Spectral Analysis of Relativistic Atoms – Interaction with the Quantized Radiation Field

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Abstract. This is the the second part of a series of two papers, which investigate spectral properties of Dirac operators with singular potentials. We will provide a spectral analysis of a relativistic one-electron atom in interaction with the second quantized radiation field and thus extend the work of Bach, Fröhlich, and Sigal [5] and Hasler, Herbst, and Huber [19] to such systems. In particular, we show that the lifetime of excited states in a relativistic hydrogen atom coincides with the life time given by Fermi's Golden Rule in the non-relativistic case. We will rely on the technical preparations derived in the first part [25] of this work.

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1 Introduction

We continue our study of resonances for relativistic electrons and apply the results about one-particle Dirac operators with singular potentials in [25] to a relativistic Pauli-Fierz model. We prove upper and lower bounds on the lifetime of excited states for a relativistic hydrogen (-like) atom coupled to the quantized radiation field and show that it is described by Fermi's Golden Rule and coincides with the non-relativistic result in leading order in the fine structure constant α.
The spectral analysis of non-relativistic atoms in interaction with the radiation field was initiated by Bach, Fröhlich, and Sigal [4, 5]. It was carried on by Griesemer, Lieb and Loss [16], by Fröhlich, Griesemer and Schlein (see for

example [15]) and many others (see for example Hiroshima [22], Arai and Hirokawa [3], Dereziński and Gérard [12], Hiroshima and Spohn [21]), Loss, Miyao and Spohn [32] or Hasler and Herbst [18, 17]). Recently, Miyao and Spohn [35] showed the existence of a groundstate for a semi-relativistic electron coupled to the quantized radiation field.
Bach, Fröhlich, and Sigal [5] proved a lower bound on the lifetime of excited states in non-relativistic QED. Later, an upper bound was proven by Hasler, Herbst, and Huber [19] (see also [24]) and by Abou Salem et al. [1]. As in [4, 5, 19] we use the method of complex dilation. Since the corresponding operators are not normal, we are going to apply the Feshbach projection method, which was introduced in non-relativistic QED by Bach et al. [4, 5].
We describe the electron by the Coulomb-Dirac operator, projected onto its positive spectral subspace. Note that this choice is not gauge invariant. Our analysis will work for other potentials as well, as long as condition (26) holds for the difference between fine structure components, and as long the eigenfunctions have a exponential decay uniform in the velocity of light.
On a technical level the relativistic model is more difficult to handle than the nonrelativistic Pauli-Fierz model. One reason is the fine structure splitting of the eigenvalues. Moreover, due to the use of complex dilation one has to make sense of the notion of a positive spectral subspace for a non-selfadjoint operator. Finally, a factor of α is missing in front of the radiation field.
We would like to mention that the Feshbach method is named after the physicist Herman Feshbach, which used the method to deal with resonances in nuclear physics [14, Equation (2.14)]. Also Howland [23] used the Feshbach operator calling it "Livšic matrix", since Livšic [31, 30] used the method in scattering theory. Moreover, the method is known under the name "Schur complement". This name is due to Haynsworth [20], who used it in honor of the Schur determinant formula. Also Menniken and Motovilov [34, 33] use the Schur complement to treat resonances of 2×2-operator matrices. They call it "transfer function", however. For a detailed overview over the history of the Schur complement, we refer the reader to [40]. For some more references about resonances in general and the spectral analysis of (non-relativistic QED) we refer the reader to [25].

2 Model and Definitions

The (initial) Hilbert space of our model is $\mathcal{H}' := \mathcal{H}_{\mathrm{el}} \otimes \mathcal{F}$, where $\mathcal{H}_{\mathrm{el}} := L^2(\mathbb{R}^3; \mathbb{C}^4)$ is the Hilbert space for a relativistic electron and

$$\mathcal{F} := \bigoplus_{N=0}^{\infty} \mathcal{S}_N L^2[(\mathbb{R}^3 \times \mathbb{Z}_2)]^N$$

is the Fock space (with vacuum Ω) of the quantized electromagnetic field taking into account the two polarizations of the photon. $\mathcal{S}_N$ is the projection onto the subspace of functions which are symmetric under exchange of variables.

The Coulomb-Dirac operator with velocity of light $\mathfrak{c}$, Planck constant $\hbar$, electron mass m, elementary charge e, atomic number $\mathfrak{Z}$ and permittivity of the vacuum ϵ_0 is in SI units

$$D' := -\mathrm{i}\,\hbar\mathfrak{c}\boldsymbol{\alpha}\cdot\nabla + \beta mc^2 - \frac{e^2\mathfrak{Z}}{4\pi\epsilon_0}\frac{1}{|\cdot|}.$$

This operator is self-adjoint on the domain $H^1(\mathbb{R}^3;\mathbb{C}^4)$ for $\frac{e^2\mathfrak{Z}}{4\pi\epsilon_0} < \frac{\sqrt{3}}{2}c$. In the following, we will always assume that this condition is fulfilled. Actually, for technical reasons, we are even going to impose some more restrictive conditions later on (see for example Theorem 3).
We denote the positive spectral projection of this operator by $\Lambda'^{(+)}$. We will restrict the operator to its positive spectral subspace and couple it to the quantized radiation field $A'_{\kappa'}(x) := A'_{\kappa'}(x)_+ + A'_{\kappa'}(x)_-$, where $A'_{\kappa'}(x)_+$ and $A'_{\kappa'}(x)_-$ are defined as in the non-relativistic case by

$$A'_{\kappa'}(x)_+ := \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \mathrm{d}k\,\kappa'(|k|)\sqrt{\frac{\hbar}{2\epsilon_0\mathfrak{c}|k|(2\pi)^3}}\varepsilon'_\mu(k)e^{-ik\cdot x}a'^*_\mu(k) \qquad (1)$$

$$A'_{\kappa'}(x)_- := \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \mathrm{d}k\,\kappa'(|k|)\sqrt{\frac{\hbar}{2\epsilon_0\mathfrak{c}|k|(2\pi)^3}}\varepsilon'_\mu(k)e^{\mathrm{i}\,\alpha k\cdot x}a'_\mu(k). \qquad (2)$$

Here $\varepsilon'_\mu(k)$, $\mu = 1,2$ are the polarization vectors of the photons, which depend only on the direction of k.
If we add the operator H'_f for the kinetic energy of the photons

$$H'_\mathrm{f} := \hbar\mathfrak{c}\sum_{\mu=1,2}\int_{k\in\mathbb{R}^3}\mathrm{d}k\,|k|a'^*_\mu(k)a'_\mu(k), \qquad (3)$$

we obtain (cf. [11, B-V.1., Formula (35) through (39), page 431])

$$H' := \Lambda'^{(+)}[\mathfrak{c}\boldsymbol{\alpha}\cdot(-\mathrm{i}\,\hbar\nabla - eA'_{\kappa'}(x)) + \beta m\mathfrak{c}^2 - \frac{e^2\mathfrak{Z}}{4\pi\epsilon_0}\frac{1}{|\cdot|}]\Lambda'^{(+)} + H'_f.$$

In principle, one could define the operator without restriction to the positive spectral subspace. For this case it is at least known that selfadjoint realizations exist [2, Theorem 1.2], which are, however, not explicitly known. Moreover the expression for the inverse life lifetime (see equation (21)) without UV cutoff would diverge in this case so the investigation of this operator with regard to the lifetime of excited states would not make any sense. We would like to mention that for a certain class of potentials – which does not include the Coulomb potential – it is known that the operator without projections is essentially self-adjoint on a suitable domain. (see Stockmeyer and Zenk [38] and Arai [2]).
Similar to the non-relativistic case [19, 5] we set $a_0 := \alpha^{-1}(\frac{\hbar}{m\mathfrak{c}})$ (Bohr radius), $\zeta := a_0$ and $\xi^{-1} := \frac{\alpha}{a_0}$ and scale the operator according to $x \to \zeta x$ and $k \to \xi^{-1}k$. We denote the corresponding unitary transformation by U. In

this scaling we can expect to be able to treat the operator similarly as in the non-relativistic case. We have to make the replacements

$$\hbar\nabla \to \alpha m\mathfrak{c}\nabla \qquad eA'_{\kappa'}(x) \to \alpha^{5/2} m\mathfrak{c} A_\kappa(\alpha x)$$
$$\frac{e^2}{4\pi\epsilon_0}\frac{1}{|\cdot|} \to \alpha^2 m\mathfrak{c}^2 \frac{1}{|\cdot|} \qquad H'_{\mathrm{f}} \to \alpha^2 m\mathfrak{c}^2 H_{\mathrm{f}}$$

and obtain

$$\tilde{H}'_{\alpha,\mathfrak{Z}} := UH'U^* = \alpha^2 m\mathfrak{c}^2 \left[\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}[D_{\alpha^{-1},\mathfrak{Z}} - \sqrt{\alpha}\boldsymbol{\alpha}\cdot A_\kappa(\alpha x)]\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} + H_{\mathrm{f}}\right]. \quad (4)$$

Here

$$D_{\alpha^{-1},\mathfrak{Z}} := -\mathrm{i}\,\alpha^{-1}\boldsymbol{\alpha}\cdot\nabla + \alpha^{-2}\beta - \frac{\mathfrak{Z}}{|\cdot|}$$

with $\alpha\mathfrak{Z} < \sqrt{3}/2$ is the scaled version of the Dirac operator D'. $\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}$ is the positive spectral projection of the operator $D_{\alpha^{-1},\mathfrak{Z}}$, where α^{-1} plays the role of the velocity of light after the scaling and $\mathfrak{Z}$ the role of the coupling constant. We denote the eigenvalues of this operator by $\tilde{E}_{n,l}(\alpha^{-1},\mathfrak{Z})$, where n is the principal quantum number and l numbers the eigenvalues belonging to the principal quantum number n by size *not* counting multiplicities. We have $n \in \mathbb{N}$ and $l \in \mathbb{N}$ with $l \leq n$. We set

$$E_{n,l}(\alpha^{-1},\mathfrak{Z}) := \tilde{E}_{n,l}(\alpha^{-1},\mathfrak{Z}) - c^2, \quad E_n(\infty,\mathfrak{Z}) := -\frac{\mathfrak{Z}^2}{2n^2}, \quad (5)$$

where $E_n(\infty,\mathfrak{Z})$ is the n-th eigenvalue (not counting multiplicities) of the Schrödinger operator which we obtain in the limit $\alpha \to 0$ (see [25, Section 8]). We abbreviate $E_n := E_n(\infty,\mathfrak{Z})$ and $E_{n,l}(\alpha) := E_{n,l}(\alpha^{-1},\mathfrak{Z})$ for $n \in \mathbb{N}$ and for $1 \leq l \leq n$.

H_{f} and $A_\kappa(x)$ are given by

$$H_{\mathrm{f}} := \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \mathrm{d}k\,|k| a^*_\mu(k) a_\mu(k) \quad (6)$$

and $A_\kappa(x) := A_\kappa(x)_+ + A_\kappa(x)_-$ with

$$A_\kappa(x)_+ := \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \frac{\mathrm{d}k\,\kappa(|k|)}{\sqrt{4\pi^2|k|}}\varepsilon_\mu(k) e^{-ik\cdot x} a^*_\mu(k) \quad (7)$$

$$A_\kappa(x)_- := \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \frac{\mathrm{d}k\,\kappa(|k|)}{\sqrt{4\pi^2|k|}}\varepsilon_\mu(k) e^{ik\cdot x} a_\mu(k) \quad (8)$$

as in the non-relativistic case.

In the following, we will consider the operator

$$H_\alpha := \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}[D_{\alpha^{-1},\mathfrak{Z}} - \alpha^{-2} - \sqrt{\alpha}\boldsymbol{\alpha}\cdot A_\kappa(\alpha x)]\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} + H_{\mathrm{f}} \quad (9)$$

on $\mathcal{H} := \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} L^2(\mathbb{R}^3;\mathbb{C}^4) \otimes \mathcal{F}$, where we omit trivial factors $\otimes \mathbf{1}_{\mathrm{f}}$ or $\mathbf{1}_{\mathrm{el}}\otimes$.
In order to apply the methods of the non-relativistic case (see Bach, Fröhlich, and Sigal [5] and Hasler, Herbst, and Huber [19]) with a minimal amount of changes, and in order to apply the results about the non-relativistic limit obtained in [25], we subtract the rest energy α^{-2}. As in the non-relativistic case we define the perturbation parameter $g := \alpha^{3/2} > 0$ and the perturbation operator

$$W^{(\alpha)} := \sqrt{\alpha}\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} \boldsymbol{\alpha} \cdot A_\kappa(\alpha x)\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}$$

as well as the free operator

$$H_{\alpha,0} := \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} D_{\alpha^{-1},\mathfrak{Z}} \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} + H_{\mathrm{f}} - \alpha^{-2}$$

and the electronic operator

$$H^{(\alpha)}_{\mathrm{el}} := \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}[D_{\alpha^{-1},\mathfrak{Z}} - \alpha^{-2}]\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}.$$

We will always assume $\mathfrak{Z} > 0$.
We will prove the self-adjointness of these operators in Section 3. Note that contrary to the non-relativistic case also the free operator depends on α. We suppress the dependence of the operators H_α, $H_{\alpha,0}$ and $H^{(\alpha)}_{\mathrm{el}}$ on the atomic number $\mathfrak{Z}$, since we will treat it as a fixed parameter.
Note that the prefactor of the photonic field in (9) is $\sqrt{\alpha}$ only and not $\alpha^{3/2}$ as in the non-relativistic case. Moreover, $D_{\alpha,\mathfrak{Z}}$ depends on the fine structure constant. The limit $\alpha \to 0$ corresponds in this scaling to the non-relativistic limit. In the treatment of the resonances for this operator the distance of neighbouring eigenvalues may vanish as $\alpha \to 0$ so that the estimates on the Feshbach operator (see below) have to be improved. Nevertheless we will use the perturbation parameter $g = \alpha^{3/2}$.
As in [5, 19], we will make use of (complex) dilations of the above operators: We define

$$\begin{aligned} H^{(\alpha)}_{\mathrm{el}}(\theta) : &= \mathcal{U}_{\mathrm{el}}(\theta) H^{(\alpha)}_{\mathrm{el}} \mathcal{U}_{\mathrm{el}}(\theta)^{-1},\ H_g(\theta) := \mathcal{U}(\theta) H_g \mathcal{U}(\theta)^{-1} \text{ and} \\ W_g(\theta) : &= \mathcal{U}(\theta) W_g \mathcal{U}(\theta)^{-1} \end{aligned} \tag{10}$$

for real θ, where $\mathcal{U}(\theta)$ is the unitary group associated to the generator of dilations. It is defined in such a way that the coordinates of the electron are dilated as $x_j \mapsto e^\theta x_j$ and the momenta of the photons as $k \mapsto e^{-\theta}k$. In this way we obtain the operator

$$H^{(\alpha)}_{\mathrm{el}}(\theta) := \mathcal{U}_{\mathrm{el}}(\theta) H^{(\alpha)}_{\mathrm{el}} \mathcal{U}_{\mathrm{el}}(\theta)^{-1} = \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)[D_{\alpha^{-1},\mathfrak{Z}}(\theta) - \alpha^{-2}]\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)$$

on $\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) L^2(\mathbb{R}^3;\mathbb{C}^4)$, which is selfadjoint on $\mathrm{Dom}(H^{(\alpha)}_{\mathrm{el}}(\theta)) =$

$= \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) H^1(\mathbb{R}^3;\mathbb{C}^4)$, as well as the operators

$$\begin{aligned}
H_\alpha(\theta) &:= \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)[D_{\alpha^{-1},\mathfrak{Z}}(\theta) - \alpha^{-2} - \sqrt{\alpha}\boldsymbol{\alpha}\cdot A^{(\theta)}_\kappa(\alpha x)]\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) + e^{-\theta} H_{\mathrm{f}}\\
W^{(\alpha)}(\theta) &:= \sqrt{\alpha}\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)[\boldsymbol{\alpha}\cdot A^{(\theta)}_\kappa(\alpha x)]\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)\\
H_{\alpha,0}(\theta) &:= \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) D_{\alpha^{-1},\mathfrak{Z}}(\theta)\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) + e^{-\theta} H_{\mathrm{f}} - \alpha^{-2}
\end{aligned}$$

on $\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) L^2(\mathbb{R}^3;\mathbb{C}^4)\otimes\mathcal{F}$, where $\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)$ has been defined in [25] even for non-real θ. Here $A^{(\theta)}_\kappa(x) := A^{(\theta)}_\kappa(x)_+ + A^{(\theta)}_\kappa(x)_-$, where

$$A^{(\theta)}_\kappa(x)_+ := \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \mathrm{d}k\, G^{(\bar\theta)}_x(k,\mu)^* a^*_\mu(k)$$

and

$$A^{(\theta)}_\kappa(x)_- := \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \mathrm{d}k\, G^{(\theta)}_x(k,\mu) a_\mu(k)$$

with

$$G^{(\theta)}_x(k,\mu) := \frac{e^{-\theta}\kappa(e^{-\theta}|k|)}{\sqrt{4\pi^2|k|}} e^{\mathrm{i}\,k\cdot x}\epsilon_\mu(k).$$

We will show in Section 3 that these operators admit a holomorphic continuation to certain values of θ. Moreover, we define

$$W^{(\alpha)}_{1,0}(\theta) := \sqrt{\alpha}\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)\left[\boldsymbol{\alpha}\cdot A^{(\theta)}_\kappa(\alpha x)_+\right]\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) \tag{11}$$

$$W^{(\alpha)}_{0,1} := \sqrt{\alpha}\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)\left[\boldsymbol{\alpha}\cdot A^{(\theta)}_\kappa(\alpha x)_-\right]\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) \tag{12}$$

$$w_{0,1}(k,\mu;\theta) := \sqrt{\alpha}\boldsymbol{\alpha}\cdot G^{(\theta)}_{\alpha x}(k,\mu) \tag{13}$$

$$w_{1,0}(k,\mu;\theta) := w_{0,1}(k,\mu;\bar\theta)^*. \tag{14}$$

Using the notation from [25, Section 5] we define the projections

$$\begin{aligned}
P^{(\alpha)}_{\mathrm{el},n,l}(\theta) &:= P_{n,l}(\alpha^{-1},\mathfrak{Z};\theta) & P^{(\alpha)}_{\mathrm{el},n,l} &:= P_{n,l}(\alpha^{-1},\mathfrak{Z};0)\\
P^{(\alpha)}_{\mathrm{el},n}(\theta) &:= P_{n}(\alpha^{-1},\mathfrak{Z};\theta) & P^{(\alpha)}_{\mathrm{el},n} &:= P_{n}(\alpha^{-1},\mathfrak{Z};0)\\
\overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta) &:= \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) - P^{(\alpha)}_{\mathrm{el},n}(\theta) & \overline{P}^{(\alpha)}_{\mathrm{el},n} &:= \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} - P^{(\alpha)}_{\mathrm{el},n}\\
\overline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta) &:= P^{(\alpha)}_{\mathrm{el},n}(\theta) - P^{(\alpha)}_{\mathrm{el},n,l}(\theta) & \overline{P}^{(\alpha)}_{\mathrm{el},n,l} &:= P^{(\alpha)}_{\mathrm{el},n} - P^{(\alpha)}_{\mathrm{el},n,l}\\
\underline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta) &:= \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) - P^{(\alpha)}_{\mathrm{el},n,l}(\theta) & \underline{P}^{(\alpha)}_{\mathrm{el},n,l} &:= \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} - P^{(\alpha)}_{\mathrm{el},n,l}
\end{aligned}$$

as operators on $\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) L^2(\mathbb{R}^3;\mathbb{C}^4)$ and $\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} L^2(\mathbb{R}^3;\mathbb{C}^4)$, respectively. Moreover, we need for a $\eta > 0$ such that $\tilde E_{\tilde n,\tilde n}(\alpha^{-1},\mathfrak{Z}) < \alpha^{-2} - \eta$ and $\tilde E_{\tilde n+1,1}(\alpha^{-1},\mathfrak{Z}) > \alpha^{-2} - \eta$ for some $\tilde n \in \mathbb{N}$ (see [25, Section 7]) the projections

$$P_{\mathrm{disc}}(\alpha;\theta) := P_{\mathrm{disc},\tilde n}(\alpha^{-1},\mathfrak{Z};\theta) = \sum_{1\le n'\le\tilde n} P_{n'}(\alpha^{-1},\mathfrak{Z};\theta) \tag{15}$$

and

$$\bar{P}_{\mathrm{disc}}(\alpha;\theta) := \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) - P_{\mathrm{disc}}(\alpha;\theta) \tag{16}$$

as operators on $\mathrm{Ran}\,\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)L^2(\mathbb{R}^3;\mathbb{C}^4)$ as well. η is chosen in such a way that $\tilde{n} > n$, where n is the principal quantum number whose life-time we are interested in.

For $\rho > 0$ (to be specified later) we define the projections

$$P_{n,l}(\theta) := P^{(\alpha)}_{\mathrm{el},n,l} \otimes \chi_{H_\mathrm{f}\le\rho}, \quad \overline{P}_{n,l}(\theta) := \mathbf{1} - P_{n,l}(\theta)$$

and for $R > 0$

$$\overline{P}_{n,l}(\theta;R) := P^{(\alpha)}_{\mathrm{el},n,l}(\theta) \otimes \chi_{H_\mathrm{f}+R>\rho} + \underline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta) \otimes \mathbf{1}_\mathrm{f}$$

as operators on $\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)L^2(\mathbb{R}^3;\mathbb{C}^4)\otimes\mathcal{F}$.

As in [5, 19], the main technical tool in our analysis is the Feshbach operator

$$\begin{aligned}\mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta)-z) := P_{n,l}(\theta)(H_\alpha(\theta)-z)P_{n,l}(\theta) - P_{n,l}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l}(\theta)\\ \times [\overline{P}_{n,l}(\theta)(H_\alpha(\theta)-z)\overline{P}_{n,l}(\theta)]^{-1}\overline{P}_{n,l}(\theta)W^{(\alpha)}(\theta)P_{n,l}(\theta),\end{aligned} \tag{17}$$

which we define as an operator on $\mathrm{Ran}\,P_{n,l}(\theta)$. Note that we need the Feshbach operator for *each* fine structure component of the considered principal quantum number n, i.e. for all $1 \le l \le n$. Note moreover that we do not distinguish between the operators PAP and $PAP|_{\mathrm{Ran}\,P}$ when we write PAP, where A is a closed operator an P a projection with $\mathrm{Dom}\,A \subset \mathrm{Ran}\,P$. The meaning of this expression will be clear from the context.

We will show below that the Feshbach operator can be approximated in a certain sense by the operators

$$\begin{aligned}Z^{od}_{n,l,\pm}(\alpha) := \lim_{\epsilon\downarrow 0}\sum_{\mu=1,2}\int_{k\in\mathbb{R}^3}\mathrm{d}k\,P^{(\alpha)}_{\mathrm{el},n,l}w_{0,1}(k,\mu;0)\\ \times \underline{P}^{(\alpha)}_{\mathrm{el},n,l}\left[\underline{P}^{(\alpha)}_{\mathrm{el},n,l}H^{(\alpha)}_{\mathrm{el}} - E_{n,l}(\alpha) + |k| \pm \mathrm{i}\,\epsilon\right]^{-1}\underline{P}^{(\alpha)}_{\mathrm{el},n,l}w_{1,0}(k,\mu;0)P^{(\alpha)}_{\mathrm{el},n,l}\end{aligned} \tag{18}$$

and

$$Z^d_{n,l}(\alpha) := \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3}\frac{\mathrm{d}k}{|k|}P^{(\alpha)}_{\mathrm{el},n,l}w_{0,1}(k,\mu;0)P^{(\alpha)}_{\mathrm{el},n,l}w_{1,0}(k,\mu;0)P^{(\alpha)}_{\mathrm{el},n,l} \tag{19}$$

as well as

$$Z_{n,l,\pm}(\alpha) := Z^d_{n,l}(\alpha) + Z^{od}_{n,l,\pm}(\alpha), \tag{20}$$

defined as operators on $\mathrm{Ran}\,P^{(\alpha)}_{\mathrm{el},n,l}$. These operators are the relativistic analoga of [19, Equations (3) and (4)]. Note that $\mathcal{U}_{\mathrm{el}}(\theta)$ restricted to $\mathrm{Ran}\,P^{(\alpha)}_{\mathrm{el},n,l}$ is a similarity transformation ([25, Lemma 9]).

It is easy to see that the imaginary part of $Z_{n,l,\pm}(\alpha)$ is given by (cf. Equation (11) in Remark 1 in [19])

$$\begin{aligned}\operatorname{Im} Z_{n,l,\pm}(\alpha) = \mp\,\pi \sum_{\substack{n',l':\\ E_{n',l'}(\alpha)<E_{n,l}(\alpha)}} \sum_{\mu=1,2} \int_{|\omega|=1} d\omega\, (E_{n',l'}(\alpha)-E_{n,l}(\alpha))^2 \\ \times P^{(\alpha)}_{\mathrm{el},n,l} w_{0,1}((E_{n,l}(\alpha)-E_{n',l'}(\alpha))\omega,\mu;0) P^{(\alpha)}_{\mathrm{el},n',l'} \\ \times w_{1,0}((E_{n,l}(\alpha)-E_{n',l'}(\alpha))\omega,\mu;0) P^{(\alpha)}_{\mathrm{el},n,l}.\end{aligned}$$

It will turn out that the lifetime in lowest order in the fine structure constant α is given by the same expression as in the non-relativistic case (see Lemma 10). Therefore, we define (cf. [19, Equation (12)])

$$Z_{n,l,\mathrm{im}} = g^2 \frac{2}{3} \sum_{\substack{1\le n'<n\\ 1\le l\le n}} (-E_{n'}+E_n)^3 \times \kappa(|-E_{n'}+E_n|)^2 P^{(0)}_{\mathrm{el},n,l} x P^{(0)}_{\mathrm{el},n',l'} x P^{(0)}_{\mathrm{el},n,l} \tag{21}$$

and

$$Y_{n,l,\pm}(\alpha) := \mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{Z};0)^{-1} \operatorname{Re} Z_{n,l}(\alpha) \mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{Z};0) \mp \mathrm{i}\, Z_{n,l,\mathrm{im}} \tag{22}$$

as operators on $\operatorname{Ran} P^{(0)}_{\mathrm{el},n,l}$, where we defined $\operatorname{Ran} P^{(0)}_{\mathrm{el},n,l} := \mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{Z};0)^{-1} \times P^{(\alpha)}_{\mathrm{el},n,l} \mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{Z};0)$. $\mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{Z};0)$ is the unitary transformation which corresponds to taking the non-relativistic limit (see [25, Section 8]). We set

$$Z_{n,l}(\alpha) := Z_{n,l,-}(\alpha), \quad Y_{n,l}(\alpha) := Y_{n,l,-}(\alpha).$$

Note that contrary to [19] the coupling constant g is contained in the definition of the objects $Z_{n,l}(\alpha)$, $Y_{n,l}(\alpha)$ and so on. We see from Equation (21) that transitions between fine structure components of a principal quantum number do not play a role in lowest order in α.

Note that we remove the dependence on α only from the imaginary part, since a discussion of the real part, which yields the Lamb shift [28, 6], does not make sense without an UV renormalization. Moreover, the Lamb shift is not important for lifetime measurements using the so called "beam-foil"-method [10, 13, 7, 8].

We can now formulate our main result: Fix $n > 2$. Since $Z_{n,l,\mathrm{im}}$ is obtained from the corresponding matrix in the nonrelativistic case by restricting the corresponding quadratic form to $\operatorname{Ran} P^{(0)}_{\mathrm{el},n,l}$, we see immediately that in this case $Z_{n,l,\mathrm{im}}$ is strictly positive for all $1 \le l \le n$ (see [19, Appendix B.3]). Note that this is not the case for $n = 2$ due to the metastability of the 2s-sates of hydrogen. Indeed we will need in our proof the Feshbach operator and the matrices $Z_{n,l,\pm}(\alpha)$ and $Y_{n,l,\pm}(\alpha)$ for *all* fine structure components of the corresponding principal quantum number and not only for the fine structure component, whose lifetime we are interested in.

Theorem 1. *Let $n > 2$ and $\phi(\alpha)$ a normalized eigenvector of $H_{\mathrm{el}}^{(\alpha)}$ with eigenvalue $E_{n,l}(\alpha)$, $\psi(\alpha) := \phi(\alpha)\otimes\Omega$ and $\phi(0) := \mathcal{U}_{\mathrm{NR}}(\alpha^{-1}, \mathfrak{Z}; 0)^{-1}\phi(\alpha)$. Then there is a $C > 0$ such that for all $\alpha > 0$ small enough and all $s \geq 0$*

$$\langle\psi(\alpha), e^{-isH_\alpha}\psi(\alpha)\rangle = \langle\phi(0), e^{-is(E_{n,l}(\alpha)-Y_{n,l}(\alpha))}\phi(0)\rangle + b(g, s)$$

holds, where $|b(g, s)| \leq C\sqrt{\alpha}$.

We will prove Theorem 1 in Section 7.

Remark 1. *If we compare Definition* (22) *of $Y_{n,l}(\alpha)$ with [19, Formula (12)] we see that the lifetime of an excited state in the relativistic model is the same as in the Pauli-Fierz model. Thus relativistic effects play a minor role for electric dipole transitions. But there seems to be a small relativistic contribution for the decay of the metastable* $2s$*-state of hydrogen (see Breit and Teller [9]).*

3 Selfadjointness and Dilation Analyticity

Before we can turn to the operator H_α in the following sections we have to prove its selfadjointness and the holomorphicity properties of the operators $H_\alpha(\theta)$.

Theorem 2. *Let $0 < \alpha\mathfrak{Z} < \sqrt{3}/2$. Then the following holds: The operator*

$$H_\alpha : \mathcal{D} \subset (\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} L^2(\mathbb{R}^3; \mathbb{C}^4)) \otimes \mathrm{Dom}(H_{\mathrm{f}}) \to (\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} L^2(\mathbb{R}^3; \mathbb{C}^4)) \otimes \mathcal{F}$$

is on $\mathcal{D} := \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} H^1(\mathbb{R}^3; \mathbb{C}^4))\hat{\otimes}\,\mathrm{Dom}(H_{\mathrm{f}})$ essentially selfadjoint, where $\hat{\otimes}$ denotes the algebraic tensor product.

Proof. Because of [39, Theorem 4.4] the operator $H_{\mathrm{el}}^{(\alpha)} + \alpha^{-2}$ is selfadjoint and positive on the domain $\mathrm{Dom}(H_{\mathrm{el}}^{(\alpha)}) = \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} H^1(\mathbb{R}^3; \mathbb{C}^4)$. Since H_{f} is selfadjoint and positive on a suitable domain $\mathrm{Dom}(H_{\mathrm{f}})$, it follows from [36, Theorem VIII.33] that $H_{\alpha,0} + \alpha^{-2}$ is essentially selfadjoint and positive on the (algebraic) tensor product $\mathcal{D} = \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} H^1(\mathbb{R}^3; \mathbb{C}^4)\hat{\otimes}\,\mathrm{Dom}(H_{\mathrm{f}})$. We have for all $\psi \in \mathcal{D}$ and all $\epsilon > 0$ with a $C > 0$ (see for example [5, Proof of Lemma 1.1])

$$\begin{aligned}&\|W^{(\alpha)}\psi\| \leq C\sqrt{\alpha}\|(H_{\mathrm{f}} + 1)^{1/2}\psi\| \leq C\sqrt{\alpha}(\|\psi\| + \sqrt{\|\psi\|\|H_{\mathrm{f}}\psi\|})\\ &\leq C\sqrt{\alpha}[(1 + \frac{1}{2\epsilon})\|\psi\| + \frac{\epsilon}{2}\|H_{\mathrm{f}}\psi\|] \leq C\sqrt{\alpha}[(1 + \frac{1}{2\epsilon})\|\psi\| + \frac{\epsilon}{2}\|(H_{\alpha,0} + \alpha^{-2})\psi\|].\end{aligned}$$

Thus $W^{(\alpha)}$ is infinitesimally $(H_{\alpha,0}+\alpha^{-2})$-bounded, and in turn $H_\alpha+\alpha^{-2}$ (and thus also H_α) is essentially selfadjoint on $\mathrm{Dom}(H_{\alpha,0})$. □

We denote the operators defined in Theorem 2 again by H_α and $H_{\alpha,0}$ respectively.

We turn to the operators $H_\alpha(\theta)$ and $H_{\alpha,0}(\theta)$ on the domain $\mathrm{Dom}(H_\alpha(\theta)) = \mathrm{Dom}(H_{\alpha,0}(\theta)) = \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) H^1(\mathbb{R}^3;\mathbb{C}^4)\hat{\otimes}\,\mathrm{Dom}(H_\mathrm{f})$. In the following theorem we show that the families of operators

$$\begin{aligned} &U_\mathrm{DL}(\alpha^{-1},\mathfrak{Z};\theta)H_\alpha(\theta)U_\mathrm{DL}(\alpha^{-1},\mathfrak{Z};\theta)^{-1},\\ &\qquad\qquad U_\mathrm{DL}(\alpha^{-1},\mathfrak{Z};\theta)H_{\alpha,0}(\theta)U_\mathrm{DL}(\alpha^{-1},\mathfrak{Z};\theta)^{-1}, \end{aligned} \tag{23}$$

defined on the Hilbert space $\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}} L^2(\mathbb{R}^3;\mathbb{C}^4)\otimes\mathcal{F}$ with domain $U_\mathrm{DL}(\alpha^{-1},\mathfrak{Z};\theta) \times \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)H^1(\mathbb{R}^3;\mathbb{C}^4)\hat{\otimes}\,\mathrm{Dom}(H_\mathrm{f})$, are holomorphic families of type (B) on a suitable domain. Here $U_\mathrm{DL}(\alpha^{-1},\mathfrak{Z};\theta)$ is the transformation function between positive spectral projections of $D_{\alpha^{-1},\mathfrak{Z}}$ and $D_{\alpha^{-1},\mathfrak{Z}}(\theta)$ defined in [25, Theorem 6]. We will write $U_\mathrm{DL}(\alpha^{-1},\mathfrak{Z};\theta)$ for the operator $U_\mathrm{DL}(\alpha^{-1},\mathfrak{Z};\theta)\otimes\mathbf{1}_\mathrm{f}$.

Theorem 3. *Let $\theta\in S_{\pi/4}$, $2\alpha\mathfrak{Z}C(\mathrm{Im}\,\theta)<1$, $C_\mathrm{DL}|\theta|<q$ and $C_\mathrm{DLS}|\theta|<q$ for some $0<q<1$, where the constants C_DL and C_DLS are defined in [25, Section 6] and $C(\mathrm{Im}\,\theta)$ is defined in [25, Section 4]. Then there is a $\theta_0>0$ independent of $0<\alpha\le 1$ such that for all $|\theta|\le\theta_0$ the operators* (23) *define holomorphic families of operators $\tilde{H}_\alpha(\theta)$ bzw. $\tilde{H}_{\alpha,0}(\theta)$ of type (B) on a suitable domain $\mathrm{Dom}(\tilde{H}_\alpha(\theta)) = \mathrm{Dom}(\tilde{H}_{\alpha,0}(\theta))$. These operators are m-sectorial.*

Proof. The expression $q_{\alpha^{-1},0}(\psi) := \langle\psi,(D_{\alpha^{-1},\mathfrak{Z}}\otimes\mathbf{1}+\mathbf{1}\otimes H_\mathrm{f})\psi\rangle$ for $\psi\in\mathcal{D}$ is a positive closable quadratic form whose closure $\tilde{q}_{\alpha^{-1},0}$ defines a selfadjoint operator which coincides with the operator $H_{\alpha,0}$ defined in Theorem 2. We have $\mathrm{Dom}(\tilde{q}_{\alpha^{-1},0}) = \mathrm{Dom}((H_{\alpha,0}+\alpha^{-2})^{1/2})$. In particular, for $\psi\in\mathrm{Dom}(\tilde{q}_{\alpha^{-1},0})$ the estimate

$$\begin{aligned} \||D_{\alpha^{-1},\mathfrak{Z}}|^{1/2}\psi\| &= \|(\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}D_{\alpha^{-1},\mathfrak{Z}}\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}})^{1/2}\psi\|\\ &\le \|(\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}D_{\alpha^{-1},\mathfrak{Z}}\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}\otimes\mathbf{1}+\mathbf{1}\otimes H_\mathrm{f})^{1/2}\psi\|<\infty \end{aligned}$$

holds, and in the same way we see $\|(H_\mathrm{f}+1)^{1/2}\psi\|<\infty$.
Thus, we find for $\psi\in\mathrm{Dom}(\tilde{q}_{\alpha^{-1},0})$

$$\begin{aligned} &\langle\psi,U_\mathrm{DL}(\alpha^{-1};\theta)D_{\alpha^{-1},\mathfrak{Z}}(\theta)U_\mathrm{DL}(\alpha^{-1};\theta)^{-1}\psi\rangle\\ &=\langle|D_{\alpha^{-1},\mathfrak{Z}}|^{1/2}\psi,|D_{\alpha^{-1},\mathfrak{Z}}|^{-1/2}|D_{\alpha^{-1},0}|^{1/2}\\ &\quad\times|D_{\alpha^{-1},0}|^{-1/2}U_\mathrm{DL}(\alpha^{-1};\theta)D_{\alpha^{-1},\mathfrak{Z}}(\theta)U_\mathrm{DL}(\alpha^{-1};\theta)^{-1}|D_{\alpha^{-1},0}|^{-1/2}\\ &\quad\times|D_{\alpha^{-1},0}|^{1/2}|D_{\alpha^{-1},\mathfrak{Z}}|^{-1/2}|D_{\alpha^{-1},\mathfrak{Z}}|^{1/2}\psi\rangle. \end{aligned} \tag{24}$$

[25, Lemma 5 and Lemma 6] imply

$$|\langle\psi,U_\mathrm{DL}(\alpha^{-1};\theta)D_{\alpha^{-1},\mathfrak{Z}}(\theta)U_\mathrm{DL}(\alpha^{-1};\theta)^{-1}\psi\rangle-\langle\psi,D_{\alpha^{-1},\mathfrak{Z}}\psi\rangle|\le C|\theta|\langle\psi,D_{\alpha^{-1},\mathfrak{Z}}\psi\rangle$$

with some $C>0$ independent of α and θ. Moreover, $|e^{-\theta}\langle\psi,H_\mathrm{f}\psi\rangle-\langle\psi,H_\mathrm{f}\psi\rangle|\le B|\theta|\langle\psi,H_\mathrm{f}\psi\rangle$ with $B:=e^{\pi/4}$. Since $\|W^{(\alpha)}(\theta)(H_\mathrm{f}+1)^{-1/2}\|\le\sqrt{\alpha}C_1$ with some

$C_1 > 0$ independent of θ and α (see for example [5, Proof of Lemma 1.1]) we obtain for all $\epsilon > 0$

$$\begin{aligned}|\langle\psi, U_{\mathrm{DL}}(\alpha^{-1};\theta)W^{(\alpha)}(\theta)U_{\mathrm{DL}}(\alpha^{-1};\theta)^{-1}\psi\rangle| \\ \leq \sqrt{\alpha}C_1(1+C_{\mathrm{DL}}|\theta|)^2[(1/\epsilon^2+\epsilon^2)\|\psi\|^2+\epsilon^2\langle\psi, H_{\mathrm{f}}\psi\rangle]. \quad (25)\end{aligned}$$

It follows that the quadratic form $p_{\alpha^{-1};\theta}(\psi) := \langle\psi, (H_\alpha+\alpha^{-2})\psi\rangle$ for $\psi \in \mathrm{Dom}(\tilde{q}_{\alpha^{-1},0})$ is well defined for sufficiently small $|\theta|$. If we choose $|\theta|$ so small that $(C+B)|\theta| < 1$ holds, and then in (25) $\epsilon > 0$ small enough (depending on θ), we see that the quadratic form $p_{\alpha^{-1};\theta} - \tilde{q}_{\alpha,0}$ is relatively $\tilde{q}_{\alpha,0}$-bounded with form bound smaller than 1.
Because of [27, Theorem VI-1.33] the quadratic form $p_{\alpha^{-1};\theta}$ is closed with $\mathrm{Dom}(p_{\alpha^{-1};\theta}) = \mathrm{Dom}(\tilde{q}_{\alpha^{-1},0})$ and sectorial. Moreover,

$$\begin{aligned}&|D_{\alpha^{-1},0}|^{-1/2}U_{\mathrm{DL}}(\alpha^{-1};\theta)D_{\alpha^{-1},\mathfrak{Z}}(\theta)U_{\mathrm{DL}}(\alpha^{-1};\theta)^{-1}|D_{\alpha^{-1},0}|^{-1/2}\\ =&|D_{\alpha^{-1},0}|^{-1/2}U_{\mathrm{DL}}(\alpha^{-1};\theta)|D_{\alpha^{-1},0}|^{1/2}|D_{\alpha^{-1},0}|^{-1/2}D_{\alpha^{-1},\mathfrak{Z}}(\theta)|D_{\alpha^{-1},0}|^{-1/2}\\ &\times|D_{\alpha^{-1},0}|^{1/2}U_{\mathrm{DL}}(\alpha^{-1};\theta)^{-1}|D_{\alpha^{-1},0}|^{-1/2}.\end{aligned}$$

Using Equation (24) and [25, Theorem 6 c)] we see that the expression $\langle\psi, U_{\mathrm{DL}}(\alpha^{-1};\theta)D_{\alpha^{-1},\mathfrak{Z}}(\theta)U_{\mathrm{DL}}(\alpha^{-1};\theta)^{-1}\psi\rangle$ for all $\psi \in \mathrm{Dom}(p_{\alpha^{-1};\theta})$ is a holomorphic function of θ. It is easy to see that

$$(H_{\mathrm{f}}+1)^{-1/2}U_{\mathrm{DL}}(\alpha^{-1};\theta)W^{(\alpha)}(\theta)U_{\mathrm{DL}}(\alpha^{-1};\theta)^{-1}(H_{\mathrm{f}}+1)^{-1/2}$$

is bounded-holomorphic. Thus $\langle\psi, U_{\mathrm{DL}}(\alpha^{-1};\theta)W^{(\alpha)}(\theta)U_{\mathrm{DL}}(\alpha^{-1};\theta)^{-1}\psi\rangle$ is holomorphic function of θ. It follows that $p_{\alpha^{-1};\theta}(\psi)$ is a holomorphic function of θ for all $\psi \in \mathrm{Dom}(p_{\alpha^{-1};\theta}) = \mathrm{Dom}(\tilde{q}_{\alpha^{-1},0})$. The family of m-sectorial operators defined by these quadratic forms is a holomorphic family of type (B) (see [27, Chapter VII-4.2]). The proof for the operator without interaction works analogously. Since $\|W^{(\alpha)}(\theta)(H_{\mathrm{f}}+1)^{-1/2}\| \leq \sqrt{\alpha}C_1$ (see above), is infinitesimally operator bounded with respect to the free operator which implies the equality of the domains. □

Remark 2. *The above proof also shows that the operators* $U_{\mathrm{DL}}(\alpha^{-1};\theta)D_{\alpha^{-1},\mathfrak{Z}}(\theta)U_{\mathrm{DL}}(\alpha^{-1};\theta)^{-1}|_{\mathrm{Ran}\,\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}}$ *on the space* $\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}L^2(\mathbb{R}^3;\mathbb{C}^4)$ *are sectorial for sufficiently small* $|\theta|$. *In particular, the assumptions of the Ichinose Lemma (see [37, Corollary 2 on page 183] or [26]) are fulfilled so that*

$$\begin{aligned}\sigma(D_{\alpha^{-1},\mathfrak{Z}}(\theta)|_{\mathrm{Ran}\,\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)}\otimes\mathbf{1}_{\mathrm{f}}+e^{-\theta}\mathbf{1}_{\mathrm{el}}\otimes H_{\mathrm{f}}) =\\ =\sigma(D_{\alpha^{-1},\mathfrak{Z}}(\theta)|_{\mathrm{Ran}\,\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)})+e^{-\theta}\sigma(H_{\mathrm{f}})\end{aligned}$$

holds.

In the following, we will consider $U_{\mathrm{DL}}(\alpha^{-1},\mathfrak{Z};\theta)^{-1}\tilde{H}_\alpha(\theta)U_{\mathrm{DL}}(\alpha^{-1},\mathfrak{Z};\theta)$ and $U_{\mathrm{DL}}(\alpha^{-1},\mathfrak{Z};\theta)^{-1}\tilde{H}_{\alpha,0}(\theta)U_{\mathrm{DL}}(\alpha^{-1},\mathfrak{Z};\theta)$ on the respective domains $U_{\mathrm{DL}}(\alpha^{-1},\mathfrak{Z};\theta)^{-1}\,\mathrm{Dom}(\tilde{H}_\alpha(\theta))$ and $U_{\mathrm{DL}}(\alpha^{-1},\mathfrak{Z};\theta)^{-1}\,\mathrm{Dom}(\tilde{H}_{\alpha,0}(\theta))$. We will denote these operators by $H_\alpha(\theta)$ and $H_{\alpha,0}(\theta)$ again.

4 Technical Lemmata

In this section we will formulate and prove all technical statements which we will need to show the existence of the Feshbach operator and in order to approximate it by suitable operators.
Using the dilation analyticity we can restrict to $\theta = \mathrm{i}\,\vartheta$ with $0 < \vartheta < \theta_0$. We choose θ_0 so small that the statements of Theorem 3 as well as the statements in [25, Appendix A] hold. Moreover, we choose for this θ_0 a $\alpha_0 > 0$ so small that the statements about the "nonrelativistic limit" of the operator $D_{\alpha^{-1},\mathfrak{Z}}(\theta)$ (proven in [25, Section 8]) and inequality (26) hold. In particular, all projections occurring in the following are uniformly bounded in α and θ.
We put

$$\delta_{n,l,\pm}(\alpha) := \begin{cases} |E_{n,l}(\alpha) - E_{n,l\pm 1}(\alpha)|/2 & 1 < l < n \\ |E_{n,l}(\alpha) - E_{n,l+1}(\alpha)|/2 & l = 1 \\ |E_{n,l}(\alpha) - E_{n,l-1}(\alpha)|/2 & l = n \end{cases}$$

$$\delta_{n,l}(\alpha) := \min\{\delta_{n,l,+}(\alpha), \delta_{n,l,-}(\alpha)\}, \qquad \delta_{n,\pm} := |E_n - E_{n\pm 1}|/2,$$

$$\delta_n := \min\{\delta_{n,+}, \delta_{n,-}\}.$$

Note that $\delta_{n,l}(\alpha) = \delta_{n,l,\pm}(\alpha)$ holds for $l = 1$ or $l = n$. We will suppress the dependence of these quantities on α in certain places in order to simplify notation. It follows from the explicit form of the eigenvalues (see [29]) that for all $\alpha < \alpha_0$ with $\alpha_0 > 0$ small enough the inequality

$$c_1\alpha^2 \le \delta_{n,l,\pm}(\alpha) \le c_2\alpha^2 \tag{26}$$

holds with two constants $0 < c_1 < c_2$ independent of α and l.
We choose $\rho, \sigma > 0$ and define the sets (see Figure 1)

$$\mathcal{A}^{<}_{n,l}(\alpha,\sigma) := [E_{n,l}(\alpha) - \delta_{n,l,-}(\alpha), E_{n,l}(\alpha) + \delta_{n,l,+}(\alpha)] + \mathrm{i}\,[-\sigma,\infty), \quad 1 \le l \le n$$

and

$$\mathcal{A}_{n,l}(\alpha,\sigma) := \begin{cases} [E_{n,l}(\alpha) - \delta_{n,l,-}(\alpha), E_{n,l}(\alpha) + \delta_{n,l,+}(\alpha)] + \mathrm{i}\,[-\sigma,\infty) & 1 < l < n \\ [E_n - \delta_{n,-}, E_{n,l}(\alpha) + \delta_{n,l,+}(\alpha)] + \mathrm{i}\,[-\sigma,\infty) & l = 1 \\ [E_n - \delta_{n,l,-}(\alpha), E_n + \delta_{n,+}] + \mathrm{i}\,[-\sigma,\infty) & l = n \end{cases}.$$

Note that for $1 < l < n$ the identity $\mathcal{A}^{<}_{n,l}(\alpha,\sigma) = \mathcal{A}_{n,l}(\alpha,\sigma)$ holds. Moreover, following [5] we define $B_\theta(\rho) := \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)[H^{(\alpha)}_{\mathrm{el}}(\theta) - E_{n,l}(\alpha) + e^{-\theta}(H_{\mathrm{f}} + \rho)]\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)$ as an operator on the Hilbert space $\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)L^2(\mathbb{R}^3;\mathbb{C}^4) \otimes \mathcal{F}$ with domain $\mathrm{Dom}(B_\theta(\rho)) = U_{\mathrm{DL}}(\alpha^{-1},\mathfrak{Z};\theta)^{-1}\,\mathrm{Dom}(\tilde{H}_{\alpha,0}(\theta))$. The operator is a densely defined and closed operator (cf. Theorem 3 and the remarks following it). It follows that $B_\theta(\rho)^*$ is densely defined as well and we have $B_\theta(\rho)^{**} = B_\theta(\rho)$. Note that the adjoint is to be taken with respect to the scalar product on the Hilbert space $\Lambda^{(+)}_{\alpha^{-1},3}(\theta)L^2(\mathbb{R}^3;\mathbb{C}^4)\otimes\mathcal{F}$. In particular, $B_\theta(\rho)^*\neq B_{\bar\theta}(\rho)$. As in the Pauli-Fierz model $B_\theta(\rho)$ is only an auxiliary object, which saves some combinatorics. In principle, one could prove all statements without using $B_\theta(\rho)$. Note that all norms, scalar products and adjoints are to be understood in the sense of $\Lambda^{(+)}_{\alpha^{-1},3}(\theta)L^2(\mathbb{R}^3;\mathbb{C}^4)$ or $\Lambda^{(+)}_{\alpha^{-1},3}(\theta)L^2(\mathbb{R}^3;\mathbb{C}^4)\otimes\mathcal{F}$. We will choose ρ and σ later on as suitable functions of the coupling constant g. At the moment, we assume only that σ and ρ are nonnegative and bounded by some constant from above.

In the proofs in this and the following section, C denotes a generic, positive constant, which does not depend on α and z, but perhaps on ϑ.

In the following lemmas we will prove some estimates on the resolvents of the free operator $H_{\alpha,0}$ and of the electronic operator $H^{(\alpha)}_{\mathrm{el}}$. The lemmas generalize similar statements and their proofs [5]. Due to the fine structure splitting and the missing power of α some additional difficulties have to be addressed.

Lemma 1. *Let $0<\vartheta<\theta_0$. Then the following statements hold:*

a) *There is a $C>0$ such that for all $\alpha\le\alpha_0$, all $\sigma\le\frac{\delta_{n,l}(\alpha)\sin\vartheta}{2\cos\vartheta}$, all $R>0$ and all $z\in\mathcal{A}_{n,l}(\alpha,\sigma)$*

$$\Big\|\big[\overline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\mathbf{1}_{\mathrm{f}}(H^{(\alpha)}_{\mathrm{el}}(\theta)+e^{-\theta}(H_{\mathrm{f}}+R)-z)\overline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\mathbf{1}_{\mathrm{f}}\big]^{-1} \times\overline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\mathbf{1}_{\mathrm{f}}\Big\|\le\frac{C}{\delta_{n,l}(\alpha)\sin\vartheta}\qquad(27)$$

holds.

b) *There is a $C>0$ such that for all $\rho>0$, all $\sigma\le\frac{\rho\sin\vartheta}{2}$, all $R\ge\rho$ and all $z\in\mathcal{A}_{n,l}(\alpha,\sigma)$*

$$\Big\|\big[\overline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\mathbf{1}_{\mathrm{f}}(H^{(\alpha)}_{\mathrm{el}}(\theta)+e^{-\theta}(H_{\mathrm{f}}+R)-z)\overline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\mathbf{1}_{\mathrm{f}}\big]^{-1} \times\overline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\mathbf{1}_{\mathrm{f}}\Big\|\le\frac{C}{R\sin\vartheta}\qquad(28)$$

holds.

c) *There is a $C>0$ such that for all $\alpha\le\alpha_0$, all $\sigma\le\frac{\delta_n\sin\vartheta}{2\cos\vartheta}$, all $R>0$ and all $z\in\mathcal{A}_{n,l}(\alpha,\sigma)$*

$$\Big\|\big[\overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)\otimes\mathbf{1}_{\mathrm{f}}(H^{(\alpha)}_{\mathrm{el}}(\theta)+e^{-\theta}(H_{\mathrm{f}}+R)-z)\overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)\otimes\mathbf{1}_{\mathrm{f}}\big]^{-1} \times\overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)\otimes\mathbf{1}_{\mathrm{f}}\Big\|\le\frac{C}{\delta_n\sin\vartheta}.\qquad(29)$$

Proof.
a) We split the projection and the resolvent according to the formula $\overline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta) = \sum_{1\leq l'\leq n,\, l'\neq l} P^{(\alpha)}_{\mathrm{el},n,l'}(\theta)$ and use the representation (spectral theorem) in which H_{f} is the multiplication with the variable r. In order to simplify the notation we will suppress the dependence of the eigenvalues $E_{n,l'}(\alpha)$ on α. Note that for $E_{n,l'} < E_{n,l}$

$$|E_{n,l'} - z + e^{-\theta}(r+R)| \geq \mathrm{Im}\,(e^{\theta}(z - E_{n,l'})) \geq \\ \geq -(\cos\vartheta)\sigma + \sin\vartheta(\mathrm{Re}\, z - E_{n,l'}) \geq \frac{\sin\vartheta\delta_{n,l}(\alpha)}{2} \quad (30)$$

and for $E_{n,l'} > E_{n,l}$

$$|E_{n,l'} - z + e^{-\theta}(r+R)| \geq \mathrm{Re}\,(E_{n,l'} - z + e^{-\theta}(r+R)) \geq \delta_{n,l}(\alpha) \quad (31)$$

holds, which proves the claim together with [25, Corollary 5]. For $l = 1$ and $l = n$ the estimates (30) and (31) respectively are not needed. We used in the first estimate that $(\cos\vartheta)\sigma \leq \frac{\sin\vartheta\delta_{n,l}(\alpha)}{2}$.
b) We estimate $\mathrm{Im}\,(-E_{n,l'} + z - e^{-\theta}(r+R)) \geq -\sigma + \sin\vartheta(r+R) \geq \frac{\sin\vartheta R}{2}$, where we used $\sigma \leq \frac{\sin\vartheta R}{2}$.
c) We split the projection $\overline{P}^{(\alpha)}_{\mathrm{el},n} = \bar{P}_{\mathrm{disc}}(\alpha;\theta) + \sum_{\substack{1\leq n'\leq \tilde{n}\\ n'\neq n}} P^{(\alpha)}_{\mathrm{el},n'}(\theta)$ according to (15) and obtain analogously to the proof of a) the estimate $|\frac{1}{E_{n,l'}-z+e^{-\theta}(r+R)}| \leq \frac{C}{\delta_n \sin\vartheta}$ and with [25, Corollary 4]

$$\left\| \left[\bar{P}_{\mathrm{disc}}(\alpha;\theta)(H^{(\alpha)}_{\mathrm{el}}(\theta) + e^{-\theta}(H_{\mathrm{f}} + R) - z)\bar{P}_{\mathrm{disc}}(\alpha;\theta) \otimes \mathbf{1}_{\mathrm{f}}\right]^{-1} \bar{P}_{\mathrm{disc}}(\alpha;\theta) \right\| \\ \leq \sup_{r>0} \frac{C}{-\eta - (\mathrm{Re}\, z - (r+R))} \leq \frac{C}{\delta_n}.$$

□

LEMMA 2. *Let $0 < \vartheta < \theta_0$. Then the following statements hold:*

a) There is a $C > 0$ such that for all $\alpha \leq \alpha_0$, all $R > 0$, all $\sigma \leq \min\{\frac{\delta_{n,l}(\alpha)\sin\vartheta}{2\cos\vartheta}, \frac{\delta_n \sin\vartheta}{2\cos\vartheta}, 1/2\rho\sin\vartheta\}$ and all $z \in \mathcal{A}_{n,l}(\alpha,\sigma)$

$$\left\| \left[\overline{P}_{n,l}(\theta;R)(H^{(\alpha)}_{\mathrm{el}}(\theta) + e^{-\theta}(H_{\mathrm{f}} + R) - z)\overline{P}_{n,l}(\theta;R)\right]^{-1} \overline{P}_{n,l}(\theta;R) \right\| \leq \\ \leq \frac{C}{\min\{\delta_n, \delta_{n,l}(\alpha), \rho\}\sin\vartheta} \quad (32)$$

b) There is a $C > 0$ such that for all $\alpha \leq \alpha_0$, all $\sigma \leq \min\{\frac{\delta_{n,l}(\alpha)\sin\vartheta}{2\cos\vartheta}$,

$\frac{\delta_n \sin\vartheta}{2\cos\vartheta}, 1/2\rho\sin\vartheta\}$ *and all* $z \in \mathcal{A}_{n,l}(\alpha,\sigma)$

$$\left\| \left[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\right]^{-1}\overline{P}_{n,l}(\theta)B_\theta(\rho)\right\| \leq$$
$$\leq \frac{C}{\sin\vartheta}\left(1+\frac{\rho}{\min\{\delta_{n,l}(\alpha),\delta_n\}}\right) \quad (33)$$

holds.

Proof.
a) We split the projection

$$\overline{P}_{n,l}(\theta;R) = \overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)\otimes \mathbf{1}_{\mathrm{f}} + \overline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes \mathbf{1}_{\mathrm{f}} + P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\chi_{H_{\mathrm{f}}+R\geq\rho}.$$

For $r+R\geq\rho$ we estimate as follows: $\mathrm{Im}\,(-E_{n,l}+z-e^{-\theta}(r+R)) \geq -\sigma + \sin\vartheta(r+R) \geq \frac{\sin\vartheta\rho}{2}$. We used here $\sigma \leq 1/2\rho\sin\vartheta$ and $r+R\geq\rho$. This shows the claim together with (27) and (29) in Lemma 1.

b) As before, we split $\overline{P}_{n,l}(\theta) = \overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)\otimes\mathbf{1}_{\mathrm{f}} + \overline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\mathbf{1}_{\mathrm{f}} + P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\chi_{H_{\mathrm{f}}\geq\rho}$. We start with

$$\left\|\left[P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\chi_{H_{\mathrm{f}}\geq\rho}(H_{\alpha,0}(\theta)-z)\right]^{-1}P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\chi_{H_{\mathrm{f}}\geq\rho}B_\theta(\rho)\right\|$$
$$= \sup_{r\geq\rho}\left|\frac{e^{-\theta}(r+\rho)}{E_{n,l}-z+e^{-\theta}r}\right|\|P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\| \leq \frac{4}{\sin\vartheta}\|P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\|,$$

where we used the inequality

$$\mathrm{Im}\,(-E_{n,l}+z-e^{-\theta}r) \geq -\sigma+\sin\vartheta r \geq \frac{\sin\vartheta r}{2}, \quad (34)$$

which follows from $\sigma \leq \frac{\sin\vartheta\rho}{2}$ and $\rho\leq r$.
Using Equations (30) and (31) from the proof of Lemma 1 as well as Equation (34) we obtain with some $C>0$ (independent of α)

$$\left|\frac{E_{n,l'}-E_{n,l}+e^{-\theta}(r+\rho)}{E_{n,l'}-z+e^{-\theta}r}\right| \leq \left|\frac{E_{n,l'}-E_{n,l}+e^{-\theta}\rho}{E_{n,l'}-z+e^{-\theta}r}\right| + \left|\frac{e^{-\theta}r}{E_{n,l'}-z+e^{-\theta}r}\right|$$
$$\leq C\frac{2(\alpha^2+\rho)}{\sin\vartheta\delta_{n,l}(\alpha)} + \begin{cases}\frac{2\rho}{\sin\vartheta\delta_{n,l}(\alpha)}, & r\leq\rho\\ \frac{2r}{\sin\vartheta r}, & r>\rho\end{cases}$$
$$\leq C\frac{4}{\sin\vartheta}(1+\frac{\rho}{\delta_{n,l}(\alpha)}).$$

Analogously, we obtain for $n'\neq n$ the estimate $\left|\frac{E_{n',l'}-E_{n,l}+e^{-\theta}(r+\rho)}{E_{n',l'}-z+e^{-\theta}r}\right| \leq$

$C\frac{4}{\sin\vartheta}(1+\frac{\rho}{\delta_n})$. Eventually we find

$$\begin{aligned}
&\left\|\left[\bar{P}_{\mathrm{disc}}(\alpha;\theta)(H_{\alpha,0}(\theta)-z)\right]^{-1}\bar{P}_{\mathrm{disc}}(\alpha;\theta)B_\theta(\rho)\right\| \\
\le&\|\bar{P}_{\mathrm{disc}}(\alpha;\theta)\| + \sup_{r\ge 0}\left\|\frac{z-E_{n,l}-e^{-\theta}\rho}{\bar{P}_{\mathrm{disc}}(\alpha;\theta)(H_{\mathrm{el}}^{(\alpha)}(\theta)-z+e^{-\theta}(r+\rho_0))}\bar{P}_{\mathrm{disc}}(\alpha;\theta)\right\| \\
\le&\|\bar{P}_{\mathrm{disc}}(\alpha;\theta)\| + \sup_{r\ge 0}\frac{\max\{\delta_{n,-},\delta_{n,+}\}+\rho}{\mathrm{Re}\,(-\eta-z)+\cos\vartheta r} \le \frac{C}{\delta_n},
\end{aligned}$$

using [25, Corollary 4]. □

Part b) of the above lemma and the following lemmas are preparations for the proof of relative bounds on the interaction.

Corollary 1. *Let $0<\vartheta<\theta_0$. Then there is a $C>0$ such that for all $\alpha\le\alpha_0$, all $\sigma\le\min\{\frac{\delta_{n,l}(\alpha)\sin\vartheta}{2\cos\vartheta},\frac{\delta_n\sin\vartheta}{2\cos\vartheta},1/2\rho\sin\vartheta\}$ and all $z\in\mathcal{A}_{n,l}(\alpha,\sigma)$*

$$\||B_\theta(\rho)|^{1/2}[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)]^{-1}\overline{P}_{n,l}(\theta)|B_\theta(\rho)^*|^{1/2}\| \le \frac{C}{\sin\vartheta}$$

holds.

Proof. We find

$$\begin{aligned}
&\left\||B_\theta(\rho)|\left[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\right]^{-1}\overline{P}_{n,l}(\theta)\right\| \\
&=\left\|\left[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\right]^{-1}\overline{P}_{n,l}(\theta)|B_\theta(\rho)^*|\right\|.
\end{aligned}$$

The claim follows by complex interpolation and using Lemma 2 b). □

Lemma 3. *Let $0<\vartheta<\theta_0$. Then there is a $C>0$ such that for all $0<\alpha<\alpha_0$ and all $\rho>0$ the following statements hold:*

a)

$$\left\|\overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)B_\theta(\rho)^{-1}\right\| \le \frac{C}{\sin\vartheta} \tag{35}$$

b)

$$\left\|B_\theta(\rho)^{-1}\right\| \le \frac{C}{\sin\vartheta}\left(1+\frac{1}{\rho}\right) \tag{36}$$

c)

$$\left\|H_{\mathrm{f}}B_\theta(\rho)^{-1}\right\| \le \frac{C}{\sin\vartheta} \tag{37}$$

Proof.
a) We estimate using [25, Corollary 4]

$$\begin{aligned}&\|\bar{P}_{\mathrm{disc}}(\alpha;\theta)B_\theta(\rho)^{-1}\|\\ &\leq \sup_{r\geq 0}\frac{C}{-\eta - E_{n,l}+\cos\vartheta(r+\rho)}\leq\frac{C}{\delta_n}\end{aligned}$$

and note that analogously to the Formulas (30) and (31) we find for $n'<n$

$$\begin{aligned}|E_{n',l'}-E_{n,l}+e^{-\theta}(r+\rho)| &\geq \operatorname{Im}(e^{\theta}(z-E_{n,l}))\\ &\geq -(\cos\vartheta)\sigma+\sin\vartheta(E_{n,l}-E_{n',l'})\geq \sin\vartheta\delta_n \qquad (38)\end{aligned}$$

and for $n'>n$

$$|E_{n',l'}-E_{n,l}+e^{-\theta}(r+\rho)| \geq \operatorname{Re}(E_{n',l'}-E_{n,l}+e^{-\theta}(r+\rho))\geq\delta_n, \qquad (39)$$

which proves the claim.
b) In view of part a) it suffices to show the estimate on $\operatorname{Ran} P^{(\alpha)}_{\mathrm{el},n}(\theta)$. We find for all $1\leq n'\leq\tilde{n}$ and all $1\leq l\leq n'$, in particular for $n'=n$,

$$|E_{n',l'}(\alpha)-E_{n,l}(\alpha)+e^{-\theta}(r+\rho)|\geq\sin\vartheta(r+\rho)\geq\rho\sin\vartheta, \qquad (40)$$

which proves the claim.
c) Using Formula (40) we obtain for all $1\leq n'\leq\tilde{n}$ and all $1\leq l\leq n'$

$$\frac{r}{|E_{n',l'}(\alpha)-E_{n,l}(\alpha)+e^{-\theta}(r+\rho)|}\leq\frac{r}{\sin\vartheta(r+\rho)}\leq\frac{1}{\sin\vartheta},$$

which prove the claim on $\operatorname{Ran} P_{\mathrm{disc}}(\alpha;\theta)$. Using [25, Corollary 4] we find on $\operatorname{Ran}\bar{P}_{\mathrm{disc}}(\alpha;\theta)$

$$\left\|H_{\mathrm{f}}B_\theta(\rho)^{-1}\bar{P}_{\mathrm{disc}}(\alpha;\theta)\right\|\leq C\sup_{r\geq0}\frac{r}{\eta-E_{n,l}+\cos\vartheta(r+\rho)}\leq C\frac{1}{\cos\vartheta}.$$

Note that $|\sin\vartheta|<\cos\vartheta$ for $|\vartheta|<\pi/4$. ☐

Corollary 2. *Let $0<\vartheta<\theta_0$. Then there is a $C>0$ such that for all $0<\alpha<\alpha_0$ and all $\rho>0$ the following estimates hold:*

a)

$$\|\overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)|B_\theta(\rho)|^{-1/2}\|\leq\frac{C}{\sqrt{\sin\vartheta}},\quad \||B_\theta(\rho)^*|^{-1/2}\overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)\|\leq\frac{C}{\sqrt{\sin\vartheta}}$$

b)

$$\||B_\theta(\rho)|^{-1/2}\|\leq\frac{C}{\sqrt{\sin\vartheta}}\left(1+\frac{1}{\sqrt{\rho}}\right),\quad \||B_\theta(\rho)^*|^{-1/2}\|\leq\frac{C}{\sqrt{\sin\vartheta}}\left(1+\frac{1}{\sqrt{\rho}}\right)$$

c)

$$\|H_{\mathrm{f}}^{1/2}|B_\theta(\rho)|^{-1/2}\| \le \frac{C}{\sqrt{\sin\vartheta}}, \qquad \|H_{\mathrm{f}}^{1/2}|B_\theta(\rho)^*|^{-1/2}\| \le \frac{C}{\sqrt{\sin\vartheta}}$$

Proof.
a) We find $\|\overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)|B_\theta(\rho)|^{-1/2}\|^2 \le \|\overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)\|\|\overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)B_\theta(\rho)^{-1}\|$ as well as $\||B_\theta(\rho)^*|^{-1/2}\overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)\|^2 \le \|\overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)^*\|\|B_\theta(\rho)^{-1}\overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)\|$. The claim follows now from Lemma 3.
b) This follows immediately from the spectral theorem for self-adjoint operators.
c) From Formula (37) in Lemma 3 we obtain for all $\psi \in \mathrm{Dom}(B_\theta(\rho))$ the estimate $\|H_{\mathrm{f}}\psi\| \le \frac{C}{\min\{\sin\vartheta,\cos\vartheta\}}\|B_\theta(\rho)\psi\|$. Taking the square root of this operator inequality, the claim follows. The second inequality follows analogously using the identity $\|H_{\mathrm{f}}B_\theta(\rho)^{-1}\| = \|[B_\theta(\rho)^*]^{-1}H_{\mathrm{f}}\| = \|H_{\mathrm{f}}[B_\theta(\rho)^*]^{-1}\|$. □

In the last two lemmas in this section, we prove relative bounds on the interaction. In comparison to the non-relativistic case, we have the additional difficulty that the factor in front of the interaction is $\sqrt{\alpha}$ only. To circumvent this problem, we use the statements about the non-relativistic limit shown in [25].

Lemma 4. *Let $0 < \vartheta < \theta_0$. Then there is a $C > 0$ such that for all $0 < \alpha < \alpha_0$ and all $\rho > 0$ the estimate*

$$\left\||B_\theta(\rho)^*|^{-1/2}W^{(\alpha)}(\theta)|B_\theta(\rho)|^{-1/2}\right\| \le \frac{C}{\sin\vartheta}\sqrt{\alpha}\left[1+\alpha\left(1+\frac{1}{\rho^{1/2}}\right)\right]$$

holds.

Proof. We split the projection according to $\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta) = P_1(\theta)+P_2(\theta)$, where $P_1(\theta) = P^{(\alpha)}_{\mathrm{el},n}(\theta)\otimes\mathbf{1}_{\mathrm{f}}$, $P_2(\theta) = \overline{P}^{(\alpha)}_{\mathrm{el},n}(\theta)\otimes\mathbf{1}_{\mathrm{f}}$. Since the estimate with $A^{(\theta)}_\kappa(\alpha x)_+$ works analogously, we consider $A^{(\theta)}_\kappa(\alpha x)_-$ only. We find for $\psi,\psi' \in \Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)L^2(\mathbb{R}^3;\mathbb{C}^4)\otimes\mathcal{F}$ and $i,j\in\{1,2\}$

$$\begin{aligned}&\left|\left\langle\psi', |B_\theta(\rho)^*|^{-1/2}P_i(\theta)\boldsymbol{\alpha}\cdot A^{(\theta)}_\kappa(\alpha x)_- P_j(\theta)|B_\theta(\rho)|^{-1/2}\psi\right\rangle\right| \\ &\qquad \le \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3}\frac{\mathrm{d}k\,|\kappa(e^{-\theta}|k|)|}{\sqrt{4\pi^2|k|}}\left\|P_i(\theta)^*|B_\theta(\rho)^*|^{-1/2}\psi'\right\| \\ &\qquad\quad \times\left\|P_i(\theta)\boldsymbol{\alpha}\cdot\varepsilon_\mu(k)e^{\mathrm{i}\,\alpha k\cdot x}P_j(\theta)\right\|\left\|a_\mu(k)P_j(\theta)|B_\theta(\rho)|^{-1/2}\psi\right\|. \qquad (41)\end{aligned}$$

We have to make a case distinction:

Case 1: $i = j = 2$. Using Corollary 2 a) we find $\| |B_\theta(\rho)^*|^{-1/2} P_i(\theta)\| \leq C$. Moreover $\|P_i(\theta)\boldsymbol{\alpha} \cdot \varepsilon_\mu(k) e^{\mathrm{i}\,\alpha k \cdot x} P_j(\theta)\| \leq C$. The r.h.s. of Formula (41) can be estimated by

$$C\|\psi'\|\|\psi\|\|H_{\mathrm{f}}^{1/2}|B_\theta(\rho)|^{-1/2}\| \leq \frac{C}{\sqrt{\sin\vartheta}}\|\psi'\|\|\psi\| \tag{42}$$

with a generic $C > 0$, where we used Corollary 2 c) in the last step.
Case 2: All other combinations of i and j. ¿From [25, Lemma 10 or Theorem 11] it follows that $\|P_i(\theta)\boldsymbol{\alpha} \cdot \varepsilon_\mu(k) e^{\mathrm{i}\,\alpha k \cdot x} P_j(\theta)\| \leq C\alpha(1 + \alpha|k|)$, and from Corollary 2 a) and b) that $\| |B_\theta(\rho)^*|^{-1/2} P_i(\theta)\| \leq \frac{C}{\sqrt{\sin\vartheta}}(1 + \frac{1}{\rho^{1/2}})$. The r.h.s. of Formula (41) can be estimated by

$$\alpha\frac{C}{\sqrt{\sin\vartheta}}\left(1 + \frac{1}{\rho^{1/2}}\right)\|\psi'\|\sqrt{\sum_{\mu=1,2}\int_{k\in\mathbb{R}^3}\frac{\mathrm{d}k\,|\kappa(e^{-\theta}|k|)|^2(1+\alpha|k|)^2}{|k|^2}} \tag{43}$$

$$\times\sqrt{\sum_{\mu=1,2}\int_{k\in\mathbb{R}^3}\mathrm{d}k\,|k|\|a_\mu(k)P_j(\theta)|B_\theta(\rho)|^{-1/2}\psi\|^2} \leq \alpha\frac{C}{\sin\vartheta}\left(1+\frac{1}{\rho^{1/2}}\right)\|\psi'\|\|\psi\|$$

in this case with a generic $C > 0$. □

LEMMA 5. *Let $0 < \vartheta < \theta_0$. Then there is a $C > 0$ such that for all $0 < \alpha < \alpha_0$ and all $\rho > 0$ the following estimates hold:*

a)

$$\left\| |B_\theta(\rho)^*|^{-1/2} W_{1,0}^{(\alpha)}(\theta) P_{n,l}(\theta)\right\| \leq \frac{C}{\sqrt{\sin\vartheta}} g \tag{44}$$
$$\left\| P_{n,l}(\theta) W_{0,1}^{(\alpha)}(\theta) |B_\theta(\rho)|^{-1/2}\right\| \leq \frac{C}{\sqrt{\sin\vartheta}} g$$

b)

$$\left\| |B_\theta(\rho)^*|^{-1/2} W_{0,1}^{(\alpha)}(\theta) P_{n,l}(\theta)\right\| \leq \frac{C}{\sqrt{\sin\vartheta}} g\rho^{1/2} \tag{45}$$
$$\left\| P_{n,l}(\theta) W_{1,0}^{(\alpha)}(\theta) |B_\theta(\rho)|^{-1/2}\right\| \leq \frac{C}{\sqrt{\sin\vartheta}} g\rho^{1/2}$$

c)

$$\left\| W_{0,1}^{(\alpha)}(\theta) P_{n,l}(\theta)\right\| \leq Cg\rho, \qquad \left\| P_{n,l}(\theta) W_{1,0}^{(\alpha)}(\theta)\right\| \leq Cg\rho \tag{46}$$

Proof. We begin with

$$\left\| |B_\theta(\rho)^*|^{-1/2} W_{0,1}^{(\alpha)}(\theta) P_{n,l}(\theta)\right\|$$
$$\leq \sqrt{\alpha}\left\| |B_\theta(\rho)^*|^{-1/2}\right\| \left\| \Lambda_{\alpha^{-1},\mathfrak{Z}}^{(+)}(\theta)\boldsymbol{\alpha} \cdot A_\kappa^{(\theta)}(\alpha x)_- P_{n,l}(\theta)\right\|$$

and find with [25, Theorem 11] similarly as in [4, Lemma IV.9.]

$$\begin{aligned}
&\left|\left\langle \psi', \Lambda^{(+)}_{\alpha^{-1},3}(\theta)\boldsymbol{\alpha}\cdot A^{(\theta)}_{\kappa}(\alpha x)_{-}P_{n,l}(\theta)\psi\right\rangle\right| \\
&\le \sum_{\mu=1}^{2}\int_{k\in\mathbb{R}^3}\frac{\mathrm{d}k\,|\kappa(e^{-\theta}|k|)|}{\sqrt{4\pi^2|k|}}\left|\left\langle \psi', \Lambda^{(+)}_{\alpha^{-1},3}(\theta)\boldsymbol{\alpha}\cdot\varepsilon_\mu(k)e^{\mathrm{i}\,\alpha k\cdot x}P^{(\alpha)}_{\mathrm{el},n,l}a_\mu(k)\chi_{H_{\mathrm{f}}\le\rho}\psi\right\rangle\right| \\
&\le C\alpha\sqrt{\sum_{\mu=1}^{2}\int_{|k|\le\rho}\mathrm{d}k\,\frac{|\kappa(e^{-\theta}|k|)|^2(1+\alpha|k|)^2}{|k|^2}}\sqrt{\sum_{\mu=1,2}\int_{k\in\mathbb{R}^3}|k|\|a_\mu(k)\chi_{H_{\mathrm{f}}\le\rho}\psi\|^2} \\
&\qquad\qquad\le \alpha\rho\|\psi\|\|\psi'\|. \qquad (47)
\end{aligned}$$

For $\|P_{n,l}(\theta)W^{(\alpha)}_{1,0}(\theta)|B_\theta(\rho)|^{-1/2}\|$ one shows a similar estimate such that the claim in b) follows from Corollary 2. Formula (47) and an analogous calculation for $W^{(\alpha)}_{1,0}(\theta)$ prove the claim in c).
To show a) we estimate similarly as in Formula (47)

$$\begin{aligned}
&|\langle\psi', P_{n,l}(\theta)W^{(\alpha)}_{0,1}(\theta)|B_\theta(\rho)|^{-1/2}\psi\rangle \\
&\qquad\le C\sqrt{\alpha}\alpha\|\psi'\|\|H_{\mathrm{f}}^{1/2}|B_\theta(\rho)|^{-1/2}\|\|\psi\| \le \frac{C}{\sqrt{\sin\theta}}g\|\psi'\|\|\psi\|.
\end{aligned}$$

The estimate on $\||B_\theta(\rho)^*|^{-1/2}W^{(\alpha)}_{1,0}(\theta)P_{n,l}(\theta)\|$ follows analogously. $\square$

5 Existence and approximation of the Feshbach Operator

We set now $\rho_0 = g^{4/3} = \alpha^2$ and $\sigma_0 = g^{5/3} = \alpha^{5/2}$ and use the estimates from Section 4 for $\rho = \rho_0$ and $\sigma = \sigma_0$.
We apply the strategy from [5], but have to overcome additional difficulties. First, we generalize [5, Lemma 3.14] to the relativistic case and show the existence of the inverse $[\overline{P}_{n,l}(\theta)(H_\alpha(\theta)-z)\overline{P}_{n,l}(\theta)]^{-1}$.

LEMMA 6. *Let $0<\vartheta<\theta_0$. Then there is a $C>0$ such that for all sufficiently small $\alpha>0$ the following holds: The operator $\overline{P}_{n,l}(\theta)(H_\alpha(\theta)-z)\overline{P}_{n,l}(\theta)$ is for all $z\in\mathcal{A}_{n,l}(\alpha,\sigma_0)$ invertible on $\operatorname{Ran}\overline{P}_{n,l}(\theta)$, and we have*

$$\|[\overline{P}_{n,l}(\theta)(H_\alpha(\theta)-z)\overline{P}_{n,l}(\theta)]^{-1}\overline{P}_{n,l}(\theta)\| \le \frac{C}{\sin^2\vartheta\rho_0}.$$

Proof. The claim follows from the series expansion

$$\begin{aligned}
&\left\|\left[\overline{P}_{n,l}(\theta)(H_\alpha(\theta)-z)\overline{P}_{n,l}(\theta)\right]^{-1}\overline{P}_{n,l}(\theta)\right\| \\
&=\Big\|\sum_{n=0}^{\infty}\left[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\right]^{-1}\overline{P}_{n,l}(\theta) \\
&\quad\times\left[-W^{(\alpha)}(\theta)\left[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\right]^{-1}\overline{P}_{n,l}(\theta)\right]^n\Big\|
\end{aligned}$$

$$
\begin{aligned}
=& \Big\| \sum_{n=0}^{\infty} |B_\theta(\rho_0)|^{-1/2} \\
&\times |B_\theta(\rho_0)|^{+1/2} \big[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\big]^{-1}\overline{P}_{n,l}(\theta)|B_\theta(\rho_0)^*|^{1/2} \\
&\times \Big[-|B_\theta(\rho_0)^*|^{-1/2} W^{(\alpha)}(\theta)|B_\theta(\rho_0)|^{-1/2} \\
&\times |B_\theta(\rho_0)|^{+1/2} \big[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\big]^{-1}\overline{P}_{n,l}(\theta)|B_\theta(\rho_0)^*|^{1/2}\Big]^n \\
&\times |B_\theta(\rho_0)^*|^{-1/2}\Big\| \\
\le& \Big\||B_\theta(\rho_0)|^{-1/2}\Big\| \Big\||B_\theta(\rho_0)^*|^{-1/2}\Big\| \\
&\times \Big\||B_\theta(\rho_0)|^{+1/2} \big[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\big]^{-1}\overline{P}_{n,l}(\theta)|B_\theta(\rho_0)^*|^{1/2}\Big\| \\
&\times \sum_{n=0}^{\infty} \Big[\Big\||B_\theta(\rho_0)^*|^{-1/2} W^{(\alpha)}(\theta)|B_\theta(\rho_0)|^{-1/2}\Big\| \\
&\times \Big\||B_\theta(\rho_0)|^{+1/2} \big[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\big]^{-1}\overline{P}_{n,l}(\theta)|B_\theta(\rho_0)^*|^{1/2}\Big\| \Big]^n \\
\le& \frac{C}{\sin^2\vartheta\sqrt{\rho_0}\sqrt{\rho_0}} \sum_{n=0}^{\infty} \Big[\frac{C}{\sin^2\vartheta}\sqrt{\alpha}\big(1+\alpha\big(1+\frac{1}{\sqrt{\rho_0}}\big)\big)\Big]^n \\
\le& \frac{C}{\sin^2\vartheta\rho_0} \sum_{n=0}^{\infty} \left(\frac{C}{\sin^2\vartheta}\sqrt{\alpha}\right)^n
\end{aligned}
$$

with a generic $C > 0$ independent of z and α. We used Corollary 2 b), Corollary 1 and Lemma 4. □

We turn now to the existence of the Feshbach operator and generalize [5, Lemma 3.15].

Lemma 7. *Let $0 < \vartheta < \theta_0$ small enough. Then there is a $C > 0$ such that for all sufficiently small $\alpha > 0$ and all $z \in \mathcal{A}_{n,l}(\alpha, \sigma_0)$ the following estimates hold:*

a)

$$
\Big\|P_{n,l}(\theta)W^{(\alpha)}(\theta)[\overline{P}_{n,l}(\theta)(H_\alpha(\theta)-z)\overline{P}_{n,l}(\theta)]^{-1}\overline{P}_{n,l}(\theta)\Big\| \le g\frac{C}{\sin^2\vartheta\sqrt{\rho_0}}. \tag{48}
$$

$$
\Big\|[\overline{P}_{n,l}(\theta)(H_\alpha(\theta)-z)\overline{P}_{n,l}(\theta)]^{-1}\overline{P}_{n,l}(\theta)W^{(\alpha)}(\theta)P_{n,l}(\theta)\Big\| \le g\frac{C}{\sin^2\vartheta\sqrt{\rho_0}}. \tag{49}
$$

b) For all $1 \le l, l', l'' \le n$ we have

$$
\begin{aligned}
\Big\|P_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l}(\theta)[\overline{P}_{n,l}(\theta)(H_\alpha(\theta)-z)\overline{P}_{n,l}(\theta)]^{-1}& \\
\times \overline{P}_{n,l}(\theta)W^{(\alpha)}(\theta)P_{n,l''}(\theta)\Big\| \le \frac{C}{(\sin\vartheta)^2}g^2&
\end{aligned} \tag{50}
$$

c) *The Feshbach operator, defined in equation* (17)*, exists for all* $z \in \mathcal{A}_{n,l}(\alpha,\sigma_0)$ *and fulfills the equation*

$$\begin{aligned}
&(H_\alpha(\theta)-z)^{-1} = \\
&= \Big[P_{n,l}(\theta) - \overline{P}_{n,l}(\theta)\big(\overline{P}_{n,l}(\theta)H_\alpha(\theta)\overline{P}_{n,l}(\theta)-z\big)^{-1}\overline{P}_{n,l}(\theta)W^{(\alpha)}(\theta)P_{n,l}(\theta)\Big] \\
&\qquad \times \big[\mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta)-z)\big]^{-1} \\
&\times \Big[P_{n,l}(\theta) - P_{n,l}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l}(\theta)\big(\overline{P}_{n,l}(\theta)H_\alpha(\theta)\overline{P}_{n,l}(\theta)-z\big)^{-1}\overline{P}_{n,l}(\theta)\Big] \\
&\qquad + \overline{P}_{n,l}(\theta)\big(\overline{P}_{n,l}(\theta)H_\alpha(\theta)\overline{P}_{n,l}(\theta)-z\big)^{-1}\overline{P}_{n,l}(\theta),
\end{aligned} \tag{51}$$

where the l.h.s. exists if and only if the r.h.s. exists.

Proof.
a) We obtain as in the proof of Lemma 6

$$\begin{aligned}
&\Big\|P_{n,l}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l}(\theta)\big[\overline{P}_{n,l}(\theta)(H_\alpha(\theta)-z)\overline{P}_{n,l}(\theta)\big]^{-1}\overline{P}_{n,l}(\theta)\Big\| \\
\leq&\Big\|P_{n,l}(\theta)W^{(\alpha)}(\theta)|B_\theta(\rho)|^{-1/2}\Big\| \\
&\times \Big\||B_\theta(\rho)|^{+1/2}\big[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\big]^{-1}\overline{P}_{n,l}(\theta)|B_\theta(\rho)^*|^{1/2}\Big\| \\
&\times \sum_{n=0}^{\infty}\Big[\||B_\theta(\rho)^*|^{-1/2}W^{(\alpha)}(\theta)|B_\theta(\rho)|^{-1/2}\| \\
&\times \||B_\theta(\rho)|^{+1/2}\big[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\big]^{-1}\overline{P}_{n,l}(\theta)|B_\theta(\rho)^*|^{1/2}\|\Big]^n\| \\
&\times \||B_\theta(\rho)^*|^{-1/2}\| \\
\leq& g\frac{C}{\sin^2\vartheta\sqrt{\rho_0}}\sum_{n=0}^{\infty}\left(\frac{C}{\sin^2\vartheta}\sqrt{\alpha}\right)^n,
\end{aligned}$$

where we used additionally Lemma 5 a) and b). The other estimate follows analogously.
b) Follows similarly as in a).
c) This follows from Lemma 6 and Part a) of [4, Theorem IV.1]. □

Having shown the existence of the Feshbach operator, we can turn now to its approximation by suitable other operators. The aim is to control its numerical range and gain thus information about its invertability.
We define the operator

$$\begin{aligned}
Q^{(\alpha)}_{n,l}(z;\theta) := \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3}\mathrm{d}k\, &P_{n,l}(\theta)[w_{0,1}(k,\mu;\theta)\otimes \mathbf{1}_{\mathrm{f}}] \\
&\times\left[\frac{\overline{P}_{n,l}(\theta;|k|)}{H^{(\alpha)}_{\mathrm{el}}(\theta)+e^{-\mathrm{i}\,\vartheta}(H_{\mathrm{f}}+|k|)-z}\right][w_{1,0}(k,\mu;\theta)\otimes\mathbf{1}_{\mathrm{f}}]P_{n,l}(\theta)
\end{aligned}$$

as operator on $\operatorname{Ran} P_{n,l}(\theta)$ for $z \in \mathcal{A}_{n,l}(\alpha,\sigma_0)$. Futhermore, we define θ-dependent versions of the operators $Z_{n,l,\pm}(\alpha)$ (cf. [19, Equation (8)]). We set for $\operatorname{Im}\theta \neq 0$

$$\begin{aligned} Z_{n,l}(\alpha;\theta) &:= \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \mathrm{d}k\, P^{(\alpha)}_{\mathrm{el},n,l}(\theta) w_{0,1}(k,\mu;\theta) \underline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta) \\ &\times \left[\underline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta) H^{(\alpha)}_{\mathrm{el}}(\theta) - E_{n,l}(\alpha) + e^{-\theta}|k|\right]^{-1} \underline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta) w_{1,0}(k,\mu;\theta) P^{(\alpha)}_{\mathrm{el},n,l}(\theta) \\ &+ \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \frac{\mathrm{d}k}{e^{-\theta}|k|} P^{(\alpha)}_{\mathrm{el},n,l}(\theta) w_{0,1}(k,\mu;\theta) P^{(\alpha)}_{\mathrm{el},n,l}(\theta) w_{1,0}(k,\mu;\theta) P^{(\alpha)}_{\mathrm{el},n,l}(\theta). \end{aligned}$$

We have $Z_{n,l}(\alpha;\theta) = \mathcal{U}_{\mathrm{el}}(\theta) Z_{n,l,-}(\alpha)\mathcal{U}_{\mathrm{el}}(\theta)^{-1}$ for $\operatorname{Im}\theta > 0$ and $Z_{n,l}(\alpha;\theta) = \mathcal{U}_{\mathrm{el}}(\theta) Z_{n,l,+}(\alpha)\mathcal{U}_{\mathrm{el}}(\theta)^{-1}$ for $\operatorname{Im}\theta < 0$. Moreover, we define the following remainder terms:

$$\begin{aligned} &\mathrm{Rem}_0 := \\ &P_{n,l}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l}(\theta)[\overline{P}_{n,l}(\theta)(H_\alpha(\theta)-z)\overline{P}_{n,l}(\theta)]^{-1}\overline{P}_{n,l}(\theta)W^{(\alpha)}(\theta)P_{n,l}(\theta) \\ &-P_{n,l}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l}(\theta)[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)]^{-1}\overline{P}_{n,l}(\theta)W^{(\alpha)}(\theta)P_{n,l}(\theta) \end{aligned}$$

$$\begin{aligned} &\mathrm{Rem}_1 := \\ &P_{n,l}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l}(\theta)[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)]^{-1}\overline{P}_{n,l}(\theta)W^{(\alpha)}(\theta)P_{n,l}(\theta) \\ &-P_{n,l}(\theta)W^{(\alpha)}_{0,1}(\theta)\overline{P}_{n,l}(\theta)[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)]^{-1}\overline{P}_{n,l}(\theta)W^{(\alpha)}_{1,0}(\theta)P_{n,l}(\theta) \end{aligned}$$

$$\begin{aligned} &\mathrm{Rem}_2 := \\ &= P_{n,l}(\theta)W^{(\alpha)}_{0,1}(\theta)\overline{P}_{n,l}(\theta)[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)]^{-1}\overline{P}_{n,l}(\theta)W^{(\alpha)}_{1,0}(\theta)P_{n,l}(\theta) \\ &\qquad - Q^{(\alpha)}_{n,l}(z;\theta) \end{aligned}$$

$$\mathrm{Rem}_3 := P_{n,l}(\theta)W^{(\alpha)}(\theta)P_{n,l}(\theta)$$

We generalize Lemma [5, Lemma 3.16] (see also [19, Lemma A.7]).

Lemma 8. *Let $0 < \vartheta < \theta_0$. Then there is a $C > 0$ such that for all sufficiently small $\alpha > 0$ and all $z \in \mathcal{A}_{n,l}(\alpha,\sigma_0)$ the estimate*

$$\|[\mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta)-z)-(H^{(\alpha)}_{\mathrm{el}}(\theta)-z+e^{-\theta}H_{\mathrm{f}}-Q^{(\alpha)}_{n,l}(z;\theta))P_{n,l}(\theta)]\| \leq \frac{C}{\sin^4\vartheta} g^2\sqrt{\alpha}$$

holds.

Proof. We begin with the estimate on Rem_0:

$$\|\mathrm{Rem}_0\| \leq \sum_{n=1}^{\infty} \left\| P_{n,l}(\theta)W^{(\alpha)}(\theta)|B_\theta(\rho_0)|^{-1/2}\right.$$

$$
\begin{aligned}
&\times |B_\theta(\rho_0)|^{+1/2}\big[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\big]^{-1}\overline{P}_{n,l}(\theta)|B_\theta(\rho_0)^*|^{1/2}\\
&\times \Big[-|B_\theta(\rho_0)^*|^{-1/2}W^{(\alpha)}(\theta)|B_\theta(\rho_0)|^{-1/2}\\
&\times |B_\theta(\rho_0)|^{+1/2}\big[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\big]^{-1}\overline{P}_{n,l}(\theta)|B_\theta(\rho_0)^*|^{1/2}\Big]^n\\
&\times |B_\theta(\rho_0)^*|^{-1/2}W^{(\alpha)}(\theta)P_{n,l}(\theta)\Big\|\\
\le&\big\|P_{n,l}(\theta)W^{(\alpha)}(\theta)|B_\theta(\rho_0)|^{-1/2}\big\|\big\||B_\theta(\rho_0)^*|^{-1/2}W^{(\alpha)}(\theta)P_{n,l}(\theta)\big\|\\
&\times \Big\||B_\theta(\rho_0)|^{+1/2}\big[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\big]^{-1}\overline{P}_{n,l}(\theta)|B_\theta(\rho_0)^*|^{1/2}\Big\|\\
&\times \sum_{n=1}^{\infty}\big[\big\||B_\theta(\rho_0)^*|^{-1/2}W^{(\alpha)}(\theta)|B_\theta(\rho_0)|^{-1/2}\big\|\\
&\times \big\||B_\theta(\rho_0)|^{+1/2}\big[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\big]^{-1}\overline{P}_{n,l}(\theta)|B_\theta(\rho_0)^*|^{1/2}\big\|\big]^n\\
\le&\frac{C}{\sin^2\vartheta}g^2\sum_{n=1}^{\infty}\Big(\frac{C}{\sin^2\vartheta}\sqrt{\alpha}\Big)^n\le\frac{C}{\sin^4\vartheta}g^2\sqrt{\alpha}
\end{aligned}
$$

We used here Lemma 5 a) and b), Lemma 4 and Corollary 1. For Rem_1 we find

$$
\begin{aligned}
\|\mathrm{Rem}_1\|\le&\Big\||B_\theta(\rho_0)|^{+1/2}\big[\overline{P}_{n,l}(\theta)(H_{\alpha,0}(\theta)-z)\overline{P}_{n,l}(\theta)\big]^{-1}\overline{P}_{n,l}(\theta)|B_\theta(\rho_0)^*|^{1/2}\Big\|\\
&\times\Big(\big\|P_{n,l}(\theta)W^{(\alpha)}_{1,0}(\theta)|B_\theta(\rho_0)|^{-1/2}\big\|\big\||B_\theta(\rho_0)^*|^{-1/2}W^{(\alpha)}_{1,0}(\theta)P_{n,l}(\theta)\big\|\\
&+\big\|P_{n,l}(\theta)W^{(\alpha)}_{1,0}(\theta)|B_\theta(\rho_0)|^{-1/2}\big\|\big\||B_\theta(\rho_0)^*|^{-1/2}W^{(\alpha)}_{0,1}(\theta)P_{n,l}(\theta)\big\|\\
&+\big\|P_{n,l}(\theta)W^{(\alpha)}_{0,1}(\theta)|B_\theta(\rho_0)|^{-1/2}\big\|\big\||B_\theta(\rho_0)^*|^{-1/2}W^{(\alpha)}_{0,1}(\theta)P_{n,l}(\theta)\big\|\Big)\\
\le&\frac{C}{\sin^2\vartheta}g^2\rho_0^{1/2}=\frac{C}{\sin^2\vartheta}g^2\alpha
\end{aligned}
$$

using Corollary 1 and Lemma 5 a) and b).

For Rem_2 we use the pull-through formula [4, Lemma IV.8]: We have

$$
\begin{aligned}
\mathrm{Rem}_2=&\alpha\sum_{\mu,\mu'=1,2}\int_{k\in\mathbb{R}^3}\mathrm{d}k\int_{k'\in\mathbb{R}^3}\mathrm{d}k'\,P_{n,l}(\theta)\boldsymbol{\alpha}\cdot G^{(\theta)}_{\alpha x}(k,\mu)a^*_{\mu'}(k')\\
&\times\frac{P^{(\alpha)}_{\mathrm{el},n,l}\otimes\mathbf{1}_{\mathrm{f}}+P^{(\alpha)}_{\mathrm{el},n,l}\otimes\chi_{H_{\mathrm{f}}+|k|+|k'|\ge\rho_0}}{H^{(\alpha)}_{\mathrm{el}}(\theta)+e^{-\theta}(H_{\mathrm{f}}+|k|+|k'|)-z}\boldsymbol{\alpha}\cdot G^{(\theta)}_{\alpha x}(k',\mu')a_\mu(k)P_{n,l}(\theta).
\end{aligned}
$$

Using Lemma 2 (for the resolvent) and [25, Theorem 11] (for the expectation values of the Dirac matrix) we obtain

$$
\begin{aligned}
&\big|\langle\psi,\mathrm{Rem}_2\psi'\rangle\big|\le C\alpha\sum_{\mu,\mu'=1}^{2}\int_{|k|\le\rho_0}\mathrm{d}k\int_{|k'|\le\rho_0}\mathrm{d}k'\,\frac{|\kappa(e^{-\theta}|k|)|}{\sqrt{|k|}}\frac{|\kappa(e^{-\theta}|k'|)|}{\sqrt{|k'|}}\\
&\times\big\|P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\boldsymbol{\alpha}\cdot\epsilon_\mu(k)e^{\mathrm{i}\,\alpha x\cdot k}\Lambda^{(+)}_{\alpha^{-1},3}(\theta)\big\|\big\|\Lambda^{(+)}_{\alpha^{-1},3}(\theta)\boldsymbol{\alpha}\cdot\epsilon_\mu(k')e^{-\mathrm{i}\,\alpha x\cdot k'}P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\big\|
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\| \frac{P_{\mathrm{el},n,l}^{(\alpha)}(\theta) \otimes \mathbf{1}_{\mathrm{f}} + P_{\mathrm{el},n,l}^{(\alpha)}(\theta) \otimes \chi_{H_{\mathrm{f}}+|k|+|k'|\geq\rho_0}}{H_{\mathrm{el}}^{(\alpha)}(\theta) + e^{-\theta}(H_{\mathrm{f}} + |k| + |k'|) - z} \right\| \\
&\times \left\| a_\mu(k)\chi_{H_{\mathrm{f}}\leq\rho_0}\psi \right\| \left\| a_{\mu'}(k')\chi_{H_{\mathrm{f}}\leq\rho_0}\psi \right\| \\
\leq & \frac{Cg^2}{\sin\vartheta\rho_0} \sum_{\mu,\mu'=1}^{2} \int_{|k|\leq\rho_0} \mathrm{d}k\, \frac{|\kappa(e^{-\theta}|k|)|(1+\alpha|k|)}{\sqrt{|k|}\sqrt{|k|}} \sqrt{|k|} \left\| a_\mu(k)\chi_{H_{\mathrm{f}}\leq\rho_0}\psi \right\| \\
&\times \int_{|k'|\leq\rho_0} \mathrm{d}k'\, \frac{|\kappa(e^{-\theta}|k'|)|(1+\alpha|k'|)}{\sqrt{|k'|}\sqrt{|k'|}} \sqrt{|k'|} \left\| a_{\mu'}(k')\chi_{H_{\mathrm{f}}\leq\rho_0}\psi' \right\| \\
\leq & \frac{Cg^2}{\sin\vartheta\rho_0} \Big(\int_{|k|\leq\rho_0} \mathrm{d}k\, \frac{1}{|k|^2} \Big) \left\| H_{\mathrm{f}}^{1/2}\chi_{H_{\mathrm{f}}\leq\rho_0}\psi' \right\| \left\| H_{\mathrm{f}}^{1/2}\chi_{H_{\mathrm{f}}\leq\rho_0}\psi \right\| \\
\leq & \frac{Cg^2}{\sin\vartheta\rho_0} \rho_0^2 \|\psi'\| \|\psi\| = \frac{C}{\sin\vartheta} g^2\alpha^2 \|\psi'\| \|\psi\|
\end{aligned}
$$

with a generic $C > 0$.

Finally, we consider $\mathrm{Rem}_3 := P_{n,l}(\theta)W^{(\alpha)}(\theta)P_{n,l}(\theta)$, where we show the estimate with $A_\kappa^{(\theta)}(\alpha x)_-$ only. The other estimate works analogously. We find using [25, Lemma 10]

$$
\begin{aligned}
&\sqrt{\alpha} \left| \langle \psi', P_{\mathrm{el},n,l}^{(\alpha)}(\theta) \otimes \chi_{H_{\mathrm{f}}\leq\rho_0} \boldsymbol{\alpha} \cdot A_\kappa^{(\theta)}(\alpha x)_- P_{\mathrm{el},n,l}^{(\alpha)}(\theta) \otimes \chi_{H_{\mathrm{f}}\leq\rho_0}\psi \rangle \right| \\
\leq & \sqrt{\alpha} \sum_{\mu,\mu'=1}^{2} \int_{|k|\leq\rho_0} \mathrm{d}k\, \frac{|\kappa(e^{-\theta}|k|)|}{\sqrt{|k|}} \\
&\times \left\| P_{\mathrm{el},n,l}^{(\alpha)}(\theta) \boldsymbol{\alpha} \cdot \epsilon_\mu(k) e^{\mathrm{i}\,\alpha x\cdot k} P_{\mathrm{el},n,l}^{(\alpha)}(\theta) \right\| \|\psi'\| \left\| a_\mu(k)\chi_{H_{\mathrm{f}}\leq\rho_0}\psi \right\| \\
\leq & Cg \sqrt{\int_{|k|\leq\rho_0} \mathrm{d}k\, \frac{1}{|k|^2}} \|\psi'\| \left\| H_{\mathrm{f}}^{1/2}\chi_{H_{\mathrm{f}}\leq\rho_0}\psi \right\| \leq Cg\rho_0 \|\psi'\| \|\psi\| = C\sqrt{\alpha} g^2 \|\psi'\| \|\psi\|.
\end{aligned}
$$

□

Note that the following Lemma 9 holds only for $z \in \mathcal{A}_{n,l}^{<}(\alpha,\sigma_0)$, contrary to Lemma 8. It generalizes [5, Lemma 3.16] (see also [19, Lemma A.8]).

Lemma 9. *Let $0 < \vartheta < \theta_0$. Then there is a $C > 0$ such that for all $\alpha > 0$ sufficiently small and all $z \in \mathcal{A}_{n,l}^{<}(\alpha,\sigma_0)$ the estimate*

$$
\left\| Q_{n,l}^{(\alpha)}(z;\theta) - Z_{n,l}(\alpha;\theta) \right\| \leq \frac{C}{\sin^2\vartheta} g^2\alpha
$$

holds.

Proof. We split $Q_{n,l}^{(\alpha)}(z;\theta) - Z_{n,l}(\alpha;\theta) = \mathrm{Rem}_{4a} + \mathrm{Rem}_{4b}$ with

$$\begin{aligned}\mathrm{Rem}_{4a} := & \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \mathrm{d}k\, P_{n,l}(\theta)[w_{0,1}(k,\mu;\theta)\otimes \mathbf{1}_{\mathrm{f}}] \\ & \times \left[\frac{P_{\mathrm{el},n,l}^{(\alpha)}(\theta)\otimes \chi_{H_{\mathrm{f}}+|k|\geq\rho_0}}{H_{\mathrm{el}}^{(\alpha)}(\theta)+e^{-\theta}(H_{\mathrm{f}}+|k|)-z}\right][w_{1,0}(k,\mu;\theta)\otimes \mathbf{1}_{\mathrm{f}}]P_{n,l}(\theta) \\ & - \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \frac{\mathrm{d}k}{e^{-\theta}|k|} P_{\mathrm{el},n,l}^{(\alpha)}(\theta) w_{0,1}(k,\mu;\theta) P_{\mathrm{el},n,l}^{(\alpha)}(\theta) w_{1,0}(k,\mu;\theta) P_{\mathrm{el},n,l}^{(\alpha)}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Rem}_{4b} := & \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \mathrm{d}k\, P_{n,l}(\theta)[w_{0,1}(k,\mu;\theta)\otimes \mathbf{1}_{\mathrm{f}}] \\ & \times \left[\frac{\underline{P}_{\mathrm{el},n,l}^{(\alpha)}(\theta)\otimes \mathbf{1}_{\mathrm{f}}}{H_{\mathrm{el}}^{(\alpha)}(\theta)+e^{-\theta}(H_{\mathrm{f}}+|k|)-z}\right][w_{1,0}(k,\mu;\theta)\otimes \mathbf{1}_{\mathrm{f}}]P_{n,l}(\theta) \\ & - \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \mathrm{d}k\, P_{\mathrm{el},n,l}^{(\alpha)} \otimes \chi_{H_{\mathrm{f}}\leq\rho_0} w_{0,1}(k,\mu;\theta) \\ & \times \left[\frac{\underline{P}_{\mathrm{el},n,l}^{(\alpha)}(\theta)}{H_{\mathrm{el}}^{(\alpha)}(\theta) - E_{n,l} + e^{-\theta}|k|}\right] w_{1,0}(k,\mu;\theta) P_{\mathrm{el},n,l}^{(\alpha)}(\theta)\otimes \chi_{H_{\mathrm{f}}\leq\rho_0}.\end{aligned}$$

We start with Rem_{4a}: As in the proof of Lemma 2 a) one shows for $\rho_0 \leq r+|k|$ the inequalities

$$|E_{n,l}(\alpha)+e^{-\theta}(r+|k|)-z)| \geq -\sigma_0 + \sin\vartheta(r+|k|) \geq \frac{|k|\sin\vartheta}{2} \tag{52}$$

and

$$|E_{n,l}(\alpha)+e^{-\theta}(r+|k|)-z)| \geq -\sigma_0 + \sin\vartheta(r+|k|) \geq \frac{\rho_0\sin\vartheta}{2}, \tag{53}$$

since we have $\sigma_0 \leq \frac{\rho_0 \sin\vartheta}{2} \leq \frac{(r+|k|)\sin\vartheta}{2}$ for sufficiently small $\alpha > 0$.
As in the proof of Lemma 4 one obtains using [25, Lemma 10] the inequality

$$\|P_{\mathrm{el},n,l}^{(\alpha)}(\theta) w_{0,1}(k,\mu;\theta) P_{\mathrm{el},n,l}^{(\alpha)}(\theta)\| \leq Cg\frac{|\kappa(e^{-\theta}|k|)|}{\sqrt{|k|}}. \tag{54}$$

We find after a little transformation of Rem_{4a}

$$\begin{aligned}\big\|\mathrm{Rem}_{4a}\big\| = & \Big\| \sum_{\mu=1,2}\int_{k\in\mathbb{R}^3} \mathrm{d}k\, P_{n,l}(\theta)[w_{0,1}(k,\mu;\theta)\otimes \mathbf{1}_{\mathrm{f}}] P_{\mathrm{el},n,l}^{(\alpha)}(\theta) \\ & \times \left[\frac{\big(e^{-\theta}H_{\mathrm{f}} + E_{n,l}(\alpha) - z\big)\,\chi_{H_{\mathrm{f}}+|k|\geq\rho_0}\chi_{H_{\mathrm{f}}\leq\rho_0}}{(E_{n,l}(\alpha)+e^{-\theta}(H_{\mathrm{f}}+|k|)-z)\,e^{-\theta}|k|}\right]\end{aligned}$$

$$
\begin{aligned}
&\times P^{(\alpha)}_{\mathrm{el},n,l}(\theta)[w_{1,0}(k,\mu;\theta)\otimes \mathbf{1}_{\mathrm{f}}]P_{n,l}(\theta)\\
&-\sum_{\mu=1,2}\int_{k\in\mathbb{R}^3}\mathrm{d}k\, P^{(\alpha)}_{\mathrm{el},n,l}(\theta)w_{0,1}(k,\mu;\theta)P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\\
&\times\frac{\chi_{H_{\mathrm{f}}\le\rho_0}\chi_{H_{\mathrm{f}}+|k|\le\rho_0}}{e^{-\theta}|k|}w_{1,0}(k,\mu;\theta)P^{(\alpha)}_{\mathrm{el},n,l}\Big\|\\
\le&\frac{C}{\sin\vartheta}g^2(\alpha^2+\rho_0)\Big(\frac{1}{\rho_0}\int_{|k|\le\rho_0}\mathrm{d}k\,\frac{|\kappa(e^{-\theta}|k|)|^2}{|k|^2}+\int_{|k|\ge\rho_0}\mathrm{d}k\,\frac{|\kappa(e^{-\theta}|k|)|^2}{|k|^3}\Big)\\
&+g^2\int_{|k|\le\rho_0}\mathrm{d}k\,\frac{|\kappa(e^{-\theta}|k|)|^2}{|k|^2}\\
\le&\frac{C}{\sin\vartheta}g^2(\alpha^2\frac{\rho_0}{\rho_0}+\alpha^2\ln\rho_0^{-1}+\rho_0)\le\frac{C}{\sin\vartheta}g^2\alpha.
\end{aligned}
$$

Here, we split the integration in the first summand in the regions $|k|\le\rho_0$ and $|k|>\rho_0$. We use inequality (53) in the first region, and inequality (52) in the second region.
The estimate on Rem_{4b} is more difficult. We split the projection $\underline{P}^{(\alpha)}_{\mathrm{el},n,l} = \overline{P}^{(\alpha)}_{\mathrm{el},n} + \overline{P}^{(\alpha)}_{\mathrm{el},n,l}$ and obtain for $P=\overline{P}^{(\alpha)}_{\mathrm{el},n}$ as well as for $P=\overline{P}^{(\alpha)}_{\mathrm{el},n,l}$

$$
\begin{aligned}
&\Big\|\sum_{\mu=1,2}\int_{k\in\mathbb{R}^3}\mathrm{d}k\, P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\chi_{H_{\mathrm{f}}\le\rho_0}[w_{0,1}(k,\mu;\theta)\otimes\mathbf{1}_{\mathrm{f}}]\\
&\times\left[\frac{P\otimes\mathbf{1}_{\mathrm{f}}}{H^{(\alpha)}_{\mathrm{el}}(\theta)+e^{-\theta}(H_{\mathrm{f}}+|k|)-z}\right][w_{1,0}(k,\mu;\theta)\otimes\mathbf{1}_{\mathrm{f}}]P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\chi_{H_{\mathrm{f}}\le\rho_0}\\
&-\sum_{\mu=1,2}\int_{k\in\mathbb{R}^3}\mathrm{d}k\, P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\chi_{H_{\mathrm{f}}\le\rho_0}w_{0,1}(k,\mu;\theta)\\
&\times\left[\frac{P\otimes\mathbf{1}_{\mathrm{f}}}{H^{(\alpha)}_{\mathrm{el}}(\theta)-E_{n,l}+e^{-\theta}|k|}\right]w_{1,0}(k,\mu;\theta)P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\otimes\chi_{H_{\mathrm{f}}\le\rho_0}\Big\|\\
\le&\alpha\sum_{\mu=1,2}\int_{k\in\mathbb{R}^3}\mathrm{d}k\,\frac{|\kappa(e^{-\theta}|k|)|^2}{|k|}\\
&\times\big\|P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\boldsymbol{\alpha}\cdot\epsilon_\mu(k)e^{\mathrm{i}\,\alpha x\cdot k}\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)\big\|\big\|\Lambda^{(+)}_{\alpha^{-1},\mathfrak{Z}}(\theta)\boldsymbol{\alpha}\cdot\epsilon_\mu(k')e^{-\mathrm{i}\,\alpha x\cdot k'}P^{(\alpha)}_{\mathrm{el},n,l}(\theta)\big\|\\
&\times\Big[\Big\|\frac{P\otimes\mathbf{1}_{\mathrm{f}}}{H^{(\alpha)}_{\mathrm{el}}(\theta)+e^{-\theta}(H_{\mathrm{f}}+|k|)-z}\Big\|\Big\|\frac{P}{H^{(\alpha)}_{\mathrm{el}}(\theta)-E_{n,l}+e^{-\theta}|k|}\Big\|\\
&\times\big(|E_{n,l}-z|+\|H_{\mathrm{f}}\chi_{H_{\mathrm{f}}\le\rho_0}\|\big)\Big]\\
\le&Cg^2\alpha^2\int_{k\in\mathbb{R}^3}\mathrm{d}k\,\frac{|\kappa(e^{-\theta}|k|)|^2(1+\alpha|k|)^2}{|k|}\\
&\times\Big\|\frac{P\otimes\mathbf{1}_{\mathrm{f}}}{H^{(\alpha)}_{\mathrm{el}}(\theta)+e^{-\theta}(H_{\mathrm{f}}+|k|)-z}\Big\|\Big\|\frac{P}{H^{(\alpha)}_{\mathrm{el}}(\theta)-E_{n,l}+e^{-\theta}|k|}\Big\|.
\end{aligned}
$$

We used [25, Theorem 11]. Note that all estimates on $\|\frac{P\otimes\mathbf{1}_{\mathrm{f}}}{H_{\mathrm{el}}^{(\alpha)}(\theta)+e^{-\theta}(H_{\mathrm{f}}+|k|)-z}\|$ in Lemma 1 hold also for $\|[PH_{\mathrm{el}}^{(\alpha)}(\theta)-E_{n,l}+e^{-\theta}|k|]^{-1}P\|$, since the operator under the norm in the second expression is the projection of the operator in the first expression on the vacuum sector with $z=E_{n,l}$.

Case 1: $P=\overline{P}_{\mathrm{el},n,l}^{(\alpha)}$. We split the integration in the regions $B_1:=\{k\in\mathbb{R}^3||k|\le\rho_0\}$ and $B_2:=\{k\in\mathbb{R}^3||k|>\rho_0\}$. Using Formula (27) in Lemma 1 a), the integral over B_1 can be estimated by

$$\frac{C}{\sin^2\vartheta}g^2\alpha^2\frac{1}{\delta_{n,l}(\alpha)^2}\int_{k\in B_1}\mathrm{d}k\,\frac{1}{|k|}\le\frac{C}{\sin^2\vartheta}g^2\alpha^{-2}\rho_0^2=\frac{C}{\sin^2\vartheta}g^2\alpha^2.$$

With Formula (28) in Lemma 1 b) we estimate the integral over B_2 by

$$\frac{C}{\sin^2\vartheta}g^2\alpha^2\int_{k\in B_2}\mathrm{d}k\,\frac{|\kappa(e^{-\theta}|k|)|^2(1+\alpha|k|)^2}{|k|^3}\le\frac{C}{\sin^2\vartheta}g^2\alpha^2\ln\rho_0^{-1}.$$

Case 2: $P=\overline{P}_{\mathrm{el},n}^{(\alpha)}$. We estimate the resolvents with Lemma 1 c) and obtain the estimate

$$\frac{C}{\delta_n^2\sin^2\vartheta}g^2\alpha^2\int_{k\in\mathbb{R}^3}\mathrm{d}k\,\frac{|\kappa(e^{-\theta}|k|)|^2(1+\alpha|k|)^2}{|k|}\le\frac{C}{\delta_n^2\sin^2\vartheta}g^2\alpha^2=\frac{C}{\delta_n^2\sin^2\vartheta}g^2\alpha^2.$$

□

The following Lemma generalizes [19, Corollary A.9]. Note, however, that we do not remove the α-dependence of the real part.

LEMMA 10. *There is a constant $C>0$ such that for all sufficiently small $\alpha>0$ the estimate*

$$\|\mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0)^{-1}Z_{n,l,\pm}(\alpha)\mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0)-Y_{n,l,\pm}(\alpha)\|\le Cg^2\alpha$$

holds.

Proof. We consider the case with the minus sign only. It suffices to show

$$\|\mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0)^{-1}\mathrm{Im}\,Z_{n,l,-}(\alpha)\mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0)-Z_{n,l,im}\|\le Cg^2\alpha.$$

Because of $[x,H_{\mathrm{el}}^{(\alpha)}]=\mathrm{i}\,\alpha^{-1}\boldsymbol{\alpha}$ and $|e^{\mathrm{i}\,\alpha k\cdot x}-1|\le\alpha|k||x|$ we obtain from [25, Lemma 10 and Lemma 12]

$$\begin{aligned}\|\mathrm{Im}\,Z_{n,l,-}(\alpha)-g^2\pi&\sum_{\substack{n',l':\\E_{n',l'}(\alpha)<E_{n,l}(\alpha)}}\sum_{\mu=1,2}\int_{|\omega|=1}\mathrm{d}\omega\,(E_{n',l'}(\alpha)-E_{n,l}(\alpha))\\&\times\frac{\kappa(|E_{n',l'}(\alpha)-E_{n,l}(\alpha)|)^2}{4\pi^2}P_{\mathrm{el},n,l}^{(\alpha)}\epsilon_\mu(\omega)\cdot xP_{\mathrm{el},n',l'}^{(\alpha)}\epsilon_\mu(\omega)\cdot xP_{\mathrm{el},n,l}^{(\alpha)}\|\le g^2\alpha.\end{aligned}$$

The integral over ω and the sum over the polarizations can be done in the same way as in the non-relativistic case (see [19, Remark 1]). If we take additionally into account that $|E_{n,l'}(\alpha) - E_{n,l}(\alpha)| \leq C\alpha^2$, we obtain

$$\|\mathrm{Im}\, Z_{n,l,-}(\alpha) - g^2 \frac{2}{3} \sum_{n',l':n'<n} (E_{n',l'}(\alpha) - E_{n,l}(\alpha)) \times \frac{\kappa(|E_{n',l'}(\alpha) - E_{n,l}(\alpha)|)^2}{4\pi^2} P^{(\alpha)}_{\mathrm{el},n,l} x P^{(\alpha)}_{\mathrm{el},n',l'} x P^{(\alpha)}_{\mathrm{el},n,l}\| \leq g^2\alpha.$$

[25, Lemma 8] implies $\mathcal{U}_{\mathrm{NR}}(\alpha^{-1}, \mathfrak{z}; 0)^{-1} P^{(\alpha)}_{\mathrm{el},n,l} = P^{(0)}_{\mathrm{el},n,l} \mathcal{U}_{\mathrm{NR}}(\alpha^{-1}, \mathfrak{z}; 0)$. The claim follows together with [25, Lemma 7], [25, Equation (76) in Lemma 8] and [25, Lemma 11]. Note that κ admits an analytic continuation. □

6 Estimates on the Numerical Range

The estimates in Section 5 allow us to control the numerical range of the Feshbach operator. But since $\mathrm{Re}\, Z_{n,l,\pm}(\alpha)$ depends on α, we have to prove that $Z_{n,l,\pm}(\alpha)$ is of order g^2:

Lemma 11. *Let $0 < \vartheta < \theta_0$ and $n > 2$. Then the following holds:*

a) There is a $C > 0$ such that for all sufficiently small $\alpha > 0$ the estimate

$$\|Z_{n,l,\pm}(\alpha)\| \leq Cg^2$$

holds.

b) There is a $c > 0$ such that for all sufficiently small $\alpha > 0$ the estimates

$$\mathrm{Im}\, Z_{n,l,-}(\alpha) \geq cg^2 + \mathcal{O}(g^2\alpha)$$
$$\mathrm{Im}\, Z_{n,l,+}(\alpha) \leq -cg^2 + \mathcal{O}(g^2\alpha)$$

hold.

Proof.
b) follows immediately from Lemma 10, since by [19, Theorem B.1] there is a $c > 0$ such that the estimates $\mathrm{Im}\, Y_{n,l,-}(\alpha) \geq cg^2$ and $\mathrm{Im}\, Y_{n,l,+}(\alpha) \leq -cg^2$ hold (cf. the Definition (22) of $\mathrm{Im}\, Y_{n,l,\pm}(\alpha)$ as well as the remark before Theorem 1).
a) As in the estimates on Rem_{4a} in the proof of Lemma 9 we find

$$\|\sum_{\mu=1,2} \int_{k\in\mathbb{R}^3} \frac{\mathrm{d}k}{e^{-\theta}|k|} P^{(\alpha)}_{\mathrm{el},n,l}(\theta) w_{0,1}(k,\mu;\theta) P^{(\alpha)}_{\mathrm{el},n,l}(\theta) w_{1,0}(k,\mu;\theta) P^{(\alpha)}_{\mathrm{el},n,l}\| \leq Cg^2.$$

Moreover, we obtain

$$
\begin{aligned}
&\| \sum_{\mu=1,2} \int_{k\in\mathbb{R}^3} \mathrm{d}k\, P^{(\alpha)}_{\mathrm{el},n,l}(\theta) \otimes \chi_{H_\mathrm{f}\leq\rho_0} w_{0,1}(k,\mu;\theta) \\
&\times \underline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta) [\underline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta) H^{(\alpha)}_{\mathrm{el}}(\theta) - E_{n,l} + e^{-\theta}|k|]^{-1} \\
&\times \underline{P}^{(\alpha)}_{\mathrm{el},n,l}(\theta) w_{1,0}(k,\mu;\theta) P^{(\alpha)}_{\mathrm{el},n,l}(\theta) \otimes \chi_{H_\mathrm{f}\leq\rho_0} \| \leq Cg^2.
\end{aligned}
$$

To see this, we proceed as in the estimate on Rem_{4b} in the proof of Lemma 9: In Case 1 we can estimate the integral over B_1 by

$$
\frac{C}{\sin\vartheta} g^2 \frac{1}{\delta_{n,l}(\alpha)} \int_{k\in B_1} \mathrm{d}k\, \frac{|\kappa(e^{-\theta}|k|)|^2(1+\alpha|k|)^2}{|k|} \leq \frac{C}{\sin\vartheta} g^2 \alpha^{-2}\rho_0^2 = \frac{C}{\sin^2\vartheta} g^2\alpha^2
$$

and the integral over B_2 by

$$
\frac{C}{\sin\vartheta} g^2 \int_{k\in B_2} \mathrm{d}k\, \frac{|\kappa(e^{-\theta}|k|)|^2(1+\alpha|k|)^2}{|k|^2} \leq \frac{C}{\sin\vartheta} g^2.
$$

In Case 2 we obtain the estimate

$$
\frac{C}{\delta_n \sin\vartheta} g^2 \int_{k\in\mathbb{R}^3} \mathrm{d}k\, \frac{|\kappa(e^{-\theta}|k|)|^2(1+\alpha|k|)^2}{|k|} \leq \frac{C}{\delta_n\sin\vartheta} g^2.
$$

[25, Lemma 9] yields the claim. □

This lemma implies in particular that the numerical range of $Z_{n,l,\pm}(\alpha)$ is contained in a ball around 0 with radius $\mathcal{O}(g^2)$. In particular, this holds for the real part $\mathrm{Re}\, Z_{n,l,\pm}(\alpha) = \mathrm{Re}\, Y_{n,l,\pm}(\alpha)$. As in [19], there are constants $a, b > 0$ such that $\mathrm{NumRan}\, Y_{n,l,\pm}(\alpha) \subset g^2 A(c,a,b)$ with $A(c,a,b) := ic + ([-a,a] + \mathrm{i}\,[0,b])$. As in the non-relativistic case, we set $\nu := \min\{\vartheta, \arctan(c/(2a))\}$. Since we are interested only in $n \leq \tilde{n}$, we can choose the set $A(c,a,b)$ and the angle ν independent of n and l.

Thus, we can control the inverse of the Feshbach operator $\mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta) - z)$ for $z \in \mathcal{A}^<_{n,l}(\alpha,\sigma_0)$ analogously to the non-relativsitic case (see [19, Lemma 6]) as follows (see Figure 1):

Lemma 12. *Let $0 < \vartheta < \theta_0$ and $0 < g \ll \vartheta$ small enough. Then the following estimates hold:*

*a) There are constants $C_1, C_2 > 0$ such that $\mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta) - z)$ has a bounded inverse for all $z \in \mathcal{A}^<_{n,l}(\alpha,\sigma_0) \setminus D(\mathrm{NumRan}(E_{n,l}(\alpha) - Y_{n,l}(\alpha) \otimes \mathbf{1}_\mathrm{f} + e^{-\theta}\mathbf{1}_\mathrm{el} \otimes H_\mathrm{f})|_{\mathrm{Ran}\, P^{(0)}_{\mathrm{el},n,l}}, C_1 \cdot g^2\sqrt{\alpha})$, and for $\lambda \in [E_{n,l}(\alpha) - \delta_{n,l,-}(\alpha), E_{n,l}(\alpha)$
$+ \delta_{n,l,+}(\alpha)]$ the esimate*

$$
\left\| \mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta) - \lambda)^{-1} \right\| \leq \frac{C_2}{\sin\nu \sqrt{(E_{n,l}(\alpha) - \lambda)^2 + cg^4}} \tag{55}
$$

holds.

b) *There are constants* $C_1, C_2 > 0$ *such that for all* $z \in \mathbb{C} \setminus D(\mathrm{NumRan}(E_{n,l}(\alpha) - Y_{n,l}(\alpha))|_{\mathrm{Ran}\, P^{(0)}_{\mathrm{el},n,l}(0)}, C_1 \cdot g^2\alpha)$ *the operator* $(E_{n,l}(\alpha) - z - Z_{n,l}(\alpha;\theta))|_{\mathrm{Ran}\, P^{(\alpha)}_{\mathrm{el},n,l}(\theta)}$ *defined on* $\mathrm{Ran}\, P^{(\alpha)}_{\mathrm{el},n,l}(\theta)$ *has a bounded inverse which fulfills the estimate*

$$\begin{aligned}\|[(E_{n,l}(\alpha) - z - Z_{n,l}(\alpha;\theta))|_{\mathrm{Ran}\, P^{(\alpha)}_{\mathrm{el},n,l}(\theta)}]^{-1}\| \\ \leq \frac{C}{\mathrm{dist}(z, \mathrm{NumRan}(E_{n,l}(\alpha) - Y_{n,l}(\alpha))|_{\mathrm{Ran}\, P^{(\alpha)}_{\mathrm{el},n,l}(0)})}, \quad (56)\end{aligned}$$

and in particular (55).

Proof. This can be shown using Lemmas 8, 9 and 10 exactly as in the proof of [19, Lemma 6]. □

For $l = 1$ or $l = n$, the set $\mathcal{A}^{<}_{n,l}(\alpha, \sigma_0)$ is strictly interior of the set $\mathcal{A}_{n,l}(\alpha, \sigma_0)$, such that we need a relativistic analogon of [19, Lemma 7] in this case.

Lemma 13. *Let* $0 < \vartheta < \theta_0$ *and* $0 < g \ll \vartheta$ *small enough. Let moreover* $l = 1$ *or* $l = n$. *Then the following statements hold: The Feshbach operator* $\mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta) - z)$ *is bounded invertible for all* $z \in \mathcal{A}_{n,l}(\alpha, \sigma_0) \setminus \mathcal{A}^{<}_{n,l}(\alpha, \sigma_0)$ *and there is a* $C > 0$ *such that for* $\lambda \in [E_n - \delta_{n,-}, E_{n,1}(\alpha) - \delta_{n,1,-}(\alpha)]$ *respectively* $\lambda \in [E_{n,n}(\alpha) + \delta_{n,n,+}(\alpha), E_n + \delta_{n,+}]$ *the estimate*

$$\|\mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta) - \lambda)^{-1}\| \leq \frac{C}{\sin\vartheta|\lambda - E_{n,l}(\alpha)| - Cg^2}$$

holds with $l = 1$ *or* $l = n$, *respectively. The same estimate holds for* $[E_{n,l}(\alpha) - \lambda - Q^{(\alpha)}_{n,l}(\lambda;\theta)]^{-1}$.

Proof. This follows analogously to the non-relativistic case (see the proof of [19, Lemma 7]) from Lemma 7 b). For the claim on $[E_{n,l}(\alpha) - \lambda - Q^{(\alpha)}_{n,l}(\lambda;\theta)]^{-1}$, note additionally Lemma 8 and the proof thereof. □

Corollary 3. *Let* $0 < \vartheta < \theta_0$ *and* $0 < g \ll \vartheta$ *small enough. The for all* $1 \leq l \leq n$ *the following holds:*

$$\begin{aligned}&\sigma(H_\alpha(\theta)) \cap \mathcal{A}_{n,l}(\alpha, \sigma_0) \\ &\subset D(\mathrm{NumRan}(E_{n,l}(\alpha) - Y_{n,l}(\alpha) \otimes \mathbf{1}_{\mathrm{f}} + e^{-\theta}\mathbf{1}_{\mathrm{el}} \otimes H_{\mathrm{f}})|_{\mathrm{Ran}\, P^{(0)}_{\mathrm{el},n,l}}, C_1 \cdot g^2\sqrt{\alpha}),\end{aligned}$$

where C_1 *was defined in Lemma 12. In particular,* $[E_n - \delta_{n,-}, E_n + \delta_{n,+}] \subset \rho(H_\alpha(\theta))$.

Proof. This follows because of Lemma 7 c) immediately from Lemma 12 and Lemma 13. □

Remark 3. *The estimates above hold as in the non-relativistic case (cf. [19, Remark 5]) also for $-\theta_0 < \vartheta < 0$, if one reflects the sets $\mathcal{A}_{n,l}(\alpha, \sigma_0)$ and $\mathcal{A}^<_{n,l}(\alpha, \sigma_0)$ about the real axis and replaces $Y_{n,l}(\alpha) = Y_{n,l,-}(\alpha)$ by $Y_{n,l,+}(\alpha)$ for the localization of the numerical range.*

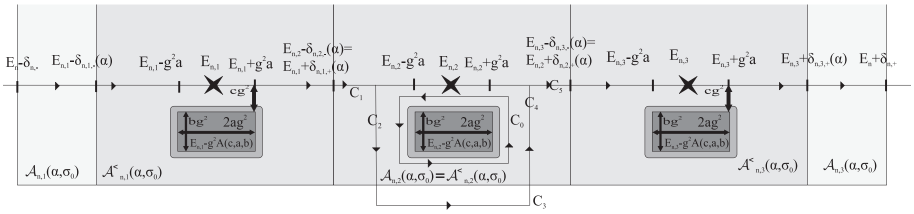

Figure 1: The integration contour in the relativistic model for the principal quantum number $n = 3$.

7 Lifetime of Excited States

We are now able to prove Theorem 1 similarly as in the non-relativistic case. The fine structure splitting induces some differences, however: Since a spectral cutoff around the fine structure component considered would converge to zero as α^2, we introduce a spectral cutoff around all the fine structure components of the corresponding principal quantum number so that additionally contributions of the other components have to be estimated.

Proof of Theorem 1.
Step 1: We pick a function $\tilde{F} \in C_0^\infty(\mathbb{R})$ with $\tilde{F}(x) = 0$ for $|x| \geq 1$ and $\tilde{F}(x) = 1$ for $|x| \leq 1/2$ and define a cutoff function $F(x) := \tilde{F}(\delta_n^{-1}(x - E_n))$. As in the non-relativistic case (see step 1 in the proof of [19, Theorem 1]) one shows $|\langle \psi(\alpha), e^{-isH_\alpha} F(H_\alpha)\psi(\alpha)\rangle - \langle \psi(\alpha), e^{-isH_\alpha}\psi(\alpha)\rangle| \leq C\sqrt{\alpha}$ uniformly in $s \geq 0$.
Step 2: We write

$$\begin{aligned}
&\langle \psi(\alpha), e^{-isH_\alpha} F(H_\alpha)\psi(\alpha)\rangle \\
&\quad = -\frac{1}{2\pi \mathrm{i}} \lim_{\epsilon \downarrow 0} \int \mathrm{d}\lambda\, e^{-\mathrm{i}\,\lambda s} F(\lambda)[f(0, \lambda - \mathrm{i}\,\epsilon) - f(0, \lambda + \mathrm{i}\,\epsilon)] \\
&\quad = -\frac{1}{2\pi \mathrm{i}} \int \mathrm{d}\lambda\, e^{-\mathrm{i}\,\lambda s} F(\lambda)[f(\overline{\theta}, \lambda) - f(\theta, \lambda)],
\end{aligned}$$

where $f(\theta, \lambda) := \langle \psi(\alpha; \overline{\theta}), \frac{1}{H_\alpha(\theta) - \lambda}\psi(\alpha; \theta)\rangle$ with $\psi(\alpha; \theta) := \phi(\alpha; \theta) \otimes \Omega$ and $\phi(\alpha; \theta) := \mathcal{U}_{\mathrm{el}}(\theta)\phi(\alpha)$. (We choose $\operatorname{Im}\theta > 0$.) In the first step, we used [36, Theorem VII.13]. In the second step, we used the dilation analyticity of $H_\alpha(\theta)$ (see Theorem 3) and the fact that $H_\alpha(\theta)$ has no spectrum in the interval

$[E_n - \delta_{n,-}, E_n + \delta_{n,+}/2]$ (see Corollary 3). We split the integration into several intervals:

$$\begin{aligned}
&-\frac{1}{2\pi \mathrm{i}} \int \mathrm{d}\lambda\, e^{-\mathrm{i}\,\lambda s} F(\lambda)[f(\overline{\theta},\lambda) - f(\theta,\lambda)] \\
&= -\frac{1}{2\pi \mathrm{i}} \Bigg\{ \sum_{l'=1}^{n} \int_{E_{n,l'}(\alpha)-\delta_{n,l',-}(\alpha)}^{E_{n,l'}(\alpha)+\delta_{n,l',+}(\alpha)} \mathrm{d}\lambda\, e^{-\mathrm{i}\,\lambda s}[f(\overline{\theta},\lambda) - f(\theta,\lambda)] \\
&\quad + \int_{E_n-\delta_{n,-}}^{E_{n,1}(\alpha)-\delta_{n,1,-}(\alpha)} \mathrm{d}\lambda\, e^{-\mathrm{i}\,\lambda s} F(\lambda)[f(\overline{\theta},\lambda) - f(\theta,\lambda)] \\
&\quad + \int_{E_{n,n}(\alpha)+\delta_{n,n,+}(\alpha)}^{E_n+\delta_{n,+}} \mathrm{d}\lambda\, e^{-\mathrm{i}\,\lambda s} F(\lambda)[f(\overline{\theta},\lambda) - f(\theta,\lambda)] \Bigg\}
\end{aligned}$$

We used here $F(\lambda) = 1$ for $\lambda \in [E_{n,1}(\alpha) - \delta_{n,1,-}(\alpha), E_{n,n}(\alpha) + \delta_{n,n,+}(\alpha)] \subset [E_n - \delta_n/2, E_n + \delta_n/2]$.

Step 3: For $\lambda \in [E_{n,l}(\alpha) - \delta_{n,l,-}(\alpha), E_{n,l}(\alpha) + \delta_{n,l,+}(\alpha)]$ we observe that Equation (51) in Lemma 7 implies

$$\langle \psi(\alpha;\overline{\theta}), \frac{1}{H_g(\theta) - \lambda} \psi(\alpha;\theta)\rangle = \langle \psi(\alpha;\overline{\theta}), \mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta) - \lambda)^{-1}\psi(\alpha;\theta)\rangle$$

and find

$$\begin{aligned}
f(\theta,\lambda) &= \langle \psi(\alpha;\overline{\theta}), \mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta) - \lambda)^{-1}\psi(\alpha;\theta)\rangle \\
&= \langle \phi(\alpha;\overline{\theta}), [E_{n,l}(\alpha) - \lambda - Z_{n,l}(\alpha;\theta)]^{-1}\phi(\alpha;\theta)\rangle \\
&\quad - \langle \psi(\alpha;\overline{\theta}), [E_{n,l}(\alpha) - \lambda - Z_{n,l}(\alpha;\theta)]^{-1} \\
&\quad \times [\mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta) - \lambda) - (E_{n,l}(\alpha) - \lambda + e^{-\theta}\mathbf{1}_{\mathrm{el}} \otimes H_{\mathrm{f}} - Z_{n,l}(\alpha;\theta))P_{n,l}(\theta)] \\
&\quad \times \mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta) - \lambda)^{-1}\psi(\alpha;\theta)\rangle =: \hat{f}(\theta,\lambda) + B_1(\theta,\lambda)
\end{aligned}$$

using the second resolvent equation. Here $\hat{f}(\theta,\lambda)$ is the first term in the sum. Using the dilation analyticity and the resolvent identity once again, we obtain

$$\begin{aligned}
\hat{f}(\theta,\lambda) &= \langle \phi(\alpha), [E_{n,l}(\alpha) - \lambda - Z_{n,l,-}(\alpha)]^{-1}\phi(\alpha)\rangle \\
&= \langle \phi(0), [E_{n,l}(\alpha) - \lambda - \mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0)^{-1} Z_{n,l,-}(\alpha)\mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0)]^{-1}\phi(0)\rangle \\
&= \langle \phi(0), [E_{n,l}(\alpha) - \lambda - Y_{n,l,-}(\alpha)]^{-1}\phi(0)\rangle \\
&\quad - \langle \phi(0), [E_{n,l}(\alpha) - \lambda - Y_{n,l,-}(\alpha)]^{-1} \\
&\quad \times [\mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0)^{-1} Z_{n,l,-}(\alpha)\mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0) - Y_{n,l,-}(\alpha)] \\
&\quad \times [E_{n,l}(\alpha) - \lambda - \mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0)^{-1} Z_{n,l,-}(\alpha)\mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0)]^{-1}\phi(0)\rangle \\
&=: \tilde{f}_-(\lambda) + B_{2,-}(\lambda),
\end{aligned}$$

where $\tilde{f}_-(\lambda)$ is the first term in the sum. We set $B(\theta,\lambda) := B_1(\theta,\lambda) + B_{2,-}(\lambda)$.

Accordingly we obtain

$$
\begin{aligned}
&\hat{f}(\bar{\theta},\lambda) = \langle \phi(\alpha), [E_{n,l}(\alpha) - \lambda - Z_{n,l,+}(\alpha)]^{-1}\phi(\alpha)\rangle \\
=&\langle \phi(0), [E_{n,l}(\alpha) - \lambda - Y_{n,l,+}(\alpha)]^{-1}\phi(0)\rangle \\
&- \langle \phi(0), [E_{n,l}(\alpha) - \lambda - Y_{n,l,+}(\alpha)]^{-1} \\
&\times [\mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0)^{-1} Z_{n,l,+}(\alpha)\mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0) - Y_{+}(\alpha)] \\
&\times [E_{n,l}(\alpha) - \lambda - \mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0)^{-1} Z_{n,l,+}(\alpha)\mathcal{U}_{\mathrm{NR}}(\alpha^{-1},\mathfrak{z};0)]^{-1}\phi(0)\rangle \\
=:& \tilde{f}_{+}(\lambda) + B_{2,+}(\lambda),
\end{aligned}
$$

where $\tilde{f}_{+}(\lambda)$ is the first term in the sum. We set $B(\bar{\theta},\lambda) := B_1(\bar{\theta},\lambda) + B_{2,+}(\lambda)$. As in the non-relativistic case, we move the contour for $\tilde{f}_{\pm}(\lambda)$ and estimate the terms $B(\theta,\lambda)$ and $B(\bar{\theta},\lambda)$ on the real axis. We find

$$
\begin{aligned}
&\int_{E_{n,l}(\alpha)-\delta_{n,l,-}(\alpha)}^{E_{n,l}(\alpha)+\delta_{n,l,+}(\alpha)} \mathrm{d}\lambda\, e^{-\mathrm{i}\,\lambda s}[f(\bar{\theta},\lambda) - f(\theta,\lambda)] \\
&= \int_{E_{n,l}(\alpha)-\delta_{n,l,-}(\alpha)}^{E_{n,l}(\alpha)+\delta_{n,l,+}(\alpha)} \mathrm{d}\lambda\, e^{-\mathrm{i}\,\lambda s}[B(\overline{\theta},\lambda) - B(\theta,\lambda)] \\
&+ \int_{C_1+C_5} \mathrm{d}z\, e^{-\mathrm{i}\,zs}[\tilde{f}_{+}(z) - \tilde{f}_{-}(z)] \\
&+ \int_{C_2+C_3+C_4} \mathrm{d}z\, e^{-\mathrm{i}\,zs}[\tilde{f}_{+}(z) - \tilde{f}_{-}(z)] - \int_{C_0} \mathrm{d}z\, e^{-\mathrm{i}\,zs}[\tilde{f}_{+}(z) - \tilde{f}_{-}(z)],
\end{aligned}
$$

where $C := C_1 + C_2 + C_3 + C_4 + C_5$ with $C_1 := [E_{n,l}(\alpha) - \delta_{n,l,-}(\alpha), E_{n,l}(\alpha) - \delta_{n,l,-}(\alpha)/2]$, $C_2 := [E_{n,l}(\alpha) - \delta_{n,l,-}(\alpha)/2, E_{n,l}(\alpha) - \delta_{n,l,-}(\alpha)/2 - \mathrm{i}\,\delta_{n,l}(\alpha)]$, $C_3 := [E_{n,l}(\alpha) - \delta_{n,l,-}(\alpha)/2 - \mathrm{i}\,\delta_{n,l}(\alpha), E_{n,l}(\alpha) + \delta_{n,l,+}(\alpha)/2 - \mathrm{i}\,\delta_{n,l}(\alpha)]$, $C_4 := [E_{n,l}(\alpha) + \delta_{n,l,+}(\alpha)/2 - \mathrm{i}\,\delta_{n,l}(\alpha), E_{n,l}(\alpha) + \delta_{n,l,+}(\alpha)/2]$ and $C_5 := [E_{n,l}(\alpha) + \delta_{n,l,+}(\alpha)/2, E_{n,l}(\alpha) + \delta_{n,l,+}(\alpha)]$. Note that this contour lies partially *outside* $\mathcal{A}_{n,l}(\alpha,\sigma_0)$, which is possible since we do not consider any integrals which contain $Q_{n,l}^{(\alpha)}(z;\theta)$. C_0 is a suitable contour to pick a pole contribution of $\tilde{f}(\theta,z)$. We choose as in the non-relativistic case $C_0 = [E_{n,l}(\alpha) + g^2(-(a+c/2) - ic/2), E_{n,l}(\alpha) + g^2((a+c/2) - ic/2)] + [E_{n,l}(\alpha) + g^2((a+c/2) - ic/2), E_{n,l}(\alpha) + g^2((a+c/2) - \mathrm{i}\,(b+3c/2))] + [E_{n,l}(\alpha) + g^2((a+c/2) - \mathrm{i}\,(b+3c/2)), E_{n,l}(\alpha) + g^2(-(a+c/2) - \mathrm{i}\,(b+3c/2))] + [E_{n,l}(\alpha) + g^2(-(a+c/2) - \mathrm{i}\,(b+3c/2)), E_{n,l}(\alpha) + g^2(-(a+c/2) - \mathrm{i}\,(c/2))]$.

Estimates on the real axis: We show the estimate on $B_1(\theta,\lambda)$. Using Lemma 8, Lemma 9 and Lemma 12 we obtain $|B_1(\theta,\lambda)| \le C\nu^{-2} \cdot \frac{g^2\sqrt{\alpha}}{(E_{n,l}(\alpha)-\lambda)^2 + c^2g^4}$. It is easy to see that $\int \mathrm{d}\lambda \frac{g^2\sqrt{\alpha}}{(E_{n,l}(\alpha)-\lambda)^2+c^2g^4}$ is $\mathcal{O}(\sqrt{\alpha})$. The same estimates hold for $B_1(\bar{\theta},\lambda)$. The estimates on $B_{2,\pm}(\lambda)$ work analogously using Lemma 10 and Lemma 12.

Estimates on the contour C: We estimate the integral $\int_C |e^{-isz}||\tilde{f}_{+}(z) -$

$\tilde{f}_-(z)||\mathrm{d}z|$: Note that

$$\tilde{f}_-(z) = \frac{1}{E_{n,l}(\alpha) - z}\langle\phi(0), \phi(0)\rangle + \langle\phi(0), \frac{1}{E_{n,l}(\alpha) - z} Y_{n,l,-}(\alpha) \frac{1}{E_{n,l}(\alpha) - z - Y_{n,l,-}(\alpha)}\phi(0)\rangle$$

holds. Accordingly, the leading terms of $\tilde{f}_-(z)$ and $\tilde{f}_+(z)$ cancel, and it suffices to show that the remaining terms are at least of order $\sqrt{\alpha}$. It follows from Equation (22) and Lemma 11 that $\|Y_{n,l,\pm}(\alpha)\| \le Cg^2$. Thus we can estimate

$$|\langle\phi(0), \frac{1}{E_{n,l}(\alpha) - (\lambda - \mathrm{i}\,\delta_{n,l}(\alpha))} Y_{n,l,-}(\alpha) \times \frac{1}{E_{n,l}(\alpha) - (\lambda - \mathrm{i}\,\delta_{n,l}(\alpha)) - Y_{n,l,-}(\alpha)}\phi(0)\rangle| \le C \cdot \frac{g^2}{(E_{n,l}(\alpha) - \lambda)^2 + \delta_{n,l}(\alpha)^2}.$$

Since the contour C_3 has length $\mathcal{O}(\alpha^2)$, we estimate the integral over the expression above by $C\alpha$. Similar estimates hold on C_1, C_2, C_4 and C_5. The integral over $\tilde{f}_+(z)$ can be estimated analogously.
Pole-Term: The integral along C_0 over $f_-(z)$ yields the claimed leading term, the integral over $f_+(z)$ is zero.
Step 4: For $\lambda \in [E_{n,l'}(\alpha) - \delta_{n,l',-}(\alpha), E_{n,l'}(\alpha) + \delta_{n,l',+}(\alpha)]$ with $l' \neq l$ we observe that $\phi(\alpha) \in \operatorname{Ran} P^{(\alpha)}_{\mathrm{el},n,l}$ implies

$$\overline{P}_{n,l'}(\theta)\psi(\alpha;\theta) = (P^{(\alpha)}_{\mathrm{el},n,l'}(\theta) \otimes \chi_{H_\mathrm{f} \ge \rho_0} + \underline{P}^{(\alpha)}_{\mathrm{el},n,l'}(\theta) \otimes \mathbf{1}_\mathrm{f})\psi(\alpha;\theta) = \psi(\alpha;\theta)$$

and $P_{n,l'}(\theta)\psi(\alpha;\theta) = 0$, which in turn shows

$$\begin{aligned}
&f(\theta,\lambda) = \\
&= \big\langle \psi(\alpha;\bar\theta), \overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_\alpha(\theta)\overline{P}_{n,l'}(\theta) - \lambda\big)^{-1}\overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)P_{n,l'}(\theta) \\
&\times \big[\mathcal{F}_{P_{n,l'}(\theta)}(H_\alpha(\theta) - \lambda)\big]^{-1} \\
&\times P_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l}(\theta)H_\alpha(\theta)\overline{P}_{n,l'}(\theta) - \lambda\big)^{-1}\overline{P}_{n,l'}(\theta)\psi(\alpha;\theta)\big\rangle \\
&+ \big\langle \psi(\alpha;\bar\theta), \overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_\alpha(\theta)\overline{P}_{n,l'}(\theta) - \lambda\big)^{-1}\overline{P}_{n,l'}(\theta)\psi(\alpha;\theta)\big\rangle \\
&=: f_1(\theta,\lambda) + f_2(\theta,\lambda)
\end{aligned}$$

using (51) in Lemma 7, where $f_1(\theta,\lambda)$ is the first summand. Using the resolvent identity we find $f_1(\theta,\lambda) = f_{1,a}(\theta,\lambda) + f_{1,b}(\theta,\lambda) + f_{1,c}(\theta,\lambda)$ with

$$\begin{aligned}
&f_{1,a}(\theta,\lambda) := \big\langle \psi(\alpha;\bar\theta), \overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_{\alpha,0}(\theta)\overline{P}_{n,l'}(\theta) - \lambda\big)^{-1} \\
&\times \overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)P_{n,l'}(\theta)\mathcal{F}_{P_{n,l'}(\theta)}(H_\alpha(\theta) - \lambda)^{-1}P_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta) \\
&\times \big(\overline{P}_{n,l'}(\theta)H_{\alpha,0}(\theta)\overline{P}_{n,l'}(\theta) - \lambda\big)^{-1}\overline{P}_{n,l'}(\theta)\psi(\alpha;\theta)\big\rangle,
\end{aligned}$$

$$
\begin{aligned}
f_{1,b}(\theta,\lambda) := &-\big\langle \psi(\alpha;\bar\theta), \overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_{\alpha,0}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\\
&\times \overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_{\alpha}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\\
&\times \overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)P_{n,l'}(\theta)\big[\mathcal{F}_{P_{n,l'}(\theta)}(H_\alpha(\theta)-\lambda)\big]^{-1}P_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta)\\
&\times \big(\overline{P}_{n,l'}(\theta)H_{\alpha,0}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\overline{P}_{n,l'}(\theta)\psi(\alpha;\theta)\big\rangle\\
&-\big\langle \psi(\alpha;\bar\theta), \overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_{\alpha,0}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\\
&\times \overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)P_{n,l'}(\theta)\big[\mathcal{F}_{P_{n,l'}(\theta)}(H_\alpha(\theta)-\lambda)\big]^{-1}P_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta)\\
&\times \big(\overline{P}_{n,l'}(\theta)H_{\alpha}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta)\\
&\times \big(\overline{P}_{n,l'}(\theta)H_{\alpha,0}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\overline{P}_{n,l'}(\theta)\psi(\alpha;\theta)\big\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
&f_{1,c}(\theta,\lambda) := \big\langle \psi(\alpha;\bar\theta), \overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_{\alpha,0}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\\
&\times \overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_{\alpha}\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\\
&\times \overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)P_{n,l'}(\theta)\big[\mathcal{F}_{P_{n,l'}(\theta)}(H_\alpha(\theta)-\lambda)\big]^{-1}P_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta)\\
&\times \big(\overline{P}_{n,l'}(\theta)H_{\alpha}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta)\\
&\times \big(\overline{P}_{n,l'}(\theta)H_{\alpha,0}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\overline{P}_{n,l'}(\theta)\psi(\alpha;\theta)\big\rangle.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
f_{1,a}(\theta,\lambda) = &\frac{1}{(E_{n,l}(\alpha)-\lambda)^2}\langle\psi(\alpha;\bar\theta), W_{0,1}^{(\alpha)}(\theta)P_{n,l'}(\theta)\\
&\times\mathcal{F}_{P_{n,l'}(\theta)}(H_\alpha(\theta)-\lambda)^{-1}P_{n,l'}(\theta)W_{1,0}^{(\alpha)}(\theta)\psi(\alpha;\theta)\rangle.
\end{aligned}
$$

Lemma 5 c) and Lemma 12 a) imply

$$|f_{1,a}(\theta,\lambda)| \le \frac{1}{|E_{n,l}(\alpha)-\lambda|^2}\frac{g^2\rho_0^2}{g^2},$$

which shows

$$\int_{E_{n,l'}(\alpha)-\delta_{n,l',-}(\alpha)}^{E_{n,l'}(\alpha)+\delta_{n,l',+}(\alpha)} \mathrm{d}\lambda\, |f_{1,a}(\theta,\lambda)| \le \frac{\rho_0^2}{\alpha^2} = \mathcal{O}(\alpha^2).$$

In order to estimate $f_{1,b}(\theta,\lambda)$ it suffices to consider the first summand, which can be estimated according to

$$
\begin{aligned}
&\frac{1}{|E_{n,l}(\alpha)-\lambda|^2}|\langle\psi(\alpha;\bar\theta),\\
&\times P_{n,l}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_\alpha(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)P_{n,l'}(\theta)\\
&\times\big[\mathcal{F}_{P_{n,l'}(\theta)}(H_\alpha(\theta)-\lambda)\big]^{-1}P_{n,l'}(\theta)W_{1,0}^{(\alpha)}(\theta)\psi(\alpha;\theta)\rangle|\\
&\le \frac{C}{|E_{n,l}(\alpha)-\lambda|^2}\frac{g^2 g\rho_0}{g^2},
\end{aligned}
$$

where we used Lemma 5 c), Lemma 12 a) and Lemma 7 b) in the last step. It follows that

$$\int_{E_{n,l'}(\alpha)-\delta_{n,l',-}(\alpha)}^{E_{n,l'}(\alpha)+\delta_{n,l',+}(\alpha)} \mathrm{d}\lambda\, |f_{1,b}(\theta,\lambda)| \leq C\frac{g\rho_0}{\alpha^2} = \mathcal{O}(g) = \mathcal{O}(\alpha^{3/2}).$$

Eventually, we obtain by Lemma 12 a) and Lemma 7 b)

$$\begin{aligned}
|f_{1,c}(\theta,\lambda)| &= \frac{1}{|E_{n,l}(\alpha)-\lambda|^2}|\langle\psi(\alpha;\bar{\theta}), P_{n,l}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta) \\
&\times \big(\overline{P}_{n,l'}(\theta)H_\alpha(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)P_{n,l'}(\theta) \\
&\times \left[\mathcal{F}_{P_{n,l'}(\theta)}(H_\alpha(\theta)-\lambda)\right]^{-1} \\
&\times P_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta) \\
&\times \big(\overline{P}_{n,l'}(\theta)H_\alpha\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)P_{n,l}(\theta)\psi(\alpha;\theta)\rangle| \\
&\leq C\frac{1}{|E_{n,l}(\alpha)-\lambda|^2}\frac{g^4}{g^2}.
\end{aligned}$$

Integration yields

$$\int_{E_{n,l'}(\alpha)-\delta_{n,l',-}(\alpha)}^{E_{n,l'}(\alpha)+\delta_{n,l',+}(\alpha)} \mathrm{d}\lambda\, |f_{1,c}(\theta,\lambda)| \leq C\frac{g^2}{\alpha^2} = \mathcal{O}(\alpha).$$

Now, we have to treat the term $f_2(\theta,\lambda)$. Using the resolvent identity we find $f_2(\theta,\lambda) = f_{2,a}(\theta,\lambda) + f_{2,b}(\theta,\lambda) + f_{3,c}(\theta,\lambda)$, with

$$\begin{aligned}
f_{2,a}(\theta,\lambda) &:= \langle\psi(\alpha;\bar{\theta}), \overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_{\alpha,0}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\overline{P}_{n,l'}(\theta)\psi(\alpha;\theta)\rangle, \\
f_{2,b}(\theta,\lambda) &:= -\langle\psi(\alpha;\bar{\theta}), \overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_{\alpha,0}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1} \\
&\quad\times \overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta) \\
&\quad\times \big(\overline{P}_{n,l'}(\theta)H_{\alpha,0}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\overline{P}_{n,l'}(\theta)\psi(\alpha;\theta)\rangle
\end{aligned}$$

and

$$\begin{aligned}
f_{2,c}(\theta,\lambda) &:= \langle\psi(\alpha;\bar{\theta}), \overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_{\alpha,0}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1} \\
&\times \overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_\alpha(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1} \\
&\times \overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta)\big(\overline{P}_{n,l'}(\theta)H_{\alpha,0}(\theta)\overline{P}_{n,l'}(\theta)-\lambda\big)^{-1}\overline{P}_{n,l'}(\theta)\psi(\alpha;\theta)\rangle.
\end{aligned}$$

Using the dilation analyticity we obtain

$$f_{2,a}(\theta,\lambda) = \frac{1}{E_{n,l}(\alpha)-\lambda}\langle\psi(\alpha;0),\psi(\alpha;0)\rangle,$$

which implies $f_{2,a}(\bar{\theta},\lambda) - f_{2,a}(\theta,\lambda) = 0$. Moreover, we have

$$f_{2,b}(\theta,\lambda) = -\frac{1}{(E_{n,l}(\alpha)-\lambda)^2}\langle\psi(\alpha;\bar{\theta}), W^{(\alpha)}(\theta)\psi(\alpha;\theta)\rangle = 0$$

and

$$\begin{aligned}|f_{2,c}(\theta,\lambda)| = & \frac{1}{|E_{n,l}(\alpha)-\lambda|^2}|\langle\psi(\alpha;\bar{\theta}), P_{n,l}(\theta)W^{(\alpha)}(\theta)\overline{P}_{n,l'}(\theta) \\ & \times(\overline{P}_{n,l'}(\theta)H_\alpha(\theta)\overline{P}_{n,l'}(\theta)-\lambda)^{-1}\overline{P}_{n,l'}(\theta)W^{(\alpha)}(\theta)P_{n,l}(\theta)\psi(\alpha;\theta)\rangle| \\ & \leq C\frac{g^2}{|E_{n,l}(\alpha)-\lambda|^2},\end{aligned}$$

where we used Lemma 7 b) in the last step. Integration yields

$$\int_{E_{n,l'}(\alpha)-\delta_{n,l',-}(\alpha)}^{E_{n,l'}(\alpha)+\delta_{n,l',+}(\alpha)} \mathrm{d}\lambda\,|f_{2,c}(\theta,\lambda)| \leq C\frac{g^2}{\alpha^2} = \mathcal{O}(\alpha).$$

Step 5: For $\lambda \in [E_{n,n}(\alpha)+\delta_{n,n,+}(\alpha), E_n+\delta_{n,+}]$ and also for $\lambda \in [E_n - \delta_{n,-}, E_{n,1}(\alpha)-\delta_{n,1,-}(\alpha)]$ we have to proceed somewhat differently: We consider the first case only and make a case distinction.
1st Case: $1 < l \leq n$. Lemma 13 with $l' = 1$ implies $\|\mathcal{F}_{P_{n,l'}(\theta)}(H_\alpha(\theta)-z)^{-1}\| \leq \frac{C}{\sin\vartheta|\lambda - E_{n,l'}(\alpha)|-Cg^2} \leq \frac{C}{\alpha^2}$, which we use to estimate $f_1(\theta,\lambda)$. $f_2(\theta,\lambda)$ can be estimated as in Step 4. Note that for both the estimates on $f_1(\theta,\lambda)$ and on $f_2(\theta,\lambda)$ the integration limits have to be changed accordingly. Thus, we obtain as in Step 4

$$\Big|\int_{E_n-\delta_{n,-}}^{E_{n,1}(\alpha)-\delta_{n,1,-}(\alpha)} \mathrm{d}\lambda\, e^{-\mathrm{i}\,\lambda s}F(\lambda)[f(\overline{\theta},\lambda)-f(\theta,\lambda)]\Big| = \mathcal{O}(\alpha).$$

2nd case: $l = 1$. Using the resolvent identity we find

$$\begin{aligned}f(\theta,\lambda) = & \langle\psi(\alpha;\overline{\theta}), \mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta)-\lambda)^{-1}\psi(\alpha;\theta)\rangle \\ = & \langle\psi(\alpha;\overline{\theta}), [E_{n,l}(\alpha)-\lambda-Q_{n,l}^{(\alpha)}(\lambda;\theta)]^{-1}\psi(\alpha;\theta)\rangle \\ & - \langle\psi(\alpha;\overline{\theta}), [E_{n,l}(\alpha)-\lambda-Q_{n,l}^{(\alpha)}(\lambda;\theta)]^{-1} \\ & \times[\mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta)-\lambda) - (E_{n,l}(\alpha)-\lambda+e^{-\theta}\mathbf{1}_{\mathrm{el}}\otimes H_{\mathrm{f}} - Q_{n,l}^{(\alpha)}(\lambda;\theta))P_{n,l}(\theta)] \\ & \times\left[\mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta)-\lambda)\right]^{-1}\psi(\alpha;\theta)\rangle =: \tilde{f}(\theta,\lambda)+B(\theta,\lambda),\end{aligned}$$

where $\tilde{f}(\theta,\lambda)$ is the first summand. Lemma 13 yields the estimate $\|\mathcal{F}_{P_{n,l}(\theta)}(H_\alpha(\theta)-\lambda)^{-1}\| \leq \frac{C}{\sin\vartheta|\lambda-E_{n,l}(\alpha)|-Cg^2}$ and the same estimate for $[E_{n,l}(\alpha)-\lambda-Q_{n,l}^{(\alpha)}(\lambda;\theta)]^{-1}$. Thus, Lemma 8 implies

$$|B(\theta,\lambda)| \leq \frac{Cg^2\sqrt{\alpha}}{(\sin\vartheta|\lambda-E_{n,l}(\alpha)|-Cg^2)^2}$$

and finally

$$\int_{E_n-\delta_{n,-}}^{E_{n,1}(\alpha)-\delta_{n,1,-}(\alpha)} \mathrm{d}\lambda\, F(\lambda)|B(\theta,\lambda)| = \mathcal{O}(g) = \mathcal{O}(\alpha^{3/2})$$

with the same reasoning as in the non-relativistic case (Proof of [19, Theorem 1], Step 2). The same holds for $B(\bar{\theta}, \lambda)$.
To estimate $\tilde{f}(\theta, \lambda)$,we use

$$\begin{aligned}\tilde{f}(\theta,\lambda) &= \langle \psi(\alpha;\bar{\theta}), [E_{n,1}(\alpha)-\lambda]^{-1}\psi(\alpha;\theta)\rangle + \\ &+ \langle \psi(\alpha;\bar{\theta}), [E_{n,l}(\alpha)-\lambda]^{-1} Q_{n,1}^{(\alpha)}(\lambda;\theta)[E_{n,1}(\alpha)-\lambda-Q_{n,1}^{(\alpha)}(\lambda;\theta)]^{-1}\psi(\alpha;\theta)\rangle.\end{aligned}$$

The first summand cancels with the corresponding summand of $\tilde{f}(\bar{\theta}, \lambda)$. The second summand can be estimated by $g^2 \frac{C}{|E_{n,1}(\alpha)-\lambda|\,(\sin\vartheta|\lambda-E_{n,l}(\alpha)|-Cg^2)}$, which implies

$$\int_{E_n-\delta_{n,-}}^{E_{n,1}(\alpha)-\delta_{n,1,-}(\alpha)} \mathrm{d}\lambda\, F(\lambda)|\tilde{f}(\theta,\lambda)-\tilde{f}(\bar{\theta},\lambda)| = \mathcal{O}(g^{2/3}) = \mathcal{O}(\alpha)$$

as above. □

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