The Critical Values of Certain Dirichlet Series

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Abstract. We investigate the values of several types of Dirichlet series $D(s)$ for certain integer values of s, and give explicit formulas for the value $D(s)$ in many cases. The easiest types of D are Dirichlet L-functions and their variations; a somewhat more complex case involves elliptic functions. There is one new type that includes $\sum_{n=1}^{\infty}(n^2+1)^{-s}$ for which such values have not been studied previously.

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Introduction

By a Dirichlet character modulo a positive integer d we mean as usual a $\mathbf{C}$-valued function χ on $\mathbf{Z}$ such that $\chi(x) = 0$ if x is not prime to d, and χ induces a character on $(\mathbf{Z}/d\mathbf{Z})^{\times}$. In this paper we always assume that χ is primitive and nontrivial, and so $d > 1$. For such a χ we put

$$L(s,\,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}. \tag{0.1}$$

It is well known that if k is a positive integer such that $\chi(-1) = (-1)^k$, then $L(k,\,\chi)$ is π^k times an algebraic number, or equivalently, $L(1-k,\,\chi)$ is an algebraic number. In fact, there is a well-known formula, first proved by Hecke in [3]:

$$kd^{1-k}L(1-k,\,\chi) = -\sum_{a=1}^{d-1} \chi(a)B_k(a/d), \tag{0.2}$$

where $B_k(t)$ is the Bernoulli polynomial of degree k. Actually Hecke gave the result in terms of $L(k,\,\chi)$, but here we state it in the above form. Hecke's proof is based on a classical formula

$$B_k(t) = -k!(2\pi i)^{-k} \sum_{0\neq h\in\mathbf{Z}} h^{-k}\mathbf{e}(ht) \qquad (0 < k \in \mathbf{Z},\ 0 < t < 1). \tag{0.3}$$

There is also a well-known proof of (0.2), which is essentially the functional equation of $L(s, \chi)$ combined with a proof of (0.3). We will not discuss it in the present paper, as it is not particularly inspiring.

In [9] we gave many formulas for $L(1-k, \chi)$ different from (0.2). The primary purpose of the present paper is to give elementary proofs for some of them, as well as (0.2), and discuss similar values of a few more types of Dirichlet series. The point of our new proofs can be condensed to the following statement: *We find infinite sum expressions for $L(s, \chi)$, which are valid for all $s \in \mathbf{C}$ and so can be evaluated at $s = 1-k$, whereas the old proof of Hecke and our proofs in [9]* employ calculations at $s = k$ and involve the Gauss sum of $\overline{\chi}$.

To make our exposition smooth we put

$$\mathbf{e}(z) = \exp(2\pi i z) \qquad (z \in \mathbf{C}), \tag{0.4}$$

$$H = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}. \tag{0.5}$$

The three additional types of Dirichlet series we consider naturally involve a complex variable s, and are defined as follows:

$$D^{\nu}(s;\, a,\, p) = \sum_{-a \neq n \in \mathbf{Z}} (n+a)^{\nu} |n+a|^{-\nu-s} \mathbf{e}\big(p(n+a)\big), \tag{0.6}$$

where $a \in \mathbf{R}$, $p \in \mathbf{R}$, and $\nu = 0$ or 1,

$$\mathcal{L}_k(s,\, z) = \sum_{m \in \mathbf{Z}} \mathbf{e}(mr)(z+m)^{-k}|z+m|^{-2s} \qquad (k \in \mathbf{Z},\ r \in \mathbf{Q},\ z \in H), \tag{0.7}$$

$$\varphi_{\nu}(u,\, s;\, L) = \sum_{\alpha \in L} (u+\alpha)^{-\nu} |u+\alpha|^{\nu-2s}, \tag{0.8}$$

where L is a lattice in $\mathbf{C}$, $0 \leq \nu \in \mathbf{Z}$, and $u \in \mathbf{C}, \notin L$. We should also note

$$\mathfrak{E}(z,\, s) = \mathrm{Im}(z)^s \sum_{(m,\, n)} (mz+n)^{-k}|mz+n|^{-2s} \qquad (0 \leq k \in \mathbf{Z},\, z \in H), \tag{0.9}$$

where (m, n) runs over the nonzero elements of $\mathbf{Z}^2$. The value $\mathfrak{E}(z, \mu)$ for an integer μ such that $1-k \leq \mu \leq 0$ was already discussed in [9], and so it is not the main object of study in this paper, but we mention it because (0.8) is a natural analogue of (0.9). We will determine in Section 3 the value $\varphi_{\nu}(u, \kappa/2; L)$ for an integer κ such that $2-\nu \leq \kappa \leq \nu$, which may be called a *nearly holomorphic elliptic function.* Now (0.6) is closely connected with $L(s, \chi)$. In [9, Theorem 4.2] we showed that $D^{\nu}(k; a, p)$ for $0 < k \in \mathbf{Z}$ is elementary factors times the value of a generalized Euler polynomial $E_{c,k-1}(t)$ at $t = p$. In Section 2 we will reformulate this in terms of $D^{\nu}(1-k; a, p)$. Finally, the nature of the series of (0.7) is quite different from the other types. We will show in Section 4 that $i^k\mathcal{L}_k(\beta, z)$ is a $\mathbf{Q}$-rational expression in π, $\mathbf{e}(z/N)$, and $\mathrm{Im}(z)$, if $\beta \in \mathbf{Z}$ and $-k < \beta \leq 0$, where N is the smallest positive integer such that $Nr \in \mathbf{Z}$. Similar results will also be given under other conditions on β. In the final section we will make some comments in the case where the base field is an algebraic number field.

1. $L(1-k,\,\chi)$

1.1. We start with an elementary proof of (0.2). Strange as it may sound, the main idea is the binomial theorem. We first note

$$B_n(t)=\sum_{\nu=0}^{n}\binom{n}{\nu}B_\nu t^{n-\nu}\qquad (0\le n\in\mathbf{Z}),\tag{1.1}$$

$$B_0=1,\quad \zeta(0)=-1/2=B_1,\tag{1.2a}$$

$$n\zeta(1-n)=-B_n\qquad (1<n\in\mathbf{Z}),\tag{1.2b}$$

where B_n is the nth Bernoulli number. Formulas (1.1) and (1.2a) are well-known; (1.2b) is usually given only for even n, but actually true also for odd n, since $\zeta(-2m)=0=B_{2m+1}$ for $0<m\in\mathbf{Z}$.

To prove (0.2), we first make a trivial calculation:

$$\begin{aligned}L(s,\,\chi)-\sum_{a=1}^{d-1}\chi(a)a^{-s}&=\sum_{m=1}^{\infty}\sum_{a=1}^{d-1}\chi(dm+a)(dm+a)^{-s}\\&=\sum_{m=1}^{\infty}\sum_{a=1}^{d-1}\chi(a)(dm)^{-s}\left(1+\frac{a}{dm}\right)^{-s}.\end{aligned}$$

Now we apply the binomial theorem to $(1+X)^{-s}$. Thus the last double sum equals

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{a=1}^{d-1}\chi(a)(dm)^{-s}\sum_{r=0}^{\infty}\binom{-s}{r}\left(\frac{a}{dm}\right)^{r}\\&=\sum_{r=0}^{\infty}\binom{-s}{r}\sum_{m=1}^{\infty}m^{-s-r}d^{-s}\sum_{a=1}^{d-1}\chi(a)(a/d)^{r},\end{aligned}\tag{1.3}$$

where

$$\binom{\tau}{r}=\frac{\tau(\tau-1)\ldots(\tau-r+1)}{r!},$$

which is of course understood to be 1 if $r=0$. So far our calculation is formal, but can be justified at least for $\mathrm{Re}(s)>1$. Indeed, put $\mathrm{Re}(s)=\sigma$ and $|s|=\alpha$. Then

$$\left|\binom{-s}{r}\right|\le\frac{\alpha(\alpha+1)\ldots(\alpha+r-1)}{r!}=(-1)^r\binom{-\alpha}{r}.\tag{1.4}$$

Therefore the triple sum obtained from (1.3) by taking the absolute value of each term is majorized by

$$\begin{aligned}&\sum_{a=1}^{d-1}\sum_{r=0}^{\infty}\sum_{m=1}^{\infty}m^{-\sigma-r}d^{-\sigma}\binom{-\alpha}{r}\left(\frac{-a}{d}\right)^{r}\\&\le\zeta(\sigma)d^{-\sigma}\sum_{a=1}^{d-1}\sum_{r=0}^{\infty}\binom{-\alpha}{r}\left(\frac{-a}{d}\right)^{r}=\zeta(\sigma)d^{-\sigma}\sum_{a=1}^{d-1}\left(1-\frac{a}{d}\right)^{-\alpha}\end{aligned}$$

if $\sigma > 1$. Thus, for $\mathrm{Re}(s) > 1$, (1.3) can be justified, and so

$$L(s, \chi) - \sum_{a=1}^{d-1} \chi(a)a^{-s} = \sum_{r=0}^{\infty} \binom{-s}{r} \zeta(s+r)d^{-s} \sum_{a=1}^{d-1} \chi(a)(a/d)^r. \tag{1.5}$$

We can show that the last sum $\sum_{r=0}^{\infty}$ defines a meromorphic function in s on the whole $\mathbf{C}$. For that purpose given $s \in \mathbf{C}$, take a positive integer μ so that $\mathrm{Re}(s) > -\mu$ and decompose the sum into $\sum_{r=0}^{\mu+1}$ and $\sum_{r=\mu+2}^{\infty}$. There is no problem about the first sum, as it is finite. As for the latter, we have $|\zeta(s+r)| \le \zeta(2)$ for $r \ge \mu + 2$. Putting $\varepsilon = (d-1)/d$, we have $|\sum_{a=1}^{d-1} \chi(a)(a/d)^r| \le (d-1)\varepsilon^r$. Therefore for $\mathrm{Re}(s) > -\mu$ the infinite sum $\sum_{r=\mu+2}^{\infty}$ can be majorized by

$$d^{\mu}(d-1)\zeta(2) \sum_{r=0}^{\infty} \binom{-\alpha}{r} (-\varepsilon)^r = d^{\mu}(d-1)\zeta(2)(1-\varepsilon)^{-\alpha}.$$

This proves the desired meromorphy of the right-hand side of (1.5).

Now, for $0 < k \in \mathbf{Z}$, without assuming that $\chi(-1) = (-1)^k$, we evaluate (1.5) at $s = 1-k$. We easily see that $\binom{-s}{r} = 0$ and $\zeta(s+r)$ is finite for $s = 1-k$ if $r > k$. We have to be careful about the term for $r = k$, as $\zeta(s+k)$ has a pole at $s = 1-k$. Since

$$\lim_{s \to 1-k} \binom{-s}{k} \zeta(s+k) = \lim_{s \to 1-k} \binom{-s}{k} \frac{1}{s-1+k} = \frac{-1}{k}, \tag{1.6}$$

the term for $r = k$ at $s = 1-k$ produces $-(d^{k-1}/k)\sum_{a=1}^{d-1} \chi(a)(a/d)^k$. Thus the evaluation of (1.5) at $s = 1-k$ gives

$$\begin{aligned} kd^{1-k}L(1-k, \chi) &= k\sum_{a=1}^{d-1} \chi(a)(a/d)^{k-1} \\ &- \sum_{a=1}^{d-1} \chi(a)(a/d)^k + \sum_{r=0}^{k-1} k \binom{k-1}{r} \zeta(1+r-k) \sum_{a=1}^{d-1} \chi(a)(a/d)^r. \end{aligned} \tag{1.7}$$

By (1.2b) we have, for $0 \le r < k-1$,

$$k\binom{k-1}{r}\zeta(1+r-k) = \frac{-k}{k-r}\binom{k-1}{r} B_{k-r} = -\binom{k}{r} B_{k-r}.$$

The term for $r = k-1$ produces $k\zeta(0)\sum_{a=1}^{d-1} \chi(a)(a/d)^{k-1}$, which combined with the first term on the right-hand side of (1.7) gives $-kB_1 \sum_{a=1}^{d-1} \chi(a)(a/d)^{k-1}$. Thus we obtain

$$kd^{1-k}L(1-k, \chi) = -\sum_{r=0}^{k} \binom{k}{r} B_{k-r} \sum_{a=1}^{d-1} \chi(a)(a/d)^r,$$

which together with (1.1) proves (0.2). Notice that we did not assume that $\chi(-1) = (-1)^k$, and so we proved (0.2) for every positive integer k. If $\chi(-1) = (-1)^{k-1}$, we have $L(1-k, \chi) = 0$, which means that the right-hand side of (0.2) is 0 if $\chi(-1) = (-1)^{k-1}$. This can be proved more directly; see [9, (4.28)].

In the above calculation the term for $r = 0$ actually vanishes, as $\sum_{a=1}^{d-1} \chi(a) = 0$. However, we included the term for the following reason. In later subsections we will consider similar infinite sums with r ranging from 0 to ∞, of which the terms for $r = 0$ are not necessarily zero.

1.2. By the same technique as in §1.1 (that is, employing the binomial theorem) we will express $L(1-k,\,\chi)$ explicitly in terms of a polynomial Φ_{k-1} of degree $k-1$. Writing n for $k-1$, the polynomial is defined by

$$\Phi_n(t) = t^n - \sum_{\nu=1}^{[(n+1)/2]} \binom{n}{2\nu-1} (2^{2\nu}-1)\frac{B_{2\nu}}{\nu} t^{n+1-2\nu} \quad (0 \le n \in \mathbf{Z}), \tag{1.8}$$

where B_ν denotes the Bernoulli number as before. We understand that $\Phi_0(t) = 1$. We will eventually show that Φ_n is the classical Euler polynomial of degree n, but we prove Theorem 1.4 below with this definition of Φ_n, with no knowledge of the Euler polynomial. We first prove:

Lemma 1.3. *Let χ be a primitive Dirichlet character of conductor $4d_0$ with $0 < d_0 \in \mathbf{Z}$. Then $\chi(a - 2d_0) = -\chi(a)$ for every $a \in \mathbf{Z}$.*

Proof. We may assume that a is prime to $2d_0$, as the desired equality is trivial otherwise. Then we can find an integer b such that $ab - 1 \in 4d_0\mathbf{Z}$, and we have $\chi(a - 2d_0) = \chi(a)\chi(1 - 2d_0b)$. Since $(1 - 2d_ob)^2 - 1 \in 4d_0\mathbf{Z}$, we have $\chi(1 - 2d_ob) = \pm 1$. Suppose $\chi(1 - 2d_ob) = 1$; let $x = 1 - 2d_0y$ with $y \in \mathbf{Z}$. Then $x^b - (1 - 2d_0b)^y \in 4d_0\mathbf{Z}$, and so $\chi(x)^b = 1$. Thus $\chi(x) = 1$, as b is odd. This shows that the conductor of χ is a divisor of $2d_0$, a contradiction. Therefore $\chi(1 - 2d_0b) = -1$, which proves the desired fact.

Theorem 1.4. *Let χ be a nontrivial primitive Dirichlet character modulo d, and let k be a positive integer such that $\chi(-1) = (-1)^k$.*

(i) *If $d = 2q + 1$ with $0 < q \in \mathbf{Z}$, then*

$$L(1-k,\,\chi) = \frac{d^{k-1}}{2^k\chi(2) - 1} \sum_{b=1}^{q} (-1)^b \chi(b) \Phi_{k-1}(b/d). \tag{1.9}$$

(ii) *If $d = 4d_0$ with $1 < d_0 \in \mathbf{Z}$, then*

$$L(1-k,\,\chi) = (2d_0)^{k-1} \sum_{a=1}^{d_0-1} \chi(a) \Phi_{k-1}(2a/d). \tag{1.10}$$

Before proving these, we note that these formulas are *better than* (0.2) in the sense that $\Phi_{k-1}(t)$ is of degree $k-1$, whereas $B_k(t)$ is of degree k.

Proof. We first put

$$Z(s) = \sum_{n=1}^{\infty} (-1)^n n^{-s}, \qquad \Lambda(s) = \sum_{n=1}^{\infty} (-1)^n \chi(n) n^{-s}.$$

We easily see that

$$\Lambda(s) + L(s, \chi) = 2\sum_{n=1}^{\infty} \chi(2n)(2n)^{-s} = \chi(2)2^{1-s}L(s, \chi),$$

and a similar equality holds for $Z(s)$. Thus

$$Z(s) = \zeta(s)(2^{1-s} - 1), \qquad \Lambda(s) = L(s, \chi)\bigl\{\chi(2)2^{1-s} - 1\bigr\}.$$

We prove (i) by computing $\Lambda(1-k)$ for a given k in the same elementary way as we did in §1.1. With q as in (i) we observe that every positive integer m not divisible by d can be written uniquely $m = nd + a$ with $0 \le n \in \mathbf{Z}$ or $m = nd - a$ with $0 < n \in \mathbf{Z}$, where in either case a is in the range $0 < a \le q$. Therefore

$$\Lambda(s) = \sum_{a=1}^{q}(-1)^a\chi(a)a^{-s}$$
$$+\sum_{a=1}^{q}\sum_{n=1}^{\infty}\Bigl\{(-1)^{nd+a}\chi(nd+a)(nd+a)^{-s} + (-1)^{nd-a}\chi(nd-a)(nd-a)^{-s}\Bigr\}.$$

The last double sum can be written

$$\sum_{a=1}^{q}\sum_{n=1}^{\infty}(-1)^{n+a}d^{-s}n^{-s}\left\{\chi(a)\left(1+\frac{a}{nd}\right)^{-s} + \chi(-a)\left(1-\frac{a}{nd}\right)^{-s}\right\}.$$

Applying the binomial theorem to $(1 \pm X)^{-s}$, we obtain

$$\begin{aligned}
&\Lambda(s) - \sum_{a=1}^{q}(-1)^a\chi(a)a^{-s}\\
&\quad= \sum_{n=1}^{\infty}\sum_{r=0}^{\infty}(-1)^n(dn)^{-r-s}\binom{-s}{r}\bigl\{1+(-1)^{r+k}\bigr\}\sum_{a=1}^{q}(-1)^a\chi(a)a^r\\
&\quad= \sum_{r=0}^{\infty}d^{-s}\binom{-s}{r}Z(s+r)\bigl\{1+(-1)^{r+k}\bigr\}\sum_{a=1}^{q}(-1)^a\chi(a)(a/d)^r.
\end{aligned}$$

By the same technique as in §1.1, we can justify this for $\mathrm{Re}(s) > 1$. We can even show that the last sum $\sum_{r=0}^{\infty}$ is absolutely convergent for every $s \in \mathbf{C}$ as follows. We first note that Z is an entire function. Take a positive integer μ and s so that $\mathrm{Re}(s) > -\mu$. Then for $r \ge \mu+2$ we have $|Z(s+r)| \le \zeta(2)$. Put $|s| = \alpha$. We have also $|\sum_{a=1}^{q}(-1)^a\chi(a)(a/d)^r| \le 2^{-r}q$. Therefore for $\mathrm{Re}(s) > -\mu$ the infinite sum $\sum_{r=\mu+2}^{\infty}$ can be majorized by

$$2d^{\mu}\zeta(2)q\sum_{r=0}^{\infty}\binom{-\alpha}{r}(-2)^{-r} = 2d^{\mu}\zeta(2)q(1-2^{-1})^{-\alpha}.$$

This proves the desired convergence of $\sum_{r=0}^{\infty}$. Substituting $1-k$ for s in the above equality, we obtain $\Lambda(1-k)$ as an infinite sum, which is actually a finite sum, because $\binom{k-1}{r} = 0$ if $r \ge k$. (This time, the term $r = k$ causes no problem.) Also, we need only those r such that $k-r \in 2\mathbf{Z}$. Putting $k-r = 2\nu$, we find that

$$\begin{aligned}
&d^{1-k}L(1-k,\,\chi)\bigl\{\chi(2)2^k-1\bigr\} = d^{1-k}\Lambda(1-k)\\
&=\sum_{a=1}^{q}(-1)^a\chi(a)(a/d)^{k-1}+2\sum_{\nu=1}^{[k/2]}\binom{k-1}{2\nu-1}Z(1-2\nu)\sum_{a=1}^{q}(-1)^a\chi(a)(a/d)^{k-2\nu}.
\end{aligned}$$

From (1.2b) we obtain

$$2\nu Z(1-2\nu)=2\nu(2^{2\nu}-1)\zeta(1-2\nu)=(1-2^{2\nu})B_{2\nu}. \tag{1.11}$$

Using this expression for $Z(1-2\nu)$, we obtain a formula for $L(1-k,\,\chi)$. Then comparison of it with our definition of Φ_n proves (1.9).

Next, let $d=4d_0$ with $1<d_0\in\mathbf{Z}$ as in (ii). Observe that the set of all positive integers greater than d_0 and not divisible by d_0 is the disjoint union of the sets

$$\bigl\{4\nu d_0\pm a\,\big|\,0<a<d_0,\ 0<\nu\in\mathbf{Z}\bigr\}\sqcup\bigl\{(4\nu+2)d_0\pm a\,\big|\,0<a<d_0,\ 0\le\nu\in\mathbf{Z}\bigr\}.$$

Clearly $\chi(4\nu d_0\pm a)=\chi(\pm a)$; also $\chi\bigl((4\nu+2)d_0\pm a\bigr)=-\chi(\pm a)$ by Lemma 1.3. Therefore we have

$$\begin{aligned}
L(s,\,\chi)&=\sum_{a=1}^{d_0-1}\chi(a)a^{-s}\\
&+\sum_{\nu=1}^{\infty}\sum_{a=1}^{d_0-1}\bigl\{\chi(a)(4\nu d_0+a)^{-s}+\chi(-a)(4\nu d_0-a)^{-s}\bigr\}\\
&-\sum_{\nu=0}^{\infty}\sum_{a=1}^{d_0-1}\bigl\{\chi(a)\bigl((4\nu+2)d_0+a\bigr)^{-s}+\chi(-a)\bigl((4\nu+2)d_0-a\bigr)^{-s}\bigr\}.
\end{aligned}$$

Employing the binomial theorem in the same manner as before, we have

$$\begin{aligned}
&L(s,\,\chi)-\sum_{a=1}^{d_0-1}\chi(a)a^{-s}\\
&=\sum_{\nu=1}^{\infty}\sum_{r=0}^{\infty}\binom{-s}{r}(4\nu d_0)^{-s-r}\bigl\{1+(-1)^{k+r}\bigr\}\sum_{a=1}^{d_0-1}\chi(a)a^r\\
&-\sum_{\nu=0}^{\infty}\sum_{r=0}^{\infty}\binom{-s}{r}\bigl((4\nu+2)d_0\bigr)^{-s-r}\bigl\{1+(-1)^{k+r}\bigr\}\sum_{a=1}^{d_0-1}\chi(a)a^r.
\end{aligned}$$

Notice that $\sum_{\nu=1}^{\infty}(4\nu)^{-s}-\sum_{\nu=0}^{\infty}(4\nu+2)^{-s}=2^{-s}Z(s)$. Therefore

$$L(s,\,\chi)=\sum_{a=1}^{d_0-1}\chi(a)a^{-s}+\sum_{r=0}^{\infty}\binom{-s}{r}(2d_0)^{-s-r}Z(s+r)\bigl\{1+(-1)^{k+r}\bigr\}\sum_{a=1}^{d_0-1}\chi(a)a^r.$$

The validity of this formula for all $s\in\mathbf{C}$ can be proved in the same way as in the previous case. The last infinite sum $\sum_{r=0}^{\infty}$ evaluated at $s=1-k$ becomes a finite sum $\sum_{r=0}^{k-1}$, which is actually extended only over those r such that $k-r=2\nu$ with $\nu\in\mathbf{Z}$. Therefore, using (1.11), we obtain (1.10).

1.5. Let us now show that $\varPhi_n$ coincides with the classical Euler polynomial. In [9, (4.2)] we defined polynomials $E_{c,\,n}(t)$ for $c = -\mathbf{e}(\alpha)$ with $\alpha \in \mathbf{R}, \notin \mathbf{Z}$, by

$$\frac{(1+c)e^{tz}}{e^z+c} = \sum_{n=0}^{\infty} \frac{E_{c,n}(t)}{n!} z^n. \tag{1.12}$$

If $c = 1$, the polynomial $E_{1,n}(t)$ is the classical Euler polynomial of degree n. Our task is to prove

$$E_{1,n} = \varPhi_n. \tag{1.13}$$

We first note here some basic formulas:

$$E_{c,n}(t) = (1+c^{-1})n!(2\pi i)^{-n-1} \sum_{h\in\mathbf{Z}} (h+\alpha)^{-n-1}\mathbf{e}\big((h+\alpha)t\big) \tag{1.14}$$

$$(c = -\mathbf{e}(\alpha),\ \alpha \in \mathbf{R},\ \notin \mathbf{Z};\ 0 < t < 1 \text{ if } n = 0;\ 0 \le t \le 1 \text{ if } 0 < n \in \mathbf{Z}),$$

$$E_{c,n}(t+r) = \sum_{k=0}^{n} \binom{n}{k} E_{c,k}(r)t^{n-k} \qquad (0 \le n \in \mathbf{Z}), \tag{1.15}$$

$$E_{1,0}(0) = 1, \quad E_{1,n}(0) = 2(1-2^{n+1})(n+1)^{-1}B_{n+1} \quad (0 < n \in \mathbf{Z}). \tag{1.16}$$

Formula (1.14) was given in [9, (4.5)]; the sum $\sum_{h\in\mathbf{Z}}$ means $\lim_{m\to\infty}\sum_{|h|\le m}$ if $n = 0$. Replacing t in (1.12) by $t+r$ and making an obvious calculation, we obtain (1.15). We have $E_{c,0}(t) = 1$ as noted in [9, (4.3h)]. Clearly $E_{1,n}(0) = 0$ if n is even. Assuming n to be odd, take $t = 0$ and $\alpha = 1/2$ in (1.14), and recall that $2\cdot m!(2\pi i)^{-m}\zeta(m) = -B_m$ if $0 < m \in 2\mathbf{Z}$. Then we obtain $E_{1,n}(0)$ as stated in (1.16). Taking $r = 0$ in (1.15) and using (1.16), we obtain (1.13). The value $E_{c,n}(0)$ for an arbitrary c is given in [9, (4.6)].

1.6. In [9, Theorem 4.14] we proved, for χ, d, and k as in Theorem 1.4,

$$L(1-k,\,\chi) = \frac{d^{k-1}}{2^k - \overline{\chi}(2)} \sum_{a=1}^{q} \chi(a)E_{1,k-1}(2a/d), \tag{1.17}$$

where $q = [(d-1)/2]$, and derived (i) and (ii) above, with $E_{1,k-1}$ in place of $\varPhi_{k-1}$, from (1.17). In fact, (i) and (ii) combined are equivalent to (1.17). Though this is essentially explained in [9, p. 36], here let us show that (1.17) for even d follows from (ii). With $d = 4d_0$ as before, we have $[(d-1)/2] = 2d_0 - 1$ and

$$\begin{aligned}\sum_{a=1}^{2d_0-1} &\chi(a)\varPhi_{k-1}(2a/d)\\ &= \sum_{a=1}^{d_0-1}\Big\{\chi(a)\varPhi_{k-1}(2a/d) + \chi(2d_0-a)\varPhi_{k-1}\big(2(2d_0-a)/d\big)\Big\}.\end{aligned}$$

We have $E_{1,n}(1-t) = (-1)^n E_{1,n}(t)$ as noted in [9, (4.3f)]. This combined with (1.13) shows that $\varPhi_{k-1}(1-t) = (-1)^{k-1}\varPhi_{k-1}(t)$. By Lemma 1.3, we have $\chi(2d_0-a) = -\chi(-a) = (-1)^{k+1}\chi(a)$, and so the last sum equals

$$2\sum_{a=1}^{d_0-1}\chi(a)\varPhi_{k-1}(2a/d).$$

Therefore (1.17) follows from (1.10) if $d=4d_0$. Similarly we can derive(1.17) for odd d from (1.9), which, in substance, is shown in the last paragraph of [9, p. 36].

1.7. Our technique is applicable even to $\zeta(1-k)$. Instead of $\zeta(s)$ we consider $W(s)=\sum_{m=0}^{\infty}(2m+1)^{-s}$. We have clearly

$$\begin{aligned}W(s)&=1+\sum_{m=1}^{\infty}(2m+1)^{-s}=1+\sum_{m=1}^{\infty}(2m)^{-s}\Bigl(1+\frac{1}{2m}\Bigr)^{-s}\\&=1+\sum_{m=1}^{\infty}(2m)^{-s}\sum_{r=0}^{\infty}\binom{-s}{r}(2m)^{-r}=1+\sum_{r=0}^{\infty}\zeta(s+r)\binom{-s}{r}2^{-s-r}.\end{aligned}$$

We evaluate this at $s=1-k$ with $0<k\in\mathbf{Z}$. Our calculation is similar to that of §1.1; we use (1.6) for determining the term for $r=k$, which produces $-(2k)^{-1}$. Thus

$$(1-2^{k-1})\zeta(1-k)=W(1-k)=1-\frac{1}{2k}+\sum_{r=0}^{k-1}\binom{k-1}{r}2^{k-1-r}\zeta(1-k+r).$$

Taking $k=1$, we find a well-known fact $\zeta(0)=-1/2$. Also, $\zeta(1-k)$ appears on both sides. Therefore, putting $k-r=t$ and rearranging our sum, we obtain

$$(1-2^{k})\zeta(1-k)=\frac{k-1}{2k}+\sum_{t=2}^{k-1}\binom{k-1}{t-1}2^{t-1}\zeta(1-t).$$

This holds for every even or odd integer $k>1$. Recall that $\zeta(-m)=0$ for $0<m\in 2\mathbf{Z}$. Thus, taking $k=2n$ with $0<n\in\mathbf{Z}$, we obtain a formula for $\zeta(1-2n)$ as a linear combination of $\zeta(1-2\nu)$ for $1\le\nu<n$ (which is 0 if $n=1$) plus a constant as follows:

$$(1-2^{2n})\zeta(1-2n)=\frac{2n-1}{4n}+\sum_{\nu=1}^{n-1}\binom{2n-1}{2\nu-1}2^{2\nu-1}\zeta(1-2\nu).\tag{1.18}$$

Similarly, taking $k=2n+1$ and putting $t=2\nu$, we obtain

$$\sum_{\nu=1}^{n}\binom{2n}{2\nu-1}2^{2\nu-1}\zeta(1-2\nu)=\frac{-n}{2n+1}.\tag{1.19}$$

Either of these equalities (1.18) and (1.19) expresses $\zeta(1-2n)$ as a $\mathbf{Q}$-linear combination of $\zeta(1-2\nu)$ for $1\le\nu<n$ plus a constant. The two expressions are different, as can easily be seen.

In [9, (11.8)] we gave a similar recurrence formula which can be written

$$4(1-2^{n+1})\zeta(-n)=1+2\sum_{k=2}^{n}\binom{n}{k-1}(2^k-1)\zeta(1-k)\qquad(0<n\in\mathbf{Z}).\tag{1.20}$$

Taking n to be even or odd, we again obtain two different recurrence formulas for $\zeta(1-2n)$. It should be noted that the technique of using the binomial theorem is already in §68 of Landau [5], in which $(s-1)\zeta(s)$ is discussed, while we employ $W(s)$.

2. Extending the parameters c and n in $E_{c,n}$

2.1. The function $E_{c,n}(t)$ is a polynomial in t of degree n, and involves $c = -\mathbf{e}(\alpha)$ with $\alpha \in \mathbf{R}$. We now extend this in two ways: first, we take $\alpha \in \mathbf{C}, \notin \mathbf{Z}$; second, we consider $(h+\alpha)^{-s}$ instead of $(h+\alpha)^{-n-1}$. The first case is simpler. Since $E_{c,n}(t)$ is a polynomial in t and $(1+c)^{-1}$ as noted in [9, p. 26], we can define a function $\mathcal{E}_n(\alpha, t)$ by

$$\mathcal{E}_n(\alpha,\, t) = E_{c,n}(t), \quad c = -\mathbf{e}(\alpha),\;\; \alpha \in \mathbf{C}, \notin \mathbf{Z},\;\; 0 \le n \in \mathbf{Z}. \tag{2.1}$$

This is a polynomial in t, whose coefficients are holomorphic functions in $\alpha \in \mathbf{C}, \notin \mathbf{Z}$. Now equality (1.14) can be extended to

$$\mathcal{E}_n(\alpha,\, t) = \big(1-\mathbf{e}(-\alpha)\big)n!(2\pi i)^{-n-1}\sum_{h\in\mathbf{Z}}(h+\alpha)^{-n-1}\mathbf{e}\big((h+\alpha)t\big) \tag{2.2}$$

for all $\alpha \in \mathbf{C}, \notin \mathbf{Z}$, where $0 < t < 1$ if $n = 0$, and $0 \le t \le 1$ if $n > 0$. Indeed, if $n > 0$, the right-hand side is absolutely convergent, and defines a holomorphic function. Since (2.2) holds for $\alpha \in \mathbf{R}, \notin \mathbf{Z}$, we obtain (2.2) as expected. If $n = 0$, we have to consider $\lim_{m\to\infty}\sum_{|h|\le m}(h+\alpha)^{-1}\mathbf{e}\big((h+\alpha)t\big)$. Clearly

$$\sum_{h=-m}^{m}\frac{\mathbf{e}(ht)}{\alpha+h} = \frac{1}{\alpha} + \sum_{h=1}^{m}\frac{2\alpha\cdot\cos(2\pi ht)}{\alpha^2-h^2} + 2i\sum_{h=1}^{m}\frac{h\cdot\sin(2\pi ht)}{h^2-\alpha^2}.$$

The last sum on the right-hand side equals

$$\sum_{h=1}^{m}\frac{\sin(2\pi ht)}{h} + \sum_{h=1}^{m}\frac{\sin(2\pi ht)\alpha^2}{h(h^2-\alpha^2)}.$$

It is well-known that the first sum tends to a finite value as $m \to \infty$. Obviously the last sum converges to a holomorphic function in $\alpha \in \mathbf{C}, \notin \mathbf{Z}$ as $m \to \infty$. Thus we can justify (2.2) for $n = 0$.

Formula (2.2) for $n = 0$ (with $-\alpha$ in place of α) can be written

$$\frac{\mathbf{e}(t\alpha)}{1-\mathbf{e}(\alpha)} = \frac{1}{2\pi i}\sum_{h\in\mathbf{Z}}\frac{\mathbf{e}(th)}{h-\alpha} \qquad (\alpha \in \mathbf{C}, \notin \mathbf{Z},\; 0 < t < 1). \tag{2.3}$$

This was first given by Kronecker [4].

2.2. We next ask if the power $(h+a)^{-n-1}$ in (1.14) can be replaced by $(h+a)^{-s}$ with a complex parameter s. Since $h+a$ can be negative, $(h+a)^{-s}$ is not suitable. Thus, for $s \in \mathbf{C}$, $a \in \mathbf{R}$, $p \in \mathbf{R}$, and $\nu = 0$ or 1 we put

$$D^\nu(s;\, a,\, p) = \sum_{-a\ne n\in\mathbf{Z}}(n+a)^\nu|n+a|^{-\nu-s}\mathbf{e}\big(p(n+a)\big), \tag{2.4}$$

$$T^\nu(s;\,a,\,p) = \Gamma\big((s+\nu)/2\big)\pi^{-(s+\nu)/2}D^\nu(s;\,a,\,p). \tag{2.5}$$

Clearly the infinite series of (2.4) is absolutely convergent for $\mathrm{Re}(s) > 1$, and defines a holomorphic function of s there. Notice that if $k - \nu \in 2\mathbf{Z}$, then $D^\nu(k;\,0,\,t) = \sum_{0\neq n\in\mathbf{Z}} n^{-k}\mathbf{e}(nt)$, which is the infinite sum of (0.3). Thus the Bernoulli polynomials are included in our discussion.

Theorem 2.3. *The function $T^\nu(s;\,a,\,p)$ can be continued as a meromorphic function of s to the whole $\mathbf{C}$. It is entire if $\nu = 1$. If $\nu = 0$, then $T^0(s;\,a,\,p)$ is*

$$\frac{-2\delta(a)}{s} + \frac{2\mathbf{e}(ap)\delta(p)}{s-1}$$

plus an entire function, where $\delta(x) = 1$ if $x \in \mathbf{Z}$ and $\delta(x) = 0$ if $x \notin \mathbf{Z}$. Moreover,

$$T^\nu(1-s;\,a,\,p) = i^{-\nu}\mathbf{e}(ap)T^\nu(s;\,-p,\,a). \tag{2.6}$$

Proof. Put $\varphi(x) = x^\nu\mathbf{e}(-x^2z^{-1}/2 + px)$ for $x \in \mathbf{R}$ and $z \in H$. Denote by $\widehat{\varphi}$ the Fourier transform of φ. Then from [9, (2.25)] we easily obtain $\widehat{\varphi}(x) = i^{-\nu}(-iz)^\kappa(x-p)^\nu\mathbf{e}\big((x-p)^2z/2\big)$, where $\kappa = \nu + 1/2$. Put also

$$f(z) = \sum_{n\in\mathbf{Z}}(n+a)^\nu\mathbf{e}\big((n+a)^2z/2 + p(n+a)\big),$$

and $f^\#(z) = (-iz)^{-\kappa}f(-z^{-1})$. Then $f(-z^{-1}) = \sum_{n\in\mathbf{Z}}\varphi(n+a)$, which equals $\sum_{m\in\mathbf{Z}}\mathbf{e}(ma)\widehat{\varphi}(m)$ by virtue of the Poisson summation formula. In this way we obtain

$$f^\#(z) = i^{-\nu}\sum_{m\in\mathbf{Z}}\mathbf{e}(ma)(m-p)^\nu\mathbf{e}\big((m-p)^2z/2\big).$$

Now $T^\nu(2s-\nu;\,a,\,p)$ is the Mellin transform of $f(iy)$, and so we obtain our theorem by the general principle of Hecke, which is given as Theorem 3.2 in [9].

Theorem 2.4. *For $\nu = 0$ or 1, $0 \le a \le 1$, and a positive integer k such that $k - \nu \in 2\mathbf{Z}$ we have*

$$D^0(0;\,a,\,p) = -\delta(a), \tag{2.7}$$

$$D^\nu(\nu - 2m;\,a,\,p) = 0 \quad \text{if} \quad 0 < m \in \mathbf{Z}, \tag{2.8}$$

$$D^\nu(1-k;\,a,\,p) = 2(2\pi i)^{-k}(k-1)!\mathbf{e}(ap)D^\nu(k;\,-p,\,a), \tag{2.9}$$

$$D^\nu(1-k;\,a,\,p) = -2\mathbf{e}(ap)B_k(a)/k \quad \text{if} \quad p \in \mathbf{Z}, \tag{2.10}$$

$$D^\nu(1-k;\,a,\,p) = \frac{2\mathbf{e}(ap)}{1-\mathbf{e}(p)}E_{c,k-1}(a) \quad \text{if} \quad p \notin \mathbf{Z}, \tag{2.11}$$

where $c = -\mathbf{e}(-p)$, and we have to assume that $0 < a < 1$ in (2.10) and (2.11) if $k = 1$.

Proof. By Theorem 2.3, $\big[sT^0(s;\,a,\,p)\big]_{s=0} = -2\delta(a)$, from which we obtain (2.7). Next, let $0 < m \in \mathbf{Z}$. Since $\Gamma\big((s+\nu)/2\big)D^\nu(s;\,a,\,p)$ is finite and

$\Gamma\big((s+\nu)/2\big)$ has a pole at $s = \nu - 2m$, we obtain (2.8). We easily see that $\Gamma(1/2-m) = \pi^{1/2}(-2)^m \prod_{t=1}^{m}(2t-1)^{-1}$. Therefore from (2.6) we obtain (2.9). If $p \in \mathbf{Z}$, then $D^\nu(k;\, -p,\, a) = \sum_{0\neq n\in\mathbf{Z}} n^{-k}\mathbf{e}(an)$. The well-known classical formula, stated in [9, (4.9)] (and also as (0.3)), shows that the last sum equals $-(2\pi i)^k B_k(a)/k!$ for $0 \le a \le 1$ if $k > 1$, and for $0 < a < 1$ if $k = 1$. If $p \notin \mathbf{Z}$, then $D^\nu(k;\, -p,\, a) = \sum_{n\in\mathbf{Z}}(n-p)^{-k}\mathbf{e}\big(a(n-p)\big)$. By (1.14), this equals $(1+c^{-1})^{-1}(2\pi i)^k E_{c,k-1}(a)/(k-1)!$, where $c = -\mathbf{e}(-p)$, under the same condition on a. Combining these with (2.9), we obtain (2.10) and (2.11).

We note here a special case of (2.10):

$$D^\nu(1-k;\, 0,\, 0) = \begin{cases} -2B_k/k & \text{if } k > 1, \\ 0 & \text{if } k = 1. \end{cases} \tag{2.12}$$

It should be noted that $D^1(s;\, 0,\, 0) = 0$.

3. Nearly holomorphic elliptic functions

3.1. Let L be a lattice in $\mathbf{C}$. As an analogue of (2.4) we put

$$\varphi_\nu(u,\, s;\, L) = \sum_{\alpha\in L}(u+\alpha)^{-\nu}|u+\alpha|^{\nu-2s} \tag{3.1}$$

for $0 \le \nu \in \mathbf{Z},\ u \in \mathbf{C},\notin L$, and $s \in \mathbf{C}$. Clearly

$$\varphi_\nu(\lambda u,\, s;\, \lambda L) = \lambda^{-\nu}|\lambda|^{\nu-2s}\varphi_\nu(u,\, s;\, L) \quad \text{for every} \quad \lambda \in \mathbf{C}^\times, \tag{3.2a}$$

$$\varphi_\nu(u+\alpha,\, s;\, L) = \varphi_\nu(u,\, s;\, L) \quad \text{for every} \quad \alpha \in L. \tag{3.2b}$$

If L is a $\mathbf{Z}$-lattice in an imaginary quadratic field K and $u \in K$, (3.1) is the same as the series of [9, (7.1)]. The analytic properties of the series that we proved there can easily be extended to the case of (3.1). First of all, the right-hand side of (3.1) is absolutely convergent for $\mathrm{Re}(s) > 1$, and defines a holomorphic function of s there.

Theorem 3.2. *Put $\Phi(u,\, s) = \pi^{-s}\Gamma(s+\nu/2)\varphi_\nu(u,\, s;\, L)$. Then $\Phi(u,\, s)$ can be continued to the whole s-plane as a meromorphic function in s, which is entire if $\nu > 0$. If $\nu = 0$, then $\Phi(u,\, s)$ is an entire function of s plus $v(L)^{-1}/(s-1)$, where $v(L) = \mathrm{vol}(\mathbf{C}/L)$. Moreover, $\Phi(u,\, s)$ is a C^∞ function in u, except when $\nu = 0$ and $s = 1$, and each derivative $(\partial/\partial u)^a(\partial/\partial\overline{u})^b\Phi(u,\, s)$ is meromorphic in s on the whole $\mathbf{C}$.*

Proof. This can be proved by the same argument as in [9, §7.2], except for the differentiability with respect to u and the last statement about the derivatives, which can be shown as follows. As shown in the proof of [9, Theorem 3.2], the product $\pi^{-s}\Gamma(s)\varphi_\nu(u,\, s-\nu/2;\, L)$ minus the pole part can be written

$$\int_p^\infty F(u,\, y)y^{s-1}dy + \int_p^\infty G(u,\, y)y^{\nu-s}dy,$$

where

$$F(u,\, y) = \sum_{\alpha\in L}(\overline{u}+\alpha)^\nu \exp\bigl(-\pi|u+\alpha|^2 y\bigr),$$

$$G(u,\, y) = A\sum_{\beta\in B}\exp\bigl(\pi i(\beta\overline{u}+\overline{\beta}u)\bigr)\sum_{\xi-\beta\in M}\xi^\nu\exp\bigl(-\pi|\xi|^2 y\bigr)$$

with a constant A, a finite subset B of $\mathbf{C}$, a positive constant p, and lattices L and M in $\mathbf{C}$. Therefore the differentiability and the last statement follow from the standard fact on differentiation under the integral sign.

3.3. Before stating the next theorem, we note a few elementary facts. Take L in the form $L = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ with complex numbers ω_1 and ω_2 such that $\omega_1/\omega_2 \in H$. We put then $v(\omega_1,\, \omega_2) = v(L)$. It can easily be seen that

$$v(\omega_1,\, \omega_2) = |\omega_2|^2\mathrm{Im}(\omega_1/\omega_2) = (2i)^{-1}(\omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2), \tag{3.3}$$

and in particular, $v(z,\, 1) = \mathrm{Im}(z)$. We also recall the function ζ of Weierstrass defined by

$$\zeta(u) = \zeta(u;\, \omega_1,\, \omega_2) = \frac{1}{u} + \sum_{0\neq\alpha\in L}\left\{\frac{1}{u-\alpha} + \frac{1}{\alpha} + \frac{u}{\alpha^2}\right\}, \tag{3.4}$$

where $L = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$. It is well known that

$$\zeta(-u) = -\zeta(u), \qquad (\partial/\partial u)\zeta(u;\, \omega_1,\, \omega_2) = -\wp(u;\, \omega_1,\, \omega_2) \tag{3.5}$$

with the Weierstrass function $\wp$. We put as usual

$$\eta_\mu(\omega_1,\, \omega_2) = 2\zeta(\omega_\mu/2) \qquad (\mu = 1,\, 2). \tag{3.6a}$$

Then

$$\zeta(u+\omega_\mu) = \zeta(u) + \eta_\mu(\omega_1,\, \omega_2). \tag{3.6b}$$

We also need the classical nonholomorphic Eisenstein series E_2 of weight 2, which can be given by

$$E_2(z) = \frac{1}{8\pi y} - \frac{1}{24} + \sum_{n=1}^\infty\Biggl(\sum_{0<d|n} d\Biggr)\mathbf{e}(nz). \tag{3.7}$$

We are interested in the value of $\varphi_\nu(u,\, s;\, L)$ at $s = \nu/2$, which is meaningful for every $\nu \in \mathbf{Z}, > 0$, by Theorem 3.2. The results can be given as follows.

THEOREM 3.4. *For $L = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ with $\omega_1/\omega_2 \in H$ we have*

$$\varphi_\nu(u,\, \nu/2;\, L) = \frac{(-1)^\nu}{(\nu-1)!}\frac{\partial^{\nu-2}}{\partial u^{\nu-2}}\wp(u;\, \omega_1,\, \omega_2) \qquad (2 < \nu \in \mathbf{Z}), \tag{3.8}$$

$$\varphi_2(u,\, 1;\, L) = \wp(u;\, \omega_1,\, \omega_2) - 8\pi^2\omega_2^{-2}E_2(\omega_1/\omega_2), \tag{3.9}$$

$$\varphi_1(u,\, 1/2;\, L) = \zeta(u) + 8\pi^2\omega_2^{-2}E_2(\omega_1/\omega_2)u - \pi v(L)^{-1}\overline{u}, \tag{3.10}$$

$$\eta_\mu(\omega_1,\,\omega_2) = \pi\overline{\omega}_\mu v(L)^{-1} - 8\pi^2\omega_2^{-2}E_2(\omega_1/\omega_2)\omega_\mu \qquad (\mu = 1, 2). \tag{3.11}$$

Proof. If $\nu > 2$, then clearly $\varphi_\nu(u,\,\nu/2;\,L) = \sum_{\alpha\in L}(u+\alpha)^{-\nu}$, from which we obtain (3.8). The cases $\nu = 2$ and $\nu = 1$ are more interesting. We first note that

$$(\partial/\partial u)\varphi_\nu(u,\,s;\,L) = (-s-\nu/2)\varphi_{\nu+1}(u,\,s+1/2;\,L), \tag{3.12a}$$

$$(\partial/\partial \overline{u})\varphi_\nu(u,\,s;\,L) = (-s+\nu/2)\varphi_{\nu-1}(u,\,s+1/2;\,L), \tag{3.12b}$$

at least for sufficiently large $\mathrm{Re}(s)$. Since both sides of (3.12a, b) are meromorphic in s on the whole $\mathbf{C}$, we obtain (3.12a, b) for every s. The first formula with $\nu = 2$ produces

$$(\partial/\partial u)\varphi_2(u,\,1;\,L) = -2\varphi_3(u,\,3/2;\,L) = (\partial/\partial u)\wp(u;\,\omega_1,\,\omega_2),$$

from which we obtain $\varphi_2(u,\,1;\,L) = \wp(u;\,\omega_1,\,\omega_2) + c(\overline{u})$ with an anti-holomorphic function $c(\overline{u})$. Since (3.12b) shows that $\varphi_2(u,\,1;\,L)$ is holomorphic in u, we see that $c(\overline{u})$ does not involve u or $\overline{u}$, that is, it is a constant depending only on L. Suppose $L = \mathbf{Z}z+\mathbf{Z}$ with $z \in H$. For $0 < N \in \mathbf{Z}$ and $(p,\,q) \in \mathbf{Z}^2,\notin N\mathbf{Z}^2$ define a standard Eisenstein series $\mathfrak{E}_\nu^N(z,\,s;\,p,\,q)$ of level N by [9, (9.1)]. Then we easily see that

$$\varphi_\nu\big((pz+q)/N,\,s;\,L\big) = N^{2s}y^{\nu/2-s}\mathfrak{E}_\nu^N(z,\,s-\nu/2;\,p,\,q),$$
$$\varphi_\nu\big((pz+q)/N,\,\nu/2;\,L\big) = N^{\nu}\mathfrak{E}_\nu^N(z,\,0;\,p,\,q).$$

Define F_ν and $\mathcal{F}_2$ as in [9, (10.10b, c, d)]. Taking $\nu = 2$, we obtain

$$\varphi_2\big((pz+q)/N,\,1;\,L\big) = N^2\mathfrak{E}_2^N(z,\,0;\,p,\,q) = (2\pi i)^2\mathcal{F}_2(z;\,p/N,\,q/N).$$

By [9, (10.13)], $\mathcal{F}_2(z;\,a,\,b) = (2\pi i)^{-2}\wp\big(az+b;\,z,\,1\big) + 2E_2(z)$ with E_2 of (3.7). Therefore we can conclude that

$$\varphi_2(u,\,1;\,\mathbf{Z}z+\mathbf{Z}) = \wp(u;\,z,\,1) - 8\pi^2E_2(z). \tag{3.13}$$

More generally, using (3.2a) we obtain (3.9).

We next consider the case $\nu = 1$. Since $(\partial/\partial u)\zeta(u;\,\omega_1,\,\omega_2) = -\wp(u;\,\omega_1,\,\omega_2)$, from (3.9) and (3.12a) we obtain

$$\begin{aligned}(\partial/\partial u)\varphi_1(u,\,1/2;\,L) &= -\varphi_2(u,\,1;\,L)\\ &= (\partial/\partial u)\zeta(u;\,\omega_1,\,\omega_2) + 8\pi^2\omega_2^{-2}E_2(\omega_1/\omega_2).\end{aligned}$$

We have also

$$(\partial/\partial\overline{u})\varphi_1(u,\,1/2;\,L) = \lim_{\sigma\to 1}(1-\sigma)\varphi_0(u,\,\sigma;\,L) = -\pi/v(L),$$

since the residue of $\pi^{-s}\Gamma(s)\varphi_0(u,\,s;\,L)$ at $s = 1$ is $v(L)^{-1}$ as shown in Theorem 3.2. Therefore $\varphi_1(u,\,1/2;\,L) = -\pi\overline{u}/v(L) + g(u)$ with a function g holomorphic in u. Clearly $\partial g/\partial u = (\partial/\partial u)\varphi_1(u,\,1/2;\,L)$, and so we can conclude that

$$\varphi_1(u,\,1/2;\,L) = \zeta(u) + 8\pi^2\omega_2^{-2}E_2(\omega_1/\omega_2)u - \pi v(L)^{-1}\overline{u} + \xi(L) \tag{3.14}$$

with a constant $\xi(L)$ independent of u. From (3.2a) we obtain $\varphi_1(-u,\,s;\,L) = -\varphi_1(u,\,s;\,L)$. Also $\zeta(-u) = -\zeta(u)$. Thus $\xi(L) = 0$, and consequently we obtain (3.10). Since $\varphi_1(u,\,1/2;\,L)$ is invariant under $u \mapsto u + \omega_\mu$, we obtain (3.11) from (3.10) and (3.6b).

3.5. In [9] we discussed the value of an Eisenstein series $E(z,\,s)$ of weight k at $s = -m$ for an integer m such that $0 \le m \le k-1$, and observed that it is nearly holomorphic in the sense that it is a polynomial in y^{-1} with holomorphic functions as coefficients; for a precise statement, see [9, Theorem 9.6]. As an analogue we investigate $\varphi_\nu(u,\,\kappa/2;\,L)$ for an integer κ such that $2-\nu \le \kappa \le \nu$ and $\kappa - \nu \in 2\mathbf{Z}$. From (3.12b) we obtain, for $0 \le a \in \mathbf{Z}$,

$$(\partial/\partial\overline{u})^a\varphi_\nu(u,\,(\nu/2)-a;\,L) = a!\cdot\varphi_{\nu-a}(u,\,(\nu-a)/2;L). \tag{3.15}$$

THEOREM 3.6. *Let κ be an integer such that $2-\nu \le \kappa \le \nu$ and $\kappa-\nu \in 2\mathbf{Z}$. Then $\varphi_\nu(u,\,\kappa/2;L)$ is a polynomial in $\overline{u}$ of degree d with holomorphic functions in u as coefficients, where $d = (\nu-\kappa)/2$ if $\nu+\kappa \ge 4$ and $d = (\nu-\kappa+2)/2$ if $\nu+\kappa = 2$. The leading term is $\overline{u}^d\varphi_{(\nu+\kappa)/2}(u,\,(\nu+\kappa)/4;\,L)$ or $-\pi d^{-1}v(L)^{-1}\overline{u}^d$ according as $\nu+\kappa \ge 4$ or $\nu+\kappa = 2$.*

Proof. Given κ as in the theorem, put $a = (\nu-\kappa)/2$. Then $(\nu/2)-a = \kappa/2$ and $\nu - a = (\nu+\kappa)/2 \ge 1$. If $\nu - a \ge 2$, then by Theorem 3.4, $\varphi_{\nu-a}(u,\,(\nu-a)/2;L)$ is holomorphic in u, and so (3.15) shows that $\varphi_\nu(u,\,\kappa/2;\,L)$ is a polynomial in $\overline{u}$ of degree a with holomorphic functions in u as coefficients. If $\nu-a = 1$, the function $\varphi_1(u,\,1/2;\,L)$ is linear in $\overline{u}$ as given in (3.10). Therefore we obtain our theorem.

Thus, we may call $\varphi_\nu(u,\,\kappa/2;\,L)$ a *nearly holomorphic elliptic function.* In the higher-dimensional case it is natural to consider theta functions instead of periodic functions. For details of the basic ideas and results on this the reader is referred to [6] and [7].

4. THE SERIES WITH A PARAMETER IN H

4.1. To state the following lemma, we first define a confluent hypergeometric function $\tau(y;\,\alpha,\,\beta)$ for $y > 0$ and $(\alpha,\,\beta) \in \mathbf{C}^2$ by

$$\tau(y;\,\alpha,\,\beta) = \int_0^\infty e^{-yt}(1+t)^{\alpha-1}t^{\beta-1}dt. \tag{4.1}$$

This is convergent for $\mathrm{Re}(\beta) > 0$. It can be shown that $\Gamma(\beta)^{-1}\tau(y;\,\alpha,\,\beta)$ can be continued to a holomorphic function in $(\alpha,\,\beta)$ on the whole $\mathbf{C}^2$; see [9, Section A3], for example. Also, for $v \in \mathbf{C}^\times$ and $\alpha \in \mathbf{C}$ we define v^α by

$$v^\alpha = \exp(\alpha\,\log(v)), \qquad -\pi < \mathrm{Im}[\log(v)] \le \pi. \tag{4.2}$$

LEMMA 4.2. *For* $\alpha, \beta \in \mathbf{C}$ *such that* $\mathrm{Re}(\alpha+\beta) > 1$, $0 \le r < 1$, *and* $z = x+iy \in H$ *we have*

$$\begin{aligned} i^{\alpha-\beta}(2\pi)^{-\alpha-\beta}\Gamma(\alpha)\Gamma(\beta)\sum_{m\in\mathbf{Z}}\mathbf{e}(mr)(z+m)^{-\alpha}(\overline{z}+m)^{-\beta} \\ = \sum_{n=1}^{\infty}\mathbf{e}\big((n-r)z\big)(n-r)^{\alpha+\beta-1}\tau\big(4\pi(n-r)y;\,\alpha,\,\beta\big) \\ + \sum_{n=1}^{\infty}\mathbf{e}\big(-(n+r)\overline{z}\big)(n+r)^{\alpha+\beta-1}\tau\big(4\pi(n+r)y;\,\beta,\,\alpha\big) \\ + \begin{cases} (4\pi y)^{1-\alpha-\beta}\Gamma(\alpha+\beta-1) & \text{if } \ r=0, \\ \mathbf{e}(-r\overline{z})r^{\alpha+\beta-1}\tau\big(4\pi ry;\,\beta,\,\alpha\big) & \text{if } \ r\neq 0. \end{cases} \end{aligned}$$

Proof. If $r = 0$, this is Lemma A3.4 of [9]. The case with nontrivial r can be proved in the same way as follows. Define two functions $f(x)$ and $f_1(x)$ of $x \in \mathbf{R}$ by $f(x) = (x+iy)^{-\alpha}(x-iy)^{-\beta}$ with a fixed $y > 0$ and $f_1(x) = \mathbf{e}(rx)f(x)$. Then $\hat{f}_1(x) = \hat{f}(x-r)$, and so the Poisson summation formula (see [9, (2.9)]) shows that

$$\mathbf{e}(-rx)\sum_{m\in\mathbf{Z}}f_1(x+m) = \mathbf{e}(-rx)\sum_{n\in\mathbf{Z}}\hat{f}_1(n)\mathbf{e}(nx) = \sum_{n\in\mathbf{Z}}\mathbf{e}\big((n-r)x\big)\hat{f}(n-r).$$

In [9, p. 133] we determined $\hat{f}$ explicitly in terms of τ as follows:

$$i^{\alpha-\beta}(2\pi)^{-\alpha-\beta}\Gamma(\alpha)\Gamma(\beta)\hat{f}(t) = \begin{cases} \mathbf{e}(ity)t^{\alpha+\beta-1}\tau(4\pi ty;\,\alpha,\,\beta) & (t>0), \\ \mathbf{e}(-ity)|t|^{\alpha+\beta-1}\tau(4\pi|t|y;\,\beta,\,\alpha) & (t<0), \\ (4\pi y)^{1-\alpha-\beta}\Gamma(\alpha+\beta-1) & (t=0). \end{cases}$$

Therefore we obtain our lemma.

4.3. We now need an elementary result:

$$\sum_{n=1}^{\infty}n^{k-1}x^n = \frac{xP_k(x)}{(1-x)^k} \qquad (1 \le k \in \mathbf{Z}). \tag{4.3}$$

Here x is an indeterminate and P_k is a polynomial. We have $P_1 = P_2 = 1$ and $P_{k+1} = (kx-x+1)P_k - (x^2-x)P_k'$ for $k \ge 2$. Thus P_k is of degree $k-2$ for $k \ge 2$. These are easy; see [9, p. 17]. We also need two formulas and an estimate given as (A3.11), (A3.14), and Lemma A3.2 in [9]:

$$\tau(y;\,n,\,\beta) = \sum_{\mu=0}^{n-1}\binom{n-1}{\mu}\Gamma(\beta+\mu)y^{-\mu-\beta} \qquad (0 < n \in \mathbf{Z}), \tag{4.4}$$

$$\big[\tau(y;\,\alpha,\,\beta)/\Gamma(\beta)\big]_{\beta=0} = 1, \tag{4.5}$$

(4.6) $\Gamma(\beta)^{-1}y^{\beta}\tau(y;\,\alpha,\,\beta)$ *is bounded when* (α,β) *belongs to a compact subset of* $\mathbf{C}^2$ *and* $y > c$ *with a positive constant* c.

Our principal aim of this section is to study the nature of the series

$$\mathcal{L}_k(s,\,z)=\sum_{m\in\mathbf{Z}}\mathbf{e}(mr)(z+m)^{-k}|z+m|^{-2s},\tag{4.7}$$

for certain integer values of s. Here $k \in \mathbf{Z}$, $r \in \mathbf{R}$, $s \in \mathbf{C}$, and $z \in H$. The sum depends only on r modulo $\mathbf{Z}$, and so we may assume that $0 \le r < 1$. Clearly this series is absolutely convergent for $\mathrm{Re}(2s + k) > 1$, and defines a holomorphic function of s there.

THEOREM 4.4. *The function $\mathcal{L}_k(s, z)$ can be continued as a meromorphic function of s to the whole $\mathbf{C}$, which is entire if $r \notin \mathbf{Z}$. If $r \in \mathbf{Z}$, the locations of the poles of $\mathcal{L}_k(s, z)$ are the same as those of $\Gamma(2s+k-1)/\{\Gamma(s+k)\Gamma(s)\}$.*

Proof. Our function is the infinite series of Lemma 4.2 defined with $\alpha = s+k$ and $\beta = s$. Therefore our assertion can easily be verified by means of the formula of Lemma 4.2 and the estimate given by (4.6).

THEOREM 4.5. *Assuming that $r \in \mathbf{Q}$, let N be the smallest positive integer such that $Nr \in \mathbf{Z}$ and let $\beta \in \mathbf{Z}$. Then the following assertions hold:*

(i) *If $\beta > 0$ or $\beta + k > 0$, then $\mathcal{L}_k(s, z)$ is finite at $s = \beta$ and $i^k\mathcal{L}_k(\beta, z)$ is a rational function in π, $\mathbf{e}(z/N)$, $\mathbf{e}(-\overline{z}/N)$, and $\mathrm{Im}(z)$ with coefficients in $\mathbf{Q}$.*

(ii) *If $-k < \beta \le 0$, then $i^k\{1 - \mathbf{e}(z)\}^{k+\beta}\mathcal{L}_k(\beta, z)$ is a polynomial in π, $\mathbf{e}(z/N)$, $\mathrm{Im}(z)$, and $\mathrm{Im}(z)^{-1}$ with coefficients in $\mathbf{Q}$.*

(iii) *If $0 < \beta \le -k$, then $i^k\{1 - \mathbf{e}(-\overline{z})\}^{\beta}\mathcal{L}_k(\beta, z)$ is a polynomial in π, $\mathbf{e}(-\overline{z}/N)$, $\mathrm{Im}(z)$, and $\mathrm{Im}(z)^{-1}$ with coefficients in $\mathbf{Q}$.*

Proof. As we already said, we may assume that $0 \le r < 1$. Put $\alpha = \beta + k$. We first have to study the nature of $\Gamma(2s + k - 1)/\{\Gamma(s + k)\Gamma(s)\}$ at $s = \beta$. This is clearly finite at $s = \beta$ if $\alpha + \beta > 1$. Suppose $\alpha + \beta \le 1$; then $\alpha \le 0$ if $\beta > 0$, and $\beta \le 0$ if $\alpha > 0$. In all cases the value is finite, and in fact is a rational number. We now evaluate the formula of Lemma 4.2 divided by $\Gamma(\alpha)\Gamma(\beta)$. If $\alpha > 0$, we have, by (4.4),

$$\tau\bigl(4\pi(n-r)y;\,\alpha,\,\beta\bigr)/\Gamma(\beta)=\sum_{\mu=0}^{\alpha-1}\binom{\alpha-1}{\mu}(n-r)^{-\mu-\beta}(4\pi y)^{-\mu-\beta}\prod_{\kappa=0}^{\mu-1}(\beta+\kappa).$$

Thus an infinite sum of the form $\sum_{n=1}^{\infty}\mathbf{e}\bigl((n-r)z\bigr)(n-r)^{\alpha-\mu-1}$ appears. Applying the binomial theorem to the power of $n - r$, we see that the sum is a $\mathbf{Q}$-linear combination of $\mathbf{e}(-rz)\sum_{n=1}^{\infty}\mathbf{e}(nz)n^{\nu}$ for $0 \le \nu \le \alpha - \mu - 1$. We can handle $\tau\bigl(4\pi(n + r)y;\,\beta,\,\alpha\bigr)/\Gamma(\alpha)$ in a similar way if $\beta > 0$. Put $\mathbf{q} = \mathbf{e}(z)$ and $\mathbf{q}_r = \mathbf{e}(rz)$. Then, assuming that $0 < r < 1$, $\alpha > 0$, and $\beta > 0$, we have

$$\begin{aligned}i^k\mathcal{L}_k(\beta,\,z)=\mathbf{q}_r^{-1}&\sum_{\mu=0}^{\alpha-1}\sum_{\nu=0}^{\alpha-\mu-1}a_{\mu\nu}\pi^{\alpha-\mu}y^{-\mu-\beta}\sum_{n=1}^{\infty}n^{\nu}\mathbf{q}^n\\&+\overline{\mathbf{q}}_r\sum_{\mu=0}^{\beta-1}\sum_{\nu=0}^{\beta-\mu-1}b_{\mu\nu}\pi^{\beta-\mu}y^{-\mu-\alpha}\sum_{n=0}^{\infty}n^{\nu}\overline{\mathbf{q}}^n,\end{aligned}$$

where $a_{\mu\nu}$ and $b_{\mu\nu}$ are rational numbers depending on β, k, and r. Applying (4.3) to $\sum_{n=1}^{\infty} n^{\nu}X^n$ with $X = \mathbf{q}$ and $X = \overline{\mathbf{q}}$, we obtain (i). Suppose $\beta \le 0$ and $\beta + k > 0$; then the sum involving $\tau\bigl(4\pi(n+r)y;\, \beta,\, \alpha\bigr)/\bigl\{\Gamma(\alpha)\Gamma(\beta)\bigr\}$ vanishes and we obtain (ii). The case in which $\beta > 0$ and $\beta + k \le 0$ is similar and produces (iii). If $r = 0$, the constant term of $i^k\mathcal{L}_k(\beta,\, z)$ is $2\pi(2y)^{1-\alpha-\beta}\Gamma(\alpha+\beta-1)/[\Gamma(a)\Gamma(\beta)]$, which causes no problem. This completes the proof.

One special case is worthy of attention. Taking $\beta = 0$ and $1 < k = \alpha \in \mathbf{Z}$, and using (4.5), we obtain, for $0 \le r < 1$,

$$\sum_{m\in\mathbf{Z}} \frac{\mathbf{e}\bigl(r(z+m)\bigr)}{(z+m)^k} = \frac{(2\pi i)^k}{(k-1)!}\sum_{\nu=1}^{k}\binom{k-1}{\nu-1} r^{k-\nu}\frac{\mathbf{q}P_\nu(\mathbf{q})}{(\mathbf{q}-1)^\nu}, \tag{4.8}$$

where $\mathbf{q} = \mathbf{e}(z)$. We assume $0 < r < 1$ and $\sum_{m\in\mathbf{Z}} = \lim_{h\to\infty}\sum_{|m|\le h}$ when $k = 1$. In (4.7) we take z in H, but in (4.8) we can take $z \in \mathbf{C}, \notin \mathbf{Z}$, since both sides of (4.8) are meaningful for such z. If $k = 1$, the result is the same as (2.3).

We can mention another special case. Namely, take $z = ia$ with a positive rational number a. Then we see that the values

$$\sum_{m\in\mathbf{Z}} (a^2+m^2)^{-\beta} \tag{4.9}$$

for $0 < \beta \in \mathbf{Z}$ belong to the field generated by π and $e^{-2\pi a}$ over $\mathbf{Q}$, and therefore any three such values satisfy a nontrivial algebraic equation over $\mathbf{Q}$.

5. The rationality over a totally real base field

5.1. Throughout this section we fix a totally real algebraic number field F. The algebraicity of $\pi^{-k}L(k,\, \chi)$ can be generalized to the case of L-functions over F, but there is no known formulas similar to (0.2), (1.9), (1.10), except that Siegel proved some such formulas in [10] and [11] when $[F:\mathbf{Q}] = 2$. The paper [1] of Hecke may be mentioned in this connection. In this section we merely consider a generalization of (2.4) and prove an algebraicity result on its critical values, without producing explicit expressions.

We denote by $\mathfrak{g}$, D_F, and $\mathbf{a}$ the maximal order of F, the discriminant of F, and the set of archimedean primes of F. We also put $\mathrm{Tr}(x) = \mathrm{Tr}_{F/\mathbf{Q}}(x)$ for $x \in F$ and $[F:\mathbf{Q}] = g$. For $\alpha \in F$ and a fractional ideal $\mathfrak{a}$ in F we put $\alpha + \mathfrak{a} = \bigl\{\, \alpha + x \bigm| x \in \mathfrak{a} \,\bigr\}$ and $\widetilde{\mathfrak{a}} = \bigl\{\xi \in F \mid \mathrm{Tr}(\xi\mathfrak{a}) \subset \mathbf{Z}\bigr\}$.

Given α and $\mathfrak{a}$ as above, $\xi \in F$, $0 < \mu \in \mathbf{Z}$, and a (sufficiently small) subgroup U of $\mathfrak{g}^\times$ of finite index, we put

$$D_\mu(s;\, \xi,\, \alpha,\, \mathfrak{a}) = r_U \sum_{0\ne h\in U\backslash(\alpha+\mathfrak{a})} \mathbf{e}_{\mathbf{a}}(h\xi)h^{-\mu\mathbf{a}}|h|^{(\mu-s)\mathbf{a}}, \tag{5.1}$$

$$r_U = [\mathfrak{g}^\times : U]^{-1}, \tag{5.1a}$$

where $\mathbf{e_a}(\xi) = \mathbf{e}\bigl(\sum_{v\in\mathbf{a}}(\xi_v)\bigr)$ for $\xi \in F$ and $x^{t\mathbf{a}} = \prod_{v\in\mathbf{a}} x_v^t$ for $x \in \mathbf{C}^{\mathbf{a}}$ and $U\backslash X$ means a complete set of representatives for X modulo multiplication by the elements of U. We have to take U so small that the sum of (5.1) is meaningful. For instance, it is sufficient to take

$$U \subset \bigl\{u \in \mathfrak{g}^\times \bigm| u^{\mathbf{a}} = 1,\, u\xi - \xi \in \alpha^{-1}\widetilde{\mathfrak{g}} \cap \widetilde{\mathfrak{a}}\bigr\}.$$

The factor r_U makes the quantity of (5.1) independent of the choice of U. Clearly the sum is convergent for $\mathrm{Re}(s) > 1$. Now $D_\mu(s;\, \xi,\, \alpha,\, \mathfrak{a})$ is a special case of the series of [8, (18.1)], and so from Lemma 18.2 of [8] we see that it can be continued as a holomorphic function in s to the whole $\mathbf{C}$.

Theorem 5.2. *For $0 < \mu \in \mathbf{Z}$ we have*

$$(2\pi i)^{-\mu g} D_F^{1/2} D_\mu(\mu;\, \xi,\, 0,\, \mathfrak{a}) \in \mathbf{Q}, \tag{5.2}$$

$$D_\mu(1-\mu;\, 0,\, \alpha,\, \mathfrak{a}) \in \mathbf{Q}. \tag{5.3}$$

Proof. The last formula is a restatement of Proposition 18.10(2) of [8]. To prove (5.2), let $\mathfrak{b} = \bigl\{x \in \mathfrak{a} \bigm| \mathbf{e_a}(x\xi) = 1\bigr\}$ and let R be a complete set of representatives for $\mathfrak{a}/\mathfrak{b}$. Then

$$D_\mu(s;\, \xi,\, 0,\, \mathfrak{a}) = \sum_{\beta\in R} \mathbf{e_a}(\beta\xi) D_\mu(s; 0,\, \beta,\, \mathfrak{b}).$$

Put $Q_\mu(\beta,\, \mathfrak{b}) = (2\pi i)^{-\mu g} D_F^{1/2} D_\mu(\mu;\, 0,\, \beta,\, \mathfrak{b})$. Then the quantity of (5.2) equals $\sum_{\beta\in R} \mathbf{e_a}(\beta\xi) Q_\mu(\beta,\, \mathfrak{b})$. Let $t \in \prod_p \mathbf{Z}_p^\times$ and let σ be the image of t under the canonical homomorphism of $\mathbf{Q}_{\mathbf{A}}^\times$ onto $\mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q})$. Our task is to show that the last sum is invariant under σ. By [8, Proposition 18.10(1)] we have $Q_\mu(\beta,\, \mathfrak{b})^\sigma = Q_\mu(\beta_1,\, \mathfrak{b})$ with $\beta_1 \in F$ such that $(t\beta_1 - \beta)_v \in \mathfrak{b}_v$ for every nonarchimedean prime v of F. For $\beta \in R$ there is a unique $\beta_1 \in R$ with that property. Now $\mathbf{e}(c)^\sigma = \mathbf{e}(t^{-1}c)$ for every $c \in \mathbf{Q}/\mathbf{Z} = \prod_p(\mathbf{Q}_p/\mathbf{Z}_p)$; see [8, (8.2)]. Since $\mathbf{e_a}(\beta\xi) = \mathbf{e}\bigl(\mathrm{Tr}(\beta\xi)\bigr)$, we easily see that $\mathbf{e_a}(\beta\alpha)^\sigma = \mathbf{e_a}(\beta_1\alpha)$, which gives the desired fact.

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