On Proper $\mathbb{R}$-Actions on Hyperbolic Stein Surfaces

Christian Miebach, Karl Oeljeklaus[(1)]

Received: April 18, 2009
Revised: August 3, 2009

Communicated by Thomas Peternell

Abstract. In this paper we investigate proper $\mathbb{R}$–actions on hyperbolic Stein surfaces and prove in particular the following result: Let $D \subset \mathbb{C}^2$ be a simply-connected bounded domain of holomorphy which admits a proper $\mathbb{R}$–action by holomorphic transformations. The quotient $D/\mathbb{Z}$ with respect to the induced proper $\mathbb{Z}$–action is a Stein manifold. A normal form for the domain D is deduced.

2000 Mathematics Subject Classification: 32E10; 32M05; 32Q45; 32T05
Keywords and Phrases: Stein manifolds; bounded domains of holomorphy; proper actions; quotient by a discrete group

1. Introduction

Let X be a Stein manifold endowed with a real Lie transformation group G of holomorphic automorphisms. In this situation it is natural to ask whether there exists a G–invariant holomorphic map $\pi\colon X \to X/\!\!/G$ onto a complex space $X/\!\!/G$ such that $\mathcal{O}_{X/\!\!/G} = (\pi_*\mathcal{O}_X)^G$ and, if yes, whether this quotient $X/\!\!/G$ is again Stein. If the group G is compact, both questions have a positive answer as is shown in [Hei91].

For non-compact G even the existence of a complex quotient in the above sense of X by G cannot be guaranteed. In this paper we concentrate on the most basic and already non-trivial case $G = \mathbb{R}$. We suppose that G acts properly on X. Let $\Gamma = \mathbb{Z}$. Then X/Γ is a complex manifold and if, moreover, it is Stein, we can define $X/\!\!/G := (X/\Gamma)/\!\!/(G/\Gamma)$. The following was conjectured by Alan Huckleberry.

Let X be a contractible bounded domain of holomorphy in $\mathbb{C}^n$ with a proper action of $G = \mathbb{R}$. Then the complex manifold $X/\mathbb{Z}$ is Stein.

(1) The authors would like to thank Peter Heinzner and Jean-Jacques Loeb for numerous discussions on the subject.

In [FI01] this conjecture is proven for the unit ball and in [Mie08] for arbitrary bounded homogeneous domains in $\mathbb{C}^n$. In this paper we make a first step towards a proof in the general case by showing

Theorem 1.1. — *Let D be a simply-connected bounded domain of holomorphy in $\mathbb{C}^2$. Suppose that the group $\mathbb{R}$ acts properly by holomorphic transformations on D. Then the complex manifold $D/\mathbb{Z}$ is Stein. Moreover, $D/\mathbb{Z}$ is biholomorphically equivalent to a domain of holomorphy in $\mathbb{C}^2$.*

As an application of this theorem we deduce a normal form for domains of holomorphy whose identity component of the automorphism group is non-compact as well as for proper $\mathbb{R}$–actions on them. Notice that we make no assumption on smoothness of their boundaries.
We first discuss the following more general situation. Let X be a hyperbolic Stein manifold with a proper $\mathbb{R}$–action. Then there is an induced local holomorphic $\mathbb{C}$–action on X which can be globalized in the sense of [HI97]. The following result is central for the proof of the above theorem.

Theorem 1.2. — *Let X be a hyperbolic Stein surface with a proper $\mathbb{R}$–action. Suppose that either X is taut or that it admits the Bergman metric and $H^1(X,\mathbb{R}) = 0$. Then the universal globalization X^* of the induced local $\mathbb{C}$–action is Hausdorff and $\mathbb{C}$ acts properly on X^*. Furthermore, for simply-connected X one has that $X^* \to X^*/\mathbb{C}$ is a holomorphically trivial $\mathbb{C}$–principal bundle over a simply-connected Riemann surface.*

Finally, we discuss several examples of hyperbolic Stein manifolds X with proper $\mathbb{R}$–actions such that $X/\mathbb{Z}$ is not Stein. If one does not require the existence of an $\mathbb{R}$–action, there are bounded Reinhardt domains in $\mathbb{C}^2$ with proper $\mathbb{Z}$–actions for which the quotients are not Stein.

2. Hyperbolic Stein $\mathbb{R}$–manifolds

In this section we present the general set-up.

2.1. The induced local $\mathbb{C}$–action and its globalization. — Let X be a hyperbolic Stein manifold. It is known that the group $\mathrm{Aut}(X)$ of holomorphic automorphisms of X is a real Lie group with respect to the compact-open topology which acts properly on X (see [Kob98]). Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a closed one parameter subgroup of $\mathrm{Aut}(D)$. Consequently, the action $\mathbb{R}\times X \to X$, $t\cdot x := \varphi_t(x)$, is proper. By restriction, we obtain also a proper $\mathbb{Z}$–action on X. Since every such action must be free, the quotient $X/\mathbb{Z}$ is a complex manifold. This complex manifold $X/\mathbb{Z}$ carries an action of $S^1 \cong \mathbb{R}/\mathbb{Z}$ which is induced by the $\mathbb{R}$–action on X.
Integrating the holomorphic vector field on X which corresponds to this $\mathbb{R}$–action we obtain a local $\mathbb{C}$–action on X in the following sense. There are an open neighborhood $\Omega \subset \mathbb{C}\times X$ of $\{0\}\times X$ and a holomorphic map $\Phi\colon \Omega \to X$, $\Phi(t,x) =: t\cdot x$, such that the following holds:

(1) For every $x \in X$ the set $\Omega(x) := \{t \in \mathbb{C};\ (t,x) \in \Omega\} \subset \mathbb{C}$ is connected;
(2) for all $x \in X$ we have $0 \cdot x = x$;
(3) we have $(t+t') \cdot x = t \cdot (t' \cdot x)$ whenever both sides are defined.

Following [Pal57] (compare [HI97] for the holomorphic setting) we say that a globalization of the local $\mathbb{C}$–action on X is an open $\mathbb{R}$–equivariant holomorphic embedding $\iota\colon X \hookrightarrow X^*$ into a (not necessarily Hausdorff) complex manifold X^* endowed with a holomorphic $\mathbb{C}$–action such that $\mathbb{C}\cdot\iota(X) = X^*$. A globalization $\iota\colon X \hookrightarrow X^*$ is called *universal* if for every $\mathbb{R}$–equivariant holomorphic map $f\colon X \to Y$ into a holomorphic $\mathbb{C}$–manifold Y there exists a holomorphic $\mathbb{C}$–equivariant map $F\colon X^* \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "\iota"] \arrow[dr, "f"'] & & X^* \arrow[dl, "F"] \\ & Y & \end{tikzcd}$$

commutes. It follows that a universal globalization is unique up to isomorphism if it exists.
Since X is Stein, the universal globalization X^* of the induced local $\mathbb{C}$–action exists as is proven in [HI97]. We will always identify X with its image $\iota(X) \subset X^*$. Then the local $\mathbb{C}$–action on X coincides with the restriction of the global $\mathbb{C}$–action on X^* to X.
Recall that X is said to be orbit-connected in X^* if for every $x \in X^*$ the set $\Sigma(x) := \{t \in \mathbb{C};\ t \cdot x \in X\}$ is connected. The following criterion for a globalization to be universal is proven in [CTIT00].

Lemma 2.1. — Let X^* be any globalization of the induced local $\mathbb{C}$–action on X. Then X^* is universal if and only if X is orbit-connected in X^*.

Remark. — The results about (universal) globalizations hold for a bigger class of groups ([CTIT00]). However, we will need it only for the groups $\mathbb{C}$ and $\mathbb{C}^*$ and thus will not give the most general formulation.

For later use we also note the following

Lemma 2.2. — The $\mathbb{C}$–action on X^* is free.

Proof. — Suppose that there exists a point $x \in X^*$ such that $\mathbb{C}_x$ is non-trivial. Because of $\mathbb{C} \cdot X = X^*$ we can assume that $x \in X$ holds. Since $\mathbb{C}_x$ is a non-trivial closed subgroup of $\mathbb{C}$, it is either a lattice of rank 1 or 2, or $\mathbb{C}$. The last possibility means that x is a fixed point under $\mathbb{C}$ which is not possible since $\mathbb{R}$ acts freely on X.
We observe that the lattice $\mathbb{C}_x$ is contained in the connected $\mathbb{R}$–invariant set $\Sigma(x) = \{t \in \mathbb{C};\ t \cdot x \in X\}$. By $\mathbb{R}$–invariance $\Sigma(x)$ is a strip. Since X is hyperbolic, this strip cannot coincide with $\mathbb{C}$. The only lattice in $\mathbb{C}$ which can possibly be contained in such a strip is of the form $\mathbb{Z}r$ for some $r \in \mathbb{R}$. Since this contradicts the fact that $\mathbb{R}$ acts freely on X, the lemma is proven. □

Note that we do not know whether X^* is Hausdorff. In order to guarantee the Hausdorff property of X^*, we make further assumptions on X. The following result is proven in [Ian03] and [IST04].

Theorem 2.3. — *Let X be a hyperbolic Stein manifold with a proper $\mathbb{R}$–action. Suppose in addition that X is taut or admits the Bergman metric. Then X^* is Hausdorff. If X is simply-connected, then the same is true for X^*.*

We refer the reader to Chapter 5 in [Kob98] for the definition and examples of tautness. For the reader's convenience we describe here the construction of the Bergman metric for an arbitrary n–dimensional complex manifold X. For more details see Chapter 4.10 in [Kob98]. The space $\mathcal{A}^2(X)$ of square integrable holomorphic n–forms on X is a separable complex Hilbert space with respect to the inner product $\langle\omega_1,\omega_2\rangle := i^{n^2}\int_X \omega_1\wedge\overline{\omega_2}$. Let $\omega_1,\omega_2,\ldots$ be an orthonormal basis of $\mathcal{A}^2(X)$ and define $B_X := \sum_{j\geq 1} i^{n^2}\omega_j\wedge\overline{\omega_j}$. The non-negative (n,n)–form B_X is independent of the chosen basis and is called the Bergman kernel form of X. Suppose that B_X is positive, i. e. that for every $x\in X$ there exists $\omega\in\mathcal{A}^2(X)$ with $\omega_x\neq 0$. Then we may define the map $\iota\colon X\to\mathbb{P}\big(\mathcal{A}^2(X)^*\big)$ which associates to each $x\in X$ the hyperplane consisting of forms in $\mathcal{A}^2(X)$ which vanish at x. By definition one says that X *admits the Bergman metric* if this map ι is an immersion. The Bergman metric of X is then defined as the pull-back of the Fubini-Study metric of $\mathbb{P}\big(\mathcal{A}^2(X)^*\big)$.

Remark. — Every bounded domain in $\mathbb{C}^n$ admits the Bergman metric.

2.2. The quotient $X/\mathbb{Z}$. — We assume from now on that X fulfills the hypothesis of Theorem 2.3. Since X^* is covered by the translates $t\cdot X$ for $t\in\mathbb{C}$ and since the action of $\mathbb{Z}$ on each domain $t\cdot X$ is proper, we conclude that the quotient $X^*/\mathbb{Z}$ fulfills all axioms of a complex manifold except for possibly not being Hausdorff.

We have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X^* \arrow[d] \\ X/\mathbb{Z} \arrow[r] & X^*/\mathbb{Z}. \end{tikzcd}$$

Note that the group $\mathbb{C}^* = (S^1)^{\mathbb{C}} \cong \mathbb{C}/\mathbb{Z}$ acts on $X^*/\mathbb{Z}$. Concretely, if we identify $\mathbb{C}/\mathbb{Z}$ with $\mathbb{C}^*$ via $\mathbb{C}\to\mathbb{C}^*$, $t\mapsto e^{2\pi i t}$, the quotient map $p\colon X^*\to X^*/\mathbb{Z}$ fulfills $p(t\cdot x) = e^{2\pi i t}\cdot p(x)$.

Lemma 2.4. — *The induced map $X/\mathbb{Z}\hookrightarrow X^*/\mathbb{Z}$ is the universal globalization of the local $\mathbb{C}^*$–action on $X/\mathbb{Z}$.*

Proof. — The open embedding $X\hookrightarrow X^*$ induces an open embedding $X/\mathbb{Z}\hookrightarrow X^*/\mathbb{Z}$. This embedding is S^1–equivariant and we have $\mathbb{C}^*\cdot X/\mathbb{Z} = X^*/\mathbb{Z}$. This implies that $X^*/\mathbb{Z}$ is a globalization of the local $\mathbb{C}^*$–action on $X/\mathbb{Z}$.

In order to prove that this globalization is universal, by the globalization theorem in [CTIT00] it is enough to show that $X/\mathbb{Z}$ is orbit-connected in $X^*/\mathbb{Z}$. Hence, we must show that for every $[x] \in X/\mathbb{Z}$ the set $\Sigma\bigl([x]\bigr) := \{t \in \mathbb{C}^*;\ t\cdot[x] \in X/\mathbb{Z}\}$ is connected in $\mathbb{C}^*$. For this we consider the set $\Sigma(x) = \{t \in \mathbb{C};\ t\cdot x \in X\}$. Since the map $X \to X/\mathbb{Z}$ intertwines the local $\mathbb{C}$– and $\mathbb{C}^*$–actions, we conclude that $t \in \Sigma(x)$ holds if and only if $e^{2\pi i t} \in \Sigma\bigl([x]\bigr)$ holds. Since X^* is universal, $\Sigma(x)$ is connected which implies that $\Sigma\bigl([x]\bigr)$ is likewise connected. Thus $X^*/\mathbb{Z}$ is universal. □

Remark. — The globalization $X^*/\mathbb{Z}$ is Hausdorff if and only if $\mathbb{Z}$ or, equivalently, $\mathbb{R}$ act properly on X^*. As we shall see in Lemma 3.3, this is the case if X is taut.

2.3. A sufficient condition for $X/\mathbb{Z}$ to be Stein.

— If $\dim X = 2$, we have the following sufficient condition for $X/\mathbb{Z}$ to be a Stein surface.

Proposition 2.5. — If the $\mathbb{C}$–action on X^* is proper and if the Riemann surface $X^*/\mathbb{C}$ is not compact, then $X/\mathbb{Z}$ is Stein.

Proof. — Under the above hypothesis we have the $\mathbb{C}$–principal bundle $X^* \to X^*/\mathbb{C}$. If the base $X^*/\mathbb{C}$ is not compact, then this bundle is holomorphically trivial, i. e. X^* is biholomorphic to $\mathbb{C} \times R$ where R is a non-compact Riemann surface. Since R is Stein, the same is true for X^* and for $X^*/\mathbb{Z} \cong \mathbb{C}^* \times R$. Since $X/\mathbb{Z}$ is locally Stein, see [Mie08], in the Stein manifold $X^*/\mathbb{Z}$, the claim follows from [DG60]. □

Therefore, the crucial step in the proof of our main result consists in showing that $\mathbb{C}$ acts properly on X^* under the assumption $\dim X = 2$.

3. Local properness

Let X be a hyperbolic Stein $\mathbb{R}$–manifold. Suppose that X is taut or that it admits the Bergman metric and $H^1(X, \mathbb{R}) = \{0\}$. We show that then $\mathbb{C}$ acts locally properly on X^*.

3.1. Locally proper actions.

— Recall that the action of a Lie group G on a manifold M is called locally proper if every point in M admits a G–invariant open neighborhood on which G acts properly.

Lemma 3.1. — Let $G \times M \to M$ be locally proper.

(1) *For every $x \in M$ the isotropy group G_x is compact.*
(2) *Every G–orbit admits a geometric slice.*
(3) *The orbit space M/G is a smooth manifold which is in general not Hausdorff.*
(4) *All G–orbits are closed in M.*
(5) *The G–action on M is proper if and only if M/G is Hausdorff.*

Proof. — The first claim is elementary to check. The second claim is proven in [DK00]. The third one is a consequence of (2) since the slices yield charts on M/G which are smoothly compatible because the transitions are given by the smooth action of G on M. Assertion (4) follows from (3) because in locally Euclidian topological spaces points are closed. The last claim is proven in [Pal61]. $\square$

Remark. — Since $\mathbb{R}$ acts properly on X and $\mathbb{C} \cdot X = X^*$, the $\mathbb{R}$-action on X^* is locally proper.

3.2. Local properness of the $\mathbb{C}$–action on X^*. — Recall that we assume that

$$X \text{ is taut} \tag{3.1}$$

or that

$$X \text{ admits the Bergman metric and } H^1(X, \mathbb{R}) = \{0\}. \tag{3.2}$$

We first show that assumption (3.1) implies that $\mathbb{C}$ acts locally properly on X^*.

Since X^* is the universal globalization of the induced local $\mathbb{C}$–action on X, we know that X is orbit-connected in X^*. This means that for every $x \in X^*$ the set $\Sigma(x) = \{t \in \mathbb{C};\ t \cdot x \in X\}$ is a strip in $\mathbb{C}$. In the following we will exploit the properties of the thickness of this strip.

Since $\Sigma(x)$ is $\mathbb{R}$–invariant, there are "numbers" $u(x) \in \mathbb{R} \cup \{-\infty\}$ and $o(x) \in \mathbb{R} \cup \{\infty\}$ for every $x \in X^*$ such that

$$\Sigma(x) = \big\{t \in \mathbb{C};\ u(x) < \mathrm{Im}(t) < o(x)\big\}.$$

The functions $u\colon X^* \to \mathbb{R} \cup \{-\infty\}$ and $o\colon X^* \to \mathbb{R} \cup \{\infty\}$ so obtained are upper and lower semicontinuous, respectively. Moreover, u und o are $\mathbb{R}$–invariant and $i\mathbb{R}$–equivariant:

$$u(it \cdot x) = u(x) - t \quad \text{and} \quad o(it \cdot x) = o(x) - t.$$

Proposition 3.2. — The functions $u, -o\colon X^* \to \mathbb{R} \cup \{-\infty\}$ are plurisubharmonic. Moreover, u and o are continuous on $X^* \setminus \{u = -\infty\}$ and $X^* \setminus \{o = \infty\}$, respectively.

Proof. — It is proven in [For96] that u and $-o$ are plurisubharmonic on X. By equivariance, we obtain this result for X^*.

Now we prove that the function $u\colon X \setminus \{u = -\infty\} \to \mathbb{R}$ is continuous which was remarked without complete proof in [Ian03]. For this let (x_n) be a sequence in X which converges to $x_0 \in X \setminus \{u = -\infty\}$. Since u is upper semi-continuous, we have $\limsup_{n\to\infty} u(x_n) \leq u(x_0)$. Suppose that u is not continuous in x_0. Then, after replacing (x_n) by a subsequence, we find $\varepsilon > 0$ such that $u(x_n) \leq u(x_0) - \varepsilon < u(x_0)$ holds for all $n \in \mathbb{N}$. Consequently, we have $\Sigma(x_0) = \big\{t \in \mathbb{C};\ u(x_0) < \mathrm{Im}(t) < o(x_0)\big\} \subset \Sigma := \big\{t \in \mathbb{C};\ u(x_0) - \varepsilon < \mathrm{Im}(t) < o(x_0)\big\} \subset \Sigma(x_n)$ for all n and hence obtain the sequence of holomorphic functions $f_n\colon \Sigma \to X$, $f_n(t) := t \cdot x_n$. Since X is taut and $f_n(0) = x_n \to x_0$, the sequence (f_n) has a subsequence which compactly converges to a holomorphic function $f_0\colon \Sigma \to X$. Because of $f_0\big(iu(x_0)\big) = \lim_{n\to\infty} f_n\big(iu(x_0)\big) = \lim_{n\to\infty} iu(x_0)\cdot x_n = iu(x_0)\cdot x_0 \notin X$ we arrive at a contradiction. Thus the function $u\colon X\setminus\{u=-\infty\}\to\mathbb{R}$ is continuous. By $(i\mathbb{R})$–equivariance, u is also continuous on $X^*\setminus\{u=-\infty\}$. A similar argument shows continuity of $-o\colon X^*\setminus\{o=\infty\}\to\mathbb{R}$. ◻

Let us consider the sets

$$\mathcal{N}(o) := \big\{x\in X^*;\ o(x)=0\big\} \quad\text{and}\quad \mathcal{P}(o) := \big\{x\in X^*;\ o(x)=\infty\big\}.$$

The sets $\mathcal{N}(u)$ and $\mathcal{P}(u)$ are similarly defined. Since $X = \big\{x\in X^*;\ u(x) < 0 < o(x)\big\}$, we can recover X from X^* with the help of u and o.

Lemma 3.3. — The action of $\mathbb{R}$ on X^* is proper.

Proof. — Let $\partial^* X$ denote the boundary of X in X^*. Since the functions u and $-o$ are continuous on $X^*\setminus\mathcal{P}(u)$ and $X^*\setminus\mathcal{P}(o)$ one verifies directly that $\partial^*X = \mathcal{N}(u)\cup\mathcal{N}(o)$ holds. As a consequence, we note that if $x\in\partial^*X$, then for every $\varepsilon>0$ the element $(i\,\varepsilon)\cdot x$ is not contained in ∂^*X.
Let (t_n) and (x_n) be sequences in $\mathbb{R}$ and X^* such that $(t_n\cdot x_n, x_n)$ converges to (y_0,x_0) in $X^*\times X^*$. We may assume without loss of generality that x_0 and hence x_n are contained in X for all n. Consequently, we have $y_0\in X\cup\partial^*X$. If $y_0\in\partial^*X$ holds, we may choose an $\varepsilon>0$ such that $(i\,\varepsilon)\cdot y_0$ and $(i\,\varepsilon)\cdot x_0$ lie in X. Since the $\mathbb{R}$–action on X is proper, we find a convergent subsequence of (t_n) which was to be shown. ◻

Lemma 3.4. — We have:

1. *$\mathcal{N}(u)$ and $\mathcal{N}(o)$ are $\mathbb{R}$–invariant.*
2. *We have $\mathcal{N}(u)\cap\mathcal{N}(o)=\emptyset$.*
3. *The sets $\mathcal{P}(u)$ and $\mathcal{P}(o)$ are closed, $\mathbb{C}$–invariant and pluripolar in X^*.*
4. *$\mathcal{P}(u)\cap\mathcal{P}(o)=\emptyset$.*

Proof. — The first claim follows from the $\mathbb{R}$–invariance of u and o.
The second claim follows from $u(x)<o(x)$.
The third one is a consequence of the $\mathbb{R}$–invariance and $i\mathbb{R}$–equivariance of u and o.
If there was a point $x\in\mathcal{P}(u)\cap\mathcal{P}(o)$, then $\mathbb{C}\cdot x$ would be a subset of X which is impossible since X is hyperbolic. ◻

Lemma 3.5. — If o is not identically ∞, then the map

$$\varphi\colon i\mathbb{R}\times\mathcal{N}(o)\to X^*\setminus\mathcal{P}(o),\quad \varphi(it,z)=it\cdot z,$$

is an $i\mathbb{R}$–equivariant homeomorphism. Since $\mathbb{R}$ acts properly on $\mathcal{N}(o)$, it follows that $\mathbb{C}$ acts properly on $X^*\setminus\mathcal{P}(o)$. The same holds when o is replaced by u.

Proof. — The inverse map φ^{-1} is given by $x\mapsto\big(-io(x), io(x)\cdot x\big)$. ◻

Corollary 3.6. — The $\mathbb{C}$–action on X^* is locally proper. If $\mathcal{P}(o)=\emptyset$ or $\mathcal{P}(u)=\emptyset$ hold, then $\mathbb{C}$ acts properly on X^*.

From now on we suppose that X fulfills the assumption (3.2). Recall that the Bergman form ω is a Kähler form on X invariant under the action of $\mathrm{Aut}(X)$. Let ξ denote the complete holomorphic vector field on X which corresponds to the $\mathbb{R}$–action, i. e. we have $\xi(x) = \frac{\partial}{\partial t}\big|_0 \varphi_t(x)$. Hence, $\iota_\xi\omega = \omega(\cdot,\xi)$ is a 1–form on X and since $H^1(X,\mathbb{R}) = \{0\}$ there exists a function $\mu^\xi \in \mathcal{C}^\infty(X)$ with $d\mu^\xi = \iota_\xi\omega$.

Remark. — This means that μ^ξ is a momentum map for the $\mathbb{R}$–action on X.

Lemma 3.7. — The map $\mu^\xi\colon X \to \mathbb{R}$ is an $\mathbb{R}$–invariant submersion.

Proof. — The claim follows from $d\mu^\xi(x)J\xi_x = \omega_x(J\xi_x,\xi_x) > 0$. □

Proposition 3.8. — The $\mathbb{C}$–action on X^* is locally proper.

Proof. — Since μ^ξ is a submersion, the fibers $(\mu^\xi)^{-1}(c)$, $c \in \mathbb{R}$, are real hypersurfaces in X. Then

$$\frac{d}{dt}\bigg|_0 \mu^\xi(it\cdot x) = \omega_x(J\xi_x,\xi_x) > 0$$

implies that every $i\mathbb{R}$–orbit intersects $(\mu^\xi)^{-1}(c)$ transversally. Since X is orbit-connected in X^*, the map $i\mathbb{R}\times(\mu^\xi)^{-1}(c) \to X^*$ is injective and therefore a diffeomorphism onto its open image. Together with the fact that $(\mu^\xi)^{-1}(c)$ is $\mathbb{R}$–invariant this yields the existence of differentiable local slices for the $\mathbb{C}$–action. □

3.3. A necessary condition for $X/\mathbb{Z}$ to be Stein. — We have the following necessary condition for $X/\mathbb{Z}$ to be a Stein manifold.

Proposition 3.9. — If the quotient manifold $X/\mathbb{Z}$ is Stein, then X^* is Stein and the $\mathbb{C}$–action on X^* is proper.

Proof. — Suppose that $X/\mathbb{Z}$ is a Stein manifold. By [CTIT00] this implies that X^* is Stein as well.
Next we will show that the $\mathbb{C}^*$–action on $X^*/\mathbb{Z}$ is proper. For this we will use as above a moment map for the S^1–action on $X^*/\mathbb{Z}$.
By compactness of S^1 we may apply the complexification theorem from [Hei91] which shows that $X^*/\mathbb{Z}$ is also a Stein manifold and in particular Hausdorff. Hence, there exists a smooth strictly plurisubharmonic exhaustion function $\rho\colon X^*/\mathbb{Z} \to \mathbb{R}^{>0}$ invariant under S^1. Consequently, $\omega := \frac{i}{2}\partial\overline{\partial}\rho \in \mathcal{A}^{1,1}(X^*)$ is an S^1–invariant Kähler form. Associated to ω we have the S^1–invariant moment map

$$\mu\colon X^*/\mathbb{Z} \to \mathbb{R},\quad \mu^\xi(x) := \frac{d}{dt}\bigg|_0 \rho\big(\exp(it\xi)\cdot x\big),$$

where ξ is the complete holomorphic vector field on $X^*/\mathbb{Z}$ which corresponds to the S^1–action. Now we can apply the same argument as above in order to deduce that $\mathbb{C}^*$ acts locally properly on $X^*/\mathbb{Z}$.

We still must show that $(X^*/\mathbb{Z})/\mathbb{C}^*$ is Hausdorff. To see this, let $\mathbb{C}^* \cdot x_j$, $j = 0, 1$, be two different orbits in $X^*/\mathbb{Z}$. Since $\mathbb{C}^*$ acts locally properly, these are closed and therefore there exists a function $f \in \mathcal{O}(X^*/\mathbb{Z})$ with $f|_{\mathbb{C}^* \cdot x_j} = j$ for $j = 0, 1$. Again we may assume that f is S^1– and consequently $\mathbb{C}^*$–invariant. Hence, there is a continuous function on $(X^*/\mathbb{Z})/\mathbb{C}^*$ which separates the two orbits, which implies that $(X^*/\mathbb{Z})/\mathbb{C}^*$ is Hausdorff. This proves that $\mathbb{C}^*$ acts properly on $X^*/\mathbb{Z}$.
Since we know already that the $\mathbb{C}$–action on X^* is locally proper, it is enough to show that $X^*/\mathbb{C}$ is Hausdorff. But this follows from the properness of the $\mathbb{C}^*$–action on $X^*/\mathbb{Z}$ since $X^*/\mathbb{C} \cong (X^*/\mathbb{Z})/\mathbb{C}^*$ is Hausdorff. □

4. Properness of the $\mathbb{C}$–action

Let X be a hyperbolic Stein $\mathbb{R}$–manifold. Suppose that X fulfills (3.1) or (3.2). We have seen that $\mathbb{C}$ acts locally properly on X^*. In this section we prove that under the additional assumption $\dim X = 2$ the orbit space $X^*/\mathbb{C}$ is Hausdorff. This implies that $\mathbb{C}$ acts properly on X^* if $\dim X = 2$.

4.1. Stein surfaces with $\mathbb{C}$–actions.

— For every function $f \in \mathcal{O}(\Delta)$ which vanishes only at the origin, we define

$$X_f := \{(x, y, z) \in \Delta \times \mathbb{C}^2;\ f(x)y - z^2 = 1\}.$$

Since the differential of the defining equation of X_f is given by $\big(f'(x)y\ f(x)\ -2z\big)$, we see that 1 is a regular value of $(x, y, z) \mapsto f(x)y - z^2$. Hence, X_f is a smooth Stein surface in $\Delta \times \mathbb{C}^2$.
There is a holomorphic $\mathbb{C}$–action on X_f defined by

$$t \cdot (x, y, z) := \big(x, y + 2tz + t^2 f(x), z + tf(x)\big).$$

Lemma 4.1. — The $\mathbb{C}$–action on X_f is free, and all orbits are closed.

Proof. — Let $t \in \mathbb{C}$ such that $\big(x, y + 2tz + t^2 f(x), z + tf(x)\big) = (x, y, z)$ for some $(x, y, z) \in X_f$. If $f(x) \neq 0$, then $z + tf(x) = z$ implies $t = 0$. If $f(x) = 0$, then $z \neq 0$ and $y + 2tz = y$ gives $t = 0$.
The map $\pi\colon X_f \to \Delta$, $(x, y, z) \mapsto x$, is $\mathbb{C}$–invariant. If $a \in \Delta^*$, then $f(a) \neq 0$ and we have

$$\frac{z}{f(a)} \cdot \big(a, f(a)^{-1}, 0\big) = (a, y, z) \in X_f,$$

which implies $\pi^{-1}(a) = \mathbb{C} \cdot \big(a, f(a)^{-1}, 0\big)$. A similar calculation gives $\pi^{-1}(0) = \mathbb{C} \cdot p_1 \cup \mathbb{C} \cdot p_2$ with $p_1 = (0, 0, i)$ and $p_2 = (0, 0, -i)$. Consequently, every $\mathbb{C}$–orbit is closed. □

Remark. — The orbit space $X_f/\mathbb{C}$ is the unit disc with a doubled origin and in particular not Hausdorff.

We calculate slices at the point p_j, $j = 1, 2$, as follows. Let $\varphi_j \colon \Delta \times \mathbb{C} \to X_f$ be given by $\varphi_1(z,t) := t \cdot (z, 0, i)$ and $\varphi_2(w,s) = s \cdot (w, 0, -i)$. Solving the equation $s \cdot (w, 0, -i) = t \cdot (z, 0, i)$ for (w, s) yields the transition function $\varphi_{12} = \varphi_2^{-1} \circ \varphi_1 \colon \Delta^* \times \mathbb{C} \to \Delta^* \times \mathbb{C}$,

$$(z,t) \mapsto \left(z, t + \frac{2i}{f(z)} \right).$$

The function $\frac{1}{f}$ is a meromorphic function on Δ without zeros and with the unique pole 0.

Lemma 4.2. — *Let $\mathbb{R}$ act on X_f via $\mathbb{R} \hookrightarrow \mathbb{C}$, $t \mapsto ta$, for some $a \in \mathbb{C}^*$. Then there is no $\mathbb{R}$–invariant domain $D \subset X_f$ with $D \cap \mathbb{C} \cdot p_j \neq \emptyset$ for $j = 1, 2$ on which $\mathbb{R}$ acts properly.*

Proof. — Suppose that $D \subset X_f$ is an $\mathbb{R}$–invariant domain with $D \cap \mathbb{C} \cdot p_j \neq \emptyset$ for $j = 1, 2$. Without loss of generality we may assume that $p_1 \in D$ and $\zeta \cdot p_2 = (0, -2\zeta i, -i) \in D$ for some $\zeta \in \mathbb{C}$. We will show that the orbits $\mathbb{R} \cdot p_1$ and $\mathbb{R} \cdot (\zeta \cdot p_2)$ cannot be separated by $\mathbb{R}$–invariant open neighborhoods.
Let $U_1 \subset D$ be an $\mathbb{R}$–invariant open neighborhood of p_1. Then there are $r, r' > 0$ such that $\Delta_r^* \times \Delta_{r'} \times \{i\} \subset U_1$ holds. Here, $\Delta_r = \{z \in \mathbb{C};\ |z| < r\}$. For $(\varepsilon_1, \varepsilon_2) \in \Delta_r^* \times \Delta_{r'}$ and $t \in \mathbb{R}$ we have

$$t \cdot (\varepsilon_1, \varepsilon_2, i) = \big(\varepsilon_1, \varepsilon_2 + 2(ta)i + (ta)^2 f(\varepsilon_1), i + (ta) f(\varepsilon_1)\big) \in U_1.$$

We have to show that for all $r_2, r_3 > 0$ there exist $(\widetilde{\varepsilon_2}, \widetilde{\varepsilon_3}) \in \Delta_{r_2} \times \Delta_{r_3}$, $(\varepsilon_1, \varepsilon_2) \in \Delta_r^* \times \Delta_{r'}$ and $t \in \mathbb{R}$ such that

$$\big(\varepsilon_1, \varepsilon_2 + 2(ta)i + (ta)^2 f(\varepsilon_1), i + (ta) f(\varepsilon_1)\big) = (\varepsilon_1, -2\zeta i + \widetilde{\varepsilon_2}, -i + \widetilde{\varepsilon_3}) \tag{4.1}$$

holds.
Let $r_2, r_3 > 0$ be given. From (4.1) we obtain $\widetilde{\varepsilon_3} = ta f(\varepsilon_1) + 2i$ or, equivalently, $ta = \frac{\widetilde{\varepsilon_3} - 2i}{f(\varepsilon_1)}$. Setting $\widetilde{\varepsilon_2} = \varepsilon_2$ we obtain from $2(ta)i + (ta)^2 f(\varepsilon_1) = -2\zeta i$ the equivalent expression

$$f(\varepsilon_1) = -2i \frac{\zeta + ta}{(ta)^2}. \tag{4.2}$$

for $t \neq 0$. Choosing a real number $t \gg 1$, we find an $\varepsilon_1 \in \Delta_r^*$ such that (4.2) is fulfilled. After possibly enlarging t we have $\widetilde{\varepsilon_3} := ta f(\varepsilon_1) + 2i = -2i \frac{\zeta}{ta} \in \Delta_{r_3}$. Together with $\varepsilon_2 = \widetilde{\varepsilon_2}$ equation (4.1) is fulfilled and the proof is finished. □

Thus, the Stein surface X_f cannot be obtained as globalization of the local $\mathbb{C}$–action on any $\mathbb{R}$–invariant domain $D \subset X_f$ on which $\mathbb{R}$ acts properly.

4.2. The quotient $X^*/\mathbb{C}$ is Hausdorff. — Suppose that $X^*/\mathbb{C}$ is not Hausdorff and let $x_1, x_2 \in X$ be such that the corresponding $\mathbb{C}$–orbits cannot be separated in $X^*/\mathbb{C}$. Since we already know that $\mathbb{C}$ acts locally proper on X^* we find local holomorphic slices $\varphi_j \colon \Delta \times \mathbb{C} \to U_j \subset X$, $\varphi_j(z,t) = t \cdot s_j(z)$ at each $\mathbb{C} \cdot x_j$ where $s_j \colon \Delta \to X$ is holomorphic with $s_j(0) = x_j$. Consequently, we obtain the transition function $\varphi_{12} \colon (\Delta \setminus A) \times \mathbb{C} \to (\Delta \setminus A) \times \mathbb{C}$ for some closed subset $A \subset \Delta$ which must be of the form $(z,t) \mapsto \big(z, t+f(z)\big)$ for some $f \in \mathcal{O}(\Delta \setminus A)$. The following lemma applies to show that A is discrete and that f is meromorphic on Δ. Hence, we are in one of the model cases discussed in the previous subsection.

Lemma 4.3. — Let Δ_1 and Δ_2 denote two copies of the unit disk $\{z \in \mathbb{C};\ |z| < 1\}$. Let $U \subset \Delta_j$, $j = 1,2$, be a connected open subset and $f\colon U \subset \Delta_1 \to \mathbb{C}$ a non-constant holomorphic function on U. Define the complex manifold

$$M := (\Delta_1 \times \mathbb{C}) \cup (\Delta_2 \times \mathbb{C})/\!\sim,$$

where $\sim$ is the relation $(z_1,t_1) \sim (z_2,t_2) :\Leftrightarrow z_1 = z_2 =: z \in U$ and $t_2 = t_1 + f(z)$.
Suppose that M is Hausdorff. Then the complement A of U is discrete and f extends to a meromorphic function on Δ_1.

Proof. — We first prove that for every sequence (x_n), $x_n \in U$, with $\lim_{n\to\infty} x_n = p \in \partial U$, one has $\lim_{n\to\infty}|f(x_n)| = \infty \in \mathbb{P}_1(\mathbb{C})$. Assume the contrary, i.e. there is a sequence (x_n), $x_n \in U$, with $\lim_{n\to\infty} x_n = p \in \partial U$ such that $\lim_{n\to\infty} f(x_n) = a \in \mathbb{C}$. Choose now $t_1 \in \mathbb{C}$, consider the two points $(p,t_1) \in \Delta_1 \times \mathbb{C}$ and $(p, t_1 + a) \in \Delta_2 \times \mathbb{C}$ and note their corresponding points in M as q_1 and q_2. Then $q_1 \neq q_2$. The sequences $(x_n,t_1) \in \Delta_1 \times \mathbb{C}$ and $(x_n, t_1 + f(x_n)) \in \Delta_2 \times \mathbb{C}$ define the same sequence in M having q_1 and q_2 as accumulation points. So M is not Hausdorff, a contradiction.
In particular we have proved that the zeros of f do not accumulate to ∂U in Δ_1. So there is an open neighborhood V of ∂U in Δ_1 such that the restriction of f to $W := U \cap V$ does not vanish. Let $g := 1/f$ on W. Then g extends to a continuous function on V taking the value zero outside of U. The theorem of Rado implies that this function is holomorphic on V. It follows that the boundary ∂U is discrete in Δ_1 and that f has a pole in each of the points of this set, so f is a meromorphic function on Δ_1. □

Theorem 4.4. — The orbit space $X^*/\mathbb{C}$ is Hausdorff. Consequently, $\mathbb{C}$ acts properly on X^*.

Proof. — By virtue of the above lemma, in a neighborhood of two non-separable $\mathbb{C}$–orbits X is isomorphic to a domain in one of the model Stein surfaces discussed in the previous subsection. Since we have seen there that these surfaces are never globalizations, we arrive at a contradiction. Hence, all $\mathbb{C}$–orbits are separable. □

5. Examples

In this section we discuss several examples which illustrate our results.

5.1. Hyperbolic Stein surfaces with proper $\mathbb{R}$–actions. — Let R be a compact Riemann surface of genus $g \geq 2$. It follows that the universal covering of R is given by the unit disc $\Delta \subset \mathbb{C}$ and hence that R is hyperbolic. The fundamental group $\pi_1(R)$ of R contains a normal subgroup N such that $\pi_1(R)/N \cong \mathbb{Z}$. Let $\widetilde{R} \to R$ denote the corresponding normal covering. Then $\widetilde{R}$ is a hyperbolic Riemann surface with a holomorphic $\mathbb{Z}$–action such that $\widetilde{R}/\mathbb{Z} = R$. Note that $\mathbb{Z}$ is not contained in a one parameter group of automorphisms of $\widetilde{R}$.

We have two mappings

$$\begin{tikzcd} X := \mathbb{H} \times_{\mathbb{Z}} \widetilde{R} \arrow[r, "q"] \arrow[d, "p"'] & \widetilde{R}/\mathbb{Z} = R \\ \mathbb{H}/\mathbb{Z} \cong \Delta \setminus \{0\}. & \end{tikzcd}$$

The map $p\colon X \to \Delta \setminus \{0\}$ is a holomorphic fiber bundle with fiber $\widetilde{R}$. Since the Serre problem has a positive answer if the fiber is a non-compact Riemann surface ([Mok82]), the suspension $X = \mathbb{H} \times_{\mathbb{Z}} \widetilde{R}$ is a hyperbolic Stein surface. The group $\mathbb{R}$ acts on $\mathbb{H} \times \widetilde{R}$ by $t \cdot (z, x) = (z + t, x)$ and this action commutes with the diagonal action of $\mathbb{Z}$. Consequently, we obtain an action of $\mathbb{R}$ on X.

Lemma 5.1. — The universal globalization of the local $\mathbb{C}$–action on X is given by $X^* = \mathbb{C} \times_{\mathbb{Z}} \widetilde{R}$. Moreover, $\mathbb{C}$ acts properly on X^*.

Proof. — One checks directly that $t \cdot [z, x] := [z + t, x]$ defines a holomorphic $\mathbb{C}$–action on $X^* = \mathbb{C} \times_{\mathbb{Z}} \widetilde{R}$ which extends the $\mathbb{R}$–action on X. We will show that X is orbit-connected in X^*: Since $[z + t, x]$ lies in X if and only if there exist elements $(z', x') \in \mathbb{H} \times \widetilde{R}$ and $m \in \mathbb{Z}$ such that $(z+t, x) = \bigl(z' + m, m \cdot x'\bigr)$, we conclude $\mathbb{C}[z, x] = \bigl\{t \in \mathbb{C};\ \mathrm{Im}(t) > -\mathrm{Im}(z)\bigr\}$ which is connected.

In order to show that $\mathbb{C}$ acts properly on X^* it is sufficient to show that $\mathbb{C} \times \mathbb{Z}$ acts properly on $\mathbb{C} \times \widetilde{R}$. Hence, we choose sequences $\{t_n\}$ in $\mathbb{C}$, $\{m_n\}$ in $\mathbb{Z}$ and $\bigl\{(z_n, x_n)\bigr\}$ in $\mathbb{C} \times \widetilde{R}$ such that

$$\begin{aligned} &\bigl((t_n, m_n) \cdot (z_n, x_n), (z_n, x_n)\bigr) = \\ &\qquad = \bigl((z_n + t_n + m_n, m_n \cdot x_n), (z_n, x_n)\bigr) \to \bigl((z_1, x_1), (z_0, x_0)\bigr) \end{aligned}$$

holds. Since $\mathbb{Z}$ acts properly on $\widetilde{R}$, it follows that $\{m_n\}$ has a convergent subsequence, which in turn implies that $\{t_n\}$ has a convergent subsequence. Hence, the lemma is proven. □

Proposition 5.2. — The quotient $X/\mathbb{Z} \cong \Delta^* \times R$ is not holomorphically separable and in particular not Stein. The quotient $X^*/\mathbb{C}$ is biholomorphically equivalent to $\widetilde{R}/\mathbb{Z} = R$.

Proof. — It is sufficient to note that the map $\Phi\colon X = \mathbb{H} \times_{\mathbb{Z}} \widetilde{R} \to \Delta^* \times R$, $\Phi[z, x] := \bigl(e^{2\pi i z}, [x]\bigr)$, induces a biholomorphic map $X/\mathbb{Z} \to \Delta^* \times R$. □

Thus we have found an example for a hyperbolic Stein surface X endowed with a proper $\mathbb{R}$–action such that the associated $\mathbb{Z}$–quotient is not holomorphically separable. Moreover, the $\mathbb{R}$–action on X extends to a proper $\mathbb{C}$–action on a Stein manifold X^* containing X as an orbit-connected domain such that $X^*/\mathbb{C}$ is any given compact Riemann surface of genus $g \geq 2$.

5.2. Counterexamples with domains in $\mathbb{C}^n$. — There is a bounded Reinhardt domain D in $\mathbb{C}^2$ endowed with a holomorphic action of $\mathbb{Z}$ such that $D/\mathbb{Z}$ is not Stein. However, this $\mathbb{Z}$–action does not extend to an $\mathbb{R}$–action. We give quickly the construction.
Let $\lambda := \frac{1}{2}(3+\sqrt{5})$ and

$$D := \{(x,y) \in \mathbb{C}^2 \mid |x| > |y|^\lambda, |y| > |x|^\lambda\}.$$

It is obvious that D is a bounded Reinhardt domain in $\mathbb{C}^2$ avoiding the coordinate hyperplanes. The holomorphic automorphism group of D is a semidirect product $\Gamma \ltimes (S^1)^2$, where the group $\Gamma \simeq \mathbb{Z}$ is generated by the automorphism $(x,y) \mapsto (x^3y^{-1}, x)$ and $(S^1)^2$ is the rotation group. Therefore the group Γ is not contained in a one-parameter group. Furthermore the quotient D/Γ is the (non-Stein) complement of the singular point in a 2-dimensional normal complex Stein space, a so-called "cusp singularity". These singularities are intensively studied in connection with Hilbert modular surfaces and Inoue-Hirzebruch surfaces, see e.g. [vdG88] and [Zaf01].
In the rest of this subsection we give an example of a hyperbolic domain of holomorphy in a 3–dimensional Stein solvmanifold endowed with a proper $\mathbb{R}$–action such that the $\mathbb{Z}$–quotient is not Stein. While this domain is not simply-connected, its fundamental group is much simpler than the fundamental groups of our two-dimensional examples.
Let $G := \left\{ \left(\begin{smallmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right);\ a,b,c \in \mathbb{C} \right\}$ be the complex Heisenberg group and let us consider its discrete subgroup

$$\Gamma := \left\{ \begin{pmatrix} 1 & m & \frac{m^2}{2} + 2\pi i k \\ 0 & 1 & m + 2\pi i l \\ 0 & 0 & 1 \end{pmatrix};\ m,k,l \in \mathbb{Z} \right\}.$$

Note that Γ is isomorphic to $\mathbb{Z}_m \ltimes \mathbb{Z}^2_{(k,l)}$. We let Γ act on $\mathbb{C}^2$ by

$$(z,w) \mapsto \left(z + mw - \frac{m^2}{2} - 2\pi i k, w - m - 2\pi i l \right).$$

Proposition 5.3. — The group Γ acts properly and freely on $\mathbb{C}^2$, and the quotient manifold $\mathbb{C}^2/\Gamma$ is holomorphically separable but not Stein.

Proof. — Since $\Gamma' \cong \mathbb{Z}^2$ is a normal subgroup of Γ, we obtain $\mathbb{C}^2/\Gamma \cong (\mathbb{C}^2/\Gamma')/(\Gamma/\Gamma')$. The map $\mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}^*$, $(z,w) \mapsto \big(\exp(z), \exp(w)\big)$, identifies $\mathbb{C}^2/\Gamma'$ with $\mathbb{C}^* \times \mathbb{C}^*$. The induced action of $\Gamma/\Gamma' \cong \mathbb{Z}$ on $\mathbb{C}^* \times \mathbb{C}^*$ is given by

$$(z,w) \mapsto \left(e^{-m^2/2} z w^m, e^{-m} w \right)$$

which shows that Γ acts properly and freely on $\mathbb{C}^2$. Moreover, we obtain the commutative diagram

$$\begin{tikzcd} \mathbb{C}^* \times \mathbb{C}^* \arrow[r] \arrow[d, "(z,w)\mapsto w"'] & Y := (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z} \arrow[d] \\ \mathbb{C}^* \arrow[r] & T := \mathbb{C}^*/\mathbb{Z}. \end{tikzcd}$$

The group $\mathbb{C}^*$ acts by multiplication in the first factor on $\mathbb{C}^* \times \mathbb{C}^*$ and this action commutes with the $\mathbb{Z}$–action. One checks directly that the joint $(\mathbb{C}^* \times \mathbb{Z})$–action on $\mathbb{C}^* \times \mathbb{C}^*$ is proper which implies that the map $Y \to T$ is a $\mathbb{C}^*$–principal bundle. Consequently, Y is not Stein.

In order to show that Y is holomorphically separable, note that by [Oel92] this $\mathbb{C}^*$–principal bundle $Y \to T$ extends to a line bundle $p\colon L \to T$ with first Chern class $c_1(L) = -1$. Therefore the zero section of $p\colon L \to T$ can be blown down and we obtain a singular normal Stein space $\overline{Y} = Y \cup \{y_0\}$ where $y_0 = \mathrm{Sing}(\overline{Y})$ is the blown down zero section. Thus Y is holomorphically separable. $\square$

Let us now choose a neighborhood of the singularity $y_0 \in \overline{Y}$ biholomorphic to the unit ball and let U be its inverse image in $\mathbb{C}^2$. It follows that U is a hyperbolic domain with smooth strictly Levi-convex boundary in $\mathbb{C}^2$ and in particular Stein. In order to obtain a proper action of $\mathbb{R}$ we form the suspension $D = \mathbb{H} \times_\Gamma U$ where Γ acts on $\mathbb{H} \times U$ by $(t, z, w) \mapsto (t + m, z + mw - \frac{m^2}{2} - 2\pi ik, w - m - 2\pi il)$.

Proposition 5.4. — The suspension $D = \mathbb{H} \times_\Gamma U$ is isomorphic to a Stein domain in the Stein manifold G/Γ.

Proof. — We identify $\mathbb{H} \times U$ with the $\mathbb{R} \times \Gamma$–invariant domain

$$\Omega := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} ;\ \mathrm{Im}(a) > 0, (c, b) \in U \right\}$$

in G.

Since $\mathbb{H} \times U$ is Stein, it follows that $\mathbb{H} \times_\Gamma U$ is locally Stein in G/Γ. Hence, by virtue of [DG60] we only have to show that G/Γ is Stein.

For this we note first that G is a closed subgroup of $\mathrm{SL}(2, \mathbb{C}) \ltimes \mathbb{C}^2$ which implies that G/Γ is a closed complex submanifold of $X := \big(\mathrm{SL}(2, \mathbb{C}) \ltimes \mathbb{C}^2\big)/\Gamma$. By [Oel92] the manifold X is holomorphically separable, hence G/Γ is holomorphically separable. Since G is solvable, a result of Huckleberry and Oeljeklaus ([HO86]) yields the Steinness of G/Γ.

One checks directly that the action of $\mathbb{R} \times \Gamma$ on $\mathbb{H} \times U$ is proper which implies that $\mathbb{R}$ acts properly on $\mathbb{H} \times_\Gamma U$. $\square$

Because of $(\mathbb{H} \times_\Gamma U)/\mathbb{Z} \cong \Delta^* \times (U/\Gamma)$ this quotient manifold is not Stein but holomorphically separable.

6. Bounded domains with proper $\mathbb{R}$–actions

In this section we give the proof of our main result.

6.1. Proper $\mathbb{R}$–actions on D.

— Let $D \subset \mathbb{C}^n$ be a bounded domain and let $\mathrm{Aut}(D)^0$ be the connected component of the identity in $\mathrm{Aut}(D)$.

Lemma 6.1. — A proper $\mathbb{R}$–action by holomorphic transformations on D exists if and only if the group $\mathrm{Aut}(D)^0$ is non-compact.

Proof. — We note first that effective $\mathbb{R}$–actions by holomorphic transformations on D correspond bijectively to one parameter subgroups $\mathbb{R} \hookrightarrow \mathrm{Aut}(D)^0$, $t \mapsto \varphi_t$, where the correspondence is given by $t \cdot z = \varphi_t(z)$ for $t \in \mathbb{R}$ and $z \in D$. Since the group $\mathrm{Aut}(D)^0$ acts properly on D, proper $\mathbb{R}$–actions correspond to closed embeddings $\mathbb{R} \hookrightarrow \mathrm{Aut}(D)^0$. If $\mathrm{Aut}(D)^0$ admits such an embedding, it cannot be compact.

Conversely, suppose that $\mathrm{Aut}(D)^0$ is not compact. By Theorem 3.1 in [Ho65] there are a maximal compact subgroup K of $\mathrm{Aut}(D)^0$ and a linear subspace V of the Lie algebra of $\mathrm{Aut}(D)^0$ such that the map $K \times V \to \mathrm{Aut}(D)^0$, $(k, \xi) \mapsto k \exp(\xi)$, is a diffeomorphism. Since $\mathrm{Aut}(D)^0$ is not compact, the vector space V has positive dimension and the map $t \mapsto \varphi_t := \exp(t\xi)$, for some $0 \neq \xi \in V$, defines a closed embedding of $\mathbb{R}$ into $\mathrm{Aut}(D)^0$ and hence a proper $\mathbb{R}$–action by holomorphic transformations on D. □

6.2. Steinness of $D/\mathbb{Z}$.

— Now we give the proof of our main result.

Theorem 6.2. — Let D be a simply-connected bounded domain of holomorphy in $\mathbb{C}^2$. Suppose that the group $\mathbb{R}$ acts properly by holomorphic transformations on D. Then the complex manifold $D/\mathbb{Z}$ is biholomorphically equivalent to a domain of holomorphy in $\mathbb{C}^2$.

Proof. — Let $D \subset \mathbb{C}^2$ be a simply-connected bounded domain of holomorphy. Since the Serre problem is solvable if the fiber is D, see [Siu76], the universal globalization D^* is a simply-connected Stein surface, [CTIT00]. Moreover, we have shown in Theorem 4.4, that $\mathbb{C}$ acts properly on D^*. Since the Riemann surface $D^*/\mathbb{C}$ is also simply-connected, it must be Δ, $\mathbb{C}$ or $\mathbb{P}_1(\mathbb{C})$. In all three cases the bundle $D^* \to D^*/\mathbb{C}$ is holomorphically trivial. So we can exclude the case that $D^*/\mathbb{C}$ is compact and it follows that $D/\mathbb{Z} \cong \mathbb{C}^* \times (D^*/\mathbb{C})$ is a Stein domain in $\mathbb{C}^2$. □

6.3. A normal form for domains with non-compact $\mathrm{Aut}(D)^0$.

— Let $D \subset \mathbb{C}^2$ be a simply-connected bounded domain of holomorphy such that the identity component of its automorphism group is non-compact. As we have seen, this yields a proper $\mathbb{R}$–action on D by holomorphic transformations and the universal globalization of the induced local $\mathbb{C}$–action on D is isomorphic to $\mathbb{C} \times S$ where S is either Δ or $\mathbb{C}$ and where $\mathbb{C}$ acts by translation in the first factor.

Moreover, there are plurisubharmonic functions $u, -o\colon \mathbb{C} \times S \to \mathbb{R} \cup \{-\infty\}$ which fulfill

$$u\big(t \cdot (z_1, z_2)\big) = u(z_1, z_2) - \operatorname{Im}(t) \quad \text{and} \quad o\big(t \cdot (z_1, z_2)\big) = o(z_1, z_2) - \operatorname{Im}(t)$$

such that $D = \big\{(z_1, z_2) \in \mathbb{C} \times S;\ u(z_1, z_2) < 0 < o(z_1, z_2)\big\}$. From this we conclude $u(z_1, z_2) = u(0, z_2) - \operatorname{Im}(z_1)$, $o(z_1, z_2) = o(0, z_2) - \operatorname{Im}(z_1)$ and define $u'(z_2) := u(0, z_2)$, $o'(z_2) := o(0, z_2)$.
We summarize our remarks in the following

Theorem 6.3. — Let D be a simply-connected bounded domain of holomorphy in $\mathbb{C}^2$ admitting a non-compact connected identity component of its automorphism group. Then D is biholomorphic to a domain of the form

$$\widetilde{D} = \big\{(z_1, z_2) \in \mathbb{C} \times S;\ u'(z_2) < \operatorname{Im}(z_1) < o'(z_2)\big\},$$

where the functions $u', -o'$ are subharmonic in S.

Remark. — As a consequence of this normal form we see that the domain D admits a continuous fibration over the contractible domain S such that every fiber is a strip in $\mathbb{C}$. Hence, it follows a posteriori that the simply-connected domain of holomorphy D is contractible.

References

[CTIT00] E. Casadio Tarabusi, A. Iannuzzi & S. Trapani – "Globalizations, fiber bundles, and envelopes of holomorphy", *Math. Z.* 233 (2000), no. 3, p. 535–551.

[DG60] F. Docquier & H. Grauert – "Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten", *Math. Ann.* 140 (1960), p. 94–123.

[DK00] J. J. Duistermaat & J. A. C. Kolk – *Lie groups*, Universitext, Springer-Verlag, Berlin, 2000.

[FI01] C. de Fabritiis & A. Iannuzzi – "Quotients of the unit ball of $\mathbb{C}^n$ for a free action of $\mathbb{Z}$", *J. Anal. Math.* 85 (2001), p. 213–224.

[For96] F. Forstneric – "Actions of $(\mathbf{R}, +)$ and $(\mathbf{C}, +)$ on complex manifolds", *Math. Z.* 223 (1996), no. 1, p. 123–153.

[Hei91] P. Heinzner – "Geometric invariant theory on Stein spaces", *Math. Ann.* 289 (1991), no. 4, p. 631–662.

[HI97] P. Heinzner & A. Iannuzzi – "Integration of local actions on holomorphic fiber spaces", *Nagoya Math. J.* 146 (1997), p. 31–53.

[Ho65] G. Hochschild – *The structure of Lie groups*, Holden-Day, Inc., San Francisco-London-Amsterdam, 1965.

[HO86] A. T. Huckleberry & E. Oeljeklaus – "On holomorphically separable complex solv-manifolds ", *Ann. Inst. Fourier (Grenoble)* 36 (1986), no. 3, p. 57–65.

[Ian03] A. Iannuzzi – "Induced local actions on taut and Stein manifolds", *Proc. Amer. Math. Soc.* 131 (2003), no. 12, p. 3839–3843 (electronic).

[IST04] A. Iannuzzi, A. Spiro & S. Trapani – "Complexifications of holomorphic actions and the Bergman metric", *Internat. J. Math.* 15 (2004), no. 8, p. 735–747.

[Kob98] S. Kobayashi – *Hyperbolic complex spaces*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 318, Springer-Verlag, Berlin, 1998.

[Mie08] C. Miebach – "Quotients of bounded homogeneous domains by cyclic groups", 2008, arxiv:math.CV/0803.4476v2.

[Mok82] N. Mok – "The Serre problem on Riemann surfaces", *Math. Ann.* 258 (1981/82), no. 2, p. 145–168.

[Oel92] K. Oeljeklaus – "On the holomorphic separability of discrete quotients of complex Lie groups", *Math. Z.* 211 (1992), no. 4, p. 627–633.

[Pal57] R. S. Palais – "A global formulation of the Lie theory of transformation groups", *Mem. Amer. Math. Soc. No.* 22 (1957), p. iii+123.

[Pal61] ______, "On the existence of slices for actions of non-compact Lie groups", *Ann. of Math. (2)* 73 (1961), p. 295–323.

[Siu76] Y. T. Siu – "Holomorphic fiber bundles whose fibers are bounded Stein domains with zero first Betti number", *Math. Ann.* 219 (1976), no. 2, p. 171–192.

[vdG88] G. van der Geer – *Hilbert modular surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 16, Springer-Verlag, Berlin, 1988.

[Zaf01] D. Zaffran – "Serre problem and Inoue-Hirzebruch surfaces", *Math. Ann.* 319 (2001), no. 2, p. 395–420.

Christian Miebach
LATP-UMR(CNRS) 6632
CMI-Université d'Aix-Marseille I
39 rue Joliot-Curie
F-13453 Marseille Cedex 13
France
miebach@cmi.univ-mrs.fr

Karl Oeljeklaus
LATP-UMR(CNRS) 6632
CMI-Université d'Aix-Marseille I
39 rue Joliot-Curie
F-13453 Marseille Cedex 13
France
karloelj@cmi.univ-mrs.fr