $\bigl\{a \bigm| 1 \le a < d\bigr\} = \bigl\{a \bigm| 1 \le a \le q\bigr\} \sqcup \bigl\{d-a \bigm| 1 \le a \le q\bigr\}$, we have

$$\Lambda(s)=\sum_{a=1}^{q}\sum_{n=0}^{\infty}(-1)^{nd+a}\chi(a)(nd+a)^{-s}+\sum_{a=1}^{q}\sum_{n=0}^{\infty}(-1)^{nd+d-a}\chi(-a)(nd+d-a)^{-s},$$

and so $d^s\Lambda(s)$ equals

$$\sum_{a=1}^{q}(-1)^a\chi(a)\Biggl\{\sum_{n=0}^{\infty}(-1)^n\Bigl(n+\frac{a}{d}\Bigr)^{-s} - \chi(-1)\sum_{n=0}^{\infty}(-1)^n\Bigl(n+\frac{d-a}{d}\Bigr)^{-s}\Biggr\}$$
$$= \sum_{a=1}^{q}(-1)^a\chi(a)\bigl\{\zeta(s;\,a/d,\,-1) - (-1)^k\zeta(s;\,1-a/d,\,-1)\bigr\}.$$

Putting $s = 1-k$ and employing (0.4) and (0.3a), we obtain (0.6).

Our next task is to prove (0.7). Let $d = 4d_0$ with $1 < d_0 \in \mathbf{Z}$ as in Theorem 1.3(ii). We note an easy fact(see [S08, Lemma 1.3]):

$$\chi(2d_0 + a) = -\chi(a) \quad \text{for every} \quad a \in \mathbf{Z}. \tag{1.7}$$

Observe that $\{x \in \mathbf{Z} \mid x > 0,\, d_0 \nmid x\}$ is a disjoint union

$$(*) \quad \{nd + a \mid 0 < a < d_0,\ 0 \le n \in \mathbf{Z}\} \sqcup \{nd + 2d_0 + a \mid 0 < a < d_0,\ 0 \le n \in \mathbf{Z}\}$$
$$\sqcup \{nd - a \mid 0 < a < d_0,\ 0 < n \in \mathbf{Z}\} \sqcup \{nd + 2d_0 - a \mid 0 < a < d_0,\ 0 \le n \in \mathbf{Z}\}.$$

The sum of $\sum \chi(x)x^{-s}$ for x belonging to the first two sets equals

$$\sum_{a=1}^{d_0-1} \left\{ \sum_{\nu=0}^{\infty} \chi(a)(4\nu d_0 + a)^{-s} + \sum_{\nu=0}^{\infty} \chi(2d_0 + a)\bigl(2(2\nu+1)d_0 + a\bigr)^{-s} \right\}.$$

Employing (1.7), we see that this equals

$$\sum_{a=1}^{d_0-1} \sum_{m=1}^{\infty} (-1)^m \chi(a)(2md_0 + a)^{-s} = (2d_0)^{-s} \sum_{a=1}^{d_0-1} \chi(a)\zeta(s;\, 2a/d, -1). \tag{1.8}$$

Similarly, from the last two sets of $(*)$ we obtain

$$\sum_{a=1}^{d_0-1} \left\{ \sum_{\nu=1}^{\infty} \chi(-a)(4\nu d_0 - a)^{-s} + \sum_{\nu=0}^{\infty} \chi(2d_0 - a)\bigl(2(2\nu+1)d_0 - a\bigr)^{-s} \right\}$$
$$= -\sum_{a=1}^{d_0-1} \sum_{m=0}^{\infty} \chi(-a)(-1)^m (2md_0 + 2d_0 - a)^{-s}$$
$$= -(2d_0)^{-s} \sum_{a=1}^{d_0-1} \chi(-a)\zeta(s;\, 1 - 2a/d,\, -1)$$

by (1.7). Thus, adding (1.8) to this and putting $s = 1-k$, from (0.4) we obtain

$$(2d_0)^{1-k} L(1-k,\, \chi) = 2^{-1} \sum_{a=1}^{d_0-1} \chi(a)\bigl\{E_{1,k-1}(2a/d) - \chi(-1)E_{1,k-1}(1 - 2a/d)\bigr\}.$$

Suppose $\chi(-1) = (-1)^k$; then applying (0.3a) to $E_{1,k-1}(1 - 2a/d)$, we obtain (0.7). The proof of Theorem 0.3 is now complete

1.5. The case $d = 4$ is excluded in Theorem 0.3(ii). In this case, however, the matter is simpler. Indeed, for $\mu_4(n) = \left(\dfrac{-1}{n}\right)$ we have

$$L(s,\, \mu_4) = \sum_{m=0}^{\infty} (-1)^m (2m+1)^{-s}$$
$$= 2^{-s} \sum_{m=0}^{\infty} (-1)^m (m + 1/2)^{-s} = 2^{-s}\zeta(s;\, 1/2,\, -1),$$

and so by (0.4) we obtain

$$L(1-k,\, \mu_4) = 2^{k-2} E_{1,k-1}(1/2) = E_{1,k-1}/2 \tag{1.9}$$

for every odd positive integer k, where $E_{1,n}$ denotes the nth Euler number. This is classical, except that the result is usually given in terms of $L(k,\, \mu_4)$ instead of $L(1-k,\, \mu_4)$.

1.6. Let us now show that a special case of Theorem 0.3(ii) can be given in a somewhat different way. Let ψ be a primitive character whose conductor d is odd, μ_4 the primitive character modulo 4 as above, and k a positive integer such that $\psi(-1) = (-1)^{k+1}$; put $m = (d-1)/2$. Then

$$L(1-k,\,\psi\mu_4) = (-1)^m(2d)^{k-1}\sum_{j=1}^{m}(-1)^j\psi(2j)E_{1,k-1}\bigl(\tfrac{1}{2}+\tfrac{j}{d}\bigr). \tag{1.10}$$

This was given in [S07, (6.2)], if in terms of $L(k,\,\psi\mu_4)$, but we can derive it also from Theorem 0.3(ii) as follows. Take $\chi = \psi\mu_4$ in (0.7). Then the sum on the right-hand side of (0.7) is $\sum_{a=1}^{2m}\chi(a)E_{1,k-1}(a/2d)$, which equals $\sum_{b=1}^{2m}\chi(d-b)E_{1,k-1}\bigl((d-b)/2d\bigr)$. Since the bth term is nonvanishing only for even b, employing (0.3a), we see that the last sum equals $\psi(-1)$ times $\sum_{j=1}^{m}(\psi\mu_4)(d-2j)E_{1,k-1}\bigl((d+2j)/2d\bigr)$. Since $d-2j = 2(m-j)+1$, we have $\mu_4(d-2j) = (-1)^{m-j}$, and so we obtain (1.10).

1.7. We can show that $\zeta(-m;\,a,\,\gamma)$ for $0 \le m \in \mathbf{Z}$ is a polynomial in a by a formal calculation as follows. Assuming that $|\gamma| < 1$, we have

$$\zeta(-m;\,a,\,\gamma) = a^m + \sum_{n=1}^{\infty}\gamma^n(n+a)^m = a^m + \sum_{\nu=0}^{m}\binom{m}{\nu}a^{m-\nu}c_\nu$$

with $c_\nu = \sum_{n=1}^{\infty}n^\nu\gamma^n$, and so $\zeta(-m;\,a,\,\gamma)$ is a polynomial in a at least for $|\gamma| < 1$, and so Theorem 0.1 guarantees the same in a larger domain as described in that theorem

We have $c_{\nu-1} = \gamma(1-\gamma)^{-\nu}P_\nu(\gamma)$ for $1 \le \nu \in \mathbf{Z}$ with a polynomial P_ν introduced in [S07, (2.16)]; see also [S08, (4.3)]. We showed that $P_{\nu+1}(\gamma) = (\gamma-1)^\nu E_{-\gamma,\nu}(0)$ for $\nu > 0$ in [S07, (4.6)] and that $\gamma^{n-2}P_n(\gamma^{-1}) = P_n(\gamma)$ in [S07, (2.19)]; also, $E_{c,n}(t) = \sum_{k=0}^{n}\binom{n}{k}E_{c,k}(0)t^{n-k}$ by [S08, (1.15)]. Combining these together, we obtain (0.4).

Though clearly this is not the best way to prove (0.4), at least it explains an elementary aspect of the nature of the problem. In Section 5, we will return to this idea in our discussion in the higher-dimensional case.

2. The functional equation for $\zeta(s;\,a,\,\gamma)$

2.1. For $\mathrm{Re}(s) > 1$ Hurwitz proved (see [Hu, p. 93, 1)] and [WW, p. 269])

$$\begin{aligned}\zeta(1-s,\,a)&\\ &= \frac{2\Gamma(s)}{(2\pi)^s}\biggl\{\cos(\pi s/2)\sum_{n=1}^{\infty}\frac{\cos(2\pi na)}{n^s} + \sin(\pi s/2)\sum_{n=1}^{\infty}\frac{\sin(2\pi na)}{n^s}\biggr\}.\end{aligned} \tag{2.1}$$

Therefore it is natural to ask if there is a meaningful formula for $\zeta(1-s,\,a;\,\gamma)$. Though this was essentially done by Lerch in [Le], here we take a different approach. For $s \in \mathbf{C}$, $a \in \mathbf{R}$, $p \in \mathbf{R}$, and $\nu = 0$ or 1 we put

$$D^\nu(s;\,a,\,p) = \sum_{-a\neq n\in\mathbf{Z}}(n+a)^\nu|n+a|^{-\nu-s}\mathbf{e}\bigl(p(n+a)\bigr), \tag{2.2}$$

(2.3) $$T^\nu(s;\, a,\, p) = g_\nu(s) D^\nu(s;\, a,\, p), \quad g_\nu(s) = \pi^{-(s+\nu)/2}\Gamma\big((s+\nu)/2\big).$$

These were introduced in [S08]. In particular, we proved

(2.4) $$T^\nu(1-s;\, a,\, p) = i^{-\nu}\mathbf{e}(ap)T^\nu(s;\, -p,\, a).$$

Let $\gamma = \mathbf{e}(p)$ and $b = 1-a$ with $0 < a < 1$. Then it is easy to verify that

(2.5a) $$D^0(s;\, a,\, p) = \mathbf{e}(ap)\zeta(s;\, a,\, \gamma) + \mathbf{e}(-bp)\zeta(s;\, b,\, \gamma^{-1}),$$

(2.5b) $$D^1(s;\, a,\, p) = \mathbf{e}(ap)\zeta(s;\, a,\, \gamma) - \mathbf{e}(-bp)\zeta(s;\, b,\, \gamma^{-1}),$$

and so

(2.6) $$2\mathbf{e}(ap)\zeta(s;\, a,\, \gamma) = D^0(s;\, a,\, p) + D^1(s;\, a,\, p).$$

Employing (2.3) and (2.4), we obtain

(2.7) $$2\zeta(1-s;\, a,\, \gamma) = \frac{g_0(s)}{g_0(1-s)}D^0(s;\, -p,\, a) - \frac{ig_1(s)}{g_1(1-s)}D^1(s;\, -p,\, a).$$

From (2.5a, b) we see that $D^\nu(s;\, -p,\, a)$ is a linear combination of $\zeta(s;\, -p,\, \delta)$ and $\zeta(s;\, 1+p,\, \delta^{-1})$, where $\delta = \mathbf{e}(a)$. Thus, for $-1 < p < 0$ we have

$$2\zeta(1-s;\, a,\, \gamma) = \mathbf{e}(-ap)A\zeta(s;\, -p,\, \delta) + \mathbf{e}(-a-ap)B\zeta(s;\, 1+p,\, \delta^{-1})$$

with

(2.8) $$A = \frac{g_0(s)}{g_0(1-s)} - \frac{ig_1(s)}{g_1(1-s)}, \qquad B = \frac{g_0(s)}{g_0(1-s)} + \frac{ig_1(s)}{g_1(1-s)}.$$

Recalling that $\Gamma(s/2)\Gamma((s+1)/2) = 2^{1-s}\pi^{1/2}\Gamma(s)$ and $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, we find that

(2.9) $$A = 2^{1-s}\pi^{-s}\mathbf{e}(-s/4)\Gamma(s), \qquad B = 2^{1-s}\pi^{-s}\mathbf{e}(s/4)\Gamma(s),$$

and so we obtain

(2.10) $$\zeta(1-s;\, a,\, \gamma) = \frac{\mathbf{e}(-ap)\Gamma(s)}{(2\pi)^s}\Big\{\mathbf{e}(-s/4)\zeta(s;\, -p,\, \delta) + \mathbf{e}(s/4-a)\zeta(s;\, p+1,\, \delta^{-1})\Big\}.$$

at least when $-1 < p < 0$ and $0 < a < 1$, where $\gamma = \mathbf{e}(p)$ and $\delta = \mathbf{e}(a)$.

This does not apply to the case $\gamma = 1$. In this case we have

$$2\zeta(s;\, a,\, 1) = D^0(s;\, a,\, 0) + D^1(s;\, a,\, 0) \quad \text{for} \quad 0 < a \le 1,$$
$$D^\nu(s;\, 0,\, a) = \delta\zeta(s;\, 1,\, \delta) + (-1)^\nu\delta^{-1}\zeta(s;\, 1,\, \delta^{-1}).$$

Repeating the same argument as in the case $\gamma \neq 1$, we find that

$$2\zeta(1-s;\, a,\, 1) = \delta A\zeta(s;\, 1,\, \delta) + \delta^{-1}B\zeta(s;\, 1,\, \delta^{-1})$$

with the same A and B as in (2.8) and (2.9), and so

(2.11) $$\zeta(1-s;\, a,\, 1) = \frac{\Gamma(s)}{(2\pi)^s}\bigg\{\mathbf{e}\Big(a-\frac{s}{4}\Big)\zeta(s;\, 1,\, \delta) + \mathbf{e}\Big(\frac{s}{4}-a\Big)\zeta(s;\, 1,\, \delta^{-1})\bigg\}.$$

where $\delta = \mathbf{e}(a)$, $0 < a \le 1$. This gives (2.1). Indeed, we have

$$\mathbf{e}(\pm a)\zeta(s;\,1,\,\delta^{\pm 1}) = \sum_{m=1}^{\infty}\frac{\delta^{\pm m}}{m^s} = \sum_{m=1}^{\infty}\frac{\cos(2\pi ma) \pm i\sin(2\pi ma)}{m^s},$$

and so a simple calculation transforms (2.11) into (2.1).

Returning to (2.10) in which we assumed $-1 < p < 0$, put $q = -p$. Then $0 < q < 1$ and

$$\begin{aligned}(2.12)\qquad &\frac{(2\pi)^s}{\Gamma(s)}\zeta\bigl(1-s;\,a,\,\mathbf{e}(q)^{-1}\bigr)\\ &= \mathbf{e}(-s/4)\sum_{h=0}^{\infty}\frac{\mathbf{e}\bigl(a(h+q)\bigr)}{(h+q)^s} + \mathbf{e}(s/4)\sum_{h=0}^{\infty}\frac{\mathbf{e}\bigl(a(q-h-1)\bigr)}{(h+1-q)^s}.\end{aligned}$$

Then employing (0.4), for $0 < s = k \in \mathbf{Z}$ we obtain

$$(2.13)\qquad \frac{(2\pi i)^k}{(k-1)!(1+c^{-1})}E_{c,k-1}(a) = \sum_{h\in\mathbf{Z}}\frac{\mathbf{e}\bigl(a(h+q)\bigr)}{(h+q)^k},$$

where $c = -\mathbf{e}(q)$. This formula was given in [S07, (4.5)]. Thus we have given a proof of (2.13) different from that of [S07]. (The case $k = 1$ must be handled carefully; see [S07, pp. 26–27].) In [S08] we asked the question whether the parameter k in (2.13) can be extended to a complex variable, and presented $D^{\nu}(s;\,a,\,p)$ as an answer. Since (2.13) is a special case of (2.12), we can now say that (2.12) is another answer to that question.

3. The case of a totally real number field

3.1. Let F be a totally real algebraic number field. We ask whether we can define a function similar to $\zeta(s;\,a,\,\gamma)$ by taking the totally positive integers in F in place of n in (0.1). We are going to give a partially affirmative answer to this question. We let $\mathfrak{g}$ denote the maximal order of F, $\mathfrak{d}$ the different of F relative to $\mathbf{Q}$, and $\mathbf{a}$ the set of all archimedean primes of F. For each $v \in \mathbf{a}$ we denote by F_v the v-completion of F, identified with $\mathbf{R}$. In other words, v defines an injection of F into $\mathbf{R}$, and for $\xi \in F$ we denote by ξ_v the image of ξ under this injection. We put $[F:\mathbf{Q}] = g$, $F_{\mathbf{a}} = \prod_{v\in\mathbf{a}} F_v$, and $F_{\mathbf{a}}^{\times} = \prod_{v\in\mathbf{a}} F_v^{\times}$. Then $F_{\mathbf{a}}$ can be identified with $\mathbf{R}^g$, and for $\xi \in F$ the map $\xi \mapsto (\xi_v)_{v\in\mathbf{a}}$ defines an injection of F into $F_{\mathbf{a}}$. We then put

$$\mathbf{e}_{\mathbf{a}}(\xi) = \mathbf{e}\bigl(\textstyle\sum_{v\in\mathbf{a}}\xi_v\bigr) \qquad (\xi \in F_{\mathbf{a}}),$$

$$\xi^k = \prod_{v\in\mathbf{a}}\xi_v^{k_v}, \qquad \xi^{\mathbf{a}} = \prod_{v\in\mathbf{a}}\xi_v, \qquad (k = (k_v)_{v\in\mathbf{a}} \in \mathbf{Z}^{\mathbf{a}},\ \xi \in F_{\mathbf{a}}^{\times}).$$

We also put $N(\xi) = N_{F/\mathbf{Q}}(\xi)$ for $\xi \in F$. Then $N(\xi\mathfrak{g}) = |N(\xi)| = |\xi^{\mathbf{a}}|$ for $\xi \in F^{\times}$. We write $\xi \gg 0$ if $\xi_v > 0$ for every $v \in \mathbf{a}$ and put $\mathfrak{g}_+^{\times} = \bigl\{u \in \mathfrak{g}^{\times} \bigm| u \gg 0\bigr\}$.

Now for $s \in \mathbf{C}$, a fractional ideal $\mathfrak{b}$, and $a,\,p \in F$ we put

$$(3.1)\qquad \zeta(s;\,\mathfrak{b},\,a,\,p) = [\mathfrak{g}_+^{\times} : U]^{-1}\sum_{0\ll\xi\in\mathfrak{b}+a \pmod{U}}|N(\xi)|^{-s}\mathbf{e}_{\mathbf{a}}(p\xi).$$

Here the sum is taken over $F^{\times}/U$ under the condition that $\xi \gg 0$ and $\xi - a \in \mathfrak{b}$. We take a subgroup U of $\mathfrak{g}_+^{\times}$ of finite index such that $u - 1 \in a^{-1}\mathfrak{b} \cap$

$p^{-1}(\mathfrak{b}^{-1}\mathfrak{d}^{-1} \cap a^{-1}\mathfrak{d}^{-1})$ for every $u \in U$. Clearly such a U exists, the sum of (3.1) is meaningful, and $\zeta(s;\, \mathfrak{b},\, a,\, p)$ is well defined independently of the choice of U.

This is an analogue of (0.1), but it should be remembered that here both a and p belong to F. Thus (1.5) is a special case of (3.1).

Next, let $k \in \mathbf{Z}^{\mathbf{a}}$. With s, $\mathfrak{b}$, a, and p as above, we put

$$D_k(s;\, \mathfrak{b},\, a,\, p) = [\mathfrak{g}_+^\times : U]^{-1} \sum_{0 \neq \xi \in \mathfrak{b}+a \pmod{U}} \xi^{-k}|\xi|^{k-s\mathbf{a}}\mathbf{e}_{\mathbf{a}}(p\xi), \tag{3.2}$$

where the summation is the same as in (3.1) except that this time we do not impose the condition $\xi \gg 0$. This is well defined and both (3.1) and (3.2) are convergent for $\mathrm{Re}(s) > 1$.

Lemma 3.2. *Let $k \in \mathbf{Z}^{\mathbf{a}}$ with $k_v = 0$ or 1 for every $v \in \mathbf{a}$. Then the product*

$$D_k(s;\, \mathfrak{b},\, a,\, p)\prod_{v\in\mathbf{a}} \Gamma\big((s+k_v)/2\big)$$

can be continued to a meromorphic function of s on the whole s-plane that is holomorphic except for possible simple poles at $s = 0$ and 1, which occur only when $k = 0$. The pole at $s = 0$ occurs if and only if $k = 0$ and $a \in \mathfrak{b}$.

This is included in [S00, Lemma 18.2], as $D_k(s;\, \mathfrak{b},\, a,\, p)$ is a special case of the series $D_k(s,\, \kappa)$ of [S00, (18.1)]. Notice that $D_k(s;\, \mathfrak{b},\, a,\, p)$ is finite at $s = 0$ for every k.

Lemma 3.3. *For $0 < \mu \in \mathbf{Z}$ and $k \in \mathbf{Z}^{\mathbf{a}}$ with $k_v = 0$ or 1 the following assertions hold:*

(i) *$D_k(1-\mu;\, \mathfrak{b},\, a,\, p) = 0$ if $k_v - \mu \notin 2\mathbf{Z}$ for some $v \in \mathbf{a}$.*

(ii) *Suppose $k_v - \mu \in 2\mathbf{Z}$ for every $v \in \mathbf{a}$; then $D_k(1-\mu;\, \mathfrak{b},\, a,\, p) \in \mathbf{Q}_{\mathrm{ab}}$, where $\mathbf{Q}_{\mathrm{ab}}$ denotes the maximal abelian extension of $\mathbf{Q}$; in particular, $D_k(1-\mu;\, \mathfrak{b},\, a,\, 0) \in \mathbf{Q}$.*

Proof. Suppose $k_v - \mu \notin 2\mathbf{Z}$ for one particular v. Then $\Gamma\big((s+k_v)/2\big)$ has a pole at $s = 1-\mu$, and so (i) follows from Lemma 3.2. As for (ii), that $D_k(1-\mu;\, \mathfrak{b},\, a,\, 0) \in \mathbf{Q}$ is given in Proposition 18.10(ii) of [S00]. For any $p \in F$, we see that $D_k(s;\, \mathfrak{b},\, a,\, p)$ is a finite $\mathbf{Q}_{\mathrm{ab}}$-linear combination of $D_k(s;\, \mathfrak{b}',\, a',\, 0)$ with several $(\mathfrak{b}',\, a')$, and so $D_k(1-\mu;\, \mathfrak{b},\, a,\, p) \in \mathbf{Q}_{\mathrm{ab}}$.

Now our principal result on $\zeta(s;\, \mathfrak{b},\, a,\, p)$ can be stated as follows.

Theorem 3.4. (i) *$\zeta(s;\, \mathfrak{b},\, a,\, p)$ can be continued to a meromorphic function of s on the whole s-plane, which is holomorphic except for a possible simple pole at $s = 1$.*

(ii) *Let $0 < \mu \in \mathbf{Z}$. Then $\zeta(1-\mu;\, \mathfrak{b},\, a,\, p) \in \mathbf{Q}_{\mathrm{ab}}$, and in particular, $\zeta(1-\mu;\, \mathfrak{b},\, a,\, 0) \in \mathbf{Q}$.*

Proof. For $\xi \in F^\times$ we note an easy fact

$$\sum_{k\in\mathbf{Z}^{\mathbf{a}}/2\mathbf{Z}^{\mathbf{a}}} \xi^{-k}|\xi|^k = \begin{cases} 2^g & \text{if } \xi \gg 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $g = [F : \mathbf{Q}]$. Therefore we obtain

$$2^g\zeta(s;\, \mathfrak{b},\, a,\, p) = \sum_{k\in\mathbf{Z}^{\mathbf{a}}/2\mathbf{Z}^{\mathbf{a}}} D_k(s;\, \mathfrak{b},\, a,\, p), \tag{3.3}$$

and so assertion (i) follows immediately from Lemma 3.2. Take μ as in (ii). By Lemma 3.3(i), the terms on the right-hand side of (3.3) vanish except for the term with k such that $k_v - \mu \in 2\mathbf{Z}$ for every v. Therefore we obtain the desired result from Lemma 3.3(ii), and our proof is complete.

4. Some explicit expressions for Gauss sums

4.1. There are two kinds of formulas for the critical values of $L(s,\, \chi)$: one is for $L(k,\, \chi)$ and the other for $L(1-k,\, \chi)$. The former involves π and the Gauss sum of χ, whereas the latter does not. In a sense $L(1-k,\, \chi)$ is conceptually more natural than $L(k,\, \chi)$, but there is an interesting aspect in the computation of $L(k,\, \chi)$, since it allows us to find an explicit expression for a certain Gauss sum. This can be achieved by computing $L(k,\, \chi)$ in two different ways, which involve two *different* Gauss sums. Let us begin with the definition of a Gauss sum and an easy lemma.

Given a primitive or an imprimitive Dirichlet character χ' modulo a positive integer, we take the primitive character χ associated with χ', and define the Gauss sum $G(\chi')$ to be the same as the Gauss sum $G(\chi)$ of χ, given by

$$G(\chi) = \sum_{a=1}^{d} \chi(a)\mathbf{e}(a/d), \tag{4.1}$$

where d is the conductor of χ. Now we have an elementary

Lemma 4.2. (i) *Let $\chi_1,\, \dots,\, \chi_m$ be Dirichlet characters. Then the number $G(\chi_1)\cdots G(\chi_m)/G(\chi_1\cdots\chi_m)$ belongs to the field generated by the values of $\chi_1,\, \dots,\, \chi_m$ over $\mathbf{Q}$.*

(ii) *Let ψ and χ be primitive characters of conductor c and d, respectively. If c and d are relatively prime, then*

$$G(\psi\chi) = \psi(d)\chi(c)G(\psi)G(\chi). \tag{4.2}$$

Proof. For the proof of (i) see [S78, Proposition 4.12], which generalizes [S76, Lemma 8]. In the setting of (ii) take $r,\, s \in \mathbf{Z}$ so that $cr + ds = 1$. Then for $x,\, y \in \mathbf{Z}$ the map $(x,\, y) \mapsto xds + ycr$ gives a bijection of $(\mathbf{Z}/c\mathbf{Z}) \times (\mathbf{Z}/d\mathbf{Z})$ onto $\mathbf{Z}/cd\mathbf{Z}$, and so

$$\begin{aligned} G(\psi\chi) &= \sum_{x=1}^{c}\sum_{y=1}^{d} \mathbf{e}\big((xds + ycr)/cd\big)\psi(xds)\chi(ycr) \\ &= \sum_{x=1}^{c} \psi(xds)\mathbf{e}(xs/c) \sum_{y=1}^{d} \chi(ycr)\mathbf{e}(yr/d), \end{aligned}$$

which proves (4.2).

Theorem 4.3. *Let χ be a primitive character of conductor d such that $\chi(-1) = -1$, and let $\lambda(m) = \left(\frac{3}{m}\right)$. Suppose d is odd and $0 < d/9 \in \mathbf{Z}$. Then*

$$\frac{G(\chi\lambda)}{G(\chi)} = \Big\{2\sqrt{3}\chi(2)\sum_{a=1}^{d-1}(\chi\lambda)(a)\Big\}\Big/\Big\{\sum_{j=1-g}^{d-g}(-1)^j\chi(j)\Big\}, \tag{4.3}$$

where $g = [d/6] + 1$.

Proof. In [S07, Theorem 6.3(v)] we gave a formula for $L(k,\, \chi\lambda)$. Taking $k = 1$ and $\bar{\chi}$ in place of χ, we obtain

$$2\sqrt{3}\chi(2)(\pi i)^{-1}G(\chi)L(1,\, \bar{\chi}\lambda) = \sum_{j=1-g}^{d-g}(-1)^j\chi(j),$$

since $E_{1,0}(t) = 1$. Now $\bar{\chi}\lambda$ is primitive and has conductor $4d$, and so the formula [S07, (4.34)] applied to $\bar{\chi}\lambda$ produces

$$(\pi i)^{-1}G(\chi\lambda)L(1,\, \bar{\chi}\lambda) = \sum_{a=1}^{d-1}(\chi\lambda)(a).$$

Taking the quotient of these two formulas, we obtain (4.3).

In [S07, Theorem 6.3] we gave eight formulas for $L(k,\, \chi\lambda)$, where λ is a "constant" character and χ is a "variable" character. In the above theorem we employed only one of those formulas. We can actually state results about $G(\chi\lambda)/G(\chi)$ in the other seven cases, but they are not so interesting, since we can apply (4.2) to $\chi\lambda$ in those cases, and the case we employed in the above theorem is the only case to which (4.2) is not applicable. Even in that case, the significance of (4.3) is rather obscure. Still, the formula is clear-cut and nontrivial, and we state it here with the hope that future researchers will be able to clarify its nature in a better perspective.

We end our discussion of this subject by showing the quantity of (4.3) can be determined in a different way. We begin with some preliminary results.

Lemma 4.4. *Let χ and ψ be primitive characters of conductor p^m and p^n, respectively, where p is a prime number and m, n are positive integers. Suppose $m \geq n$ and $\chi\psi$ has conductor p^m. Then*

$$G(\chi)G(\psi) = G(\chi\psi)\sum_{a=1}^{p^n}\chi(1 - p^{m-n}a)\psi(a). \tag{4.4}$$

This was given in [S76, (4.2)].

Lemma 4.5. *Let χ be a primitive character of conductor c, where $c = 3^m$ with $m > 1$ or $c = 2^m$ with $m > 3$, and let μ_3 and μ_4 denote the primitive characters of conductor 3 and 4, respectively. Let $\chi' = \chi\mu_3$ if $c = 3^m$ and $\chi' = \chi\mu_4$ if $c = 2^m$. Then $G(\chi') = \varepsilon G(\chi)$ with $\varepsilon = \pm 1$ determined by $\chi(1 - 3^{m-1}) = \mathbf{e}(\varepsilon/3)$ if $c = 3^m$ and $\chi(1 - 2^{m-2}) = \mathbf{e}(\varepsilon/4)$ if $c = 2^m$.*

Proof. We first consider the case $c = 2^m$. By (4.4), $G(\chi)G(\mu_4)/G(\chi') = \beta - \gamma$ with $\beta = \chi(1-2^{m-2})$ and $\gamma = \chi(1+2^{m-2})$. Since $m > 3$, we have $(1-2^{m-2})^2 \equiv 1 - 2^{m-1} \pmod{2^m}$, $(1-2^{m-2})(1+2^{m-2}) \equiv 1 \pmod{2^m}$, and $(1-2^{m-2})^4 \equiv 1 \pmod{2^m}$, and so $\beta^4 = \beta\gamma = 1$. Suppose $\beta = \pm 1$. Then $\chi(1 - 2^{m-1}) = 1$, and so $\chi(1 - 2^{m-1}a) = \chi(1 - 2^{m-1})^a = 1$ for every $a \in \mathbf{Z}$, which means that χ

has conductor $\leq 2^{m-1}$, a contradiction. Thus $\beta \neq \pm 1$, and so $\beta = \varepsilon i$ with $\varepsilon = \pm 1$. Since $G(\mu_4) = 2i$, we have $G(\chi)/G(\chi') = (\beta - \beta^{-1})/2i = \varepsilon$, which proves the case $c = 2^m$.

If $c = 3^m$, we have similarly $G(\chi)G(\mu_3)/G(\chi') = \beta - \gamma$ with $\beta = \chi(1-3^{m-1})$ and $\gamma = \chi(1+3^{m-1})$. We easily see that $\beta\gamma = \beta^3 = 1$ and $\beta \neq 1$. Since $G(\mu_3) = \sqrt{3}i$, we have $G(\chi)/G(\chi') = (\beta - \beta^{-1})/\sqrt{3}i = \varepsilon$, and ε is determined by $\beta = \mathbf{e}(\varepsilon/3)$. This completes the proof.

4.6. Returning to the setting of Theorem 4.3, put $d = 3^m f$ with $m \in \mathbf{Z}$ and $0 < f \in \mathbf{Z}$, $3 \nmid f$; put also $\chi = \chi_0\chi_1$ with characters χ_0 and χ_1 of conductor 3^m and f, respectively. Since $\lambda = \mu_3\mu_4$, we have $\chi\lambda = \chi_0\mu_3\chi_1\mu_4$. By (4.2) we have $G(\chi_1\mu_4) = 2i\chi_1(4)\mu_4(f)G(\chi_1)$, and so

$$\begin{aligned} G(\chi\lambda) &= (\chi_1\mu_4)(3^m)(\chi_0\mu_3)(4f)G(\chi_0\mu_3)G(\chi_1\mu_4) \\ &= 2i(-1)^m\chi_0(f)\chi(4)\chi_1(3^m)\lambda(f)G(\chi_0\mu_3)G(\chi_1). \end{aligned}$$

Also, $G(\chi) = \chi_0(f)\chi_1(3^m)G(\chi_0)G(\chi_1)$. Therefore

$$G(\chi\lambda)/G(\chi) = 2i(-1)^m\lambda(f)\chi(4)G(\chi_0\mu_3)/G(\chi_0).$$

Applying Lemma 4.5 to $G(\chi_0\mu_3)/G(\chi_0)$, we thus obtain

$$G(\chi\lambda)/G(\chi) = 2i(-1)^m\lambda(f)\chi(4)\varepsilon, \tag{4.5}$$

where $\varepsilon = \pm 1$ is determined by $\chi_0(1-3^{m-1}) = \mathbf{e}(\varepsilon/3)$.

5. The case of a domain of positivity

5.1. There is a natural analogue of $\zeta(s;\, a,\, \gamma)$ defined on the space of symmetric matrices. To be explicit, with a positive integer n we denote by V the set of all real symmetric matrices of size n, and write $h > 0$ resp. $h \geq 0$ for $h \in V$ when h is positive definite resp. nonnegative. We put $V_\mathbf{C} = V \otimes_\mathbf{R} \mathbf{C}$,

$$\kappa = (n+1)/2, \quad P = \{h \in V \mid h > 0\}, \quad \text{and}$$
$$\mathfrak{H} = \{x + iy \in V_\mathbf{C} \mid x \in V,\, y \in P\}.$$

For $h \in V$ we denote by $\lambda(h)$ and $\mu(h)$ the maximum and minimum absolute value of eigenvalues of h, respectively. We easily see that V is a normed space with $\lambda(h)$ as the norm of h, that is, $\lambda(ch) = |c|\lambda(h)$ for $c \in \mathbf{R}$ and $\lambda(h+k) \leq \lambda(h) + \lambda(k)$. For $0 \leq d \in \mathbf{Z}$ we denote by S_d the space of all $\mathbf{C}$-valued homogeneous polynomial functions on V of degree d. Here are two easy facts:

$$\mathrm{tr}(gh) \geq \lambda(h)\mu(g) \quad \text{if} \quad g,\, h \in P. \tag{5.1}$$

(5.2) $|\xi(h)| \leq c_\xi\lambda(h)^d$ for every $\xi \in S_d$ and $h \in V$ with a positive constant c_ξ that depends only on ξ.

To prove (5.1), we may assume that g is diagonal. For $g = \mathrm{diag}[\mu_1, \ldots, \mu_n]$ we have $\mathrm{tr}(gh) = \sum_{i=1}^n \mu_i h_{ii} \geq \mu(g)\mathrm{tr}(h) \geq \mu(g)\lambda(h)$, since $\mathrm{tr}(h)$ is the sum of all eigenvalues of h. As for (5.2), given $h \in V$, take an orthogonal matrix p so that $h = {}^t p \cdot \mathrm{diag}[\kappa_1, \ldots, \kappa_n]p$ with $\kappa_i \in \mathbf{R}$. Since $|p_{ij}| \le 1$ and $|\kappa_i| \le \lambda(h)$, we see that $|h_{ij}| \le n\lambda(h)$, from which we obtain (5.2).

We now consider three types of infinite series:

$$\Phi(s;\, L,\, a,\, z) = \sum_{h \in L} \det\bigl(2\pi i(h-z)\bigr)^{-s}\mathbf{e}\bigl(\mathrm{tr}(a(h-z))\bigr), \tag{5.3}$$

$$F(s;\, L,\, a,\, z) = \sum_{h\in L,\, h+a>0} \det(h+a)^{-s}\mathbf{e}\bigl(\mathrm{tr}(hz)\bigr), \tag{5.4}$$

$$F_0(s;\, L,\, a,\, z) = \sum_{0 \le h \in L} \det(h+a)^{-s}\mathbf{e}\bigl(\mathrm{tr}(hz)\bigr). \tag{5.5}$$

Here $s \in \mathbf{C}$, L is a lattice in V, $a \in V$, $z \in \mathfrak{H}$, and the sum of (5.4) is extended over all $h \in L$ such that $h + a \in P$; the sum of (5.3) is simply over all $h \in L$; in (5.5) we assume that $a > 0$ and the sum is extended over all nonnegative h in L. For $z \in \mathfrak{H}$ and $s \in \mathbf{C}$ we define $\det(-2\pi i z)^s$ so that it coincides with $\det(2\pi p)^s$ if $z = ip$ with $p \in P$.

Lemma 5.2. *For every $\xi \in S_d$ the infinite series*

$$\sum_{h\in L,\, h+a>0} \det(h+a)^s\xi(h+a)\mathbf{e}\bigl(\mathrm{tr}(hz)\bigr) \tag{5.6}$$

converges absolutely and locally uniformly for $(s,\, a,\, z) \in \mathbf{C} \times V \times \mathfrak{H}$.

Proof. For a fixed positive number α the number of $h \in L$ such that $\lambda(h) \le \alpha$ is finite, as $\bigl\{h \in V \,\big|\, \lambda(h) \le \alpha\bigr\}$ is compact and L is discrete in V. Thus, to prove the convergence of (5.6), we can restrict h to those that satisfy $\lambda(h) > \lambda(a)+1$ for every $a \in A$, where A is a fixed compact subset of V. For such an h we have $1 \le \lambda(h+a) < 2\lambda(h)$ and $\det(h+a) \le \lambda(h+a)^n \le 2^n\lambda(h)^n$. Also, by (5.2), $|\xi(h+a)| \le c_\xi \lambda(h+a)^d \le c_\xi 2^d \lambda(h)^d$. Thus for $\mathrm{Re}(s) = \sigma$ we have $|\det(h+a)^s\xi(h+a)| \le 2^{n\sigma+d}c_\xi\lambda(h)^{n\sigma+d}$. On the other hand, for $0 < N \in \mathbf{Z}$ the number of $h \in L$ such that $N \le \lambda(h) < N+1$ is less than CN^b with positive constants C and b. Put $g = 2\pi\mathrm{Im}(z)$. Then $\bigl|\mathbf{e}\bigl(\mathrm{tr}(hz)\bigr)\bigr| = e^{-\mathrm{tr}(gh)}$. Since $-\mathrm{tr}(gh) \le -\lambda(h)\mu(g)$ by (5.1), our partial sum can be majorized by $2^{n\sigma+d}c_\xi C\sum_{N=1}^{\infty}(N+1)^{n\sigma+d+b}e^{-N\mu(g)}$, which proves our lemma.

Theorem 5.3. (i) *The infinite series of (5.3)* converges absolutely and locally uniformly in $(s,\, a,\, z) \in \bigl\{s \in \mathbf{C} \,\big|\, \mathrm{Re}(s) > n\bigr\} \times V \times \mathfrak{H}$. Thus it defines a holomorphic function in s for $\mathrm{Re}(s) > n$.

(ii) *The infinite series of (5.4)* and *(5.5) converge absolutely and locally uniformly in $(s,\, a,\, z) \in \mathbf{C} \times V \times \mathfrak{H}$, and so they define entire functions of s.*

(iii) *For* $\mathrm{Re}(s) > n$ *we have*

$$\mathrm{vol}(V/L)F(\kappa - s;\, L,\, a,\, z) = \Gamma_n(s)\Phi(s;\, L',\, a,\, z), \tag{5.7}$$

where $L' = \bigl\{x \in V \,\big|\, \mathrm{tr}(xL) \subset \mathbf{Z}\bigr\}$ *and*

$$\Gamma_n(s) = \pi^{n(n-1)/4}\prod_{k=0}^{n-1}\Gamma\bigl(s-(k/2)\bigr).$$

(iv) *For* $0 \le m \in \mathbf{Z}$ *the values* $F(-m;\, L,\, a,\, z)$, $F_0(-m;\, L,\, a,\, z)$, *and* $\Phi(\kappa + m;\, L,\, a,\, z)$ *are polynomial functions of* a *of degree* $\le mn$, *whose coefficients depend on* L *and* z.

Statement (iv) for F_0 can be taken literally, but the cases of F and Φ require some clarifications, which will be given in the proof.

Proof. Assertion (i) is included in [S82, Lemma 1.3]; (ii) follows from Lemma 5.2. To prove (iii), given $s \in \mathbf{C}$ and $p \in P$, we consider a function g on V defined by

$$g(u) = \begin{cases} e^{-\mathrm{tr}(up)} \det(u)^{s-\kappa} & (u \in P), \\ 0 & (u \notin P), \end{cases}$$

and define its Fourier transform $\hat{g}$ by

$$\hat{g}(t) = \int_V \mathbf{e}\big(- \mathrm{tr}(ut)\big) g(u) du \qquad (t \in V),$$

where $du = \prod_{i \le j} du_{ij}$. As shown in [S82, (1.22)], $\hat{g}(t) = \Gamma_n(s) \det(p + 2\pi i t)^{-s}$, provided $\mathrm{Re}(s) > \kappa - 1$. Then the Poisson summation formula establishes equality (5.7) with both sides multiplied by $\mathbf{e}\big(\mathrm{tr}(az)\big)$ for $z = (-2\pi i)^{-1} p$ when the series of (5.3) and (5.4) are convergent, which is the case at least when $\mathrm{Re}(s) > n$. Since both sides of (5.7) are holomorphic in z if $\mathrm{Re}(s) > n$, we obtain (iii) as stated.

To prove (iv), take a $\mathbf{C}$-basis B of $\sum_{d=0}^{mn} S_d$. For $0 \le m \in \mathbf{Z}$ we have

$$F_0(-m;\, L,\, a,\, z) = \sum_{0 \le h \in L} \det(h + a)^m \mathbf{e}\big(\mathrm{tr}(hz)\big). \tag{5.8}$$

We can put $\det(h+a)^m = \sum_{\beta \in B} \beta(a) f_\beta(h)$ with polynomial functions f_β, and so

$$F_0(-m;\, L,\, a,\, z) = \sum_{\beta \in B} \beta(a) G_\beta(z) \quad \text{with} \tag{5.8a}$$

$$G_\beta(z) = \sum_{0 \le h \in L} f_\beta(h) \mathbf{e}\big(\mathrm{tr}(hz)\big). \tag{5.8b}$$

Thus $F_0(-m;\, L,\, a,\, z)$ is a polynomial in a as stated in (iv). This argument is basically valid for F in place of F_0, but the functions corresponding to G_β in that case may depend on a. To avoid that difficulty, we first take a compact subset A of V and restrict a to A. As shown in the proof of Lemma 5.2, we can find a subset M of L independent of a such that

$$\big\{ h \in L \,\big|\, h > 0,\, h + a > 0 \big\} = M \sqcup K_a,$$

with a finite set K_a for each $a \in A$. Then, taking F in place of F_0, we obtain, for $a \in A$,

$$F(-m;\, L,\, a,\, z) = \sum_{\beta \in B} \beta(a) H_\beta(z) \quad \text{with} \tag{5.9a}$$

$$H_\beta(z) = \sum_{h \in K_a} f_\beta(h) \mathbf{e}\big(\mathrm{tr}(hz)\big) + \sum_{h \in M} f_\beta(h) \mathbf{e}\big(\mathrm{tr}(hz)\big). \tag{5.9b}$$

Thus the statement about $F(-m;\, L,\, a,\, z)$ in (iv) must be understood in the sense of (5.9a, b). It is a polynomial in a whose coefficients $H_\beta(z)$ is the sum of a "principal part" that is independent of a and a finite sum depending on a.

As for the value of Φ, from (5.7) we see that Φ is an entire functions of s. Also $\Gamma_n(s)^{-1}$ is nonzero for $(n-1)/2 < s \in \mathbf{R}$. Thus $\Phi(\kappa + m;\, L',\, a,\, z)$ is a nonzero constant times $F(-m;\, L,\, a,\, z)$, and so is a polynomial in a in the sense explained above.

5.4. Let us add some remarks. Clearly F_0 of (5.5) is a natural generalization of (0.1), but we introduced F and Φ as in (5.4) and (5.3), as we think they are natural objects of study closely related to (5.5). It must be remembered, however, that (5.5) includes (0.1) as a special case only if $|\gamma| < 1$. To define something like (5.5) that includes (0.1) with $|\gamma| = 1$ is one of the open problems in this area.

Next, there are four classical types of domains of positivity associated with tube domains discussed in [S82]. Our V, P, and $\mathfrak{H}$ in this section belong to the easiest type. We can in fact define the analogues of (5.3), (5.4), and (5.5) for all three other types of domains, and prove the results similar to Theorem 5.3 in those cases.

As to the nature of the polynomials in the variable a obtained in Theorem 5.3(iv), we do not have their description as explicit as what we know about $E_{c,m}(t)$. Still, we can show that they are of a rather special kind. For that purpose we need a matrix of differential operators $\partial_a = (\partial_{ij})_{i,j=1}^n$ on V as folows. Taking a variable symmetric matrix $a = (a_{ij})$ on V, we put $\partial_{ii} = \partial/\partial a_{ii}$ and $\partial_{ij} = 2^{-1}\partial/\partial a_{ij}$ for $i \neq j$. Then for every $\varphi \in S_d$ we can define a diffrential operator $\varphi(\partial_a)$. In particular, taking $\varphi(a) = \det(a)$, we put

$$\Delta_a = \det(\partial_a) = \sum_\sigma \operatorname{sgn}(\sigma)\partial_{1\sigma(1)}\cdots\partial_{n\sigma(n)}, \tag{5.10}$$

where σ runs over all permutations of $\{1, \ldots, n\}$. It is well known that

$$\Delta_a\bigl(\det(a)^s\bigr) = \prod_{k=0}^{n-1}(s + k/2)\cdot\det(a)^{s-1}. \tag{5.11}$$

This is a special case of a general formula on $\varphi(\partial_a)\det(z)^s$ for $\varphi \in S_d$ given in [S84].

Fixing L and z, for $0 \le m \in \mathbf{Z}$ put

$$\mathscr{E}_m(a) = F_0(-m;\, L,\, a,\, z). \tag{5.12}$$

We have shown that $\mathscr{E}_m$ is a polynomial of degree $\le mn$. Notice that $\mathscr{E}_0(a) = \sum_{0\le h\in L}\mathbf{e}\bigl(\operatorname{tr}(hz)\bigr)$. From (5.8) and (5.11) we obtain

$$\Delta_a\mathscr{E}_m(a) = \prod_{k=0}^{n-1}(m + k/2)\cdot\mathscr{E}_{m-1}(a). \tag{5.13}$$

This is a generalization of the formula $(d/dt)E_{c,m}(t) = mE_{c,m-1}(t)$, noted in [S07, (4.3c)].

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