Vanishing of Hochschild Cohomology for Affine Group Schemes and Rigidity of Homomorphisms between Algebraic Groups

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Abstract. Let k be an algebraically closed field. If $\mathbf{G}$ is a linearly reductive k–group and $\mathbf{H}$ is a smooth algebraic k–group, we establish a rigidity property for the set of group homomorphisms $\mathbf{G} \to \mathbf{H}$ up to the natural action of $\mathbf{H}(k)$ by conjugation. Our main result states that this set remains constant under any base change K/k with K algebraically closed. This is proven as consequence of a vanishing result for Hochschild cohomology of affine group schemes.

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Contents

1. Introduction.

Our original goal was to prove a strong version of a rigidity principle for homomorphisms between algebraic groups which is part of the area's folklore. The general philosophy is that if $\mathbf{G}$ and $\mathbf{H}$ are algebraic groups over an algebraically closed field k, then the set $\mathrm{Hom}_{k-gr}(\mathbf{G}, \mathbf{H})$ modulo the adjoint action of $\mathbf{H}$ should remain constant under any base change K/k with K algebraically closed. Our result is as follows.

Theorem 1.1. *Let k be an algebraic closed field. Let $\mathbf{G}$ be a linearly reductive (affine) algebraic k–group, and $\mathbf{H}$ a smooth algebraic k–group scheme. Then for every algebraically closed field extension K/k, the natural map*

$$\mathrm{Hom}_{k-gr}(\mathbf{G}, \mathbf{H})/\mathbf{H}(k) \to \mathrm{Hom}_{K-gr}(\mathbf{G}_K, \mathbf{H}_K)/\mathbf{H}(K)$$

is bijective.

When k is of characteristic 0 and $\mathbf{G}$ and $\mathbf{H}$ are both reductive this result has been established by Vinberg [19, prop. 10] by reducing to the case where $\mathbf{G} = \mathbf{GL}_N$ and $\mathbf{H}$ is connected. Our proof is very different in spirit than Vinberg's, and the main result more general. The proof we give is based on the deformation theory à la Demazure-Grothendieck described in [17], which is itself linked to the analytic viewpoint later taken by Richardson on similar problems [12] [13] [16]. The main auxiliary statement we use is case (i) of the following Theorem, a vanishing result for Hochschild cohomology of affine group schemes which is of its own interest.

Theorem 1.2. *Let R be a commutative ring. Let $\mathbf{G}$ be a flat affine group scheme over* $\mathrm{Spec}(R)$. *Assume that the fibers of $\mathbf{G}$ over all closed points of* $\mathrm{Spec}(R)$ *are linearly reductive groups (as affine groups over the corresponding residue fields. See §3.1 below for the relevant definitions and references). Let L be a $\mathbf{G}$-R–module (see §2.1 below). Assume that one of the following two conditions holds:*

(i) R is noetherian,

(ii) the group $\mathbf{G}$ is of finite presentation as an R-scheme, and L is a direct limit $\mathbf{G}$-R–modules which are finitely presented as R-modules.

Then

$$H^i(\mathbf{G}, L) = 0 \text{ for all } i > 0.$$

This result extends a theorem of Grothendieck for R–groups of multiplicative type [17, IX.3.1].

At this point we recall some standard notation that will be used throughout the paper. Let S be scheme, and $\mathbf{G}$ a group scheme over S. For all scheme morphism $S \to T$ we will denote as it is customary the T-group $\mathbf{G} \times_S T$ by $\mathbf{G}_T$. If $T = \mathrm{Spec}(R)$ we write $\mathbf{G}_R$ instead of $\mathbf{G}_T$, and $\mathbf{G}(R)$ instead of $\mathbf{G}(T)$. Group schemes over a given scheme S will for brevity and convenience sometimes be refereed to simply as S–groups, or R–groups in the case when $S = \mathrm{Spec}(R)$.

2. Generalities on Hochschild cohomology

In this paper, we deal with Hochschild cohomology of a flat affine group scheme $\mathbf{G}$ over an affine base $\mathfrak{X} = \mathrm{Spec}(R)$, and their corresponding $\mathbf{G}$-$\mathcal{O}_{\mathfrak{X}}$–modules [17, I 4.7]. This set up is equivalent to that of $\mathbf{G}$-R-modules as we now explain. Let $\mathbf{G} = \mathrm{Spec}(R[\mathbf{G}])$. The group structure of $\mathbf{G}$ gives the R-algebra $R[\mathbf{G}]$ a coassociative and counital Hopf algebra structure. We have thus a comultiplication $\Delta_{\mathbf{G}} : R[\mathbf{G}] \to R[\mathbf{G}] \otimes_R R[\mathbf{G}]$, a counit $\epsilon : R[\mathbf{G}] \to R$ and an antipode map $\iota : R[\mathbf{G}] \to R[\mathbf{G}]$.
For any ring homomorphism $R \to S$ recall that the S-group $\mathbf{G} \times_R S$ obtained by base change is denoted by $\mathbf{G}_S$. This is an affine S–group with $S[\mathbf{G}] = S \otimes_R R[\mathbf{G}]$ as its Hopf algebra. Similarly, for any R–module L we denote the S–module $L \otimes_R S$ by L_S.

2.1. Definition and basic properties. Let L be an R-module, and $\rho : \mathbf{G} \to \mathbf{GL}(L)$ a linear representation of $\mathbf{G}$. This amounts to give for each R–algebra S an S–linear representation ρ_S of the abstract group $\mathbf{G}(S)$ on the S–module L_S in such a way that the family (ρ_S) is "functorial on S." We also then say that L is a (left) $\mathbf{G}$-R–module. Because $\mathbf{G}$ is affine, to give L a $\mathbf{G}$-R–module structure is equivalent to give L a (right) $R[\mathbf{G}]$-comodule structure, that is an R–linear map

$$\Delta_L : L \to L \otimes_R R[\mathbf{G}]$$

satisfying the two following natural axioms:

(CM1) The following diagram is commutative

$$\begin{tikzcd}
L \arrow[r, "\Delta_L"] \arrow[d, "\Delta_L"'] & L \otimes_R R[\mathbf{G}] \arrow[d, "\mathrm{id}_L \otimes \Delta_{\mathbf{G}}"] \\
L \otimes_R R[\mathbf{G}] \arrow[r, "\Delta_L \otimes \mathrm{id}_{R[\mathbf{G}]}"] & L \otimes_R R[\mathbf{G}] \otimes_R R[\mathbf{G}]
\end{tikzcd}$$

(CM2) The composite map

$$L \xrightarrow{\Delta_L} L \otimes_R R[\mathbf{G}] \xrightarrow{\mathrm{id}_L \otimes \epsilon} L$$

is the identity map id_L.

The flatness condition on $\mathbf{G}/R$ is natural within the present context since the category of $\mathbf{G}$-R–modules is then abelian. See [15, prop. 2].[1] Recall that the fixed points of L under $\mathbf{G}$ are defined by

$$L^{\mathbf{G}} := \big\{ f \in L \;\mid\; \Delta_L(f) = f \otimes 1 \big\}.$$

This is an R–submodule of L. Because of the assumption on flatness, the Hochschild cohomology groups $H^n(\mathbf{G}, L)$ are the derived functors of the "fixed point" functor $\mathbf{G}\text{-}R - mod \to R - mod$ given by $L \to L^{\mathbf{G}}$ [17, I 5.3.1]. The $H^n(\mathbf{G}, L)$ can thus be computed as the cohomology groups of the complex [4, II §3.3.1]

[1] The existence of the unit section of $\mathbf{G}$, more precisely of the counit ϵ, shows that $R[\mathbf{G}]$ is in fact a faithfully flat R-algebra.

$$(2.1) \qquad L \xrightarrow{\partial_0} L \otimes_R R[\mathbf{G}] \xrightarrow{\partial_1} L \otimes_R R[\mathbf{G}^2] \xrightarrow{\partial_2} L \otimes_R R[\mathbf{G}^3] \to \cdots$$

where as usual $R[\mathbf{G}^n] = R[\mathbf{G} \times \cdots \times \mathbf{G}] \simeq R[\mathbf{G}] \otimes_R \cdots \otimes_R R[\mathbf{G}]$, and both products and tensor products are taken n-times. We denote as it is customary $\ker(\partial_i)$ by $Z^i(\mathbf{G}, L)$ (the cocycles), and $\mathrm{Im}(\partial_{i-1})$ by $B^i(\mathbf{G}, L)$ (the coboundaries). In particular we have the exact sequence

$$(2.2) \qquad 0 \to L^{\mathbf{G}} = H^0(\mathbf{G}, L) \to L \xrightarrow{\partial_0} Z^1(\mathbf{G}, L) \to H^1(\mathbf{G}, L) \to 0.$$

The following four properties easily follow from the resolution (2.1).

LEMMA 2.1. *Let L be a $\mathbf{G}$-R–module.*

(1) Let I be an ideal of R which annihilates L. Then $L_{R/I} = L \otimes_R R/I$ is naturally a $\mathbf{G}_{R/I}$-R/I–module, and $H^n(\mathbf{G}, L) \xrightarrow{\sim} H^n(\mathbf{G}_{R/I}, L_{R/I})$ for all $n \geq 0$.

(2) If S/R is a flat extension of rings, then

$$H^n(\mathbf{G}, L) \otimes_R S \xrightarrow{\sim} H^n(\mathbf{G}_S, L_S) \text{ for all } n \geq 0.$$

(3) Let $L = \varinjlim_i L_i$ be the inductive limit of $\mathbf{G}$-R–modules. Then

$$\varinjlim_i H^n(\mathbf{G}, L_i) \xrightarrow{\sim} H^n(\mathbf{G}, L) \text{ for all } n \geq 0.$$

(4) Let $S = \varinjlim_\alpha S_\alpha$ be an inductive limit of R–rings. Then

$$\varinjlim_\alpha H^n(\mathbf{G}_{S_\alpha}, L_{S_\alpha}) \xrightarrow{\sim} H^n(\mathbf{G}_S, L_S) \text{ for all } n \geq 0.$$

Proof. (1) The natural map $L \to L \otimes_R R/I$ is an isomorphism of both R and R/I–modules. We have R and R/I–module isomorphisms

$$L \otimes_R R[\mathbf{G}^n] \simeq L \otimes_R R/I \otimes_R R[\mathbf{G}^n] \simeq L \otimes_R R/I[\mathbf{G}^n] \simeq$$
$$\simeq L \otimes_R R/I \otimes_{R/I} R/I[\mathbf{G}^n] \simeq L_{R/I} \otimes_{R/I} R/I[\mathbf{G}^n].$$

Now (1) follows from the fact that $H^n(\mathbf{G}, L)$ and $H^n(\mathbf{G}_{R/I}, L_{R/I})$ are computed by the cohomology of the same complex. This is also a special case of [17, III 1.1.2].

(2) See [10, I.4.13].

(3) See [10, I.4.17].

(4) The terms of the complex (2.1) for the $\mathbf{G}_S$-S–module L_S are

$$L_S \otimes_S S[\mathbf{G}^n] = (L \otimes_R S) \otimes_S (S \otimes_R R[\mathbf{G}^n]) \simeq \big(L \otimes_R R[\mathbf{G}^n]\big) \otimes_R S$$

So this complex reads

$$L \otimes_R S \to \big(L \otimes_R R[\mathbf{G}]\big) \otimes_R S \to \big(L \otimes_R R[\mathbf{G}^2]\big) \otimes_R S \to \big(L \otimes_R R[\mathbf{G}^3]\big) \otimes_R S \cdots$$

which is the inductive limit over the S_α of the complexes

$$L \otimes_R S_\alpha \to \big(L \otimes_R R[\mathbf{G}]\big) \otimes_R S_\alpha \to \big(L \otimes_R R[\mathbf{G}^2]\big) \otimes_R S_\alpha \to \big(L \otimes_R R[\mathbf{G}^3]\big) \otimes_R S_\alpha \cdots$$

whence the statement. □

The third property in the last Proposition is useful in view of the following fact.

Proposition 2.2. *(Serre) Assume that one of the following hypothesis holds.*

(i) R is noetherian,

(ii) $\mathbf{G}$ is essentially free over R (see §6).

Let L be a $\mathbf{G}$-R–module. Then L is the inductive limit of its $\mathbf{G}$-R submodules which are of finite type as R–modules.

Proof. (i) See [14, prop. 2].
(ii) See [17, VI_B 11.10].

□

We also recall the following application of erasing functors.

Lemma 2.3. *Let $d > 0$ be a positive integer such that $H^d(\mathbf{G}, L) = 0$ for all $\mathbf{G}$-R–modules L. Then $H^{d+i}(\mathbf{G}, L) = 0$ for all $\mathbf{G}$-R–modules L and for all $i \geq 0$.*

Proof. It is enough to prove the vanishing for $d+1$. Let $\mathbf{e}_R$ be the trivial R–group, and view L as a (necessarily trivial) $\mathbf{e}_R$-R–module. We also view L as a trivial $\mathbf{G}$-R–module which we denote by L^0 to avoid any possible confusion. Now we embed L into the induced $\mathbf{G}$-R–module $\mathrm{ind}_{\mathbf{e}_R}^{\mathbf{G}}(L) = L^0 \otimes_R R[\mathbf{G}]$ via the comodule map Δ_L, and denote by Q the resulting quotient. We know that the Shapiro lemma holds [10, I.4.6], namely that

$$H^i(\mathbf{G}, \mathrm{ind}_{\mathbf{e}_R}^{\mathbf{G}}(L)) \xrightarrow{\sim} H^i(\mathbf{e}_R, L) = 0 \ \forall \ i > 0.$$

The long exact sequence for cohomology for $0 \to L \to \mathrm{ind}_1^{\mathbf{G}}(L) \to Q \to 0$ yields an isomorphism $H^d(\mathbf{G}, Q) \xrightarrow{\sim} H^{d+1}(\mathbf{G}, L)$, whence the result. □

3. Vanishing of Hochschild cohomology

The proof of Theorem 1.2 proceeds by considering successively the cases of fields, artinian rings, complete noetherian rings and local rings. We begin by recalling and collecting a few facts about linearly reductive groups.

3.1. Linearly reductive groups. Let k be a field. A k–group $\mathbf{G}$ is *linearly reductive* if it is affine and its corresponding category $\mathrm{Rep}_k(\mathbf{G})$ of finite dimensional linear representations is semisimple. We recall the following criterion.

Proposition 3.1. *Let $\mathbf{G}$ be an affine k-group. Then the following are equivalent:*

1. $\mathbf{G}$ *is linearly reductive.*
2. *Every linear representation of* $\mathbf{G}$ *is semisimple.*
3. $H^1(\mathbf{G}, V) = 0$ *for any finitely dimensional* $\mathbf{G}$-k*–module* V*.*
4. $H^1(\mathbf{G}, V) = 0$ *for any* $\mathbf{G}$-k*–module* V*.*
5. $H^i(\mathbf{G}, V) = 0$ *for any* $\mathbf{G}$-k*–module* V *and all* $i > 0$*.*

(6) *k is a direct summand of the $\mathbf{G}$-k–module structure on $k[\mathbf{G}]$ corresponding to the right regular representation.*
(7) *k is an injective $\mathbf{G}$-k–module.*

Proof. For the equivalence of the first five assertions, see [4, II prop. 3.3.7].

(2) $\Longrightarrow$ (6): This follows from the fact that k is a submodule of $k[\mathbf{G}]$.

(6) $\Longrightarrow$ (7): See [10, I.3.10].

(7) $\Longrightarrow$ (3): If k is an injective $\mathbf{G}$-k–module, the group $H^1(\mathbf{G}, V) = \mathrm{Ext}^1_{\mathbf{G}}(k, V)$ vanishes for each finite dimensional $\mathbf{G}$-k–module V (by duality). □

The property of being linearly reductive behaves well with respect to base change.

PROPOSITION 3.2. *Let $\mathbf{G}$ be an affine algebraic k–group. Let K/k be a field extension. For the K-group $\mathbf{G}_K$ to be linearly reductive it is necessary and sufficient that $\mathbf{G}$ be linearly reductive. In particular, if $\overline{k}$ is an algebraic closure of k and $\mathbf{G}_{\overline{k}}$ is a linearly reductive $\overline{k}$-group, then $\mathbf{G}$ is linearly reductive.*

This result is certainly known. We give three different proofs for the sake of completeness.

Proof. (1) As observed by S. Donkin in §2 of [5], $\mathbf{G}$ is linearly reductive if and only if the injective envelope $E_{\mathbf{G}}(k)$ of the trivial $\mathbf{G}$-k–module k coincides with k. One also knows [*ibid.* eq. (1)] that $E_{\mathbf{G}_K}(K) = E_{\mathbf{G}}(k) \otimes_k K$. The proposition follows.

(2) Assume that the k–group $\mathbf{G}$ is linearly reductive. By the criterion (6) of Proposition 3.1, k is a direct summand of $k[\mathbf{G}]$. Hence K is a direct summand of $K[\mathbf{G}]$ and therefore $\mathbf{G}_K$ is linearly reductive. Conversely if $\mathbf{G}_K$ is linearly reductive and V is a $\mathbf{G}$-k–module, then by Lemma 2.1.2 we have $H^1(\mathbf{G}, V) \otimes_k K \simeq H^1(\mathbf{G}_K, V_K) = 0$.

(3) The argument depends on the characteristic of k. One uses [4] IV prop. 3.3 in characteristic 0, and Nagata's theorem (*ibid.* théorème 3.6) if the characteristic is positive.

□

REMARK 3.3. Let $\mathbf{G}$ be an affine algebraic group over a field k. Let S be a scheme over k, and consider the S-group scheme $\mathbf{G}_S = \mathbf{G} \times_k S$. The fibers of $\mathbf{G}_S$ are then affine algebraic groups over the corresponding residue fields. It follows from the previous proposition that if *any* of the fibers is linearly reductive, then *all* fibers are linearly reductive.

The following useful statement seems to have gone unnoticed in the literature.

PROPOSITION 3.4. *Let $1 \to \mathbf{G}_1 \to \mathbf{G}_2 \to \mathbf{G}_3 \to 1$ be an exact sequence of affine algebraic k–groups. Then the following are equivalent:*

(1) *$\mathbf{G}_2$ is linearly reductive ,*
(2) *$\mathbf{G}_1$ and $\mathbf{G}_3$ are linearly reductive.*

Proof. (1) $\Longrightarrow$ (2): Since $\mathbf{G}_2/\mathbf{G}_1$ is affine, we know that the induction functor $\mathrm{ind}_{\mathbf{G}_1}^{\mathbf{G}_2}$ is exact [10, I.5.13], and therefore Shapiro's lemma hence holds (*ibid.* I.4.6). Thus

$$H^*(\mathbf{G}_2, \mathrm{ind}_{\mathbf{G}_1}^{\mathbf{G}_2}(V_1)) \xrightarrow{\sim} H^*(\mathbf{G}_1, V_1)$$

for any $\mathbf{G}_1$-k–module V_1. Thus $H^i(\mathbf{G}_1, V_1) = 0$ for $i > 0$ and Proposition 3.1 shows that $\mathbf{G}_1$ is linearly reductive. Since the functor $\mathrm{ind}_{\mathbf{G}_1}^{\mathbf{G}_2}$ is exact we can use the Hochschild-Serre spectral sequence in this framework (*ibid.* I.6.6.) Given a finite dimensional representation V_3 of $\mathbf{G}_3$, this spectral sequence reads as follows

$$E_{p,q}^2 = H^p(\mathbf{G}_3, H^q(\mathbf{G}_1, V_3)) \Longrightarrow H^{p+q}(\mathbf{G}_2, V_3).$$

Since $\mathbf{G}_1$ is linearly reductive $H^q(\mathbf{G}_1, V_3)$ vanishes for all $q \geq 1$, hence $H^n(\mathbf{G}_3, V_3) \xrightarrow{\sim} H^n(\mathbf{G}_2, V_3)$ for all $n \geq 0$. Since $H^1(\mathbf{G}_2, V_3) = 0$, $H^1(\mathbf{G}_3, V_3) = 0$ and we conclude that $\mathbf{G}_3$ is linearly reductive by Proposition 3.1.

(2) $\Longrightarrow$ (1): Assume that $\mathbf{G}_1$ and $\mathbf{G}_3$ are linearly reductive. Let us check that $\mathbf{G}_2$ is linearly reductive by again appealing to Proposition 3.1. Let V_2 be a finitely dimensional representation of $\mathbf{G}_2$. Again we can use the Hochschild-Serre spectral sequence which now reads as follows

$$E_{p,q}^2 = H^p(\mathbf{G}_3, H^q(\mathbf{G}_1, V_2)) \Longrightarrow H^{p+q}(\mathbf{G}_2, V_2).$$

The only non zero E_2-term is $H^0(\mathbf{G}_3, H^0(\mathbf{G}_1, V_2)) = H^0(\mathbf{G}_2, V_2)$. Hence $H^i(\mathbf{G}_2, V_2) = 0$ for $i > 0$. Thus $\mathbf{G}_2$ is linearly reductive. □

Note that Proposition 3.4 agrees with Nagata's theorem characterizing linearly reductive groups over an algebraically closed field [11].

Proposition 3.5. *Let* $\mathbf{G}$ *be an affine algebraic* k*–group which admits a composition series where each of the factors is of one of the following types:*

(i) algebraic k*-groups of multiplicative type,*

(ii) finite étale k*-group whose order is invertible in* k*,*

(iii) reductive k*-group if* k *is of characteristic zero.*

Then $\mathbf{G}$ *is linearly reductive.*

Proof. By Proposition 3.4 we are reduced to verifying the result for each of the given types. Proposition 3.2 permits us to assume that the base field k is algebraically closed. Case (i) is then that of a diagonalizable k–group [17, th. I.5.3.3]. Case (ii) is the case of a finite constant group of invertible order (Maschke's theorem, see [11]). Case (iii) is a classical result due to H. Weyl (see [18, th. 27.3.3]). □

3.2. Finiteness considerations. Recall that for arbitrary groups schemes $\mathbf{G}$ and $\mathbf{H}$ over a scheme S the functor $\mathbf{Hom}_{S-\mathrm{gp}}(\mathbf{G},\mathbf{H}) : Sch/S \to Sets$ is defined by

$$T \mapsto \mathbf{Hom}_{S-gr}(\mathbf{G},\mathbf{H})(T) = \mathrm{Hom}_{T-gr}(\mathbf{G}_T,\mathbf{H}_T)$$

for all schemes T/S.
The following observations will be repeatedly used in the proofs of our main results. For the remainder of this section we assume that $\mathbf{G}$ and $\mathbf{H}$ are *finitely presented* group schemes over S.
Assume that $T = \mathrm{Spec}(B)$ is an affine scheme (in the absolute sense) over S. In what follows we will encounter ourselves several times in the situation where B is given to us as an inductive limit

$$B = \varinjlim_{\lambda\in\Lambda} B_\lambda \tag{3.1}$$

over some directed set Λ. Note that the $\mathrm{Spec}(B_\lambda)$ do not in general have any natural structure of schemes over S.
Under these assumptions the group schemes $\mathbf{G}_B$ and $\mathbf{H}_B$ are defined over some B_μ by [17, VI_B 10.10.3], i.e. there exists $\mu \in \Lambda$ and finitely presented B_μ–group schemes $\mathbf{G}_\mu$ and $\mathbf{H}_\mu$ such that

$$\mathbf{G}_B = \mathbf{G}_\mu \times_{B_\mu} B \text{ and } \mathbf{H}_B = \mathbf{H}_\mu \times_{B_\mu} B. \tag{3.2}$$

Furthermore if either $\mathbf{G}$ is affine (resp. flat, smooth), so is $\mathbf{G}_\mu$ by [9] 8.10.5 (resp. 11.2.6, 17.7.8). Similarly for $\mathbf{H}$.
It follows from the very definition that
(3.3)

$$\mathbf{Hom}_{S-gp}(\mathbf{G},\mathbf{H})(B) = \mathbf{Hom}_{B-gp}(\mathbf{G}_B,\mathbf{H}_B) = \mathbf{Hom}_{B_\mu-gp}(\mathbf{G}_\mu,\mathbf{H}_\mu)(B)$$

For all $\lambda \geq \mu$ define $\mathbf{G}_\lambda = \mathbf{G}_\mu \times_{B_\mu} B_\lambda$ and $\mathbf{H}_\lambda = \mathbf{H}_\mu \times_{B_\mu} B_\lambda$. Then the canonical map

$$\varinjlim_{\lambda\geq\mu} \mathrm{Hom}_{B_\lambda-gr}(\mathbf{G}_\lambda,\mathbf{H}_\lambda) \to \mathrm{Hom}_{B-gr}(\mathbf{G}_B,\mathbf{H}_B). \tag{3.4}$$

is bijective by [17, VI_B 10.10.2] (see also [9, théorème 8.8.2]).

Remark 3.6. From the foregoing it follows that if $u, v : \mathbf{G} \to \mathbf{H}$ are two homomorphisms of S-group schemes, then there exist $\mu \in \Lambda$ such that u_B and v_B are obtained by the base change $B_\mu \to B$ from group homomorphisms $u_\mu, v_\mu \in \mathrm{Hom}_{B_\mu-gp}(\mathbf{G}_\mu,\mathbf{H}_\mu)$.

Lemma 3.7. *Let L be a $\mathbf{G}_B$-B-module which is of finite presentation as a B-module. Then there exists an index μ and a $\mathbf{G}_\mu$-B_μ-module L_μ which is finitely presented as a B_μ–module such that $L = L_\mu \otimes_{B_\mu} B$.*

Proof. According to (3.2) and proposition 8.9.1 (ii) of [9] we can find an index α, a B_α–group $\mathbf{G}_\alpha$ and a finitely presented B_α-module L_α such that $\mathbf{G}_\alpha \times_{B_\alpha} B = \mathbf{G}_B$ and $L_\alpha \otimes_{B_\alpha} B = L$. For $\lambda \geq \alpha$, we set $\mathbf{G}_\lambda = \mathbf{G}_\alpha \times_{B_\alpha} B_\lambda$ and $L_\lambda = L \otimes_{B_\alpha} B_\lambda$.

By [9] 8.5.2.2, we have an isomorphism

$$\varinjlim_{\lambda\geq\alpha} \mathrm{Hom}_{B_\lambda}\big(L_\lambda, L_\lambda\otimes_{B_\lambda} B_\lambda[\mathbf{G}_\lambda]\big) \xrightarrow{\sim} \mathrm{Hom}_B\Big(\varinjlim_{\lambda\geq\alpha} L_\lambda, \varinjlim_{\lambda\geq\alpha} L_\lambda\otimes_{B_\lambda} B_\lambda[\mathbf{G}_\lambda]\Big) = \\ = \mathrm{Hom}_B(L, L\otimes_B B[\mathbf{G}]).$$

It follows that there exists $\lambda\geq\alpha$ such that the B–module homomorphisms $\Delta_L : L\to L\otimes_B B[\mathbf{G}]$ is obtained by the base change $B_\lambda\to B$ from a B_λ–module homomorphism $\Delta_{L_\lambda} : L_\lambda \to L_\lambda\otimes_{B_\lambda} B_\lambda[\mathbf{G}_\lambda]$. The same reasoning applied to $\mathrm{Hom}_B(L, L\otimes_B B[\mathbf{G}]\otimes_B B[\mathbf{G}])$ and $\mathrm{Hom}_B(L,L)$ show that there exists $\mu\geq\lambda$ such that Δ_{L_λ} satisfies conditions (CM1) and (CM2) after applying the base change $B_\lambda\to B_\mu$. □

3.3. Proof of Theorem 1.2. We assume throughout that $i>0$.
Case (i) *R a noetherian ring*: The proof is a classical dévissage argument [8, §7.2.7)].

Case of R a field: The result follows from Proposition 3.1.

Case of R local artinian: Let $\mathfrak{m}$ be the maximal ideal of R, and k the corresponding residue field. By our assumption on the closed fibers of $\mathbf{G}$ the k–group $\mathbf{G}_k = \mathbf{G}\times_R k$ is linearly reductive.
Fix an integer $e\geq 2$ such that $\mathfrak{m}^e = 0$. Thus there exists a smallest integer $j=j(L)$ such that $0<j\leq e$ and $\mathfrak{m}^j L = 0$. We reason by induction on j.
If $j=1$ then $\mathfrak{m}L=0$. By Lemma 2.1.1, we have $H^i(\mathbf{G},L)\cong H^i(\mathbf{G}_k,L_k)$ for all i, and $H^i(\mathbf{G}_k,L_k)$ vanishes since $\mathbf{G}_k$ is linearly reductive. Assume now that $H^i(\mathbf{G},M)=0$ for all $\mathbf{G}$-R–modules M satisfying $\mathfrak{m}^jM=0$. If $\mathfrak{m}^{j+1}L=0$, we consider the exact sequence

$$0\to\mathfrak{m}L\to L\to L'\to 0$$

of $\mathbf{G}$-R–modules. Observe that $\mathfrak{m}^j(\mathfrak{m}L)=0$ and that $\mathfrak{m}L'=0$. This sequence gives rise to the long exact sequence of cohomology [10, I.4.2]

$$\cdots\to H^i(\mathbf{G},\mathfrak{m}L)\to H^i(\mathbf{G},L)\to H^i(\mathbf{G},L')\to\cdots$$

We have $H^i(\mathbf{G},L')=0$ by the case $j=1$ and $H^i(\mathbf{G},\mathfrak{m}L)=0$ by the induction hypothesis. Thus $H^i(\mathbf{G},L)=0$ as desired.

Case of R local and complete: We denote by $\mathfrak{m}$ the maximal ideal of R, and set $R_n = R/\mathfrak{m}^{n+1}$ for all $n\geq 0$.
By Lemma 2.3 it will suffice to establish the case $i=1$. Furthermore, Proposition 2.2 together with Lemma 2.1.3 allows us to assume that L is finitely generated over R. By the Artin-Rees lemma [7, cor. 0.7.3.3] we have a natural isomorphism

$$L\xrightarrow{\sim}\varprojlim_n L_n$$

where $L_n = L\otimes_R R_n$. We are given a cocycle $z\in Z^1(\mathbf{G},L)$ and our goal is to show by using approximation that z is a coboundary. Since R_n is a local artinian ring we have $H^1(\mathbf{G}_n, L_n) = 0$ where $\mathbf{G}_n = \mathbf{G} \times_R R_n$.[2] We consider the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathbf{G}, L) & \longrightarrow & L & \xrightarrow{\partial_0} & Z^1(\mathbf{G}, L) & \longrightarrow & H^1(\mathbf{G}, L) \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^0(\mathbf{G}_n, L_n) & \longrightarrow & L_n & \xrightarrow{\partial_{0,n}} & Z^1(\mathbf{G}_n, L_n) & \longrightarrow & H^1(\mathbf{G}_n, L_n) = 0.
\end{array}$$

Then the images z_n of z in the $Z^1(\mathbf{G}_n, L_n)$ define elements $b_n \in B^1(\mathbf{G}_n, L_n) \subset Z^1(\mathbf{G}, L_n)$. We look now

$$0 \longrightarrow H^0(\mathbf{G}_n, L_n) \longrightarrow L_n \xrightarrow{\partial_{0,n}} B^1(\mathbf{G}_n, L_n) \longrightarrow 0.$$

Since $H^0(\mathbf{G}_n, L_n)$ is a finitely generated R_n-module, it is artinian. Hence the system $\big(H^0(\mathbf{G}, L_n)\big)_{n \geq 0}$ satisfies the Mittag-Leffler condition [7, cor. 0.13.2.2]. We get then an exact sequence (*ibid.* prop. 13.2.2)

$$0 \longrightarrow \varprojlim_n H^0(\mathbf{G}_n, L_n) \longrightarrow \varprojlim_n L_n \longrightarrow \varprojlim_n B^1(\mathbf{G}_n, L_n) \longrightarrow 0.$$

It follows that there exists $l \in L$ such that $z = \partial_0(l)$ modulo $\mathfrak{m}^{n+1}$ for all $n \geq 0$. Thus $z = \partial_0(l)$ and therefore the image of z in $H^1(\mathbf{G}, L)$ vanishes.

Case of R local: We know that the completion $\widehat{R}$ of R is local noetherian and faithfully flat over R [7, cor. 0.7.3.5]). By Lemma 2.1.2, we have

$$H^i(\mathbf{G}, L) \otimes_R \widehat{R} \xrightarrow{\sim} H^i(\mathbf{G}_{\widehat{R}}, L_{\widehat{R}}).$$

The right hand side vanishes by the local complete case, hence $H^i(\mathbf{G}, L) = 0$ by faithfully flat descent.

Case of R arbitrary noetherian: By the same reasoning used in the previous case we have $H^i(\mathbf{G}, L) \otimes_R R_{\mathfrak{m}} = 0$ for any maximal ideal $\mathfrak{m}$ of R. Thus $H^i(\mathbf{G}, L) = 0$.

Case (ii) *The group $\mathbf{G}$ is finitely presented as an R-scheme and L is a direct limit of $\mathbf{G}$-R-modules which are finitely presented as R–modules*: By Lemma 2.1.3 we may assume that L is a finitely presented R–module. The same reasoning used in the final step of the noetherian case allows us to assume that R is a local ring. Let $\mathfrak{m}$ be the maximal ideal of R and k its residue field. We consider the standard filtration $R = \varinjlim_\lambda R_\lambda$ of R by its finitely generated (hence noetherian) $\mathbb{Z}$–subalgebras. For each λ, we consider the prime ideal $\mathfrak{p}_\lambda := \mathfrak{p} \cap R_\lambda$ of R_λ, and the corresponding local ring $R'_\lambda := (R_\lambda)_{\mathfrak{p}_\lambda}$ whose maximal ideal $\mathfrak{p}_\lambda R'_\lambda$ we denote by $\mathfrak{m}_\lambda$. Note that the residue field k_λ of R'_λ is a subfield of k. We have $R = \varinjlim_\lambda R'_\lambda$ and the following commutative diagram

[2]One of course verifies that the R_n-groups $\mathbf{G}_n$ satisfy the assumptions of the theorem. Similar considerations apply to the reductions that follow.

$$\begin{array}{ccc} R'_\lambda & \longrightarrow & R \\ \big\downarrow & & \big\downarrow \\ k_\lambda = R'_\lambda/\mathfrak{m}_\lambda & \longrightarrow & R/\mathfrak{m} = k. \end{array}$$

We now apply the considerations of §3.2 to the case when $S = \operatorname{Spec}(R)$, $B = R$ and $B_\lambda = R'_\lambda$. This yield the existence of an R'_μ, an affine, flat and finitely presented R'_μ– group scheme $\mathbf{G}_\mu$ and a $\mathbf{G}_\mu$-R'_μ-module L_μ such that $\mathbf{G} = \mathbf{G}_\mu \times_{R'_\mu} R$ and $L = L_\mu \otimes_{R'_\mu} R$. By Lemma 2.1.4, we have

$$H^i(\mathbf{G}, L) = \varinjlim_{\lambda \geq \mu} H^i(\mathbf{G}_\mu \times_{R'_\mu} R'_\lambda, L_\mu \otimes_{R'_\mu} R'_\lambda). \tag{3.5}$$

We also have by the transitivity of base change that

$$(\mathbf{G}_\mu \times_{R'_\mu} k_\mu) \times_{k_\mu} k \simeq \mathbf{G}_\mu \times_{R'_\mu} k \simeq (\mathbf{G}_\mu \times_{R'_\mu} R) \times_R k = \mathbf{G} \times_R k. \tag{3.6}$$

From our assumptions on the R–group $\mathbf{G}$ it follows that the k–group $\mathbf{G} \times_R k$ is affine algebraic and linearly reductive. It then follows from (3.6) and Proposition 3.2 that the k_μ–algebraic group $\mathbf{G}_\mu \times_{R'_\mu} k_\mu$ is linearly reductive as well. This shows that the R'_μ–group $\mathbf{G}_\mu$ satisfies the assumption of the first part of the theorem. Similar considerations apply to the R'_λ–group $\mathbf{G}_\mu \times_{R'_\mu} R'_\lambda$ for all $\lambda \geq \mu$. Thus the noetherian case that we have already established shows, with the aid of (3.5), that $H^i(\mathbf{G}, L) = 0$. □

4. Rigidity and deformation theory

4.1. Locally finitely presented S–functors. Let S be a scheme and $F : Sch/S \to Sets$ a contravariant functor. We recall the following definitions:

- F is locally of finite presentation over S if for every filtered inverse system of affine S-schemes $\operatorname{Spec}(B_i)$, the canonical morphism

$$\varinjlim F(B_i) \to F(\varinjlim B_i)$$

is an isomorphism [3, §8.3].[3]

- F is formally smooth (resp. formally unramified, formally étale) if for any affine scheme $\operatorname{Spec}(B)$ over S and any subscheme $\operatorname{Spec}(B_0)$ of $\operatorname{Spec}(B)$ defined by a nilpotent ideal I of B, the map

$$F(B) \to F(B_0)$$

is surjective (resp. injective, bijective) [17, XI.1.1].

Note that all these definitions are stable by an arbitrary base change $T \to S$. In the second definition, we can require furthermore that $I^2 = 0$. The following lemma is elementary.

[3]This reference has assumptions on the nature of S related to Artin's approximation theorem which are relevant to their work, but not to ours. As it is customary, given an affine scheme $\operatorname{Spec}(B)$ over S, we write $F(B)$ instead of $F\big(\operatorname{Spec}(B)\big)$.

Lemma 4.1. *Assume that F is locally of finite presentation over S. Consider a field extension K/k over S, that is morphisms* $\mathrm{Spec}(K) \to \mathrm{Spec}(k) \to S$. *Assume that k is separably closed and K is a separable field extension of k. Then the map $F(k) \to F(K)$ is injective.*

Remark 4.2. *If k is algebraically closed, any field extension K/k is separable, hence the Lemma applies.*

Proof. We may assume without loss of generality that $S = \mathrm{Spec}(k)$. We are given two elements $\alpha, \beta \in F(k)$ with same image in $F(K)$. Since K is the inductive limit of its finitely generated subalgebras, there exists a finitely generated k–algebra A such that α and β have same image in $F(A)$. Since K/k is separable, the finitely generated k–subalgebras of K are separable over k. Hence A is integral and absolutely reduced (i.e. $A \otimes_k \overline{k}$ is reduced), and therefore the affine variety $\mathrm{Spec}(A)$ admits a k–point [1, AG.13.3]. In other words, the ring homomorphim $k \to A$ admits a section. This, in turn, induces a section of the group homomorphism $F(k) \to F(A)$, hence $\alpha = \beta$ in $F(k)$. □

4.2. Formal étalness. We recall the following crucial statement of deformation theory for group scheme homomorphisms due to Demazure.

Theorem 4.3. *(*[17, cor. III.2.6]*) Let $\mathbf{G}$ and $\mathbf{H}$ be group schemes over a scheme S. Assume that $\mathbf{G}$ is affine (in the absolute sense) and flat, and that $\mathbf{H}$ is smooth. Let S_0 be a closed subscheme of S defined by an ideal I of $\mathcal{O}_S$ such that $I^2 = 0$. We set $\mathbf{G}_0 = \mathbf{G} \times_S S_0$ and $\mathbf{H}_0 = \mathbf{H} \times_S S_0$. Let $f_0 : \mathbf{G}_0 \to \mathbf{H}_0$ be a homomorphism of S_0-groups, and let $\mathbf{G}_0$ act on* $\mathrm{Lie}(\mathbf{H}_0)$ *via f_0 and the adjoint representation of $\mathbf{H}_0$. Then*

(1) *If $H^2\big(\mathbf{G}_0, \mathrm{Lie}(\mathbf{H}_0) \otimes_{\mathcal{O}_{S_0}} I\big) = 0$ the homomorphism f_0 lifts to an S–group homomorphism $f : \mathbf{G} \to \mathbf{H}$.*
(2) *If $H^1\big(\mathbf{G}_0, \mathrm{Lie}(\mathbf{H}_0) \otimes_{\mathcal{O}_{S_0}} I\big) = 0$, then any two liftings f and f' of f_0 as in (1) are conjugate under an element of* $\ker\big(\mathbf{H}(S) \to \mathbf{H}(S_0)\big)$. *More precisely $f' = \mathrm{int}(h)f$ for some $h \in \ker\big(\mathbf{H}(S) \to \mathbf{H}(S_0)\big)$.* □

Combined with the vanishing result given by Theorem 1.2 we are very close to the completion of the proof of our main result. The missing ingredient is some detailed information pertaining to the nature of certain functors related to homomorphisms between group schemes.

Let $\mathbf{G}$ and $\mathbf{H}$ be group schemes over a scheme S. The functor $\mathbf{Hom}_{S-\mathrm{gp}}(\mathbf{G},\mathbf{H})$ was already defined in §3.2. Any element $h \in \mathbf{H}(T)$ defines an inner automorphism $\mathrm{int}(t) \in \mathrm{Aut}_{T-gr}(\mathbf{H}_T)$, and this last group acts naturally on the set $\mathbf{Hom}_{S-gr}(\mathbf{G},\mathbf{H})(T)$. This allows us to define a new functor $\overline{\mathbf{Hom}}_{S-gr}(\mathbf{G},\mathbf{H}) : Sch/S \to Sets$ by

$$T \mapsto \overline{\mathbf{Hom}}_{S-gr}(\mathbf{G},\mathbf{H})(T) = \mathrm{Hom}_{T-gr}(\mathbf{G}_T,\mathbf{H}_T)/\mathbf{H}(T).$$

The final functor which is relevant to us is the transporter of two elements of $\mathrm{Hom}_{S-gr}(\mathbf{G},\mathbf{H})$. Let $u, v : \mathbf{G} \to \mathbf{H}$ two homomorphisms of S-group schemes.

Recall the subfunctor $\mathbf{Transp}(u,v)$ of $\mathbf{H}$ defined by

$$T \to \mathbf{Transp}(u,v)(T) = \{h \in \mathbf{H}(T) \mid u_T = \mathrm{int}(h)\, v_T\}.$$

We begin with an easy observation.

Lemma 4.4. *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be finitely presented group schemes over* S, *and let* $u, v \in \mathrm{Hom}_{S-gr}(\mathbf{G},\mathbf{H})$. *The* S*-functors* $\mathbf{Hom}_{S-gr}(\mathbf{G},\mathbf{H})$, $\overline{\mathbf{Hom}}_{S-gr}(\mathbf{G},\mathbf{H})$ *and* $\mathbf{Transp}(u,v)$ *are locally of finite presentation.*

Proof. For every filtered inverse system of affine schemes $\mathrm{Spec}(B_\lambda)$ over S based on some directed set Λ, and we have to show that the canonical morphisms

$$\varinjlim \mathbf{Hom}_{S-gr}(\mathbf{G},\mathbf{H})(B_\lambda) \to \mathbf{Hom}_{S-gr}(\mathbf{G},\mathbf{H})(\varinjlim B_\lambda) = \mathbf{Hom}_{S-gr}(\mathbf{G},\mathbf{H})(B),$$

$$\varinjlim \overline{\mathbf{Hom}}_{S-gr}(\mathbf{G},\mathbf{H})(B_\lambda) \to \overline{\mathbf{Hom}}_{S-gr}(\mathbf{G},\mathbf{H})(\varinjlim B_\lambda) = \overline{\mathbf{Hom}}_{S-gr}(\mathbf{G},\mathbf{H})(B),$$

and

$$\varinjlim \mathbf{Transp}(u,v)(B_\lambda) \to \mathbf{Transp}(u,v)(\varinjlim B_\lambda) = \mathbf{Transp}(u,v)(B)$$

are bijective. Taking into account (3.2), (3.3) and (4.1) we may replace S by $\mathrm{Spec}(B_\mu)$ for some suitable index $\mu \in \Lambda$, and replace Λ by the subset of Λ consisting of all indices $\lambda \geq \mu$. Denote $\mathbf{G} \times_{B_\mu} B_\lambda$ and $\mathbf{H} \times_{B_\mu} B_\lambda$ by $\mathbf{G}_\lambda$ and $\mathbf{H}_\lambda$ respectively, just as we did in §3.2. Then (3.4) shows that $\mathbf{Hom}_{S-gr}(\mathbf{G},\mathbf{H})$ is locally of finite presentation.
As for the second assertion, we look in view of (3.3) at the map

$$\varinjlim \mathrm{Hom}_{B_\lambda-gr}(\mathbf{G}_\lambda,\mathbf{H}_\lambda)/\mathbf{H}_\lambda(B_\lambda) \to \mathrm{Hom}_{B-gr}(\mathbf{G}_B,\mathbf{H}_B)/\mathbf{H}_B(B)$$

which is already known to be surjective. For the injectivity, we are given $\phi_\alpha, \phi'_\alpha \in \mathrm{Hom}_{B_\alpha-gr}(\mathbf{G}_\alpha,\mathbf{H}_\alpha)$ for some $\alpha \geq \mu$ whose images ϕ, ϕ' in $\mathrm{Hom}_{B-gr}(\mathbf{G}_B,\mathbf{H}_B)$ are conjugated under $\mathbf{H}_B(B) = \mathbf{H}(B)$. Since $\mathbf{H}$ is of finite presentation $\varinjlim \mathbf{H}(B_\lambda) \xrightarrow{\sim} \mathbf{H}(B)$. So there exists $\beta \geq \alpha$ and $h_\beta \in \mathbf{H}(B_\beta)$ such that $\phi = \mathrm{int}(h)\phi'$ where h stands for the image of h_β in $\mathbf{H}(B)$. By (3.4) there exists $\gamma \geq \beta$ such that $\phi_\alpha \times_{B_\alpha} \mathrm{id}_{B_\gamma} = \mathrm{int}(h_\gamma)(\phi'_\alpha \times_{B_\alpha} \mathrm{id}_{B_\gamma})$, where h_γ is the image of h_β in $\mathbf{H}(B_\gamma)$. In other words, $\phi_\alpha, \phi'_\alpha$ map to the same element of $\mathrm{Hom}_{B_\gamma-gr}(\mathbf{G}_\gamma,\mathbf{H}_\gamma)/\mathbf{H}_\gamma(B_\gamma)$, hence define the same element in the inductive limit $\varinjlim \mathrm{Hom}_{B_\lambda-gr}(\mathbf{G}_\lambda,\mathbf{H}_\lambda)/\mathbf{H}_\lambda(B_\lambda)$. We conclude that $\overline{\mathbf{Hom}}_{S-gr}(\mathbf{G},\mathbf{H})$ is locally of finite presentation.
Finally we look at the case of the transporter. Assume that $h \in \mathbf{H}(B)$ is such that $u_B = \mathrm{int}(h)v_B$. Since $\mathbf{H}$ is finitely presented there exists $\alpha \geq \mu$ and an element $h_\alpha \in \mathbf{H}(B_\alpha)$ whose image in $\mathbf{H}(B)$ is h. Then the two elements u_α and $\mathrm{int}(h_\alpha)v_\alpha$ of $\mathrm{Hom}_{B_\alpha}(\mathbf{G}_\alpha,\mathbf{H}_\alpha)$ map to the same element of $\mathrm{Hom}_B(\mathbf{G}_B,\mathbf{H}_B)$. By (3.4) there exists $\beta \geq \alpha$ such that $u_\beta = \mathrm{int}(h_\beta)v_\beta$ (where the subindex β denotes the image of the element in question under the map $B_\alpha \to B_\beta$). This shows that our map is surjective. Note that from the definition of the transporter it follows that

$$(4.1) \qquad \mathbf{Transp}(u,v)(B) = \mathbf{Transp}(u_B,v_B)(B) = \mathbf{Transp}(u_\mu,v_\mu)(B)$$

Injectivity is clear since for all $\lambda \geq \mu$ we have $\mathbf{Transp}(u_\mu, v_\mu)(B_\lambda) \subset \mathbf{H}_\mu(B_\lambda)$ and $\mathbf{H}_\mu$ is of finite presentation. □

Theorem 4.5. *Let S be a scheme and let $\mathbf{G}$ and $\mathbf{H}$ be finitely presented group schemes over S. Assume that $\mathbf{G}$ is affine (in the absolute sense) and flat, and that $\mathbf{H}$ is smooth. Assume that for all $s \in S$ the fiber $\mathbf{G}_s$ is linearly reductive (as an affine algebraic group over the residue field $\kappa(s)$ of s). Then*

(1) The functor $\mathbf{Hom}(\mathbf{G}, \mathbf{H})$ is formally smooth.

(2) The functor $\overline{\mathbf{Hom}}(\mathbf{G}, \mathbf{H})$ is formally étale.

(3) If $u, v : \mathbf{G} \to \mathbf{H}$ are two homomorphisms of S-group schemes, the subfunctor $\mathbf{Transp}(u, v)$ of $\mathbf{H}$ is formally smooth.

The case when $\mathbf{G}$ is of multiplicative type is an important result of Grothendieck [17, XI prop. 2.1]. If S is of characteristic zero and $\mathbf{G}$ is reductive, the first statement is due to Demazure [17, XXIV prop. 7.3.1.a].

Proof. We note that if $\mathbf{Hom}_{S-gr}(\mathbf{G}, \mathbf{H})$ is formally smooth, then $\overline{\mathbf{Hom}}_{S-gr}(\mathbf{G}, \mathbf{H})$ is formally smooth as well. As a consequence, to establish (1) and (2) it will suffice to prove that $\mathbf{Hom}_{S-gr}(\mathbf{G}, \mathbf{H})$ is formally smooth and that $\overline{\mathbf{Hom}}_{S-gr}(\mathbf{G}, \mathbf{H})$ is formally unramified. We are given an affine scheme $\mathrm{Spec}(B)$ over S, and a closed subscheme $\mathrm{Spec}(B_0)$ defined by an ideal I of B satisfying $I^2 = 0$, and we need to show that

(I) $\mathbf{Hom}_{S-gr}(\mathbf{G}, \mathbf{H})(B) \to \mathbf{Hom}_{S-gr}(\mathbf{G}, \mathbf{H})(B_0)$ is surjective,

(II) $\overline{\mathbf{Hom}}_{S-gr}(\mathbf{G}, \mathbf{H})(B) \to \overline{\mathbf{Hom}}_{S-gr}(\mathbf{G}, \mathbf{H})(B_0)$ is injective, and

(III) $\mathbf{Transp}(u, v)(B) \to \mathbf{Transp}(u, v)(B_0)$ is surjective.

Proof of (I) *and* (II): By the first equality of (3.3) we may assume with no loss of generality that $S = \mathrm{Spec}(B)$. We claim that, with the obvious adaptations to the notation of Theorem 4.3,

$$H^i\big(\mathbf{G}_0, \mathrm{Lie}(\mathbf{H}_0) \otimes_{B_0} I\big) = 0 \text{ for all } i > 0. \tag{4.2}$$

Write $B = \varinjlim B_\lambda$ where the limit is taken over all finitely generated $\mathbb{Z}$–subalgebras (hence noetherian) B_λ of B. Then $J_\lambda := I \cap B_\lambda$ is an ideal of B_λ such that $J_\lambda^2 = 0$ and $I = \varinjlim J_\lambda$. Consider the trivial $\mathbf{G}_0$-B_0 module

$$I_\lambda := J_\lambda \otimes_{B_\lambda} B_0.$$

Since J_λ is a B_λ-module of finite presentation, I_λ is a B_0-module of finite presentation. We have an isomorphism of B_0-modules

$$\varinjlim I_\lambda \xrightarrow{\sim} I$$

hence an isomorphism of $\mathbf{G}_0$-B_0–modules

$$\varinjlim \big(\mathrm{Lie}(\mathbf{H}_0) \otimes_{B_0} I_\lambda\big) \xrightarrow{\sim} \mathrm{Lie}(\mathbf{H}_0) \otimes_{B_0} I.$$

Because $\mathbf{H}_0$ is a smooth B_0–group $\mathrm{Lie}(\mathbf{H}_0)$ is a finitely presented B_0–module (see [4] II §4.8). Since the tensor product of finitely presented modules is finitely presented, $\mathrm{Lie}(\mathbf{H}_0) \otimes_{B_0} I$ is a direct limit of $\mathbf{G}_0$-B_0–modules which are finitely presented as B_0-modules. It is clear that the B_0-groups $\mathbf{G}_0$ and $\mathbf{H}_0$ satisfy the assumptions of Theorem 1.2.2. This shows that (4.2) holds, and we can now apply Theorem 4.3 to obtain (I) and (II)

Proof of (III): For convenience we denote $\mathbf{Transp}(u,v)$ by $\mathbf{T}(u,v)$. Consider the B-group homomorphisms $u_B, v_B \in \mathrm{Hom}_{B-gr}(\mathbf{G}_B, \mathbf{H}_B)$ induced by the base change $\mathrm{Spec}(B) \to S$. By the definition of the transporter we see that $\mathbf{T}(u,v)(B) = \mathbf{T}(u_B,v_B)(B)$ and $\mathbf{T}(u,v)(B_0) = \mathbf{T}(u_B,v_B)(B_0)$ where $\mathrm{Spec}(B_0) \to \mathrm{Spec}(B)$ is the natural map. From this it follows that to establish (III) we may assume without loss of generality that $S = \mathrm{Spec}(B)$.
Let u_0 and v_0 be the elements of $\mathrm{Hom}_{B_0-gr}(\mathbf{G}_0, \mathbf{H}_0)$ induced by the base change $B \to B_0$. Let $h_0 \in \mathbf{Transp}(u_0,v_0)(B_0)$, so that $u_0 = \mathrm{int}(h_0)v_0$. Lift h_0 to an element $h' \in \mathbf{H}(B)$ (which is possible since $\mathbf{H}$ is smooth), and set $u' = \mathrm{int}(h')v_B$. Then u' and u_B map to the same element of $\mathrm{Hom}_{B_0-gr}(\mathbf{G}_0, \mathbf{H}_0)$, namely u_0. By II there exists $h'' \in \mathbf{H}(B)$ such that $u_B = \mathrm{int}(h'')u'$. Furthermore, because of (4.2) we may assume that $h'' \in \ker\big(\mathbf{H}(B) \to \mathbf{H}(B_0)\big)$. Then $h = h''h' \in \mathbf{H}(B)$ maps to h_0 and satisfies $u_B = \mathrm{int}(h)v_B$. □

REMARK 4.6. The assumption on the fibers of $\mathbf{G}$ is not superfluous. Let B = $\mathbb{C}[\epsilon]$ be the ring of dual numbers over $\mathbb{C}$, and let $S = \mathrm{Spec}(B)$. If $I = \mathbb{C}\epsilon$, then $B_0 = \mathbb{C}$. Consider now the case when $\mathbf{G} = \mathbf{G}_a$ and $\mathbf{H} = \mathbf{G}_m$ (the additive and multiplicative groups over B.)
It is well-known that $\mathbf{Hom}_{B-gr}(\mathbf{G},\mathbf{H})(B_0) = \mathrm{Hom}_{\mathbb{C}-gr}(\mathbf{G}_{a,\mathbb{C}}, \mathbf{G}_{m,\mathbb{C}})$ is trivial. On the other hand $\mathbf{Hom}_{B-gr}(\mathbf{G},\mathbf{H})(B)$ is infinite; it consists of the homomorphisms $\{\phi_z : z \in \mathbb{C}\}$ which under Yoneda correspond to the B–Hopf algebra homomorphisms $\phi_z^* : \underline{B[t^{\pm 1}]} \to B[x]$ given by $\phi_z^* : t \mapsto 1 + z\epsilon x$. Since $\mathbf{H}$ is abelian the functors $\overline{\mathbf{Hom}}_{B-gr}(\mathbf{G},\mathbf{H})$ and $\mathbf{Hom}_{B-gr}(\mathbf{G},\mathbf{H})$ coincide. The above discussion shows that $\overline{\mathbf{Hom}}_{B-gr}(\mathbf{G},\mathbf{H})$ is not formally étale.

LEMMA 4.7. *If* $\mathbf{G}$ *is essentially free over* S *(see §6), the functor* $\mathbf{Transp}(u,v)$ *is representable by a closed* S*–subscheme of* $\mathbf{H}$

Proof. Consider the two morphisms $q_1, q_2 : \mathbf{H} \to \mathbf{Hom}(\mathbf{G},\mathbf{H})$ which for all schemes T/S and $h \in \mathbf{H}(T)$ are given by and $q_1(h) = u_T$ and $q_2(h) = \mathrm{int}(h)v_T$. Since $\mathbf{G}$ is assumed essentially free over S and $\mathbf{H}$ is separated over S, Grothendieck's criterion [17, VIII.6.5.b] applied to $\mathbf{X} = \mathbf{H}$, $\mathbf{Y} = \mathbf{G}$, $\mathbf{Z} = \mathbf{H}$ shows the representability of $\mathbf{Transp}(u_1,u_2)$ by a closed S–subscheme of $\mathbf{H}$. □

COROLLARY 4.8. *Under the assumptions of Theorem 4.5, assume furthermore that* $\mathbf{G}$ *is essentially free over* S*. Let* $u, v : \mathbf{G} \to \mathbf{H}$ *be two homomorphisms of* S*-group schemes. Then the* S*-functor* $\mathbf{Transp}(u,v)$ *is representable by a smooth closed* S*-subscheme of* $\mathbf{H}$*. In particular, if* $u = v$*, then the centralizer subfunctor* $\mathbf{Centr}(u)$ *of* $\mathbf{H}$ *is representable by a smooth closed subscheme of* $\mathbf{H}$*.*

Proof. By the last Lemma the S-functor $\mathbf{Transp}(u,v)$ is representable by a closed subscheme of $\mathbf{H}$, which is locally of finite presentation by Lemma 4.4

and [9] 8.14.2, and formally smooth by Theorem 4.5. Thus $\mathbf{Transp}(u,v)$ is a smooth scheme over S. □

Corollary 4.9. *Under the assumptions of Theorem 4.5, assume furthermore that* $\mathbf{G}$ *is essentially free over* S *and that* $S = \mathrm{Spec}(B)$ *where* B *is a henselian local ring of residue field* k*. Then the map* $\overline{\mathbf{Hom}}(\mathbf{G},\mathbf{H})(B) \to \overline{\mathbf{Hom}}(\mathbf{G},\mathbf{H})(k)$ *is injective. Thus two homomorphisms* $u, v : \mathbf{G} \to \mathbf{H}$ *of* S*-group schemes are conjugate under* $\mathbf{H}(B)$ *if and only if* $u_k, v_k : \mathbf{G} \times_B k \to \mathbf{H} \times_B k$ *are conjugate under* $\mathbf{H}(k)$*.*

Proof. By Corollary 4.8, the B–functor $\mathbf{Transp}(u,v)$ is representable by a smooth B–scheme. By Hensel's lemma [3, §2.3] the map

$$\mathbf{Transp}(u,v)(B) \to \mathbf{Transp}(u,v)(k)$$

is surjective. Thus if $u_k, v_k : \mathbf{G} \times_B k \to \mathbf{H} \times_B k$ are such that $u_k = \mathrm{int}(h_0)v_k$ for some $h_0 \in \mathbf{H}(k)$, then there exists $h \in \mathbf{H}(B)$ such that $u = \mathrm{int}(h)v$. □

5. Applications

5.1. Rigidity. Our first result establishes Theorem 1.1

Theorem 5.1. *Let* k *be a field. Let* $\mathbf{G}$ *be a linearly reductive algebraic* k*–group and let* $\mathbf{H}$ *be a smooth algebraic* k*–group. Let* K/k *be a field extension such that* k *and* K *are both separably closed and* K *is separable over* k *(for example if both* k *and* K *are algebraically closed).*
Then the map

$$\overline{\mathbf{Hom}}_{k-gr}(\mathbf{G},\mathbf{H})(k) \to \overline{\mathbf{Hom}}_{k-gr}(\mathbf{G},\mathbf{H})(K) = \overline{\mathbf{Hom}}_{K-gr}(\mathbf{G}_K,\mathbf{H}_K)(K)$$

is bijective.

Proof. By Lemma 4.1.1 and Lemma 4.4 the map $\overline{\mathbf{Hom}}_{k-gr}(\mathbf{G},\mathbf{H})(k) \to \overline{\mathbf{Hom}}_{k-gr}(\mathbf{G},\mathbf{H})(K)$ is injective. Conversely we are given an element $u \in \mathrm{Hom}_{K-gr}(\mathbf{G}_K,\mathbf{H}_K)$ and we want to show that there exists $v_0 \in \mathrm{Hom}_{k-gr}(\mathbf{G},\mathbf{H})$ and $h \in \mathbf{H}(K)$ such that $v_0 \times_k \mathrm{id}_K = \mathrm{int}(h)u$.
The homomorphism $u : \mathbf{G}_K \to \mathbf{H}_K$ arises by base change from some A-group scheme homomorphism $v \in \mathrm{Hom}_{A-gr}(\mathbf{G}_A,\mathbf{H}_A)$, i.e. $u = v_K$, where $A \subset K$ is a finitely generated k–algebra. Under our assumption on k we may assume, by considering a basic open affine subsheme of $\mathrm{Spec}(A)$ if needed, that A is smooth over k. In particular, A is normal.
Since A is separable over k and k is separably closed, there exists a maximal ideal $\mathfrak{m}$ of A such that $A/\mathfrak{m} = k$. Then v gives rise to a k–homomorphism $v_0 : \mathbf{G} \to \mathbf{H}$. Denote by B the (strict) henselization of the local ring $A_\mathfrak{m}$. Then B is noetherian and may be identified with a subring of a separable closure of the fraction field of A [M, I.4.10, 11]. In particular B can be assumed to embed into K. By Proposition 3.2 and Remark 3.3 the group $\mathbf{G}_B$ (which is clearly affine and free of finite rank over B) satisfies the assumption on the fibers of Theorem 4.5. By Corollary 4.9, $v_0 \times_k \mathrm{id}_B = \mathrm{int}(h)(v \times_A \mathrm{id}_B)$ for some $h \in \mathbf{H}(B)$. Thus $v_0 \times_k \mathrm{id}_K = \mathrm{int}(h_K)u$ as desired. □

Remark 5.2. The assumption that $\mathbf{G}$ be linearly reductive is not superfluous. Recall (see §3.1) that $H^1(\mathbf{G}, V) = \mathrm{Ext}^1_{\mathbf{G}}(k, V)$. Assume that k is algebraically closed of positive characteristic, and let K/k be an arbitrary field extension. One knows from Nagata's work that for each non-trivial semisimple k-group $\mathbf{G}$ there exists a non-trivial finite dimensional irreducible $\mathbf{G}$-k–module V such that $H^1(\mathbf{G}, V) \neq 0$. This implies that Theorem 5.1 fails for $\overline{\mathbf{Hom}}_{k-gr}(\mathbf{G}, \mathbf{GL}_n)$ if $n = \dim(V) + 1$. Indeed $\overline{\mathbf{Hom}}_{k-gr}(\mathbf{G}, \mathbf{GL}_n)(k)$ measures the equivalence classes of n–dimensional linear representations of $\mathbf{G}$. We know that $\mathrm{Ext}^1_{\mathbf{G}}(k, V)$, which by assumption is a non-trivial k-space, can be identified with the subset of $\overline{\mathbf{Hom}}_{k-gr}(\mathbf{G}, \mathbf{GL}_n)(k)$ that corresponds to those representations of $\mathbf{G}$ that are extensions of k by V. Similar considerations apply to $\overline{\mathbf{Hom}}_{K-gr}(\mathbf{G}_K, \mathbf{GL}_{n,K})(K)$. Since $H^1(\mathbf{G}, V) \otimes_k K = H^1(\mathbf{G}_K, V_K)$ the foregoing discussion shows that the natural map $\overline{\mathbf{Hom}}_{k-gr}(\mathbf{G}, \mathbf{H})(k) \to \overline{\mathbf{Hom}}_{k-gr}(\mathbf{G}, \mathbf{H})(K)$ is not surjective whenever $k \neq K$.

Remark 5.3. Let $\mathbf{H}$ be a simple Chevalley $\mathbb{Z}$–group of adjoint type. In [2] Borel, Friedman and Morgan provide a considerable amount of information about the set of conjugacy classes of n-tuples $\mathbf{x} = (x_1, \cdots, x_n)$ of commuting elements of finite order of $\mathbf{H}(\mathbb{C})$.[4] The methods used in [2] are topological and analytic in nature, and do not immediately translate to other algebraically closed fields of characteristic 0. One of the reasons why this problem is relevant is because of its applications to infinite dimensional Lie theory. The interested reader can consult [6] for details and further references about this topic.
Fix an n-tuple $\mathbf{m} = (m_1, \cdots, m_n)$ of positive integers, and let $\mathbf{F_m}$ be the finite constant $\overline{\mathbb{Q}}$–group corresponding to the finite group $\mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_n\mathbb{Z}$. Because of the nature of our base field the group $\mathbf{F_m}$ is diagonalizable, hence linearly reductive. Let K be an algebraically closed field of characteristic 0. The conjugacy classes of n-tuples $\mathbf{x} = (x_1, \cdots, x_n)$ of commuting elements of $\mathbf{H}(K)$ where the x_i satisfy $x_i^{m_i} = 1$ are parametrized by $\overline{\mathbf{Hom}}_{K-gr}(\mathbf{F_{m,K}}, \mathbf{H}_K)(K)$. By Theorem 1.1 we have natural bijections

$$\overline{\mathbf{Hom}}_{K-gr}(\mathbf{F_{m,K}}, \mathbf{H}_K)(K) \simeq \overline{\mathbf{Hom}}_{\overline{\mathbb{Q}}-gr}(\mathbf{F_m}, \mathbf{H}_{\overline{\mathbb{Q}}})(\overline{\mathbb{Q}})) \simeq \\ \simeq \overline{\mathbf{Hom}}_{\mathbb{C}-gr}(\mathbf{F}_{\mathbf{m},\mathbb{C}}, \mathbf{H}_{\mathbb{C}})(\mathbb{C}).$$

This allows us to translate the relevant information within [2] to the group $\mathbf{H}(K)$.

5.2. Lie algebras. Assume henceforth that the base scheme S is of "characteristic zero", i.e. that S is a scheme over $\mathbb{Q}$. Let $\mathbf{G}/S$ be a semisimple group scheme and let $\mathbf{H}/S$ be an affine smooth group scheme. In this case, we already know that the functor $\mathbf{Hom}_{S-gp}(\mathbf{G}, \mathbf{H})$ is representable by a smooth affine S-scheme of finite presentation [17, XXIV.7.3.1]. Furthermore, if $\mathbf{G}/S$ is simply connected, we have an S-scheme isomorphism

$$\mathbf{Hom}_{S-gr}(\mathbf{G}, \mathbf{H}) \xrightarrow{\sim} \mathbf{Hom}_{S-Lie}\big(Lie(\mathbf{G}), Lie(\mathbf{H})\big).$$

[4]When lifted to the simply connected cover of $\mathbf{H}(\mathbb{C})$ the n-tuples will be comprised of "almost commuting" elements.

From this and Theorem 4.5 it follows that the functor

$$T \mapsto \mathrm{Hom}_{T-Lie}\big(\mathrm{Lie}(\mathbf{G}_T), \mathrm{Lie}(\mathbf{H}_T)\big)/\mathbf{H}(T)$$

is formally étale.

Corollary 5.4. *Let k be an algebraically closed field of characteristic zero. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over k. Let $\mathbf{H}$ be a smooth algebraic k–group. If K is an algebraically closed field extension of k, then the map*

$$\mathrm{Hom}_{k-\mathrm{Lie}}(\mathfrak{g}, \mathrm{Lie}(\mathbf{H}))/\mathbf{H}(k) \to \mathrm{Hom}_{K-\mathrm{Lie}}(\mathfrak{g} \otimes_k K, \mathrm{Lie}(\mathbf{H}) \otimes_k K)/\mathbf{H}(K)$$

is bijective. □

6. Appendix: Affine group schemes which are essentially free

Definition 6.1. [17, VIII 6] *A morphism of schemes X/S is essentially free if there exists an open covering of S by affine schemes $S_i = \mathrm{Spec}(A_i)$, and for all i an faithfully flat extension $S_i' = \mathrm{Spec}(A_i') \to S_i$ such that each $X \times_S S_i'$ admits an open covering by affine schemes $(\mathrm{Spec}(B_{i,j}'))$ such that $B_{i,j}'$ is a free A_i'–module for all j.*

Note that an essentially free morphism is flat. Furthermore this property is stable by arbitrary base change and is local with respect to the Zariski and the *fpqc* topology. Recall that that a sequence

$$1 \to \mathbf{G}_1 \to \mathbf{G}_2 \to \mathbf{G}_3 \to 1$$

of S–group schemes is said to be exact if it is exact as a sequence of *fpqc*-sheaves over S [17, VI_B 9].

Lemma 6.2. (1) *Let $\mathbf{G}/S$ be a flat group scheme which is essentially free over S. Let $X \to S$ be a $\mathbf{G}$–torsor which is locally trivial for the fpqc–topology. Then X is essentially free over S.*

(2) *Let $1 \to \mathbf{G}_1 \to \mathbf{G}_2 \to \mathbf{G}_3 \to 1$ be an exact sequence of S–group schemes. If $\mathbf{G}_1$ and $\mathbf{G}_3$ are essentially free over S, then $\mathbf{G}_2$ is essentialy free over S.*

Proof. (1) Since we can reason locally for the Zariski and for the *fpqc* topology, we can assume that X is the trivial torsor, namely $X = \mathbf{G}$.

(2) Similarly, we can assume that $S = \mathrm{Spec}(A)$ and that $\mathbf{G}_1$ (resp. $\mathbf{G}_3$) is covered by open affines subschemes $\mathrm{Spec}(B_j)$ (resp. $\mathrm{Spec}(C_l)$) for the *fpqc* topology such that the B_j and the C_l are free A–modules. Up to refining the second *fpqc* covering, we can furthermore assume that

$$\mathbf{G}_2 \times_{\mathbf{G}_3} \mathrm{Spec}(C_l) \xrightarrow{\sim} \mathbf{G}_1 \times_S \mathrm{Spec}(C_l).$$

It follows that the $\mathrm{Spec}(B_l \otimes_A C_l)$'s form a *fpqc*–cover of $\mathbf{G}_2$ where the $B_l \otimes_A C_l$ are free A–modules. Thus $\mathbf{G}_2$ is essentially free over S as desired. □

Several well-known affine group schemes are essentially free.

Proposition 6.3. *Let* $\mathbf{G}/S$ *be an affine* S*–group scheme which admits locally for the fpqc topology a composition series with factors of the following kind:*

(i) S*-group schemes of multiplicative type,*

(ii) twisted finite constant S*–group schemes,*

(iii) smooth S*-group schemes with connected geometric fibers.*

Then $\mathbf{G}$ *is essentially free over* S*.*

Note that the last case includes reductive group schemes over S and their parabolic subgroups.

Proof. By Lemma 6.2 it suffices to verify each of the three cases locally for the *fpqc* topology. Case (i) is then the case of diagonalizable groups which are essentially free over S by definition. Case (ii) is that of finite constant S–group schemes which are also essentially free over S by definition. Case (iii) has been noticed by Seshadri [15, Lemma 1 p. 230] using a result of Raynaud. □

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