Laplace Transform Representations and Paley–Wiener Theorems for Functions on Vertical Strips

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Received: July 1, 2009

Communicated by Thomas Peternell

Abstract. We consider the problem of representing an analytic function on a vertical strip by a bilateral Laplace transform. We give a Paley–Wiener theorem for weighted Bergman spaces on the existence of such representations, with applications. We generalise a result of Batty and Blake, on abscissae of convergence and boundedness of analytic functions on halfplanes, and also consider harmonic functions. We consider analytic continuations of Laplace transforms, and uniqueness questions: if an analytic function is the Laplace transform of functions f_1, f_2 on two disjoint vertical strips, and extends analytically between the strips, when is $f_1 = f_2$? We show that this is related to the uniqueness of the Cauchy problem for the heat equation with complex space variable, and give some applications, including a new proof of a Maximum Principle for harmonic functions.

2010 Mathematics Subject Classification: Primary 44A10; Secondary 30E20, 31A10, 35K05

Keywords and Phrases: Laplace transform, Paley–Wiener theorem, heat equation, Maximum Principle, analytic continuation, Hardy spaces, Bergman spaces

1 Introduction and notation

We are concerned with Laplace transforms: for an analytic function F on $\{a < \mathrm{Re}(z) < b\}$, we would like to know when

$$F(z) = \mathcal{L}h(z) \sim \int_{t=-\infty}^{\infty} e^{-zt} h(t)\, dt \sim \int_{t=-\infty}^{\infty} e^{-xt} h(t) e^{-iyt}\, dt$$

for some h, in some sense: either as an absolutely convergent Lebesgue integral, or as the L^2 or tempered distribution Fourier transform of $e^{-xt}h(t)$. Our normalisation of the Fourier transform is

$$\widehat{f}(\omega) \sim \int_{t=-\infty}^{\infty} e^{-i\omega t} f(t)\, dt, \qquad f(t) \sim \frac{1}{2\pi} \widehat{\widehat{f}}(-t).$$

In Section 2 we give a fairly general Paley–Wiener theorem which guarantees the existence of such an h for analytic functions F in certain weighted Bergman spaces, with applications. In Section 3 we generalise a result of C. Batty and M. D. Blake concerning bounded functions on halfplanes; we obtain the same result, but under weaker assumptions, as well as a similar result for harmonic functions.

In Section 4 we consider the *uniqueness* problem, which is important because analytic functions can sometimes be represented by Laplace transforms of different h on disjoint vertical strips. We obtain an explicit formula for analytic continuation under quite mild conditions, and relate this to the heat equation. Thus uniqueness theorems on the heat equation immediately give uniqueness theorems for boundary values of harmonic functions; see Corollaries 4.5, 4.6. Finally, Sections 5, 6 contain some longer proofs.

The problem of existence of Laplace transform representations for functions in certain spaces has been studied extensively; for example, see [4], [5], [9], [12], [20], [27], [29].

Given any domain $\Omega \subseteq \mathbb{C}$ and Banach space E, we write $\mathrm{Hol}(\Omega, E)$ for the set of all analytic functions $F : \Omega \to E$, or just $\mathrm{Hol}(\Omega)$ when $E = \mathbb{C}$. We need the theory of *Hardy spaces*: see [1], [10], [21], [25] and [26].

Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geqslant 0\}$. For any Banach space E, $1 \leqslant p < \infty$ and $F \in \mathrm{Hol}(\mathbb{C}_+, E)$, define

$$\|F\|_{H^p(\mathbb{C}_+,E)} = \sup_{r>0} \left(\int_{-\infty}^{\infty} \|F(r+iy)\|_E^p \frac{dy}{2\pi} \right)^{1/p}.$$

The set of all F with $\|F\| < \infty$ is the *Hardy space* $H^p(\mathbb{C}_+, E)$. When $E = \mathbb{C}$ we write simply $H^p(\mathbb{C}_+)$. We mainly use the case $p = 2$ with E a Hilbert space.

The classical Paley–Wiener Theorem says that $\mathcal{L} : L^2(\mathbb{R}_+, E) \to H^2(\mathbb{C}_+, E)$ is a *unitary operator* from L^2 *onto* H^2, *provided* that E is a Hilbert space:

$$\|f\|_{L^2(\mathbb{R}_+,E)} = \left(\int_{t=0}^{\infty} \|f(t)\|_E^2 \, dt \right)^{1/2} = \|\mathcal{L}f\|_{H^2(\mathbb{C}_+,E)},$$

and $\mathcal{L}^{-1} : H^2(\mathbb{C}_+, E) \to L^2(\mathbb{R}_+, E)$ is well–defined. Here, we are thinking of $H^2(\mathbb{C}_+, E) \subset L^2(i\mathbb{R}, E)$ in terms of a.e. boundary values.

2 Hilbert space Paley–Wiener type results

Theorem 2.1 *Let* $-\infty \leqslant a < b \leqslant +\infty$*, let* E *be a Hilbert space, and let* $\Omega = \{z \in \mathbb{C} : a < \mathrm{Re}(z) < b\}$.

Suppose that $v : (a, b) \to [0, +\infty]$ *is Lebesgue measurable, with* $v > 0$ *almost everywhere. For any* $F \in \mathrm{Hol}(\Omega, E)$*, define*

$$\|F\|_{L^2(\Omega,v,E)}^2 = \frac{1}{2\pi} \int_{x=a}^{b} \int_{y=-\infty}^{\infty} \|F(x+iy)\|_E^2 \, v(x) \, dy \, dx.$$

For any $h : \mathbb{R} \to E$ strongly measurable, define

$$N_v(h)^2 = \int_{t\in\mathbb{R}} \|h(t)\|_E^2 \left(\int_{x=a}^{b} e^{-2xt} v(x)\, dx \right) dt.$$

Then*, whenever $N_v(h) < \infty$, we have $N_v(h) = \|\mathcal{L}h\|_{L^2(\Omega,v,E)}$.*
Conversely*, let $F \in \mathrm{Hol}(\Omega, E)$ with $\|F\|_{L^2(\Omega,v,E)} < \infty$.* Assume also *that:*

$$\forall\, a < \alpha < \beta < b, \quad \exists\, \varepsilon(\alpha,\beta) > 0 \text{ such that } \int_\alpha^\beta v(x)^{-\varepsilon(\alpha,\beta)} dx < \infty. \tag{1}$$

Then: *$\exists\, h$ such that $F = \mathcal{L}h$ on Ω, and $N_v(h) = \|F\|_{L^2(\Omega,v,E)}$. Furthermore, $F \in L^2(c + i\mathbb{R})$ for* every *$a < c < b$, so h is given by the standard Bromwich Inversion Formula*

$$h(t) \sim \frac{1}{2\pi} \int_{y=-\infty}^{\infty} F(c+iy) e^{ct} e^{iyt}\, dy \sim \frac{1}{2\pi i} \int_{c+i\mathbb{R}} F(z) e^{zt}\, dz,$$

in the sense of $L^2(\mathbb{R}, E)$ Fourier transforms.

The paper [11] proves this result in the special case $a = 0$, $b = \infty$ and $v(x) = x^r$ with $r \geqslant 0$, and gives some applications. However, their method is different and probably cannot be generalised (*the conformal transformation $\frac{1-z}{1+z}$ induces an isometric isomorphism with a weighted Bergman space on the disc, for which $(z^n)_{n\geqslant 0}$ is an orthogonal basis*). Other related results and examples are given in Section 2 of [19].

Proof: The proof that $N_v(h) < \infty$ implies $\mathcal{L}h \in L^2(\Omega, v, E)$ with the same norm is not hard: by Fubini's Theorem, $\int_{x=a}^{b} \|e^{-xt}h(t)\|^2_{L^2(\mathbb{R},E)} v(x)\, dx < \infty$. Thus $e^{-xt}h(t) \in L^2$ for a.e. $x \in (a,b)$, because $v > 0$ a.e. Now the Plancherel Theorem can be applied to the function $e^{-xt}h$, for a.e. x, and integrating with $v(x)dx$ gives the result.

For the converse: first, let $a < \alpha < \beta < b$. We must show that F is *bounded* on $\{x + iy : \alpha \leqslant x \leqslant \beta\}$. Let $r > 0$ be sufficiently small, so that $a < \alpha - r < \alpha < \beta < \beta + r < b$. Fix $\varphi \in E^*$ and consider $F_\varphi(z) = \varphi(F(z))$. We have the following result, which is a substitute for the lack of subharmonicity of $|F_\varphi|^p$ when $p < 1$. See Lemma 2, p. 172 of [14], there attributed to Hardy and Littlewood; the proof is given also on p. 185 of [23]:

$$\forall\, p > 0, \qquad |F_\varphi(\lambda)| \leqslant C_p \left(\frac{1}{\pi r^2} \int_{|z-\lambda|<r} |F_\varphi(z)|^p\, dA(z) \right)^{1/p}, \tag{2}$$

with some $C_p < \infty$. (*This is true more generally for harmonic functions in several variables. The case $p \geqslant 1$ is trivial by the Mean Value Property*). By assumption, $\int_{\alpha-r}^{\beta+r} v(x)^{-\varepsilon} dx < \infty$ for some $\varepsilon > 0$. Now let $p = 2\varepsilon(1+\varepsilon)^{-1}$. Apply Hölder's inequality with exponent $2/p$ to obtain that

$$\int_{|z-\lambda|<r} \|F(z)\|^p\, dA(z) = \int_{|z-\lambda|<r} \|F(z)\|^p v(x)^{p/2} v(x)^{-p/2}\, dA(z)$$

is bounded by a multiple of $\left(\int_{|z-\lambda|<r}\|F(z)\|^2 v(x)\,dA(z)\right)^{p/2}$, independently of λ with $\alpha \leqslant \mathrm{Re}(\lambda) \leqslant \beta$. By (2) and (1), we now have $|F_\varphi(\lambda)| \leqslant K\|\varphi\|_{E^*}$, so indeed F is bounded on $\{\alpha \leqslant x \leqslant \beta\}$ as required.
Second, suppose that $\int_{-\infty}^{\infty}\|F(x+iy)\|^2\,dy < \infty$ for $x = \alpha, \beta$, where $a < \alpha < \beta < b$. Thus for each $Y > 0$, Cauchy's Integral Formula gives

$$F(\lambda) = \frac{1}{2\pi i}\int_{\partial R_Y}\frac{F(z)}{z-\lambda}dz \quad \text{for all } \lambda \in R_Y = (\alpha,\beta)\times(-Y,Y).$$

But F is bounded on R_Y, uniformly in Y, by above; so we can let $Y \to \infty$ for each fixed λ to obtain

$$F(\lambda) = \frac{1}{2\pi i}\left(\int_{\beta+i\mathbb{R}} - \int_{\alpha+i\mathbb{R}}\right)\frac{F(z)}{z-\lambda}dz, \quad \text{whenever } \mathrm{Re}(\lambda) \in (\alpha,\beta).$$

Now $\int_{\alpha+i\mathbb{R}}\frac{F(z)}{z-(\alpha+\omega)}dz$, as a function of $\omega \in \mathbb{C}_+$, is the Szegö projection of the $L^2(i\mathbb{R},E)$ function $F(iy+\alpha)$ onto the Hardy space $H^2(\mathbb{C}_+,E)$, and so by the Paley–Wiener Theorem it can be represented as $\mathcal{L}f_1$ for some $f_1 \in L^2(\mathbb{R}_+,E)$. We can consider similarly $\int_{\beta+i\mathbb{R}}\frac{F(z)}{z-(\beta-\omega)}dz$. Thus

$$F(\lambda) = -\int_{t=0}^{\infty} e^{-(\lambda-\alpha)t} f_1(t)\,dt + \int_{t=0}^{\infty} e^{-(\beta-\lambda)t} f_2(t)\,dt,$$

and so $F(\lambda) = \mathcal{L}h(\lambda) = \int_{t=-\infty}^{\infty} e^{-\lambda t}h(t)\,dt$ on $\{\alpha < \mathrm{Re}(\lambda) < \beta\}$, with

$$\int_0^\infty \|e^{-\alpha t}h(t)\|^2\,dt, \int_0^\infty \|e^{\beta t}h(-t)\|^2\,dt < \infty.$$

This shows that $e^{-ct}h(t) \in L^2(\mathbb{R},E)$ for each $\alpha < c < \beta$.
Now $v > 0$ a.e., so $\int_{-\infty}^{\infty}\|F(x+iy)\|^2\,dy < \infty$ for a.e. $x \in (a,b)$. So choose sequences $(\alpha_j) \searrow a$ and $(\beta_j) \nearrow b$ such that this holds with $x = \alpha_j, \beta_j$. Then $F = \mathcal{L}h_j$ on $\{\alpha_j < \mathrm{Re}(\lambda) < \beta_j\}$ for each j. By uniqueness of the Fourier transform we must have $h_j \equiv h_1 = h$ a.e.
So finally $F = \mathcal{L}h$ on $\{a < \mathrm{Re}(\lambda) < b\}$, and Plancherel's Theorem gives

$$\frac{1}{2\pi}\int_{y=-\infty}^{\infty}\|F(x+iy)\|^2 dy = \int_{t=-\infty}^{\infty} e^{-2xt}\|h(t)\|^2\,dt$$

for each $a < x < b$. Hence $N_v(h) = \|F\|_{L^2(\Omega,v,E)}$.
□
Similarly, with the Hausdorff–Young theorem and Paley–Wiener theorem for H^p, we can easily obtain the following result:

THEOREM 2.2 *Let $F \in \mathrm{Hol}\{a < \mathrm{Re}(z) < b\}$, let $1 < p \leqslant 2$, let v satisfy the same conditions as Theorem 2.1, and suppose that*

$$\int_{x=a}^{b}\int_{y=-\infty}^{\infty}|F(x+iy)|^p\,v(x)\,dy\,dx < \infty. \tag{3}$$

Then there exists some h *such that* $F = \mathcal{L}h$ *and*

$$\int_{x=a}^{b} \left(\int_{t=-\infty}^{\infty} e^{-p'xt} |h(t)|^{p'} \, dt \right)^{p-1} v(x) \, dx < \infty. \tag{4}$$

We can consider Dirichlet–type norms also; for example:

COROLLARY 2.3 *Let* $F \in \mathrm{Hol}(\mathbb{C}_+, E)$*, for a Hilbert space* E*. Then*

$$\iint_{\mathbb{C}_+} \|F'(z)\|^2 x \, dx \, dy < \infty \qquad \Longleftrightarrow \qquad F \in H^2(\mathbb{C}_+, E) + \{\textit{constants}\}.$$

This is obvious, since $\int_0^\infty \|h(t)\|^2 dt/t^2 < \infty$ if and only if $h(t)/t \in L^2(\mathbb{R}_+, E)$ if and only if $\mathcal{L}(h(t)/t) \in H^2$.

COROLLARY 2.4 *Let* $F \in \mathrm{Hol}\{0 < \mathrm{Re}(z) < R\}$ *be bounded, for some* $0 < R \leqslant +\infty$*. Then* $\exists\, g : \mathbb{R} \to \mathbb{C}$ *such that* $F(z) = z\mathcal{L}g(z)$*, and*

$$\int_{-\infty}^{0} e^{2R|t|} |g(t)|^2 \, dt < \infty, \qquad \sup_{T > 1/R} \left(\frac{1}{T} \int_0^T |g(t)|^2 \, dt \right) < \infty.$$

Also $\sup_{0 \leqslant c \leqslant R} \|e^{-ct} g\|_{BMO(\mathbb{R})} < \infty$*. In particular,*

$$\int_{t=0}^{\infty} \frac{|g(t)| + e^{-Rt}|g(-t)|}{1+t^2} dt < \infty.$$

In the case $R = +\infty$*, we have* $g(t) = 0$ *for all* $t < 0$*.*

$BMO(\mathbb{R})$ is the very important Bounded Mean Oscillation space, discussed in [1], [16], [23] and many other books, which often serves as a useful substitute for $L^\infty(\mathbb{R})$. For locally integrable $f : \mathbb{R} \to \mathbb{C}$ we have

$$\|f\|_{BMO(\mathbb{R})} = \sup_I |f - f_I|_I, \quad \text{where} \quad f_I = \frac{1}{|I|} \int_I f(t) \, dt,$$

I ranges over all *bounded intervals* of $\mathbb{R}$, and $|I|$ is the *length*.
PROOF: The existence of g is immediate from Theorem 2.1, if we consider $G(z) = F(z)/z$ and take, e.g. $v(x) = x/(1 + x^3)$. The estimates follow from Plancherel's Theorem:

$$\int_{-\infty}^{\infty} e^{-2xt} |g(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{F(x+iy)}{x+iy} \right|^2 \, dy \ll \frac{1}{x}.$$

For the estimate with $t > 0$, let $T > 1/R$ and consider $\int_{t=0}^T$ only with $x = 1/T$. For $\int_{t=-\infty}^0$ we just let $x \nearrow R$.

For the BMO result, let $0 < c < R$. Then $\frac{F(c+iy)}{c+iy}$ is an L^2 function of $y \in \mathbb{R}$, with $\left|\frac{F(c+iy)}{c+iy}\right| \leqslant \frac{\sup|F|}{|y|}$. We have

$$e^{-ct}g(t) \sim \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(c+iy)}{c+iy} e^{iyt}\, dy.$$

Now apply Lemma 2.5 below to get $\|e^{-ct}g\|_{BMO(\mathbb{R})} \leqslant K$ for some K independent of c. For $c = 0$ and $c = R$, choose sequences $c_j \searrow 0$ and $c_j \nearrow R$ and use Dominated Convergence: for each interval I, $(e^{-c_j t}g)_I \to g_I$ or $(e^{-Rt}g)_I$ as appropriate. Then $|e^{-c_j t}g - (e^{-c_j t}g)_I|_I$ also converges appropriately; since the BMO norm is given by a supremum over all I, we have the result.
$\square$

LEMMA 2.5 *If* $f \in L^2(\mathbb{R})$ *then* $\left\|\hat{f}\right\|_{BMO(\mathbb{R})} \leqslant C \sup_{\beta\in\mathbb{R}} |\beta f(\beta)|$, *where* C *is a universal constant independent of* f.

PROOF: By considering the restrictions of f to $\mathbb{R}_+$ and $\mathbb{R}_-$ separately, it is enough to consider $f \in L^2(\mathbb{R}_+)$ with $|\beta f(\beta)| \leqslant 1$. Take $u \in L^2(\mathbb{R}_+)$ and consider the convolution $(k * u)(\alpha) = \int_{s=0}^{\alpha} k(\alpha - s)u(s)\, ds$. By Hardy's Inequality (see [18]),

$$\begin{aligned}\int_{\alpha=0}^{\infty} |(\beta f * u)(\alpha)|^2 \frac{d\alpha}{\alpha^2} &\leqslant \int_{\alpha=0}^{\infty} \left(\frac{1}{\alpha}\int_{s=0}^{\alpha} |u(s)|\, ds\right)^2 d\alpha \\ &\leqslant 4\int_{s=0}^{\infty} |u(s)|^2 ds.\end{aligned}$$

Taking Laplace transforms and using (the easy half of) Theorem 2.1 gives

$$\iint_{\mathbb{C}_+} |\mathcal{L}(\beta f)(z)\mathcal{L}u(z)|^2 x\, dx\, dy \leqslant K\|\mathcal{L}u\|^2_{H^2(\mathbb{C}_+)}.$$

But this says exactly that $|\mathcal{L}(\beta f)(z)|^2 x\, dx\, dy = |(\mathcal{L}f)'(z)|^2 x\, dx\, dy$ is a *Carleson Measure* on $\mathbb{C}_+$. Hence $\mathcal{L}f \in \mathrm{Hol}(\mathbb{C}_+)$ is the *Poisson integral* of some function $U \in BMO$, by [13]. But also $\mathcal{L}f \in H^2(\mathbb{C}_+)$, and so $\mathcal{L}f$ is the Poisson integral of its boundary function $\hat{f}$. Hence $\hat{f} = U \in BMO$ as required.
$\square$

In Theorem 3.1 below we obtain further results on g, assuming extra conditions on F (*decay behaviour on a vertical line*).

3 RESULTS ASSUMING DECAY ON A VERTICAL LINE

The following theorem generalises the main result of [3].

THEOREM 3.1 *Let* $0 < R \leqslant +\infty$ *and* $\Omega = \{z : 0 < \mathrm{Re}(z) < R\}$. *Let* E *be a Banach space, and let* $F \in \mathrm{Hol}(\Omega, E)$ *be bounded. Assume that* $\exists\, 0 < c < R$, $0 < \delta \leqslant 1$ *and* $\nu > 1$ *such that*

$$\forall\, \varphi \in E^*, \qquad \int_{y=-\infty}^{\infty} \frac{\left|\varphi(F(c+iy))\right|^\nu}{(1+|y|)^{1-\delta}} dy < \infty. \tag{5}$$

Then *there exists some continuous* $g : \mathbb{R} \to E$ *with* $F(z) = z\mathcal{L}g(z)$ *for all* $z \in \Omega$, *such that*

$$\|g(t)\| \leqslant \begin{cases} M(1+t) & \text{for } t > 0, \\ Me^{Rt}(1+|t|) & \text{for } t < 0. \end{cases} \tag{6}$$

In the case $R = +\infty$, *we have* $g(t) = 0$ *for all* $t \leqslant 0$. *Also* g *satisfies local Hölder estimates: there is some* $M < \infty$ *such that*

$$\|g(t+s) - g(s)\| \leqslant Me^{cs}t^{\delta/\nu} \qquad (\forall\, s \in \mathbb{R},\ 0 < t < 1). \tag{7}$$

The proof is given in Section 5. Of course we can get additional information about $|\varphi(g(t))|^2$ by applying Corollary 2.4 above to $\varphi \circ F$.
In [3] the main result was the estimate (6) for the case $R = +\infty$ only, assuming the much stronger condition

$$F = \mathcal{L}f \quad \text{with} \quad \int_0^\infty \|e^{-rt}f(t)\|^p dt < \infty, \quad p > 1, r > 0. \tag{8}$$

[3] also explains that (8) is not sufficient in the case $p = 1$. Under assumption (8), we would have $g(t) = \int_0^t f(s)\,ds$. By increasing r if necessary and using Hölder's inequality, we could take $1 < p \leqslant 2$ without loss of generality. Then the Hausdorff–Young Theorem would give (5) for $c = r$ with $\nu = (1 - 1/p)^{-1} = p' \geqslant 2$ and $\delta = 1$. The estimate (6) is best possible in general, even under the extra assumption (8), as shown in [2].
Additionally (7), which is a *conclusion* of our theorem, would follow automatically from the assumption (8).
In the case $R = +\infty$, we have a similar result for *harmonic* functions:

Theorem 3.2 *Let* $F : \mathbb{C}_+ \to E$ *be a bounded* harmonic *function, where* E *is a Banach space. Assume that* (5) *holds with* $c > 0$.
Then: *there exist* $g_j : \mathbb{R}_+ \to E$ *continuous,* $j = 1, 2$, *such that*

$$g_j(0) = 0, \quad \|g_j(t)\| \leqslant K(1+t^2),$$
$$F(z) = z\,\mathcal{L}g_1(z) + \bar{z}\mathcal{L}g_2(\bar{z}) \text{ on } \mathbb{C}_+,$$

and g_1, g_2 *satisfy the same Hölder estimate* (7) *from Theorem 3.1.*

See Section 6 for the proof. Unfortunately, the case $R < \infty$ is unsatisfactory. For example, there is no function g such that $z + a = z\mathcal{L}g(z)$, with $a \in \mathbb{C}$ constant. Thus $2\mathrm{Re}(z) = z + \bar{z}$ is harmonic and bounded on $\{0 < \mathrm{Re}(z) < 1\}$ but *cannot* be written as $z\mathcal{L}g_1(z) + \bar{z}\mathcal{L}g_2(\bar{z})$ for any functions g_1, g_2.

4 Uniqueness conditions

It is natural to consider *uniqueness*: if $\mathcal{L}f_1 = \mathcal{L}f_2$ on $\{a < \mathrm{Re}(z) < b\}$, in any reasonable sense, then $f_1 = f_2$ by uniqueness of Fourier transforms. However, this does not answer the following:

QUESTION 4.1 *Let* $a_1 < b_1 < a_2 < b_2$ *and* $F \in \mathrm{Hol}\{a_1 < \mathrm{Re}(z) < b_2\}$*, with*

$$\sup_{a_j < c < b_j} \int_{-\infty}^{\infty} |F(c+iy)|^2 \, dy < \infty, \quad \textit{for } j = 1, 2,$$

so that $F = \mathcal{L} f_j$ *on* $\{a_j < \mathrm{Re}(z) < b_j\}$ *for some (uniquely determined)* f_1, f_2*, by Theorem 2.1. When do we have* $f_1 = f_2$*?*

In contrast to Laurent series on concentric annuli $\{r_j < |z| < R_j\}$, it is possible to have $f_1 \neq f_2$. The paper [24] considers

$$G \in \mathrm{Hol}(\mathbb{C}), \qquad G(z) = \int_{t=0}^{\infty} e^{zt} t^{-t} dt.$$

Then G is entire, and bounded on $\{|\mathrm{Im}(z)| > \pi/2 + \delta\}$ for each $\delta > 0$. Define $F(z) = -iG(iz)$. By Cauchy's Theorem as in [24] we obtain

$$F(z) = \int_{s=0}^{\infty} e^{-zs} \exp\left(-is \log s + \frac{\pi s}{2}\right) ds, \qquad \mathrm{Re}(z) > \frac{\pi}{2}.$$

Since $G(\bar{z}) = \overline{G(z)}$, we have $F(-\bar{z}) = -\overline{F(z)}$. Thus

$$F(z) = -\int_{s=-\infty}^{0} e^{-zs} \exp\left(-is \log(-s) - \frac{\pi s}{2}\right) ds, \qquad \mathrm{Re}(z) < -\frac{\pi}{2}.$$

So F is entire and represented by *different* bilateral Laplace transforms on $\{x > \pi/2\}$, $\{x < -\pi/2\}$, even though (*using Plancherel's Theorem*)

$$\int_{-\infty}^{\infty} |F(x+iy)|^2 \, dy \leqslant M \left(|x| - \frac{\pi}{2}\right)^{-1} \quad \text{whenever } |x| > \pi/2.$$

Thus by rescaling, for any $\epsilon > 0$ the "gap" $\{|x| < \epsilon\}$ is "unsafe": crossing the gap can change the Laplace transform function. However, we shall prove below that the gap $\{x = 0\}$ can be safely crossed under quite mild restrictions. First we derive an *explicit formula* for analytic continuation of Laplace transforms.

THEOREM 4.2 *Let* $\Omega = \{z : a < \mathrm{Re}(z) < b\}$ *and* $F \in \mathrm{Hol}(\Omega, E)$*, with* E *a Banach space. Assume that* $a < c < b$*,* $\kappa > 0$*, and*

$$\int_{-\infty}^{\infty} \|F(c+iy)\| \exp(-\kappa y^2) \, dy < \infty. \tag{9}$$

Define $F_\sigma \in \mathrm{Hol}(\mathbb{C}, E)$*, for sufficiently small* $\sigma > 0$*, by*

$$\begin{aligned} F_\sigma(z) = F_{\sigma,c}(z) &= \int_{\lambda \in c + i\mathbb{R}} F(\lambda) \exp\left(\frac{(\lambda - z)^2}{2\sigma^2}\right) \frac{d\lambda}{i\sigma\sqrt{2\pi}} \\ &= \int_{y=-\infty}^{\infty} F(c+iy) \exp\left(\frac{(c+iy-z)^2}{2\sigma^2}\right) \frac{dy}{\sigma\sqrt{2\pi}}. \end{aligned}$$

Then $\sup_C |F_\sigma - F| \leqslant K(C)\sigma^2$ *for each fixed compact* $C \subset \Omega$*. In particular,* $F_\sigma \to F$ *locally uniformly on* Ω*, as* $\sigma \to 0^+$*.*

Proof: Define $G(\lambda, z) = \frac{F(\lambda)}{i\sigma\sqrt{2\pi}} \exp\left(\frac{(\lambda-z)^2}{2\sigma^2}\right)$, so that

$$\|G(\lambda, x+iy)\| = \frac{\|F(\lambda)\|}{\sigma\sqrt{2\pi}} \exp\left(\frac{|x-\mathrm{Re}(\lambda)|^2 - |y-\mathrm{Im}(\lambda)|^2}{2\sigma^2}\right).$$

For each fixed $z \in \mathbb{C}$, $F_\sigma(z)$ is the integral of $G(\lambda, z)$ over the contour $c + i\mathbb{R}$, which converges for $1/2\sigma^2 > \kappa$ by condition (9). Since $G(\lambda, z)$ is an analytic function of λ, we can use Cauchy's Theorem with the same contour as in Theorem 3.1. Pick $\omega = \omega_1 + i\omega_2 \in \Omega$ and fix a square $\Sigma = \{|x-\omega_1|, |y-\omega_2| \leqslant \delta\} \subset \Omega$. Let Y be large, much larger than δ, and consider the contours

$$\begin{aligned} \Gamma(x) &= \{\mathrm{Re}(\lambda) = x, & |\mathrm{Im}(\lambda) - \omega_2| \leqslant Y\}, \\ \Gamma_Y^\pm(x) &= \{\mathrm{Re}(\lambda) \in [x, c], & \mathrm{Im}(\lambda) = \omega_2 \pm Y\}, \\ \Gamma' &= \{\mathrm{Re}(\lambda) = c, & |\mathrm{Im}(\lambda) - \omega_2| \geqslant Y\}. \end{aligned}$$

For $\lambda \in \Gamma_Y^\pm(x)$, we have

$$\|G(\lambda, z)\| \leqslant \sup_{\mu \in I} \|F(\mu)\| \cdot \sigma^{-1} \exp\left(\frac{M - Y^2/2}{2\sigma^2}\right),$$

uniformly for $z \in \Sigma$, where $I = \Gamma_Y^\pm(\omega_1 - \delta)$ or $I = \Gamma_Y^\pm(\omega_1 + \delta)$ as appropriate (*depending on whether* $\omega_1 < c$ *or* $\omega_1 > c$). We are using $(Y-y)^2 > Y^2/2$ and $(c-x)^2 < M$.
Thus $\int_{\Gamma_Y^\pm(x)} G\, d\lambda \to 0$ rapidly as $\sigma \to 0$, uniformly in z, as long as Y is large enough. By condition (9) again, also $\int_{\Gamma'} G\, d\lambda \to 0$ rapidly as $\sigma \to 0$, uniformly for $z \in \Sigma$. Finally, the integral over $\Gamma(x)$ is a standard Gaussian convolution approximation to $F(z)$:

$$\int_{\Gamma(x)} G(\lambda, x+iy)\, d\lambda = \int_{\omega_2 - Y}^{\omega_2 + Y} F(x+iu) \exp\left(\frac{-(y-u)^2}{2\sigma^2}\right) \frac{du}{\sigma\sqrt{2\pi}}.$$

After Y is chosen, $F(t+iu)$ is then bounded on the tall, narrow rectangle $|t-\omega_1| \leqslant \delta$, $|u-\omega_2| \leqslant Y$. If we approximate $F(x+iu)$ by its Taylor series about $x+iy$, it is now routine to verify that $\int_{\Gamma(x)} G(\lambda, x+iy)\, d\lambda = F(x+iy) + O(\sigma^2)$, *uniformly* for $|x-\omega_1|, |y-\omega_2| < \delta/2$, say. The errors from the other contours are much smaller, being $O(\exp(-\nu/\sigma^2))$ for some $\nu > 0$.
□

Corollary 4.3 *With F as in Theorem 4.2 and $E = \mathbb{C}$, suppose that*

$$\exists\, 1 \leqslant p \leqslant 2 \quad \textit{such that} \quad \int_{-\infty}^{\infty} |F(c+iy)|^p\, dy < \infty.$$

Then

$$F(z) = \lim_{\sigma \to 0^+} \int_{t=-\infty}^{\infty} e^{-zt} h(t) \exp(-\sigma^2 t^2/2)\, dt \tag{10}$$

locally uniformly for $z \in \Omega$, for some measurable h satisfying

$$\int_{-\infty}^{\infty} |h(t)| e^{-\delta t^2}\, dt < \infty \quad \forall\, \delta > 0.$$

PROOF: We use the Hausdorff–Young Theorem. Set

$$h(t)e^{-ct} \sim \frac{1}{2\pi} \widehat{F(c+iy)}(-t) \in L^{p'}(\mathbb{R}),$$

for $p' = (1-1/p)^{-1}$ the conjugate exponent to p. This is well–defined for a.e. $t \in \mathbb{R}$. Now $\int \widehat{h(t)e^{-ct}}(y) g(y)\, dy = \int h(t)e^{-ct}\widehat{g}(t)\, dt$ for every Schwartz function g. Now put $F(c+iy) \sim \widehat{he^{-ct}}(y)$ in the definition of $F_{\sigma,c}(z)$ and calculate.
□

Notice that (10) is just a weak kind of Laplace transform representation for F. It says that a particular Abelian summability method assigns the value $F(z)$ to the formal integral "$\int_{-\infty}^{\infty} e^{-zt}h(t)\, dt$", even though this integral may diverge. See [17] for much more on these topics; unfortunately the classical results given there appear to be inadequate for our problem.

COROLLARY 4.4 *Let $\Omega = \{z : a < \mathrm{Re}(z) < b\}$ and $F \in \mathrm{Hol}(\Omega)$. Suppose that there exist $a < c_1 < c_2 < b$ and f_1, f_2 such that*

$$F(c_j + iy) \sim \int_{t=-\infty}^{\infty} f_j(t)e^{-c_j t} \exp(-iyt)\, dt \qquad (y \in \mathbb{R}, \quad j = 1, 2),$$

as Fourier transforms of $f_j(t)e^{-c_j t} \in L^2(\mathbb{R})$. Define

$$H(z, v) = \int_{t=-\infty}^{\infty} \big(f_1(t) - f_2(t) \big) \exp(izt - vt^2) dt$$

for $z \in \mathbb{C}$, $v > 0$. Then H has a continuous extension $H : i\Omega \times [0, \infty) \to \mathbb{C}$ satisfying

$$\frac{\partial^2 H}{\partial z^2} = \frac{\partial H}{\partial v}, \qquad H(z, 0) \equiv 0 \quad (z \in i\Omega).$$

PROOF: By Corollary 4.3, equation (10) holds for both $h = f_1$ and $h = f_2$. Therefore, $H(z, v) \to F(-iz) - F(-iz) = 0$ as $v \to 0^+$, for each $z \in i\Omega$. Because this convergence is locally uniform, we have the required continuity of H. Finally, the complex heat equation $\frac{\partial^2 H}{\partial z^2} = \frac{\partial H}{\partial v}$ follows immediately by differentiating under the integral sign.
□

The letter t is normally used for the time variable, but we use $v = \sigma^2/2$ (for *variance*, with an extra factor of 2). Now we can apply known results on the heat equation. The papers [6], [30] prove many results about functions on discs. The following corollaries are closely related (after applying a conformal transformation), but our proofs are easier and quite different.

COROLLARY 4.5 *Let* $F \in \mathrm{Hol}(\{-1 < x < 1\})$ *and* $A, B, r > 0$, *with*

$$\int_{-\infty}^{\infty} |F(x+iy)|^2 dy \leqslant A \exp(B|x|^{-r}), \quad \forall\, x \neq 0.$$

Then $\sup_{-1<x<1} \int_{-\infty}^{\infty} |F(x+iy)|^2 dy < \infty$. *In particular,* F *is bounded on* $|x| < 1-\epsilon$, *for each* $\epsilon > 0$.

Notice that, a priori, it is not obvious that any estimate for $|F(iy)|$ is possible: $\exp(B|x|^{-r})$ grows so rapidly as $|x| \to 0$ that any simple attempt based on the Mean Value Property must fail.

PROOF: First, by Theorem 3.1, we know that $F = \mathcal{L}f_+$ on $\{0 < x < 1\}$ and $F = \mathcal{L}f_-$ on $\{-1 < x < 0\}$, with $\int_{-\infty}^{\infty} e^{-2\delta t}|f_+(t)|^2\, dt \ll \exp(B\delta^{-r})$, and similarly for f_-. Now consider $\varphi = f_+ - f_-$. Then

$$\int_{-\infty}^{\infty} e^{-2\delta|t|}|\varphi(t)|^2\, dt \ll \exp(B\delta^{-r}).$$

Following Corollary 4.4, define

$$H(y, v) = \int_{-\infty}^{\infty} \exp(iyt - vt^2)\varphi(t)\, dt$$

for $y \in \mathbb{R}$ and $v > 0$. Then H satisfies the heat equation and extends to be continuous on $\{v \geqslant 0\}$, with $H(y, 0) \equiv 0$. We calculate

$$\begin{aligned}|H(y,v)| &\leqslant \left(\int_{-\infty}^{\infty} e^{-\delta|t|}|\varphi(t)|\, dt\right) \sup_{\tau\in\mathbb{R}} \exp(\delta|\tau| - v\tau^2) \\ &\ll \delta^{-1/2} \exp(B\delta^{-r}/2) \exp(\delta^2/4v) \leqslant \exp\left(C\delta^{-r} + \frac{\delta^2}{4v}\right),\end{aligned}$$

for any $0 < \delta < 1$. We have used the Cauchy–Schwarz inequality and $\delta^{-1/2} < \exp(\delta^{-1/2}) \leqslant \exp(\delta^{-r})$, as long as $r \geqslant 1/2$, which we could clearly assume from the start. Notice that C does not depend on δ.

Now choose $\delta = v^\alpha$ with $\alpha = 1/(r+2)$, so that $\alpha r = 1 - 2\alpha$, to obtain

$$|H(y,v)| \leqslant A' \exp\left(\frac{C + 4^{-1}}{v^{1-2\alpha}}\right) = A' \exp(C'/v^\eta), \tag{11}$$

for all $y \in \mathbb{R}$, $0 < v < 1$, with $0 < \eta < 1$. Since $H = H(y, v)$ is a solution to the heat equation with $H(y, 0) \equiv 0$, condition (11) implies that $H \equiv 0$. See [8], [15]. In general the condition $|H| \leqslant A(\epsilon) \exp(\epsilon/v)$ for each $\epsilon > 0$ is not sufficient, as shown in [7], but our proof works because we have an estimate for $A(\epsilon)$. Therefore $\varphi = 0$ and $f_+ = f_-$ almost everywhere, as required.
$\square$

COROLLARY 4.6 *Let* $\Omega = \{x + iy : 0 < x < 1\}$. *Let* E *be a Banach space,* $F : \overline{\Omega} \to E$ *continuous, harmonic on* Ω*, and* $F \in L^\infty(\partial\Omega)$*. Suppose that* $A, B, r > 0$ *satisfy*

$$\|F(x + iy)\| \leqslant A \exp(B[x(1 - x)]^{-r}) \qquad \forall\, 0 < x < 1, y \in \mathbb{R}.$$

Then $F \in L^\infty(\Omega)$ *with* $\sup_\Omega \|F\| = \sup_{\partial\Omega} \|F\|$.

This is a Maximum Principle, similar in some ways (*but quite different in other ways*) to the Phragmén–Lindelöf theorems.

PROOF: By considering $\varphi \circ F$ for each $\varphi \in E^*$, it is enough to consider the case $E = \mathbb{C}$; by considering $\mathrm{Re}(F), \mathrm{Im}(F)$, we can take $E = \mathbb{R}$. Now let $\widetilde{F}$ be the unique *bounded* harmonic function with $F = \widetilde{F}$ on $\partial\Omega$. For example, we could obtain $\widetilde{F}$ by conformal mapping and the well–known Poisson Formula for the disc. By considering $F - \widetilde{F}$, *we only need to prove the special case where* F *is real–valued, and zero on* $\partial\Omega$.

By the Schwarz Reflection Principle, we can extend F to be harmonic on $\mathbb{C}_+$ and continuous on $i\mathbb{R}$, by defining $F(n + x + iy) = -F(n - x + iy)$ repeatedly for $x \in [0, 1]$, $y \in \mathbb{R}$ and $n = 1, 2, 3, \ldots$.

Thus $|F| \ll \exp(C \cdot \mathrm{dist}(x, \mathbb{Z})^{-r})$, where dist means *distance*. We have $F = g + \bar{g}$ for some $g \in \mathrm{Hol}(\mathbb{C}_+)$. Now

$$g'(\lambda) = \frac{1}{2\pi r} \int_0^{2\pi} F(\lambda + re^{i\theta}) e^{-i\theta} d\theta,$$

so that $|g'| \ll \exp(C' \mathrm{dist}(x, \mathbb{Z})^{-r})$, by simple estimates for F with the Mean Value Property. Also $\int_{n-1/2}^{n+1/2} |g'(t)|\, dt$ is independent of n, because of the reflection process used to extend F. Thus

$$\begin{aligned} |g(z)| &= \left| g(1) + \int_1^x g'(t)\, dt + i \int_0^y g'(x + is)\, ds \right| \\ &\leqslant A'(1 + |z|) \exp\left(\frac{C'}{\mathrm{dist}(x, \mathbb{Z})^r} \right). \end{aligned}$$

Now consider $h(z) = g(z)(1+z)^{-1}$. By applying Corollary 4.5 to h repeatedly on the domains $\{|x - n| < 1 - \epsilon\}$ (with trivial rescaling), we obtain $\int_{-\infty}^{\infty} |h(x + iy)|^2 dy \leqslant M(\epsilon)$ for all $x > \epsilon$, i.e. $h \in H^2(\{\mathrm{Re}(z) > \epsilon\})$ for each $\epsilon > 0$. Thus $h = \mathcal{L}u$ on $\mathbb{C}_+$ for some u on $\mathbb{R}_+$ with $\int_0^\infty e^{-\delta t} |u(t)|^2 dt < \infty$ for all $\delta > 0$. But now

$$0 = \frac{F(n)}{1 + n} = 2\mathrm{Re}[h(n)] = \int_0^\infty e^{-nt} 2(\mathrm{Re}\, u)(t)\, dt$$

for all $n = 1, 2, 3, \ldots$. So $\mathcal{L}(\mathrm{Re}\, u)$ is bounded and analytic on $\{\mathrm{Re}(z) > 1/2\}$, with a zero at each n, and thus identically zero everywhere by the Blaschke condition for zero sequences of Hardy space functions. Thus $\mathrm{Re}\, u = 0$ almost everywhere.

So $\bar{u} = -u$ a.e., and now $\overline{h(x)} = -h(x)$ for all $x > 0$, so that $F(x) = g(x) + \overline{g(x)} = 0$. Now the proof is finished; we have shown that $F = 0$ on $\{0 < x < 1\}$. Applying this to $F_\alpha = F(z + i\alpha)$ for each $\alpha \in \mathbb{R}$, we obtain that $F \equiv 0$, as required.
□

We remark, omitting the details, that Corollary 4.6 can be used to prove Corollary 4.5, so they are equivalent: given an analytic F on $\{|x| < 1\}$, consider $U(z) = F(z) - F(-\bar{z})$ on $\{0 \leqslant x \leqslant 1/2\}$.

5 proof of Theorem 3.1

We first prove (6). First consider the scalar case $E = \mathbb{C}$. By Theorem 2.1 applied to $F(z)/z$, we see immediately that $F(z)/z = \mathcal{L}g(z)$ for some g, given by

$$g(t) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(c+iy)}{c+iy} e^{(c+iy)t}\, dy \sim \frac{1}{2\pi i} \int_{c+i\mathbb{R}} \frac{F(z)}{z} e^{zt}\, dz.$$

If $R = +\infty$ then Theorem 2.1 also gives $g(t) \equiv 0$ for $t \leqslant 0$. But $\int_{c+i\mathbb{R}} \left| \frac{F(z)}{z} \right| |dz| < \infty$ by Hölder's inequality, so in fact $g : \mathbb{R} \to \mathbb{C}$ is *continuous* (after changing g on a set of measure zero).

The estimate $|g(t)| \leqslant M(1 + t)$ for $t > 0$ was already proved in [3] for the special case $R = +\infty$ and $F \in L^q(c + i\mathbb{R})$ for some $q > 1$. But that proof needed only the estimate $\int_{|z|>\kappa} \frac{|F(z)|}{|z|} |dz| = O(\kappa^{-\epsilon})$ for some $0 < \epsilon < 1$, which follows from (5) by Hölder's inequality. The proof also applies without change when $R < \infty$. For $t < 0$, we can simply apply the result to $F(R - z)$.

The Hölder estimate (7) follows by direct calculation: we have

$$|g(t+s) - g(s)| \ll \int_{c+i\mathbb{R}} \left| \frac{F(z)}{z} \right| e^{cs} |e^{zt} - 1|\, |dz|.$$

By Hölder's inequality, this is

$$\ll e^{cs} \left(\int_{c+i\mathbb{R}} \frac{|F(z)|^\nu}{|z|^{1-\delta}} dy \right)^{1/\nu} \left(\int_{c+i\mathbb{R}} \frac{|e^{zt} - 1|^{\nu'}}{|z|^\alpha} |dz| \right)^{1/\nu'},$$

where $\alpha = \left(1 - \frac{1-\delta}{\nu}\right) \nu' = 1 + \delta \frac{\nu'}{\nu} > 1$. Since $|z| \approx c + |y| \approx 1 + |y|$, the second integral above is

$$\begin{aligned}
&\ll \int_{\mathbb{R}} \frac{|e^{(c+iy)it} - 1|^{\nu'}}{(1+|y|)^\alpha} dy \\
&\ll t^{\nu'} \int_{|c+iy|<t^{-1}} \frac{|c+iy|^{\nu'}}{(1+|y|)^\alpha} dy + \int_{|c+iy|>t^{-1}} \frac{dy}{(1+|y|)^\alpha} \\
&\ll t^{\nu'} \int_{|y|<At^{-1}} (1+|y|)^{\nu'-\alpha} dy + \int_{|y|>Bt^{-1}} \frac{dy}{(1+|y|)^\alpha} \\
&\ll t^{\nu'} (1/t)^{\nu'-\alpha+1} + (1/t)^{1-\alpha} \ll t^{\alpha-1}.
\end{aligned}$$

Here t is small, $A, B > 0$ depend on c, and $|e^\lambda - 1| \ll |\lambda|$ for λ bounded; note also that $\nu' > \alpha$. Since $(\alpha - 1)/\nu' = \delta/\nu$, we obtain (7) as required, *in the special case where* $E = \mathbb{C}$.

Now let E be a general Banach space. A standard Closed Graph Theorem argument shows that

$$\left(\int_{c+i\mathbb{R}} \frac{|\varphi(F(z))|^\nu}{|z|^{1-\delta}} dy\right)^{1/\nu} \leqslant K\|\varphi\|_{E^*}$$

for all $\varphi \in E^*$, with some constant $K < \infty$. For each $\varphi \in E^*$ we consider $\varphi(F(z))$ and apply the scalar–valued case, to obtain a continuous $g_\varphi : \mathbb{R} \to \mathbb{C}$ such that

$$\varphi(F(z)) = z\mathcal{L}g_\varphi(z), \qquad g_\varphi(t) = \frac{1}{2\pi i}\int_{c+i\mathbb{R}} \frac{\varphi(F(z))}{z} e^{zt}\, dz.$$

Examining the above proof carefully, we find that the constant M in (6) is bounded by an absolute constant multiple of

$$\int_{c+i\mathbb{R}} \frac{|\varphi(F(z))|}{|z|} |dz| + \sup_{z\in\Omega} |\varphi(F(z))| \ll \|\varphi\|_{E^*}.$$

Thus $|g_\varphi(t)| \leqslant M'\|\varphi\|_{E^*}(1 + t)$ with some $M' < \infty$ for $t > 0$, and similarly for $t < 0$, so we can define $g : \mathbb{R} \to E^{**}$ by $g(t)(\varphi) = g_\varphi(t)$. As usual, regard $E \subseteq E^{**}$ via the canonical embedding. We also have a similar estimate to (7) for

$$g_\varphi(t + s) - g_\varphi(s) = [g(t + s) - g(s)](\varphi),$$

which gives (7) with $\|g(t+s) - g(s)\|_{E^{**}}$ instead of $\|\cdot\|_E$. Crucially, this also shows that $g : \mathbb{R} \to E^{**}$ is *continuous*.

But now $\varphi(F(z)) = z(\mathcal{L}g_\varphi)(z) = [z\mathcal{L}g(z)](\varphi)$, so that $F(z) = z\mathcal{L}g(z)$, considered as an E^{**}-valued function; note that $\mathcal{L}g$ converges because we have an estimate for $\|g(t)\|_{E^{**}}$. Thus all is finished, except that $g(t) \in E^{**}$ instead of E. Put

$$H : \mathbb{R} \to E, \qquad H(t) = \frac{1}{2\pi i}\int_{c+i\mathbb{R}} \frac{F(z)}{z^2} e^{zt}\, dz,$$

which is well–defined and continuous because $\int_{c+i\mathbb{R}} \frac{\|F(z)\|_E}{|z|^2}|dz| < \infty$. Now $(\varphi \circ H)'(t) = (\varphi \circ g)(t)$ for each $\varphi \in E^*$ and $t \in \mathbb{R}$. Because g, $\varphi \circ g$ are continuous, we have

$$\varphi(H(t)) - \varphi(H(0)) = \int_0^t \varphi(g(\tau))\, d\tau.$$

Hence $H(t) = H(0) + \int_0^t g(\tau)\, d\tau$ as an E^{**}-valued integral, so $H'(t) = g(t)$ for all $t \in \mathbb{R}$, again by continuity of $g : \mathbb{R} \to E^{**}$. Thus finally $g(t) \in E$ as required, because H is E-valued.

6 Proof of Theorem 3.2

Now $F : \mathbb{C}_+ \to E$ is *harmonic*; so there exist *analytic* $F_j \in \mathrm{Hol}(\mathbb{C}_+, E)$, with $j = 1, 2$, such that $F(z) = F_1(z) + F_2(\bar{z})$. The functions F_1, F_2 are unique up to additive constants. We will show that F_1, F_2 can be chosen to satisfy (5) in Theorem 3.1, and that F_1, F_2 are almost bounded (*with only logarithmic unboundedness*); the result will then follow by a similar proof to Theorem 3.1.
F is bounded, so we can represent F on $\{\mathrm{Re}(z) > c\}$ by its Poisson integral:

$$\forall\, u > 0, \quad F(c + u + iv) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(c + iy) \frac{u}{u^2 + (v - y)^2}\, dy.$$

Now we define

$$G_1(\lambda) = \frac{1}{2\pi i} \int_{c+i\mathbb{R}} \frac{F(z)}{\lambda - z}\, dz, \qquad G_2(\lambda) = \frac{1}{2\pi i} \int_{c+i\mathbb{R}} \frac{F(z)}{\lambda - \bar{z}}\, dz$$

for all $\mathrm{Re}(\lambda) > c$, so that

$$G_j \in \mathrm{Hol}(\{\mathrm{Re}(\lambda) > c\}, E), \qquad F(\lambda) = G_1(\lambda) + G_2\left(\bar{\lambda}\right).$$

Because $(G_1 - F_1)(\lambda) = -(G_2 - F_2)(\bar{\lambda})$ on $\mathrm{Re}(\lambda) > c$, the functions $G_1 - F_1 \equiv F_2 - G_2$ are constant; so we have analytic continuations $G_j \in \mathrm{Hol}(\mathbb{C}_+, E)$ for $j = 1, 2$.
We use the standard theory of the Weighted Hilbert Transform, found in [22], [16] and many other sources. The famous *Muckenhoupt weight condition* $w \in A_\nu(\mathbb{R})$ for $w : \mathbb{R} \to [0, +\infty]$, $1 < \nu < \infty$ is

$$\sup_{\text{bounded intervals } I \subset \mathbb{R}} \left(\frac{1}{|I|} \int_I w(t)\, dt \right) \left(\frac{1}{|I|} \int_I w(s)^{-1/(\nu-1)}\, ds \right)^{\nu - 1} < \infty.$$

Now $w \in A_\nu(\mathbb{R})$ is *equivalent* to the Hilbert transform being bounded on $L^\nu(w)$:

$$\mathcal{H}f(t) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{y \in \mathbb{R},\ |y - t| > \varepsilon} \frac{f(y)}{t - y}\, dy \quad \text{exists for a.e. } t \in \mathbb{R},$$

whenever $f \in L^\nu(w)$, i.e. $\int_{\mathbb{R}} |f|^\nu w\, dt < \infty$, and furthermore $\|\mathcal{H}f\|_{L^\nu(w)} \leqslant C_w \|f\|_{L^\nu(w)}$ for some constant $C_w < \infty$ depending only on w.
For our problem, we easily check that $w(y) = |y|^{-\varsigma}$ satisfies $w \in A_\nu(\mathbb{R})$ for any $0 < \varsigma < 1$. Assume for the moment that $E = \mathbb{C}$. Define

$$H_\varepsilon(\alpha) = \int_{|y - \alpha| > \varepsilon} \frac{F(c + iy)}{\alpha - y}\, dy,$$

so that by above $H_\varepsilon(\alpha) \to H(\alpha)$ as $\varepsilon \to 0$, for almost every $\alpha \in \mathbb{R}$ and some $H \in L^\nu(|\alpha|^{-(1-\delta)})$. Fix $\alpha \in \mathbb{R}$ such that $H_\varepsilon(\alpha) \to H(\alpha)$ does hold. For R large, the condition (5) gives $\int_{|y| > R} \frac{|F(c+iy)|}{|y|}\, dy \ll R^{-\eta}$, and so

$$H_\varepsilon(\alpha) = \int_{\varepsilon < |y - \alpha| < R} \frac{F(c + iy)}{\alpha - y}\, dy + O(R^{-\eta})$$

as $R \to \infty$, for some unimportant $\eta > 0$. But F is harmonic and thus smooth on $\mathbb{C}_+$, so $\int_{|y-\alpha|<1} \left| \frac{F(c+iy)-F(c+i\alpha)}{\alpha-y} \right| dy < \infty$. Also $\int_{\varepsilon<|y-\alpha|<R} \frac{dy}{\alpha-y} = 0$, so we can write

$$\begin{aligned} H_\varepsilon(\alpha) &= \int_{\varepsilon<|y-\alpha|<R} \frac{F(c+iy)-F(c+i\alpha)}{\alpha-y}\, dy + O(R^{-\eta}) \\ &= \int_{|y-\alpha|<R} \frac{F(c+iy)-F(c+i\alpha)}{\alpha-y}\, dy + o_R + o_\varepsilon \end{aligned}$$

where $o_\varepsilon \to 0$ as $\varepsilon \to 0^+$, uniformly in $R > 1$, and similarly $o_R \to 0$ as $R \to \infty$, uniformly in $0 < \varepsilon < 1$. Now, for all $0 < \xi < 1$,

$$\begin{aligned} G_1(c+i\alpha+\xi) &= \frac{1}{2\pi} \int_{|y-\alpha|<R} \frac{F(c+iy)}{\xi + i(\alpha-y)}\, dy + O(R^{-\eta}) \\ &= \frac{1}{2\pi} \int_{|y-\alpha|<R} \frac{F(c+iy)-F(c+i\alpha)}{\xi+i(\alpha-y)}\, dy + o_R \\ &\quad + F(c+i\alpha) I(R,\xi), \end{aligned}$$

where

$$I(R,\xi) = \frac{1}{2\pi} \int_{|y-\alpha|<R} \frac{dy}{\xi+i(\alpha-y)} = \frac{\tan^{-1}(R/\xi)}{\pi}.$$

Now fix R and let $\xi \to 0^+$. Then $I(R,\xi) \to \frac{1}{2}$ and $G_1(c+\xi+i\alpha) \to G_1(c+i\alpha)$, simply because $G_1 \in \mathrm{Hol}(\mathbb{C}_+)$, so that

$$G_1(c+i\alpha) = \frac{1}{2} F(c+i\alpha) + \frac{1}{2\pi i} H_\varepsilon(\alpha) + o_R + o_\varepsilon$$

by Dominated Convergence, because $\int_{|y-\alpha|<R} \left| \frac{F(c+iy)-F(c+i\alpha)}{\alpha-y} \right| dy < \infty$. Finally let $\varepsilon \to 0$ and $R \to \infty$, to give $G_1(c+i\alpha) = \frac{1}{2}F(c+i\alpha) + \frac{1}{2\pi i} H(\alpha)$. This is true for almost every α, and $F(c+i\alpha)$, $H(\alpha)$ are both in $L^\nu(|\alpha|^{-(1-\delta)})$, and thus so is $G_1(c+i\alpha)$.

Now we have $\int_{c+i\mathbb{R}} \frac{|G_1|^\nu}{|z|^{1-\delta}} dy < \infty$, in the special case $E = \mathbb{C}$. In general, we get the same for $\varphi \circ G_1$, for each $\varphi \in E^*$. This is the estimate (5) we need in Theorem 3.1, for G_1 instead of F.

Theorem 3.1 does not apply to G_1 because G_1 may be unbounded. However, the unboundedness is at most *logarithmic*, by two simple calculations:

LEMMA 6.1 *Let $F : \Omega \to E$ be harmonic, for some domain Ω. Then, for every $z \in \Omega$ and $r > 0$ such that $\{\lambda : |\lambda - z| \leqslant r\} \subset \Omega$, we have*

$$\frac{\partial F}{\partial z} = \frac{1}{2\pi i} \oint_{|\lambda-z|=r} \frac{F(\lambda)}{(\lambda-z)^2} d\lambda, \qquad \left\| \frac{\partial F}{\partial z} \right\| \leqslant \frac{\max_{|\lambda-z|=r} \|F(\lambda)\|}{r}.$$

The proof is immediate, from power series representations.

LEMMA 6.2 *Let $F : \mathbb{C}_+ \to E$ be harmonic and bounded, with $F(z) = G_1(z) + G_2(\bar{z})$ for $G_1, G_2 \in \mathrm{Hol}(\mathbb{C}_+, E)$. Then there exist constants $M_j < \infty$, $j = 1, 2$, such that*

$$\|G_j(x+iy)\|_E \leqslant M_j\big(1 + |\log x| + \log(1+|y|)\big). \tag{12}$$

PROOF: Given $u + iv \in \mathbb{C}_+$, we have

$$G_j(u+iv) - G_j(1) = \left(\int_{\substack{x\in[1,S],\\ y=0}} + \int_{\substack{x=S,\\ y\in[0,v]}} + \int_{\substack{x\in[S,u],\\ y=v}}\right) G_j'(z)\,dz,$$

for any $S > 1$. But $G_1' = \partial F/\partial z$, so with $r = x/2$ in Lemma 6.1 we obtain $\|G_1'(x+iy)\| \leqslant 2\left(\sup_{\mathbb{C}_+}\|F\|\right)/x$. Thus

$$\|G_1(u+iv)\| \leqslant \|G_1(1)\| + 2\sup_{\mathbb{C}_+}\|F\|\left(\log S + \frac{|v|}{S} + |\log S - \log u|\right).$$

Now letting $S = |v| + 1$ gives the result for G_1, and the proof for G_2 is similar. □

The logarithmic terms are unavoidable; e.g. $2\theta = -i(\log z - \log\bar{z})$ is harmonic and bounded on $\mathbb{C}_+ = \{re^{i\theta} : r > 0, |\theta| < \pi/2\}$.

Finally, to complete the proof of Theorem 3.2: G_1 satisfies (12), and also the vertical estimate (5) on $c + i\mathbb{R}$. Similarly, or by considering $F(\bar{z})$ instead, G_2 also satisfies the same estimates. The local Hölder estimate (7) follows from the proof of Theorem 3.1 without change, since only (5) is needed.

To estimate $\|g_j(t)\|$ for large $t > 0$, we use the same method as Theorem 3.1 (which in fact is the method used in [3]), but with additional logarithmic estimates. As usual, consider $\varphi \circ F$ for each $\varphi \in E^*$. In the contour integral formula

$$\varphi \circ g_j(t) = \frac{1}{2\pi i}\int_{c+i\mathbb{R}} \frac{\varphi \circ G_j(z)}{z} e^{zt}\,dz,$$

use Cauchy's Theorem to replace $c + i\mathbb{R}$ by the contours $\{c + iy : |y| \geqslant \kappa\}$, $\{x \pm i\kappa : t^{-1} < x < c\}$ and $\{t^{-1} + iy : |y| \leqslant \kappa\}$, for t large. Estimating $\left|\frac{\varphi\circ G_j(z)}{z}e^{zt}\right|$ on each of these contours finally gives that $|\varphi \circ g_j(t)|/\|\varphi\|_{E^*}$ is

$$\ll e^{ct}\kappa^{-\epsilon} + (1 + \log t + \log(1+\kappa))\big[e^{ct}\kappa^{-1} + \exp(t^{-1}\cdot t)\log(\kappa t)\big]$$

for t large and some $0 < \epsilon < 1$, which is $\ll t^2$ upon taking $\kappa = \exp(ct/\epsilon)$.

7 ACKNOWLEDGEMENTS

The author thanks the Department of Mathematics, Chulalongkorn University, Bangkok for past financial support, and Yemon Choi, Matt Daws and Jonathan Partington for various mathematical discussions.

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