The Sato-Tate Conjecture for Modular Forms of Weight 3

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Received: August 26, 2009
Revised: December 15, 2009

Communicated by Don Blasius

Abstract. We prove a natural analogue of the Sato-Tate conjecture for modular forms of weight 2 or 3 whose associated automorphic representations are a twist of the Steinberg representation at some finite place.

2000 Mathematics Subject Classification: 11F33.

Contents

[1]The author was partially supported by NSF grant DMS-0841491

1. Introduction

The Sato-Tate conjecture is a conjecture about the distribution of the number of points on an elliptic curve over finite fields. Specifically, if E is an elliptic curve over $\mathbb{Q}$ without CM, then for each prime l such that E has good reduction at l we set

$$a_l := 1 + l - \#E(\mathbb{F}_l).$$

Then the Sato-Tate conjecture states that the quantities $\cos^{-1}(a_l/2\sqrt{l})$ are equidistributed with respect to the measure

$$\frac{2}{\pi}\sin^2\theta d\theta$$

on $[0,\pi]$. Alternatively, by the Weil bounds for E, the polynomial

$$X^2 - a_l X + l = (X - \alpha_l l^{1/2})(X - \beta_l l^{1/2})$$

satisfies $|\alpha_l| = |\beta_l| = 1$, and there is a well-defined conjugacy class $x_{E,l}$ in $SU(2)$, the conjugacy class of the matrix

$$\begin{pmatrix} \alpha_l & 0 \\ 0 & \beta_l \end{pmatrix}.$$

The Sato-Tate conjecture is then equivalent to the statement that the classes $x_{E,l}$ are equidistributed with respect to the Haar measure on $SU(2)$.

Tate observed that the conjecture would follow from properties of the symmetric power L-functions of E, specifically that these L-functions (suitably normalised) should have nonvanishing analytic continuation to the region $\Re s \geq 1$. This would follow (given the modularity of elliptic curves) from the Langlands conjectures (specifically, it would be a consequence of the symmetric power functoriality from GL_2 to GL_n for all n). Unfortunately, proving this functoriality appears to be well beyond the reach of current techniques. However, Harris, Shepherd-Baron and Taylor observed that the required analytic properties would follow from a proof of the potential automorphy of the symmetric power L-functions (that is, the automorphy of the L-functions after base change to some extension of $\mathbb{Q}$), and were able to use Taylor's potential automorphy techniques to prove the Sato-Tate conjecture for all elliptic curves E with non-integral j-invariant (see [HSBT09]).

There are various possible generalisations of the Sato-Tate conjecture; if one wishes to be maximally ambitious, one could consider equidistribution results for the Satake parameters of rather general automorphic representations (see for example section 2 of [Lan79]). Again, such results appear to be well beyond the range of current technology. There is, however, one special case that does seem to be reasonable to attack, which is the case of Hilbert cuspidal eigenforms of regular weight. In this paper, we prove a natural generalisation of the Sato-Tate conjecture for modular newforms (over $\mathbb{Q}$) of weight 2 or 3, subject to the natural analogue of the condition that an elliptic curve has non-integral j-invariant. We note that previously the only modular forms for which the conjecture was known were those corresponding to elliptic curves; in particular, there were no examples of weight 3 modular forms for which the conjecture was known. After this paper was made available, the conjecture was proved for all modular forms of weight at least 2 in [BLGHT09], by rather different methods. Our approach is similar to that of [HSBT09], and we are fortunate in being able to quote many of their results. Indeed, it is straightforward to check that Tate's argument shows that the conjecture would follow from the potential automorphy of the symmetric powers of the l-adic Galois representations associated to a modular form. One might then hope to prove this potential automorphy in the style of [HSBT09]; one would proceed by realising the symmetric powers of the mod l Galois representation geometrically in such a way that their potential automorphy may be established, and then deduce the potential automorphy of the l-adic representations by means of the modularity lifting theorems of [CHT08] and [Tay08].

It turns out that this simple strategy encounters some significant obstacles. First and foremost, it is an unavoidable limitation of the known potential automorphy methods that they can only deduce that a mod l Galois representation is automorphic of minimal weight (which we refer to as "weight 0"). However, the symmetric powers of the Galois representations corresponding to modular forms of weight greater than 2 are never automorphic of minimal weight, so one has no hope of directly proving their potential automorphy in the fashion outlined above without some additional argument. If, for example, one knew the weight part of Serre's conjecture for GL_n (or even for unitary groups) one would be able to deduce the required results, but this appears to be an extremely difficult problem in general. There is, however, one case in which the analysis of the Serre weights is rather easier, which is the case that the l-adic Galois representations are ordinary. It is this observation that we exploit in this paper.

In general, it is anticipated that for a given newform f of weight $k \geq 2$, there is a density one set of primes l such that there is an ordinary l-adic Galois representation corresponding to f. Unfortunately, if $k > 3$ then it is not even known that there is an infinite set of such primes; this is the reason for our restriction to $k = 2$ or 3. In these cases, one may use the Ramanujan conjecture and Serre's form of the Cebotarev density theorem (see [Ser81]) to prove that the set of l which are "ordinary" in this sense has density one, via an argument that is presumably well-known to the experts (although we have not been able to find the precise argument that we use in the literature). We note that it is important for us to be able to choose l arbitrarily large in certain arguments (in order to satisfy the hypotheses of the automorphy lifting theorems of [Tay08]), so it does not appear to be possible to apply our methods to any modular forms of weight greater than 3. Similarly, we cannot prove anything for Hilbert modular forms of parallel weight 3 over any field other than $\mathbb{Q}$.

We now outline our arguments in more detail, and explain exactly what we prove. The early sections of the paper are devoted to proving the required potential automorphy results. In section 2 we recall some basic definitions and results from [CHT08] on the existence of Galois representations attached to regular automorphic representations of GL_n over totally real and CM fields, subject to suitable self-duality hypotheses and to the existence of finite places at which the representations are square integrable. Section 3 recalls some standard results on the Galois representations attached to modular forms, and proves the result mentioned above on the existence of a density one set of primes for which there is an ordinary Galois representation.

In section 4 we prove the potential automorphy in weight 0 of the symmetric powers of the residual Galois representations associated to a modular form, under the hypotheses that the residual Galois representation is ordinary and irreducible, and the automorphic representation corresponding to the modular form is an unramified twist of the Steinberg representation at some finite place. The latter condition arises because of restrictions of our knowledge as to when there are Galois representations associated to automorphic representations on unitary groups, and it is anticipated that it will be possible to remove it in the near future. That would then allow us to prove our main theorems for any modular forms of weights 2 or 3 which are not of CM type. (Note added in proof: such results are now available, cf. [Shi09], [CH09], [Gue09], and it is thus an easy exercise to deduce our main results without any Steinberg assumption.)

One approach to proving the potential automorphy result in weight 0 would be to mimic the proofs for elliptic curves in [HSBT09]. In fact we can do better than this, and are able to directly utilise their results. We are reduced to proving that after making a quadratic base change and twisting, the mod l representation attached to our modular form is, after a further base change, congruent to a mod l representation arising from a certain Hilbert-Blumenthal abelian variety. This is essentially proved in [Tay02], and we only need to make minor changes to the proofs in [Tay02] in order to deduce the properties we need. We can then directly apply one of the main results of [HSBT09] to deduce the automorphy of the even-dimensional symmetric powers of the Hilbert-Blumenthal abelian variety, and after twisting back we deduce the required potential automorphy of our residual representations. Note that apart from resulting in rather clean proofs, the advantage of making an initial congruence to a Galois representation attached to an abelian variety and then using the potential automorphy of the symmetric powers of this abelian variety is that we are able to obtain local-global compatibility at all finite places (including those dividing the residue characteristic). This compatibility is not yet available for automorphic representations on unitary groups in general, and is needed in our subsequent arguments. In particular, it tells us that the automorphic representations of weight 0 which correspond to the symmetric powers of the l-adic representations coming from our Hilbert-Blumenthal abelian variety are ordinary at l.

In section 5 we exploit this ordinarity to deduce that the even-dimensional symmetric powers of the mod l representations are potentially automorphic of the "correct" weight. This is a basic consequence of Hida theory for unitary groups, but we are not aware of any reference that proves the precise result we need. Accordingly, we provide a proof in the style of the arguments of [Tay88].

There is nothing original in this section, and as the arguments are somewhat technical the reader may wish to skip it on a first reading.
The results of the preceding sections are combined in section 6 to establish the required potential automorphy results for l-adic (rather than mod l) representations. This essentially comes down to checking the hypotheses of the modularity lifting theorem that we wish to apply from [Tay08], which follow from the analogous arguments in [HSBT09] together with the conditions that we have imposed in our potential automorphy arguments. It is here that we need the freedom to choose l to be arbitrarily large, which results in our restriction to weights 2 and 3.
Finally, in section 7 we deduce the form of the Sato-Tate conjecture mentioned above. As in [HSBT09] we have only proved the potential automorphy of the even-dimensional symmetric powers of the l-adic representations associated to our modular form, and we deduce the required analytic properties for the L-functions attached to odd-dimensional symmetric powers via an argument with Rankin-Selberg convolutions exactly analogous to that of [HSBT09]. In fact, we need to prove the same results for the L-functions of certain twists of our representations by finite-order characters, but this is no more difficult.
We now describe the form of the final result, which is slightly different from that for elliptic curves, because our modular forms may have non-trivial nebentypus (and indeed are required to do so if they have weight 3). Suppose that the newform f has level N, nebentypus χ_f and weight k; then the image of χ_f is precisely the m-th roots of unity for some m. Then if $p \nmid N$ is a prime, we know that if

$$X^2 - a_p X + p^{k-1}\chi_f(p) = (X - \alpha_p p^{(k-1)/2})(X - \beta_p p^{(k-1)/2})$$

where a_p is the eigenvalue of f for the Hecke operator T_p, then the matrix

$$\begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}$$

defines a conjugacy class $x_{f,p}$ in $U(2)_m$, the subgroup of $U(2)$ of matrices with determinant an m-th root of unity. Then our main result is

Theorem. *If f has weight 2 or 3 and the associated automorphic representation is a twist of the Steinberg representation at some finite place, then the conjugacy classes $x_{f,p}$ are equidistributed with respect to the Haar measure on $U(2)_m$ (normalised so that $U(2)_m$ has measure 1).*

One can make this more concrete by restricting to primes p such that $\chi_f(p)$ is a specific m-th root of unity; see the remarks at the end of section 7.
We would like to thank Thomas Barnet-Lamb, David Geraghty and Richard Taylor for various helpful discussions during the writing of this paper.

2. Notation and assumptions

We let ϵ denote the l-adic cyclotomic character, regarded as a character of the absolute Galois group of a number field or of a completion of a number field at

a finite place. We sometimes use the same notation for the mod l cyclotomic character; it will always be clear from the context which we are referring to. We denote Tate twists in the usual way, i.e. $\rho(n) := \rho \otimes \epsilon^n$. We write $\bar{K}$ for a separable closure of a field K. If x is a finite place of a number field F, we will write I_x for the inertia subgroup of $\mathrm{Gal}(\overline{F}_x/F_x)$. We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and regard all finite extensions of $\mathbb{Q}$ as being subfields of $\overline{\mathbb{Q}}$. We also fix algebraic closures $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ for all primes p, and embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.
We need several incarnations of the local Langlands correspondence. Let K be a finite extension of $\mathbb{Q}_p$, and $l \neq p$ a prime. We have a canonical isomorphism

$$\mathrm{Art}_K : K^\times \to W_K^{ab}$$

normalised so that geometric Frobenius elements correspond to uniformisers. Let $\mathrm{Irr}(\mathrm{GL}_n(K))$ denote the set of isomorphism classes of irreducible admissible representations of $\mathrm{GL}_n(K)$ over $\mathbb{C}$, and let $\mathrm{WDRep}_n(W_K)$ denote the set of isomorphism classes of n-dimensional Frobenius semi-simple complex Weil-Deligne representations of the Weil group W_K of K. The main result of [HT01] is that there is a family of bijections

$$\mathrm{rec}_K : \mathrm{Irr}(\mathrm{GL}_n(K)) \to \mathrm{WDRep}_n(W_K)$$

satisfying a number of properties that specify them uniquely (see the introduction to [HT01] for a complete list). Among these properties are:

- If $\pi \in \mathrm{Irr}(\mathrm{GL}_1(K))$ then $\mathrm{rec}_K(\pi) = \pi \circ \mathrm{Art}_K^{-1}$.
- $\mathrm{rec}_K(\pi^\vee) = \mathrm{rec}_K(\pi)^\vee$.
- If $\chi_1, \dots, \chi_n \in \mathrm{Irr}(\mathrm{GL}_1(K))$ are such that the normalised induction n-Ind$(\chi_1, \dots, \chi_n)$ is irreducible, then

$$\mathrm{rec}_K(\text{n-Ind}(\chi_1, \dots, \chi_n)) = \oplus_{i=1}^n \mathrm{rec}_K(\chi_i).$$

We will often just write rec for rec_K when the choice of K is clear from the context. After choosing an isomorphism $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$ one obtains bijections rec_l from the set of isomorphism classes of irreducible admissible representations of $\mathrm{GL}_n(K)$ over $\overline{\mathbb{Q}}_l$ to the set of isomorphism classes of n-dimensional Frobenius semi-simple Weil-Deligne $\overline{\mathbb{Q}}_l$-representations of W_K. We then define $r_l(\pi)$ to be the l-adic representation of $\mathrm{Gal}(\overline{K}/K)$ associated to $\mathrm{rec}_l(\pi^\vee \otimes |\cdot|^{(1-n)/2})$ whenever this exists (that is, whenever the eigenvalues of $\mathrm{rec}_l(\pi^\vee \otimes |\cdot|^{(1-n)/2})(\phi)$ are l-adic units, where ϕ is a Frobenius element). We will, of course, only use this notation where it makes sense. It is useful to note that

$$r_l(\pi)^\vee(1-n) = r_l(\pi^\vee).$$

Let M denote a CM field with maximal totally real subfield F (by "CM field" we always mean "imaginary CM field"). We denote the nontrivial element of $\mathrm{Gal}(M/F)$ by c. Following [CHT08] we define a RACSDC (regular, algebraic, conjugate self dual, cuspidal) automorphic representation of $\mathrm{GL}_n(\mathbb{A}_M)$ to be a cuspidal automorphic representation π such that

- $\pi^\vee \cong \pi^c$, and

- π_∞ has the same infinitesimal character as some irreducible algebraic representation of $\mathrm{Res}_{M/\mathbb{Q}}\,\mathrm{GL}_n$.

We say that $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(M,\mathbb{C})}$ is a weight if

- $a_{\tau,1} \geq \cdots \geq a_{\tau,n}$ for all $\tau \in \mathrm{Hom}(M,\mathbb{C})$, and
- $a_{\tau c,i} = -a_{\tau,n+1-i}$.

For any weight a we may form an irreducible algebraic representation W_a of $\mathrm{GL}_n^{\mathrm{Hom}(M,\mathbb{C})}$, the tensor product over τ of the irreducible algebraic representations of GL_n with highest weight a_τ. We say that π has weight a if it has the same infinitesimal character as $W_a^\vee$; note that any RACSDC automorphic representation has some weight. Let S be a non-empty finite set of finite places of M. For each $v \in S$, choose an irreducible square integrable representation ρ_v of $\mathrm{GL}_n(M_v)$ (in this paper, we will in fact only need to consider the case where each ρ_v is the Steinberg representation). We say that an RACSDC automorphic representation π has type $\{\rho_v\}_{v\in S}$ if for each $v \in S$, π_v is an unramified twist of $\rho_v^\vee$. There is a compatible family of Galois representations associated to such a representation in the following fashion.

PROPOSITION 2.1. *Let $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Suppose that π is an RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_M)$ of type $\{\rho_v\}_{v\in S}$ for some nonempty set of finite places S. Then there is a continuous semisimple representation*

$$r_{l,\iota}(\pi) : \mathrm{Gal}(\overline{M}/M) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

such that

(1) *For each finite place $v \nmid l$ of M, we have*

$$r_{l,\iota}(\pi)|^{ss}_{\mathrm{Gal}(\overline{M}_v/M_v)} = r_l(\iota^{-1}\pi_v)^\vee(1-n)^{ss}.$$

(2) $r_{l,\iota}(\pi)^c = r_{l,\iota}(\pi)^\vee \epsilon^{1-n}$.

Proof. This follows from Proposition 4.2.1 of [CHT08] (which in fact also gives information on $r_{l,\iota}|_{\mathrm{Gal}(\overline{M}_v/M_v)}$ for places $v|l$). □

The representation $r_{l,\iota}(\pi)$ may be conjugated to be valued in the ring of integers of a finite extension of $\mathbb{Q}_l$, and we may reduce it modulo the maximal ideal of this ring of integers and semisimplify to obtain a well-defined continuous representation

$$\bar{r}_{l,\iota}(\pi) : \mathrm{Gal}(\overline{M}/M) \to \mathrm{GL}_n(\overline{\mathbb{F}}_l).$$

Let $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(M,\overline{\mathbb{Q}}_l)}$, and let $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Define $\iota_* a \in (\mathbb{Z}^n)^{\mathrm{Hom}(M,\mathbb{C})}$ by $(\iota_* a)_{\iota\tau,i} = a_{\tau,i}$. Now let ρ_v be a discrete series representation of $\mathrm{GL}_n(M_v)$ over $\overline{\mathbb{Q}}_l$ for each $v \in S$. If $r : \mathrm{Gal}(\overline{M}/M) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$, we say that r is automorphic of weight a and type $\{\rho_v\}_{v\in S}$ if $r \cong r_{l,\iota}(\pi)$ for some RACSDC automorphic representation π of weight $\iota_* a$ and type $\{\iota\rho_v\}_{v\in S}$. Similarly, if $\bar{r} : \mathrm{Gal}(\overline{M}/M) \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$, we say that $\bar{r}$ is automorphic of weight a and type $\{\rho_v\}_{v\in S}$ if $\bar{r} \cong \bar{r}_{l,\iota}(\pi)$ for some RACSDC automorphic representation π with π_l unramified, of weight $\iota_* a$ and type $\{\iota\rho_v\}_{v\in S}$.

We now consider automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$. We say that a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ is RAESDC (regular, algebraic, essentially self dual, cuspidal) if

- $\pi^\vee \cong \chi\pi$ for some character $\chi : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ with $\chi_v(-1)$ independent of $v|\infty$, and
- π_∞ has the same infinitesimal character as some irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$.

We say that $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\mathbb{C})}$ is a weight if

$$a_{\tau,1} \geq \cdots \geq a_{\tau,n}$$

for all $\tau \in \mathrm{Hom}(F,\mathbb{C})$. For any weight a we may form an irreducible algebraic representation W_a of $\mathrm{GL}_n^{\mathrm{Hom}(F,\mathbb{C})}$, the tensor product over τ of the irreducible algebraic representations of GL_n with highest weight a_τ. We say that an RAESDC automorphic representation π has weight a if it has the same infinitesimal character as $W_a^\vee$. In this case, by the classification of algebraic characters over a totally real field, we must have $a_{\tau,i} + a_{\tau,n+1-i} = w_a$ for some w_a independent of τ. Let S be a non-empty finite set of finite places of F. For each $v \in S$, choose an irreducible square integrable representation ρ_v of $\mathrm{GL}_n(M_v)$. We say that an RAESDC automorphic representation π has type $\{\rho_v\}_{v\in S}$ if for each $v \in S$, π_v is an unramified twist of $\rho_v^\vee$. Again, there is a compatible family of Galois representations associated to such a representation in the following fashion.

Proposition 2.2. *Let $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Suppose that π is an RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, of type $\{\rho_v\}_{v\in S}$ for some nonempty set of finite places S, with $\pi^\vee \cong \chi\pi$. Then there is a continuous semisimple representation*

$$r_{l,\iota}(\pi) : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

such that

(1) *For each finite place $v \nmid l$ of F, we have*

$$r_{l,\iota}(\pi)|^{ss}_{\mathrm{Gal}(\overline{F}_v/F_v)} = r_l(\iota^{-1}\pi_v)^\vee(1-n)^{ss}.$$

(2) $r_{l,\iota}(\pi)^\vee = r_{l,\iota}(\chi)\epsilon^{n-1}r_{l,\iota}(\pi)$.

Here $r_{l,\iota}(\chi)$ is the l-adic Galois representation associated to χ via ι (see Lemma 4.1.3 of [CHT08]*).*

Proof. This is Proposition 4.3.1 of [CHT08] (which again obtains a stronger result, giving information on $r_{l,\iota}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ for places $v|l$). □

Again, the representation $r_{l,\iota}(\pi)$ may be conjugated to be valued in the ring of integers of a finite extension of $\mathbb{Q}_l$, and we may reduce it modulo the maximal ideal of this ring of integers and semisimplify to obtain a well-defined continuous representation

$$\bar{r}_{l,\iota}(\pi) : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_l).$$

Let $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)}$, and let $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Define $\iota_* a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\mathbb{C})}$ by $(\iota_* a)_{\iota\tau,i} = a_{\tau,i}$. Let ρ_v be a discrete series representation of $\mathrm{GL}_n(M_v)$ over $\overline{\mathbb{Q}}_l$ for each $v \in S$. If $r : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$, we say that r is automorphic of weight a and type $\{\rho_v\}_{v\in S}$ if $r \cong r_{l,\iota}(\pi)$ for some RAESDC automorphic representation π of weight $\iota_* a$ and type $\{\iota\rho_v\}_{v\in S}$. Similarly, if $\bar{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$, we say that $\bar{r}$ is automorphic of weight a and type $\{\rho_v\}_{v\in S}$ if $\bar{r} \cong \bar{r}_{l,\iota}(\pi)$ for some RAESDC automorphic representation π with π_l unramified, of weight $\iota_* a$ and type $\{\iota\rho_v\}_{v\in S}$.
As in [HSBT09] we denote the Steinberg representation of $\mathrm{GL}_n(K)$, K a nonarchimedean local field, by $\mathrm{Sp}_n(1)$.

3. Modular forms

3.1. Let f be a cuspidal newform of level $\Gamma_1(N)$, nebentypus χ_f, and weight $k \geq 2$. Suppose that for each prime $p \nmid N$ we have $T_p f = a_p f$. Then each a_p is an algebraic integer, and the set $\{a_p\}$ generates a number field K_f with ring of integers $\mathcal{O}_f$. We will view K_f as a subfield of $\mathbb{C}$. It is known that K_f contains the image of χ_f. For each place $\lambda | l$ of $\mathcal{O}_f$ there is a continuous representation

$$\rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{f,\lambda})$$

which is determined up to isomorphism by the property that for all $p \nmid Nl$, $\rho_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is unramified, and the characteristic polynomial of $\rho_{f,\lambda}(\mathrm{Frob}_p)$ is $X^2 - a_p X + p^{k-1}\chi_f(p)$ (where Frob_p is a choice of a geometric Frobenius element at p).
Assume from now on that f is not of CM type.

Definition 3.1. Let λ be a prime of $\mathcal{O}_f$ lying over a rational prime l. Then we say that f is ordinary at λ if $\lambda \nmid a_l$. We say that f is ordinary at l if it is ordinary at λ for some $\lambda | l$.

Lemma 3.2. *If $k = 2$ or 3, then the set of primes l such that f is ordinary at l has density one.*

Proof. The proof is based on an argument of Wiles (see the final lemma of [Wil88]). Let S be the finite set of primes which either divide N or which are ramified in $\mathcal{O}_f$. Suppose that f is not ordinary at $p \notin S$. By definition we have that $\lambda | a_p$ for each prime λ of $\mathcal{O}_f$ lying over p. Since p is unramified in $\mathcal{O}_f$, $(p) = \prod_{\lambda|p} \lambda$, so $p | a_p$. Write $a_p = p b_p$ with $b_p \in \mathcal{O}_f$.
Since $p \nmid N$, the Weil bounds (that is, the Ramanujan-Petersson conjecture) tell us that for each embedding $\iota : K_f \hookrightarrow \mathbb{C}$ we have $|\iota(a_p)| \leq 2p^{(k-1)/2}$. Since $k \leq 3$, this implies that $|\iota(b_p)| \leq 2$ for all ι. Let T be the set of $y \in \mathcal{O}_f$ such that $|\iota(y)| \leq 2$ for all ι. This is a finite set, because one can bound the absolute values of the coefficients of the characteristic polynomial of such a y.
From the above analysis, it is sufficient to prove that for each $y \in T$, the set of primes p for which $a_p = py$ has density zero. However, by Corollaire 1 to Théorème 15 of [Ser81], the number of primes $p \leq x$ for which $a_p = py$ is

$O(x/(\log x)^{5/4-\delta})$ for any $\delta > 0$, which immediately shows that the density of such primes is zero, as required. □

The following result is well known, and follows from, for example, [Sch90] and Theorem 2 of [Wil88].

LEMMA 3.3. *If f is ordinary at a place $\lambda | l$ of $\mathcal{O}_f$, and $l \nmid N$, then the Galois representation $\rho_{f,\lambda}$ is crystalline, and furthermore it is ordinary; that is,*

$$\rho_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)} \cong \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \epsilon^{1-k} \end{pmatrix}$$

where ψ_1 and ψ_2 are unramified characters of finite order. In addition, ψ_1 takes Frob_l *to the unit root of $X^2 - a_l X + \chi_f(l) l^{k-1}$.*

3.2. Let $\overline{\rho}_{f,\lambda}$ denote the semisimplification of the reduction mod λ of $\rho_{f,\lambda}$; this makes sense because $\rho_{f,\lambda}$ may be conjugated to take values in $\mathrm{GL}_2(\mathcal{O}_{f,\lambda})$, and it is independent of the choice of lattice. It is valued in $\mathrm{GL}_2(k_{f,\lambda})$, where $k_{f,\lambda}$ is the residue field of $K_{f,\lambda}$.

DEFINITION 3.4. We say that $\overline{\rho}_{f,\lambda}$ has *large image* if

$$\mathrm{SL}_2(k) \subset \overline{\rho}_{f,\lambda}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \subset k_{f,\lambda}^{\times} \mathrm{GL}_2(k)$$

for some subfield k of $k_{f,\lambda}$.

We will need to know that the residual Galois representations $\overline{\rho}_{f,\lambda}$ frequently have large image. The following result is essentially due to Ribet (see [Rib75], which treats the case $N = 1$; for a concrete reference, which also proves the corresponding result for Hilbert modular forms, see [Dim05]).

LEMMA 3.5. *For all but finitely many primes λ of $\mathcal{O}_f$, $\overline{\rho}_{f,\lambda}$ has large image.*

3.3. We let $\pi(f)$ be the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ corresponding to f, normalised so that $\pi(f)$ is RAESDC of weight $(k-2, 0)$ (it is essentially self dual because

$$\pi(f)^{\vee} \cong \chi \pi(f)$$

where $\chi = |\cdot|^{k-2} \chi_f^{-1}$). Let $\lambda | l$ be a place of $\mathcal{O}_f$, and choose an isomorphism $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ and a compatible embedding $K_{f,\lambda} \hookrightarrow \overline{\mathbb{Q}}_l$; that is, an embedding such that the diagram

$$\begin{tikzcd} K_f \arrow[r] \arrow[d] & \mathbb{C} \\ K_{f,\lambda} \arrow[r] & \overline{\mathbb{Q}}_l \arrow[u, "\iota"'] \end{tikzcd}$$

commutes. Assume that $\pi_{f,v}$ is square integrable for some finite place v. Then by Proposition 2.2 there is a Galois representation

$$r_{l,\iota}(\pi(f)) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$$

associated to π_f, and it follows from the definitions that

$$r_{l,\iota}(\pi(f)) \cong \rho_{f,\lambda} \otimes_{K_{f,\lambda}} \overline{\mathbb{Q}}_l.$$

Definition 3.6. We say that f is Steinberg at a prime q if $\pi(f)_q$ is an unramified twist of the Steinberg representation.

Definition 3.7. We say that f is potentially Steinberg at a prime q if $\pi(f)_q$ is a (possibly ramified) twist of the Steinberg representation.

Note that if f is (potentially) Steinberg at q for some q then it is not CM. Note also that if f is potentially Steinberg at q then there is a Dirichlet character θ such that $f \otimes \theta$ is Steinberg at q.

4. Potential automorphy in weight 0

4.1. Let l be an odd prime, and let f be a modular form of weight $2 \leq k < l$ and level N, $l \nmid N$. Assume that f is Steinberg at q. Suppose that $\lambda | l$ is a place of $\mathcal{O}_f$ such that f is ordinary at λ. Assume that $\overline{\rho}_{f,\lambda}$ is absolutely irreducible. By Lemma 3.3 we have

$$\overline{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)} \cong \begin{pmatrix} \overline{\psi}_1 & * \\ 0 & \overline{\psi}_2 \epsilon^{1-k} \end{pmatrix}$$

where $\overline{\psi}_1$ and $\overline{\psi}_2$ are unramified characters. We wish to prove that the symmetric powers of $\overline{\rho}_{f,\lambda}$ are potentially automorphic of some weight. To do so, we use a potential modularity argument to realise $\overline{\rho}_{f,\lambda}$ geometrically, and then appeal to the results of [HSBT09].

The potential modularity result that we need is almost proved in [Tay02]; the one missing ingredient is that we wish to preserve the condition of being Steinberg at q. This is, however, easily arranged, and rather than repeating all of the arguments of [Tay02], we simply indicate the modifications required.

We begin by recalling some definitions from [Tay02]. Let N be a totally real field. Then an N-HBAV over a field K is a triple (A, i, j) where

- A/K is an abelian variety of dimension $[N : \mathbb{Q}]$,
- $i : \mathcal{O}_N \hookrightarrow \mathrm{End}(A/K)$, and
- $j : \mathcal{O}_N^+ \xrightarrow{\sim} \mathcal{P}(A, i)$ is an isomorphism of ordered invertible $\mathcal{O}_N$-modules.

For the definitions of ordered invertible $\mathcal{O}_N$-modules and of $\mathcal{O}_N^+$ and $\mathcal{P}(A, i)$, see page 133 of [Tay02].

Choose a totally real quadratic field F in which l is inert and q is unramified and which is linearly disjoint from $\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\lambda})}$ over $\mathbb{Q}$, a finite extension $k/k_{f,\lambda}$ and a character $\overline{\theta} : \mathrm{Gal}(\overline{F}/F) \to k^\times$ which is unramified at q such that

$$\det \overline{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{F}/F)} = \epsilon^{-1}\overline{\theta}^{-2}$$

and $(\overline{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{F}/F)} \otimes \overline{\theta})(\mathrm{Frob}_w)$ has eigenvalues 1, $\#k(w)$, where $w|q$ is a place of F. This is possible as the obstruction to taking a square root of a character is in the 2-part of the Brauer group, and because any class in the Brauer group of a local field splits over an unramified extension. Let $\overline{\rho} = \overline{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{F}/F)} \otimes \overline{\theta}$:

$\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(k)$, so that $\det \overline{\rho} = \epsilon^{-1}$. If x is the place of F lying over l, then we may write (for some character $\overline{\chi}_x$ of $\mathrm{Gal}(\overline{F_x}/F_x)$)

$$\overline{\rho}|_{\mathrm{Gal}(\overline{F_x}/F_x)} \cong \begin{pmatrix} \overline{\chi}_x^{-1} & * \\ 0 & \overline{\chi}_x \epsilon^{-1} \end{pmatrix}$$

with $\overline{\chi}_x^2|_{I_x} = \epsilon^{2-k}$.

THEOREM 4.1. *There is a finite totally real Galois extension E/F which is linearly disjoint from $\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\lambda})}$ over $\mathbb{Q}$ and in which the unique prime of F dividing l splits completely, a totally real field N, a place $\lambda'|l$ of N, a place $v_q|q$ of E, and an N-HBAV $(A,i,j)/E$ with potentially good reduction at all places dividing l such that*

- *the representation of $\mathrm{Gal}(\bar{E}/E)$ on $A[\lambda']$ is equivalent to $(\overline{\rho}|_{\mathrm{Gal}(\bar{E}/E)})^\vee$,*
- *at each place $x|l$ of E, the action of $\mathrm{Gal}(\overline{E_x}/E_x)$ on $T_{\lambda'}A \otimes \mathbb{Q}_l$ is of the form*
$$\begin{pmatrix} \chi_x^{-1}\epsilon & * \\ 0 & \chi_x \end{pmatrix}$$
with χ_x a tamely ramified lift of $\overline{\chi}_x$, and
- *A has multiplicative reduction at v_q.*

Proof. As remarked above, this is essentially proved in [Tay02]. Indeed, if $k > 2$ then with the exception of the fact that E can be chosen to be linearly disjoint from $\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\lambda})}$ over $\mathbb{Q}$, and the claim that A can be chosen to have multiplicative reduction at some place over q, the result is obtained on page 136 of [Tay02] (the existence of A with $A[\lambda']$ equivalent to $(\overline{\rho}|_{\mathrm{Gal}(\bar{E}/E)})^\vee$ is established in the second paragraph on that page, and the form of the action of $\mathrm{Gal}(\overline{E_x}/E_x)$ for $x|l$ follows from Lemma 1.5 of *loc. cit.*).
We now indicate the modifications needed to the arguments of [Tay02] to obtain the slight strengthening that we require. Suppose firstly that $k > 2$. Rather than employing the theorem of Moret-Bailly stated as Theorem G of [Tay02], we use the variant given in Proposition 2.1 of [HSBT09]. This immediately allows us to assume that E is linearly disjoint from $\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\lambda})}$ over $\mathbb{Q}$, so we only need to ensure that A has multiplicative reduction at some place dividing q. Let X be the moduli space defined in the first paragraph of page 136 of [Tay02]. Let v be a place of F lying over q. It is enough to check that there is a non-empty open subset Ω_v of $X(F_v)$ such that for each point of Ω_v, the corresponding N-HBAV has multiplicative reduction. Let Ω_v denote the set of *all* points of $X(F_v)$ such that the corresponding N-HBAV has multiplicative reduction; this is an open subset of $X(F_v)$, and it is non-empty (by the assumptions on $\bar{\theta}$ at places of F dividing q, and the assumption that $\pi(f)$ is an unramified twist of the Steinberg representation, we see that $\overline{\rho}(\mathrm{Frob}_v)$ has eigenvalues 1 and $\#k(v)$, and is congruent to a Galois representation attached to an unramified twist of a Steinberg representation, so any N-HBAV with multiplicative reduction suffices), as required.

If $k = 2$, then the only additional argument needed is one to ensure that if $\overline{\chi}_x^2 = 1$, then the abelian variety can be chosen to have good reduction rather than multiplicative reduction. This follows easily from the fact that $\overline{\rho}|_{\mathrm{Gal}(\overline{F_x}/F_x)}$ is finite flat (cf. the proof of Theorem 2.1 of [KW08], which establishes a very similar result). $\square$

Let M be a totally real field, and let $(A, i, j)/M$ be an N-HBAV. Fix an embedding $N \subset \mathbb{R}$. We recall some definitions from section 4 of [HSBT09]. For each finite place v of M there is a two dimensional Weil-Deligne representation $\mathrm{WD}_v(A, i)$ defined over $\overline{N}$ such that if $\mathfrak{p}$ is a place of N of residue characteristic p different from the residue characteristic of v, we have

$$\mathrm{WD}(H^1(A \times \overline{M}, \mathbb{Q}_p)|_{\mathrm{Gal}(\overline{M}_v/M_v)} \otimes_{N_p} \overline{N}_{\mathfrak{p}}) \cong \mathrm{WD}_v(A, i) \otimes_{\overline{N}} \overline{N}_{\mathfrak{p}}.$$

Definition 4.2. We say that $\mathrm{Sym}^m A$ is automorphic of type $\{\rho_v\}_{v \in S}$ if there is an RAESDC representation π of $\mathrm{GL}_{m+1}(\mathbb{A}_M)$ of weight 0 and type $\{\rho_v\}_{v \in S}$ such that for all finite places v of M,

$$\mathrm{rec}(\pi_v)|\,\mathrm{Art}_{M_v}^{-1}|^{-m/2} = \mathrm{Sym}^m \mathrm{WD}_v(A, i).$$

Theorem 4.3. *Let E, A be as in the statement of Theorem 4.1. Let $\mathcal{N}$ be a finite set of even positive integers. Then there is a finite Galois totally real extension F'/E and a place $w_q|q$ of F' such that*

- *for any $n \in \mathcal{N}$, $\mathrm{Sym}^{n-1} A$ is automorphic over F' of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$,*
- *The primes of E dividing l are unramified in F', and*
- *F' is linearly disjoint from $\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\lambda})}$ over $\mathbb{Q}$.*

Proof. This is essentially Theorem 4.1 of [HSBT09]. In particular, the proof in [HSBT09] establishes that there is a Galois totally real extension F'/E, and a place w_q of F' lying over q such that for any $n \in \mathcal{N}$, $\mathrm{Sym}^{n-1} A$ is automorphic over F' of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$. Note that the l used in their argument is *not* the l used here. To complete the proof, we need to establish that it is possible to obtain an F' in which l is unramified, and which is linearly disjoint from $\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\lambda})}$ over $\mathbb{Q}$. The latter point causes no difficulty, but the first point requires some minor modifications of the arguments of [HSBT09]. We now outline the necessary changes.

To aid comparison to [HSBT09], for the rest of this proof we will refer to our l as s; all references to l will be to primes of that name in the proofs of various theorems in [HSBT09]. We begin by choosing a finite solvable totally real extension L of E, linearly disjoint from $\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\lambda})}$ over $\mathbb{Q}$, such that the base change of A to L has good reduction at all places dividing s. Choose a prime l as in the proof of Theorem 4.1 of [HSBT09]. We then apply a slight modification of Theorem 4.2 of *loc.cit.*, with the conclusion strengthened to include the hypothesis that s is unramified in F'. To prove this, in the proof of Theorem 4.2 of *loc.cit.*, note that $F_1 = E$. Choose all auxiliary primes not to divide s. Rather than constructing a moduli space X_W over E, construct the analogous space over L, and consider the restriction of scalars $Y = \mathrm{Res}_{L/E}(X_W)$. Applying Proposition 2.1 of [HSBT09] to Y, rather than X_W, we may find a finite totally real Galois extension $F^{(1)}/E$ in which s is unramified, such that Y has an $F^{(1)}$-point. Furthermore, we may assume that $F^{(1)}$ is linearly disjoint from $\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\lambda})}$ over $\mathbb{Q}$. Note that an $F^{(1)}$-point of Y corresponds to an $F^{(1)}L$-point of X_W. We now make a similar modification to the proof of Theorem 3.1 of [HSBT09], replacing the schemes T_{W_i} over F with $\mathrm{Res}_{LF/F}\,T_{W_i}$. We conclude that there is a finite Galois totally real extension F'/E in which s is unramified, which is linearly disjoint from $\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\lambda})}$ over $\mathbb{Q}$, such that for any $n \in \mathcal{N}$, $\mathrm{Sym}^{n-1} A$ is automorphic over $F'L$ of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$. Since the extension $F'L/F'$ is solvable, it follows from solvable base change (e.g. Lemma 1.3 of [BLGHT09]) that in fact for any $n \in \mathcal{N}$, $\mathrm{Sym}^{n-1} A$ is automorphic over F' of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$, as required. $\square$

We may now twist $\overline{\rho}$ by $\overline{\theta}^{-1}$ in order to deduce results about $\overline{\rho}_{f,\lambda}$. Let N and λ be as in the statement of Theorem 4.1. Fix an embedding $N_{\lambda'} \hookrightarrow \overline{\mathbb{Q}}_l$. Let θ be the Teichmüller lift of $\overline{\theta}$, and let ρ_n denote the action of $\mathrm{Gal}(\bar{E}/E)$ on

$$\mathrm{Sym}^{n-1}(H^1(A \times \overline{E}, \mathbb{Q}_l) \otimes_{N_l} N_{\lambda'} \otimes \theta^{-1}) \otimes_{N'_\lambda} \overline{\mathbb{Q}}_l.$$

By construction, ρ_n is a lift of $\mathrm{Sym}^{n-1}\,\overline{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{E}/E)} \otimes_{k_{f,\lambda}} \bar{\mathbb{F}}_l$ (where the embedding $k_{f,\lambda} \hookrightarrow \overline{\mathbb{F}}_l$ is determined by the embedding $k \hookrightarrow \overline{\mathbb{F}}_l$ induced by the embedding $N_{\lambda'} \hookrightarrow \overline{\mathbb{Q}}_l$). Note also that (again by construction) at each place $x|l$ of E,

$$\rho_2|_{\mathrm{Gal}(\overline{E_x}/E_x)} \cong \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \omega^{2-k} \epsilon^{-1} \end{pmatrix}$$

with ψ_1, ψ_2 unramified lifts of $\overline{\psi}_1|_{\mathrm{Gal}(\overline{E_x}/E_x)}$ and $\overline{\psi}_2|_{\mathrm{Gal}(\overline{E_x}/E_x)}$ respectively, and ω the Teichmüller lift of ϵ.

COROLLARY 4.4. *Let $\mathcal{N}$ be a finite set of even positive integers. Then there is a Galois totally real extension F'/E and a place $w_q|q$ of F' such that*

- *for any $n \in \mathcal{N}$, $\rho_n|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F')}$ is automorphic of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$,*
- *every prime of E dividing l is unramified in F' (so that l is unramified in F'), and*
- *F' is linearly disjoint from $\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\lambda})}$ over $\mathbb{Q}$.*

Let $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, and for $n \in \mathcal{N}$ let π_n be the RAESDC representation of $\mathrm{GL}_n(\mathbb{A}_{F'})$ with $r_{l,\iota}(\pi_n) \cong \rho_n|_{\mathrm{Gal}(\overline{\mathbb{Q}}_l/F')}$. If $k = 2$ then $\pi_{n,x}$ is unramified for each $x|l$, and if $k > 2$ then for each place $x|l$ of F', $\pi_{n,x}$ is a principal series representation $\text{n-Ind}_{B_n(F'_x)}^{\mathrm{GL}_n(F'_x)}(\chi_1, \ldots, \chi_n)$ with $\iota^{-1}\chi_i \circ \mathrm{Art}_{F'_x}^{-1}|_{I_x} = \omega^{(i-1)(2-k)}$ and $v_l(\iota^{-1}\chi_i(l)) = [F'_x : \mathbb{Q}_l]\left(i - 1 + \frac{1-n}{2}\right)$, where v_l is the l-adic valuation on $\overline{\mathbb{Q}}_l$ with $v_l(l) = 1$.

Proof. This is a straightforward consequence of Theorem 4.3. The only part that needs to be checked is the assertion about the form of $\pi_{n,x}$ for $x|l$ when $k > 2$. Without loss of generality, we may assume that $2 \in \mathcal{N}$. Note firstly that any principal series representation of the given form is irreducible, so that we need only check that

$$\iota^{-1} \operatorname{rec}(\pi_{n,x}) = \bigoplus_{i=1}^{n} \omega^{(i-1)(2-k)} \alpha_i,$$

where α_i is an unramified character with $v_l(\alpha_i(l)) = [F'_x : \mathbb{Q}_l] \left(i - 1 + \frac{1-n}{2}\right)$. By Definition 4.2 and Theorem 4.3 we see that $\operatorname{rec}(\pi_{n,x}) = \operatorname{Sym}^{n-1} \operatorname{rec}(\pi_{2,x})$, so it suffices to establish the result in the case $n = 2$, or rather (because of the compatibility of rec with twisting) it suffices to check the corresponding result for $\mathrm{WD}_v(A, i)$ at places $v|l$. This is now an immediate consequence of local-global compaitibility, and follows at once from, for example, Lemma B.4.1 of [CDT99], together with the computations of the Weil-Deligne representations associated to characters in section B.2 of *loc. cit.*

□

5. CHANGING WEIGHT

5.1. We now explain how to deduce from the results of the previous section that $\operatorname{Sym}^n \overline{\rho}_{f,\lambda}$ is potentially automorphic of the correct weight (that is, the weight of the conjectural automorphic representation corresponding to $\operatorname{Sym}^n \rho_{f,\lambda}$), rather than potentially automorphic of weight 0. We accomplish this as a basic consequence of Hida theory; note that we simply need a congruence, rather than a result about families, and the result follows from a straightforward combinatorial argument. This result is certainly known to the experts, but as we have been unable to find a reference which provides the precise result we need, we present a proof in the spirit of the arguments of [Tay88].

5.2. For each n-tuple of integers $a = (a_1, \ldots, a_n)$ with $a_1 \geq \cdots \geq a_n$ there is an irreducible representation of the algebraic group GL_n defined over $\mathbb{Q}_l$, with highest weight (with respect to the Borel subgroup of upper-triangular matrices) given by

$$\operatorname{diag}(t_1, \ldots, t_n) \mapsto \prod_{i=1}^{n} t_i^{a_i}.$$

We will need an explicit model of this representation, for which we follow section 2 of [Che04].

Let K be an algebraic extension of $\mathbb{Q}_l$, N the subgroup of $\mathrm{GL}_n(K)$ consisting of upper triangular unipotent matrices, $\overline{N}$ the subgroup of lower triangular unipotent matrices, and T the subgroup of diagonal matrices. Let $R := K[\mathrm{GL}_n] = K[\{X_{i,j}\}_{1 \leq i,j \leq n}, \det(X_{i,j})^{-1}]$. We have commuting natural actions of $\mathrm{GL}_n(K)$ on R by left and right multiplication. For an element

$g \in \mathrm{GL}_n(K)$ we denote these actions by g_l and g_r respectively, so that if we let M denote the matrix $(X_{i,j})_{i,j} \in M_n(R)$, we have

$$(g_l.X)_{i,j} = g^{-1}M$$

and

$$(g_r.X)_{i,j} = Mg.$$

If $(t_1, \dots, t_n) \in \mathbb{Z}^n$, we say that an element $f \in R$ is of left weight t (respectively of right weight t) if for all $d \in T$ we have $d_l f = t^{-1}(d)f$ (respectively $d_r f = t(d)f$) where

$$t(\mathrm{diag}(x_1, \dots, x_n)) = \prod_{i=1}^{n} x_i^{t_i}.$$

For each $1 \le i \le n$ and each i-tuple $j = (j_1, \dots, j_i)$, $1 \le j_1 < \cdots < j_i \le n$, we let $Y_{i,j}$ be the minor of order i of M obtained by taking the entries from the first i rows and columns $j_1, \dots, j_i$. Let $R^{\overline{N}}$ denote the subalgebra of R of elements fixed by the g_l-action of $\overline{N}$; it is easy to check that $Y_{i,j} \in R^{\overline{N}}$. Because T normalises $\overline{N}$ it acts on $R^{\overline{N}}$ on the left, and we let $R_t^{\overline{N}}$ be the sub K-vector space of elements of left weight t; this has a natural action of $\mathrm{GL}_n(K)$ induced by g_r.

Proposition 5.1. *Suppose that $t_1 \ge \cdots \ge t_n$. Then $R_t^{\overline{N}}$ is a model of the irreducible algebraic representation of $\mathrm{GL}_n(K)$ of highest weight t. Furthermore, it is generated as a K-vector space by the monomials in $Y_{i,j}$ of left weight t, and a highest weight vector is given by the unique monomial in $Y_{i,j}$ of left and right weight t.*

Proof. This follows from Proposition 2.2.1 of [Che04]. □

Assume that in fact $t_1 \ge \cdots \ge t_n \ge 0$, and let X_t denote the free $\mathcal{O}_K$-module with basis the monomials in $Y_{i,j}$ of left weight t. By Proposition 5.1, X_t is a $\mathrm{GL}_n(\mathcal{O}_K)$-stable lattice in $R_t^{\overline{N}}$. Let T^+ be the submonoid of T consisting of elements of the form

$$\mathrm{diag}(l^{b_1}, \dots, l^{b_n})$$

with $b_1 \ge \cdots \ge b_n \ge 0$; then X_t is certainly also stable under the action of T^+. Let $\alpha = \mathrm{diag}(l^{b_1}, \dots, l^{b_n}) \in T^+$. We wish to determine the action of α on X_t.

Lemma 5.2. *If $Y \in X_t$ is a monomial in the $Y_{i,j}$, then $\alpha(Y) \subset l^{\sum_{i=1}^n b_i t_{n+1-i}} X_t$. If in fact $b_1 > \cdots > b_n$ then $\alpha(Y) \subset l^{1+\sum_{i=1}^n b_i t_{n+1-i}} X_t$ unless Y is the unique lowest weight vector.*

Proof. If Y has (right) weight $(v_1, \dots, v_n)$, then $\alpha(Y) = l^{\sum_{i=1}^n b_i v_i} Y$. The unique lowest weight vector has weight $(t_n, \dots, t_1)$, so it suffices to prove that for any other Y of weight $(v_1, \dots, v_n)$ which occurs in $R_t^{\overline{N}}$, the quantity $\sum_{i=1}^n b_i v_i$ is at least as large, and is strictly greater if $b_1 > \cdots > b_n$. However, by standard weight theory we know that we may obtain $(v_1, \dots, v_n)$ from $(t_n, \dots, t_1)$ by successively adding vectors of the form

$(0,\dots,1,0\dots,0,-1,0,\dots,0)$, and it is clear that the addition of such a vector does not decrease the sum, and in fact increases it if $b_1 > \dots > b_n$, as required. $\square$

We define a new action of T^+ on X_t, which we denote by $\cdot_{twist}$, by multiplying the natural action of $\mathrm{diag}(l^{b_1},\dots,l^{b_n})$ by $l^{-\sum_{i=1}^n b_i t_{n+1-i}}$; this is legitimate by Lemma 5.2.

5.3. Fix for the rest of this section a choice of isomorphism $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Assume for the rest of this section that F' is a totally real field in which each l is unramified, and π' is an RAESDC representation of $\mathrm{GL}_n(\mathbb{A}_{F'})$ of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$ for some place $w_q|q$ of F', with $(\pi')^\vee = \chi\pi'$. Suppose furthermore that there is an integer $k > 2$ such that

- for each place $x|l$, π'_x is a principal series $\text{n-Ind}_{B_n(F_x)}^{\mathrm{GL}_n(F_x)}(\chi_1,\dots,\chi_n)$ with $v_l(\iota^{-1}\chi_i(l)) = [F'_x : \mathbb{Q}_l]\left(i-1+\frac{1-n}{2}\right)$ and $\iota^{-1}\chi_i \circ \mathrm{Art}_{F_x}^{-1}|_{I_x} = \omega^{(i-1)(2-k)}$.

(See Corollary 4.4 for an example of such a representation.) We transfer to a unitary group, following section 3.3 of [CHT08]. Firstly, we make a quartic totally real Galois extension F/F', linearly disjoint from $\overline{\mathbb{Q}}^{\ker \overline{r}_{l,\iota}(\pi)}$ over $\mathbb{Q}$, such that w_q and all primes dividing l split in F. Let $S(B)$ be the set of places of F lying over w_q. Let E be a imaginary quadratic field in which l and q split, such that E is linearly disjoint from $\overline{\mathbb{Q}}^{\ker \overline{r}_{l,\iota}(\pi)}$ over $\mathbb{Q}$. Let $M = FE$. Let c denote the nontrivial element of $\mathrm{Gal}(M/F)$. Let S_l denote the places of F dividing l, and let $\tilde{S}_l$ denote a set of places of M dividing l such that the natural map $\tilde{S}_l \to S_l$ is a bijection. If $v|l$ is a place of F then we write $\tilde{v}$ for the corresponding place in $\tilde{S}_l$.

LEMMA 5.3. *There is a finite order character $\phi : M^\times\backslash\mathbb{A}_M^\times \to \mathbb{C}^\times$ such that*

- *$\phi \circ N_{M/F} = \chi \circ N_{M/F}$, and*
- *ϕ is unramified at all places lying over $S(B)$ and at all places in $\tilde{S}_l$.*

Proof. By Lemma 4.1.1 of [CHT08] (or more properly its proof, which shows that the character produced may be arranged to have finite order) there is a finite order character $\psi : M^\times\backslash\mathbb{A}_M^\times \to \mathbb{C}^\times$ such that for each $v \in S_l$, $\psi|_{M_{\tilde{v}}{}^\times} = 1$ and $\psi|_{M_{c\tilde{v}}{}^\times} = \chi|_{F_v^\times}$, and such that ψ is unramified at each place in $S(B)$. It now suffices to prove the result for the character $\chi(\psi|_{\mathbb{A}_F^\times})^{-1}$, which is unramified at $S(B) \cup S_l$, and the result now follows from Lemma 4.1.4 of [CHT08]. $\square$

Now let $\pi = \pi'_M \otimes \phi$, which is an RACSDC representation of $\mathrm{GL}_n(\mathbb{A}_M)$, satisfying:

- π has weight 0.
- π has type $\{\mathrm{Sp}_n(1)\}_{w|w_q}$.
- for each place $x \in \tilde{S}_l$, π_x is a principal series $\text{n-Ind}_{B_n(M_x)}^{\mathrm{GL}_n(M_x)}(\chi_1,\dots,\chi_n)$ with $v_l(\iota^{-1}\chi_i(l)) = [F'_x : \mathbb{Q}_l]\left(i-1+\frac{1-n}{2}\right)$ and $\iota^{-1}\chi_i \circ \mathrm{Art}_{M_x}^{-1}|_{I_x} = \omega^{(i-1)(2-k)}$ with $k > 2$.

5.4. Choose a division algebra B with centre M such that

- B splits at all places not dividing a place in $S(B)$.
- If w is a place of M lying over a place in $S(B)$, then B_w is a division algebra.
- $\dim_M B = n^2$.
- $B^{op} \cong B \otimes_{M,c} M$.

For any involution $\ddagger$ on B with $\ddagger|_M = c$, we may define a reductive algebraic group $G_\ddagger/F$ by

$$G_\ddagger(R) = \{g \in B \otimes_F R : g^{\ddagger\otimes 1}g = 1\}$$

for any F-algebra R. Because $[F : \mathbb{Q}]$ is divisible by 4 and $\#S(B)$ is even, we may (by the argument used to prove Lemma 1.7.1 of [HT01]) choose $\ddagger$ such that

- If $v \notin S(B)$ is a finite place of F then $G_\ddagger(F_v)$ is quasi-split, and
- If $v|\infty$, $G_\ddagger(F_v) \cong U(n)$.

Fix such a choice of $\ddagger$, and write G for $G_\ddagger$. We wish to work with algebraic modular forms on G; in order to do so, we need an integral model for G. We obtain such a model by fixing an order $\mathcal{O}_B$ in B such that $\mathcal{O}_B^\ddagger = \mathcal{O}_B$ and $\mathcal{O}_{B,w}$ is a maximal order for all primes w which are split over M (see section 3.3 of [CHT08] for a proof that such an order exists). We now regard G as an algebraic group over $\mathcal{O}_F$, by defining

$$G(R) = \{g \in \mathcal{O}_B \otimes_{\mathcal{O}_F} R : g^{\ddagger\otimes 1}g = 1\}$$

for all $\mathcal{O}_F$-algebras R.

We may identify G with GL_n at places not in $S(B)$ which split in M in the following way. Let $v \notin S(B)$ be a place of F which splits in M. Choose an isomorphism $i_v : \mathcal{O}_{B,v} \xrightarrow{\sim} M_n(\mathcal{O}_{M_v})$ such that $i_v(x^\ddagger) = {}^t i_v(x)^c$ (where ^{t} denotes matrix transposition). Choosing a prime $w|v$ of M gives an isomorphism

$$\begin{aligned} i_w : G(F_v) &\xrightarrow{\sim} \mathrm{GL}_n(M_w) \\ i_v^{-1}(x, {}^t x^{-c}) &\mapsto x. \end{aligned}$$

This identification satisfies $i_w G(\mathcal{O}_{F,v}) = \mathrm{GL}_n(\mathcal{O}_{M,w})$. Similarly, if $v \in S(B)$ then v splits in M, and if $w|v$ then we obtain an isomorphism

$$i_w : G(F_v) \xrightarrow{\sim} B_w^\times$$

with $i_w G(\mathcal{O}_{F,v}) = \mathcal{O}_{B,w}^\times$.

Now let $K = \overline{\mathbb{Q}}_l$. Write $\mathcal{O}$ for the ring of integers of K, and k for the residue field $\overline{\mathbb{F}}_l$.

Let $I_l = \mathrm{Hom}(F, K)$, and let $\tilde{I}_l$ be the subset of elements of $\mathrm{Hom}(M, K)$ such that the induced place of M is in $\tilde{S}_l$. Let $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(M,K)}$; we assume that

- $a_{\tau,1} \geq \cdots \geq a_{\tau,n} \geq 0$ if $\tau \in \tilde{I}_l$, and
- $a_{\tau c,i} = -a_{\tau,n+1-i}$.

Consider the constructions of section 5.2 applied to our choice of K. Then we have an $\mathcal{O}$-module

$$Y_a = \otimes_{\tau \in \tilde{I}_l} X_{a_\tau}$$

which has a natural action of $G(\mathcal{O}_{F,l})$, where $g \in G(\mathcal{O}_{F,l})$ acts on X_{a_τ} by $\tau(i_\tau g_\tau)$. From now on, if $v|l$ is a place of F, we will identify $G(\mathcal{O}_{F_v})$ with $\mathrm{GL}_n(\mathcal{O}_{M_{\tilde{v}}})$ via the map $i_{\tilde{v}}$ without comment.

We say that an open compact subgroup $U \subset G(A_F^\infty)$ is sufficiently small if for some place v of F the projection of U to $G(F_v)$ contains no nontrivial elements of finite order. Assume from now on that U is sufficiently small, and in addition that we may write $U = \prod_v U_v$, $U_v \subset G(\mathcal{O}_{F_v})$, such that

- if $v \in S(B)$ and $w|v$ is a place of M, then $i_w(U_v) = \mathcal{O}_{B,w}^\times$, and
- if $v|l$ then U_v is the Iwahori subgroup of matrices which are upper-triangular mod l.

If $v|l$, let U_v' denote the pro-l subgroup of U_v corresponding to the group of matrices which are (upper-triangular) unipotent mod l, and let

$$\chi_v : U_v/U_v' \to \mathcal{O}^\times$$

be a character. Let $\chi = \otimes \chi_v : \prod_{v|l} U_v \to \mathcal{O}^\times$, and write

$$Y_{a,\chi} = Y_a \otimes_{\mathcal{O}} \chi,$$

a $\prod_{v|l} U_v$-module.

Let A be an $\mathcal{O}$-algebra. Then we define the space of algebraic modular forms

$$S_{a,\chi}(U, A)$$

to be the space of functions

$$f : G(F)\backslash G(\mathbb{A}_F^\infty) \to A \otimes_{\mathcal{O}} Y_{a,\chi}$$

satisfying

$$f(gu) = u^{-1} f(g)$$

for all $u \in U$, $g \in G(\mathbb{A}_F^\infty)$, where the action of U on $A \otimes_{\mathcal{O}} Y_{a,\chi}$ is inherited from the action of $\prod_{v|l} U_v$ on $Y_{a,\chi}$. Note that because U is sufficiently small we have

$$S_{a,\chi}(U, A) = S_{a,\chi}(U, \mathcal{O}) \otimes_{\mathcal{O}} A.$$

More generally, if V is any U''-module with U'' a sufficiently small compact open subgroup, we define the space of algebraic modular forms

$$S(U'', V)$$

to be the space of functions

$$f : G(F)\backslash G(\mathbb{A}_F^\infty) \to V$$

satisfying

$$f(gu) = u^{-1} f(g)$$

for all $u \in U''$, $g \in G(\mathbb{A}_F^\infty)$.

Let T_l^+ denote the monoid of elements of $G(\mathbb{A}_F^\infty)$ which are trivial outside of places dividing l, and at places dividing l correspond to matrices

$\operatorname{diag}(l^{b_1}, \ldots, l^{b_n})$ with $b_1 \geq \cdots \geq b_n \geq 0$. In addition to the action of U on $Y_{a,\chi}$, we can also allow T_l^+ to act. We define the action of T_l^+ via the action $\cdot_{twist}$ on X_t defined above. This gives us an action of the monoid $\langle U, T_l^+ \rangle$ on $Y_{a,\chi}$. Now suppose that g is an element of $G(\mathbb{A}_F^\infty)$ with either $g_l \in G(\mathcal{O}_{F,l})$ or $g \in T_l^+$; then we write

$$UgU = \coprod_i g_i U,$$

a finite union of cosets, and define a linear map

$$[UgU] : S_{a,\chi}(U, A) \to S_{a,\chi}(U, A)$$

by

$$([UgU]f)(h) = \sum_i g_i f(hg_i).$$

We now introduce some notation for Hecke algebras. Let v be a place of F which splits in M, and suppose that $v \notin S(B)$ and that $U_v = G(\mathcal{O}_{F_v})$ (so, in particular $v \nmid l$). Suppose that $w|v$ is a place of M, so that we may regard $G(\mathcal{O}_{F_v})$ as $\mathrm{GL}_n(\mathcal{O}_{M_w})$ via i_w. Then we let $T_w^{(j)}$, $1 \leq j \leq n$ denote the Hecke operator given by

$$[U \operatorname{diag}(\varpi_w, \ldots, \varpi_w, 1, \ldots, 1) U]$$

where ϖ_w is a uniformiser of M_w, and there are j occurrences of it in this matrix. We let $\mathbb{T}_{a,\chi}(U, A)$ denote the commutative A-subalgebra of $\operatorname{End}(S_{a,\chi}(U, A))$ generated by the operators $T_w^{(j)}$ and $(T_w^{(n)})^{-1}$ for all w, j as above. Note that $\mathbb{T}_{a,\chi}(U, A)$ commutes with $[UgU]$ for all $g \in T_l^+$. More generally, let $\mathbb{T}(U)$ denote the polynomial ring over $\mathcal{O}$ in the formal variables $T_w^{(j)}$ and $(T_w^n)^{-1}$, which we may think of as acting on $S_{a,\chi}(U, A)$ via the obvious map $\mathbb{T}(U) \to \mathbb{T}_{a,\chi}(U, A)$.

We also wish to consider the Hecke operator $U_l = [UuU]$, where $u \in T_l^+$ has $u_v = \operatorname{diag}(l^{n-1}, \ldots, l, 1)$ for each $v|l$. As usual, we can define a Hida idempotent

$$e_l = \lim_{n \to \infty} U_l^{n!},$$

which has the property that U_l is invertible on $e_l S_{a,\chi}(U, \mathcal{O})$ and is topologically nilpotent on $(1 - e_l) S_{a,\chi}(U, \mathcal{O})$. We write

$$S_{a,\chi}^{ord}(U, A) := e_l S_{a,\chi}(U, A).$$

Let $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(M,K)}$ be a weight, and let $\chi_a = \otimes_{v|l} \chi_{a,v}$, where $\chi_{a,v} : U_v / U'_v \cong ((\mathcal{O}_{M_v}/\mathfrak{m}_{M_v})^\times)^n \to \mathcal{O}^\times$ is given by the character $(x_1, \ldots, x_n) \mapsto \prod_\tau \prod_i \tau(\tilde{x}_i)^{a_{\tilde{v},n+1-i}}$, where $\tilde{x}_i$ is the Teichmüller lift of x_i, and the product is over the embeddings $\tau \in \tilde{I}_l$ which give rise to v.

The main lemma we require is the following.

Lemma 5.4. *Let a be a weight. Then there is a $\mathbb{T}(U)$-equivariant isomorphism*

$$S_{a,\chi}^{ord}(U, k) \to S_{0,\chi\chi_a}^{ord}(U, k).$$

Proof. Note firstly that there is a natural projection map j from $Y_{a,\chi}$ to the $\mathcal{O}$-module given by the tensor product $Z_{a,\chi}$ of the lowest weight vectors. This is a map of $\prod_{v|l} U_v$-modules, and by Lemma 5.2 we see that j induces an isomorphism

$$u \cdot_{twist} Y_{a,\chi} \otimes_{\mathcal{O}} k \to u \cdot_{twist} Z_{a,\chi} \otimes_{\mathcal{O}} k.$$

Note also that by definition we have an isomorphism of $\langle U, T_l^+\rangle$-modules $Z_{a,\chi} \to Y_{0,\chi\chi_a}$. It thus suffices to prove that the induced map

$$j : S^{ord}_{a,\chi}(U,k) \to S^{ord}(U, Z_{a,\chi} \otimes_{\mathcal{O}} k) \ (= S^{ord}_{0,\chi\chi_a}(U,k))$$

is an isomorphism.
We claim that there is a diagram

$$\begin{tikzcd}
S_{a,\chi}(U,k) \arrow[r, "j"] & S(U, Z_{a,\chi} \otimes_{\mathcal{O}} k) \arrow[r, "u\cdot_{twist}"] & S(U \cap uUu^{-1}, u \cdot_{twist} Z_{a,\chi} \otimes_{\mathcal{O}} k) \arrow[d, "j^{-1}"] \\
& S_{a,\chi}(U \cap uUu^{-1}, k) \arrow[ul, "\mathrm{cor}"] & S(U \cap uUu^{-1}, u \cdot_{twist} Y_{a,\chi} \otimes_{\mathcal{O}} k) \arrow[l, "i"']
\end{tikzcd}$$

such that the maps

$$\mathrm{cor} \circ i \circ j^{-1} \circ u \cdot_{twist} \circ j : S_{a,\chi}(U,k) \to S_{a,\chi}(U,k)$$

and

$$j \circ \mathrm{cor} \circ i \circ j^{-1} \circ u \cdot_{twist} : S(U, Z_{a,\chi} \otimes_{\mathcal{O}} k) \to S(U, Z_{a,\chi} \otimes_{\mathcal{O}} k)$$

are both given by U_l. Since U_l is an isomorphism on $S^{ord}_{a,\chi}(U,k)$, the result will follow.
In fact, the construction of the diagram is rather straightforward. The maps j, j^{-1} are just the natural maps on the coefficients (note that both are maps of U-modules). The map $u \cdot_{twist}$ is given by

$$(u \cdot_{twist} f)(h) = u \cdot_{twist} f(hu).$$

The map i is given by the inclusion of U-modules $u \cdot_{twist} Y_{a,\chi} \otimes_{\mathcal{O}} k \hookrightarrow Y_{a,\chi} \otimes_{\mathcal{O}} k$. Finally, the map cor is defined in the following fashion. We may write

$$U = \coprod u_i (U \cap uUu^{-1}),$$

and we define

$$(\mathrm{cor}\, f)(h) = \sum u_i f(hu_i).$$

The claims regarding the compositions of these maps follow immediately from the observation that

$$UuU = \coprod u_i uU.$$

□

5.5. We now recall some results on tamely ramified principal series representations of GL_n from [Roc98]. Let L be a finite extension of $\mathbb{Q}_p$ for some p, and let π_L be an irreducible smooth complex representation of $\mathrm{GL}_n(L)$. Let I denote the Iwahori subgroup of $\mathrm{GL}_n(\mathcal{O}_L)$ consisting of matrices which are upper-triangular mod $\mathfrak{m}_L$, and let I_1 denote its Sylow pro-l subgroup. Let l be the residue field of L, and let ϖ_L denote a uniformiser of L. Then there is a natural isomorphism $I/I_1 \cong (l^\times)^n$. If $\chi = (\chi_1, \ldots, \chi_n) : (l^\times)^n \to \mathbb{C}^\times$ is a character, then we let $\pi_L^{I,\chi}$ denote the space of vectors in π_L which are fixed by I_1 and transform by χ under the action of I/I_1. The space $\pi_L^{I,\chi}$ has a natural action of the Hecke algebra $\mathcal{H}(I,\chi)$ of compactly supported χ^{-1}-spherical functions on $\mathrm{GL}_n(L)$. We consider the commutative subalgebra $\mathbb{T}(I,\chi)$ of $\mathcal{H}(I,\chi)$ generated by double cosets $[I\alpha I]$ where $\alpha = \mathrm{diag}(\varpi_L^{b_1}, \ldots, \varpi_L^{b_n})$ with $b_1 \geq \cdots \geq b_n \geq 0$.

If $\chi : (\mathcal{O}_L^\times)^n \to \mathbb{C}^\times$ is tamely ramified, then we let $\pi_L^{I,\chi}$ denote $\pi_L^{I,\bar{\chi}}$, where $\bar{\chi}$ is the character $(l^\times)^n \to \mathbb{C}^\times$ determined by χ. Let δ denote the modulus character of $\mathrm{GL}_n(L)$, so that

$$\delta(\mathrm{diag}(a_1, \ldots, a_n)) = |a_1|^{n-1}|a_2|^{n-3} \ldots |a_n|^{1-n}$$

where $|\cdot|$ denotes the usual norm on L.

PROPOSITION 5.5. (1) *If $\pi_L^I \neq 0$ then π is a subquotient of an unramified principal series representation.*

(2) *If $\pi_L^{I_1} \neq 0$ then π is a subquotient of a tamely ramified principal series representation. More precisely, if $\pi_L^{I,\chi} \neq 0$ then π_L is a subquotient of a tamely ramified principal series representation* $\text{n-Ind}_{B_n(L)}^{\mathrm{GL}_n(L)}(\chi_1', \ldots, \chi_n')$ *with χ_i' extending χ_i for each i.*

(3) *If $\pi_L = \text{n-Ind}_{B_n(L)}^{\mathrm{GL}_n(L)}(\chi)$ with χ tamely ramified, then*

$$\pi_L^{I,\chi} \cong \oplus_w \chi\delta^{-1/2}$$

as a $\mathbb{T}(I,\chi)$-module, where the sum is over the elements w of the Weyl group of GL_n with $\chi^w = \chi$; that is, the double coset $[I\alpha I]$ acts via $(\chi\delta^{-1/2})(\alpha)$ on $\pi_L^{I,\chi}$.

Proof. The first two parts follow from Lemma 3.1.6 of [CHT08] and its proof. All three parts follow at once from Theorem 7.7 and Remark 7.8 of [Roc98] (which are valid for GL_n without any restrictions on L - see the proof of Lemma 3.1.6 of [CHT08]), together with the standard calculation of the Jacquet module of a principal series representation, for which see for example Theorem 6.3.5 of [Cas95] (although note that there is a missing factor of $\delta^{1/2}$ (or rather $\delta_\Omega^{1/2}$ in the notation of *loc. cit.*) in the formula given there). $\square$

5.6. Keep our running assumptions on π. Suppose that $U = \prod_v U_v$ is a sufficiently small subgroup of $G(\mathbb{A}_F)$. Assume further that U has been chosen such that if $v \notin S(B)$, $v = ww^c$ splits completely in M, and U_v is a maximal compact subgroup of $G(F_v)$, then π_w is unramified. Recall that we have fixed an isomorphism $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. There is a maximal ideal $\mathfrak{m}_{\iota,\pi}$ of $\mathbb{T}(U)$ determined by π in the following fashion. For each place $v = ww^c$ as above the Hecke operators $T_w^{(i)}$ act via scalars $\alpha_{w,i}$ on $(\pi_w)^{\mathrm{GL}_n(\mathcal{O}_{M_w})}$. The $\alpha_{w,i}$ are all algebraic integers, so that $\iota^{-1}(\alpha_{w,i}) \in \mathcal{O}$. Then $\mathfrak{m}_{\iota,\pi}$ is the maximal ideal of $\mathbb{T}(U)$ containing all the $T_w^{(i)} - \iota^{-1}(\alpha_{w,i})$. Let $\sigma_k \in (\mathbb{Z}^n)^{\mathrm{Hom}(M,K)}$ be the weight determined by $(\sigma_k)_{\tau,i} = (k-2)(n-i)$ for each $\tau \in \tilde{I}_l$.

LEMMA 5.6. *Suppose that π is a RACSDC representation of $\mathrm{GL}_n(\mathbb{A}_M)$ of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{S(B)}$. Suppose that for each place $x \in \tilde{S}_l$, π_x is a principal series* n-$\mathrm{Ind}_{B_n(M_x)}^{\mathrm{GL}_n(M_x)}(\chi_{x,1}, \dots, \chi_{x,n})$ *with $\iota^{-1}\chi_{x,i} \circ \mathrm{Art}_{F_x}^{-1}|_{I_x} = \omega^{(i-1)(2-k)}$. Then there is a sufficiently small compact open subgroup U of $G(\mathbb{A}_F)$ such that U satisfies the requirements above (in particular, $U = \prod_v U_v$ where U_v is an Iwahori subgroup of $\mathrm{GL}_n(F_v)$ for each $v|l$) and $S_{0,\chi_{\sigma_k}}(U,\mathcal{O})_{\mathfrak{m}_{\iota,\pi}} \neq 0$. If we assume furthermore that $v_l(\iota^{-1}\chi_{x,i}(l)) = [M_x : \mathbb{Q}_l]\left(i - 1 + \frac{1-n}{2}\right)$ for all i (and all $x \in \tilde{S}_l$) then $S^{ord}_{0,\chi_{\sigma_k}}(U,\mathcal{O})_{\mathfrak{m}_{\iota,\pi}} \neq 0$.*

Proof. This is a consequence of Proposition 3.3.2 of [CHT08]. The only issues are at places dividing l and places in $S(B)$. For the latter, it is enough to note that under the Jacquet-Langlands correspondence, $\mathrm{Sp}_n(1)$ corresponds to the trivial representation. For the first part, we also need to check that at each place $x \in \tilde{S}_l$, $\pi_x^{I_x,\chi_x} \neq 0$, where I_x is the standard Iwahori subgroup of $\mathrm{GL}_n(M_x)$, and $\chi_x = (\chi_{x,1}, \dots, \chi_{x,n})$. This follows at once from Proposition 5.5.

For the second part, we must check in addition that if the Hecke operator $[I_x u_x I_x]$ (where $u_x = \mathrm{diag}(l^{n-1}, \dots, 1)$) acts via the scalar α_x on $\pi_x^{I_x,\chi_x}$, then $\iota^{-1}(\alpha_x)$ is an l-adic unit. This is straightforward; by Proposition 5.5(3), $\alpha_x = \chi_x(u)\delta^{-1/2}(u)$. Thus

$$\begin{aligned}
v_l(\iota^{-1}(\alpha_x)) &= v_l(\iota^{-1}(\chi_x(u)\delta^{-1/2}(u))) \\
&= \sum_{i=1}^{n}(n-i)v_l(\iota^{-1}\chi_{x,i}(l)) + \sum_{i=1}^{n}(n-i)v_l((l^{-[M_x:\mathbb{Q}_l]})^{-(n+1-2i)/2)}) \\
&= \sum_{i=1}^{n}(n-i)([M_x:\mathbb{Q}_l]\left(i-1+\frac{1-n}{2}\right))+ \\
&\qquad\qquad + \sum_{i=1}^{n}[M_x:\mathbb{Q}_l](n-i)(n+1-2i)/2 \\
&= \frac{[M_x:\mathbb{Q}_l]}{2}\sum_{i=1}^{n}(n-i)((2i-1-n)+(n+1-2i)) \\
&= 0,
\end{aligned}$$

as required. □

LEMMA 5.7. *Keep (all) the assumptions of Lemma 5.6. Then there is an RACSDC representation π'' of $\mathrm{GL}_n(\mathbb{A}_M)$ of weight $\iota_*\sigma_k$, type $\{\mathrm{Sp}_n(1)\}_{\{S(B)\}}$ and with π''_l unramified such that $\overline{r}_{l,\iota}(\pi'') \cong \overline{r}_{l,\iota}(\pi)$.*

Proof. This is essentially a consequence of Lemma 5.6, Lemma 5.4, and Proposition 5.5, together with Proposition 3.3.2 of [CHT08]. Indeed, Lemma 5.4 and Lemma 5.6 show that $S^{ord}_{\sigma_k,1}(U,\mathcal{O})_{\mathfrak{m}_{\iota,\pi}} \neq 0$, which by Proposition 5.5(1) and Proposition 3.3.2 of [CHT08] gives us a π'' satisfying all the properties we claim, except that we only know that for each $x|l$, π''_x is a subquotient of an unramified principal series representation. We claim that this unramified principal series is irreducible, so that π''_x is unramified. To see this, note that the fact that we know that $S^{ord}_{\sigma_k,1}(U,\mathcal{O})_{\mathfrak{m}_{\iota,\pi}} \neq 0$ (rather than merely $S_{\sigma_k,1}(U,\mathcal{O})_{\mathfrak{m}_{\iota,\pi}} \neq 0$) means that we can choose π'' so that for each $x \in \tilde{S}_l$, π''_x is a subquotient of an unramified principal series representation $\text{n-Ind}_{B_n(M_x)}^{\mathrm{GL}_n(M_x)}(\chi_{x,1},\dots,\chi_{x,n})$ with

$$v_l(\iota^{-1}\chi_{x,i}(l)) = [M_x:\mathbb{Q}_l]\,((i-1)(k-1)+(1-n)/2)$$

(this follows from the comparison of the Hecke actions on $(\pi''_x)^{I_x}$ and $S_{\sigma_k,1}(U,\mathcal{O})$, noting that the latter action is defined in terms of $\cdot_{twist}$). Now, if the principal series $\text{n-Ind}_{B_n(M_x)}^{\mathrm{GL}_n(M_x)}(\chi_{x,1},\dots,\chi_{x,n})$ were reducible, there would be i, j with $\chi_{x,i} = \chi_{x,j}|\cdot|$, so that $\chi_{x,i}(l)l^{[M_x:\mathbb{Q}_l]} = \chi_{x,j}(l)$, which is a contradiction because $k > 2$. The result follows. $\square$

Combining Corollary 4.4 with Lemma 5.7, we obtain

PROPOSITION 5.8. *Let l be an odd prime, and let f be a modular form of weight $2 \leq k < l$ and level coprime to l. Assume that f is Steinberg at q, and that for some place $\lambda|l$ of $\mathcal{O}_f$, f is ordinary at λ and $\overline{\rho}_{f,\lambda}$ is absolutely irreducible. Fix an embedding $K_{f,\lambda} \hookrightarrow \overline{\mathbb{Q}}_l$. Let $\mathcal{N}$ be a finite set of even positive integers. Then there is a Galois totally real extension $F/\mathbb{Q}$ and a quadratic imaginary field E, together with a place $w_q|q$ of $M = FE$ such that if we choose a set $\tilde{S}_l$ of places of M consisting of one place above each place of F dividing l, and define $\sigma_k \in (\mathbb{Z}^n)^{\mathrm{Hom}(M,K)}$ by $(\sigma_k)_{\tau,i} = (k-2)(n-i)$, then*

- *for each $n \in \mathcal{N}$, there is a character $\bar{\phi}_n : \mathrm{Gal}(\overline{M}/M) \to \overline{\mathbb{F}}_l^\times$ which is unramified at all places in $\tilde{S}_l$, which satisfies*
$$\bar{\phi}_n\bar{\phi}_n^c = (\epsilon \det \overline{\rho}_{f,\lambda} \otimes \overline{\mathbb{F}}_l)^{1-n}|_{\mathrm{Gal}(\overline{M}/M)}$$
and $(\mathrm{Sym}^{n-1}\overline{\rho}_{f,\lambda} \otimes \overline{\mathbb{F}}_l)|_{\mathrm{Gal}(\overline{M}/M)} \otimes \bar{\phi}_n$ is automorphic of weight σ_k and type $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$.
- *l is unramified in M.*
- *M is linearly disjoint from $\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\lambda})}$ over $\mathbb{Q}$.*

6. POTENTIAL AUTOMORPHY

6.1. Assume as before that f is a cuspidal newform of level $\Gamma_1(N)$, weight $k \geq 2$, and nebentypus χ_f. Let $\pi(f)$ be the RAESDC representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$

corresponding to f. We will think of χ_f as an automorphic representation of $\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})$, and write $\chi_f = \otimes_p \chi_{f,p}$. We now define what we mean by the claim that the symmetric powers of f are potentially automorphic. If F is a totally real field and $v|p$ is a place of F, we write $\mathrm{rec}(\pi_{f,p})|_{F_v}$ for the restriction of the Weil-Deligne representation $\mathrm{rec}(\pi_{f,p})$ to the Weil group of F_v. Then we say that $\mathrm{Sym}^{n-1} f$ is potentially automorphic over F if there is an RAESDC representation π_n of $\mathrm{GL}_n(\mathbb{A}_F)$ such that for all primes p and all places $v|p$ of F we have

$$\mathrm{rec}(\pi_{n,v}) = \mathrm{Sym}^{n-1}(\mathrm{rec}(\pi_{f,p})|_{F_v}).$$

By a standard argument (see for example section 4 of [HSBT09]) this is equivalent to asking that $\mathrm{Sym}^{n-1} \rho_{f,\lambda}|_{\mathrm{Gal}(\overline{F}/F)}$ be automorphic for some place (equivalently for all places) λ of K_f.

Similarly, we may speak of $\mathrm{Sym}^{n-1} f$ being potentially automorphic of a specific weight and type. We then define (for each $n \geq 1$ and each integer a) the L-series

$$L(\chi_f^a \otimes \mathrm{Sym}^{n-1} f, s) = \prod_p L((\chi_{f,p}^a \circ \mathrm{Art}_{\mathbb{Q}_p}^{-1}) \otimes \mathrm{Sym}^{n-1} \mathrm{rec}(\pi_{f,p}), s + (1-n)/2).$$

We now normalise the L-functions of RAESDC automorphic representations to agree with those of their corresponding Galois representations. Specifically, if π is an RAESDC representation of $\mathrm{GL}_n(\mathbb{A}_F)$, we define

$$L(\pi, s) = \prod_{v \nmid \infty} L(\pi_v, s + (1-n)/2).$$

If π is square integrable at some finite place, then for each isomorphism $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ there is a Galois representation $r_{l,\iota}(\pi)$, and by definition we have

$$\begin{aligned}
L(\pi, s) &= \prod_{v \nmid \infty} L(\pi_v \otimes (|\cdot| \circ \det)^{(1-n)/2}, s) \\
&= \prod_{v \nmid \infty} L(\mathrm{rec}(\pi_v \otimes (|\cdot| \circ \det)^{(1-n)/2}), s) \\
&= \prod_{v \nmid \infty} L(r_l(\iota^{-1}\pi_v)^\vee(1-n), s) \\
&= L(r_{l,\iota}(\pi), s).
\end{aligned}$$

THEOREM 6.1. *Suppose that f is a cuspidal newform of level $\Gamma_1(N)$ and weight $k = 2$ or 3. Suppose that f is Steinberg at q. Let $\mathcal{N}$ be a finite set of even positive integers. Then there is a Galois totally real field F such that for any $n \in \mathcal{N}$ and any subfield $F' \subset F$ with F/F' soluble, $\mathrm{Sym}^{n-1} f$ is automorphic over F'.*

Proof. By Lemma 3.2 and Lemma 3.5 we may choose a prime $l > 3$ and a place λ of $\mathcal{O}_f$ lying over l such that

- $l \nmid N$.
- f is ordinary at λ.
- $l > \max(2n+1)_{n \in \mathcal{N}}$.

- $\overline{\rho}_{f,\lambda}$ has large image.

By Corollary 5.8 there is an embedding $K_{f,\lambda} \hookrightarrow \overline{\mathbb{Q}}_l$, a Galois totally real extension $F/\mathbb{Q}$ and a quadratic imaginary field E, together with a place $w_q|q$ of $M = FE$ such that if we choose a set $\tilde{S}_l$ of places of M consisting of one place above each place of F dividing l, and define $\sigma_k \in (\mathbb{Z}^n)^{\mathrm{Hom}(M,K)}$ by $(\sigma_k)_{\tau,i} = (k-2)(n-i)$, then

- for each $n \in \mathcal{N}$, there is a character $\bar{\phi}_n : \mathrm{Gal}(\overline{M}/M) \to \overline{\mathbb{F}}_l^\times$ which is unramified at all places in $\tilde{S}_l$ and satisfies
$$\bar{\phi}_n \bar{\phi}_n^c = (\epsilon \det \overline{\rho}_{f,\lambda} \otimes \overline{\mathbb{F}}_l)^{1-n}|_{\mathrm{Gal}(\overline{M}/M)},$$
and $(\mathrm{Sym}^{n-1} \overline{\rho}_{f,\lambda} \otimes \overline{\mathbb{F}}_l)|_{\mathrm{Gal}(\overline{M}/M)} \otimes \bar{\phi}_n$ is automorphic of weight σ_k and type $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$.
- l is unramified in M.
- M is linearly disjoint from $\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\lambda})}$ over $\mathbb{Q}$.

Fix $n \in \mathcal{N}$, and let $\rho := \mathrm{Sym}^{n-1} \rho_{f,\lambda}|_{\mathrm{Gal}(\overline{F}/F)} \otimes \overline{\mathbb{Q}}_l$. There is a crystalline character $\chi : \mathrm{Gal}(\overline{F}/F) \to \mathcal{O}_{\overline{\mathbb{Q}}_l}^\times$ which is unramified above q such that
$$\rho^\vee \cong \rho\chi\epsilon^{n-1};$$
in fact,
$$\chi = (\epsilon \det \rho_{f,\lambda} \otimes \mathcal{O}_{\overline{\mathbb{Q}}_l})^{1-n}|_{\mathrm{Gal}(\overline{F}/F)}.$$
By Lemma 4.1.6 of [CHT08] we can choose an algebraic character
$$\psi : \mathrm{Gal}(\overline{M}/M) \to \mathcal{O}_{\overline{\mathbb{Q}}_l}^\times$$
such that

- $\chi|_{\mathrm{Gal}(\overline{M}/M)} = \psi\psi^c$,
- ψ is crystalline,
- ψ is unramified at each place in $\tilde{S}_l$.
- ψ is unramified above q,
- $\bar{\psi} = \bar{\phi}_n$.

Then $\rho' = \rho|_{\mathrm{Gal}(\overline{M}/M)}\psi$ satisfies
$$(\rho')^c \cong (\rho')^\vee \epsilon^{1-n}.$$

We claim that ρ' is automorphic of weight σ_k, level prime to l and type $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$. This follows from Theorem 5.2 of [Tay08]; we now check the hypotheses of that theorem. Certainly $\bar{\rho}' \cong (\mathrm{Sym}^{n-1} \overline{\rho}_{f,\lambda} \otimes \overline{\mathbb{F}}_l)|_{\mathrm{Gal}(\overline{M}/M)} \otimes \bar{\phi}_n$ is automorphic of weight σ_k, level prime to l and type $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$. The only non-trivial conditions to check are that:

- $\overline{M}^{\ker \mathrm{ad}\, \bar{\rho}'}$ does not contain $M(\zeta_l)$, and
- The image $\bar{\rho}'(\mathrm{Gal}(\overline{M}/M(\zeta_l)))$ is big in the sense of Definition 2.5.1 of [CHT08].

These both follow from the assumption that $\overline{\rho}_{f,\lambda}$ has large image, the fact that M is linearly disjoint from $\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\lambda})}$ over $\mathbb{Q}$, Corollary 2.5.4 of [CHT08], and the fact that $\mathrm{PSL}_2(k)$ is simple if k is a finite field of cardinality greater than 3.
It follows from Lemma 4.3.3 of [CHT08] that ρ is automorphic. Then from Lemma 4.3.2 of [CHT08] we see that for each F' with F/F' soluble, $\mathrm{Sym}^{n-1}\rho_{f,\lambda}|_{\mathrm{Gal}(\overline{F}/F')}$ is automorphic, as required. □

Corollary 6.2. *Suppose that f is a cuspidal newform of level $\Gamma_1(N)$ and weight $k = 2$ or 3. Suppose that f is potentially Steinberg at q. Let $\mathcal{N}$ be a finite set of even positive integers. Then there is a Galois totally real field F such that for any $n \in \mathcal{N}$ and any subfield $F' \subset F$ with F/F' soluble,* $\mathrm{Sym}^{n-1} f$ *is automorphic over F'.*

Proof. Let θ be a Dirichlet character such that $f' = f \otimes \theta$ is Steinberg at q. The result then follows from Theorem 6.1 applied to f'. □

7. The Sato-Tate Conjecture

7.1. Let f be a cuspidal newform of level $\Gamma_1(N)$, nebentypus χ_f, and weight $k \geq 2$. Suppose that χ_f has order m, so that the image of χ_f is precisely the group μ_m of m-th roots of unity. Let $U(2)_m$ be the subgroup of $U(2)$ consisting of matrices with determinant in μ_m. For each prime $l \nmid N$, if we write

$$X^2 - a_l X + l^{k-1}\chi_f(l) = (X - \alpha_l l^{(k-1)/2})(X - \beta_l l^{(k-1)/2})$$

then (by the Ramanujan conjecture) the matrix

$$\begin{pmatrix} \alpha_l & 0 \\ 0 & \beta_l \end{pmatrix}$$

defines a conjugacy class $x_{f,l}$ in $U(2)_m$. A natural generalisation of the Sato-Tate conjecture is

Conjecture 7.1. *If f is not of CM type, then the conjugacy classes $x_{f,l}$ are equidistributed with respect to the Haar measure on $U(2)_m$ (normalised so that $U(2)_m$ has measure 1).*

The group $U(2)_m$ is compact, and its irreducible representations are given by $\det^a \otimes \mathrm{Sym}^b \mathbb{C}^2$ for $0 \leq a < m$ and $b \geq 0$. By the corollary to Theorem 2 of section I.A.2 of [Ser68] (noting the different normalisations of L-functions in force there), Conjecture 7.1 follows if one knows that for each $b \geq 1$, the functions $L(\chi_f^a \otimes \mathrm{Sym}^b f, s)$ are holomorphic and non-zero for $\Re s \geq 1 + b(k-1)/2$ (the required results for $b = 0$ are classical).

Theorem 7.2. *Suppose that f is a cuspidal newform of level $\Gamma_1(N)$, character χ_f, and weight $k = 2$ or 3. Suppose that χ_f has order m. Suppose also that f is potentially Steinberg at q for some prime q. Then for all integers $0 \leq a < m$, $b \geq 1$ the function $L(\chi_f^a \otimes \mathrm{Sym}^b f, s)$ has meromorphic continuation to the whole* *complex plane, satisfies the expected functional equation, and is holomorphic and nonzero in* $\Re s \geq 1 + b(k-1)/2$.

Proof. The argument is very similar to the proof of Theorem 4.2 of [HSBT09]. We argue by induction on b; suppose that b is odd, and the result is known for all $1 \leq b' < b$. We will deduce the result for b and for $b+1$ simultaneously. Apply Corollary 6.2 with $\mathcal{N} = \{2, b+1\}$. Let F be as in the conclusion of Corollary 6.2. By Brauer's theorem, we may write

$$1 = \Sigma_j a_j \operatorname{Ind}_{\operatorname{Gal}(F/F_j)}^{\operatorname{Gal}(F/\mathbb{Q})} \chi_j$$

where $F \supset F_j$ with F/F_j soluble, χ_j a character $\operatorname{Gal}(F/F_j) \to \mathbb{C}^\times$, and $a_j \in \mathbb{Z}$. Then for each j, $\operatorname{Sym}^b f$ is automorphic over F_j, corresponding to an RAESDC representation π_j of $\operatorname{GL}_{b+1}(\mathbb{A}_{F_j})$. In addition, f is automorphic over F_j, corresponding to an RAESDC representation σ_j of $\operatorname{GL}_2(\mathbb{A}_{F_j})$.
Then we have

$$L(\chi_f^a \otimes \operatorname{Sym}^b f, s) = \prod_j L(\pi_j \otimes (\chi_j \circ \operatorname{Art}_{F_j}) \otimes (\chi_f^a \circ N_{F_j/\mathbb{Q}}), s)^{a_j},$$

$$L(\chi_f^a \otimes \operatorname{Sym}^2 f, s) = \prod_j L((\operatorname{Sym}^2 \sigma_j) \otimes (\chi_j \circ \operatorname{Art}_{F_j}) \otimes (\chi_f^a \circ N_{F_j/\mathbb{Q}}), s)^{a_j},$$

and

$$L(\chi_f^a \otimes \operatorname{Sym}^{b+1} f, s) L(\chi_f^{a+1} \otimes \operatorname{Sym}^{b-1} f, s-k+1) = \\ = \prod_j L((\pi_j \otimes (\chi_j \circ \operatorname{Art}_{F_j}) \otimes (\chi_f^a \circ N_{F_j/\mathbb{Q}})) \times \sigma_j, s + b(k-1)/2)^{a_j}.$$

The result then follows from the main results of [CPS04] and [GJ78] (in the case $b = 1$) together with Theorem 5.1 of [Sha81]. ☐

Corollary 7.3. *Suppose that f is a cuspidal newform of level $\Gamma_1(N)$ and weight $k = 2$ or 3. Suppose also that f is potentially Steinberg at q for some prime q. Then Conjecture 7.1 holds for f.*

Finally, we note that one can make this result more concrete, as one can easily explicitly determine the Haar measure on $U(2)_m$ from that of its finite index subgroup $SU(2)$. One finds that (as already follows from Dirchlet's theorem) the classes $x_{f,l}$ are equidistributed by determinant, and that furthermore the classes with fixed determinant are equidistributed with respect to the natural analogue of the usual Sato-Tate measure. That is, suppose that $\zeta \in \mu_m$, and fix a square root $\zeta^{1/2}$ of ζ. Then any conjugacy class $x_{f,l}$ in $U(2)_m$ with determinant ζ contains a representative of the form

$$\begin{pmatrix} \zeta^{1/2} e^{i\theta_l} & 0 \\ 0 & \zeta^{1/2} e^{-i\theta_l} \end{pmatrix}$$

with $\theta_l \in [0, \pi]$, and the θ_l are equidistributed with respect to the measure $\frac{2}{\pi} \sin^2 \theta d\theta$.

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