Hyperbolic Geometry on Noncommutative Balls

Gelu Popescu[1]

Received: March 6, 2009
Revised: October 31, 2009

Communicated by Joachim Cuntz

Abstract. In this paper, we study the noncommutative balls

$$\mathcal{C}_\rho := \{(X_1, \dots, X_n) \in B(\mathcal{H})^n : \ \omega_\rho(X_1, \dots, X_n) \le 1\}, \qquad \rho \in (0, \infty],$$

where ω_ρ is the joint operator radius for n-tuples of bounded linear operators on a Hilbert space. In particular, ω_1 is the operator norm, ω_2 is the joint numerical radius, and ω_∞ is the joint spectral radius.

We introduce a Harnack type equivalence relation on $\mathcal{C}_\rho$, $\rho > 0$, and use it to define a hyperbolic distance δ_ρ on the Harnack parts (equivalence classes) of $\mathcal{C}_\rho$. We prove that the open ball

$$[\mathcal{C}_\rho]_{<1} := \{(X_1, \dots, X_n) \in B(\mathcal{H})^n : \ \omega_\rho(X_1, \dots, X_n) < 1\}, \qquad \rho > 0,$$

is the Harnack part containing 0 and obtain a concrete formula for the hyperbolic distance, in terms of the reconstruction operator associated with the right creation operators on the full Fock space with n generators. Moreover, we show that the δ_ρ-topology and the usual operator norm topology coincide on $[\mathcal{C}_\rho]_{<1}$. While the open ball $[\mathcal{C}_\rho]_{<1}$ is not a complete metric space with respect to the operator norm topology, we prove that it is a complete metric space with respect to the hyperbolic metric δ_ρ. In the particular case when $\rho = 1$ and $\mathcal{H} = \mathbb{C}$, the hyperbolic metric δ_ρ coincides with the Poincaré-Bergman distance on the open unit ball of $\mathbb{C}^n$.

We introduce a Carathéodory type metric on $[\mathcal{C}_\infty]_{<1}$, the set of all n-tuples of operators with joint spectral radius strictly less then 1, by setting

$$d_K(A, B) = \sup_p \|\Re p(A) - \Re p(B)\|, \qquad A, B \in [\mathcal{C}_\infty]_{<1},$$

where the supremum is taken over all noncommutative polynomials with matrix-valued coefficients $p \in \mathbb{C}[X_1, \dots, X_n] \otimes M_m$, $m \in \mathbb{N}$, with

[1]Research supported in part by an NSF grant

$\Re p(0) = I$ and $\Re p(X) \geq 0$ for all $X \in \mathcal{C}_1$. We obtain a concrete formula for d_K in terms of the free pluriharmonic kernel on the noncommutative ball $[\mathcal{C}_\infty]_{<1}$. We also prove that the metric d_K is complete on $[\mathcal{C}_\infty]_{<1}$ and its topology coincides with the operator norm topology.

We provide mapping theorems, von Neumann inequalities, and Schwarz type lemmas for free holomorphic functions on noncommutative balls, with respect to the hyperbolic metric δ_ρ, the Carathéodory metric d_K, and the joint operator radius ω_ρ, $\rho \in (0, \infty]$.

2000 Mathematics Subject Classification: Primary: 46L52; 46T25; 47A20; Secondary: 32Q45; 47A56

Keywords and Phrases: Noncommutative hyperbolic geometry; Noncommutative function theory; Harnack part; Hyperbolic distance; Carathéodory distance; Free holomorphic function; Free pluriharmonic function; Fock space; Creation operator; Joint operator radius; Joint numerical radius; Joint spectral radius; von Neumann inequality; Schwarz lemma.

Contents

Introduction

In [48], we provided a generalization of the Sz.-Nagy–Foiaş theory of ρ-contractions (see [54], [55], [56]), to the multivariable setting. An n-tuple $(T_1, \ldots, T_n) \in B(\mathcal{H})^n$ of bounded linear operators acting on a Hilbert space

$\mathcal{H}$ belongs to the class $\mathcal{C}_\rho$, $\rho > 0$, if there is a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and some isometries $V_i \in B(\mathcal{K})$, $i = 1, \ldots, n$, with orthogonal ranges such that

$$T_\alpha = \rho P_{\mathcal{H}} V_\alpha|_{\mathcal{H}} \quad \text{for any } \alpha \in \mathbb{F}_n^+ \backslash \{g_0\},$$

where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. Here, $\mathbb{F}_n^+$ stands for the unital free semigroup on n generators $g_1, \ldots, g_n$, and the identity g_0, while $T_\alpha := T_{i_1} T_{i_2} \cdots T_{i_k}$ if $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k} \in \mathbb{F}_n^+$ and $T_{g_0} := I_{\mathcal{H}}$, the identity on $\mathcal{H}$. According to the theory of row contractions (see [56] for the case $n = 1$, and [16], [7], [32], [33], [34], for $n \geq 2$) we have

$$\mathcal{C}_1 = [B(\mathcal{H})^n]_1^- := \left\{ (X_1, \ldots, X_n) \in B(\mathcal{H})^n : \ \|X_1 X_1^* + \cdots + X_n X_n^*\|^{1/2} \leq 1 \right\}.$$

The results in [48] (see Section 4) can be seen as the unification of the theory of isometric dilations for row contractions [54], [56], [16], [7], [32], [33], [34] (which corresponds to the case $\rho = 1$) and Berger type dilations for n-tuples $(T_1, \ldots, T_n)$ with the joint numerical radius $w(T_1, \ldots, T_n) \leq 1$ (which corresponds to the case $\rho = 2$).

Following the classical case ([19], [59]), we defined the *joint operator radius* $\omega_\rho : B(\mathcal{H})^n \to [0, \infty)$, $\rho > 0$, by setting

$$\omega_\rho(T_1, \ldots, T_n) := \inf \left\{ t > 0 : \ \left(\frac{1}{t} T_1, \ldots, \frac{1}{t} T_n \right) \in \mathcal{C}_\rho \right\}$$

and $\omega_\infty(T_1, \ldots, T_n) := \lim_{\rho \to \infty} \omega_\rho(T_1, \ldots, T_n)$. In particular, $\omega_1(T_1, \ldots, T_n)$ coincides with the norm of the row operator $[T_1 \cdots T_n]$, $\omega_2(T_1, \ldots, T_n)$ coincides with the *joint numerical radius* $w(T_1, \ldots, T_n)$, and $\omega_\infty(T_1, \ldots, T_n)$ is equal to the (*algebraic*) *joint spectral radius* (see [7], [25])

$$r(T_1, \ldots, T_n) := \lim_{k \to \infty} \left\| \sum_{|\alpha| = k} T_\alpha T_\alpha^* \right\|^{1/2k},$$

where the length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0$ and by $|\alpha| := k$ if $\alpha = g_{i_1} \cdots g_{i_k}$ and $i_1, \ldots, i_k \in \{1, \ldots, n\}$. In [48], we considered basic properties of the joint operator radius ω_ρ and we extended to the (noncommutative and commutative) multivariable setting several classical results obtained by Sz.-Nagy and Foiaş, Halmos, Berger and Stampfli, Holbrook, Paulsen, Badea and Cassier, and others (see [2], [3], [4], [5], [17], [18], [19], [20], [21], [29], [30], [55], and [59]).

In [49], we introduced a hyperbolic metric δ on the open noncommutative ball $[B(\mathcal{H})^n]_1$, which turned out to be a noncommutative extension of the Poincaré-Bergman ([6]) metric on the open unit ball $\mathbb{B}_n := \{z \in \mathbb{C}^n : \ \|z\|_2 < 1\}$. We proved that δ is invariant under the action of the group $Aut([B(\mathcal{H})^n]_1)$ of all free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$, and showed that the δ-topology and the usual operator norm topology coincide on $[B(\mathcal{H})^n]_1$. Moreover, we proved that $[B(\mathcal{H})^n]_1$ is a complete metric space with respect to the hyperbolic metric and obtained an explicit formula for δ in terms of the reconstruction operator. A Schwarz-Pick lemma for bounded free holomorphic functions on $[B(\mathcal{H})^n]_1$, with respect to the hyperbolic metric, was also obtained. In [46], we continued to study the noncommutative hyperbolic geometry on the unit ball of $B(\mathcal{H})^n$, its connections with multivariable dilation theory, and its implications to noncommutative function theory. The results from [49] and [46] make connections between noncommutative function theory (see [41], [44], [50], [47]) and classical results in hyperbolic complex analysis (see [22], [23], [24], [52], [58]).

The present paper is an attempt to extend the results [49] concerning the noncommutative hyperbolic geometry of the unit ball $[B(\mathcal{H})^n]_1$ to the more general setting of [48]. We study the noncommutative balls

$$[\mathcal{C}_\rho]_{<1} = \{(X_1,\dots,X_n)\in B(\mathcal{H})^n:\ \omega_\rho(X_1,\dots,X_n)<1\}, \qquad \rho\in(0,\infty],$$

and the Harnach parts of $\mathcal{C}_\rho$, $\rho>0$, as metric spaces with respect to a hyperbolic (resp. Carathéodory) type metric that will be introduced. We provide mapping theorems for free holomorphic functions on these noncommutative balls, extending classical results from complex analysis and hyperbolic geometry.

In Section 1, we consider some preliminaries on free holomorphic (resp. pluriharmonic) functions on the open unit ball $[B(\mathcal{H})^n]_1$, and present several characterizations for the n-tuples of operators of class $\mathcal{C}_\rho$, $\rho\in(0,\infty)$. We introduce a free pluriharmonic functional calculus for the class $\mathcal{C}_\rho$ and show that a von Neumann type inequality characterizes this class. In particular, we prove that an n-tuple of operators $(T_1,\dots,T_n)\in B(\mathcal{H})^n$ is of class $\mathcal{C}_\rho$ if and only if

$$\|p(T_1,\dots,T_n)\|\le\|\rho p(S_1,\dots,S_n)+(1-\rho)p(0)\|$$

for any noncommutative polynomial with matrix-valued coefficients $p\in\mathbb{C}[Z_1,\dots,Z_n]\otimes M_m$, $m\in\mathbb{N}$, where $S_1,\dots,S_n$ are the left creation operators on the full Fock space with n generators.

In Section 2, we introduce a preorder relation $\overset{H}{\prec}$ on the class $\mathcal{C}_\rho$. If $A:=(A_1,\dots,A_n)$ and $B:=(B_1,\dots,B_n)$ are in the class $\mathcal{C}_\rho\subset B(\mathcal{H})^n$, we say that A is Harnack dominated by B (denote $A\overset{H}{\prec}B$) if there exists $c>0$ such that

$$\Re p(A_1,\dots,A_n)+(\rho-1)\Re p(0)\le c^2\left[\Re p(B_1,\dots,B_n)+(\rho-1)\Re p(0)\right]$$

for any noncommutative polynomial with matrix-valued coefficients $p\in\mathbb{C}[X_1,\dots,X_n]\otimes M_m$, $m\in\mathbb{N}$, such that $\Re p(X):=\frac{1}{2}[p(X)^*+p(X)]\ge 0$ for any $X\in[B(\mathcal{K})^n]_1$, where $\mathcal{K}$ is an infinite dimensional Hilbert space. When we want to emphasize the constant c, we write $A\overset{H}{\underset{c}{\prec}}B$. We provide several characterizations for the Harnack domination on the noncommutative ball $\mathcal{C}_\rho$ (see Theorem 2.2), and determine the set of all elements in $\mathcal{C}_\rho$ which are Harnack dominated by 0. The results of this section will play a major role in the next sections.

The relation $\overset{H}{\prec}$ induces an equivalence relation $\overset{H}{\sim}$ on the class $\mathcal{C}_\rho$. More precisely, two n-tuples A and B are Harnack equivalent (and denote $A \overset{H}{\sim} B$) if and only if there exists $c > 1$ such that $A \underset{c}{\overset{H}{\prec}} B$ and $B \underset{c}{\overset{H}{\prec}} A$ (in this case we denote $A \underset{c}{\overset{H}{\sim}} B$). The equivalence classes with respect to $\overset{H}{\sim}$ are called Harnack parts of $\mathcal{C}_\rho$. In Section 3, we provide a Harnack type double inequality for positive free pluriharmonic functions on the noncommutative ball $\mathcal{C}_\rho$ and use it to prove that the Harnack part of $\mathcal{C}_\rho$ which contains 0 coincides with the open noncommutative ball

$$[\mathcal{C}_\rho]_{<1} := \{(X_1, \ldots, X_n) \in B(\mathcal{H})^n : \ \omega_\rho(X_1, \ldots, X_n) < 1\}.$$

We introduce a hyperbolic metric $\delta_\rho : \Delta \times \Delta \to \mathbb{R}^+$ on any Harnack part Δ of $\mathcal{C}_\rho$, by setting

$$\delta_\rho(A, B) := \ln \inf \left\{ c > 1 : \ A \underset{c}{\overset{H}{\sim}} B \right\}, \qquad A, B \in \Delta.$$

A concrete formula for the hyperbolic distance on any Harnack part of $\mathcal{C}_\rho$ is obtained. When $\Delta = [\mathcal{C}_\rho]_{<1}$, we prove that

$$\delta_\rho(A, B) = \ln \max \left\{ \left\| C_{\rho,A} C_{\rho,B}^{-1} \right\|, \left\| C_{\rho,B} C_{\rho,A}^{-1} \right\| \right\}, \qquad A, B \in [\mathcal{C}_\rho]_{<1},$$

where

$$\begin{aligned} C_{\rho,X} &:= \Delta_{\rho,X}(I - R_X)^{-1}, \\ \Delta_{\rho,X} &:= \left[\rho I + (1-\rho)(R_X^* + R_X) + (\rho - 2) R_X^* R_X\right]^{1/2}, \end{aligned}$$

and $R_X := X_1^* \otimes R_1 + \cdots + X_n^* \otimes R_n$ is the *reconstruction operator* associated with the right creation operators $R_1, \ldots, R_n$ on the full Fock space with n generators, and $X := (X_1, \ldots, X_n) \in [\mathcal{C}_\rho]_{<1}$. We recall that the reconstruction operator has played an important role in noncommutative multivariable operator theory. It appeared as a building block in the characteristic function associated to a row contraction (see [34], [45]) and also as a quantized variable (associated with the n-tuple X) in the noncommutative Cauchy, Poisson, and Berezin transform, respectively (see [41], [44], [47], [48]).

In Section 4, we study the stability of the ball $\mathcal{C}_\rho$ under contractive free holomorphic functions and provide mapping theorems, von Neumann inequalities, and Schwarz type lemmas, with respect to the hyperbolic metric δ_ρ and the operator radius ω_ρ, $\rho \in (0, \infty]$.

Let $f := (f_1, \ldots, f_m)$ be a contractive free holomorphic function with $\|f(0)\| < 1$ such that the boundary functions $\widetilde{f}_1, \ldots, \widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$ (see [36], [40]). If an n-tuple of operators $(T_1, \ldots, T_n) \in B(\mathcal{H})^n$ is of class $\mathcal{C}_\rho$, $\rho > 0$, then we prove that, under the free pluriharmonic functional calculus, the m-tuple $f(T_1, \ldots, T_n) \in B(\mathcal{H})^m$ is of class $\mathcal{C}_{\rho_f}$, where $\rho_f > 0$ is given in terms of ρ and $f(0)$.

One of the main results of this section is the following *spectral von Neumann inequality* for n-tuples of operators. If $f := (f_1, \ldots, f_m)$ satisfies the conditions above and $(T_1,\ldots,T_n)\in B(\mathcal{H})^n$ has the joint spectral radius $r(T_1,\ldots,T_n)<1$, then $r(f(T_1,\ldots,T_n))<1$.

If, in addition, $f(0)=0$ and $\delta_\rho:\Delta\times\Delta\to[0,\infty)$ is the hyperbolic metric on a Harnack part Δ of $\mathcal{C}_\rho$, then we prove that

$$\delta_\rho(f(A),f(B))\le\delta_\rho(A,B),\qquad A,B\in\Delta.$$

In particular, this holds when Δ is the open ball $[\mathcal{C}_\rho]_{<1}$. Moreover, in this setting, we show that

$$\omega_\rho(f(T_1,\ldots,T_n))<1,\qquad (T_1,\ldots,T_n)\in[\mathcal{C}_\rho]_{<1},$$

for any $\rho>0$. The general case when $f(0)\neq 0$ is also discussed.

In Section 5, we introduce a Carathéodory type metric on the set of all n-tuples of operators with joint spectral radius strictly less then 1, i.e.,

$$[\mathcal{C}_\infty]_{<1}:=\{(X_1,\ldots,X_n)\in B(\mathcal{H})^n:\ r(X_1,\ldots,X_n)<1\},$$

by setting

$$d_K(A,B)=\sup_p\|\Re p(A)-\Re p(B)\|,$$

where the supremum is taken over all noncommutative polynomials with matrix-valued coefficients $p\in\mathbb{C}[X_1,\ldots,X_n]\otimes M_m$, $m\in\mathbb{N}$, with $\Re p(0)=I$ and $\Re p(X)\ge 0$ for all $X\in[B(\mathcal{K})^n]_1$.

We obtain a concrete formula for d_K in terms of the free pluriharmonic kernel on the open unit ball $[\mathcal{C}_\infty]_{<1}$. More precisely, we show that

$$d_K(A,B)=\|P(A,R)-P(B,R)\|,\qquad A,B\in[\mathcal{C}_\infty]_{<1},$$

where

$$P(X,R):=\sum_{k=1}^\infty\sum_{|\alpha|=k}X_\alpha\otimes R_{\tilde\alpha}^*+\rho I\otimes I+\sum_{k=1}^\infty\sum_{|\alpha|=k}X_\alpha^*\otimes R_{\tilde\alpha},\qquad X\in[\mathcal{C}_\infty]_{<1},$$

and $\tilde\alpha$ is the reverse of $\alpha\in\mathbb{F}_n^+$. This is used to prove that the metric d_K is complete on $[\mathcal{C}_\infty]_{<1}$ and its topology coincides with the operator norm topology. We also prove that if $f:=(f_1,\ldots,f_m)$ is a contractive free holomorphic function with $\|f(0)\|<1$ such that the boundary functions $\widetilde{f}_1,\ldots,\widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$, then

$$d_K(f(A),f(B))\le\frac{1+\|f(0)\|}{1-\|f(0)\|}d_K(A,B),\qquad A,B\in[\mathcal{C}_\infty]_{<1}.$$

As a consequence, we deduce that the map

$$[\mathcal{C}_\infty]_{<1}\ni(X_1,\ldots,X_n)\mapsto f(X_1,\ldots,X_n)\in[\mathcal{C}_\infty]_{<1}$$

is continuous in the operator norm topology.

In Section 6, we compare the hyperbolic metric δ_ρ with the Carathéodory metric d_K, and the operator metric, respectively, on Harnack parts of the unit ball $\mathcal{C}_\rho$, $\rho>0$. In particular, we prove that the hyperbolic metric δ_ρ is complete on the open unit unit ball $[\mathcal{C}_\rho]_{<1}$, while the other two metrics, mentioned above, are not complete. On the other hand, we show the δ_ρ-topology, the d_K-topology, and the operator norm topology coincide on $[\mathcal{C}_\rho]_{<1}$.

In Section 7, we consider the single variable case ($n = 1$) and show that our Harnack domination for ρ-contractions is equivalent to the one introduced and studied by G. Cassier and N. Suciu in [9] and [10]. Consequently, we recover some of their results and, moreover, we obtain some results which seem to be new even in the single variable case.

Finally, we want to acknowledge that we were influenced in writing this paper by the work of C. Foiaş ([15]), I. Suciu ([53]), and G. Cassier and N. Suciu ([9], [10]) concerning the Harnack domination and the hyperbolic distance between two ρ-contractions. It will be interesting to see to which extent the results of this paper, concerning the hyperbolic geometry on noncommutative balls, can be extended to the Hardy algebras of Muhly and Solel (see [26], [27], [28]).

1. The noncommutative ball $\mathcal{C}_\rho$ and a free pluriharmonic functional calculus

In this section, we consider some preliminaries on free holomorphic (resp. pluriharmonic) functions on the unit ball $[B(\mathcal{H})^n]_1$, and several characterizations for the n-tuples of operators of class $\mathcal{C}_\rho$. We introduce a free pluriharmonic functional calculus for the class $\mathcal{C}_\rho$ and show that a von Neumann type inequality characterizes the class $\mathcal{C}_\rho$.

Let H_n be an n-dimensional complex Hilbert space with orthonormal basis e_1, $e_2, \ldots, e_n$, where $n = 1, 2, \ldots$, or $n = \infty$. The full Fock space of H_n is defined by

$$F^2(H_n) := \mathbb{C}1 \oplus \bigoplus_{k\geq 1} H_n^{\otimes k},$$

where $H_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of H_n. We define the left (resp. right) creation operators S_i (resp. R_i), $i = 1, \ldots, n$, acting on the full Fock space $F^2(H_n)$ by setting

$$S_i\varphi := e_i \otimes \varphi, \qquad \varphi \in F^2(H_n),$$

(resp. $R_i\varphi := \varphi \otimes e_i, \quad \varphi \in F^2(H_n)$). We recall that the noncommutative disc algebra $\mathcal{A}_n$ (resp. $\mathcal{R}_n$) is the norm closed algebra generated by the left (resp. right) creation operators and the identity. The noncommutative analytic Toeplitz algebra F_n^∞ (resp. $\mathcal{R}_n^\infty$) is the weakly closed version of $\mathcal{A}_n$ (resp. $\mathcal{R}_n$). These algebras were introduced in [36] in connection with a von Neumann type inequality [57], as noncommutative analogues of the disc algebra $A(\mathbb{D})$ and the Hardy space $H^\infty(\mathbb{D})$. For more information on theses noncommutative algebras we refer the reader to [35], [37], [38], [40], [12], and the references therein.

Let $\mathcal{H}$ be a Hilbert space and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. We identify $M_m(B(\mathcal{H}))$, the set of $m\times m$ matrices with entries from $B(\mathcal{H})$, with $B(\mathcal{H}^{(m)})$, where $\mathcal{H}^{(m)}$ is the direct sum of m copies of $\mathcal{H}$. If $\mathcal{X}$ is an operator space, i.e., a closed subspace of $B(\mathcal{H})$, we consider $M_m(\mathcal{X})$ as a subspace of $M_m(B(\mathcal{H}))$ with the induced norm. Let $\mathcal{X},\mathcal{Y}$ be operator spaces and $u:\mathcal{X}\to\mathcal{Y}$ be a linear map. Define the map $u_m: M_m(\mathcal{X})\to M_m(\mathcal{Y})$ by

$$u_m([x_{ij}]):=[u(x_{ij})].$$

We say that u is completely bounded if

$$\|u\|_{cb}:=\sup_{m\geq 1}\|u_m\|<\infty.$$

If $\|u\|_{cb}\leq 1$ (resp. u_m is an isometry for any $m\geq 1$) then u is completely contractive (resp. isometric), and if u_m is positive for all m, then u is called completely positive. For basic results concerning completely bounded maps and operator spaces we refer to [29], [31], and [13].

A few more notations and definitions are necessary. If $\omega,\gamma\in\mathbb{F}_n^+$, we say that $\omega>_l\gamma$ if there is $\sigma\in\mathbb{F}_n^+\backslash\{g_0\}$ such that $\omega=\gamma\sigma$ and set $\omega\backslash_l\gamma:=\sigma$. We denote by $\tilde\alpha$ the reverse of $\alpha\in\mathbb{F}_n^+$, i.e., $\tilde\alpha=g_{i_k}\cdots g_{i_1}$ if $\alpha=g_{i_1}\cdots g_{i_k}\in\mathbb{F}_n^+$. An operator-valued positive semidefinite kernel on the free semigroup $\mathbb{F}_n^+$ is a map $K:\mathbb{F}_n^+\times\mathbb{F}_n^+\to B(\mathcal{H})$ with the property that for each $k\in\mathbb{N}$, for each choice of vectors $h_1,\ldots,h_k$ in $\mathcal{H}$, and $\sigma_1,\ldots,\sigma_k$ in $\mathbb{F}_n^+$, the inequality

$$\sum_{i,j=1}^k\langle K(\sigma_i,\sigma_j)h_j,h_i\rangle\geq 0$$

holds. Such a kernel is called multi-Toeplitz if it has the following properties: $K(\alpha,\alpha)=I_{\mathcal{H}}$ for any $\alpha\in\mathbb{F}_n^+$, and

$$K(\sigma,\omega)=\begin{cases}K(g_0,\omega\backslash_l\sigma) & \text{if } \omega>_l\sigma\\ K(\sigma\backslash_l\omega,g_0) & \text{if } \sigma>_l\omega\\ 0 & \text{otherwise.}\end{cases}$$

An n-tuple of operators $(T_1,\ldots,T_n)$, $T_i\in B(\mathcal{H})$, belongs to the class $\mathcal{C}_\rho$, $\rho>0$, if there exist a Hilbert space $\mathcal{K}\supseteq\mathcal{H}$ and isometries $V_i\in B(\mathcal{K})$, $i=1,\ldots,n$, with orthogonal ranges, such that

$$T_\alpha=\rho P_{\mathcal{H}}V_\alpha|_{\mathcal{H}},\qquad \alpha\in\mathbb{F}_n^+\backslash\{g_0\},$$

where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. If $\mathcal{K}=\mathcal{K}_T:=\bigvee_{\alpha\in\mathbb{F}_n^+}V_\alpha\mathcal{H}$, then the n-tuple $(V_1,\ldots,V_n)$ is the minimal isometric dilation of $(T_1,\ldots,T_n)$, which is unique up to an isomorphism. Note that if $(T_1,\ldots,T_n)\in\mathcal{C}_\rho$, then the joint spectral radius $r(T_1,\ldots,T_n)\leq 1$, where

$$r(T_1,\ldots,T_n):=\lim_{k\to\infty}\left\|\sum_{|\alpha|=k}T_\alpha T_\alpha^*\right\|^{1/2k}.$$

We recall (see Corollary 1.36 from [48]) that $\bigcup_{\rho>0}\mathcal{C}_\rho$ is dense (in the operator norm topology) in the set of all n-tuples of operators with joint spectral radius $r(T_1,\ldots,T_n)\leq 1$. Moreover, any n-tuple of operators with $r(T_1,\ldots,T_n)<1$ is of class $\mathcal{C}_\rho$ for some $\rho>0$. We should add that (see Theorem 5.9 from [43])

$(T_1,\dots,T_n)\in B(\mathcal{H})^n$ has the joint spectral radius $r(T_1,\dots,T_n)<1$ if and only if it is uniformly stable, i.e., $\|\sum_{|\alpha|=k}T_\alpha T_\alpha^*\|\to 0$, as $k\to\infty$.

Since the joint spectral radius of n-tuples of operators plays an important role in the present paper, we recall (see [7], [25]) some of its properties. The joint right spectrum $\sigma_r(T_1,\dots,T_n)$ of an n-tuple $(T_1,\dots,T_n)$ of operators in $B(\mathcal{H})$ is the set of all n-tuples $(\lambda_1,\dots,\lambda_n)$ of complex numbers such that the right ideal of $B(\mathcal{H})$ generated by the operators $\lambda_1 I-T_1,\dots,\lambda_n I-T_n$ does not contain the identity operator. We know that $\sigma_r(T_1,\dots,T_n)$ is included in the closed ball of $\mathbb{C}^n$ of radius $r(T_1,\dots,T_n)$.

If we assume that $T_1,\dots,T_n\in B(\mathcal{H})$ are mutually commuting operators and $\mathcal{B}$ is a closed subalgebra of $B(\mathcal{H})$ containing $T_1,\dots,T_n$, and the identity, then the Harte spectrum $\sigma(T_1,\dots,T_n)$ is the set of all $(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$ such that

$$(\lambda_1 I-T_1)X_1+\cdots+(\lambda_n I-T_n)X_n\neq I$$

for all $X_1,\dots,X_n\in\mathcal{B}$. In this case, we have

$$r(T_1,\dots,T_n)=\max\{\|(\lambda_1,\dots,\lambda_n)\|_2:\ (\lambda_1,\dots,\lambda_n)\in\sigma(T_1,\dots,T_n)\}.$$

According to [25], the latter formula remains true if the Harte spectrum is replaced by the Taylor's spectrum for commuting operators.

According to Theorem 4.1 from [39] and Theorems 1.34 and 1.39 from [48], we have the following characterizations for the n-tuples of operators of class $\mathcal{C}_\rho$. We denote by $\mathbb{C}[Z_1,\dots,Z_n]$ the set of all noncommutative polynomials in n noncommuting indeterminates.

Theorem 1.1. *Let $T_1,\dots,T_n\in B(\mathcal{H})$ and let $\mathcal{S}\subset C^*(S_1,\dots,S_n)$ be the operator system defined by*

$$\mathcal{S}:=\{p(S_1,\dots,S_n)+q(S_1,\dots,S_n)^*:\ p,q\in\mathbb{C}[Z_1,\dots,Z_n]\}.$$

Then the following statements are equivalent:

(i) $(T_1,\dots,T_n)\in\mathcal{C}_\rho$.

(ii) *The map $\Psi:\mathcal{S}\to B(\mathcal{H})$ defined by*

$$\Psi\left(p(S_1,\dots,S_n)+q(S_1,\dots,S_n)^*\right):=p(T_1,\dots,T_n)+q(T_1,\dots,T_n)^*+(\rho-1)(p(0)+\overline{q(0)})I$$

is completely positive.

(iii) *The joint spectral radius $r(T_1\dots,T_n)\leq 1$ and the ρ-pluriharmonic kernel defined by*

$$P_\rho(rT,R):=\sum_{k=1}^\infty\sum_{|\alpha|=k}r^{|\alpha|}T_\alpha\otimes R_{\tilde\alpha}^*+\rho I\otimes I+\sum_{k=1}^\infty\sum_{|\alpha|=k}r^{|\alpha|}T_\alpha^*\otimes R_{\tilde\alpha}$$

is positive for any $0<r<1$, where the convergence is in the operator norm topology.

(iv) *The spectral radius* $r(T_1,\dots,T_n)\leq 1$ *and*

$$\rho I\otimes I+(1-\rho)r\sum_{i=1}^n (T_i\otimes R_i^*+T_i^*\otimes R_i)+(\rho-2)r^2\left(\sum_{i=1}^n T_iT_i^*\otimes I\right)\geq 0$$

for any $0<r<1$.

(v) *The multi-Toeplitz kernel* $K_{\rho,T}:\mathbb{F}_n^+\times\mathbb{F}_n^+\to B(\mathcal{H})$ *defined by*

$$K_{\rho,T}(\alpha,\beta):=\begin{cases}\frac{1}{\rho}T_{\beta\backslash_l\alpha} & \text{if } \beta>_l\alpha\\ I & \text{if } \alpha=\beta\\ \frac{1}{\rho}(T_{\alpha\backslash_l\beta})^* & \text{if } \alpha>_l\beta\\ 0 & \text{otherwise}\end{cases}$$

is positive semidefinite.

Consider $1\leq m<n$ and let $(R_1',\dots,R_m')$ and $(R_1,\dots,R_n)$ be the right creation operators on $F^2(H_m)$ and $F^2(H_n)$, respectively. According to the Wold type decomposition for isometries with orthogonal ranges [33], the m-tuple $(R_1,\dots,R_m)$ is unitarily equivalent to $(R_1'\otimes I_{\mathcal{E}},\dots,R_m'\otimes I_{\mathcal{E}})$, where $\mathcal{E}$ is equal to $F^2(H_n)\ominus F^2(H_m)$. Consequently, using Theorem 1.1, one can easily deduce the following result.

Corollary 1.2. *Let* $\rho>0$, $1\leq m<n$, *and consider an* m*-tuple* $(T_1,\dots,T_m)\in B(\mathcal{H})^m$ *and its extension* $(T_1,\dots,T_m,0,\dots,0)\in B(\mathcal{H})^n$. *Then the following statements hold:*

(i) $(T_1,\dots,T_m)\in\mathcal{C}_\rho$ *if and only if* $(T_1,\dots,T_m,0,\dots,0)\in\mathcal{C}_\rho$;
(ii) $\omega_\rho(T_1,\dots,T_m)=\omega_\rho(T_1,\dots,T_m,0,\dots,0))$;
(iii) $r(T_1,\dots,T_m)=r(T_1,\dots,T_m,0,\dots,0)$.

Throughout this paper, we assume that $\mathcal{E}$ is a separable Hilbert space. We recall [44] that a mapping $F:[B(\mathcal{H})^n]_1\to B(\mathcal{H})\bar{\otimes}_{min}B(\mathcal{E})$ is called *free holomorphic function* on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$ if there exist $A_{(\alpha)}\in B(\mathcal{E})$, $\alpha\in\mathbb{F}_n^+$, such that $\limsup_{k\to\infty}\left\|\sum_{|\alpha|=k}A_{(\alpha)}^*A_{(\alpha)}\right\|^{1/2k}\leq 1$ and

$$F(X_1,\dots,X_n)=\sum_{k=0}^\infty\sum_{|\alpha|=k}X_\alpha\otimes A_{(\alpha)},$$

where the series converges in the operator norm topology for any $(X_1,\dots,X_n)$ in the open unit ball $[B(\mathcal{H})^n]_1:=\{(X_1,\dots,X_n):\ \|X_1X_1^*+\cdots+X_nX_n\|<1\}$. The set of all free holomorphic functions on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$ is denoted by $H_{\mathbf{ball}}(B(\mathcal{E}))$. Let $H^\infty_{\mathbf{ball}}(B(\mathcal{E}))$ denote the set of all elements F in $H_{\mathbf{ball}}(B(\mathcal{E}))$ such that

$$\|F\|_\infty:=\sup\|F(X_1,\dots,X_n)\|<\infty,$$

where the supremum is taken over all n-tuples of operators $(X_1,\dots,X_n)\in[B(\mathcal{H})^n]_1$ and any Hilbert space $\mathcal{H}$. According to [44] and [47], $H^\infty_{\mathbf{ball}}(B(\mathcal{E}))$

can be identified to the operator algebra $F_n^\infty \bar{\otimes} B(\mathcal{E})$ (the weakly closed algebra generated by the spatial tensor product), via the noncommutative Poisson transform. Due to the fact that a free holomorphic function is uniquely determined by its representation on an infinite dimensional Hilbert space, we identify, throughout this paper, a free holomorphic function with its representation on a separable infinite dimensional Hilbert space.

We say that a map $u : [B(\mathcal{H})^n]_1 \to B(\mathcal{H})\bar{\otimes}_{min} B(\mathcal{E})$ is a self-adjoint *free pluriharmonic function* on $[B(\mathcal{H})^n]_1$ if $u = \Re f := \frac{1}{2}(f^* + f)$ for some free holomorphic function f. A free pluriharmonic function on $[B(\mathcal{H})^n]_1$ has the form $H := H_1 + iH_2$, where H_1, H_2 are self-adjoint free pluriharmonic functions on $[B(\mathcal{H})^n]_1$. We recall [47] that if

$$f(Z_1, \ldots, Z_n) = \sum_{k=1}^{\infty} \sum_{|\alpha|=k} Z_\alpha^* \otimes B_{(\alpha)} + I \otimes A_{(0)} + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} Z_\alpha \otimes A_{(\alpha)}$$

is a free pluriharmonic function on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$ and $(T_1, \ldots, T_n) \in B(\mathcal{H})^n$ is any n-tuple of operators with joint spectral radius $r(T_1, \ldots, T_n) < 1$, then $f(T_1, \ldots, T_n)$ is a bounded linear operator, where the corresponding series converge in norm. Moreover $\lim_{r\to 1} f(rT_1, \ldots, rT_n) = f(T_1, \ldots T_n)$ in the operator norm topology. We refer to [47] for more results on free pluriharmonic functions.

We denote by $Har^c_{\mathbf{ball}}(B(\mathcal{E}))$ the set of all free pluriharmonic functions on $[B(\mathcal{H})^n]_1$ with operator-valued coefficients in $B(\mathcal{E})$, which have continuous extensions (in the operator norm topology) to the closed ball $[B(\mathcal{H})^n]_1^-$. We assume that $\mathcal{H}$ is an infinite dimensional Hilbert space. According to Theorem 4.1 from [47], we can identify $Har^c_{\mathbf{ball}}(B(\mathcal{E}))$ with the operator space $\overline{\mathcal{A}_n(\mathcal{E})^* + \mathcal{A}_n(\mathcal{E})}^{\|\cdot\|}$, where $\mathcal{A}_n(\mathcal{E}) := \mathcal{A}_n \bar{\otimes}_{min} B(\mathcal{E})$ and $\mathcal{A}_n$ is the noncommutative disc algebra. More precisely, if $u : [B(\mathcal{H})^n]_1 \to B(\mathcal{H})\bar{\otimes}_{min} B(\mathcal{E})$, then the following statements are equivalent:

- (a) u is a free pluriharmonic function on $[B(\mathcal{H})^n]_1$ which has a continuous extension (in the operator norm topology) to the closed ball $[B(\mathcal{H})^n]_1^-$;
- (b) there exists $f \in \overline{\mathcal{A}_n(\mathcal{E})^* + \mathcal{A}_n(\mathcal{E})}^{\|\cdot\|}$ such that $u(X) = (P_X \otimes \mathrm{id})(f)$ for $X \in [B(\mathcal{H})^n]_1$, where P_X is the noncommutative Poisson transform at X;
- (c) u is a free pluriharmonic function on $[B(\mathcal{H})^n]_1$ such that $u(rS_1, \ldots, rS_n)$ converges in the operator norm topology, as $r \to 1$.

In this case, we have $f = \lim_{r\to 1} u(rS_1, \ldots, rS_n)$, where the convergence is in the operator norm topology. Moreover, the map $\Phi : Har^c_{\mathbf{ball}}(B(\mathcal{E})) \to \overline{\mathcal{A}_n(\mathcal{E})^* + \mathcal{A}_n(\mathcal{E})}^{\|\cdot\|}$ defined by $\Phi(u) := f$ is a completely isometric isomorphism of operator spaces. We call f the *model boundary function* of u.

Now, we introduce a free pluriharmonic functional calculus for the class $\mathcal{C}_\rho$.

Theorem 1.3. *Let $T := (T_1, \ldots, T_n) \in B(\mathcal{H})^n$ be of class $\mathcal{C}_\rho$, and let $u \in Har^c_{\mathbf{ball}}(B(\mathcal{E}))$ have the standard representation*

$$u(X_1, \ldots, X_n) = \sum_{k=1}^{\infty} \sum_{|\alpha|=k} X_\alpha^* \otimes B_{(\alpha)} + I \otimes A_{(0)} + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} X_\alpha \otimes A_{(\alpha)}$$

on $[B(\mathcal{H})^n]_1$, for some $A_{(\alpha)}, B_{(\alpha)} \in B(\mathcal{E})$, where the series converge in the operator norm topology. Then

$$u(T_1, \ldots, T_n) := \lim_{r\to 1} u(rT_1, \ldots, rT_1)$$

exists in the operator norm and

$$\|u(T_1, \ldots, T_n)\| \le \|\rho u + (1-\rho)u(0)\|_\infty.$$

Proof. Since $T := (T_1, \ldots, T_n) \in B(\mathcal{H})^n$ is an n-tuple of class $\mathcal{C}_\rho$, there is a minimal isometric dilation $V := (V_1, \ldots, V_n)$ of T on a Hilbert space $\mathcal{K}_T \supseteq \mathcal{H}$, satisfying the following properties: $V_i^* V_j = \delta_{ij} I$ for $i, j = 1, \ldots, n$, and $T_\alpha = \rho P_\mathcal{H} V_\alpha|_\mathcal{H}$ for any $\alpha \in \mathbb{F}_n^+ \backslash \{g_0\}$, and $\mathcal{K}_T = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H}$. Taking into account that $u \in Har^c_{\mathbf{ball}}(B(\mathcal{E}))$, we have

$$u(rV_1, \ldots, rV_n) = \sum_{k=1}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} V_\alpha^* \otimes B_{(\alpha)} + I \otimes A_{(0)} + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} V_\alpha \otimes A_{(\alpha)},$$

where the convergence is in the operator norm. Hence, and due to the fact that

$$\sum_{|\alpha|=k} r^{|\alpha|} T_\alpha^* \otimes B_{(\alpha)} = \rho (P_\mathcal{H} \otimes I) \left(\sum_{|\alpha|=k} r^{|\alpha|} V_\alpha^* \otimes B_{(\alpha)} \right) |_{\mathcal{H}\otimes\mathcal{E}}, \qquad k = 1, 2, \ldots,$$

we deduce that

$$\begin{aligned} u(rT_1, \ldots, rT_n) &:= \sum_{k=1}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} T_\alpha^* \otimes B_{(\alpha)} + I \otimes A_{(0)} + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} T_\alpha \otimes A_{(\alpha)} \\ &= \rho (P_\mathcal{H} \otimes I) u(rV_1, \ldots, rV_n)|_{\mathcal{H}\otimes\mathcal{E}} - (\rho - 1) u(0). \end{aligned}$$

exists in the operator norm topology. Now, taking into account that $\lim_{r\to 1} u(rV_1, \ldots, rV_1)$ exists in the operator norm, we deduce that $\lim_{r\to 1} u(rT_1, \ldots, rT_1)$ exists in the same topology. Consequently, we can define

$$u(T_1, \ldots, T_n) := \lim_{r\to 1} u(rT_1, \ldots, rT_1).$$

Using the considerations above, and the noncommutative von Neumann inequality, we obtain

$$\|u(T_1, \ldots, T_n)\| \le \|\rho u + (1-\rho)u(0)\|_\infty \le (\rho + |\rho - 1|)\|u\|_\infty$$

for any $(T_1, \ldots, T_n) \in \mathcal{C}_\rho$. $\square$

We will refer to the map

$$Har^c_{\mathbf{ball}}(B(\mathcal{E})) \ni u \mapsto u(T_1, \dots, T_n) \in B(\mathcal{H})\bar{\otimes}_{min} B(\mathcal{E})$$

as the free pluriharmonic functional calculus for the class $\mathcal{C}_\rho$. Since there is a completely isometric isomorphism of operator spaces $\overline{\mathcal{A}_n(\mathcal{E})^* + \mathcal{A}_n(\mathcal{E})}^{\|\cdot\|} \ni f \mapsto u \in Har^c_{\mathbf{ball}}(B(\mathcal{E}))$, given by $u = (P_X \otimes \mathrm{id})(f)$ for $X \in [B(\mathcal{H})^n]_1$, we also use the notation $f(T_1, \dots, T_n)$ for $u(T_1, \dots, T_n)$.

Now, we show that the von Neumann type inequality of Theorem 1.3 characterizes the class $\mathcal{C}_\rho$. Denote

$$\mathcal{P}(S_1, \dots, S_n) := \{p(S_1, \dots, S_n) : \ p \in \mathbb{C}[Z_1, \dots, Z_n]\},$$

where $S_1, \dots, S_n$ are the left creation operators on the full Fock space $F^2(H_n)$.

Theorem 1.4. *Let $T := (T_1, \dots, T_n) \in B(\mathcal{H})^n$ be an n-tuple of operators. Then the following statements are equivalent:*

(i) *T is of class $\mathcal{C}_\rho$;*

(ii) *the von Neumann type inequality*
$$\|p(T_1, \dots, T_n)\| \leq \|\rho p(S_1, \dots, S_n) + (1 - \rho)p(0)\|$$
holds for any noncommutative polynomial $p \in \mathbb{C}[Z_1, \dots, Z_n] \otimes M_m$, $m \in \mathbb{N}$;

(iii) *the map $\Psi_T : \mathcal{A}_n \to B(\mathcal{H})$ defined by*
$$\Psi_T\left(q(S_1, \dots, S_n)\right) := \frac{1}{\rho} q(T_1, \dots, T_n) + \left(1 - \frac{1}{\rho}\right) q(0)I$$
for $q(S_1, \dots, S_n) \in \mathcal{P}(S_1, \dots, S_n)$ is completely contractive.

Proof. The implication $(i) \implies (ii)$ follows, in particular, from Theorem 1.3. To prove the implication $(ii) \implies (iii)$, note that setting $p := \frac{1}{\rho} q + \left(1 - \frac{1}{\rho}\right) q(0)I$, where $q \in \mathbb{C}[Z_1, \dots, Z_n] \otimes M_m$, $m \in \mathbb{N}$, we have

$$\begin{aligned}\|\Psi_T\left(q(S_1, \dots, S_n)\right)\| &= \|p(T_1, \dots, T_n)\| \\ &\leq \|\rho p(S_1, \dots, S_n) + (1 - \rho)p(0)\| \\ &= \|q(S_1, \dots, S_n)\|,\end{aligned}$$

which proves that Ψ_T is completely contractive on the set of all polynomials $\mathcal{P}(S_1, \dots, S_n)$ and, consequently, extends uniquely to a completely contractive map on the noncommutative disc algebra $\mathcal{A}_n$. It remains to prove that $(iii) \implies (i)$. Due to Arveson's extension theorem, item (iii) implies the existence of a unique completely positive extension $\widetilde{\Psi}_T : \mathcal{A}_n^* + \mathcal{A}_n \to B(\mathcal{H})$ of Ψ_T. Note that

$$\begin{aligned}&\widetilde{\Psi}_T(r(S_1, \dots, S_n) + q(S_1, \dots, S_n)^*) = \\ &\qquad = \frac{1}{\rho}(r(T_1, \dots, T_n) + q(T_1, \dots, T_n)^*) + \left(1 - \frac{1}{\rho}\right)(r(0) + \overline{q(0)})I\end{aligned}$$

for any polynomials $r(S_1, \dots, S_n)$ and $q(S_1, \dots, S_n)$ in $\mathcal{P}(S_1, \dots, S_n)$. Applying Theorem 1.1 (the equivalence $(i) \leftrightarrow (ii)$), we complete the proof. □

2. Harnack domination on noncommutative balls

We introduce a preorder relation $\overset{H}{\prec}$ on the noncommutative ball $\mathcal{C}_\rho$, $\rho \in (0, \infty)$, and provide several characterizations. We determine the elements of $\mathcal{C}_\rho$ which are Harnack dominated by 0. These results will play a crucial role in the next sections.

First, we consider some preliminaries on noncommutative Poisson transforms. Let $C^*(S_1, \ldots, S_n)$ be the Cuntz-Toeplitz C^*-algebra generated by the left creation operators (see [11]). The noncommutative Poisson transform at the n-tuple $T := (T_1, \ldots, T_n) \in [B(\mathcal{H})^n]_1^-$ is the unital completely contractive linear map $P_T : C^*(S_1, \ldots, S_n) \to B(\mathcal{H})$ defined by

$$P_T[f] := \lim_{r \to 1} K_{rT}^*(I_\mathcal{H} \otimes f) K_{rT}, \qquad f \in C^*(S_1, \ldots, S_n),$$

where the limit exists in the operator norm topology of $B(\mathcal{H})$. Here, the noncommutative Poisson kernel $K_{rT} : \mathcal{H} \to \overline{\Delta_{rT}\mathcal{H}} \otimes F^2(H_n)$, $0 < r \leq 1$, is defined by

$$K_{rT} h := \sum_{k=0}^\infty \sum_{|\alpha|=k} r^{|\alpha|} \Delta_{rT} T_\alpha^* h \otimes e_\alpha, \qquad h \in \mathcal{H},$$

where $\{e_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ is the orthonormal basis for the full Fock space $F^2(H_n)$, defined by $e_\alpha := e_{i_1} \otimes \cdots \otimes e_{i_k}$ if $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$ and $e_{g_0} := 1$, and $\Delta_{rT} := (I_\mathcal{H} - r^2 T_1 T_1^* - \cdots - r^2 T_n T_n^*)^{1/2}$. We recall that $P_T[S_\alpha S_\beta^*] = T_\alpha T_\beta^*$, $\alpha, \beta \in \mathbb{F}_n^+$. When $T := (T_1, \ldots, T_n)$ is a pure row contraction, i.e., SOT-$\lim\limits_{k \to \infty} \sum_{|\alpha|=k} T_\alpha T_\alpha^* = 0$, then we have

$$P_T[f] = K_T^*(I_{\mathcal{D}_T} \otimes f) K_T, \qquad f \in C^*(S_1, \ldots, S_n) \text{ or } f \in F_n^\infty,$$

where $\mathcal{D}_T := \overline{\Delta_T \mathcal{H}}$. We refer to [41], [42], and [48] for more on noncommutative Poisson transforms on C^*-algebras generated by isometries.

A free pluriharmonic function u on $[B(\mathcal{K})^n]_1$ with operator valued coefficients is called positive, and denote $u \geq 0$, if $u(X_1, \ldots, X_n) \geq 0$ for any $(X_1, \ldots, X_n) \in [B(\mathcal{K})^n]_1$, where $\mathcal{K}$ is an infinite dimensional Hilbert space. We mention that it is enough to assume that the positivity condition holds for any finite dimensional Hilbert space $\mathcal{K}$. Indeed, for each $m \in \mathbb{N}$, consider $R^{(m)} := (R_1^{(m)}, \ldots, R_n^{(m)})$, where $R_i^{(m)}$ is the compression of the right creation operator R_i to the subspace $\mathcal{P}_m := \operatorname{span}\{e_\alpha : \alpha \in \mathbb{F}_n^+, |\alpha| \leq m\}$ of $F^2(H_n)$. We recall from [47] the following result.

Lemma 2.1. *Let u be a free pluriharmonic function on $[B(\mathcal{K})^n]_1$ with operator-valued coefficients. Then $u(X_1, \ldots, X_n) \geq 0$ for any $(X_1, \ldots, X_n) \in [B(\mathcal{K})^n]_1$ if and only if $u(R_1^{(m)}, \ldots, R_n^{(m)}) \geq 0$ for any $m \in \mathbb{N}$.*

Let $A := (A_1, \ldots, A_n)$ and $B := (B_1, \ldots, B_n)$ be n-tuples of operators in $\mathcal{C}_\rho \subset B(\mathcal{H})^n$. We say that A is Harnack dominated by B, and denote $A \overset{H}{\prec} B$, if there exists $c > 0$ such that

$$\Re p(A_1, \ldots, A_n) + (\rho - 1)\Re p(0) \leq c^2 \left[\Re p(B_1, \ldots, B_n) + (\rho - 1)\Re p(0)\right]$$

for any noncommutative polynomial with matrix-valued coefficients $p \in \mathbb{C}[X_1, \ldots, X_n] \otimes M_m$, $m \in \mathbb{N}$, such that $\Re p \geq 0$. When we want to emphasize the constant c, we write $A \overset{H}{\underset{c}{\prec}} B$.

According to Theorem 1.3, we can associate with each n-tuple $T := (T_1, \ldots, T_n) \in \mathcal{C}_\rho$ the completely positive map $\varphi_T : \overline{\mathcal{A}_n^* + \mathcal{A}_n}^{\|\cdot\|} \to B(\mathcal{H})$ defined by

$$\varphi_T(g) := \frac{1}{\rho} g(T_1, \ldots, T_n) + \left(1 - \frac{1}{\rho}\right) g(0), \tag{2.1}$$

where $g(T_1, \ldots, T_n)$ is defined by the free pluriharmonic functional calculus for the class $\mathcal{C}_\rho$.

Now, we present several characterizations for the Harnack domination in $\mathcal{C}_\rho$.

Theorem 2.2. *Let $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ and $B := (B_1, \ldots, B_n) \in B(\mathcal{H})^n$ be in the class $\mathcal{C}_\rho$ and let $c > 0$. Then the following statements are equivalent:*

(i) $A \overset{H}{\underset{c}{\prec}} B$;
(ii) $P_\rho(rA, R) \leq c^2 P_\rho(rB, R)$ *for any* $r \in [0, 1)$, *where* $P_\rho(X, R)$ *is the multi-Toeplitz kernel associated with* $X \in \mathcal{C}_\rho$;
(iii) $u(rA_1, \ldots, rA_n) + (\rho - 1)u(0) \leq c^2 \left[u(rB_1, \ldots, rB_n) + (\rho - 1)u(0)\right]$ *for any positive free pluriharmonic function* u *on* $[B(\mathcal{H})^n]_1$ *with operator-valued coefficients and any* $r \in [0, 1)$;
(iv) $K_{\rho,A} \leq c^2 K_{\rho,B}$, *where* $K_{\rho,X}$ *is the multi-Toeplitz kernel associated with* $X \in \mathcal{C}_\rho$;
(v) $c^2 \varphi_B - \varphi_A$ *is a completely positive linear map on the operator space* $\overline{\mathcal{A}_n^* + \mathcal{A}_n}^{\|\cdot\|}$, *where* φ_A, φ_B *are the c.p. maps associated with* A *and* B, *respectively.*
(vi) *there is an operator* $L_{B,A} \in B(\mathcal{K}_B, \mathcal{K}_A)$ *with* $\|L_{B,A}\| \leq c$ *such that* $L_{B,A}|_{\mathcal{H}} = I_{\mathcal{H}}$ *and*

$$L_{B,A} W_i = V_i L_{B,A}, \qquad i = 1, \ldots, n,$$

where $(V_1, \ldots, V_n)$ *on* $\mathcal{K}_A \supset \mathcal{H}$ *and* $(W_1, \ldots, W_n)$ *on* $\mathcal{K}_A \supset \mathcal{H}$ *are the minimal isometric dilations of* A *and* B, *respectively.*

Proof. First we prove that $(i) \implies (ii)$. Since $R_\alpha^{(m)} = 0$ for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| \geq m + 1$, we have

$$P_\rho(rX, R^{(m)}) = \sum_{1 \leq |\alpha| \leq m} r^{|\alpha|} X_\alpha^* \otimes R_{\tilde{\alpha}}^{(m)} + \rho I \otimes I + \sum_{1 \leq |\alpha| \leq m} r^{|\alpha|} X_\alpha \otimes R_{\tilde{\alpha}}^{(m)^*}.$$

Since $X \mapsto P_1(X, R)$ is a positive free pluriharmonic function on $[B(\mathcal{H})^n]_1$, with coefficients in $B(F^2(H_n))$, so is the map

$$X \mapsto P_1(rX, R^{(m)}) = (I \otimes P_{\mathcal{P}_m}) P_1(rX, R)|_{\mathcal{H} \otimes \mathcal{P}_m}$$

for any $r \in [0,1)$. If $A \overset{H}{\underset{c}{\prec}} B$, then we have

$$P_1(rA, R^{(m)}) + (\rho - 1)P_1(0, R^{(m)}) \le c^2 \left[P_1(rB, R^{(m)}) + (\rho - 1)P_1(0, R^{(m)})\right]$$

for any $m = 1, 2, \ldots$. Using Lemma 2.1, we deduce that

$$P_1(rA, R) + (\rho - 1)I \le c^2 \left[P_1(rB, R) + (\rho - 1)I\right]$$

for any $r \in [0,1)$. Since $P_\rho(rY, R) = P_1(rY, R) + (\rho - 1)I$ for any n-tuple $Y \in B(\mathcal{H})^n$ with spectral radius $r(Y) \le 1$ and $r \in [0,1)$, we deduce item (ii).

To prove the implication $(ii) \implies (iii)$, assume that condition (ii) holds and let u be a positive free pluriharmonic function on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$ of the form

$$u(Z_1, \ldots, Z_n) = \sum_{k=1}^\infty \sum_{|\alpha|=k} Z_\alpha^* \otimes C_{(\alpha)}^* + I \otimes C_{(0)} + \sum_{k=1}^\infty \sum_{|\alpha|=k} Z_\alpha \otimes C_{(\alpha)}.$$

It is well-known (see e.g. [29]) that if $\mathcal{S} \subseteq B(F^2(H_n))$ is an operator system and $\mu : \mathcal{S} \to B(\mathcal{K})$ is a completely bounded map, then there exists a completely bounded linear map

$$\widetilde{\mu} := \mu \otimes \mathrm{id} : \mathcal{S} \bar{\otimes}_{min} B(\mathcal{H}) \to B(\mathcal{K}) \bar{\otimes}_{min} B(\mathcal{H})$$

such that $\widetilde{\mu}(f \otimes Y) := \mu(f) \otimes Y$ for $f \in \mathcal{S}$ and $Y \in B(\mathcal{H})$. Moreover, $\|\widetilde{\mu}\|_{cb} = \|\mu\|_{cb}$ and, if μ is completely positive, then so is $\widetilde{\mu}$.

Using Corollary 5.5 from [47], we find a completely positive linear map $\nu : \mathcal{R}_n^* + \mathcal{R}_n \to B(\mathcal{E})$ such that $\nu(R_{\tilde{\alpha}}) = C_{(\alpha)}^*$ if $|\alpha| \ge 1$ and $\nu(I) = C_{(0)}$. Note that

$$\begin{aligned}
&(\mathrm{id} \otimes \nu)[c^2 P_\rho(rB, R) - P_\rho(rA, R)] \\
&\quad = (\mathrm{id} \otimes \nu) \left\{ \sum_{k=1}^\infty \sum_{|\alpha|=k} r^{|\alpha|}(c^2 B_\alpha - A_\alpha) \otimes R_{\tilde{\alpha}}^* + \rho(c^2 - 1) I \otimes I \right. \\
&\qquad\qquad \left. + \sum_{k=1}^\infty \sum_{|\alpha|=k} (c^2 B_\alpha^* - A_\alpha^*) \otimes R_{\tilde{\alpha}} \right\} \\
&\quad = \left\{ \sum_{k=1}^\infty \sum_{|\alpha|=k} r^{|\alpha|}(c^2 B_\alpha - A_\alpha) \otimes C_{(\alpha)} + \rho(c^2 - 1) I \otimes C_{(0)} \right. \\
&\qquad\qquad \left. + \sum_{k=1}^\infty \sum_{|\alpha|=k} (c^2 B_\alpha^* - A_\alpha^*) \otimes C_{(\alpha)}^* \right\} \\
&\quad = c^2 \left[u(rB_1, \ldots, rB_n) + (\rho - 1)u(0)\right] \\
&\qquad\qquad - \left[u(rA_1, \ldots, rA_n) + (\rho - 1)u(0)\right].
\end{aligned}$$

Hence, and using the fact that $c^2 P_\rho(rB, R) - P_\rho(rA, R) \ge 0$, we deduce that

$$c^2 \left[u(rB_1, \ldots, rB_n) + (\rho - 1)u(0)\right] - \left[u(rA_1, \ldots, rA_n) + (\rho - 1)u(0)\right] \ge 0,$$

which proves (iii).

Now, we prove the implication $(iii) \implies (v)$. Let $g \in \left(\overline{\mathcal{A}_n^* + \mathcal{A}_n}^{\|\cdot\|}\right) \otimes_{min} M_m$ be positive. Then, according to Theorem 4.1 from [47], the map defined by

$$g(X) := (P_X \otimes \mathrm{id})[g], \qquad X \in [B(\mathcal{H})^n]_1,$$

is a positive free pluriharmonic function. Condition (iii) implies

$$g(rA_1, \ldots, rA_n) + (\rho - 1)g(0) \le c^2 \left[g(rB_1, \ldots, rB_n) + (\rho - 1)g(0)\right]$$

for any $r \in [0, 1)$. Hence, and using relation (2.1), we get $\rho\varphi_A(g_r) \le c^2 \rho\varphi_B(g_r)$. Taking $r \to 1$, we deduce item (v).

To prove the implication $(v) \implies (i)$, let $p \in \mathbb{C}[X_1, \ldots, X_n] \otimes M_m$, $m \in \mathbb{N}$, be a noncommutative polynomial with matrix coefficients such that $\operatorname{Re} p \ge 0$. Since

$$\rho\varphi_Y(p) = p(Y_1, \ldots, Y_n) + (\rho - 1)p(0)$$

for any $Y := (Y_1, \ldots, Y_n) \in \mathcal{C}_\rho$, it is clear that (v) implies item (i).

We prove now that $(ii) \implies (iv)$. We recall that $e_\alpha := e_{i_1} \otimes \cdots \otimes e_{i_k}$ if $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$ and $e_{g_0} := 1$, and that $\{e_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ is an orthonormal basis for the full Fock space $F^2(H_n)$. First, we prove that

(2.2)

$$\left\langle P_\rho(X, rR) \left(\sum_{|\beta| \le q} h_\beta \otimes e_\beta \right), \sum_{|\gamma| \le q} h_\gamma \otimes e_\gamma \right\rangle = \rho \sum_{|\beta|, |\gamma| \le q} \langle K_{\rho, X, r}(\gamma, \beta) h_\beta, h_\gamma \rangle,$$

where the multi-Toeplitz kernel $K_{\rho, X, r} : \mathbb{F}_n^+ \times \mathbb{F}_n^+ \to B(\mathcal{H})$, $r \in (0, 1)$, is defined by

$$K_{\rho, X, r}(\alpha, \beta) := \begin{cases} \frac{1}{\rho} r^{|\beta \backslash_l \alpha|} X_{\beta \backslash_l \alpha} & \text{if } \beta >_l \alpha \\ I & \text{if } \alpha = \beta \\ \frac{1}{\rho} r^{|\alpha \backslash_l \beta|} (X_{\alpha \backslash_l \beta})^* & \text{if } \alpha >_l \beta \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\{h_\beta\}_{|\beta| \le q} \subset \mathcal{H}$, then we have

$$\begin{aligned}
&\left\langle (\rho I \otimes I \ + \sum_{k=1}^\infty \sum_{|\alpha| = k} X_\alpha^* \otimes r^k R_{\tilde{\alpha}} \right) \left(\sum_{|\beta| \le q} h_\beta \otimes e_\beta \right), \sum_{|\gamma| \le q} h_\gamma \otimes e_\gamma \right\rangle \\
&\quad = \rho \sum_{|\beta| \le q} \|h_\beta\|^2 + \sum_{k=1}^\infty \sum_{|\alpha| = k} \left\langle \sum_{|\beta| \le q} X_\alpha^* h_\beta \otimes r^k R_{\tilde{\alpha}} e_\beta, \sum_{|\gamma| \le q} h_\gamma \otimes e_\gamma \right\rangle \\
&\quad = \rho \sum_{|\beta| \le q} \|h_\beta\|^2 + + \sum_{|\alpha| \ge 1} \sum_{|\beta|, |\gamma| \le q} r^{|\alpha|} \langle e_{\beta\alpha}, e_\gamma \rangle \langle X_\alpha^* h_\beta, h_\gamma \rangle \\
&\quad = \rho \sum_{|\beta| \le q} \|h_\beta\|^2 + \sum_{\gamma > \beta;\ |\beta|, |\gamma| \le q} r^{|\gamma \backslash_l \beta|} \left\langle X_{\gamma \backslash_l \beta}^* h_\beta, h_\gamma \right\rangle \\
&\quad = \sum_{\gamma \ge \beta;\ |\beta|, |\gamma| \le q} \langle \rho K_{\rho, X, r}(\gamma, \beta) h_\beta, h_\gamma \rangle .
\end{aligned}$$

Now, taking into account that $K_{\rho,X,r}(\gamma,\beta)=K^*_{\rho,X,r}(\beta,\gamma)$, we deduce relation (2.2). Therefore, the condition $P_\rho(rA,R)\leq c^2P_\rho(rB,R)$, $r\in[0,1)$, implies

$$[K_{\rho,A,r}(\alpha,\beta)]_{|\alpha|,|\beta|\leq q}\leq c^2[K_{\rho,B,r}(\alpha,\beta)]_{|\alpha|,|\beta|\leq q}$$

for any $0<r<1$ and $q=0,1,\ldots$. Taking $r\to 1$ in the latter inequality, we obtain item (iv).

Assume now that (iv) holds. Since $c^2K_{\rho,B}-K_{\rho,A}$ is a positive semidefinite multi-Toeplitz kernel, due to Theorem 3.1 from [39] (see also the proof of Theorem 5.2 from [47]), we find a completely positive linear map $\mu : C^*(S_1,\ldots,S_n)\to B(\mathcal{E})$ such that

$$\mu(S_\alpha)=c^2K_{\rho,B}(g_0,\alpha)-K_{\rho,A}(g_0,\alpha)=\frac{1}{\rho}(c^2B_\alpha-A_\alpha)$$

for any $\alpha\in\mathbb{F}_n^+$ with $|\alpha|\geq 1$, and $\mu(I)=(c^2-1)I$. Since

$$P(rS,R):=\sum_{k=1}^\infty\sum_{|\alpha|=k}r^kS_\alpha\otimes R^*_{\tilde\alpha}+I\otimes I+\sum_{k=1}^\infty\sum_{|\alpha|=k}r^kS^*_\alpha\otimes R_{\tilde\alpha}\geq 0$$

for $r\in[0,1)$, we deduce that

$$\begin{aligned}(\mu\otimes\mathrm{id})[P(rS,R)]&=\sum_{k=1}^\infty\sum_{|\alpha|=k}\frac{1}{\rho}r^{|\alpha|}[c^2B^*_\alpha-A^*_\alpha]\otimes R_{\tilde\alpha}+(c^2-1)I\otimes I\\&\qquad+\sum_{k=1}^\infty\sum_{|\alpha|=k}\frac{1}{\rho}r^{|\alpha|}[c^2B_\alpha-A_\alpha]\otimes R^*_{\tilde\alpha}\\&=c^2P_\rho(rB,R)-P_\rho(rA,R)\geq 0,\end{aligned}$$

which implies (ii).

Let us prove that $(iv)\implies(vi)$. Assume that (iv) holds. Then we have $K_{\rho,A}\leq c^2K_{\rho,B}$, where $K_{\rho,X}$ is the multi-Toeplitz kernel associated with $X\in\mathcal{C}_\rho$. Let $V:=(V_1,\ldots,V_n)$ be the minimal isometric dilation of $A:=(A_1,\ldots,A_n)$. Then $\mathcal{K}_A=\bigvee_{\alpha\in\mathbb{F}_n^+}V_\alpha\mathcal{H}$ and $\rho P_\mathcal{H}V_\alpha|_\mathcal{H}=A_\alpha$ for any $|\alpha|\geq 1$. Similar properties hold if $W:=(W_1,\ldots,W_n)$ is the minimal isometric dilation of $B:=(B_1,\ldots,B_n)$. Hence, and taking into account that $V_1,\ldots,V_n$ and $W_1,\ldots,W_n$ are isometries with orthogonal ranges, respectively, we have

$$\begin{aligned}&\left\|\sum_{|\alpha|\leq m}V_\alpha h_\alpha\right\|^2=\\&=\sum_{\alpha>_l\beta,|\alpha|,|\beta|\leq m}\left\langle V_{\alpha\backslash_l\beta}h_\alpha,h_\beta\right\rangle+\sum_{|\alpha|\leq m}\langle h_\alpha,h_\alpha\rangle+\sum_{\beta>_l\alpha,|\alpha|,|\beta|\leq m}\left\langle V^*_{\beta\backslash_l\alpha}h_\alpha,h_\beta\right\rangle\\&=\sum_{\alpha>_l\beta,|\alpha|,|\beta|\leq m}\left\langle\frac{1}{\rho}A_{\alpha\backslash_l\beta}h_\alpha,h_\beta\right\rangle+\sum_{|\alpha|\leq m}\langle h_\alpha,h_\alpha\rangle+\sum_{\beta>_l\alpha,|\alpha|,|\beta|\leq m}\left\langle\frac{1}{\rho}A^*_{\beta\backslash_l\alpha}h_\alpha,h_\beta\right\rangle\\&=\sum_{|\alpha|\leq m,|\beta|\leq m}\langle K_{\rho,A}(\beta,\alpha)h_\alpha,h_\beta\rangle=\left\langle[K_{\rho,A}(\beta,\alpha)]_{|\alpha|,|\beta|\leq m}\mathbf{h}_m,\mathbf{h}_m\right\rangle\end{aligned}$$

for any $m \in \mathbb{N}$ and $\mathbf{h}_m := \oplus_{|\alpha|\leq m} h_\alpha \in \oplus_{|\alpha|\leq m} \mathcal{H}_\alpha$, where each $\mathcal{H}_\alpha$ is a copy of $\mathcal{H}$. Similarly, we obtain

$$\left\| \sum_{|\alpha|\leq m} W_\alpha h_\alpha \right\|^2 = \left\langle [K_{\rho,B}(\beta,\alpha)]_{|\alpha|,|\beta|\leq m} \mathbf{h}_m, \mathbf{h}_m \right\rangle .$$

Taking into account that $K_{\rho,A} \leq c^2 K_{\rho,B}$, we deduce that

$$\left\| \sum_{|\alpha|\leq m} V_\alpha h_\alpha \right\| \leq c \left\| \sum_{|\alpha|\leq m} W_\alpha h_\alpha \right\| .$$

Therefore, we can define an operator $L_{B,A} : \mathcal{K}_B \to \mathcal{K}_A$ by setting

$$L_{B,A} \left(\sum_{|\alpha|\leq m} W_\alpha h_\alpha \right) := \sum_{|\alpha|\leq m} V_\alpha h_\alpha \tag{2.3}$$

for any $m \in \mathbb{N}$ and $h_\alpha \in \mathcal{H}$, $\alpha \in \mathbb{F}_n^+$. Note that $L_{B,A}$ is a bounded operator with $\|L_{B,A}\| \leq c$. Since $L_{B,A}|_\mathcal{H} = I_\mathcal{H}$, we have $\|L_{B,A}\| \geq 1$. It is easy to see that $L_{B,A} W_i = V_i L_{B,A}$ for $i = 1, \ldots, n$. Therefore item (vi) holds.

Conversely, assume that there is an operator $L_{B,A} \in B(\mathcal{K}_B, \mathcal{K}_A)$ with norm $\|L_{B,A}\| \leq c$ such that $L_{B,A}|_\mathcal{H} = I_\mathcal{H}$ and $L_{B,A} W_i = V_i L_{B,A}$, $i = 1, \ldots, n$. Then, we deduce that $L_{B,A}\left(\sum_{|\alpha|\leq m} W_\alpha h_\alpha\right) = \sum_{|\alpha|\leq m} V_\alpha h_\alpha$ for any $m \in \mathbb{N}$ and $h_\alpha \in \mathcal{H}$, $\alpha \in \mathbb{F}_n^+$. The condition $\|L_{B,A}\| \leq c$ implies

$$\left\| \sum_{|\alpha|\leq m} V_\alpha h_\alpha \right\|^2 \leq c^2 \left\| \sum_{|\alpha|\leq m} W_\alpha h_\alpha \right\|^2 ,$$

which is equivalent to the inequality

$$\left\langle [K_{\rho,A}(\beta,\alpha)]_{|\alpha|,|\beta|\leq m} \mathbf{h}_m, \mathbf{h}_m \right\rangle \leq c^2 \left\langle [K_{\rho,B}(\beta,\alpha)]_{|\alpha|,|\beta|\leq m} \mathbf{h}_m, \mathbf{h}_m \right\rangle$$

for any $m \in \mathbb{N}$ and $\mathbf{h}_m := \oplus_{|\alpha|\leq m} h_\alpha \in \oplus_{|\alpha|\leq m} \mathcal{H}_\alpha$. Consequently, we deduce item (iv). The proof is complete. □

A closer look at the proof of Theorem 2.2 reveals that one can assume that $u(0) = I$ in part (iii), and one can also assume that $\Re p(0) = I$ in the definition of the Harnack domination $A \overset{H}{\prec} B$. We also remark that, due to Theorem 1.3, we can add an equivalence to Theorem 2.2, namely, $A \overset{H}{\underset{c}{\prec}} B$ if and only if

$$u(A_1, \ldots, A_n) + (\rho - 1) u(0) \leq c^2 \left[u(B_1, \ldots, B_n) + (\rho - 1) u(0) \right]$$

for any positive free pluriharmonic function $u \in Har^c_{\mathbf{ball}}(B(\mathcal{E}))$.

COROLLARY 2.3. *If $A, B \in \mathcal{C}_\rho$ and $A \overset{H}{\prec} B$, then*

$$\begin{aligned}\|L_{B,A}\| &= \inf\{c > 1 : \ A \underset{c}{\overset{H}{\prec}} B\} \\ &= \inf\{c > 1 : \ P_\rho(rA, R) \leq c^2 P_\rho(rB, R) \quad \text{for any} \quad r \in [0,1)\}.\end{aligned}$$

Moreover, $A \overset{H}{\prec} B$ if and only if $\sup_{r\in[0,1)} \|L_{rA,rB}\| < \infty$. In this case,

$$\|L_{A,B}\| = \sup_{r\in[0,1)} \|L_{rA,rB}\|$$

and the mapping $r \mapsto \|L_{rA,rB}\|$ is increasing on $[0,1)$.

Proof. Assume that $A \overset{H}{\prec} B$. Then, due to Theorem 2.2, $A \underset{c}{\overset{H}{\prec}} B$ if and only if there is an operator $L_{B,A} \in B(\mathcal{K}_B, \mathcal{K}_A)$ with $\|L_{B,A}\| \leq c$ such that $L_{B,A}|_{\mathcal{H}} = I_{\mathcal{H}}$ and $L_{B,A} W_i = V_i L_{B,A}$ for $i = 1, \ldots, n$. Consequently, taking $c = \|L_{B,A}\|$, we deduce that $A \underset{\|L_{B,A}\|}{\overset{H}{\prec}} B$, which is equivalent to

$$P_\rho(rA, R) \leq \|L_{B,A}\|^2 P_\rho(rB, R)$$

for any $r \in [0,1)$. Hence, we have $tA \underset{\|L_{B,A}\|}{\overset{H}{\prec}} tB$ for any $t \in [0,1)$. Applying again Theorem 2.2 to the operators tA and tB, we deduce that $\|L_{tA,tB}\| \leq \|L_{B,A}\|$. Conversely, suppose that $c := \sup_{r\in[0,1)} \|L_{rA,rB}\| < \infty$. Since $\|L_{rA,rB}\| \leq c$, Theorem 2.2 implies $rA \underset{c}{\overset{H}{\prec}} rB$ for any $r \in [0,1)$ and, therefore, $P_\rho(rtA, R) \leq c^2 P_\rho(rtB, R)$ for any $t, r \in [0,1)$. Hence, $A \underset{c}{\overset{H}{\prec}} B$ and, consequently, $\|L_{B,A}\| \leq c$. Therefore, $\|L_{A,B}\| = \sup_{r\in[0,1)} \|L_{rA,rB}\|$. The fact that $r \mapsto \|L_{rA,rB}\|$ is an increasing function on $[0,1)$ follows from the latter relation. This completes the proof. $\square$

We remark that if $1 \leq m < n$ and u is a positive free pluriharmonic function on $[B(\mathcal{K})^n]_1$, then the map

$$(X_1, \ldots, X_m) \mapsto u(X_1, \ldots, X_m, 0, \ldots, 0)$$

is a positive free pluriharmonic function on $[B(\mathcal{K})^m]_1$. Moreover, if g is a positive free pluriharmonic function on $[B(\mathcal{K})^m]_1$, then the map

$$(X_1, \ldots, X_n) \mapsto g(X_1, \ldots, X_m, 0, \ldots, 0)$$

is a positive free pluriharmonic function on $[B(\mathcal{K})^n]_1$. Consequently, using Corollary 1.2, one can easily deduce the following result.

COROLLARY 2.4. *Let $c > 0$, $\rho > 0$, and $1 \leq m < n$. Consider two n-tuples $(A_1, \ldots, A_m) \in B(\mathcal{H})^m$ and $(B_1, \ldots, B_m) \in B(\mathcal{H})^m$ in the class $\mathcal{C}_\rho$ and let*

$(A_1,\dots,A_m,0,\dots,0)$ and $(B_1,\dots,B_m,0,\dots,0)$ be their extensions in $B(\mathcal{H})^n$, respectively. Then $(A_1,\dots,A_m)\overset{H}{\underset{c}{\prec}}(B_1,\dots,B_m)$ in $\mathcal{C}_\rho\subset B(\mathcal{H})^m$ if and only if

$$(A_1,\dots,A_m,0,\dots,0)\overset{H}{\underset{c}{\prec}}(B_1,\dots,B_m,0,\dots,0)\quad \text{in } \ \mathcal{C}_\rho\subset B(\mathcal{H})^n.$$

We recall (e.g. [43]) that if $(T_1,\dots T_n)$ is an n-tuple of operators, then the joint spectral radius $r(T_1,\dots,T_n)<1$ if and only if $\lim_{k\to\infty}\left\|\sum_{|\alpha|=k}T_\alpha T_\alpha^*\right\|=0$. In what follows, we characterize the elements of $\mathcal{C}_\rho$ which are Harnack dominated by 0.

Theorem 2.5. *Let $A:=(A_1,\dots,A_n)$ be in $\mathcal{C}_\rho$. Then $A\overset{H}{\prec}0$ if and only if the joint spectral radius $r(A_1,\dots,A_n)<1$.*

Proof. Note that the map $X\mapsto P_\rho(X,R)$ is a positive free pluriharmonic function on $[B(\mathcal{H})^n]_1$ with coefficients in $B(F^2(H_n))$ and has the factorization

(2.4)
$$\begin{aligned}
&P_\rho(X,R)=\\
&=(I-R_X)^{-1}+(\rho-2)I+(I-R_X^*)^{-1}\\
&=(I-R_X^*)^{-1}\left[I-R_X+(\rho-2)(I-R_X^*)(I-R_X)+I-R_X^*\right](I-R_X)^{-1}\\
&=(I-R_X^*)^{-1}\left[\rho I+(1-\rho)(R_X^*+R_X)+(\rho-2)R_X^*R_X\right](I-R_X)^{-1},
\end{aligned}$$

where $R_X:=X_1^*\otimes R_1+\cdots+X_n^*\otimes R_n$ is the reconstruction operator associated with the n-tuple $X:=(X_1,\dots,X_n)\in[B(\mathcal{H})^n]_1$. We remark that, due to the fact that the spectral radius of R_X is equal to the joint spectral radius $r(X_1,\dots,X_n)$, the factorization above holds for any $X\in\mathcal{C}_\rho$ with $r(X_1,\dots,X_n)<1$.

Now, using Theorem 2.2 part (ii) and the above-mentioned factorization, we deduce that $A\overset{H}{\prec}0$ if and only if there exists $c>0$ such that

$$(I-R_{rA}^*)^{-1}\left[\rho I+(1-\rho)(R_{rA}^*+R_{rA})+(\rho-2)R_{rA}^*R_{rA}\right](I-R_{rA})^{-1}\le\rho c^2I$$

for any $r\in[0,1)$. Similar inequality holds if we replace the right creation operators by the left creation operators. Then, applying the noncommutative Poisson transform $\mathrm{id}\otimes P_{e^{i\theta}R}$, where $R:=(R_1,\dots,R_n)$, we obtain

(2.5)
$$\rho I+(1-\rho)(e^{-i\theta}R_{rA}^*+e^{i\theta}R_{rA})+(\rho-2)R_{rA}^*R_{rA}\le\rho c^2(I-re^{-i\theta}R_A^*)(I-re^{i\theta}R_A)$$

for any $r\in[0,1)$ and $\theta\in\mathbb{R}$.

On the other hand, since $A:=(A_1,\dots,A_n)\in\mathcal{C}_\rho$, we have $r(A_1,\dots,A_n)\le 1$. Suppose that $r(A_1,\dots,A_n)=1$. Taking into account that $r(R_A)=r(A_1,\dots,A_n)$, we can find $\lambda_0\in\mathbb{T}$ in the approximative spectrum of R_A. Consequently, there is a sequence $\{h_m\}$ in $\mathcal{H}\otimes F^2(H_n)$ such that $\|h_m\|=1$ and

$$\lambda_0h_m-R_Ah_m\to0\quad\text{as } m\to\infty. \tag{2.6}$$

In particular, relation (2.5) implies

$$\begin{aligned}(2.7)\quad &\rho\|h_m\|^2 + (1-\rho)\left[\langle \lambda_0 R_{rA}^* h_m, h_m\rangle + \left\langle \bar\lambda_0 R_{rA} h_m, h_m\right\rangle\right] + (\rho-2)\|R_{rA}h_m\|^2\\ &\qquad\leq \rho c^2\|h_m - \bar\lambda_0 R_{rA}h_m\|^2\end{aligned}$$

for any $r\in(0,1)$ and $m\in\mathbb{N}$. Note that due to (2.6) and the fact that $|\lambda_0|=1$, we have

$$\left\langle \bar\lambda_0 R_A h_m, h_m\right\rangle = \bar\lambda_0\left\langle R_A h_m - \lambda_0 h_m, h_m\right\rangle + 1 \to 1, \quad \text{ as } \ m\to\infty.$$

Since

$$\begin{aligned}\|h_m - \bar\lambda_0 R_{rA}h_m\| &\leq \|h_m - \bar\lambda_0 R_A h_m\| + \|\bar\lambda_0(R_A h_m - R_{rA}h_m)\|\\ &= \|\bar\lambda_0 h_m - R_A h_m\| + (1-r)\|R_A h_m\|\end{aligned}$$

and due to the fact that $\|R_A h_m\|\to 1$ as $m\to\infty$, we deduce that

$$\limsup_{m\to\infty}\|h_m - \bar\lambda_0 R_{rA}h_m\| \leq 1-r$$

for any $r\in(0,1)$. Now, since $R_{rA}=rR_A$ and taking $m\to\infty$ in relation (2.7), we obtain

$$\rho + 2(1-\rho)r + (\rho-2)r^2 \leq c^2\rho(1-r)^2$$

for any $r\in(0,1)$. Setting $r = 1-\frac{1}{m}$, $m\geq 2$, straightforward calculations imply $2m\leq \rho c^2 - \rho + 2$ for any $m\in\mathbb{N}$, which is a contradiction. Therefore, we must have $r(A_1,\ldots,A_n)<1$.

Conversely, assume that $A:=(A_1,\ldots,A_n)\in\mathcal{C}_\rho$ has the joint spectral radius $r(A_1,\ldots,A_n)<1$. Since $r(A_1,\ldots,A_n)=r(R_A)$, one can see that $M := \sup_{r\in(0,1)}\|(I-rR_A)^{-1}\|$ exists and $M\geq 1$. Hence

$$(2.8)\qquad M^2(I-R_{rA}^*)(I-R_{rA}) \geq I \geq I - R_{rA}^*R_{rA}$$

for any $r\in(0,1)$. Now we consider the case $\rho\geq 1$. Note that relation (2.8) implies

$$I - R_{rA}^*R_{rA} + (\rho-1)(I-R_{rA}^*)(I-R_{rA}) \leq \rho M^2(I-R_{rA}^*)(I-R_{rA}).$$

The latter inequality is equivalent to

$$\rho I + (1-\rho)(R_{rA}^* + R_{rA}) + (\rho-2)R_{rA}^*R_{rA} \leq \rho M^2(I-R_{rA}^*)(I-R_{rA}),$$

which, due to the factorization (2.4), is equivalent to

$$P_\rho(rA,R) \leq \rho M^2 = M^2 P_\rho(0,R)$$

for any $r\in[0,1)$. According to Theorem 2.2, we deduce that $A\overset{H}{\prec}0$.

Now, consider the case when $\rho\in(0,1)$. Since $\|R_{rA}\|\leq r\rho$ and $\delta-2<0$, we have

$$\begin{aligned}\rho I + (1-\rho)(R_{rA}^* + R_{rA}) + (\rho-2)R_{rA}^*R_{rA} &\leq \rho I + (1-\rho)(R_{rA}^*+R_{rA})\\ &\leq \rho I + 2(1-\rho)r\rho \leq (3\rho-2\rho^2)I.\end{aligned}$$

Using again the factorization (2.4), we deduce that

$$P_\rho(rA,R) \leq (3\rho-2\rho^2)(I-R_{rA}^*)^{-1}(I-R_{rA})^{-1}$$

for any $r \in (0,1)$. Hence and using the fact that $(I-R_{rA}^*)^{-1}(I-R_{rA})^{-1} \leq M^2 I$, we obtain

$$P_\rho(rA, R) \leq (3-2\rho)M^2 P_\rho(0, R)$$

for any $r \in (0,1)$. Using again Theorem 2.2, we get $A \overset{H}{\prec} 0$. The proof is complete. □

We mention that in the particular case when $n=1$ we can recover a result obtained by Ando, Suciu, and Timotin [1], when $\rho = 1$, and by G. Cassier and N. Suciu [9], when $\rho \neq 1$.

3. Hyperbolic metric on Harnack parts of the noncommutative ball $\mathcal{C}_\rho$

The relation $\overset{H}{\prec}$ induces an equivalence relation $\overset{H}{\sim}$ on the class $\mathcal{C}_\rho$. We provide a Harnack type double inequality for positive free pluriharmonic functions on the noncommutative ball $\mathcal{C}_\rho$ and use it to prove that the Harnack part of $\mathcal{C}_\rho$ which contains 0 coincides with the open noncommutative ball $[\mathcal{C}_\rho]_{<1}$. We introduce a hyperbolic metric on any Harnack part of $\mathcal{C}_\rho$ and obtain a concrete formula in terms of the reconstruction operator.

Since $\overset{H}{\prec}$ is a preorder relation on $\mathcal{C}_\rho$, it induces an equivalence relation $\overset{H}{\sim}$ on $\mathcal{C}_\rho$, which we call Harnack equivalence. The equivalence classes with respect to $\overset{H}{\sim}$ are called Harnack parts of $\mathcal{C}_\rho$. Let $A := (A_1, \ldots, A_n)$ and $B := (B_1, \ldots, B_n)$ be in $\mathcal{C}_\rho$. We say that A and B are Harnack equivalent (we denote $A \overset{H}{\sim} B$) if and only if there exists $c \geq 1$ such that

$$\begin{aligned} \frac{1}{c^2}\left[\Re p(B_1, \ldots, B_n) + (\rho-1)\Re p(0)\right] &\leq \Re p(A_1, \ldots, A_n) + (\rho-1)\Re p(0) \\ &\leq c^2\left[\Re p(B_1, \ldots, B_n) + (\rho-1)\Re p(0)\right] \end{aligned}$$

for any noncommutative polynomial with matrix-valued coefficients $p \in \mathbb{C}[X_1, \ldots, X_n] \otimes M_m$, $m \in \mathbb{N}$, such that $\Re p(X) \geq 0$ for any $X \in [B(\mathcal{H})^n]_1$. We also use the notation $A \underset{c}{\overset{H}{\sim}} B$ when $A \underset{c}{\overset{H}{\prec}} B$ and $B \underset{c}{\overset{H}{\prec}} A$. We remark that Theorem 2.2 can be used to provide several characterizations for the Harnack parts of $\mathcal{C}_\rho$.

The first result is an extension of Harnack inequality to positive free pluriharmonic functions on the noncommutative ball $\mathcal{C}_\rho$, $\rho > 0$.

Theorem 3.1. *If u is a positive free pluriharmonic function on $[B(\mathcal{H})^n]_1$ with operator-valued coefficients in $B(\mathcal{E})$ and $0 \leq r < 1$, then*

$$u(0)\,\frac{1 - r(2\rho-1)}{1+r} \leq u(rX_1, \ldots, rX_n) \leq u(0)\,\frac{1 + r(2\rho-1)}{1-r}$$

for any $(X_1, \ldots, X_n) \in \mathcal{C}_\rho$.

Proof. Let

$$u(Z_1,\dots,Z_n)=\sum_{k=1}^\infty\sum_{|\alpha|=k} Z_\alpha^*\otimes A_{(\alpha)}^* + I\otimes A_{(0)} + \sum_{k=1}^\infty\sum_{|\alpha|=k} Z_\alpha\otimes A_{(\alpha)}$$

be a positive free pluriharmonic function on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$. According to Theorem 1.4 from [49], for any $Y\in[B(\mathcal{H})^n]_1^-$ and $r\in[0,1)$, we have

$$u(0)\,\frac{1-r}{1+r}\le u(rY_1,\dots,rY_n)\le u(0)\,\frac{1+r}{1-r}. \tag{3.1}$$

On the other hand, let $(X_1,\dots,X_n)\in\mathcal{C}_\rho$ and let $(V_1,\dots,V_n)$ be the minimal isometric dilation of $(X_1,\dots,X_n)$ on a Hilbert space $\mathcal{K}_T\supseteq\mathcal{H}$. Since $X_\alpha=\rho P_{\mathcal{H}}V_\alpha|_{\mathcal{H}}$ for any $\alpha\in\mathbb{F}_n^+\backslash\{g_0\}$, and using the free pluriharmonic functional calculus, we have

$$\begin{aligned}
&u(rX_1,\dots,rX_n)=\\
&=\sum_{k=1}^\infty\sum_{|\alpha|=k} r^{|\alpha|}X_\alpha^*\otimes A_{(\alpha)}^* + I\otimes A_{(0)} + \sum_{k=1}^\infty\sum_{|\alpha|=k} r^{|\alpha|}X_\alpha\otimes A_{(\alpha)}\\
&=\rho(P_{\mathcal{H}}\otimes I_{\mathcal{E}})\left[\sum_{k=1}^\infty\sum_{|\alpha|=k} r^{|\alpha|}V_\alpha^*\otimes A_{(\alpha)}^*\right]|_{\mathcal{H}\otimes\mathcal{E}} + I_{\mathcal{H}}\otimes A_{(0)}\\
&\qquad + \rho(P_{\mathcal{H}}\otimes I_{\mathcal{E}})\left[\sum_{k=1}^\infty\sum_{|\alpha|=k} r^{|\alpha|}V_\alpha\otimes A_{(\alpha)}\right]|_{\mathcal{H}\otimes\mathcal{E}}\\
&=\rho(P_{\mathcal{H}}\otimes I_{\mathcal{E}})u(rV_1,\dots,rV_n)|_{\mathcal{H}\otimes\mathcal{E}}+(1-\rho)u(0),
\end{aligned}$$

where the convergence is in the operator norm topology. Due to (3.1), we have

$$u(0)\,\frac{1-r}{1+r}\le u(rV_1,\dots,V_n)\le u(0)\,\frac{1+r}{1-r}.$$

Consequently, we deduce that

$$\begin{aligned}
u(0)\left[\frac{\rho(1-r)}{1+r}+(1-\rho)\right]&\le \rho(P_{\mathcal{H}}\otimes I_{\mathcal{E}})u(rV_1,\dots,rV_n)|_{\mathcal{H}\otimes\mathcal{E}}+(1-\rho)u(0)\\
&\le u(0)\left[\frac{\rho(1+r)}{1-r}+(1-\rho)\right].
\end{aligned}$$

Since

$$u(rX_1,\dots,rX_n)=\rho(P_{\mathcal{H}}\otimes I_{\mathcal{E}})u(rV_1,\dots,rV_n)|_{\mathcal{H}\otimes\mathcal{E}}+(1-\rho)u(0),$$

the result follows. □

Now, we can determine the Harnack part of $\mathcal{C}_\rho$ which contains 0.

Theorem 3.2. *Let $A:=(A_1,\dots,A_n)$ be in $\mathcal{C}_\rho$. Then the following statements are equivalent:*

(i) $\omega_\rho(A_1,\ldots,A_n)<1$;
(ii) $A\overset{H}{\sim}0$;
(iii) $r(A_1,\ldots,A_n)<1$ *and* $P_\rho(A,R)\geq aI$ *for some constant* $a>0$.

Proof. First, we prove that $(i)\implies(ii)$. Let $A:=(A_1,\ldots,A_n)$ be in $\mathcal{C}_\rho$ and assume that $\omega_\rho(A)<1$. Then there is $r_0\in(0,1)$ such that $\omega_\rho(\frac{1}{r_0}A)=\frac{1}{r_0}\omega_\rho(A)<1$. Consequently, $\frac{1}{r_0}A\in\mathcal{C}_\rho$.
According to Theorem 3.1, we have

$$\Re p(0)\,\frac{1-r_0(2\rho-1)}{1+r_0}\leq\Re p(A_1,\ldots,A_n)\leq\Re p(0)\,\frac{1+r_0(2\rho-1)}{1-r_0}$$

for any noncommutative polynomial with matrix-valued coefficients $p\in\mathbb{C}[X_1,\ldots,X_n]\otimes M_m$, $m\in\mathbb{N}$, such that $\Re p\geq0$ on $[B(\mathcal{H})^n]_1$. Hence, we deduce that $A\overset{H}{\sim}0$.
To prove that $(ii)\implies(iii)$, assume that $A\overset{H}{\sim}0$. Due to Theorem 2.5, we have $r(A)<1$. Using now Theorem 2.2, we deduce that there exists $c>0$ such that

$$P_\rho(rA,R)\geq\frac{1}{c^2}P_\rho(0,R)=\frac{\rho}{c^2}I \tag{3.2}$$

for any $r\in[0,1)$. Since $r(A)<1$, one can prove that $\lim_{r\to1}P_\rho(rA,R)=P_\rho(A,R)$ in the operator norm topology. Consequently, taking $r\to1$ in relation (3.2), we obtain item (iii).
It remains to show that $(iii)\implies(i)$. Assume that $r(A_1,\ldots,A_n)<1$ and $P_\rho(A,R)\geq aI$ for some constant $a>0$. Note that there exists $t_0\in(0,1)$ such that the map

$$t\mapsto\left(I-\sum_{i=1}^nA_i^*\otimes tR_i\right)^{-1}+(\rho-2)I+\left(I-\sum_{i=1}^nA_i\otimes tR_i^*\right)^{-1}$$

is well-defined and continuous on $[0,1+t_0]$ in the operator norm topology. In particular, there is $\epsilon_0\in(0,t_0)$ such that

$$\|P_\rho(A,R)-P_\rho(A,tR)\|<\frac{a}{2}$$

for any $t\in(1-\epsilon_0,1+\epsilon_0)$. Consequently, if $\gamma_0\in(1,1+\epsilon_0)$, then

$$P_\rho(\gamma_0A,R)\geq P_\rho(A,R)-\|P_\rho(A,R)-P_\rho(\gamma_0A,R)\|I\geq\frac{a}{2}I>0.$$

Due to Theorem 1.1, we have $\gamma_0A\in\mathcal{C}_\rho$, which implies $\omega(\gamma_0A)\leq1$. Therefore, $\omega(A)\leq\frac{1}{\gamma_0}<1$ and item (i) holds. The proof is complete. $\square$

We remark that, when $n=1$, we recover a result obtain by Foiaş [15] if $\rho=1$, and by Cassier and Suciu [9] if $\rho>0$.

Given $A, B \in \mathcal{C}_\rho$, $\rho > 0$, in the same Harnack part of $\mathcal{C}_\rho$, i.e., $A \overset{H}{\sim} B$, we introduce

$$\Lambda_\rho(A,B) := \inf\left\{c > 1 : \; A \underset{c}{\overset{H}{\sim}} B\right\}. \tag{3.3}$$

Note that, due to Theorem 2.2, $A \overset{H}{\sim} B$ if and only if the operator $L_{B,A}$ is invertible. In this case, $L_{B,A}^{-1} = L_{A,B}$ and

$$\Lambda_\rho(A,B) = \max\{\|L_{A,B}\|, \|L_{B,A}\|\}.$$

To prove the latter equality, assume that $A \underset{c}{\overset{H}{\sim}} B$ for some $c \geq 1$. Due to the same theorem, we have $\|L_{B,A}\| \leq c$ and $\|L_{A,B}\| \leq c$. Consequently,

$$\max\{\|L_{A,B}\|, \|L_{B,A}\|\} \leq \inf\left\{c \geq 1 : \; A \underset{c}{\overset{H}{\sim}} B\right\} = \Lambda_\rho(A,B). \tag{3.4}$$

On the other hand, setting $c_0 := \|L_{B,A}\|$ and $c_0' := \|L_{A,B}\|$, Theorem 2.2 implies $A \underset{c_0}{\overset{H}{\prec}} B$ and $B \underset{c_0'}{\overset{H}{\prec}} A$. Hence, we deduce that $A \underset{d}{\overset{H}{\sim}} B$, where $d := \max\{c_0, c_0'\}$. Consequently, $\Lambda_\rho(A,B) \leq d$, which together with relation (3.4) imply $\Lambda_\rho(A,B) = \max\{\|L_{A,B}\|, \|L_{B,A}\|\}$, which proves our assertion.

Now, we can introduce a hyperbolic (*Poincaré-Bergman* type) metric $\delta_\rho : \Delta \times \Delta \to \mathbb{R}^+$ on any Harnack part Δ of $\mathcal{C}_\rho$, by setting

$$\delta_\rho(A,B) := \ln \Lambda_\rho(A,B), \qquad A, B \in \Delta. \tag{3.5}$$

Due to our discussion above, we also have

$$\delta_\rho(A,B) = \ln \max\left\{\|L_{A,B}\|, \left\|L_{A,B}^{-1}\right\|\right\}.$$

PROPOSITION 3.3. *δ_ρ is a metric on any Harnack part of $\mathcal{C}_\rho$.*

Proof. The proof is similar to that of Proposition 2.2 from [49], but uses ρ-pluriharmonic kernels. ☐

We remark that, according to Theorem 3.2, the set

$$[\mathcal{C}_\rho]_{<1} := \{(X_1, \ldots, X_n) \in B(\mathcal{H})^n : \; \omega_\rho(X_1, \ldots, X_n) < 1\}$$

is the Harnack part of $\mathcal{C}_\rho$ containing 0.

In what follows we calculate the norm of $L_{Y,X}$ with $X, Y \in [\mathcal{C}_\rho]_{<1}$, in terms of the reconstruction operators.

THEOREM 3.4. *If $X, Y \in [\mathcal{C}_\rho]_{<1}$, then $\|L_{Y,X}\| = \|C_{\rho,X} C_{\rho,Y}^{-1}\|$, where*

$$C_{\rho,X} := \Delta_{\rho,X}(I - R_X)^{-1},$$
$$\Delta_{\rho,X} := [\rho I + (1-\rho)(R_X^* + R_X) + (\rho - 2)R_X^* R_X]^{1/2}.$$

Moreover, if $X, Y \in \mathcal{C}_\rho$ is such that $X \overset{H}{\prec} Y$, then $\|L_{Y,X}\| = \sup_{r\in[0,1)} \|C_{\rho,rX} C_{\rho,rY}^{-1}\|$.

Proof. Since $X, Y \in [\mathcal{C}_\rho]_{<1}$, Theorem 3.2 implies $X \overset{H}{\sim} Y$, $r(X) < 1$, and $r(Y) < 1$. Let $c > 1$ and assume that $P_\rho(rX, R) \leq c^2 P_\rho(rY, R)$ for any $r \in [0,1)$. Since $r(X) < 1$ and $r(Y) < 1$, we can take the limit, as $r \to 1$, in the operator norm topology, and obtain $P_\rho(X, R) \leq c^2 P_\rho(Y, R)$. Conversely, if the latter inequality holds, then $P_\rho(X, S) \leq c^2 P_\rho(Y, S)$, where $S := (S_1, \ldots, S_n)$ is the n-tuple of left creation operators. Applying the noncommutative Poisson transform $\mathrm{id} \otimes P_{rR}$, $r \in [0,1)$, and taking into account that it is a positive map, we deduce that $P_\rho(rX, R) \leq c^2 P_\rho(rY, R)$ for any $r \in [0,1)$.

Therefore, due to Theorem 2.2, we have

$$P_\rho(X, R) \leq c^2 P_\rho(Y, R) \quad \text{if and only if} \quad \|L_{Y,X}\| \leq c. \tag{3.6}$$

We recall that the free pluriharmonic kernel $P_\rho(X, R)$ with $X \in [\mathcal{C}_\rho]_{<1}$, has the factorization $P(X, R) = C_{\rho,X}^* C_{\rho,X}$. Due to Theorem 3.2, $P_\rho(X, R)$ is invertible and, consequently, so is $C_{\rho,X}$. Consequently,

$$P_\rho(X, R) \leq c^2 P_\rho(Y, R) \quad \text{if and only if} \quad {C_{\rho,Y}^*}^{-1} C_{\rho,X}^* C_{\rho,X} C_{\rho,Y}^{-1} \leq c^2 I.$$

Setting $c_0 := \|C_{\rho,X} C_{\rho,Y}^{-1}\|$, we have $P_\rho(X, R) \leq c_0^2 P_\rho(Y, R)$. Now, due to relation (3.6), we obtain

$$\|L_{Y,X}\| \leq c_0 = \|C_{\rho,X} C_{\rho,Y}^{-1}\|.$$

Setting $c_0' := \|L_{Y,X}\|$ and using again (3.6), we obtain $P_\rho(X, R) \leq {c_0'}^2 P_\rho(Y, R)$. Hence, we deduce that ${C_{\rho,Y}^*}^{-1} C_{\rho,X}^* C_{\rho,X} C_{\rho,Y}^{-1} \leq {c_0'}^2 I$, which implies

$$\|C_{\rho,X} C_{\rho,Y}^{-1}\| \leq c_0' = \|L_{Y,X}\|.$$

Therefore, $\|L_{Y,X}\| = \|C_{\rho,X} C_{\rho,Y}^{-1}\|$. The last part of the theorem is now obvious. □

Combining Theorem 3.4 with the remarks preceding Proposition 3.3, we obtain a concrete formula for the hyperbolic metric δ_ρ on $[\mathcal{C}_\rho]_{<1}$ in terms of the reconstruction operator, which is the main result of this section.

Theorem 3.5. *Let $\delta_\rho : [\mathcal{C}_\rho]_{<1} \times [\mathcal{C}_\rho]_{<1} \to [0, \infty)$ be the hyperbolic metric. If $X, Y \in [\mathcal{C}_\rho]_{<1}$, then*

$$\delta_\rho(X, Y) = \ln \max \left\{ \left\| C_{\rho,X} C_{\rho,Y}^{-1} \right\|, \left\| C_{\rho,Y} C_{\rho,X}^{-1} \right\| \right\},$$

where

$$C_{\rho,X} := \Delta_{\rho,X} (I - R_X)^{-1},$$

$$\Delta_{\rho,X} := \left[\rho I + (1-\rho)(R_X^* + R_X) + (\rho - 2) R_X^* R_X\right]^{1/2},$$

and $R_X := X_1^* \otimes R_1 + \cdots + X_n^* \otimes R_n$ is the reconstruction operator associated with the right creation operators $R_1, \ldots, R_n$ and $X := (X_1, \ldots, X_n) \in [\mathcal{C}_\rho]_{<1}$.

Using Theorem 2.2, one can easily obtain the following result. Since the proof is similar to that of Lemma 2.6 from [49], we shall omit it.

LEMMA 3.6. *Let $X := (X_1, \ldots, X_n)$ and $Y := (Y_1, \ldots, Y_n)$ be in $\mathcal{C}_\rho$. Then the following properties hold.*

(i) *$X \overset{H}{\sim} Y$ if and only if $rX \overset{H}{\sim} rX$ for any $r \in [0,1)$ and $\sup_{r\in[0,1)} \Lambda_\rho(rX, rY) < \infty$. In this case,*

$$\Lambda_\rho(X, Y) = \sup_{r\in[0,1)} \Lambda_\rho(rX, rY) \quad \text{and} \quad \delta_\rho(X, Y) = \sup_{r\in[0,1)} \delta_\rho(rX, rY).$$

(ii) *If $X \overset{H}{\sim} Y$, then the functions $r \mapsto \Lambda_\rho(rX, rY)$ and $r \mapsto \delta_\rho(rX, rY)$ are increasing on $[0, 1)$.*

Putting together Theorem 3.5 and Lemma 3.6, we deduce the following result.

THEOREM 3.7. *Let $X := (X_1, \ldots, X_n)$ and $Y := (Y_1, \ldots, Y_n)$ be in $\mathcal{C}_\rho$ such that $X \overset{H}{\sim} Y$. Then the metric δ_ρ satisfies the relation*

$$\delta_\rho(X, Y) = \ln \max \left\{ \sup_{r\in[0,1)} \left\| C_{\rho,rX} C_{\rho,rY}^{-1} \right\|, \sup_{r\in[0,1)} \left\| C_{\rho,rY} C_{\rho,rX}^{-1} \right\| \right\},$$

where $C_{\rho,X} := \Delta_{\rho,X}(I - R_X)^{-1}$ and $R_X := X_1^* \otimes R_1 + \cdots + X_n^* \otimes R_n$ is the reconstruction operator.

Using the Harnack type inequality of Theorem 3.1, we obtain an upper bound for the hyperbolic distance δ_ρ on $[\mathcal{C}_\rho]_{<1}$. First, we need the following result.

PROPOSITION 3.8. *Let f be in the noncommutative disc algebra $\mathcal{A}_n$ such that $\Re f \geq 0$ and let $X := (X_1, \ldots, X_n) \in \mathcal{C}_\rho$ be with $\omega_\rho(X) < 1$. Then*

$$\rho \frac{1 - \omega_\rho(X)}{1 + \omega_\rho(X)} \Re f(0) \leq \Re f(X_1, \ldots, X_n) + (\rho - 1)\Re f(0) \leq \rho \frac{1 + \omega_\rho(X)}{1 - \omega_\rho(X)}.$$

Proof. Let $r := \omega_\rho(X)$ and define $Y := \frac{1}{r}X$. Since $\omega_\rho(Y) = \frac{1}{r}\omega_\rho(X) = 1$, we deduce that $Y \in \mathcal{C}_\rho$. Applying Theorem 3.1 to Y, we obtain

$$\frac{1 - \omega_\rho(X)(2\rho - 1)}{1 + \omega_\rho(X)} \Re f(0) \leq \Re f(X_1, \ldots, X_n) \leq \rho \frac{1 + \omega_\rho(X)(2\rho - 1)}{1 - \omega_\rho(X)}.$$

It is easy to see that the latter inequality is equivalent to the one from the proposition. □

Now, we can deduce the following upper bound for the hyperbolic distance on $[\mathcal{C}_\rho]_{<1}$.

COROLLARY 3.9. *For any $X, Y \in [\mathcal{C}_\rho]_{<1}$,*

$$\delta_\rho(X, Y) \leq \frac{1}{2} \ln \frac{(1 + \omega_\rho(X))(1 + \omega_\rho(Y))}{(1 - \omega_\rho(X))(1 - \omega_\rho(Y))}.$$

Proof. Using Theorem 2.2 and the inequality of Proposition 3.8, we deduce that

$$\Lambda_\rho(X,0) \le \left(\frac{1+\omega_\rho(X)}{1-\omega_\rho(X)}\right)^{1/2}.$$

On the other hand, since δ_ρ is a metric on $[\mathcal{C}_\rho]_{<1}$, we have $\delta(X,Y) \le \delta(X,0) + \delta_\rho(Y,0)$. Taking into account that $\delta_\rho(X,Y) = \ln \Lambda_\rho(X,Y)$, the result follows. $\square$

We remark that when $\rho = 1$, the inequality of Corollary 3.9 is sharper then the one obtained in Corollary 2.5 from [49].

Using Corollary 2.4, on can easily obtain the following result.

Corollary 3.10. *Let $\rho > 0$, and $1 \le m < n$. Consider two n-tuples $A := (A_1, \dots, A_m) \in B(\mathcal{H})^m$ and $B := (B_1, \dots, B_m) \in B(\mathcal{H})^m$ in the class $\mathcal{C}_\rho$ and their extensions $\widetilde{A} := (A_1, \dots, A_m, 0, \dots, 0)$ and $\widetilde{B} := (B_1, \dots, B_m, 0, \dots, 0)$ in $B(\mathcal{H})^n$, respectively. Then*

$$A \overset{H}{\sim} B \quad \text{if and only if} \quad \widetilde{A} \overset{H}{\sim} \widetilde{B}.$$

Moreover, in this case,

$$\delta_\rho(A,B) = \delta_\rho(\widetilde{A}, \widetilde{B}).$$

In what follows we provide a few properties for the map $\rho \mapsto \delta_\rho(A,B)$.

Lemma 3.11. *Let $A := (A_1, \dots, A_n) \in B(\mathcal{H})^n$ and $B := (B_1, \dots, B_n) \in B(\mathcal{H})^n$ be in the class $\mathcal{C}_\rho$ and let $c > 0$ and $0 < \rho \le \rho'$. Then the following statements hold.*

(i) *if $A \underset{c}{\overset{H}{\prec}} B$ in $\mathcal{C}_\rho$, then $A \underset{c}{\overset{H}{\prec}} B$ in $\mathcal{C}_{\rho'}$;*

(ii) *if $A \underset{c}{\overset{H}{\sim}} B$ in $\mathcal{C}_\rho$, then if $A \underset{c}{\overset{H}{\sim}} B$ in $\mathcal{C}_\rho$ and*

$$\delta_{\rho'}(A,B) \le \delta_\rho(A,B).$$

Proof. First recall that $\mathcal{C}_\rho \subseteq \mathcal{C}_{\rho'}$. If $A \underset{c}{\overset{H}{\prec}} B$ in $\mathcal{C}_\rho$, then

$$\Re p(A_1, \dots, A_n) + (\rho - 1)\Re p(0) \le c^2 \left[\Re p(B_1, \dots, B_n) + (\rho - 1)\Re p(0)\right]$$

for any noncommutative polynomial with matrix-valued coefficients $p \in \mathbb{C}[X_1, \dots, X_n] \otimes M_m$, $m \in \mathbb{N}$, such that $\Re p(X) \ge 0$ for any $X \in [B(\mathcal{H})^n]_1$. Hence, $c \ge 1$ and, consequently, the inequality above holds when we replace ρ with $\rho' \ge \rho$. This shows that $A \underset{c}{\overset{H}{\prec}} B$ in $\mathcal{C}_{\rho'}$. Part (ii) is a clear consequence of (i) and the definition of the hyperbolic metric. $\square$

If $A := (A_1, \dots, A_n) \in B(\mathcal{H})^n$ is a nonzero n-tuple of operators such that $A \in [C_\infty]_{<1}$, i.e., the joint spectral radius $r(A) < 1$, then

$$\rho_A := \inf\{\rho > 0 : \ A \in \mathcal{C}_\rho\} > 0.$$

Indeed, if $\rho, \rho' \in (0,\infty]$, $\rho \le \rho'$, then $\mathcal{C}_\rho \subseteq \mathcal{C}_{\rho'}$ and, moreover, we have

$$\omega_{\rho'}(A) \le \omega_\rho(A), \quad r(A) = \lim_{\rho\to\infty} \omega_\rho(A), \qquad A \in B(\mathcal{H})^n.$$

Consequently, there exists $\rho > 0$ such that $\omega_{\rho'}(A) < 1$, for any $\rho' \ge \rho$. Assume now that $\rho_A = 0$. Then $T \in \mathcal{C}_\rho$, i.e., $\omega_\rho(A) \le 1$ for any $\rho > 0$. On the other hand, we know that $\|A\| \le \rho\omega_\rho(A)$. Taking $\rho \to 0$, we deduce that $A = 0$, which is a contradiction. This proves our assertion.

Note that if $A, B \in [\mathcal{C}_\infty]_{<1}$, then

$$\rho_{A,B} := \inf\{\rho > 0 : \; A, B \in \mathcal{C}_\rho\} = \max\{\rho_A, \rho_B\}.$$

Proposition 3.12. *If $A, B \in [\mathcal{C}_\infty]_{<1}$, then the map*

$$[\rho_{A,B}, \infty) \ni \rho \mapsto \delta_\rho(A,B) \in \mathbb{R}^+$$

is continuous, decreasing, and

$$\lim_{\rho\to\infty} \delta_\rho(A,B) = 0.$$

Proof. Using Theorem 3.5 and Lemma 3.11, one can easily deduce that the map $\rho \mapsto \delta_\rho(A,B)$ is continuous and decreasing. To prove the last part of the proposition, note that since $\delta_\rho(A,B) \le \delta_\rho(A,0)+\delta_\rho(0,B)$, it is enough to show that $\lim_{\rho\to\infty} \delta_\rho(A,0) = 0$. To this end, note that Theorem 3.5, implies

$$\delta_\rho(A,0) = \ln\max\left\{\left\|C_{\rho,A}C_{\rho,0}^{-1}\right\|, \left\|C_{\rho,0}C_{\rho,A}^{-1}\right\|\right\}, \tag{3.7}$$

where

$$C_{\rho,A}C_{\rho,0}^{-1} = \frac{1}{\sqrt{\rho}}\left[\rho I + (1-\rho)(R_A^* + R_A) + (\rho-2)R_A^*R_A\right]^{1/2}(I - R_A)^{-1}.$$

Hence, we deduce that

$$\begin{aligned}
\lim_{\rho\to\infty} \|C_{\rho,A}C_{\rho,0}^{-1}\| &= \left\|\left[I - (R_A^* + R_A) + R_A^*R_A\right]^{1/2}(I - R_A)^{-1}\right\| \\
&= \left\|(I - R_A^*)^{-1}\left[I - (R_A^* + R_A) + R_A^*R_A\right](I - R_A)^{-1}\right\| \\
&= \left\|(I - R_A^*)^{-1}(I - R_A^*)(I - R_A)(I - R_A)^{-1}\right\| \\
&= 1
\end{aligned}$$

Similarly, we have $\lim_{\rho\to\infty} \|C_{\rho,0}C_{\rho,A}^{-1}\| = 1$. Using now relation (3.7), we complete the proof. $\square$

4. Mapping theorems for free holomorphic functions on noncommutative balls

In this section, we provide mapping theorems, spectral von Neumann inequalities, and Schwarz type results for free holomorphic functions on noncommutative balls, with respect to the hyperbolic metric and the operator radius ω_ρ, $\rho \in (0, \infty]$.

First, we prove the following mapping theorem for the classes $\mathcal{C}_\rho$, $\rho > 0$.

Theorem 4.1. *Let $f := (f_1, \ldots, f_m)$ be a contractive free holomorphic function with $\|f(0)\| < 1$ such that the boundary functions $\widetilde{f}_1, \ldots, \widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$. If $(T_1, \ldots, T_n) \in B(\mathcal{H})^n$ is of class $\mathcal{C}_\rho$, $\rho > 0$, then $f(T_1, \ldots, T_n)$ is of class $\mathcal{C}_{\rho_f}$, where*

$$\rho_f := \begin{cases} 1 + (\rho - 1)\frac{1-\|f(0)\|}{1+\|f(0)\|} & \text{if } \rho < 1 \\ \\ 1 + (\rho - 1)\frac{1+\|f(0)\|}{1-\|f(0)\|} & \text{if } \rho \geq 1. \end{cases} \tag{4.1}$$

Proof. Let $p \in \mathbb{C}[Z_1, \ldots, Z_m] \otimes M_k$, $k \in \mathbb{N}$, be such that $\Re p \geq 0$ on the unit ball $[B(\mathcal{H})^m]_1$. This is equivalent to $\Re p(S_1', \ldots S_m') \geq 0$, where $S_1', \ldots, S_m'$ are the left creation operators on the full Fock space $F^2(H_m)$. Applying the noncommutative Poisson transform $P_{f(X_1,\ldots,X_n)} \otimes \mathrm{id}$, which is a completely positive linear map, to the inequality $\Re p(S_1', \ldots S_m') \geq 0$, we obtain

$$\Re p(f(X_1, \ldots, X_n)) \geq 0, \qquad X \in [B(\mathcal{H})^n]_1.$$

Moreover, since the boundary functions $\widetilde{f}_1, \ldots, \widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$, we deduce that the boundary function of the composition $p \circ f$ is $p(\widetilde{f}_1, \ldots, \widetilde{f}_m) \in \mathcal{A}_n \bar{\otimes}_{min} M_k$.

Assume that $(T_1, \ldots, T_n) \in \mathcal{C}_\rho$. Using the free pluriharmonic functional calculus of Theorem 1.3 and Theorem 1.1, we deduce that

$$\Re(p \circ f)(T_1, \ldots, T_n) + (\rho - 1)\Re(p \circ f)(0) \geq 0. \tag{4.2}$$

On the other hand, according to the Harnack type inequality of Theorem 1.4 from [49] applied to the positive free pluriharmonic function $\Re p$ at the point $f(0) = (f_1(0), \ldots, f_m(0))$, we have

$$\Re p(0)\frac{1 - \|f(0)\|}{1 + \|f(0)\|} \leq \Re p(f(0)) \leq \Re p(0)\frac{1 + \|f(0)\|}{1 - \|f(0)\|}. \tag{4.3}$$

Let $\gamma > 0$ and note that

$$\Re p(f(T_1, \ldots, T_n)) + (\gamma - 1)\Re p(0) = A + B, \tag{4.4}$$

where

$$\begin{aligned} A &:= \Re p(f(T_1, \ldots, T_n)) + (\rho - 1)p(f(0)) \\ B &:= (\gamma - 1)\Re p(0) - (\rho - 1)p(f(0)). \end{aligned} \tag{4.5}$$

Assume now that $\rho \geq 1$. Using the second inequality in (4.3), we obtain

$$\begin{aligned} B &\geq (\gamma-1)\Re p(0) - (\rho-1)\Re p(0)\frac{1+\|f(0)\|}{1-\|f(0)\|} \\ &= \left[(\gamma-1) - (\rho-1)\frac{1+\|f(0)\|}{1-\|f(0)\|}\right]\Re p(0), \end{aligned}$$

which is positive if $\gamma \geq 1 + (\rho-1)\frac{1+\|f(0)\|}{1-\|f(0)\|}$. In this case, using relation (4.4) and (4.2), we obtain

$$\Re p(f(T_1,\dots,T_n)) + (\gamma-1)\Re p(0) \geq 0$$

for any $p \in \mathbb{C}[Z_1,\dots,Z_n] \otimes M_k$, $k \in \mathbb{N}$, be such that $\Re p \geq 0$ on the unit ball $[B(\mathcal{H})^m]_1$. Applying Theorem 1.1, we deduce that $f(T_1,\dots,T_n) \in \mathcal{C}_\gamma$. In particular, we have $f(T_1,\dots,T_n) \in \mathcal{C}_{\delta_f}$ where

$$\delta_f := 1 + (\rho-1)\frac{1+\|f(0)\|}{1-\|f(0)\|}.$$

Now, we consider the case $\rho \in (0,1)$. Using the first inequality in (4.3), we obtain

$$B \geq \left[(\gamma-1) - (\rho-1)\frac{1-\|f(0)\|}{1+\|f(0)\|}\right]\Re p(0),$$

which is positive if $\gamma \geq 1 + (\rho-1)\frac{1-\|f(0)\|}{1+\|f(0)\|}$. As above, using relations (4.4) and (4.2), we obtain

$$\Re p(f(T_1,\dots,T_n)) + (\gamma-1)\Re p(0) \geq 0$$

for any $p \in \mathbb{C}[Z_1,\dots,Z_n] \otimes M_k$, $k \in \mathbb{N}$, be such that $\Re p \geq 0$ on the unit ball $[B(\mathcal{H})^m]_1$. Theorem 1.1 implies $f(T_1,\dots,T_n) \in \mathcal{C}_\gamma$. In particular, we have $f(T_1,\dots,T_n) \in \mathcal{C}_{\delta_f}$ where

$$\delta_f := 1 + (\rho-1)\frac{1-\|f(0)\|}{1+\|f(0)\|}.$$

The proof is complete. $\square$

Note that under the conditions of Theorem 4.1, $\rho \leq \rho_f$ and $\rho = 1 \implies \rho_f = 1$. Moreover, if $\rho \neq 1$, then $\rho_f = \rho$ if and only if $f(0) = 0$. On can also show that $\rho_f \leq 1$ if $\rho \leq 1$.

We remark that, under the conditions of Theorem 4.1, there exists $T := (T_1,\dots,T_n) \in B(\mathcal{H})^n$ such that if $\rho > 0$ is the smallest positive number such that $(T_1,\dots,T_n) \in \mathcal{C}_\rho$, then there exists a free holomorphic function f such that ρ_f is the smallest positive number with the property that $f(T_1,\dots,T_n) \in \mathcal{C}_{\rho_f}$. Indeed, if $n \leq m$, take $f(X_1,\dots,X_n) = (X_1,\dots,X_n,0,\dots,0)$ and use Corollary 2.4. When $n > m$, take $f(X_1,\dots,X_n) = (X_1,\dots,X_m)$ and $T := (T_1,\dots,T_n,0,\dots,0)$ with $(T_1,\dots,T_n) \in \mathcal{C}_\rho$.

Corollary 4.2. *Let $f := (f_1, \dots, f_m)$ be a bounded free holomorphic function with $\|f(0)\| < \|f\|_\infty$ such that the boundary functions $\widetilde{f}_1, \dots, \widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$. If $(T_1, \dots, T_n) \in B(\mathcal{H})^n$ is of class $\mathcal{C}_\rho$, $\rho > 0$, then*

$$\omega_{\rho_f}(f(T_1, \dots, T_n)) \le \|f\|_\infty,$$

where ρ_f is given by relation (4.1). *In particular, if $f(0) = 0$ and $(T_1, \dots, T_n) \in \mathcal{C}_\rho$, then*

$$\omega_\rho(f(T_1, \dots, T_n)) \le \|f\|_\infty.$$

Proof. Applying Theorem 4.1 the function $\frac{1}{\|f\|_\infty} f$, we deduce that $\frac{1}{\|f\|_\infty} f(T_1, \dots, T_n)$ is in the class $\mathcal{C}_{\rho_f}$, which is equivalent to $\omega_{\rho_f}\left(\frac{1}{\|f\|_\infty} f(T_1, \dots, T_n)\right) \le 1$, and the first inequality of the theorem follows. Hence, and using the fact that $\rho_f = \rho$ when $f(0) = 0$, we complete the proof. □

A simple consequence of Corollary 4.2 is the following power inequality.

Corollary 4.3. *If $(T_1, \dots, T_n) \in B(\mathcal{H})^n$ is non-zero, $\rho \in (0, \infty)$, and $k \ge 1$, then*

$$\omega_\rho(T_\alpha : \ \alpha \in \mathbb{F}_n^+, |\alpha| = k) \le \omega_\rho(T_1, \dots, T_n).$$

Proof. Since $\|(T_1, \dots, T_n)\| \le \rho\omega_\rho(T_1, \dots, T_n)$, we have $\omega_\rho(T_1, \dots, T_n) \ne 0$. Applying the second part of Corollary 4.2 to the n-tuple of operators $\left(\frac{1}{\omega_\rho(T_1,\dots,T_n)} T_1, \dots, \frac{1}{\omega_\rho(T_1,\dots,T_n)} T_n\right) \in \mathcal{C}_\rho$ and to the free holomorphic function

$$f(X_1, \dots, X_n) := (X_\alpha : \ \alpha \in \mathbb{F}_n^+, |\alpha| = k), \qquad (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1,$$

we complete the proof. □

Theorem 4.4. *Let $f := (f_1, \dots, f_m)$ be a bounded free holomorphic function with $\|f(0)\| < \|f\|_\infty$ such that the boundary functions $\widetilde{f}_1, \dots, \widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$. Then, for each $r \in [0, 1)$,*

$$\sup_{T \in \mathcal{C}_\rho, \ \omega_\rho(T) \le r} \omega_{\rho_f}(f(T_1, \dots, T_n)) \le \|f(rS_1, \dots, rS_n)\|,$$

where $S_1, \dots, S_n$ are the left creation operators.

Proof. Consider the free holomorphic function f_r, defined by

$$f_r(X_1, \dots, X_n) := f(rX_1, \dots, rX_n), \qquad (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$$

and recall that $\|f_r\|_\infty = \|f(rS_1, \dots, rS_n)\|$. Applying Corollary 4.2 to f_r, we have

$$\omega_{\rho_{f_r}}(f_r(A_1, \dots, A_n)) \le \|f_r\|_\infty, \quad (A_1, \dots, A_n) \in \mathcal{C}_\rho \tag{4.6}$$

Since $f(0) = f_r(0)$, we have $\rho_f = \rho_{f_r}$. Consequently, if we assume that $\omega_\rho(T_1,\dots,T_n) \le r < 1$, then $\left(\frac{1}{r}T_1,\dots,\frac{1}{r}T_n\right) \in \mathcal{C}_\rho$ and inequality (4.6) implies

$$\omega_{\rho_f}(f(T_1,\dots,T_n)) = \omega_{\rho_f}\left(f_r\left(\frac{1}{r}T_1,\dots,\frac{1}{r}T_n\right)\right) \le \|f_r\|_\infty,$$

which completes the proof. □

COROLLARY 4.5. *Let $(T_1,\dots,T_n) \in B(\mathcal{H})^n$ be such that $\omega_\rho(T_1,\dots,T_n) < 1$, and let $f := (f_1,\dots,f_m)$ be a bounded free holomorphic function with the following properties:*

(i) *the boundary functions $\widetilde{f}_1,\dots,\widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$.*
(ii) *f_j has the standard representation of the form*

$$f_j(X_1,\dots,X_n) = \sum_{|\alpha|\ge k} a_\alpha^{(j)} X_\alpha, \qquad j = 1,\dots,m.$$

Then

$$\omega_\rho(f(T_1,\dots,T_n)) \le \omega_\rho(T_1,\dots,T_n)^k \|f\|_\infty.$$

Proof. Consider the free holomorphic function $g := \frac{1}{\|f\|_\infty} f$. Note that $\|g\|_\infty = 1$ and $g(0) = 0$. According to the Schwarz lemma for free holomorphic functions (see Theorem 2.4 from [44]), we have

$$\|g(X_1,\dots,X_n)\| \le \|(X_1,\dots,X_n)\|^k, \qquad (X_1,\dots,X_n) \in [B(\mathcal{H})^n]_1.$$

Denote $r := \omega_\rho(T_1,\dots,T_n) < 1$, $\rho > 0$, and consider

$$g_r(X_1,\dots,X_n) := g(rX_1,\dots,rX_n), \qquad (X_1,\dots,X_n) \in [B(\mathcal{H})^n]_1.$$

Note that the inequality above implies $\|g_r\|_\infty \le r^k$. Applying now Theorem 4.4 to g, and using the latter inequality, we obtain

$$\omega_\rho(g(T_1,\dots,T_n)) \le \|g_r\|_\infty \le r^k = \omega_\rho(T_1,\dots,T_n)^k.$$

Hence, the result follows. □

COROLLARY 4.6. *Let $(T_1,\dots,T_n) \in B(\mathcal{H})^n$ be such that $\omega_\rho(T_1,\dots,T_n) < 1$, and let $f : [B(\mathcal{H})^n]_1 \to B(\mathcal{H})$ be a free holomorphic function with $\Re f \le I$ and having the standard representation*

$$f(X_1,\dots,X_n) = \sum_{|\alpha|\ge k} a_\alpha X_\alpha, \qquad \textit{where} \;\; k \ge 1.$$

Then

$$\omega_\rho(f(T_1,\dots,T_n)) \le \frac{2\omega_\rho(T_1,\dots,T_n)^k}{1-\omega_\rho(T_1,\dots,T_n)^k}.$$

Proof. According to the Carathéodory type result for free holomorhic functions (see Theorem 5.1 from [51]), we have

$$\|f(X_1,\dots,X_n)\| \leq \frac{2\|\sum_{|\beta|=k} X_\beta X_\beta^*\|^{1/2}}{1-\|\sum_{|\beta|=k} X_\beta X_\beta^*\|^{1/2}}, \qquad (X_1,\dots,X_n)\in [B(\mathcal{H})^n]_1.$$

Hence, we deduce that $\|f_r\|_\infty \leq \frac{2r^k}{1-r^k}$ for any $r\in(0,1)$. Setting $r := \omega_\rho(T_1,\dots,T_n)<1$, $\rho>0$, and applying Theorem 4.4, we obtain

$$\omega_\rho(f(T_1,\dots,T_n)) \leq \|f_r\|_\infty \leq \frac{2\omega_\rho(T_1,\dots,T_n)^k}{1-\omega_\rho(T_1,\dots,T_n)^k}.$$

The proof is complete. □

Lemma 4.7. *Let $f := (f_1,\dots,f_m)$ be a contractive free holomorphic function with $\|f(0)\|<1$ such that the boundary functions $\widetilde{f}_1,\dots,\widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$. Let $A := (A_1,\dots,A_n)\in B(\mathcal{H})^n$ and $B := (B_1,\dots,B_n)\in B(\mathcal{H})^n$ be in the class $\mathcal{C}_\rho\subset B(\mathcal{H})^n$ and let $c\geq 1$. If $A\overset{H}{\underset{c}{\prec}}B$, then $f(A)$ and $f(B)$ are in $\mathcal{C}_{\rho_f}\subset B(\mathcal{H})^m$ and $f(A)\overset{H}{\underset{c}{\prec}}f(B)$, where ρ_f is given by relation* (4.1).

Proof. First, note that, due to Theorem 4.1, $f(A), f(B)$ are in $\mathcal{C}_{\rho_f}$, where ρ_f is given by relation (4.1). Let $p\in\mathbb{C}[Z_1,\dots,Z_m]\otimes M_k$, $k\in\mathbb{N}$, be such that $\Re p\geq 0$ on the unit ball $[B(\mathcal{H})^m]_1$. According to the proof of Theorem 4.1, the boundary function of the composition $p\circ f$ is $p(\widetilde{f}_1,\dots,\widetilde{f}_m)\in\mathcal{A}_n\bar{\otimes}_{min}M_k$ and $\Re(p\circ f)\geq 0$. Using the free pluriharmonic functional calculus for the class $\mathcal{C}_\rho$ and Theorem 2.2, if A,B are in $\mathcal{C}_\rho$ and $A\overset{H}{\underset{c}{\prec}}B$, $c\geq 1$, then

$$\begin{aligned}\Re(p\circ f)(A_1,\dots,A_n)+(\rho-1)\Re(p\circ f)(0)&\\ \leq c^2\left[\Re(p\circ f)(B_1,\dots,B_n)+(\rho-1)\Re(p\circ f)(0)\right].&\end{aligned}\tag{4.7}$$

Assume now that $\rho\geq 1$. Due to the Harnack type inequality (4.3), the inequality (4.7) implies

$$\Re(p\circ f)(A_1,\dots,A_n)\leq c^2\Re(p\circ f)(B_1,\dots,B_n)+(c^2-1)(\rho-1)\Re p(0)\frac{1+\|f(0)\|}{1-\|f(0)\|},$$

which is equivalent to

$$\begin{aligned}\Re(p\circ f)(A_1,\dots,A_n)+(\rho_f-1)\Re(p\circ f)(0)&\\ \leq c^2\left[\Re(p\circ f)(B_1,\dots,B_n)+(\rho_f-1)\Re(p\circ f)(0)\right],&\end{aligned}$$

where $\delta_f := 1+(\rho-1)\frac{1+\|f(0)\|}{1-\|f(0)\|}$. Applying Theorem 2.2, we deduce that $f(A)\overset{H}{\underset{c}{\prec}}f(B)$.

Now, we consider the case $\rho \in (0,1)$. The inequality (4.7) and the Harnack type inequality (4.3) imply

$$\Re(p\circ f)(A_1,\ldots,A_n) \le c^2\Re(p\circ f)(B_1,\ldots,B_n)+(c^2-1)(\rho-1)\Re p(0)\frac{1-\|f(0)\|}{1+\|f(0)\|}.$$

As above, we deduce that $f(A)\overset{H}{\underset{c}{\prec}} f(B)$ in $\mathcal{C}_{\rho_f}$, where $\delta_f := 1+(\rho-1)\frac{1-\|f(0)\|}{1+\|f(0)\|}$. This completes the proof. $\square$

Theorem 4.8. *Let $\delta_\rho : \Delta\times\Delta \to [0,\infty)$ be the hyperbolic metric on a Harnack part Δ of $\mathcal{C}_\rho$, and let $f := (f_1,\ldots,f_m)$ be a contractive free holomorphic function with $\|f(0)\| < 1$ such that the boundary functions $\widetilde{f}_1,\ldots,\widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$. Then*

$$\delta_{\rho_f}(f(A),f(B)) \le \delta_\rho(A,B), \qquad A,B\in\Delta,$$

where ρ_f is given by relation (4.1).

Proof. Let $A,B\in\Delta\subset\mathcal{C}_\rho$, i.e., there is $c\ge 1$ such that $A\overset{H}{\underset{c}{\sim}}B$. According to Theorem 4.1 and Lemma 4.7, $f(A)$ and $f(B)$ are in $\mathcal{C}_{\rho_f}$, and $f(A)\overset{H}{\underset{c}{\sim}}f(B)$ in $\mathcal{C}_{\rho_f}$, where ρ_f is given by relation (4.1). Hence and taking into account that

$$\delta_\rho(A,B) := \ln\inf\left\{c>1:\ A\overset{H}{\underset{c}{\sim}}B\right\}, \qquad A,B\in\Delta,$$

we deduce that

$$\delta_{\rho_f}(f(A),f(B)) \le \delta_\rho(A,B), \qquad A,B\in\Delta.$$

The proof is complete. $\square$

Now, we can deduce the following Schwarz type result.

Corollary 4.9. *Let $\delta_\rho : \Delta\times\Delta \to [0,\infty)$ be the hyperbolic metric on a Harnack part Δ of $\mathcal{C}_\rho$, and let $f := (f_1,\ldots,f_m)$ be a contractive free holomorphic function with $f(0)=0$ such that the boundary functions $\widetilde{f}_1,\ldots,\widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$. Then*

$$\delta_\rho(f(A),f(B)) \le \delta_\rho(A,B), \qquad A,B\in\Delta.$$

We recall that, due to Theorem 3.2, the open ball $[\mathcal{C}_\rho]_{<1}$ is the Harnack part of $\mathcal{C}_\rho$ containing 0. Consequently, Theorem 4.8 and Corollary 4.9 hold in the particular case when $\Delta := [\mathcal{C}_\rho]_{<1}$.

Ky Fan [14] showed that the von Neumann inequality [57] is equivalent to the fact that if $T\in B(\mathcal{H})$ is a strict contraction ($\|T\|<1$) and $f:\mathbb{D}\to\mathbb{D}$ is an analytic function, then $\|f(T)\|<1$. A multivariable analogue of this result was obtained in [51]. In what follows, we provide a spectral version of this result, when the norm is replaced by the joint spectral radius.

Theorem 4.10. *Let $f := (f_1, \ldots, f_m)$ be a contractive free holomorphic function with $\|f(0)\| < 1$ such that the boundary functions $\widetilde{f}_1, \ldots, \widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$. If $(T_1, \ldots, T_n) \in B(\mathcal{H})^n$ and the joint spectral radius $r(T_1, \ldots, T_n) < 1$, then*

$$r(f(T_1, \ldots, T_n)) < 1.$$

Proof. Assume that $(T_1, \ldots, T_n) \in B(\mathcal{H})^n$ has the joint spectral radius $r(T_1, \ldots, T_n) < 1$. Taking into account that $r(T_1, \ldots, T_n) = \lim_{\rho\to\infty} \omega_\rho(T_1, \ldots, T_n)$, we find $\delta > 1$ such that $\omega_\rho(T_1, \ldots, T_n) < 1$. Therefore, we have $T := (T_1, \ldots, T_n) \in \mathcal{C}_\rho$ and, due to Theorem 3.2, the n-tuple T is Harnack equivalent to 0. Consequently, $T \overset{H}{\underset{c}{\prec}} 0$ for some constant $c \geq 1$. According to Theorem 4.1, $f(T)$ and $f(0)$ are in the class $\mathcal{C}_{\rho_f}$, where ρ_f is given by relation (4.1). On the other hand, Lemma 4.7 implies $f(T) \overset{H}{\underset{c}{\prec}} f(0)$ in $\mathcal{C}_{\rho_f}$. Since $\|f(0)\| < 1$, we have the joint spectral radius $r(f(0)) < 1$. Applying Theorem 2.5, we deduce that $f(0) \overset{H}{\underset{c}{\prec}} 0$ in $\mathcal{C}_{\rho_f}$. Therefore, we have $f(T) \overset{H}{\underset{c}{\prec}} 0$ in $\mathcal{C}_{\rho_f}$. Applying again Theorem 2.5, we have $r(f(T)) < 1$. The proof is complete. ☐

An analogue of Theorem 4.10 for n-tuples of operators with joint operator radius $\omega_\rho(T_1, \ldots, T_n) < 1$ is the following.

Theorem 4.11. *Let $f := (f_1, \ldots, f_m)$ be a contractive free holomorphic function with $\|f(0)\| < 1$ such that the boundary functions $\widetilde{f}_1, \ldots, \widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$. If $(T_1, \ldots, T_n) \in B(\mathcal{H})^n$ and $\omega_\rho(T_1, \ldots, T_n) < 1$, then*

$$\omega_{\rho_f}(f(T_1, \ldots, T_n)) < 1,$$

where ρ_f is defined by relation (4.1). In particular, if $f(0) = 0$, then $\omega_\rho(f(T_1, \ldots, T_n)) < 1$.

Proof. If $T := (T_1, \ldots, T_n) \in B(\mathcal{H})^n$ and $\omega_\rho(T_1, \ldots, T_n) < 1$, then $T \in \mathcal{C}_\rho$. According to Theorem 3.2, we have

$$r(T_1, \ldots, T_n) < 1 \quad \text{and} \quad P_\rho(T, R) \geq aI$$

for some constant $a > 0$. Applying Theorem 4.1 and Theorem 4.10, we deduce that $f(T) \in \mathcal{C}_{\rho_f}$ and $r(f(T)) < 1$. Since $\omega_\rho(T) < 1$, Theorem 3.2 implies $T \overset{H}{\sim} 0$. In particular, we have $0 \overset{H}{\underset{c}{\prec}} T$ for some constant $c \geq 1$. Applying Lemma 4.7, we deduce that $f(0) \overset{H}{\underset{c}{\prec}} f(T)$ in $\mathcal{C}_{\rho_f}$, where ρ_f is given by relation (4.1). Hence, and using Theorem 2.2 (part (ii)), we get

$$P_{\rho_f}(rf(0), R) \leq c^2 P_{\rho_f}(rf(T), R), \qquad r \in [0, 1).$$

Since $r(f(0)) < 1$ and $r(f(T)) < 1$, the latter inequality implies

$$P_{\rho_f}(f(0), R) \leq c^2 P_{\rho_f}(f(T), R), \qquad r \in [0, 1). \tag{4.8}$$

On the other hand, since the mapping $X \mapsto P_1(X,R)$ is a positive free pluriharmonic function on $[B(\mathcal{H})^n]_1$, the Harnack inequality (3.1) implies

$$P_1(f(0),R) \geq P_1(0,R)\frac{1-\|f(0)\|}{1+\|f(0)\|} = \frac{1-\|f(0)\|}{1+\|f(0)\|}I.$$

Therefore, we have

$$\begin{aligned} P_{\rho_f}(f(0),R) &= P_1(f(0),R) + (\rho_f - 1)I \\ &\geq \left(\rho_f - 1 + \frac{1-\|f(0)\|}{1+\|f(0)\|}\right) I. \end{aligned}$$

Since

$$a := \rho_f - 1 + \frac{1-\|f(0)\|}{1+\|f(0)\|} = \begin{cases} \rho\frac{1-\|f(0)\|}{1+\|f(0)\|} & \text{if } \rho < 1 \\ \\ (\rho-1)\frac{1+\|f(0)\|}{1-\|f(0)\|} + \frac{1-\|f(0)\|}{1+\|f(0)\|} & \text{if } \rho \geq 1, \end{cases}$$

we have $a > 0$. Combining the latter inequality with (4.8) we obtain

$$P_{\rho_f}(f(T),R) \geq \frac{a}{c^2}I.$$

Using again Theorem 3.2, we deduce that $\omega_{\rho_f}(f(T)) < 1$. The last part of the theorem follows from Theorem 4.1. This completes the proof. □

Remark 4.12. *If $m = 1$, all the results of this section remain true when the condition $\|f(0)\| < 1$ is dropped if f is a nonconstant contractive free holomorphic function with boundary function in the noncommutative algebra $\mathcal{A}_n$.*

5. Carathéodory metric on the open noncommutative ball $[\mathcal{C}_\infty]_{<1}$ and Lipschitz mappings

In this section, we introduce a Carathéodory type metric d_K on the open ball of all n-tuples of operators $(T_1,\ldots,T_n)$ with joint spectral radius $r(T_1,\ldots,T_n) < 1$. We obtain a concrete formula for d_K in terms of the free pluriharmonic kernel on the open unit ball $[\mathcal{C}_\infty]_{<1}$. This is used to prove that the metric d_K is complete on $[\mathcal{C}_\infty]_{<1}$ and its topology coincides with the operator norm topology.

We need some notation. Consider the noncommutative balls

$$[\mathcal{C}_\rho]_{<1} := \{(X_1,\ldots,X_n) \in B(\mathcal{H})^n : \ \omega_\rho(X_1,\ldots,X_n) < 1\} \qquad \text{for } \rho \in (0,\infty],$$

where $\omega_\infty(X_1,\ldots,X_n) := r(X_1,\ldots,X_n)$ is the joint spectral radius of $(X_1,\ldots,X_n)$, and set

$$[\mathcal{C}_\rho]^{\prec 0} := \mathcal{C}_\rho \cap [\mathcal{C}_\infty]_{<1} \qquad \text{for } \rho \in (0,\infty).$$

According to Theorem 1.35 from [48], if $\rho,\rho' \in (0,\infty]$, $\rho \leq \rho'$, then $\mathcal{C}_\rho \subseteq \mathcal{C}_{\rho'}$ and, moreover, we have

$$\omega_{\rho'}(X) \leq \omega_\rho(X), \quad r(X) = \lim_{\rho\to\infty} \omega_\rho(X), \qquad X \in B(\mathcal{H})^n.$$

Consequently, we have

$$[\mathcal{C}_\rho]^{\prec 0} \subseteq [\mathcal{C}_{\rho'}]^{\prec 0}, \qquad [\mathcal{C}_\rho]_{<1} \subseteq [\mathcal{C}_{\rho'}]_{<1}.$$

Due to Theorem 2.5 and Theorem 3.2, one can easily see that

$$\{X \in \mathcal{C}_\rho : X \overset{H}{\sim} 0\} = [\mathcal{C}_\rho]_{<1} \subset [\mathcal{C}_\rho]^{\prec 0} = \left\{X \in \mathcal{C}_\rho : \ X \overset{H}{\prec} 0\right\}$$

for any $\rho \in (0, \infty)$. Note also that

$$\bigcup_{\rho>0} [\mathcal{C}_\rho]_{<1} = \bigcup_{\rho>0} [\mathcal{C}_\rho]^{\prec 0} = [\mathcal{C}_\infty]_{<1}.$$

Indeed, if $X \in [\mathcal{C}_\infty]_{<1}$, i.e., $r(X) < 1$, then taking into account that $r(X) = \lim_{\rho\to\infty} \omega_\rho(X)$, we find $\rho > 0$ such that $\omega_\rho(X) < 1$. Thus $X \in [\mathcal{C}_\rho]_{<1}$, which proves our assertion. Note also that $\bigcup_{\rho>0} [\mathcal{C}_\rho]_{<1}$ is dense (in the norm topology) in the set $\mathcal{C}_\infty$ of all n-tuples of operators $(T_1, \ldots, T_n)$ with joint spectral radius $r(T_1, \ldots, T_n) \leq 1$.

Now, we introduce the map $d_K : [\mathcal{C}_\infty]_{<1} \times [\mathcal{C}_\infty]_{<1} \to [0, \infty)$ by setting

$$d_K(A, B) = \sup_p \|\Re p(A) - \Re p(B)\|, \qquad A, B \in [\mathcal{C}_\infty]_{<1}, \tag{5.1}$$

where the supremum is taken over all polynomials $p \in \mathbb{C}[X_1, \ldots, X_n] \otimes M_m$, $m \in \mathbb{N}$, with $\Re p(0) = I$ and $\Re p \geq 0$ on $[B(\mathcal{H})^n]_1$. In what follows we will prove that d_K is a metric and obtain a concrete formula in terms of the free pluriharmonic kernel on the open unit ball $[\mathcal{C}_\infty]_{<1}$.

First, we need the following result.

Lemma 5.1. *Let G be a free pluriharmonic function on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$, such that $G(0) = I$ and $G \geq 0$. If $A, B \in [\mathcal{C}_\infty]_{<1}$, then*

$$\|G(A) - G(B)\| \leq \|P_1(A, R) - P_1(B, R)\|,$$

where where $P_1(X, R)$ is the free pluriharmonic Poisson kernel defined by

$$P_1(X, R) := \sum_{k=1}^{\infty} \sum_{|\alpha|=k} X_\alpha \otimes R_{\tilde{\alpha}}^* + I \otimes I + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} X_\alpha^* \otimes R_{\tilde{\alpha}}, \qquad X \in [\mathcal{C}_\infty]_{<1},$$

and the convergence is in the operator norm topology.

Proof. Since G is a positive free pluriharmonic function of $[B(\mathcal{H})^n]_1$ it has a unique representation of the form

$$G(X_1, \ldots, X_n) = \sum_{k=1}^{\infty} \sum_{|\alpha|=k} X_\alpha^* \otimes A_{(\alpha)}^* + I \otimes I + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} X_\alpha \otimes A_{(\alpha)}, \ X \in [B(\mathcal{H})^n]_1,$$

for some $A_{(\alpha)} \in B(\mathcal{E})$, where the series converge in the operator norm topology. Applying Theorem 5.2 from [47] to G, we find a completely positive linear map $\mu : \mathcal{R}_n^* + \mathcal{R}_n \to B(\mathcal{E})$ with $\mu(I) = I$ and $\mu(R_{\tilde{\alpha}}^*) = A_{(\alpha)}$ if $|\alpha| \geq 1$.

Since $A, B \in [\mathcal{C}_\rho]_{<1}$, we have $r(A) < 1$ and $r(B) < 1$. According to the free pluriharmonic functional calculus, $P_\rho(A, R)$, $P_\rho(B, R)$, $G(A)$, and $G(B)$

are well-defined and the corresponding series converge in the operator norm topology. Consequently, we have

$$G(A) = (\mathrm{id} \otimes \mu)(P_1(A, R)) \quad \text{and} \quad G(A) = (\mathrm{id} \otimes \mu)(P_1(A, R)).$$

Taking into account that μ is completely positive linear map with $\mu(I) = I$, we have

$$\|G(A) - G(B)\| \leq \|\mu\| \|P_1(A, R) - P_1(B, R)\| = \|P_1(A, R) - P_1(B, R)\|.$$

The proof is complete. □

According to Lemma 5.1, it makes sense to define the map $d_K' : [\mathcal{C}_\infty]_{<1} \times [\mathcal{C}_\infty]_{<1} \to [0, \infty)$ by setting

$$d_K'(A, B) := \sup_u \|u(A) - u(B)\| < \infty,$$

where the supremum is taken over all free pluriharmonic functions u on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$, such that $u(0) = I$ and $u \geq 0$.

Using the the free pluriharmonic functional calculus for for n-tuples of operators $(T_1, \ldots, T_n)$ with the joint spectral radius $r(T_1, \ldots, T_n) < 1$, one can extend Proposition 3.1 from [49] and show that for any $A, B \in [\mathcal{C}_\infty]_{<1}$,

$$d_K'(A, B) = d_K(A, B),$$

where d_K is defined by relation (5.1). Since the proof is essentially the same, we shall omit it.

Proposition 5.2. *d_K is a metric on $[\mathcal{C}_\infty]_{<1}$ satisfying relation*

$$d_K(A, B) = \|P_1(A, R) - P_1(B, R)\|, \qquad A, B \in [\mathcal{C}_\infty]_{<1}.$$

In addition, the map $[0, 1) \ni r \mapsto d_K(rA, rB) \in \mathbb{R}^+$ is increasing and

$$d_K(A, B) = \sup_{r \in [0,1)} d_K(rA, rB).$$

Proof. Using Lemma 5.1 we deduce that $d_K(A, B) \leq \|P_1(A, R) - P_1(B, R)\|$. The rest of the proof is similar to that of Proposition 3.2 from [49], so we shall omit it. □

Now, we can prove the main result of this section.

Theorem 5.3. *Let d_K be the Carathéodory metric on $[\mathcal{C}_\infty]_{<1}$. Then the following statements hold:*

(i) *the d_K-topology coincides with the norm topology on $[\mathcal{C}_\infty]_{<1}$;*
(ii) *$[\mathcal{C}_\rho]^{\prec 0}$ is a d_K-closed subset of $[\mathcal{C}_\infty]_{<1}$ for any $\rho > 0$;*
(iii) *the metric d_K is complete on $[\mathcal{C}_\infty]_{<1}$.*

Proof. We recall that the free pluriharmonic Poisson kernel is given by

$$P_1(X,R)=\sum_{k=1}^{\infty}\sum_{|\alpha|=k} X_\alpha\otimes R_{\tilde\alpha}^* + I\otimes I+\sum_{k=1}^{\infty}\sum_{|\alpha|=k} X_\alpha^*\otimes R_{\tilde\alpha}, \qquad X\in[\mathcal{C}_\infty]_{<1},$$

where the convergence is in the operator norm topology. Let $R_A := A_1^*\otimes R_1+\cdots+A_n^*\otimes R_n$ be the reconstruction operator. Note that, due to the noncommutative von Neumann inequality, we have

$$\begin{aligned}\|A-B\| &= \|R_A-R_B\|\\ &= \left\|\frac{1}{2\pi}\int_0^{2\pi} e^{it}[P_1(A,e^{it}R)-P_1(B,e^{it}R)]dt\right\|\\ &\le \sup_{t\in[0,2\pi]}\left\|P_1(A,e^{it}R)-P_1(B,e^{it}R)\right\|\\ &\le \|P_1(A,R)-P_1(B,R)\|.\end{aligned}$$

Now, Proposition 5.2 implies

$$\|A-B\|\le d_K(A,B), \qquad A,B\in[\mathcal{C}_\infty]_{<1}, \tag{5.2}$$

which shows that the d_K-topology is stronger then the norm topology on $[\mathcal{C}_\infty]_{<1}$. Conversely, to prove that the norm topology on $[\mathcal{C}_\infty]_{<1}$ is stronger than the d_K-topology, note that since $r(R_A)=r(A)<1$ and $r(R_B)=r(B)<1$, the operators $I-R_A$ and $I-R_B$ are invertible. Thus

$$d_K(A,B)=\|P_1(A,R)-P_1(B,R)\|\le 2\|(I-R_A)^{-1}-(I-R_B)^{-1}\|$$

for any $A,B\in[\mathcal{C}_\infty]_{<1}$. Hence and due to the continuity of the maps $X\mapsto I-R_X$ on $B(\mathcal{H})^n$ and $Y\mapsto Y^{-1}$ on the group of invertible elements in $B(\mathcal{H}\otimes F^2(H_n))$, in the operator norm topology, we deduce our assertion. In conclusion, the d_K-topology coincides with the norm topology on $[\mathcal{C}_\infty]_{<1}$.

Now, to prove (ii), let $\{A^{(k)}:=(A_1^{(k)},\ldots,A_n^{(k)})\}_{k=1}^\infty$ be a d_K-Cauchy sequence in $[\mathcal{C}_\rho]^{\prec 0}\subset\mathcal{C}_\rho$. Due to inequality (5.2), we deduce that $\{A^{(k)}\}_{k=1}^\infty$ is a Cauchy sequence in the norm topology of $B(\mathcal{H})^n$. Since $\mathcal{C}_\rho$ is closed in the operator norm topology, there exists $T:=(T_1,\ldots,T_n)$ in $\mathcal{C}_\rho$ such that $\|T-A^{(k)}\|\to 0$, as $k\to\infty$.

Now let us prove that the joint spectral radius $r(T)<1$. Since $\{A^{(k)}\}_{k=1}^\infty$ is a d_K-Cauchy sequence, there exists $k_0\in\mathbb{N}$ such that $d_K(A^{(k)},A^{(k_0)})\le 1$ for any $k\ge k_0$. On the other hand, since $A^{(k_0)}\in[\mathcal{C}_\rho]^{\prec 0}$, i.e., $A^{(k_0)}\overset{H}{\prec}0$, Theorem 2.2 shows that there is $c\ge 1$ such that $P_\rho(rA^{(k_0)},R)\le c^2\delta$ for any $r\in[0,1)$. Hence, and due to the noncommutative von Neumann inequality, we deduce that

$$\begin{aligned}P_\rho(rA^{(k)},R)&\le\left(\|P_\rho(rA^{(k)},R)-P_\rho(rA^{(k_0)},R)\|+\|P_\rho(rA^{(k_0)},R)\|\right)I\\ &\le\left(d_K(A^{(k)},A^{(k_0)})+\|P_\rho(rA^{(k_0)},R)\|\right)I\le(1+c^2\delta)I\end{aligned}\tag{5.3}$$

for any $k\ge k_0$ and $r\in[0,1)$.

We show now that $\lim_{k\to\infty} P_\rho(rA^{(k)}, R) = P_\rho(rT, R)$ in the operator norm topology. First, one can easily see that, since $T, A^{(k)} \in \mathcal{C}_\rho$, we have

$$\sum_{|\alpha|=p} T_\alpha T_\alpha^* \le \rho^2 I \quad \text{and} \quad \sum_{|\alpha|=p} A_\alpha^{(k)} A_\alpha^{(k)} \le \rho^2 I$$

for any $p, k = 1, 2, \ldots$. Given $\epsilon > 0$ and $r \in (0,1)$, let $m \in \mathbb{N}$ be such that $\sum_{p=m}^\infty \rho r^p < \frac{\epsilon}{2}$. Note that

$$\begin{aligned}
&\|P(rA^{(k)}, R) - P(rT, R)\| \\
&\le 2\sum_{p=1}^{m-1} \left\| \sum_{|\alpha|=p} r^{|\alpha|}(A_\alpha^{(k)} - T_\alpha) \otimes R_{\tilde\alpha}^* \right\| \\
&\qquad + 2\sum_{p=m}^{\infty} \left\| \sum_{|\alpha|=p} r^{|\alpha|} A_\alpha^{(k)} \otimes R_{\tilde\alpha}^* \right\| + 2\sum_{p=m}^{\infty} \left\| \sum_{|\alpha|=p} r^{|\alpha|} T_\alpha \otimes R_{\tilde\alpha}^* \right\| \\
&= 2\left\| \sum_{p=1}^{m-1} r^p \sum_{|\alpha|=p} (A_\alpha^{(k)} - T_\alpha)(A_\alpha^{(k)} - T_\alpha)^* \right\| \\
&\qquad + 2\left\| \sum_{p=1}^{m-1} r^p \sum_{|\alpha|=p} A_\alpha^{(k)} {A_\alpha^{(k)}}^* \right\| + 2\left\| \sum_{p=1}^{m-1} r^p \sum_{|\alpha|=p} T_\alpha {T_\alpha}^* \right\| \\
&\le 2\left\| \sum_{p=1}^{m-1} r^p \sum_{|\alpha|=p} (A_\alpha^{(k)} - T_\alpha)(A_\alpha^{(k)} - T_\alpha)^* \right\| + \epsilon
\end{aligned}$$

for any $k = 1, 2, \ldots$. Since $A^{(k)} \to T$ in the norm topology, as $k \to \infty$, and using the results above, one can easily deduce that $\lim_{k\to\infty} P_\rho(rA^{(k)}, R) = P_\rho(rT, R)$ for each $r \in [0,1)$. Now, taking $k \to \infty$ in inequality (5.3), we obtain $P_\rho(rT, R) \le (1 + c^2\delta)I$ for $r \in [0,1)$. Applying Theorem 2.2, we deduce that $T \overset{H}{\prec} 0$. Now, Theorem 2.5 implies $r(T) < 1$, which shows that T is in $[\mathcal{C}_\rho]^{\prec 0}$ and, therefore, in $[\mathcal{C}_\infty]_{<1}$, which proves part (ii).

It remains to prove part (iii). To this end, let $\{A^{(k)} := (A_1^{(k)}, \ldots, A_n^{(k)})\}_{k=1}^\infty$ be a d_K-Cauchy sequence in $[\mathcal{C}_\infty]_{<1}$. Given $\epsilon > 0$, there exists $k_0 \ge 1$ such that $d_K(A^{(k)}, A^{(j)}) < \epsilon$ for any $k, j \ge k_0$. Then we have

$$d_K(A^{(k)}, 0) \le c := d_K(A^{(k_0)}, 0) + \epsilon \quad \text{for any} \quad k \ge k_0. \tag{5.4}$$

Hence, and due to the definition of d_K, we have $\|u(A^{(k)}) - u(0)\| \le c$ and, consequently,

$$u(A^{(k)}) \le (\|u(A^{(k)} - u(0)\| + 1)I \le (c+1)u(0) \quad \text{for any} \quad k \ge k_0$$

and for any positive free pluriharmonic function u on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$ such that $u(0) = I$.

Now, for each $k \geq k_0$, fix $\rho_k \geq 1$ such that $A^{(k)} \in [\mathcal{C}_{\rho_k}]^{\prec 0}$. Note that the inequality above implies

$$u(A^{(k)}) + (\rho_k - 1)u(0) \leq \rho_k(c+1)u(0)$$

for all $k \geq k_0$. Applying Theorem 2.2 and using relation (5.4), we obtain

$$\|L_{0,A^{(k)}}\|^2 \leq d_K(A^{(k_0)}, 0) + \epsilon + 1, \qquad k \geq k_0.$$

Consequently, we have

$$1 \leq \epsilon_0 := \sup_{k \geq k_0} \|L_{0,A^{(k)}}\|^2 < \infty. \tag{5.5}$$

Since $\{A^{(k)}\}$ is a d_K-Cauchy sequence, there exists $m_0 \geq k_0$ such that $d_K(A^{(m')}, A^{(m)}) < \frac{1}{2\epsilon_0}$ for any $m, m' \geq m_0$. Using now relation (5.5), we obtain

$$d_K(A^{(m)}, A^{(m_0)}) < \frac{1}{2\|L_{0,A^{(m_0)}}\|^2}, \qquad k \geq m_0. \tag{5.6}$$

Since $A^{(m_0)} \in [\mathcal{C}_{\rho_{m_0}}]^{\prec 0}$, Theorem 2.5 implies $r(A^{(m_0)}) < 1$. On the other hand, since $\lim_{\rho \to \infty} \omega_\rho(A^{(m_0)}) = r(A^{(m_0)}) < 1$, there exists $\rho_{m_0} > 0$ such that $\omega_{\rho_{m_0}}(A^{(m_0)}) < 1$ for any $\rho \geq \rho_{m_0}$. We can assume that

$$\rho_{m_0} \geq \frac{\|L_{A^{(m_0)},0}\|^2}{\|L_{0,A^{(m_0)}}\|^2}. \tag{5.7}$$

Using Proposition 5.2 and relation (5.6), we deduce that

$$P_{\rho_{m_0}}(A^{(m_0)}, R) \leq P_{\rho_{m_0}}(A^{(k)}, R) + \frac{1}{2\|L_{0,A^{(k)}}\|^2} I, \qquad k \geq m_0. \tag{5.8}$$

On the other hand, since $\omega_{\rho_{m_0}}(A^{(m_0)}) < 1$, Theorem 3.2 implies $A^{(m_0)} \overset{H}{\sim} 0$ in $\mathcal{C}_{\rho_{m_0}}$. Consequently, we have $0 \overset{H}{\prec} A^{(m_0)}$, which due to Theorem 2.2, implies

$$\rho_{m_0} I = P_{\rho_{m_0}}(0, R) \leq \|L_{A^{(m_0)},0}\|^2 P_{\rho_{m_0}}(A^{(m_0)}, R).$$

Combining this with relation (5.7), we get

$$P_{\rho_{m_0}}(A^{(m_0)}, R) \geq \frac{1}{\|L_{0,A^{(m_0)}}\|^2} I.$$

Hence, and due to (5.8), we have

$$P_{\rho_{m_0}}(A^{(k)}, R) \geq \frac{1}{2\|L_{0,A^{(m_0)}}\|^2} I \geq \frac{1}{2\epsilon_0} I.$$

Applying Theorem 3.2, we deduce that $A^{(k)} \overset{H}{\sim} 0$ and $A^{(k)} \in \mathcal{C}_{\rho_{m_0}}$. Therefore, $A^{(k)} \in [\mathcal{C}_{\rho_{m_0}}]^{\prec 0}$ for all $k \geq m_0$ and the sequence $\{A^{(k)}\}_{k \geq m_0}$ is a d_K-Cauchy sequence in $[\mathcal{C}_{\rho_{m_0}}]^{\prec 0}$. Due to part (ii), there exists $A \in [\mathcal{C}_{\rho_{m_0}}]^{\prec 0} \subset [\mathcal{C}_\infty]_{<1}$ such that $d_K(A^{(k)}, A) \to 0$, as $k \to \infty$, which proves that d_K is a complete metric on $[\mathcal{C}_\infty]_{<1}$. The proof is complete. $\square$

We can provide now a class of Lipschitz functions with respect to the Carathéodory metric on $[\mathcal{C}_\infty]_{<1}$.

THEOREM 5.4. *Let $f := (f_1, \dots, f_m)$ be a contractive free holomorphic function with $\|f(0)\| < 1$ such that the boundary functions $\widetilde{f}_1, \dots, \widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$. Then*

$$d_K(f(A), f(B)) \le \frac{1 + \|f(0)\|}{1 - \|f(0)\|} d_K(A, B)$$

for any n-tuples $A := (A_1, \dots, A_n)$ and $B := (B_1, \dots, B_n)$ in $[\mathcal{C}_\infty]_{<1}$.

Proof. According to the maximum principle for free holomorphic functions with operator-valued coefficients (see Proposition 5.2 from [50]), the condition $\|f(0)\| < 1$ implies that $\|f(X)\| < 1$, $X \in [B(\mathcal{H})^n]_1$. If u is a free pluriharmonic function on $[B(\mathcal{H})^m]_1$, then Theorem 1.1 from [51] shows that $u \circ f$ is a free pluriharmonic function on $[B(\mathcal{H})^n]_1$. If, in addition, u is positive, then $u \circ f$ is also positive.

Assume now that A and B are in $[\mathcal{C}_\infty]_{<1}$. Due to Theorem 4.10, $f(A)$ and $f(B)$ are in $[\mathcal{C}_\infty]_{<1}$. Let $p \in \mathbb{C}[X_1, \dots, X_m] \otimes M_k$, $k \in \mathbb{N}$, be a matrix-valued noncommutative polynomial with $\Re p(0) = I$ and $\Re p \ge 0$ on $[B(\mathcal{H})^m]_1$. According to the Harnack type inequality (4.3), we have

$$\frac{1 - \|f(0)\|}{1 + \|f(0)\|} I \le \Re p(f(0)) \le \frac{1 + \|f(0)\|}{1 - \|f(0)\|} I.$$

Since $\|f(0)\| < 1$, we deduce that $\Re p(f(0))$ is a positive invertible operator of the form $I_\mathcal{H} \otimes A$ for some $A \in M_k$. Define the mapping $h : [B(\mathcal{H})^n]_1 \to B(\mathcal{H}) \bar{\otimes}_{min} M_k$ by setting

$$h(X) := [\Re p(f(0))]^{-1/2} \Re p(f(X)) [\Re p(f(0))]^{-1/2}, \qquad X \in [B(\mathcal{H})^n]_1.$$

Note that h is a positive free pluriharmonic function on $[B(\mathcal{H})^n]_1$ with coefficients in M_k with the property that $h(0) = I$. Now, using the above-mentioned Harnack type inequality, we have

$$\begin{aligned}
&\|\Re p(f(A)) - \Re p(f(B))\| \\
&\quad \le \|[\Re p(f(0))]^{1/2}\| \left\| [\Re p(f(0))]^{-1/2} \left(\Re p(f(A)) - \Re p(f(B))\right) [\Re p(f(0))]^{1/2} \right\| \\
&\qquad \cdot \|[\Re p(f(0))]^{1/2}\| \\
&\quad \le \|[\Re p(f(0))]\| \|h(A) - h(B)\| \\
&\quad \frac{1 + \|f(0)\|}{1 - \|f(0)\|} d_K(A, B).
\end{aligned}$$

Taking the supremum over all polynomials $p \in \mathbb{C}[X_1, \dots, X_m] \otimes M_k$, $k \in \mathbb{N}$, with $\Re p(0) = I$ and $\Re p \ge 0$ on $[B(\mathcal{H})^m]_1$, we obtain

$$d_K(f(A), f(B)) \le \frac{1 + \|f(0)\|}{1 - \|f(0)\|} d_K(A, B),$$

which completes the proof. □

Corollary 5.5. *Let $f := (f_1, \dots, f_m)$ be a contractive free holomorphic function with $f(0) = 0$ such that the boundary functions $\widetilde{f}_1, \dots, \widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$. Then*

$$d_K(f(A), f(B)) \le d_K(A, B)$$

for any $A, B \in [\mathcal{C}_\infty]_{<1}$.

We remark that, using Corollary 1.2 and the remarks preceding Corollary 2.4, one can easily obtain the following result, which provides a simple example when the inequality of Theorem 5.4 is an equality.

Corollary 5.6. *If $1 \le m < n$, let $A := (A_1, \dots, A_m) \in B(\mathcal{H})^m$ and $B := (B_1, \dots, B_m) \in B(\mathcal{H})^m$ be in $[\mathcal{C}_\infty]_{<1}$ and let $\widetilde{A} := (A_1, \dots, A_m, 0, \dots, 0)$ and $\widetilde{B} := (B_1, \dots, B_m, 0, \dots, 0)$ be their extensions in $B(\mathcal{H})^n$, respectively. Then*

$$d_K(A, B) = d_K(\widetilde{A}, \widetilde{B}).$$

According to Theorem 5.3, the d_K-topology coincides with the norm topology on $[\mathcal{C}_\infty]_{<1}$. Due to Theorem 5.4, we deduce the following result.

Corollary 5.7. *Let $f := (f_1, \dots, f_m)$ be a contractive free holomorphic function with $\|f(0)\| < 1$ such that the boundary functions $\widetilde{f}_1, \dots, \widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$. Then the map*

$$[\mathcal{C}_\infty]_{<1} \ni (T_1, \dots, T_n) \mapsto f(T_1, \dots, T_n) \in [\mathcal{C}_\infty]_{<1}$$

is continuous in the operator norm topology, where $[\mathcal{C}_\infty]_{<1}$ is the corresponding ball in $B(\mathcal{H})^n$ and $B(\mathcal{H})^m$, respectively.

6. Three metric topologies on Harnack parts of $\mathcal{C}_\rho$

In this section we study the relation between the δ_ρ-topology, the d_K-topology, and the operator norm topology on Harnack parts of $\mathcal{C}_\rho$. We prove that the hyperbolic metric δ_ρ is a complete metric on certain Harnack parts of $\mathcal{C}_\rho$, and that all the three topologies coincide on $[\mathcal{C}_\rho]_{<1}$. In particular, we prove that the hyperbolic metric δ_ρ is complete on the open unit unit ball $[\mathcal{C}_\rho]_{<1}$, while the other two metrics are not complete.

First, we mention another formula for the hyperbolic distance that will be used to prove the main result of this section. If $f \in \mathcal{A}_n \bar{\otimes}_{min} M_m$, $m \in \mathbb{N}$, then we call $\Re f$ strictly positive and denote $\Re f > 0$ if there exists a constant $a > 0$ such that $\Re f \ge aI$. We remark that, in this case, if $(T_1, \dots, T_n) \in \mathcal{C}_\rho$, then, using the functional calculus for the class $\mathcal{C}_\rho$, we deduce that

$$\Re f(T_1, \dots, T_n) + (\rho - 1)\Re f(0) \ge \rho a I.$$

The proof of the next result is similar to that of Proposition 3.5 from [49], but uses the functional calculus for the class $\mathcal{C}_\rho$ and Theorem 2.2 of the present paper. We shall omit it.

PROPOSITION 6.1. *Let $A := (A_1, \ldots, A_n)$ and $B := (B_1, \ldots, B_n)$ be in $\mathcal{C}_\rho$ such that $A \overset{H}{\sim} B$. Then*

$$\delta_\rho(A, B) = \frac{1}{2} \sup \left| \ln \frac{\langle [\Re f(A_1, \ldots, A_n) + (\rho - 1)\Re f(0)]x, x\rangle}{\langle [\Re f(B_1, \ldots, B_n) + (\rho - 1)\Re f(0)]x, x\rangle} \right|, \tag{6.1}$$

where the supremum is taken over all $f \in \mathcal{A}_n \otimes M_m$, $m \in \mathbb{N}$, with $\Re f > 0$ and $x \in \mathcal{H} \otimes \mathbb{C}^m$ with $x \neq 0$.

We remark that, under the conditions of Proposition 6.1, one can also prove that relation (6.1) holds if the supremum is taken over all noncommutative polynomials $f \in \mathbb{C}[X_1, \ldots, X_n] \otimes M_m$, $m \in \mathbb{N}$, with $\Re f > 0$, and $x \in \mathcal{H} \otimes \mathbb{C}^m$ with $x \neq 0$.

The main result of this section is the following.

THEOREM 6.2. *Let δ_ρ, $\rho > 0$, be the hyperbolic metric on a Harnack part Δ of $[\mathcal{C}_\rho]^{\prec 0}$. Then the following properties hold:*

(i) *δ_ρ is complete on Δ;*
(ii) *the δ_ρ-topology is stronger then the d_K-topology on Δ;*
(iii) *the δ_ρ-topology, the d_K-topology, and the operator norm topology coincide on $[\mathcal{C}_\rho]_{<1}$;*
(iv) *$[\mathcal{C}_\rho]_{<1}$ is complete relative the hyperbolic metric, but not complete with respect to the Carathéodory metric d_K and the operator metric.*

Proof. Let $A := (A_1, \ldots, A_n)$ and $B := (B_1, \ldots, B_n)$ be n-tuples in a Harnack part Δ of $[\mathcal{C}_\rho]^{\prec 0}$. Then A is Harnack equivalent to B and

$$\Re f(A_1, \ldots, A_n) + (\rho - 1)\Re f(0) \leq \Lambda_\rho(A, B)^2 [\Re f(B_1, \ldots, B_n) + (\rho - 1)\Re f(0)]$$

for any $f \in \mathcal{A}_n \bar{\otimes}_{min} M_m$ with $\Re f \geq 0$, where $\Lambda_\rho(A, B)$ is defined by (3.3). Hence, we deduce that

$$\Re f(A_1, \ldots, A_n) - \Re f(B_1, \ldots, B_n) \leq [\Lambda_\rho(A, B)^2 - 1][\Re f(B_1, \ldots, B_n) + (\rho - 1)\Re f(0)]. \tag{6.2}$$

Since $B \overset{H}{\prec} 0$, we have the joint spectral radius $r(B) < 1$, so the ρ-pluriharmonic kernel $P_\rho(B, R)$ makes sense. Due to the fact that the noncommutative Poisson transform $\mathrm{id} \otimes P_{rR}$ is completely positive, and $P_\rho(B, S) \leq \|P_\rho(B, R)\| I$, one can easily see that

$$\begin{aligned} P_\rho(rB, R) = (\mathrm{id} \otimes P_{rR})[P_\rho(B, S)] &\leq \|P_\rho(B, R)\| I \\ &= \frac{1}{\rho} \|P_\rho(B, R)\| P_\rho(0, R) \end{aligned}$$

for any $r \in [0, 1)$. Using the equivalence $(ii) \leftrightarrow (iii)$ of Theorem 2.2, when $c^2 = \frac{1}{\rho}\|P_\rho(B, R)\|$, we obtain $\Re f(rB_1, \ldots, rB_n) + (\rho - 1)\Re f(0) \leq \|P_\rho(B, R)\| \Re f(0)$ for any $r \in [0, 1)$. Letting $r \to 1$, in the operator norm topology, we deduce that

$$\Re f(B_1, \ldots, B_n) + (\rho - 1)\Re f(0) \leq \|P_\rho(B, R)\| \Re f(0).$$

Hence, and using relation (6.2), we obtain

$$\Re f(A_1,\dots,A_n) - \Re f(B_1,\dots,B_n) \le [\Lambda_\rho(A,B)^2 - 1]\|P_\rho(B,R)\|\Re f(0).$$

We can obtain a similar inequality if we interchange A with B. If, in addition, we assume that $\Re f(0) = I$, then we obtain

$$-tI \le \Re f(A_1,\dots,A_n) - \Re f(B_1,\dots,B_n) \le tI,$$

where $t := [\Lambda_\rho(A,B)^2 - 1]\max\{\|P_\rho(A,R)\|, \|P_\rho(B,R)\|\}$. On the other hand, since $\Re f(A_1,\dots,A_n) - \Re f(B_1,\dots,B_n)$ is a self-adjoint operator, we get $\|\Re f(A_1,\dots,A_n) - \Re f(B_1,\dots,B_n)\| \le t$. Hence, we deduce that $d_K(A,B) \le s$. As a consequence, we obtain

$$d_K(A,B) \le \max\{\|P_\rho(A,R)\|, \|P_\rho(B,R)\|\}\left(e^{2\delta_\rho(A,B)} - 1\right). \tag{6.3}$$

Let us prove that δ_ρ is a complete metric on Δ. To this end, let $\{A^{(k)} := (A_1^{(k)},\dots,A_n^{(k)})\}_{k=1}^\infty \subset \Delta$ be a δ_ρ-Cauchy sequence. First, we prove that the sequence $\{\|P_\rho(A^{(k)},R)\|\}_{k=1}^\infty$ is bounded. Given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\delta_\rho(A^{(k)}, A^{(p)}) < \epsilon \quad \text{for any} \quad k,p \ge k_0. \tag{6.4}$$

Let $f \in \mathcal{A}_n \bar{\otimes}_{min} M_m$ with $\operatorname{Re} f \ge 0$. Since $A^{(k_0)} \overset{H}{\prec} 0$ and

$$P_\rho(rA^{(k_0)},R) \le \frac{1}{\rho}\|P_\rho(rA^{(k_0)},R)\|P_\rho(0,R),$$

Theorem 2.2 implies

$$\Re f(A^{(k_0)}) + (\rho-1)\Re f(0) \le \frac{1}{\rho}\|P_\rho(rA^{(k_0)},R)\|[\Re f(0) + (\rho-1)\Re f(0)].$$

On the other hand, since $A^{(k)} \overset{H}{\sim} A^{(k_0)}$, Theorem 2.2 implies

$$\Re f(A^{(k)}) + (\rho-1)\Re f(0) \le \Lambda_\rho(A^{(k)},A^{(k_0)})^2[\Re f(A^{(k_0)}) + (\rho-1)\Re f(0)].$$

Combining these inequalities, we obtain

$$\Re f(A^{(k)}) + (\rho-1)\Re f(0) \le c^2\frac{1}{\rho}[\Re f(0) + (\rho-1)\Re f(0)], \tag{6.5}$$

where $c := \|P_\rho(A^{(k_0)},R)\|^{1/2}\Lambda_\rho(A^{(k)},A^{(k_0)})$, for any $f \in \mathcal{A}_n \otimes M_m$ with $\Re f \ge 0$. Consequently, due to Theorem 2.2, we have $\|P_\rho(A^{(k)},R)\| \le c^2$ for any $k \ge k_0$. Combining this with relation (6.4), we obtain

$$\|P_\rho(A^{(k)},R)\| \le \|P_\rho(A^{(k_0)},R)\|e^{2\epsilon}$$

for any $k \ge k_0$. This shows that the sequence $\{\|P_\rho(A^{(k)},R)\|\}_{k=1}^\infty$ is bounded. Consequently, inequality (6.3) implies that $\{A^{(k)}\}$ is a d_K-Cauchy sequence. Due to Theorem 5.3, there exists $A := (A_1,\dots,A_n) \in [\mathcal{C}_\rho]^{\prec 0}$ such that

$$d_K(A^{(k)},A) \to 0 \quad \text{as} \quad k \to \infty. \tag{6.6}$$

In what follows, we prove that $A \in \Delta$. Let $f \in \mathcal{A}_n \otimes M_m$ with $\Re f \geq 0$ and $\Re f(0) = I$. Taking into account relations (6.5) and (6.4), we have

$$\begin{aligned} \Re f(A^{(k)}) + (\rho - 1)\Re f(0) &\leq \Lambda_\rho(A^{(k)}, A^{(k_0)})^2 [\Re f(A^{(k_0)}) + (\rho - 1)\Re f(0)] \\ &\leq e^{2\epsilon}[\Re f(A^{(k_0)}) + (\rho - 1)\Re f(0)] \end{aligned} \tag{6.7}$$

for $k \geq k_0$. According to relation (6.6) and the definition of d_K, $\Re f(A^{(k)}) \to \Re f(A)$, as $k \to \infty$, in the operator norm topology. Consequently, relation (6.7) implies

$$\Re f(A) + (\rho - 1)\Re f(0) \leq e^{2\epsilon}[\Re f(A^{(k_0)}) + (\rho - 1)\Re f(0)]. \tag{6.8}$$

Such an inequality can be deduced in the more general case when $f \in \mathcal{A}_n \otimes M_m$ with $\Re f \geq 0$. Indeed, for each $\epsilon' > 0$ let $g := \epsilon' I + f$, $Y := \Re g(0)$, and $\varphi := Y^{-1/2} g Y^{-1/2}$. Since $\Re\varphi \geq 0$ and $\Re\varphi(0) = I$, we can apply inequality (6.8) to φ and deduce that

$$\rho\epsilon' I + \Re f(A) + (\rho - 1)\Re f(0) \leq e^{2\epsilon}\left[\rho\epsilon' I + \Re f(A^{(k_0)}) + (\rho - 1)\Re f(0)\right]$$

for any $\epsilon' > 0$. Letting $\epsilon' \to 0$, we get

$$\Re f(A) + (\rho - 1)\Re f(0) \leq e^{2\epsilon}[\Re f(A^{(k_0)}) + (\rho - 1)\Re f(0)] \tag{6.9}$$

for any $f \in \mathcal{A}_n \otimes M_m$ with $\Re f \geq 0$. Therefore,

$$A \overset{H}{\prec} A^{(k_0)}. \tag{6.10}$$

On the other hand, since $A^{(k_0)} \overset{H}{\prec} A^{(k)}$ for any $k \geq k_0$, Theorem 2.2 and relation (6.4), imply

$$\begin{aligned} \Re p(A^{(k_0)}) + (\rho - 1)\Re p(0) &\leq \Lambda_\rho(A^{(k_0)}, A^{(k)})^2 [\Re p(A^{(k)}) + (\rho - 1)\Re p(0)] \\ &\leq e^{2\epsilon}[\Re p(A^{(k)}) + (\rho - 1)\Re(0)] \end{aligned}$$

for $k \geq k_0$ and any polynomial $p \in \mathbb{C}[X_1, \ldots, X_n] \otimes M_m$, $m \in \mathbb{N}$, with $\Re p \geq 0$. According to Theorem 5.3, the d_K-topology coincides with the norm topology on $[\mathcal{C}_\rho]^{\prec 0}$. Therefore, relation (6.6) implies $A^{(k)} \to A \in [\mathcal{C}_\rho]^{\prec 0}$ in the operator norm topology. Taking the limit, as $k \to \infty$, in the inequality above, we deduce that

$$\Re p(A^{(k_0)}) + (\rho - 1)\Re p(0) \leq e^{2\epsilon}[\Re p(A) + (\rho - 1)\Re p(0)] \tag{6.11}$$

for any $p \in \mathbb{C}[X_1, \ldots, X_n] \otimes M_m$ with $\Re p \geq 0$. Consequently, we get $A^{(k_0)} \overset{H}{\prec} A$. Hence, and using relation (6.10), we obtain $A \overset{H}{\sim} A^{(k_0)}$, which proves that $A \in \Delta$. The inequalities (6.9) and (6.11) imply $\Lambda_\rho(A^{(k_0)}, A) \leq e^{2\epsilon}$. This shows that $\delta_\rho(A^{(k_0)}, A) < \epsilon$, which together with relation (6.4) imply $\delta_\rho(A^{(k)}, A) < 2\epsilon$ for any $k \geq k_0$. Therefore, $\delta_\rho(A^{(k)}, A) \to 0$, as $k \to \infty$, which proves that δ_ρ is a complete metric on the Harnack part Δ. Note that we have also proved part (ii) of this theorem.

In what follows, we prove part (iii). To this end, assume that A and B are n-tuples of operators in $[\mathcal{C}_\rho]_{<1}$. Due to Theorem 3.2, $P_\rho(B,R)$ is a positive invertible operator. Since $P_\rho(B,R)^{-1} \leq \|P_\rho(B,R)^{-1}\|$, we have $I \leq \|P_\rho(B,R)^{-1}\| P_\rho(B,R)$, which, applying the noncommutative Poisson transform, implies $I \leq \|P_\rho(B,R)^{-1}\| P_\rho(rB,R)$ for any $r \in [0,1)$. By Theorem 2.2, we deduce that $0 \overset{H}{\prec} B$ and

$$\Re f(0) \leq \|P_\rho(B,R)^{-1}\| \left[\Re f(B) + (\rho-1)\Re f(0)\right]$$

for any $f \in \mathcal{A}_n \otimes M_m$ with $\Re f \geq 0$. If, in addition, $\Re f(0) = I$, then the latter inequality implies

$$\begin{aligned}\frac{\langle [\Re f(A) + (\rho-1)\Re f(0)]x, x\rangle}{\langle [\Re f(B) + (\rho-1)\Re f(0)]x, x\rangle} - 1 &\leq \frac{\|P_\rho(B,R)^{-1}\|}{\|x\|}\left\langle \left(\Re f(A) - \Re f(B)\right)x, x\right\rangle \\ &\leq \|P_\rho(B,R)^{-1}\| d_K(A,B)\end{aligned}$$

for any $x \in \mathcal{H} \otimes \mathbb{C}^m$, $x \neq 0$. Consequently, we have

$$\ln \frac{\langle [\Re f(A) + (\rho-1)\Re f(0)]x, x\rangle}{\langle [\Re f(B) + (\rho-1)\Re f(0)]x, x\rangle} \leq \ln\left(1 + \|P_\rho(B,R)^{-1}\| d_K(A,B)\right).$$

A similar inequality can be obtained interchanging A with B. Combining these two inequalities, we get

$$\begin{aligned}&\left|\ln \frac{\langle [\Re f(A) + (\rho-1)\Re f(0)]x, x\rangle}{\langle [\Re f(B) + (\rho-1)\Re f(0)]x, x\rangle}\right| \\ &\quad\leq \ln\left(1 + \max\{\|P_\rho(B,R)^{-1}\|, \|P_\rho(A,R)^{-1}\|\} d_K(A,B)\right).\end{aligned} \tag{6.12}$$

Now, we consider the general case when $g \in \mathcal{A}_n \otimes M_m$ with $\Re g > 0$. Note that $Y := \Re g(0)$ is a positive invertible operator on $\mathcal{H} \otimes \mathbb{C}^m$ and $f := Y^{-1/2} g Y^{-1/2}$ has the properties $\Re f \geq 0$ and $\Re f(0) = I$. Applying inequality (6.12) to f when $x := Y^{-1/2}y$, $y \in \mathcal{H} \otimes \mathbb{C}^m$, and $y \neq 0$, we obtain

$$2\delta_\rho(A,B) \leq \ln\left(1 + \max\{\|P_\rho(B,R)^{-1}\|, \|P_\rho(A,R)^{-1}\|\} d_K(A,B)\right). \tag{6.13}$$

Consider a sequence $\{A^{(k)}\}_{k=1}^\infty$ of elements in $[\mathcal{C}_\rho]_{<1}$ and let $A \in [\mathcal{C}_\rho]_{<1}$ be such that $d_K(A^{(k)}, A) \to 0$, as $k \to \infty$. By Proposition 5.2, we deduce that $P_\rho(A^{(k)}, R) \to P_\rho(A,R)$ in the operator norm topology. On the other hand, due to Theorem 3.2, the operators $P(A^{(k)}, R)$ and $P(A,R)$ are invertible. Hence, and using the well-known fact that the map $Z \mapsto Z^{-1}$ is continuous on the open set of all invertible operators, we deduce that $P_\rho(A^{(k)}, R)^{-1} \to P_\rho(A,R)^{-1}$ in the operator norm topology, as $k \to \infty$. Hence, we deduce that the sequence $\{\|P_\rho(A^{(k)}, R)^{-1}\|\}_{k=1}^\infty$ is bounded. Consequently, there exists $M > 0$ with $\|P_\rho(A^{(k)}, R)^{-1}\| \leq M$ for any $k \in \mathbb{N}$. Using inequality (6.13), we obtain

$$2\delta_\rho(A^{(k)}, A) \leq \ln\left(1 + M d_K(A^{(k)}, A)\right), \qquad k \in \mathbb{N}.$$

Since $d_K(A^{(k)}, A) \to 0$, as $k \to \infty$, the latter inequality implies that $\delta_\rho(A^{(k)}, A) \to 0$. Therefore, the d_K-topology on $[\mathcal{C}_\rho]_{<1}$ is stronger than the

δ_ρ-topology. Due to the first part of this theorem, the two topologies coincide on $[\mathcal{C}_\rho]_{<1}$. Using now Theorem 5.3, we complete the proof of part (iii).

Now, we prove item (iv). Since $[\mathcal{C}_\rho]_{<1}$ is the Harnack part of 0 (see Theorem 3.2), part (i) implies its completeness with respect to the hyperbolic metric. To prove that $[\mathcal{C}_\rho]_{<1}$ is not complete with respect to the Carathéodory metric d_K and the operator metric, we consider the following example. Let $(T_1,\ldots,T_n)\in B(\mathcal{P}_1)^n$ be the n-tuple of operators defined by $T_i := P_{\mathcal{P}_1}S_i|_{\mathcal{P}_1}$, $i=1,\ldots,n$, where $\mathcal{P}_1 := \text{span}\{e_\alpha:\ |\alpha|\leq 1\}$. Note that $\|[T_1,\ldots,T_n]\|=1$ and $T_\alpha=0$ for any $\alpha\in\mathbb{F}_n^+$ with $|\alpha|\geq 2$. Set $X_i:=\rho T_i$, $i=1,\ldots,n$, and note that

$$X_\beta=\rho T_\beta=\rho P_{\mathcal{P}_1}S_\beta|_{\mathcal{P}_1},\qquad \beta\in\mathbb{F}_n^+\backslash\{g_0\}.$$

Therefore, $(X_1,\ldots,X_n)\in\mathcal{C}_\rho$, i.e., $\omega_\rho(X_1,\ldots,X_n)\leq 1$, which implies $\omega_\rho(T_1,\ldots,T_n)\leq\frac{1}{\rho}$. The reverse inequality is due to the fact that $\|[T_1,\ldots,T_n]\|\leq\rho\omega_\rho(T_1,\ldots,T_n)$. Consequently, we have

$$\omega_\rho(T_1,\ldots,T_n)=\frac{1}{\rho},\qquad \text{for } \rho\in(0,\infty).$$

On other hand, the condition $T_\alpha=0$ if $|\alpha|\geq 2$ implies $r(T_1,\ldots,T_n)=0$. Therefore, we have

$$\omega_\rho(X_1,\ldots,X_n)=1\qquad\text{and}\qquad r(X_1,\ldots,X_n)=0.$$

Now, let $c\in(0,1)$ and define $Y^{(k)}:=c^{1/k}(X_1,\ldots,X_n)$ for $k=1,2,\ldots$. Since $\omega_\rho(Y^{(k)})=c^{1/n}<1$, Theorem 3.2 implies $Y^{(k)}\overset{H}{\sim}0$ in $\mathcal{C}_\rho$ and $Y^{(k)}\in[\mathcal{C}_\rho]_{<1}$. On the other hand, since $\omega_\rho(X_1,\ldots,X_n)=1$, we have $X:=(X_1,\ldots,X_n)\notin[\mathcal{C}_\rho]_{<1}$. Now, note that

$$\begin{aligned} d_K(Y^{(k)},X)&\leq 2\|(I-R_{Y^{(k)}})^{-1}-(I-R_X)^{-1}\|\\ &=2\|R_{Y^{(k)}}-R_X\|=2\|Y^{(k)}-X\|=2\|X\|(1-c^{1/k}).\end{aligned}$$

Consequently, $Y^{(k)}\to X$ in the operator norm and $d_K(Y^{(k)},X)\to 0$, as $k\to\infty$. This shows that $[\mathcal{C}_\rho]_{<1}$ is not complete with respect to the Carathéodory metric d_K and the operator metric. The proof is complete. □

Corollary 6.3. *Let δ_ρ be the hyperbolic metric on a Harnack part Δ of $[\mathcal{C}_\rho]^{\prec 0}$. Then*

$$d_K(A,B)\leq\max\{\|P_\rho(A,R)\|,\|P_\rho(B,R)\|\}\left(e^{2\delta_\rho(A,B)}-1\right),\qquad A,B\in\Delta.$$

If, in addition $A,B\in[\mathcal{C}_\rho]_{<1}$, then

$$2\delta_\rho(A,B)\leq\ln\left(1+\max\{\|P_\rho(B,R)^{-1}\|,\|P_\rho(A,R)^{-1}\|\}d_K(A,B)\right).$$

Corollary 6.4. *Let $f:=(f_1,\ldots,f_m)$ be a contractive free holomorphic function with $\|f(0)\|<1$ such that the boundary functions $\widetilde{f}_1,\ldots,\widetilde{f}_m$ are in the noncommutative disc algebra $\mathcal{A}_n$. If Δ is a Harnack part of $[\mathcal{C}_\rho]^{\prec 0}$, then the map*

$$\Delta\ni(T_1,\ldots,T_n)\mapsto f(T_1,\ldots,T_n)\in[\mathcal{C}_{\rho_f}]^{\prec 0}$$

is continuous with respect to the hyperbolic metric δ_ρ *on* Δ *and the Carathéodory metric* d_K *on* $[\mathcal{C}_{\rho_f}]^{\prec 0}$*, where* ρ_f *is defined by relation* (4.1)*. In particular, tha map*

$$[\mathcal{C}_\rho]_{<1} \ni (T_1, \ldots, T_n) \mapsto f(T_1, \ldots, T_n) \in [\mathcal{C}_{\rho_f}]_{<1}$$

is continuous with respect to the hyperbolic metric.

7. Harnack domination and hyperbolic metric for ρ-contractions (case $n = 1$)

In this section, we consider the single variable case ($n = 1$) and show that our Harnack domination of ρ-contractions is equivalent to the one introduced and studied by Cassier and Suciu in [9]. We recover some of their results and obtain some results which seem to be new even in the single variable case.

In the particular case when $n = 1$, the free pluriharmonic Poisson kernel $P_\rho(rY, R)$, $r \in [0, 1)$, coincides with

$$Q_\rho(rY, U) := \sum_{k=1} r^k Y^{*k} \otimes U^k + \rho I \otimes I + \sum_{k=1}^{\infty} r^k Y^k \otimes U^{*k}, \qquad Y \in \mathcal{C}_\rho \subset B(\mathcal{H}),$$

where the convergence of the series is in the operator norm topology and U is the unilateral shift acting on the Hardy space $H^2(\mathbb{T})$. For each ρ-contraction $T \in B(\mathcal{H})$, consider the operator-valued Poisson kernel defined by

$$K_\rho(z, T) := \sum_{k=1}^{\infty} z^k T^{*k} + \rho I + \sum_{k=1}^{\infty} \bar{z}^k T^k, \qquad z \in \mathbb{D},$$

which was employed by Cassier and Fack in [8]. Using Theorem 2.2, in the particular case when $n = 1$, we can prove the following result.

Proposition 7.1. *Let* T *and* T' *be two* ρ*-contractions in* $B(\mathcal{H})$ *and let* $c \geq 1$*. Then the following statements are equivalent:*

(i) $T \overset{H}{\underset{c}{\prec}} T'$;
(ii) $Q_\rho(rT, U) \leq c^2 Q_\rho(rT', U)$ *for any* $r \in [0, 1)$;
(iii) $K_\rho(z, T) \leq c^2 K_\rho(z, T')$ *for any* $z \in \mathbb{D}$.

Proof. The equivalence $(i) \leftrightarrow (ii)$ follows from Theorem 2.2, when $n = 1$. To prove the implication $(ii) \implies (iii)$, we apply the noncommutative Poisson transform (when $n = 1$) at $e^{it} I$ to the inequality of part (ii). Consequently, we obtain

$$\begin{aligned} K_\rho(re^{it}, T) &= (\mathrm{id} \otimes P_{e^{it} I})[Q_\rho(rT, U)] \\ &\leq c^2 (\mathrm{id} \otimes P_{e^{it} I})[Q_\rho(rT', U)] = c^2 K_\rho(re^{it}, T') \end{aligned}$$

for any $r \in [0,1)$ and $t \in \mathbb{R}$. Now let us prove that $(iii) \implies (ii)$. Since

$$\begin{aligned}\Big\langle (T^{*k} \otimes U^k)(h_m \otimes e^{imt}), h_p \otimes e^{ipt} \Big\rangle_{\mathcal{H}\otimes H^2(\mathbb{T})} \\ = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big\langle e^{ikt} T^{*k}(e^{imt} h_m), e^{ipt} h_p \Big\rangle_{\mathcal{H}} dt\end{aligned}$$

for any $h_m, h_p \in \mathcal{H}$ and $k, m, p \in \mathbb{N}$, one can easily obtain

$$\begin{aligned}&\left\langle \left(c^2 Q_\rho(rT', U) - Q_\rho(rT, U)\right) h(e^{it}), h(e^{it})\right\rangle_{\mathcal{H}\otimes H^2(\mathbb{T})} \\ &\qquad = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\langle \left(c^2 K_\rho(re^{it}, T') - K_\rho(re^{it}, T)\right) h(e^{it}), h(e^{it})\right\rangle_{\mathcal{H}}\end{aligned}$$

for any function $e^{it} \mapsto h(e^{it})$ in $\mathcal{H}\otimes H^2(\mathbb{T})$. Now, the implication $(iii) \implies (ii)$ is clear. The proof is complete. □

Let $T, T' \in B(\mathcal{H})$ be ρ-contractions such that $T \overset{H}{\prec} T'$. Due to Proposition 7.1 and Corollary 2.3, we deduce that

$$\begin{aligned}\|L_{T',T}\| &= \inf\{c > 1 : \ Q_\rho(rT, U) \le c^2 Q_\rho(rT', U) \ \text{ for any } \ r \in [0,1)\} \\ &= \inf\{c > 1 : \ K_\rho(z, T) \le c^2 K_\rho(z, T') \ \text{ for any } \ z \in \mathbb{D}\} \\ &= \inf\{c > 1 : \ K_\rho(z, T^*) \le c^2 K_\rho(z, T'^*) \ \text{ for any } \ z \in \mathbb{D}\} = \|L_{T'^*,T^*}\|.\end{aligned}$$

Therefore $T \overset{H}{\prec} T'$ if and only if $T^* \overset{H}{\prec} T'^*$.

Theorem 7.2. *Let $T, T' \in B(\mathcal{H})$ be such that $T, T' \in [\mathcal{C}_\rho]_{<1}$. Then*

$$\|L_{T',T}\| = \sup_{z\in\mathbb{D}} \|\Delta_{\rho,T'^*}(z)^{-1}(I - \bar{z}T'^*)(I - \bar{z}T^*)^{-1}\Delta_{\rho,T^*}(z)\|,$$

where

$$\Delta_{\rho,T}(z) := [\rho I + (1-\rho)(zT^* + \bar{z}T) + (\rho - 2)TT^*]^{1/2}, \qquad z \in \mathbb{D}.$$

Moreover,

$$\delta_\rho(T, T') = \ln\max\{\|L_{T,T'}\|, \|L_{T',T}\|\}.$$

Proof. If $T, T' \in [\mathcal{C}_\rho]_{<1}$, Theorem 3.4 implies

$$\begin{aligned}\|L_{T',T}\| = \|L_{T'^*,T^*}\| &= \sup_{z\in\mathbb{D}} \|\Delta_{\rho,T^*}(z)(I - zT)^{-1}(I - zT')\Delta_{\rho,T'^*}(z)^{-1}\| \\ &= \sup_{z\in\mathbb{D}} \|\Delta_{\rho,T'^*}(z)^{-1}(I - \bar{z}T'^*)(I - \bar{z}T^*)^{-1}\Delta_{\rho,T^*}(z)\|.\end{aligned}$$

Using now Theorem 3.5, we complete the proof. □

We mention that when $\rho = 1$, we recover a result obtained by I. Suciu [53], using different methods. However, if $\rho > 0$ and $\rho \neq 1$, the result of Theorem 7.2 seems to be new. We also remark that Proposition 3.12 , Proposition 5.2, and part (i) of Theorem 5.3 are new even in the single variable case ($n = 1$).

The next result makes an interesting connection between the Harnack domination for n-tuples of operators in $\mathcal{C}_\rho$ and and the Harnack domination for ρ-contractions ($n = 1$), via the reconstruction operator.

Theorem 7.3. *Let $A := (A_1, \dots, A_n)$ and $B := (B_1, \dots, B_n)$ be in $\mathcal{C}_\rho$ and let $c > 0$. Then the following statements are equivalent:*

(i) $A \overset{H}{\underset{c}{\prec}} B$;

(ii) $R_A \overset{H}{\underset{c}{\prec}} R_B$, *where $R_X := X_1^* \otimes R_1 + \cdots + X_n^* \otimes R_n$ is the reconstruction operator associated with $X := (X_1, \dots, X_n) \in \mathcal{C}_\rho$ and the right creation operators $R_1, \dots, R_n$.*

(iii) $R_A^* \overset{H}{\underset{c}{\prec}} R_B^*$.

Proof. First, assume that item (i) holds. Due to Theorem 2.2, we have

$$P_\rho(rA, S) \le c^2 P_\rho(rB, S) \tag{7.1}$$

for any $r \in [0, 1)$, where $S := (S_1, \dots, S_n)$ is the n-tuple of left creation operators. Let U be the unilateral shift on the Hardy space $H^2(\mathbb{T})$. Since $R_i^* R_j = \delta_{ij} I$, the n-tuple $(R_1 \otimes U^*, \dots, R_n \otimes U^*)$ is a row contraction acting from $[F^2(H_n) \otimes H^2(\mathbb{T})]^n$ to $F^2(H_n) \otimes H^2(\mathbb{T})$. Applying the noncommutative Poisson transform at $(R_1 \otimes U^*, \dots, R_n \otimes U^*)$ to inequality (7.1), we obtain

$$\begin{aligned} Q_\rho(rR_A, U) &= \left(\mathrm{id} \otimes P_{(R_1 \otimes U^*, \dots, R_n \otimes U^*)}\right) [P_\rho(rA, S)] \\ &\le c^2 \left(\mathrm{id} \otimes P_{(R_1 \otimes U^*, \dots, R_n \otimes U^*)}\right) [P_\rho(rB, S)] = c^2 Q_\rho(rR_B, U) \end{aligned}$$

for any $r \in [0, 1)$. Using Proposition 7.1, we obtain that $R_A \overset{H}{\underset{c}{\prec}} R_B$. Now, assume that (ii) holds. Proposition 7.1 implies

$$K_\rho(re^{it}, R_A) \le c^2 K_\rho(re^{it}, R_B), \qquad r \in [0, 1) \text{ and } t \in \mathbb{R}. \tag{7.2}$$

Taking $t = 0$, we obtain $P_\rho(rA, R) \le c^2 P_\rho(rB, R)$ for any $r \in [0, 1)$, which, due to Theorem 2.2, implies $A \overset{H}{\underset{c}{\prec}} B$. The equivalence $(ii) \leftrightarrow (iii)$ is a consequence of Proposition 7.1 and the fact that inequality (7.2) is equivalent to

$$K_\rho(re^{it}, R_A^*) \le c^2 K_\rho(re^{it}, R_B^*), \qquad r \in [0, 1) \text{ and } t \in \mathbb{R}.$$

This completes the proof. □

We remark that, according to Theorem 3.4 and Corollary 2.3, we have

$$\|L_{B,A}\| = \|C_{\rho,A} C_{\rho,B}^{-1}\| = \inf\{c > 1 : \; P_\rho(A, R) \le c^2 P_\rho(B, R)\}$$

for any $A, B \in [\mathcal{C}_\rho]_{<1}$, where $C_{\rho,A}$ is defined in Theorem 3.4.

Corollary 7.4. *If A, B are n-tuples of operators in $[\mathcal{C}_\rho]_{<1}$, then $\|L_{B,A}\| = \|L_{R_B, R_A}\| = \|L_{R_B^*, R_A^*}\|$. Moreover, $\delta_\rho(A, B) = \delta_\rho(R_A, R_B)$.*

References

[1] T. Ando, I. Suciu, D. Timotin, Characterization of some Harnack parts of contractions, *J. Operator Theory* 2 (1979), no. 2, 233–245.

[2] C. Badea and G. Cassier, Constrained von Neumann inequalities, *Adv. Math.* 166 (2002), 260–297.

[3] C.A. Berger, A strange dilation theorem, Abstract 625–152, *Notices Amer. Math. Soc.* 12 (1965), 590.

[4] C.A. Berger and J.G. Stampfli, Mappings theorems for numerical ranges, *Amer J. Math.* 89 (1967), 1047–1055.

[5] C.A. Berger and J.G. Stampfli, Norm relations and skew dilations, *Acta Sci. Math. (Szeged)* 28 (1967), 191–195.

[6] S. Bergman, *The kernel function and conformal mapping*, Mathematical Surveys, No. V. American Mathematical Society, Providence, R.I., 1970. x+257 pp.

[7] J. W. Bunce, Models for n-tuples of noncommuting operators, *J. Funct. Anal.* 57(1984), 21–30.

[8] G. Cassier and T. Fack, Un noyau pour divers calculs fonctionnels, *C. R. Acad. Sci. Paris Sr. I Math.* 317 (1993), no. 7, 683–688.

[9] G. Cassier and N. Suciu, Mapping theorems and Harnack ordering for ρ-contractions, *Indiana Univ. Math. J.* 55 (2006), no. 2, 483–523.

[10] G. Cassier and N. Suciu, Analytic functions of a uniformly stable ρ-contraction. *Operator theory 20*, 55–73, Theta Ser. Adv. Math., 6, Theta, Bucharest, 2006.

[11] J. Cuntz, Simple C^*–algebras generated by isometries, *Commun. Math. Phys.* 57 (1977), 173–185.

[12] K. R. Davidson Free semigroup algebras. A survey. *Systems, approximation, singular integral operators, and related topics* (Bordeaux, 2000), 209–240, Oper. Theory Adv. Appl., 129, Birkhuser, Basel, 2001.

[13] E.G. Effros and Z.J. Ruan, *Operator spaces*, London Mathematical Society Monographs. New Series, 23. The Clarendon Press, Oxford University Press, New York, 2000.

[14] K. Fan, Analytic functions of a proper contraction, *Math. Z* 160 (1978), 275–290.

[15] C. Foiaş, On Harnack parts of contractions. *Rev. Roumaine Math. Pures Appl.* 19 (1974), 315–318.

[16] A. E. Frazho, Models for noncommuting operators, *J. Funct. Anal.* 48 (1982), 1–11.

[17] P.R. Halmos, *A Hilbert space problem book*, D. Van Nostrand, Princeton, NJ, 1967.

[18] P.R. Halmos, Positive definite sequences and the miracle of w, 17 pages, mimeographed.

[19] J.A.R. Holbrook, On the power-bounded operators of Sz.-Nagy and Foias, *Acta. Sci. Math. (Szeged)* 29 (1968), 299-310.

[20] J.A.R. HOLBROOK, Multiplicative propreties of the numerical radius in operator theory, *J. Reine Angew. Math.* 237 (1969), 166–174.

[21] T. KATO, Some mapping theorems for the numerical range, *Proc. Japan Acad.* 41 (1965), 652-655.

[22] S. KOBAYASHI, *Hyperbolic complex spaces*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 318. Springer-Verlag, Berlin, 1998. xiv+471 pp.

[23] S. KOBAYASHI, *Hyperbolic manifolds and holomorphic mappings*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005. xii+148 pp.

[24] S.G. KRANTZ, *Geometric function theory*, Explorations in complex analysis. Cornerstones. Birkhuser Boston, Inc., Boston, MA, 2006. xiv+314 pp.

[25] V. MÜLLER, On the joint spectral radius, *Ann. Polon. Math.* 66 (1997), 173–182.

[26] P.S. MUHLY AND B. SOLEL, Tensor algebras over C^*-correspondences: representations, dilations, and C^*-envelopes, *J. Funct. Anal.* 158 (1998), 389–457.

[27] P.S. MUHLY AND B. SOLEL, Hardy algebras, W^*-correspondences and interpolation theory, *Math. Ann.* 330 (2004), 353–415.

[28] P.S. MUHLY AND B. SOLEL, Schur Class Operator Functions and Automorphisms of Hardy Algebras, *Documenta Math.* 13 (2008), 365–411.

[29] V.I. PAULSEN, *Completely Bounded Maps and Dilations*, Pitman Research Notes in Mathematics, Vol.146, New York, 1986.

[30] C. PEARCY, An elementary proof of the power inequality for the numerical range, *Michigan Math. J.* 13 (1966), 289–291.

[31] G. PISIER, *Similarity Problems and Completely Bounded Maps*, Springer Lect. Notes Math., Vol.1618, Springer-Verlag, New York, 1995.

[32] G. POPESCU, Models for infinite sequences of noncommuting operators, *Acta. Sci. Math. (Szeged)* 53 (1989), 355–368.

[33] G. POPESCU, Isometric dilations for infinite sequences of noncommuting operators, *Trans. Amer. Math. Soc.* 316 (1989), 523–536.

[34] G. POPESCU, Characteristic functions for infinite sequences of noncommuting operators, *J. Operator Theory* 22 (1989), 51–71.

[35] G. POPESCU, Multi-analytic operators and some factorization theorems, *Indiana Univ. Math. J.* 38 (1989), 693–710.

[36] G. POPESCU, Von Neumann inequality for $(B(H)^n)_1$, *Math. Scand.* 68 (1991), 292–304.

[37] G. POPESCU, Functional calculus for noncommuting operators, *Michigan Math. J.* 42 (1995), 345–356.

[38] G. POPESCU, Multi-analytic operators on Fock spaces, *Math. Ann.* 303 (1995), 31–46.

[39] G. POPESCU, Positive-definite functions on free semigroups, *Canad. J. Math.* 48 (1996), no. 4, 887–896.

[40] G. POPESCU, Noncommutative disc algebras and their representations, *Proc. Amer. Math. Soc.* 124 (1996), 2137–2148.

[41] G. Popescu, Poisson transforms on some C^*-algebras generated by isometries, *J. Funct. Anal.* 161 (1999), 27–61.

[42] G. Popescu, Curvature invariant for Hilbert modules over free semigroup algebras, *Adv. Math.* 158 (2001), 264–309.

[43] G. Popescu, Similarity and ergodic theory of positive linear maps, *J. Reine Angew. Math.* 561 (2003), 87–129.

[44] G. Popescu, Free holomorphic functions on the unit ball of $B(\mathcal{H})^n$, *J. Funct. Anal.* 241 (2006), 268–333.

[45] G. Popescu, Characteristic functions and joint invariant suspaces, *J. Funct. Anal.* 237 (2006), 277–320.

[46] G. Popescu, Hyperbolic geometry on the unit ball of $B(\mathcal{H})^n$ and dilation theory, *Indiana Univ. Math. J.* 57 (2008), No.6, 2891-2930.

[47] G. Popescu, Noncommutative transforms and free pluriharmonic functions, *Adv. Math.* 220 (2009), 831-893.

[48] G. Popescu, Unitary invariants in multivariable operator theory, *Mem. Amer. Math. Soc.* 200 (2009), no. 941, vi+91 pp.

[49] G. Popescu, Noncommutative hyperbolic geometry on the unit ball of $B(\mathcal{H})^n$, *J. Funct. Anal.* 256 (2009), 4030-4070.

[50] G. Popescu, Free holomorphic automorphisms of the unit ball of $B(\mathcal{H})^n$, *J. Reine Angew. Math.*, to appear.

[51] G. Popescu, Free holomorphic functions on the unit ball of $B(\mathcal{H})^n$. II., *J. Funct. Anal.*, to appear.

[52] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer-verlag, New-York/Berlin, 1980.

[53] I. Suciu, Analytic formulas for the hyperbolic distance between two contractions, *Ann. Polon. Math.* 66 (1997), 239–252.

[54] B. Sz.-Nagy, Sur les contractions de l'espace de Hilbert, *Acta. Sci. Math. (Szeged)* 15 (1953), 87–92.

[55] B. Sz.-Nagy and C. Foiaş, On certain classes of power-bounded operators in Hilbert space, *Acta. Sci. Math. (Szeged)* 27 (1966), 17–25.

[56] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, New York 1970.

[57] J. von Neumann, Eine Spectraltheorie für allgemeine Operatoren eines unitären Raumes, *Math. Nachr.* 4 (1951), 258–281.

[58] K. Zhu, *Spaces of holomorphic functions in the unit ball*, Graduate Texts in Mathematics, 226. Springer-Verlag, New York, 2005. x+271 pp.

[59] J.P. Williams, Schwarz norms for operators, *Pacific J. Math.* 24 (1968), 181–188.

Gelu Popescu
Department of Mathematics
The University of Texas at San Antonio
San Antonio
TX 78249
USA
gelu.popescu@utsa.edu