Schur Class Operator Functions and Automorphisms of Hardy Algebras

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Abstract. Let E be a W^*-correspondence over a von Neumann algebra M and let $H^\infty(E)$ be the associated Hardy algebra. If σ is a faithful normal representation of M on a Hilbert space H, then one may form the dual correspondence E^σ and represent elements in $H^\infty(E)$ as $B(H)$-valued functions on the unit ball $\mathbb{D}(E^\sigma)^*$. The functions that one obtains are called Schur class functions and may be characterized in terms of certain Pick-like kernels. We study these functions and relate them to system matrices and transfer functions from systems theory. We use the information gained to describe the automorphism group of $H^\infty(E)$ in terms of special Möbius transformations on $\mathbb{D}(E^\sigma)$. Particular attention is devoted to the H^∞-algebras that are associated to graphs.

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1 Introduction

Let M be a W^*-algebra and let E be a W^*-correspondence over M. In [31] we built an operator algebra from this data that we called the Hardy algebra of E and which we denoted $H^\infty(E)$. If $M = E = \mathbb{C}$ - the complex numbers, then $H^\infty(E)$ is the classical Hardy algebra consisting of all bounded analytic functions on the open unit disc, $\mathbb{D}$ (see Example 2.4 below.) If $M = \mathbb{C}$ again, but $E = \mathbb{C}^n$, then $H^\infty(E)$ is the free semigroup algebra $\mathcal{L}_n$ studied by Davidson and Pitts [17], Popescu [32] and others (see Example 2.5.) One of the principal discoveries made in [31], and the source of inspiration for the present paper, is that attached to each faithful normal representation σ of M there is a *dual* correspondence E^σ, which is a W^*-correspondence over the commutant of $\sigma(M)$, $\sigma(M)'$, and the elements of $H^\infty(E)$ define functions on the open unit ball of E^σ, $\mathbb{D}(E^\sigma)$. Further, the value distribution theory of these functions turns out to be linked through our generalization of the Nevanlinna-Pick interpolation theorem [31, Theorem 5.3] with the positivity properties of certain Pick-like kernels of *mappings between operator spaces.*

In the setting where $M = E = \mathbb{C}$ and σ is the 1-dimensional representation of $\mathbb{C}$ on itself, then E^σ is $\mathbb{C}$ again. The representation of $H^\infty(E)$ in terms of functions on $\mathbb{D}(E^\sigma) = \mathbb{D}$ is just the usual way we think of $H^\infty(E)$. In this setting, our Nevanlinna-Pick theorem is exactly the classical theorem. If, however, σ is a representation of $\mathbb{C}$ on a Hilbert space H, $\dim(H) > 1$, then E^σ may be identified with $B(H)$ and then $\mathbb{D}(E^\sigma)$ becomes the space of strict contractions on H, i.e., all those operators of norm strictly less than 1. In this case, the value of an $f \in H^\infty(E)$ at a $T \in \mathbb{D}(E^\sigma)$ is simply $f(T)$, defined through the usual holomorphic functional calculus. Our Nevanlinna-Pick theorem gives a solution to problems such as this: given k operators $T_1, T_2, \ldots, T_k$ all of norm less than 1 and k operators, $A_1, A_2, \ldots, A_k$, determine the circumstances under which one can find a bounded analytic function f on the open unit disc of sup norm at most 1 such that $f(T_i) = A_i$, $i = 1, 2, \ldots, k$ (See [31, Theorem 6.1].) On the other hand, when $M = \mathbb{C}$, $E = \mathbb{C}^n$, and σ is one dimensional, the space E^σ is $\mathbb{C}^n$ and $\mathbb{D}(E^\sigma)$ is the unit ball $\mathbb{B}^n$. Elements in $H^\infty(E) = \mathcal{L}_n$ are realized as holomorphic functions on $\mathbb{B}^n$ that lie in a multiplier space studied in detail by Arveson [5]. More accurately, the functional representation of $H^\infty(E) = \mathcal{L}_n$ in terms of these functions expresses this space as a *quotient* of $H^\infty(E) = \mathcal{L}_n$. The Nevanlinna-Pick theorem of [31] contains those of Davidson and Pitts [18], Popescu [34], and Arias and Popescu [4], which deal with interpolation problems for these spaces of functions (possibly tensored with the bounded operators on an auxiliary Hilbert space). It also contains some of the results of Constaninescu and Johnson in [16] which treats elements of $\mathcal{L}_n$ as functions on the ball of strict row contractions with values in the operators on a Hilbert space. (See their Theorem 3.4 in particular.) This situation arises when one takes $M = \mathbb{C}$ and $E = \mathbb{C}^n$, but takes σ to be scalar multiplication on an auxiliary Hilbert space.

Our objective in the present note is basically two fold. First, we wish to identify those functions on $\mathbb{D}(E^\sigma)$ that arise from evaluating elements of $H^\infty(E)$. For this purpose, we introduce a family of functions on $\mathbb{D}(E^\sigma)$ that we call Schur class operator functions (see Definition 3.1). Roughly speaking, these functions are defined so that a Pick-like kernel that one may attach to each one is completely positive definite in the sense of Barreto, Bhat, Liebscher and Skeide [14]. In Theorem 3.3 we use their Theorem 3.2.3 to give a Kolmogorov-type representation of the kernel, from which we derive an analogue of a unitary system matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ whose transfer function

$$A + B(I - L_\eta^* D)^{-1} L_\eta^* C$$

turns out to be the given Schur class operator function. We then prove in Theorem 3.6 that each such transfer function arises by evaluating an element in $H^\infty(E)$ at points of $\mathbb{D}(E^\sigma)$ and conversely, each function in $H^\infty(E)$ has a representation in terms of a transfer function. The meaning of the notation will be made precise below, but we use it here to highlight the connection between our analysis and realization theory as it comes from mathematical systems theory. The point to keep in mind is that functions on $\mathbb{D}(E^\sigma)$ that come from elements of $H^\infty(E)$ are not, *a priori*, analytic in any ordinary sense and it is not at all clear what analytic features they have. Our Theorems 3.1 and 3.6 together with [31, Theorem 5.3] show that the Schur class operator functions are precisely the functions one obtains when evaluating functions in $H^\infty(E)$ (of norm at most 1) at points of $\mathbb{D}(E^\sigma)$. The fact that each such function may be realized as a transfer function exhibits a surprising level of analyticity that is not evident in the definition of $H^\infty(E)$.

Our second objective is to connect the usual holomorphic properties of $\mathbb{D}(E^\sigma)$ with the automorphisms of $H^\infty(E)$. As a space, $\mathbb{D}(E^\sigma)$ is the unit ball of a J^*-triple system. Consequently, every holomorphic automorphism of $\mathbb{D}(E^\sigma)$ is the composition of a Möbius transformation and a linear isometry [20]. Each of these implements an automorphism of the algebra of all bounded, *complex-valued analytic* functions on $\mathbb{D}(E^\sigma)$, but in our setting only certain of them implement automorphisms of $H^\infty(E)$ - those for which the Möbius part is determined by a "central" element of E^σ (see Theorem 4.21). Our proof requires the fact that the evaluation of functions in $H^\infty(E)$ (of norm at most 1) at points of $\mathbb{D}(E^\sigma)$ are precisely the Schur class operator functions on $\mathbb{D}(E^\sigma)$. Indeed, the whole analysis is an intricate "point - counterpoint" interplay among elements of $H^\infty(E)$, Schur class functions, transfer functions and "classical" function theory on $\mathbb{D}(E^\sigma)$. In the last section, we apply our general analysis of the automorphisms of $H^\infty(E)$ to the special case of H^∞-algebras coming from directed graphs.

In concluding this introduction, we want to note that a preprint of the present paper was posted on the arXiv on June 27, 2006. Recently, inspired in part by our preprint, Ball, Biswas, Fang and ter Horst [8] were able to realize the Fock space that we describe here in terms of the theory of completely positive definite kernels advanced by Barreto, Bhat, Liebscher and Skeide [14] that we also use (See Section 3 and, in particular, the proof of Theorem 3.3.) The analysis of Ball *et al.* makes additional ties between the theory of abstract Hardy algebras that we develop here and classical function theory on the unit disc.

2 Preliminaries

We start by introducing the basic definitions and constructions. We shall follow Lance [24] for the general theory of Hilbert C^*-modules that we shall use. Let A be a C^*-algebra and E be a right module over A endowed with a bi-additive map $\langle\cdot,\cdot\rangle : E\times E\to A$ (referred to as an A-valued inner product) such that, for $\xi,\eta\in E$ and $a\in A$, $\langle\xi,\eta a\rangle=\langle\xi,\eta\rangle a$, $\langle\xi,\eta\rangle^*=\langle\eta,\xi\rangle$, and $\langle\xi,\xi\rangle\geq 0$, with $\langle\xi,\xi\rangle=0$ only when $\xi=0$. Also, E is assumed to be complete in the norm $\|\xi\|:=\|\langle\xi,\xi\rangle\|^{1/2}$. We write $\mathcal{L}(E)$ for the space of continuous, adjointable, A-module maps on E. It is known to be a C^*-algebra. If M is a von Neumann algebra and if E is a Hilbert C^*-module over M, then E is said to be *self-dual* in case every continuous M-module map from E to M is given by an inner product with an element of E. Let A and B be C^*-algebras. A *C^*-correspondence from A to B* is a Hilbert C^*-module E over B endowed with a structure of a left module over A via a nondegenerate $*$-homomorphism $\varphi : A\to\mathcal{L}(E)$.

When dealing with a specific C^*-correspondence, E, from a C^*-algebra A to a C^*-algebra B, it will be convenient sometimes to suppress the φ in formulas involving the left action and simply write $a\xi$ or $a\cdot\xi$ for $\varphi(a)\xi$. This should cause no confusion in context.

If E is a C^*-correspondence from A to B and if F is a correspondence from B to C, then the balanced tensor product, $E\otimes_B F$ is an A,C-bimodule that carries the inner product defined by the formula

$$\langle\xi_1\otimes\eta_1,\xi_2\otimes\eta_2\rangle_{E\otimes_B F}:=\langle\eta_1,\varphi(\langle\xi_1,\xi_2\rangle_E)\eta_2\rangle_F$$

The Hausdorff completion of this bimodule is again denoted by $E\otimes_B F$.

In this paper we deal mostly with correspondences over von Neumann algebras that satisfy some natural additional properties as indicated in the following definition. (For examples and more details see [31]).

Definition 2.1 *Let M and N be von Neumann algebras and let E be a Hilbert C^*-module over N. Then E is called a* Hilbert W^*-module *over N in case E is self-dual. The module E is called a* W^*-correspondence from M to N *in case E is a self-dual C^*-correspondence from M to N such that the $*$-homomorphism $\varphi : M\to\mathcal{L}(E)$, giving the left module structure on E, is normal. If $M=N$ we shall say that E is a W^*-correspondence* over M.

We note that if E is a Hilbert W^*-module over a von Neumann algebra, then $\mathcal{L}(E)$ is not only a C^*-algebra, but is also a W^*-algebra. Thus it makes sense to talk about normal homomorphisms into $\mathcal{L}(E)$.

DEFINITION 2.2 *An* isomorphism *of a W^*-correspondence E_1 over M_1 and a W^*-correspondence E_2 over M_2 is a pair (σ, Ψ) where $\sigma : M_1 \to M_2$ is an isomorphism of von Neumann algebras, $\Psi : E_1 \to E_2$ is a vector space isomorphism preserving the σ-topology and for $e, f \in E_1$ and $a, b \in M_1$, we have $\Psi(aeb) = \sigma(a)\Psi(e)\sigma(b)$ and $\langle \Psi(e), \Psi(f) \rangle = \sigma(\langle e, f \rangle)$.*

When considering the tensor product $E \otimes_M F$ of two W^*-correspondences, one needs to take the closure of the C^*-tensor product in the σ-topology of [6] in order to get a W^*-correspondence. However, we will not distinguish notationally between the C^*-tensor product and the W^*-tensor product. Note also that given a W^*-correspondence E over M and a Hilbert space H equipped with a normal representation σ of M, we can form the Hilbert space $E \otimes_\sigma H$ by defining $\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2 \rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2 \rangle) h_2 \rangle$. Thus, H is viewed as a correspondence from M to $\mathbb{C}$ via σ and $E \otimes_\sigma H$ is just the tensor product of E and H as W^*-correspondences.

Note also that, given an operator $X \in \mathcal{L}(E)$ and an operator $S \in \sigma(M)'$, the map $\xi \otimes h \mapsto X\xi \otimes Sh$ defines a bounded operator on $E \otimes_\sigma H$ denoted by $X \otimes S$. The representation of $\mathcal{L}(E)$ that results when one lets $S = I$, is called the representation of $\mathcal{L}(E)$ *induced by* σ and is often denoted by σ^E. The composition, $\sigma^E \circ \varphi$ is a representation of M which we shall also say is induced by σ, but we shall usually denote it by $\varphi(\cdot) \otimes I$.

Observe that if E is a W^*-correspondence over a von Neumann algebra M, then we may form the tensor powers $E^{\otimes n}$, $n \geq 0$, where $E^{\otimes 0}$ is simply M viewed as the identity correspondence over M, and we may form the W^*-direct sum of the tensor powers, $\mathcal{F}(E) := E^{\otimes 0} \oplus E^{\otimes 1} \oplus E^{\otimes 2} \oplus \cdots$ to obtain a W^*-correspondence over M called the (*full*) *Fock space* over E. The actions of M on the left and right of $\mathcal{F}(E)$ are the diagonal actions and, when it is convenient to do so, we make explicit the left action by writing φ_∞ for it. That is, for $a \in M$, $\varphi_\infty(a) := diag\{a, \varphi(a), \varphi^{(2)}(a), \varphi^{(3)}(a), \cdots\}$, where for all n, $\varphi^{(n)}(a)(\xi_1 \otimes \xi_2 \otimes \cdots \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \xi_n$, $\xi_1 \otimes \xi_2 \otimes \cdots \xi_n \in E^{\otimes n}$. The *tensor algebra* over E, denoted $\mathcal{T}_+(E)$, is defined to be the *norm-closed* subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by $\varphi_\infty(M)$ and the *creation operators* T_ξ, $\xi \in E$, defined by the formula $T_\xi \eta = \xi \otimes \eta$, $\eta \in \mathcal{F}(E)$. We refer the reader to [28] for the basic facts about $\mathcal{T}_+(E)$.

DEFINITION 2.3 *([31]) Given a W^*-correspondence E over the von Neumann algebra M, the ultraweak closure of the tensor algebra of E, $\mathcal{T}_+(E)$, in $\mathcal{L}(\mathcal{F}(E))$, is called the* Hardy Algebra *of E, and is denoted $H^\infty(E)$.*

EXAMPLE 2.4 *If $M = E = \mathbb{C}$, then $\mathcal{F}(E)$ can be identified with $\ell^2(\mathbb{Z}_+)$ or, through the Fourier transform, $H^2(\mathbb{T})$. The tensor algebra then is isomorphic to the disc algebra $A(\mathbb{D})$ viewed as multiplication operators on $H^2(\mathbb{T})$ and the Hardy algebra is realized as the classical Hardy algebra $H^\infty(\mathbb{T})$.*

EXAMPLE 2.5 *If $M = \mathbb{C}$ and $E = \mathbb{C}^n$, then $\mathcal{F}(E)$ can be identified with the space $l_2(\mathbb{F}_n^+)$, where $\mathbb{F}_n^+$ is the free semigroup on n generators. The tensor* *algebra then is what Popescu refers to as the "non commutative disc algebra" $\mathcal{A}_n$ and the Hardy algebra is its w^*-closure. It was studied by Popescu [32] and by Davidson and Pitts who denoted it by $\mathcal{L}_n$ [17].*

We need to review some basic facts about the representation theory of $H^\infty(E)$ and of $\mathcal{T}_+(E)$. See [28, 31] for more details.

DEFINITION 2.6 *Let E be a W^*-correspondence over a von Neumann algebra M. Then:*

1. *A* completely contractive covariant representation *of E on a Hilbert space H is a pair (T, σ), where*
   - *(a) σ is a normal $*$-representation of M in $B(H)$.*
   - *(b) T is a linear, completely contractive map from E to $B(H)$ that is continuous in the σ-topology of [6] on E and the ultraweak topology on $B(H)$.*
   - *(c) T is a bimodule map in the sense that $T(S\xi R) = \sigma(S)T(\xi)\sigma(R)$, $\xi \in E$, and $S, R \in M$.*
2. *A completely contractive covariant representation (T, σ) of E in $B(H)$ is called* isometric *in case*
$$T(\xi)^*T(\eta) = \sigma(\langle \xi, \eta \rangle) \tag{1}$$
   for all $\xi, \eta \in E$.

It should be noted that the operator space structure on E to which Definition 2.6 refers is that which E inherits when viewed as a subspace of its linking algebra.
As we showed in [28, Lemmas 3.4–3.6] and in [31], if a completely contractive covariant representation, (T, σ), of E in $B(H)$ is given, then it determines a contraction $\tilde{T} : E \otimes_\sigma H \to H$ defined by the formula $\tilde{T}(\eta \otimes h) := T(\eta)h$, $\eta \otimes h \in E \otimes_\sigma H$. The operator $\tilde{T}$ intertwines the representation σ on H and the induced representation $\sigma^E \circ \varphi = \varphi(\cdot) \otimes I_H$ on $E \otimes_\sigma H$; i.e.
$$\tilde{T}(\varphi(\cdot) \otimes I) = \sigma(\cdot)\tilde{T}. \tag{2}$$

In fact we have the following lemma from [31, Lemma 2.16].

LEMMA 2.7 *The map $(T, \sigma) \to \tilde{T}$ is a bijection between all completely contractive covariant representations (T, σ) of E on the Hilbert space H and contractive operators $\tilde{T} : E \otimes_\sigma H \to H$ that satisfy equation (2). Given such a $\tilde{T}$ satisfying this equation, T, defined by the formula $T(\xi)h := \tilde{T}(\xi \otimes h)$, together with σ is a completely contractive covariant representation of E on H. Further, (T, σ) is isometric if and only if $\tilde{T}$ is an isometry.*

The importance of the completely contractive covariant representations of E (or, equivalently, the intertwining contractions $\tilde{T}$ as above) is that they yield all completely contractive representations of the tensor algebra. More precisely, we have the following.

Theorem 2.8 *Let E be a W^*-correspondence over a von Neumann algebra M. To every completely contractive covariant representation, (T,σ), of E there is a unique completely contractive representation ρ of the tensor algebra $\mathcal{T}_+(E)$ that satisfies*

$$\rho(T_\xi) = T(\xi) \quad \xi \in E$$

and

$$\rho(\varphi_\infty(a)) = \sigma(a) \quad a \in M.$$

The map $(T,\sigma) \mapsto \rho$ is a bijection between the set of all completely contractive covariant representations of E and all completely contractive (algebra) representations of $\mathcal{T}_+(E)$ whose restrictions to $\varphi_\infty(M)$ are continuous with respect to the ultraweak topology on $\mathcal{L}(\mathcal{F}(E))$.

Definition 2.9 *If (T,σ) is a completely contractive covariant representation of a W^*-correspondence E over a von Neumann algebra M, we call the representation ρ of $\mathcal{T}_+(E)$ described in Theorem 2.8 the* integrated form *of (T,σ) and write $\rho = \sigma \times T$.*

Remark 2.10 *One of the principal difficulties one faces in dealing with $\mathcal{T}_+(E)$ and $H^\infty(E)$ is to decide when the integrated form, $\sigma \times T$, of a completely contractive covariant representation (T,σ) extends from $\mathcal{T}_+(E)$ to $H^\infty(E)$. This problem arises already in the simplest situation, vis. when $M = \mathbb{C} = E$. In this setting, T is given by a single contraction operator on a Hilbert space, $\mathcal{T}_+(E)$ "is" the disc algebra and $H^\infty(E)$ "is" the space of bounded analytic functions on the disc. The representation $\sigma \times T$ extends from the disc algebra to $H^\infty(E)$ precisely when there is no singular part to the spectral measure of the minimal unitary dilation of T. We are not aware of a comparable result in our general context but we have some sufficient conditions. One of them is given in the following lemma. It is not a necessary condition in general.*

Lemma 2.11 *[31, Corollary 2.14] If $\|\tilde{T}\| < 1$ then $\sigma \times T$ extends to a ultraweakly continuous representation of $H^\infty(E)$.*

In [31] we introduced and studied the concepts of duality and of point evaluation (for elements of $H^\infty(E)$). These play a central role in our analysis here.

Definition 2.12 *Let E be a W^*-correspondence over a von Neumann algebra M and let $\sigma : M \to B(H)$ be a faithful normal representation of M on a Hilbert space H. Then the* σ-dual *of E, denoted E^σ, is defined to be*

$$\{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = (\varphi(a) \otimes I)\eta,\ a \in M\}.$$

An important feature of the dual E^σ is that it is a W^*-correspondence, *but over the commutant* of $\sigma(M)$, $\sigma(M)'$.

Proposition 2.13 *With respect to the action of $\sigma(M)'$ and the $\sigma(M)'$-valued inner product defined as follows, E^σ becomes a W^*-correspondence over $\sigma(M)'$: For Y and X in $\sigma(M)'$, and $\eta \in E^\sigma$, $X\cdot\eta\cdot Y := (I\otimes X)\eta Y$, and for $\eta_1, \eta_2 \in E^\sigma$, $\langle \eta_1, \eta_2\rangle_{\sigma(M)'} := \eta_1^*\eta_2$.*

In the following remark we explain what we mean by "evaluating an element of $H^\infty(E)$ at a point in the open unit ball of the dual".

Remark 2.14 *The importance of this dual space, E^σ, is that it is closely related to the representations of E. In fact, the operators in E^σ whose norm does not exceed* 1 *are precisely the adjoints of the operators of the form $\tilde{T}$ for a covariant pair (T, σ). In particular, every η in the* open *unit ball of E^σ (written $\mathbb{D}(E^\sigma)$) gives rise to a covariant pair (T, σ) (with $\eta = \tilde{T}^*$) such that $\sigma \times T$ extends to a representation of $H^\infty(E)$.*
Given $X \in H^\infty(E)$ we can apply the representation associated to η to it. The resulting operator in $B(H)$ will be denoted by $\widehat{X}(\eta^*)$. Thus

$$\widehat{X}(\eta^*) = (\sigma \times \eta^*)(X).$$

In this way, we view every element in the Hardy algebra as a $B(H)$-valued function

$$\widehat{X} : \mathbb{D}(E^\sigma)^* \to B(H)$$

on the open unit ball of $(E^\sigma)^*$. One of our primary objectives is to understand the range of the transform $X \to \widehat{X}$, $X \in H^\infty(E)$.

Example 2.15 *Suppose $M = E = \mathbb{C}$ and σ the representation of $\mathbb{C}$ on some Hilbert space H. Then it is easy to check that E^σ is isomorphic to $B(H)$. Fix an $X \in H^\infty(E)$. As we mentioned above, this Hardy algebra is the classical $H^\infty(\mathbb{T})$ and we can identify X with a function $f \in H^\infty(\mathbb{T})$. Given $S \in \mathbb{D}(E^\sigma) = B(H)$, it is not hard to check that $\widehat{X}(S^*)$, as defined above, is the operator $f(S^*)$ defined through the usual holomorphic functional calculus.*

Example 2.16 *In [17] Davidson and Pitts associate to every element of the free semigroup algebra $\mathcal{L}_n$ (see Example 2.5) a function on the open unit ball of $\mathbb{C}^n$. This is a special case of our analysis when $M = \mathbb{C}$, $E = \mathbb{C}^n$ and σ is a one dimensional representation of $\mathbb{C}$. In this case $\sigma(M)' = \mathbb{C}$ and $E^\sigma = \mathbb{C}^n$. Note, however, that our definition allows us to take σ to be the representation of $\mathbb{C}$ on an arbitrary Hilbert space H. If we do so, then E^σ is isomorphic to $B(H)^{(n)}$, the nth column space over $B(H)$, and elements of $\mathcal{L}_n$ define functions on the open unit ball of this space viewed as a correspondence over $B(H)$ with values in $B(H)$. This is the perspective adopted by Constantinescu and Johnson in [16]. In the analysis of [17] it is possible that a non zero element of $\mathcal{L}_n$ will give rise to the zero function. We shall show in Lemma 3.8 that, by choosing an appropriate H we can insure that this does not happen.*

Example 2.17 *Part of the recent work of Popescu in [35] may be cast in our framework. We will follow his notation. Fix a Hilbert space K, and let E be the column space $B(K)^n$. Take, also, a Hilbert space H and let $\sigma : B(K) \to B(K \otimes H)$ be the representation which sends $a \in B(K)$ to $a \otimes I_H$. Then, since the commutant of $\sigma(B(K))$ is naturally isomorphic to $B(H)$, it is easy to see that E^σ is the column space over $B(H)$, $B(H)^n$. It follows that $\mathbb{D}(E^\sigma)$ is the open unit ball in $B(H)^n$. A* free formal power series *with coefficients from $B(K)$ is a formal series $F = \sum_{\alpha \in \mathbb{F}_n^+} A_\alpha \otimes Z_\alpha$ where $\mathbb{F}_n^+$ is the free semigroup on n generators, the A_α are elements of $B(K)$ and where Z_α is the monomial in noncommuting indeterminates $Z_1, Z_2, \ldots, Z_n$ determined by α. If F has radius of convergence equal to 1, then one may evaluate F at points of $\mathbb{D}(E^\sigma)^*$ to get a function on $\mathbb{D}(E^\sigma)^*$ with values in $B(K \otimes H)$,* vis., *$F((S_1, S_2, \cdots S_n)) = \sum_{\alpha \in \mathbb{F}_n^+} A_\alpha \otimes S_\alpha$. See [35, Theorem 1.1]. In fact, under additional restrictions on the coefficients A_α, F may be viewed as a function X in $H^\infty(B(K)^n)$ in such a way that $F((S_1, S_2, \cdots S_n)) = \widehat{X}(S_1, S_2, \cdots S_n)$ in the sense defined in [31, p. 384] and discussed above in Remark 2.14. The space that Popescu denotes by $H^\infty(B(\mathcal{X})_1^n)$ arises when $K = \mathbb{C}$, and is naturally isometrically isomorphic to $\mathcal{L}_n$ [35, Theorem 3.1]. We noted in the preceding example that $\mathcal{L}_n$ is $H^\infty(\mathbb{C}^n)$. The point of [35], at least in part, is to study $H^\infty(B(\mathcal{X})_1^n) \simeq \mathcal{L}_n = H^\infty(\mathbb{C}^n)$ through all the representations σ of $\mathbb{C}$ on Hilbert spaces H, that is, through evaluating functions in $H^\infty(B(\mathcal{X})_1^n)$ at points the unit ball of $B(H)^n$ for all possible H's. The space $B(K)^n$ is Morita equivalent to $\mathbb{C}^n$ in the sense of [30], at least when $\dim(K) < \infty$, and, in that case the tensor algebras $\mathcal{T}_+(B(K)^n)$ and $\mathcal{T}_+(\mathbb{C}^n)$ are Morita equivalent in the sense described by [15]. The tensor algebra $\mathcal{T}_+(\mathbb{C}^n)$, in turn, is naturally isometrically isomorphic to Popescu's noncommutative disc algebra $\mathcal{A}_n$ [33]. The analysis in [15] suggests a sense in which $\mathbb{C}^n$ and $B(K)^n$ are Morita equivalent even when $\dim(K) = \infty$, and that together with [30] suggests that $H^\infty(B(K)^n)$ should be Morita equivalent to $H^\infty(B(\mathcal{X})_1^n) \simeq H^\infty(\mathbb{C}^n)$. This would suggest an even closer connection between Popescu's free power series, and all that goes with them, and the perspective we have taken in this paper, which, as we shall see, involves generalized Schur functions and transfer functions. The connection seems like a promising avenue to explore.*

In [31] we exploited the perspective of viewing elements of the Hardy algebra as $B(H)$-valued functions on the open unit ball of the dual correspondence to prove a Nevanlinna-Pick type interpolation theorem. In order to state it we introduce some notation: For operators B_1 and B_2 in $B(H)$, we write $Ad(B_1, B_2)$ for the map from $B(H)$ to itself that sends S to $B_1 S B_2^*$. Also, given elements η_1, η_2 in $\mathbb{D}(E^\sigma)$, we let θ_{η_1,η_2} denote the map, from $\sigma(M)'$ to itself that sends a to $\langle \eta_1, a\eta_2 \rangle$. That is, $\theta_{\eta_1,\eta_2}(a) := \langle \eta_1, a\eta_2 \rangle = \eta_1^* a \eta_2$, $a \in \sigma(M)'$.

Theorem 2.18 *([31, Theorem 5.3]) Let E be a W^*-correspondence over a von Neumann algebra M and let $\sigma : M \to B(H)$ be a faithful normal representation of M on a Hilbert space H. Fix k points $\eta_1, \ldots \eta_k$ in the disk $\mathbb{D}(E^\sigma)$ and choose*

$2k$ operators $B_1, \dots B_k, C_1, \dots C_k$ in $B(H)$. Then there exists an X in $H^\infty(E)$ such that $\|X\| \leq 1$ and

$$B_i \widehat{X}(\eta_i^*) = C_i$$

for $i = 1, 2, \dots, k$, if and only if the map from $M_k(\sigma(M)')$ into $M_k(B(H))$ defined by the $k \times k$ matrix

$$\big((Ad(B_i, B_j) - Ad(C_i, C_j)) \circ (id - \theta_{\eta_i, \eta_j})^{-1}\big) \tag{3}$$

is completely positive.

That is, the map T, say, given by the matrix (3) is computed by the formula

$$T((a_{ij})) = (b_{ij}),$$

where

$$b_{ij} = B_i((id - \theta_{\eta_i, \eta_j})^{-1}(a_{ij})B_j^* - C_i((id - \theta_{\eta_i, \eta_j})^{-1}(a_{ij})C_j^*$$

and

$$(id - \theta_{\eta_i, \eta_j})^{-1}(a_{ij}) = a_{ij} + \theta_{\eta_i, \eta_j}(a_{ij}) + \theta_{\eta_i, \eta_j}(\theta_{\eta_i, \eta_j}(a_{ij})) + \cdots$$

We close this section with two technical lemmas that will be needed in our analysis. Let M and N be W^*-algebras and let E be a W^*-correspondence from M to N. Given a σ-closed subcorrespondence E_0 of E we know that the orthogonal projection P of E onto E_0 is a right module map. (See [6, Consequences 1.8 (ii)]). In the following lemma we show that P also preserves the left action.

Lemma 2.19 *Let E be a W^*-correspondence from the von Neumann algebra M to the von Neumann algebra N, and let E_0 be a sub W^*-correspondence E_0 of E that is closed in the σ-topology of [6, Consequences 1.8 (ii)]. If P is the orthogonal projection from E onto E_0, then P is a bimodule map; i.e., $P(a\xi b) = aP(\xi)b$ for all $a \in M$ and $b \in N$.*

Proof. It suffices to check that $P(e\xi) = eP(\xi)$ for all $\xi \in E$ and projections $e \in M$. For $\xi, \eta \in E$ and a projection $e \in M$, we have

$$\|e\xi + f\eta\|^2 = \|\langle e\xi, e\xi\rangle + \langle f\eta, f\eta\rangle\| \leq \|\langle e\xi, e\xi\rangle\| + \|\langle f\eta, f\eta\rangle\| = \|e\xi\|^2 + \|f\eta\|^2,$$

where $f = 1 - e$. So, for every $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}(\lambda+1)^2\|fP(e\xi)\|^2 = \|fP(e\xi + \lambda fP(e\xi))\|^2 &\leq \|e\xi + \lambda fP(e\xi)\|^2 \\ &\leq \|e\xi\|^2 + \lambda^2\|fP(e\xi)\|^2.\end{aligned}$$

Hence, for every $\lambda \in \mathbb{R}$,

$$(2\lambda + 1)\|fP(e\xi)\|^2 \leq \|e\xi\|^2$$

and, thus,

$$(I-e)P(e\xi) = fP(e\xi) = 0.$$

Replacing e by $f = I - e$ we get $eP((I-e)\xi) = 0$ and, therefore,

$$P(e\xi) = eP(e\xi) = eP(\xi).$$

Since M is spanned by its projections, we are done. □

Lemma 2.20 *Let E be a W^*-correspondence over M, let σ be a faithful normal representation of M on the Hilbert space $\mathcal{E}$, and let E^σ be the σ-dual correspondence over $N := \sigma(M)'$. Then*

(i) The left action of N on E^σ is faithful if and only if E is full (i.e. if and only if the ultraweakly closed ideal generated by the inner products $\langle \xi_1, \xi_2 \rangle$, $\xi_1, \xi_2 \in E$, is all of M).

(ii) The left action of M on E is faithful if and only if E^σ is full.

Proof. We shall prove (i). Part (ii) then follows by duality (using [31, Theorem 3.6]). Given $S \in N$, $S\eta = 0$ for every $\eta \in E^\sigma$ if and only if for all $\eta \in E^\sigma$ and $g \in \mathcal{E}$, $(I \otimes S)\eta(g) = 0$. Since the closed subspace spanned by the ranges of all $\eta \in E^\sigma$ is all of $E \otimes_M \mathcal{E}$ ([31]), this is equivalent to the equation $\xi \otimes Sg = 0$ holding for all $g \in \mathcal{E}$ and $\xi \in E$. Since $\langle \xi \otimes Sg, \xi \otimes Sg \rangle = \langle g, S^* \langle \xi, \xi \rangle Sg \rangle$, we find that $SE^\sigma = 0$ if and only if $\sigma(\langle E, E \rangle)S = 0$, where $\langle E, E \rangle$ is the ultraweakly closed ideal generated by all inner products. If this ideal is all of M we find that the equation $SE^\sigma = 0$ implies that $S = 0$. In the other direction, if this is not the case, then this ideal is of the form $(I-q)M$ for some central nonzero projection q and then $S = \sigma(q)$ is different from 0 but vanishes on E^σ. □

3 Schur class operator functions and realization

Throughout this section, E will be a fixed W^*-correspondence over the von Neumann algebra M and σ will be a faithful representation of M on a Hilbert space $\mathcal{E}$. We then form the σ-dual of E, E^σ, which is a correspondence over $N := \sigma(M)'$, and we write $\mathbb{D}(E^\sigma)$ for its open unit ball. Further, we write $\mathbb{D}(E^\sigma)^*$ for $\{\eta^* \mid \eta \in \mathbb{D}(E^\sigma)\}$.

The following definition is clearly motivated by the condition appearing in Theorem 2.18 and Schur's theorem from classical function theory.

Definition 3.1 *Let Ω be a subset of $\mathbb{D}(E^\sigma)$ and let $\Omega^* = \{\omega^* \mid \omega \in \Omega\}$. A function $Z : \Omega^* \to B(\mathcal{E})$ will be called a* Schur class operator function *(with values in $B(\mathcal{E})$) if, for every k and every choice of elements $\eta_1, \eta_2, \ldots, \eta_k$ in Ω, the map from $M_k(N)$ to $M_k(B(\mathcal{E}))$ defined by the $k \times k$ matrix of maps,*

$$((id - Ad(Z(\eta_i^*), Z(\eta_j^*))) \circ (id - \theta_{\eta_i, \eta_j})^{-1}),$$

is completely positive.

Note that, when $M = E = B(\mathcal{E})$ and σ is the identity representation of $B(\mathcal{E})$ on $\mathcal{E}$, $\sigma(M)'$ is $\mathbb{C}I_{\mathcal{E}}$, E^σ is isomorphic to $\mathbb{C}$ and $\mathbb{D}(E^\sigma)^*$ can be identified with the open unit disc $\mathbb{D}$ of $\mathbb{C}$. In this case our definition recovers the classical Schur class functions. More precisely, these functions are usually defined as analytic functions Z from an open subset Ω of $\mathbb{D}$ into the closed unit ball of $B(\mathcal{E})$ but it is known that such functions are precisely those for which the Pick kernel $k_Z(z,w) = (I - Z(z)Z(w)^*)(1 - z\bar{w})^{-1}$ is positive semi-definite on Ω. The argument of [31, Remark 5.4] shows that the positivity of this kernel is equivalent, in our case, to the condition of Definition 3.1. This condition, in turn, is the same as asserting that the kernel

$$k_Z(\zeta^*, \omega^*) := (id - Ad(Z(\zeta^*), Z(\omega^*)) \circ (id - \theta_{\zeta,\omega})^{-1} \tag{4}$$

is a completely positive definite kernel on Ω^* in the sense of Definition 3.2.2 of [14].

For the sake of completeness, we record the fact that every element of $H^\infty(E)$ of norm at most one gives rise to a Schur class operator function.

THEOREM 3.2 *Let E be a W^*-correspondence over a von Neumann algebra M and let σ be a faithful normal representation of M in $B(H)$ for some Hilbert space H. If X is an element of $H^\infty(E)$ of norm at most one, then the function $\eta^* \to \widehat{X}(\eta^*)$ defined in Remark 2.14 is a Schur class operator function on $\mathbb{D}((E^\sigma))^*$ with values in $B(H)$.*

PROOF. One simply takes $B_i = I$ for all i and $C_i = \widehat{X}(\eta_i^*)$ in Theorem 2.18. $\square$

THEOREM 3.3 *Let E be a W^*-correspondence over a von Neumann algebra M. Suppose also that σ a faithful normal representation of M on a Hilbert space $\mathcal{E}$ and that q_1 and q_2 are projections in $\sigma(M)$. Finally, suppose that Ω is a subset of $\mathbb{D}((E^\sigma))$ and that Z is a Schur class operator function on Ω^* with values in $q_2 B(\mathcal{E}) q_1$. Then there is a Hilbert space H, a normal representation τ of $N := \sigma(M)'$ on H and operators A, B, C and D fulfilling the following conditions:*

(i) The operator A lies in $q_2\sigma(M)q_1$.

(ii) The operators C, B, and D, are in the spaces $B(\mathcal{E}_1, E^\sigma \otimes_\tau H)$, $B(H, \mathcal{E}_2)$, and $B(H, E^\sigma \otimes_\tau H)$, respectively, and each intertwines the representations of $N = \sigma(M)'$ on the relevant spaces (i.e. , for every $S \in N$, $CS = (S \otimes I_H)C$, $B\tau(S) = SB$ and $D\tau(S) = (S \otimes I_H)D$).

(iii) The operator matrix

$$V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5}$$

viewed as an operator from $\mathcal{E}_1 \oplus H$ to $\mathcal{E}_2 \oplus (E^\sigma \otimes_\tau H)$, is a coisometry, which is unitary if E is full.

(iv) For every η^* in Ω^*,

$$Z(\eta^*) = A + B(I - L_\eta^* D)^{-1} L_\eta^* C \tag{6}$$

where $L_\eta : H \to E^\sigma \otimes H$ is defined by the formula $L_\eta h = \eta \otimes h$ (so $L_\eta^*(\theta \otimes h) = \tau(\langle \eta, \theta \rangle)h$).

REMARK 3.4 *Before giving the proof of Theorem 3.3, we want to note that the result bears a strong resemblance to standard results in the literature. We call special attention to [1, 2, 7, 9, 10, 11, 12, 13]. Indeed, we recommend [7], which is a survey that explains the general strategy for proving the theorem. What is novel in our approach is the adaptation of the results in the literature to accommodate completely positive definite kernels.*

Since the matrix in equation (5) and the function in equation (6) are familiar constructs in mathematical systems theory, more particularly from H^∞-control theory (see, e.g., [38]), we adopt the following terminology.

DEFINITION 3.5 *Let E be a W^*-correspondence over a von Neumann algebra M. Suppose that σ is a faithful normal representation of M on a Hilbert space $\mathcal{E}$ and that q_1 and q_2 are projections in $\sigma(M)$. Then an operator matrix $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where the entries A, B, C, and D, satisfy conditions (i) and (ii) of Theorem 3.3 for some normal representation τ of $\sigma(M)'$ on a Hilbert space H, is called a* system matrix *provided V is a coisometry (that is unitary, if E is full). If V is a system matrix, then the function $A + B(I - L_\eta^* D)^{-1} L_\eta^* C$, $\eta^* \in \mathbb{D}(E^\sigma)^*$ is called the* transfer function determined by V.

PROOF. As we just remarked, the hypothesis that Z is a Schur class function on Ω^* means that the kernel k_Z in equation (4) is completely positive definite in the sense of [14]. Consequently, we may apply Theorem 3.2.3 of [14], which is a lovely extension of Kolmogorov's representation theorem for positive definite kernels, to find an N-$B(\mathcal{E})$ W^*-correspondence F and a function ι from Ω^* to F such that F is spanned by $N\iota(\Omega^*)B(\mathcal{E})$ and such that for every η_1 and η_2 in Ω^* and every $a \in N$,

$$(id - Ad(Z(\eta_1^*), Z(\eta_2^*))) \circ (id - \theta_{\eta_1,\eta_2})^{-1}(a) = \langle \iota(\eta_1), a\iota(\eta_2) \rangle.$$

It follows that for every $b \in N$ and every η_1, η_2 in Ω^*,

$$\begin{aligned} b - Z(\eta_1^*) b Z(\eta_2^*)^* &= \langle \iota(\eta_1), b\iota(\eta_2) \rangle - \langle \iota(\eta_1), \langle \eta_1, b\eta_2 \rangle \iota(\eta_2) \rangle \\ &= \langle \iota(\eta_1), b\iota(\eta_2) \rangle - \langle \eta_1 \otimes \iota(\eta_1), b\eta_2 \otimes \iota(\eta_2) \rangle. \end{aligned}$$

Thus,

$$b + \langle \eta_1 \otimes \iota(\eta_1), b\eta_2 \otimes \iota(\eta_2) \rangle = \langle \iota(\eta_1), b\iota(\eta_2) \rangle + Z(\eta_1^*) b Z(\eta_2^*)^*. \tag{7}$$

Set

$$G_1 := \overline{span}\{bZ(\eta^*)^* q_2 T \oplus b\iota(\eta) q_2 T \mid b \in N,\ \eta \in \Omega^*,\ T \in B(\mathcal{E})\ \}$$

and

$$G_2 := \overline{span}\{bq_2 T \oplus (b\eta \otimes \iota(\eta) q_2 T) \mid b \in N,\ \eta \in \Omega^*,\ T \in B(\mathcal{E})\ \}.$$

Then G_1 is a sub N-$B(\mathcal{E})$ W^*-correspondence of $B(\mathcal{E}) \oplus F$ (where we use the assumption that $q_2 Z(\eta^*) = q_2 Z(\eta^*) q_1$) and G_2 is a sub N-$B(\mathcal{E})$ W^*-correspondence of $B(\mathcal{E}) \oplus (E^\sigma \otimes_N F)$. (The closure in the definitions of G_1, G_2 is in the σ-topology of [6]. It then follows that G_1 and G_2 are W^*-correspondences [6, Consequences 1.8 (i)]). Define $v : G_1 \to G_2$ by the equation

$$v(bZ(\eta^*)^* q_2 T \oplus b\iota(\eta) q_2 T) = bq_2 T \oplus (b\eta \otimes \iota(\eta) q_2 T).$$

It follows from (7) that v is an isometry. It is also clear that it is a bimodule map. We write P_i for the orthogonal projection onto G_i, $i = 1, 2$ and $\tilde{V}$ for the map

$$\tilde{V} := P_2 v P_1 : q_1 B(\mathcal{E}) \oplus F \to q_2 B(\mathcal{E}) \oplus (E^\sigma \otimes_N F).$$

Then $\tilde{V}$ is a partial isometry and, since P_1, v and P_2 are all bimodule maps (see Lemma 2.19), so is $\tilde{V}$. We write $\tilde{V}$ matricially:

$$\tilde{V} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where $\alpha : q_1 B(\mathcal{E}) \to q_2 B(\mathcal{E})$, $\beta : F \to q_2 B(\mathcal{E})$, $\gamma : q_1 B(\mathcal{E}) \to E^\sigma \otimes F$ and $\delta : F \to E^\sigma \otimes F$ and all these maps are bimodule maps. Let H_0 be the Hilbert space $F \otimes_{B(\mathcal{E})} \mathcal{E}$ and note that $B(\mathcal{E}) \otimes_{B(\mathcal{E})} \mathcal{E}$ is isomorphic to $\mathcal{E}$ (and the isomorphism preserves the left N-action). Tensoring on the right by $\mathcal{E}$ (over $B(\mathcal{E})$) we obtain a partial isometry

$$V_0 := \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} : \begin{pmatrix} \mathcal{E}_1 \\ H_0 \end{pmatrix} \to \begin{pmatrix} \mathcal{E}_2 \\ E^\sigma \otimes H_0 \end{pmatrix}.$$

Here $A_0 = \alpha \otimes I_{\mathcal{E}}$, $B_0 = \beta \otimes I_{\mathcal{E}}$, $C_0 = \gamma \otimes I_{\mathcal{E}}$ and $D_0 = \delta \otimes I_{\mathcal{E}}$. These maps are well defined because the maps α, β, γ and δ are right $B(\mathcal{E})$-module maps. Since these maps are also left N-module maps, so are A_0, B_0, C_0 and D_0.
By the definition of V_0, its initial space is $G_1 \otimes \mathcal{E}$ and its final space is $G_2 \otimes \mathcal{E}$. In fact, V_0 induces an equivalence of the representations of N on $G_1 \otimes \mathcal{E}$ and on $G_2 \otimes \mathcal{E}$.
It will be convenient to use the notation $K_1 \preceq_N K_2$ if the Hilbert spaces K_1 and K_2 are both left N-modules and the representation of N on K_1 is equivalent to a subrepresentation of the representation of N on K_2. This means, of course, that there is an isometry from K_1 into K_2 that intertwines the two representations. If the two representations are equivalent we write $K_1 \simeq_N K_2$.

Using this notation, we can write $G_1 \otimes \mathcal{E} \simeq_N G_2 \otimes \mathcal{E}$. Form $\mathcal{M}_2 := (\mathcal{E}_2 \oplus (E^\sigma \otimes H_0)) \ominus (G_2 \otimes \mathcal{E})$, which is a left N-module, and note that $L := \mathcal{F}(E^\sigma) \otimes \mathcal{M}_2$ also is a left N-module, where the representation of N on L is the induced representation. Since $L = \mathcal{F}(E^\sigma) \otimes \mathcal{M}_2 = \bigoplus_{n=0}^{\infty}((E^\sigma)^{\otimes n} \otimes (\mathcal{M}_2))$, it is evident that $(E^\sigma \otimes L) \oplus \mathcal{M}_2 \simeq_N L$. Indeed, the isomorphisms are just the natural ones that give the associativity of the tensor products involved. Thus, $\mathcal{E}_2 \oplus (E^\sigma \otimes (H_0 \oplus L)) = \mathcal{E}_2 \oplus (E^\sigma \otimes H_0) \oplus (E^\sigma \otimes L) = G_2 \otimes \mathcal{E} \oplus \mathcal{M}_2 \oplus E^\sigma \otimes L \simeq_N G_2 \otimes \mathcal{E} \oplus L \simeq_N G_1 \otimes \mathcal{E} \oplus L \preceq_N \mathcal{E}_1 \oplus (H_0 \oplus L)$. Consequently, we obtain a coisometric operator $V : \mathcal{E}_1 \oplus (H_0 \oplus L) \to \mathcal{E}_2 \oplus E^\sigma \otimes (H_0 \oplus L)$ that intertwines the representations of N and extends V_0. Note that, if V_0 were known to be an isometry (so that $G_2 \otimes \mathcal{E} \simeq_N G_1 \otimes \mathcal{E} = \mathcal{E}_1 \oplus H_0$), then we would have equivalence above and V can be chosen to be unitary.

Assume that E is full. We also write $\mathcal{M}_1$ for $(\mathcal{E}_1 \oplus H_0) \ominus G_1 \otimes \mathcal{E}$. Since E is full, the representation ρ of N on $E^\sigma \otimes \mathcal{E}$ is faithful (Lemma 2.20) and it follows that every representation of N is quasiequivalent to a subrepresentation of ρ. Write $\mathcal{E}_\infty$ for the direct sum of infinitely many copies of $\mathcal{E}$. Then $E^\sigma \otimes \mathcal{E}_\infty$ is the direct sum of infinitely many copies of $E^\sigma \otimes \mathcal{E}$ and, thus, every representation of N is equivalent to a subrepresentation of the representation of N on $E^\sigma \otimes \mathcal{E}_\infty$. In particular, we can write $\mathcal{M}_1 \oplus \mathcal{E}_\infty \preceq_N E^\sigma \otimes \mathcal{E}_\infty$. Thus $\mathcal{E}_1 \oplus (H_0 \oplus \mathcal{E}_\infty) = (G_1 \otimes \mathcal{E}) \oplus \mathcal{M}_1 \oplus \mathcal{E}_\infty \preceq_N \mathcal{E}_2 \oplus (E^\sigma \otimes H_0) \oplus (E^\sigma \otimes \mathcal{E}_\infty) = \mathcal{E}_2 \oplus (E^\sigma \otimes (H_0 \oplus \mathcal{E}_\infty))$. So, replacing H_0 by $H_0 \oplus \mathcal{E}_\infty$, we can replace V_0 by an isometry and, using the argument just presented, we conclude that the resulting V is a unitary operator intertwining the representations of N and extending V_0.

So we let V be the coisometry just constructed (and treat it as unitary when E is full). Writing $H := H_0 \oplus L$, we can express V in the matricial form as in part (iii) of the statement of the theorem. Conditions (i) and (ii) then follow from the fact that V intertwines the indicated representations of N. It is left to prove (iv).

Setting $b = T = I$ in the definition of v above and writing v in a matricial form we see that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} Z(\eta^*)^* q_2 \\ \iota(\eta) q_2 \end{pmatrix} = \begin{pmatrix} q_2 \\ \eta \otimes \iota(\eta) q_2 \end{pmatrix}.$$

Tensoring by $I_{\mathcal{E}}$ on the right and identifying $B(\mathcal{E}) \otimes_{B(\mathcal{E})} \mathcal{E}$ with $\mathcal{E}$ as above, we find that

$$\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \begin{pmatrix} Z(\eta^*)^* g \\ \iota(\eta) \otimes g \end{pmatrix} = \begin{pmatrix} g \\ \eta \otimes (\iota(\eta) \otimes g) \end{pmatrix},$$

for $g \in \mathcal{E}_2$. Since A, B, C and D extend A_0, B_0, C_0 and D_0 respectively, we can drop the subscript 0. We also use the fact that the matrix we obtain is a coisometry, and thus its adjoint equals its inverse on its range. We conclude that

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} g \\ \eta \otimes (\iota(\eta) \otimes g) \end{pmatrix} = \begin{pmatrix} Z(\eta^*)^* g \\ \iota(\eta) \otimes g \end{pmatrix}. \tag{8}$$

Thus $\iota(\eta) \otimes g = B^* g + D^*(\eta \otimes (\iota(\eta) \otimes g)) = B^* g + D^* L_\eta(\iota(\eta) \otimes g)$ and

$$\iota(\eta) \otimes g = (I - D^* L_\eta)^{-1} B^* g.$$

Combining this equality with the other equation that we get from (8), we have

$$Z(\eta^*)^*g = A^*g + C^*L_\eta(I - D^*L_\eta)^{-1}B^*g\ ,\quad g \in \mathcal{E}.$$

Taking adjoints yields (iv). □

Thus, Theorem 3.3 asserts that every Schur class function determines a system matrix whose transfer function represents the function. The system matrix is not unique in general, but as the proof of Theorem 3.3 shows, it arises through a series of natural choices. Of course, equation (6) suggests that every Schur class function represents an element in $H^\infty(E)$. This is indeed the case, as the following converse shows.

THEOREM 3.6 *Let E be a W^*-correspondence over a W^*-algebra M, and let σ be a faithful normal representation of M on a Hilbert space $\mathcal{E}$. If $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a system matrix determined by a normal representation τ of $N := \sigma(M)'$ on a Hilbert space H, then there is an $X \in H^\infty(E)$, $\|X\| \le 1$, such that*

$$\widehat{X}(\eta^*) = A + B(I - L_\eta^* D)^{-1} L_\eta^* C,$$

for all $\eta^* \in \mathbb{D}(E^\sigma)^*$ and, conversely, every $X \in H^\infty(E)$, $\|X\| \le 1$, may be represented in this fashion for a suitable system matrix $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

PROOF. For every $n \ge 0$ we define an operator K_n from $\mathcal{E}$ to $(E^\sigma)^{\otimes n} \otimes \mathcal{E}$ as follows. For $n = 0$, we set $K_0 = A$ - an operator in $B(\mathcal{E})$. For $n = 1$, we define K_1, mapping $\mathcal{E}$ to $E^\sigma \otimes \mathcal{E}$, to be $(I_1 \otimes B)C$, where for all $k \ge 1$, I_k denotes the identity operator on $(E^\sigma)^{\otimes k}$. For $n \ge 2$, we set

$$K_n := (I_n \otimes B)(I_{n-1} \otimes D) \cdots (I_1 \otimes D)C.$$

Note, first, that it follows from the properties of A, B, C and D that, for every $n \ge 0$ and every $a \in N$, $K_n a = (\varphi_n(a) \otimes I_{\mathcal{E}})K_n$ where φ_n defines the left multiplication on $(E^\sigma)^{\otimes n}$. Thus, writing ι for the identity representation of N on $\mathcal{E}$, K_n lies in the ι-dual of $(E^\sigma)^{\otimes n}$ which, by Theorem 3.6 and Lemma 3.7 of [31], is isomorphic to $E^{\otimes n}$. Hence, for every $n \ge 0$, K_n defines a unique element ξ_n in $E^{\otimes n}$.

For every $n \ge 0$ and $\eta \in E^\sigma$ we shall write $L_n(\eta)$ for the operator from $(E^\sigma)^{\otimes n} \otimes \mathcal{E}$ to $(E^\sigma)^{\otimes(n+1)} \otimes \mathcal{E}$ given by tensoring on the left by η. Also note that, for $k \ge 1$ and $n \ge 0$, $I_k \otimes K_n$ is an operator from $(E^\sigma)^{\otimes k} \otimes \mathcal{E}$ to $(E^\sigma)^{\otimes(k+n)} \otimes \mathcal{E}$. With this notation, it is easy to see that, for all $k \ge 1$ and $n \ge 0$,

$$(I_{k+1} \otimes K_n)L_k(\eta) = L_{k+n}(\eta)(I_k \otimes K_n). \tag{9}$$

Note, too, that we can write

$$\mathcal{F}(E^\sigma) \otimes \mathcal{E} = \mathcal{E} \oplus (E^\sigma \otimes \mathcal{E}) \oplus \cdots \oplus ((E^\sigma)^{\otimes m} \otimes \mathcal{E}) \oplus \cdots$$

and every operator on $\mathcal{F}(E^\sigma)\otimes\mathcal{E}$ can be written in a matricial form with respect to this decomposition (with indices starting at 0). For every m, $0 \leq m \leq \infty$, we let S_m be the operator defined by the matrix whose i,j entry is $I_j \otimes K_{i-j}$, if $0 \leq j \leq i \leq m$, and is 0 otherwise. (For $m = \infty$, it is not clear yet that the matrix so constructed represents a bounded operator, but this will be verified later).

So far we have not used the assumption that V is a coisometry. But if we take this into account, form the product VV^*, and set it equal to $I_{\mathcal{E}\oplus(E^\sigma\otimes H)}$, we find that

$$I_{\mathcal{E}} - AA^* = BB^* \tag{10}$$

$$CC^* = I_{E^\sigma\otimes_\tau H} - DD^* \tag{11}$$

$$AC^* = -BD^* \tag{12}$$

We claim that, for $1 \leq j \leq i \leq m$, the following equations hold,

$$(I - S_mS_m^*)_{i,j} = (I_i \otimes B)(I_{i-1} \otimes D)\cdots DD^*\cdots(I_{j-1} \otimes D^*)(I_j \otimes B^*); \tag{13}$$

that for $0 < i \leq m$,

$$(I - S_mS_m^*)_{i,0} = (I_i \otimes B)(I_{i-1} \otimes D)\cdots DB^*, \tag{14}$$

and that for $i = j = 0$,

$$(I - S_mS_m^*)_{0,0} = BB^*. \tag{15}$$

Equation (15) follows immediately from (10) since $(S_m)_{0,0} = A$. For $0 < i \leq m$ we compute $(I - S_mS_m^*)_{i,0} = -(S_m)_{i,0}(S_m)_{0,0}^* = -(I_i \otimes B)(I_{i-1} \otimes D)\cdots(I_1 \otimes D)CA^* = (I_i \otimes B)(I_{i-1} \otimes D)\cdots(I_1 \otimes D)DB^*$ where, in the last equality we used (12). It is left to prove (13). Let us write $R_{i,j}$ for the left hand side of (13). (For $j = 0 < i$ we have $R_{i,0} = (I_i \otimes B)(I_{i-1} \otimes D)\cdots DB^*$ and when both are 0, $R_{0,0} = BB^*$). We have $K_0K_0^* = AA^* = I - BB^* = I - R_{0,0}R_{0,0}^*$. For $0 = j < i \leq m$ we have $K_iK_0^* = (I_i \otimes B)(I_{i-1} \otimes D)\cdots(I_1 \otimes D)CA^* = -(I_i \otimes B)(I_{i-1} \otimes D)\cdots(I_1 \otimes D)DB^* = -R_{i,0}$ and for $0 < j \leq i \leq m$, $K_iK_j^* = (I_i \otimes B)(I_{i-1} \otimes D)\cdots(I_1 \otimes D)CC^*(I_1 \otimes D^*)\cdots(I_{j-1} \otimes D^*)(I_j \otimes B^*) = (I_i \otimes B)(I_{i-1} \otimes D)\cdots(I_1 \otimes D)(I - DD^*)(I_1 \otimes D^*)\cdots(I_{j-1} \otimes D^*)(I_j \otimes B^*) = (I_i \otimes B)(I_{i-1} \otimes D)\cdots(I_1 \otimes D)(I_1 \otimes D^*)\cdots(I_{j-1} \otimes D^*)(I_j \otimes B^*) - (I_i \otimes B)(I_{i-1} \otimes D)\cdots(I_1 \otimes D)DD^*(I_1 \otimes D^*)\cdots(I_{j-1} \otimes D^*)(I_j \otimes B^*) = I_1 \otimes R_{i-1,j-1} - R_{i,j}$. We have

$$(S_mS_m^*)_{i,j} = \sum_{k=0}^{j}(S_m)_{i,k}(S_m)_{j,k} = \sum_{k=0}^{j} I_k \otimes K_{i-k}K_{j-k}^* = \sum_{l=0}^{j} I_{j-l} \otimes K_{i-j+l}K_l^*.$$

Using the computation above, we get, for $i = j \leq m$,

$$(S_mS_m^*)_{i,i} = I_i \otimes (I - R_{0,0}R_{0,0}^*) + \sum_{l=1}^{i}(I_{i-l+1} \otimes R_{l-1,l-1} - I_{i-l} \otimes R_{l,l}) = I - R_{i,i}$$

and, for $j < i \leq m$,

$$(S_m S_m^*)_{i,j} = -I_j \otimes R_{i-j,0} + \sum_{l=1}^{j} (I_{j-l+1} \otimes R_{i-j+l-1,l-1} - I_{j-l} \otimes R_{i-j+l,l}) = -R_{i,j}.$$

This completes the proof of the claim. If we let R be the operator whose matrix is $(R_{i,j})$ (letting $R_{i,j} = 0$ if i or j is larger than m) then we get $R = I - S_m S_m^*$. But it is easy to verify that R is a positive operator and, thus, $\|S_m\| \leq 1$. This holds for every m and, therefore, we can find a weak limit point of the sequence $\{S_m\}$. But this limit point it clearly equal to S_∞, showing that S_∞ is indeed a bounded operator, with norm at most 1.

Recall that the induced representation of $H^\infty(E)$ on $\mathcal{F}(E) \otimes_\sigma \mathcal{E}$ is the representation that maps $X \in H^\infty(E)$ to $\sigma^{\mathcal{F}(E)}(X) := X \otimes I_{\mathcal{E}}$. The representation is faithful and is a homeomorphism with respect to the ultraweak topologies. Its image is the ultraweakly closed subalgebra of $B(\mathcal{F}(E) \otimes \mathcal{E})$ generated by the operators $T_\xi \otimes I_{\mathcal{E}}$ and $\varphi_\infty(a) \otimes I_{\mathcal{E}}$ for $\xi \in E$ and $a \in M$. Similarly one defines the induced representation $\iota^{\mathcal{F}(E^\sigma)}$ of $H^\infty(E^\sigma)$ on $\mathcal{F}(E^\sigma) \otimes \mathcal{E}$ and its image is generated by the operators $T_\eta \otimes I$ and $\varphi_\infty(b) \otimes I$ for $\eta \in E^\sigma$ and $b \in N$. Recall also, from [31, Theorem 3.9], that there is a unitary operator $U : \mathcal{F}(E^\sigma) \otimes \mathcal{E} \to \mathcal{F}(E) \otimes \mathcal{E}$ such that

$$(\iota^{\mathcal{F}(E^\sigma)}(H^\infty(E^\sigma)))' = U^* \sigma^{\mathcal{F}(E)}(H^\infty(E))U.$$

That is, U gives an explicit representation of $H^\infty(E^\sigma)$ as the commutant of the induced algebra $\sigma^{\mathcal{F}(E)}(H^\infty(E))$. Thus, to show that $S_\infty = U^*(X \otimes I)U$ for an $X \in H^\infty(E)$, we need only show that S_∞ lies in the commutant of $\iota^{\mathcal{F}(E^\sigma)}(H^\infty(E^\sigma))$. And for this, we only have to show that it commutes with the operators $\varphi_\infty(b) \otimes I$, $b \in N$, and $T_\eta \otimes I$, $\eta \in E^\sigma$. Note that, matricially, $\varphi_\infty(b) \otimes I$ is a diagonal operator whose i,i entry is $\varphi_i(b)$. For S_∞ to commute with it we should have, for all $j \leq i$,

$$(I_j \otimes K_{i-j})(\varphi_j(b) \otimes I) = (\varphi_i(b) \otimes I)(I_j \otimes K_{i-j}).$$

This equality is obvious for $j > 0$. For $j = 0$ it amounts to the equality

$$K_i b = (\varphi_i(b) \otimes I_{\mathcal{E}})K_i$$

and, this, as was mentioned above, follows immediately from the properties of A, B, C and D. To show that S_∞ commutes with every $T_\eta \otimes I$, $\eta \in E^\sigma$, note that, matricially, the i,j entry of $T_\eta \otimes I$ vanishes unless $i = j+1$ and, in this case the entry is $L_j(\eta)$. Equation (9) then ensures that S_∞ and $T_\eta \otimes I$ commute.

Thus, by [31, Theorem 3.9], there is an element $X \in H^\infty(E)$ such that $S_\infty = U^*(X \otimes I)U$ $(= U^* \sigma^{\mathcal{F}(E)}(X)U)$. Since S_∞ has norm at most one, so does X. It remains to show that X is given by the transfer function built from V. To this end, fix $\xi \in E$ and recall that ξ defines a map $W(\xi) : \mathcal{E} \to E^\sigma \otimes \mathcal{E}$

via the formula $W(\xi)^*(\eta \otimes h) = L_\xi^* \eta(h)$, $\eta \otimes h \in E^\sigma \otimes \mathcal{E}$ (See [31, Theorem 3.6].), and that W maps E onto the ι-dual of E^σ. The desired properties follow easily from the definition of W. For every $k \geq 0$, $I_k \otimes W(\xi)^*$ is a map from $(E^\sigma)^{\otimes k+1} \otimes \mathcal{E}$ into $(E^\sigma)^{\otimes k} \otimes \mathcal{E}$. An easy computation shows that it is equal to the restriction of $U^*(T_\xi^* \otimes I_{\mathcal{E}})U$ to $(E^\sigma)^{\otimes k+1} \otimes \mathcal{E}$. (Recall from [31, Lemma 3.8] that the restriction of U to $(E^\sigma)^{\otimes k+1} \otimes \mathcal{E}$ is defined by the equation $U(\eta_1 \otimes \cdots \otimes \eta_{k+1} \otimes h) = (I_k \otimes \eta_1) \cdots (I_1 \otimes \eta_k)\eta_{k+1}(h)$.)
It then follows that the i,j entry of the matrix associated with $U^*(T_\xi \otimes I_{\mathcal{E}})U$ vanishes unless $i = j+1$ and

$$(U^*(T_\xi \otimes I_{\mathcal{E}})U)_{j+1,j} = I_j \otimes W(\xi).$$

Similarly one can show that, for $\xi \in E^{\otimes k}$, the i,j entry of the matrix associated with $U^*(T_\xi \otimes I_{\mathcal{E}})U$ vanishes unless $i = j+k$ and

$$(U^*(T_\xi \otimes I_{\mathcal{E}})U)_{j+k,j} = I_j \otimes W(\xi).$$

In the last equation, $W(\xi)$, $\xi \in E^{\otimes k}$, is a map from $\mathcal{E}$ to $(E^\sigma)^{\otimes k} \otimes \mathcal{E}$.
Recall that we defined ξ_n to be the vectors in $E^{\otimes n}$ with $W(\xi_n) = K_n$. Thus we see that the n^{th} lower diagonal in the matricial form of S_∞ is the matricial form of $U^*(T_{\xi_n} \otimes I_{\mathcal{E}})U$.
Recall from the discussion at the end of Section 2 in [31] that S_∞ is the ultraweak limit of the sequence Σ_k where

$$\Sigma_k = \sum_{j=0}^{k-1} (1 - \frac{j}{k}) U^*(T_{\xi_j} \otimes I)U.$$

Hence X is the ultraweak limit of X_k where

$$X_k = \sum_{j=0}^{k-1} (1 - \frac{j}{k}) T_{\xi_j}$$

and, for $\eta \in E^\sigma$, $\widehat{X}(\eta^*)$ is the ultraweak limit of $\widehat{X}_k(\eta^*) = \sum_{j=0}^{k-1}(1 - \frac{j}{k})\widehat{T_{\xi_j}}(\eta^*)$.
Fix $\eta \in E^\sigma$ and $k \geq 1$. Then it is easy to check that, in the notation of the theorem, $L_\eta^*(I_k \otimes B) = (I_{k-1} \otimes B)L_\eta^*$ and $L_\eta^*(I_k \otimes D) = (I_{k-1} \otimes D)L_\eta^*$, all as operators on $(E^\sigma)^{\otimes k} \otimes H$. It then follows that for $n \geq 1$,

$$(L_\eta^*)^n W(\xi_n) = (L_\eta^*)^n K_n = B(L_\eta^* D)^{n-1} L_\eta^* C$$

and

$$A + B(I - L_\eta^* D)^{-1} L_\eta^* C = A + \sum_{n=1}^{\infty} B(L_\eta^* D)^{n-1} L_\eta^* C = \sum_{n=0}^{\infty} (L_\eta^*)^n W(\xi_n).$$

(Note that the last series converges in norm). It follows from [31, Proposition 5.1] that $\widehat{T_{\xi_n}}(\eta^*) = (L_\eta^*)^n W(\xi_n)$ and, thus, we finally conclude that $\widehat{X}(\eta^*) = A + B(I - L_\eta^* D)^{-1} L_\eta^* C$.
The 'converse' portion of the Theorem is immediate from Theorems 3.2 and 3.3. □

Corollary 3.7 *Every Schur class operator function defined on a subset Ω^* of $\mathbb{D}(E^\sigma)^*$ with values in some $B(\mathcal{E})$ can be extended to a Schur class operator function defined on all of $\mathbb{D}(E^\sigma)^*$.*

Proof. Let Z be a Schur class function on Ω^* and apply Theorem 3.3 to represent Z as the restriction to Ω^* of a transfer function. The result then follows from the evident combination of Theorems 3.6 and 3.2. □

Recall that every element X in $H^\infty(E)$ with $\|X\| \leq 1$ defines a Schur class operator function by evaluation at η^* for $\eta \in \mathbb{D}(E^\sigma)$ (where σ is a suitable prescribed faithful normal representation of M) . We usually suppress reference to σ and write $\widehat{X}$ for this Schur class operator function. In general, however, the map $X \to \widehat{X}$ is not one-to-one, and whether it is or not depends on the choice of σ. Indeed, in the particular case when $M = \mathbb{C}$ and $E = \mathbb{C}^n$, so $H^\infty(E)$ is $\mathcal{L}_n$, and when σ is the identity representation of $\mathbb{C}$, Davidson and Pitts showed that the kernel of the map $X \mapsto \widehat{X}$ is precisely the commutator ideal in $\mathcal{L}_n$ [17]. We shall show in the next lemma that given E, if σ is chosen to be faithful and have *infinite uniform multiplicity*, meaning that σ is an infinite multiple of another faithful normal representation of M, then the map $X \mapsto \widehat{X}$ will be one-to-one. It will be convenient to write $K(\sigma)$ for the kernel of the map determined by σ, so that

$$\begin{aligned} K(\sigma) &= \{X \in H^\infty(E) : \widehat{X}(\eta^*) = 0, \ \ \eta \in \mathbb{D}(E^\sigma)\} \\ &= \{X \in H^\infty(E) : \sigma \times \eta^*(X) = 0, \ \ \eta \in \mathbb{D}(E^\sigma)\}. \end{aligned} \tag{16}$$

Lemma 3.8 *If σ is a faithful normal representation of M on a Hilbert space H of infinite multiplicity, then $K(\sigma) = 0$. Moreover, if $\{X_\beta\}$ is a bounded net in $H^\infty(E)$ and if there is an element $X \in H^\infty(E)$ such that for every $\eta \in \mathbb{D}(E^\sigma)$, $\widehat{X}_\beta(\eta^*) \to \widehat{X}(\eta^*)$ in the weak operator topology, then $X_\beta \to X$ ultraweakly.*

Proof. It follows from the structure of isomorphisms of von Neumann algebras that any two infinite multiples of faithful representations of a von Neumann algebra are unitarily equivalent. It follows, therefore, that to prove the lemma, we can pick a special representation with this property that is convenient for our purposes. So let π be the representation of M on $\mathcal{F}(E) \otimes_\sigma H$ defined by $\pi = \varphi_\infty \otimes I_H$. We shall see that $K(\pi) = \{0\}$. For $\xi \in E$ let $V(\xi) = T_\xi \otimes I_H$. Then (V, π) is a representation of E on $\mathcal{F}(E) \otimes_\sigma H$. The integrated form of this representation is the induced representation $\pi^{\mathcal{F}(E)}$ restricted to $H^\infty(E)$. It is a faithful representation of $H^\infty(E)$. For $0 \leq r \leq 1$, (rV, π) is also a representation of E. It follows from [31, Lemma 7.11] that, for every $X \in H^\infty(E)$, the limit in the strong operator topology of $(\pi \times rV)(X)$, as $r \to 1$, is $(\pi \times V)(X)$. Thus, for $X \neq 0$ in $H^\infty(E)$, there is an r, $0 \leq r < 1$, such that $(\pi \times rV)(X) \neq 0$. Since for such r the inequality $\|rV\| < 1$ holds, and we conclude that $K(\pi) = \{0\}$.

For the second assertion of the lemma, suppose a bounded net $\{X_\beta\}$ in $H^\infty(E)$ has the property that for every $\eta \in \mathbb{D}(E^\pi)$, $\widehat{X}_\beta(\eta^*) \to 0$. Since the net is

bounded, it has a ultraweak limit point X_0 in $H^\infty(E)$. Since "evaluation at η^*" is the same as applying a ultraweakly continuous representation, we see that $\widehat{X}_\beta(\eta^*) \to \widehat{X}_0(\eta^*)$ for every $\eta \in \mathbb{D}(E^\pi)$. But then, $\widehat{X}_0(\eta^*) = 0$ for every $\eta \in \mathbb{D}(E^\pi)$ and, consequently, $X_0 = 0$ by the first assertion of the lemma. $\square$
With this lemma in hand, we summarize the results of this section for future reference in the following corollary.

Corollary 3.9 *Let E be a W^*-correspondence over the W^*-algebra M, let σ be a faithful normal representation of M on the Hilbert space $\mathcal{E}$ and assume that σ has infinite multiplicity. Then the map $X \to \widehat{X}$ is a bijection from the closed unit ball of $H^\infty(E)$ onto the space of Schur class $B(\mathcal{E})$-valued functions on $\mathbb{D}(E^\sigma)^*$. Further, for each X in the closed unit ball of $H^\infty(E)$, $\widehat{X}$ is the transfer function associated with a system matrix $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ defined in terms of a suitable auxiliary normal representation τ of $\sigma(M)'$ on a Hilbert space H, and conversely, each such transfer function on $\mathbb{D}(E^\sigma)^*$,*

$$\eta^* \to A + B(I - L_\eta^* D)^{-1} L_\eta^* C,$$

is of the form $\widehat{X}$ for a uniquely determined $X \in H^\infty(E)$: $\widehat{X}(\eta^*) = A + B(I - L_\eta^* D)^{-1} L_\eta^* C$ for all $\eta \in \mathbb{D}(E^\sigma)$.

Proof. The proof is just the evident combination of Lemma 3.8 and Theorems 3.2, 3.3, and 3.6. $\square$

Remark 3.10 *One may well wonder why not stipulate at the outset that all σ's have uniform infinite multiplicity. It turns out that in many interesting examples, such as those coming from graphs, which we discuss in the last section, the principal σ's one wants to consider fail to have this property.*

4 Applications to automorphisms of the Hardy algebra

In this section we apply the analysis of Schur class functions to study automorphisms of $H^\infty(E)$. Our first goal is to show that under very general assumptions, the automorphisms are obtained by composition with (certain) biholomorphic automorphisms of the open unit ball of the dual correspondence. For the case were $E = \mathbb{C}^n$, so that $H^\infty(E)$ is the algebra $\mathcal{L}_n$ studied by Davidson and Pitts and by Popescu, this was shown for the dual correspondence associated with the one dimensional representation σ of $\mathbb{C}$ by Davidson and Pitts in [17].
Throughout this section we will focus on automorphisms α of $H^\infty(E)$ that are *completely isometric* and *w^*-homeomorphisms.* Also, we shall usually assume that the restriction of α to $\varphi_\infty(M)$ is the identity.
It is known that, in various settings, one can assume much less. In [17], the authors begin by assuming that α is simply an algebraic automorphism but, to get the one-to-one correspondence with automorphisms of the unit ball of

the dual, they need to impose also the assumption that the automorphism is contractive. It then follows from their results that it is, in fact, completely isometric and a w^*-homeomorphism. In [22], Katsoulis and Kribs show that in the setting when E is determined by a directed graph, G say, so $H^\infty(E)$ is the algebra they denote by $\mathcal{L}_G$, an algebraic automorphism is always norm-continuous and w^*-continuous.

As for the assumption that the restriction of α to $\varphi_\infty(M)$ is the identity, we shall see that for many purposes this is no significant restriction. However, in some situations, it can be a significant technical headache to sort out what happens if we don't impose the assumption. We will comment on this further, as we proceed. (See, in particular, Remark 4.10).

So, for the remainder of this section, unless specified otherwise, E will be a fixed W^*-correspondence over a W^*-algebra M and α will be a fixed automorphism of $H^\infty(E)$ that is completely isometric, w^*-homeomorphic and fixes $\varphi_\infty(M)$ element-wise. Also, σ will be a faithful normal $*$-representation of M on a Hilbert space H.

We think about elements of $H^\infty(E)$ as functions on $\mathbb{D}(E^\sigma)^*$ via the functional representation developed in the preceding section and we want to study the transposed action of α on $\mathbb{D}(E^\sigma)^*$. For every $\eta \in \mathbb{D}(E^\sigma)$, let $\tau(\eta) : H \to E\otimes_\sigma H$ be defined by the equation

$$\tau(\eta)^*(\xi \otimes h) = \widehat{\alpha(T_\xi)}(\eta^*)h \ (= (\sigma \times \eta^*)(\alpha(T_\xi))h\ ), \tag{17}$$

$\xi\otimes h \in E\otimes_\sigma H$. (Observe that if α is the identity automorphism of $H^\infty(E)$, then this equation implies that τ is the identity map, as it should.) The next lemma shows that $\tau(\eta)$ is well defined and is an element in the closed unit ball of E^σ. Thus τ is a map from $\mathbb{D}(E^\sigma)$ into $\overline{\mathbb{D}(E^\sigma)}$. What we would really like to show, however, is that τ carries $\mathbb{D}(E^\sigma)$ into $\mathbb{D}(E^\sigma)$, not the closure. At this stage, we can only arrange for this under special circumstances: Theorem 4.7 below. The restriction on circumstances, however, is not so limiting as to eliminate many interesting examples. We also want to show that τ is holomorphic on $\mathbb{D}(E^\sigma)$ in the usual sense of infinite dimensional holomorphy [21].

Lemma 4.1 *For each $\eta \in \mathbb{D}(E^\sigma)$, $\tau(\eta)$ is well defined and lies in the closed unit ball of E^σ.*

Proof. For $\xi \in E$, let $S(\xi) := (\sigma \times \eta^*)(\alpha(T_\xi))$. For every $a, b \in M$, $S(a\xi b) = (\sigma\times\eta^*)(\alpha(T_{a\xi b})) = (\sigma\times\eta^*)(\alpha(\varphi_\infty(a)T_\xi\varphi_\infty(b))) = (\sigma\circ\alpha)(\varphi_\infty(a))(\sigma\times\eta^*)(\alpha(T_\xi))(\sigma\circ\alpha)(\varphi_\infty(b))$. By our assumption, $\sigma\circ\alpha\circ\varphi_\infty = \sigma\circ\varphi_\infty$ and, thus, (S,σ) is a covariant pair. Also, S is a completely contractive map of E into $B(H)$ as a composition of three completely contractive maps. Thus $\tilde{S}^* = \tau(\eta)$ lies in the closed unit ball of E^σ. $\square$

To determine circumstances under which τ maps $\mathbb{D}(E^\sigma)$ into $\mathbb{D}(E^\sigma)$, we fix $\eta \in \mathbb{D}(E^\sigma)$ and determine circumstances under which $\tau(z\eta) \in \mathbb{D}(E^\sigma)$, for every $z \in \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. This will prove that τ maps $\mathbb{D}(E^\sigma)$ into itself.

So for $z \in \mathbb{D}$, we define

$$F(z) := \tau(\bar{z}\eta)^*. \tag{18}$$

Thus, $F(z)(\xi \otimes h) = (\sigma \times z\eta^*)(\alpha(T_\xi))h$ for $\xi \in E$ and $h \in H$.

Lemma 4.2 *F is an analytic function from $\mathbb{D}$ into $B(E \otimes H, H)$.*

Proof. Fix $\xi \otimes h \in E \otimes H$ with $\|\xi\| \leq 1$ and $k \in H$, and consider the expression

$$\langle F(z)(\xi \otimes h), k\rangle = \langle \widehat{\alpha(T_\xi)}(z\eta^*)h, k\rangle.$$

Since $\alpha(T_\xi) \in H^\infty(E)$ and $\|\alpha(T_\xi)\| \leq 1$, we know from Theorem 3.6 that we can write $\widehat{\alpha(T_\xi)}(z\eta^*) = A + B(I - zL_\eta^* D)^{-1} z L_\eta^* C$ for some system matrix. Thus

$$\widehat{\alpha(T_\xi)}(z\eta^*) = A + zBL_\eta^* C + \sum_{k=2}^{\infty} z^k B(L_\eta^*)^{k-1} L_\eta^* C.$$

Hence, for every $\xi \otimes h \in E \otimes H$ (even when $\|\xi\| > 1$) and $k \in H$, the function $z \mapsto \langle F(z)(\xi \otimes h), k\rangle$ is analytic. Since $\|F(z)\| \leq 1$ by Lemma 4.1, $|\langle F(z)g, k\rangle| \leq \|g\|\|k\|$ for every $g \in E \otimes H$ and $k \in H$ and it follows that, for each such g, k, the function $f_{g,k}(z) := \langle F(z)g, k\rangle$ is analytic in $\mathbb{D}$ and $|f_{g,k}(z)| \leq \|g\|\|k\|$. We can then write $f_{g,k}$ as a convergent power series $f_{g,k}(z) = \sum_{k=0}^{\infty} a_n(g,k) z^n$ and, for every $n \geq 0$, $|a_n(g,k)| \leq \|g\|\|k\|$. But then there are operators $A_n \in B(E \otimes H, H)$ with $\|A_n\| \leq 1$ such that $a_n(g,k) = \langle A_n g, k\rangle$ for $g \in E \otimes H$ and $k \in H$. Hence $F(z) = \sum_{k=0}^{\infty} z^n A_n$ where the sum converges in the weak operator topology. Since $|z| < 1$ and the norms of $\{A_n\}$ are bounded by 1, the series converges to $F(z)$, for $z \in \mathbb{D}$, in the norm topology. We conclude that $F(z)$ is analytic. $\square$

If we were dealing with scalar-valued functions, we would be able to assert that $|F(z)| < 1$ for all $z \in \mathbb{D}$, unless F is constant, by the maximum modulus theorem. Unfortunately, an unalloyed version of the maximum modulus theorem does not hold in our setting. This is what leads to the special hypotheses on τ in Theorem 4.7. The next few results, then, which lead up to Theorem 4.7 come out of our efforts to find a serviceable replacement for the maximum modulus theorem. Our first theorem in this direction, Theorem 4.4, is closely related to [36, Proposition V.2.1]. It does not seem to follow directly from this result, however. Instead, we appeal to the following lemma, which in turn is an immediate application of an operator form of the classical Pick criterion for interpolating operators at pre-assigned points by operator-valued analytic functions. As such, it may be traced back to Sz.-Nagy and Koranyi's influential paper [37]. It also is a consequence of Theorem 6.2 in [31], where it is presented as a corollary of our Nevanlinna-Pick Theorem.

Lemma 4.3 *If K, H are Hilbert spaces and if $F : \mathbb{D} \to B(K, H)$ is an analytic function satisfying $\|F(z)\| \leq 1$ for all $z \in \mathbb{D}$, then, for every $z_1, z_2 \in \mathbb{D}$, the matrix*

$$\begin{pmatrix} \frac{I_H - F(z_1)F(z_1)^*}{1-|z_1|^2} & \frac{I_H - F(z_1)F(z_2)^*}{1-z_1\bar{z_2}} \\ \frac{I_H - F(z_2)F(z_1)^*}{1-z_2\bar{z_1}} & \frac{I_H - F(z_2)F(z_2)^*}{1-|z_2|^2} \end{pmatrix}$$

is positive. In particular (setting $z_1 = z$ and $z_2 = 0$), for every $z \in \mathbb{D}$,

$$\begin{pmatrix} \frac{I_H - F(z)F(z)^*}{1-|z|^2} & I_H - F(z)F(0)^* \\ I_H - F(0)F(z)^* & I_H - F(0)F(0)^* \end{pmatrix} \geq 0. \tag{19}$$

THEOREM 4.4 *Suppose H and K are Hilbert spaces and suppose $F : \mathbb{D} \to B(K, H)$ is an analytic function that satisfies the following conditions:*

(1) $\|F(z)\| \leq 1$ for all $z \in \mathbb{D}$.

(2) There are projections P_1, P_2 in $B(H)$ that sum to I_H and projections Q_1, Q_2 in $B(K)$ that sum to I_K and satisfy:

(i) $P_1F(0)Q_2 = 0$ and $P_2F(0)Q_1 = 0$.

(ii) $P_1F(0)F(0)^*P_1 = P_1$.

(iii) $P_2F(0)F(0)^*P_2 \leq rP_2$ for some $0 < r < 1$.

Then, for every $z \in \mathbb{D}$,

(1) $P_1F(z)Q_2 = 0$.

(2) $P_1F(z)Q_1 = P_1F(0)Q_1$.

(3) There is a function $q_0(z)$ on $\mathbb{D}$, such that $0 < q_0(z) < 1$ for all $z \in \mathbb{D}$, and such that $P_2F(z)F(z)^*P_2 \leq q_0(z)P_2$.

PROOF. It will be convenient to use the projections P_1, P_2 and Q_1, Q_2 to write $F(z)$ matricially as

$$F(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

so that, by assumption,

$$F(0) = \begin{pmatrix} A(0) & 0 \\ 0 & D(0) \end{pmatrix}$$

where $A(0)A(0)^* = P_1$ and $D(0)D(0)^* \leq rP_2$.
Since F satisfies the conditions of Lemma 4.3, Equation 19 holds for all $z \in \mathbb{D}$. Compressing each entry of the matrix in (19) to the range of P_1 and using the fact that $A(0)A(0)^* = P_1$ and that $P_1F(0)Q_2 = 0$, we get

$$\begin{pmatrix} \frac{P_1 - P_1F(z)F(z)^*P_1}{1-|z|^2} & P_1 - P_1F(z)Q_1A(0)^* \\ P_1 - A(0)Q_1F(z)^*P_1 & 0 \end{pmatrix} \geq 0. \tag{20}$$

It follows that $P_1 = P_1F(z)Q_1A(0)^*$. Thus $0 \leq (P_1F(z)Q_1 - A(0))(Q_1F(z)^*P_1 - A(0)^*) = P_1F(z)Q_1F(z)^*P_1 + A(0)A(0)^* -$

$P_1F(z)Q_1A(0)^* - A(0)Q_1F(z)^*P_1 \leq 0$. Consequently, $A(0) = P_1F(z)Q_1$ (for every $z \in \mathbb{D}$).
But then $P_1F(z)Q_1F(z)^*P_1 = P_1$ and, since $P_1F(z)F(z)^*P_1 \leq P_1$, $P_1F(z)Q_2 = 0$. This proves (1) and (2).
Compress each entry of (19) to the range of P_2 to get

$$\begin{pmatrix} \frac{P_2 - P_2F(z)F(z)^*P_2}{1-|z|^2} & P_2 - P_2F(z)Q_2D(0)^* \\ P_2 - D(0)Q_2F(z)^*P_2 & P_2 - D(0)D(0)^* \end{pmatrix} \geq 0. \tag{21}$$

Write Δ for the positive square root of $P_2 - D(0)D(0)^*$ and note that Δ is invertible as an operator on the range of P_2. Equation (21) implies that

$$(P_2 - D(0)D(z)^*)\Delta^{-2}(P_2 - D(z)D(0)^*) \leq (\frac{P_2 - P_2F(z)F(z)^*P_2}{1-|z|^2}).$$

Since $D(0)D(z)^*$ lies in $B(P_2(H))$ and has norm strictly less than 1 (as $\|D(0)\| < 1$), $P_2 - D(0)D(z)^*$ is invertible in $B(P_2(H))$ and so, therefore, is $(P_2 - D(0)D(z)^*)\Delta^{-2}(P_2 - D(z)D(0)^*)$. Hence, for each $z \in \mathbb{D}$ there is a $q(z) > 0$, such that $\frac{P_2 - P_2F(z)F(z)^*P_2}{1-|z|^2} \geq (P_2 - D(0)D(z)^*)\Delta^{-2}(P_2 - D(z)D(0)^*) \geq q(z)P_2$. Thus,

$$P_2 - P_2F(z)F(z)^*P_2 \geq (1 - |z|^2)q(z)P_2,$$

which yields $P_2F(z)F(z)^*P_2 \leq (1 - q(z)(1 - |z|^2))P_2$. So, if we set $q_0(z) = (1 - q(z)(1 - |z|^2))$, we obtain a function with the desired properties. □
We return to our analysis of the special function $F : \mathbb{D} \to B(E \otimes_\sigma H, H)$ defined in equation (18).

Lemma 4.5 *The function F defined by equation (18) satisfies:*

(1) For every $z \in \mathbb{D}$ and $a \in M$, $F(z)(\varphi_E(a) \otimes I_H) = \sigma(a)F(z)$ and $F(z)F(z)^*$ commutes with $\sigma(M)$.

(2) For every $b \in \sigma(M)'$, $bF(0) = F(0)(I_E \otimes b)$ and $F(0)F(0)^* \in \mathfrak{Z}(\sigma(M))$.

Proof. Since $F(z)^* \in E^\sigma$ by Lemma 4.1, (1) holds. For (2), simply note that $bF(0)(\xi \otimes h) = b\alpha(T_\xi)(0)h = \alpha(T_\xi)(0)bh = F(0)(\xi \otimes bh) = F(0)(I_E \otimes b)(\xi \otimes h)$, where we used the fact that for every $X \in H^\infty(E)$, $X(0) \in \sigma(M)$. □

Definition 4.6 *Let τ be the map defined by equation (17). We say that $\tau(0)$ splits if there are projections P_1, P_2 in $\sigma(M)'$ such that*

(i) $P_1 + P_2 = I$,

(ii) $P_1\tau(0)^*\tau(0)P_1 = P_1$ and

(iii) $P_2\tau(0)^*\tau(0)P_2 \leq rP_2$ for some $r < 1$.

Note that $\tau(0) = F(0)^*$ so that, although F depends on a choice of $\eta \in \mathbb{D}(E^\sigma)$, $F(0)$ does not. It follows from Lemma 4.5, therefore, that $\tau(0)^*\tau(0)$ lies in the center of $\sigma(M)$, $\mathfrak{Z}(\sigma(M)) = \sigma(\mathfrak{Z}(M))$.

Note also that, if the center of M, $\mathfrak{Z}(M)$, is an atomic abelian von Neumann algebra, then $\tau(0)$ always splits. This is the case, in particular, if M is a factor or if $M = \mathbb{C}^n$. It is also the case, therefore, when E is the correspondence associated with a (countable) directed graph.

When $\tau(0)$ splits we have the following.

Theorem 4.7 *Assume that the left action map of M on E, φ_E, is injective and that $\tau(0)$ splits. Then the map τ defined in equation (17)) maps $\mathbb{D}(E^\sigma)$ into itself and satisfies the following equation*

$$(\widehat{\alpha(X)})(\eta^*) = \widehat{X}(\tau(\eta)^*),$$

for every $X \in H^\infty(E)$ and $\eta \in \mathbb{D}(E^\sigma)$.

Proof. Fix $\eta \in \mathbb{D}(E^\sigma)$ and let F be the map defined in (18). Since $\tau(0) = F(0)^*$ splits, there are projections P_1, P_2 as in Definition 4.6. Using Lemma 4.5, we see that the conditions of Theorem 4.4 are satisfied with $K = E \otimes H$ and $Q_i = I_E \otimes P_i$, $i = 1, 2$. Thus,

$$P_1F(z) = P_1F(z)(I_E \otimes P_1) = P_1F(0)(I_E \otimes P_1) = P_1F(0)$$

for all $z \in \mathbb{D}$. Consequently, for all $\xi \in E$, $P_1(\sigma \times z\eta^*)(\alpha(T_\xi)) = P_1\sigma(\alpha(T_\xi)_0)$ where, for $X \in H^\infty(E)$, X_0 is the image of X under the conditional expectation onto $\varphi_\infty(M)$. Since the representation $\sigma \times z\eta^*$ is w^*-continuous and α is surjective, we have for all $X \in H^\infty(E)$,

$$P_1(\sigma \times z\eta^*)(X) = P_1\sigma(X_0).$$

In particular, letting $X = T_\xi$, we see that $P_1(\sigma \times z\eta^*)(T_\xi) = 0$. Since, for $h \in H$, $(\sigma \times z\eta^*)(T_\xi)h = P_1\eta^*(\xi \otimes h) = 0$ we have $\eta P_1 = 0$. (Recall that $P_1 \in \sigma(M)'$ and, thus, ηP_1 is well defined since E^σ is a right module over $\sigma(M)'$).

Since η is arbitrary in $\mathbb{D}(E^\sigma)$, $E^\sigma P_1 = 0$. If $P_1 \neq 0$, it follows that E^σ is not full and, using Lemma 2.20, the map φ_E is not injective, contradicting our assumption. Thus $P_1 = 0$ and it follows from Theorem 4.4 that $\|F(z)\| < 1$ for every z. since this holds for all $\eta \in \mathbb{D}(E^\sigma)$, the conclusion of the theorem follows. □

Next we show that the map τ is holomorphic on $\mathbb{D}(E^\sigma)$. We view it as a map into $B(H, E \otimes H)$. To be holomorphic is the same as being Frechet-differentiable. If we use [21, Theorem 3.17.1] and the fact, proved in Lemma 4.1, that τ is bounded, it suffices to show that τ is (G)-differentiable in the sense of [21, Definition 3.16.2]. But if we apply [21, Theorem 3.16.1], this means that we have to show that for every $\eta_0, \eta \in \mathbb{D}(E^\sigma)$, the function $G(z) := \tau(\eta_0 + z\eta)$, defined on $D(\eta,\eta_0) := \{z \in \mathbb{C} | |z| < (1-\|\eta_0\|)/\|\eta\|\}$ is holomorphic in the sense of [21, Definition 3.10.1].

Since the set of all functionals on $B(H, E \otimes H)$ that are w^*-continuous is a determining manifold for $B(H, E \otimes H)$ in the sense of [21, Definition 2.8.2], it suffices to show that for every w^*-continuous functional w, the map $z \mapsto w(\tau(\eta_0 + z\eta))$ is holomorphic on $D(\eta,\eta_0)$. It is enough, in fact, to consider all functionals of the form $T \mapsto \langle Th, \xi \otimes k\rangle$ for $h, k \in H$ and ξ in the unit ball of E.

So we fix $\eta_0, \eta \in E^\sigma$, $h, k \in H$ and $\xi \in E$ with $\|\xi\| < 1$ and write $f(z) = \langle \tau(\eta_0 + z\eta)h, \xi \otimes k\rangle$ for $z \in D(\eta,\eta_0)$. We have

$$f(z) = \langle h, \tau(\eta_0 + z\eta)^*(\xi \otimes k)\rangle = \langle h, \widehat{\alpha(T_\xi)}(\eta_0^* + \bar{z}\eta^*)k\rangle.$$

Note that by Theorem 3.6, we can write

$$\widehat{\alpha(T_\xi)}(\eta_0^* + z\eta^*) = A + \sum_{m=1}^{\infty} B((L_{\eta_0}^* + \bar{z}L_\eta^*)D)^{m-1}(L_{\eta_0}^* + \bar{z}L_\eta^*)C$$

where A, B, C, D are from some system matrix and the sum converges in norm. Thus

$$f(z) = \langle A^*h, k\rangle + \sum_{m=1}^{\infty} \langle C^*(L_{\eta_0} + zL_\eta)(D^*(L_{\eta_0} + zL_\eta))^{m-1}B^*h, k\rangle$$

and this function is clearly holomorphic.

We can conclude:

Corollary 4.8 *The function τ is a holomorphic map from $\mathbb{D}(E^\sigma)$ to its closure.*

Theorem 4.9 *Let E be a faithful W^*-correspondence over M, let α be an automorphism of $H^\infty(E)$ that is completely isometric, is a w^*-homeomorphism and leaves $\varphi_\infty(M)$ elementwise fixed, and let σ be a faithful representation of M. Write τ for the transpose of α defined in equation (17) and write θ for the map associated similarly with α^{-1}. If both $\tau(0)$ and $\theta(0)$ split (as in Definition 4.6) then τ is a biholomorphic map of the open unit ball of E^σ, $\tau^{-1} = \theta$, and, for every $X \in H^\infty(E)$,*

$$(\widehat{\alpha(X)})(\eta^*) = \widehat{X}(\tau(\eta)^*)\ ,\ \eta \in \mathbb{D}(E^\sigma). \tag{22}$$

Proof. We already know that, under the conditions of the theorem, both τ and θ are holomorphic maps of the open unit ball. It follows from equation (17) that, for every $\xi \in E$, $h \in H$ and $\eta \in \mathbb{D}(E^\sigma)$, $\widehat{\alpha(T_\xi)}(\eta^*) = \tau(\eta)^*(\xi \otimes h)$. But $\tau(\eta)^*(\xi \otimes h) = \widehat{T_\xi}(\tau(\eta)^*)$, so that equation (22) holds for T_ξ. It also holds for $\varphi_\infty(a)$, $a \in M$, since $\alpha(\varphi_\infty(a)) = \varphi_\infty(a)$. Therefore it holds for every X in a w^*-dense subalgebra of $H^\infty(E)$. By the w^*-continuity of α, equation (22)

holds for every $X \in H^\infty(E)$. Since a similar claim holds for α^{-1} and θ, we conclude that for all $X \in H^\infty(E)$, $\widehat{X}(\eta^*) = \alpha^{-1}\widehat{(\alpha(X))}(\eta^*) = \widehat{\alpha(X)}(\theta(\eta)^*) = \widehat{X}(\tau(\theta(\eta))^*)$. Thus $\tau^{-1} = \theta$. $\Box$
A biholomorphic map τ is said to *implement* α if equation (22) holds.

REMARK 4.10 *If α is implemented by τ in the sense of equation (22), then, writing this equation when $X = \varphi_\infty(a)$, $a \in M$, shows that α leaves $\varphi_\infty(M)$ elementwise fixed. Also, inspecting the proof of Lemma 4.1, one sees that, if α does not have this property, the map τ, defined in equation (17) would map the unit ball of E^σ into the unit ball of E^π where $\pi = \sigma \circ \varphi_\infty^{-1} \circ \alpha \circ \varphi_\infty$. One can study such automorphisms by studying these maps but the situation becomes quite complicated, unless one makes a global assumption to begin with,* vis., *that σ has uniform infinite multiplicity. In that event, by properties of normal representations of von Neumann algebras, σ and π are unitarily equivalent. Say $\pi(\cdot) = u\sigma(\cdot)u^*$ for some Hilbert space isomorphism from the Hilbert space of σ to the Hilbert space of π. Then it is a straightforward calculation to see that $E^\pi = (I \otimes u)E^\sigma u^*$. It is then a straightforward matter to incorporate u into our formulas.*

As we have remarked before, $\mathbb{D}(E^\sigma)$ is the unit ball of a J^*-triple system. It results, therefore, from well-known theory [20] that the biholomorphic maps of $\mathbb{D}(E^\sigma)$ are determined by Möbius transformations (and "isometric multipliers"). As we shall, however, the Möbius transformations of $\mathbb{D}(E^\sigma)$ that implement automorphisms of $H^\infty(E)$ have to have a special form: They must be parametrized by "central" elements of $\mathbb{D}(E^\sigma)$ in the sense of the following definition. (See also Remark 2.1.3 of [14]).

DEFINITION 4.11 *Let E be a W^*-correspondence over a W^*-algebra M. The* center of E, *denoted $\mathfrak{Z}(E)$, is the set of $\xi \in E$ such that $a\xi = \xi a$ for all $a \in M$.*

LEMMA 4.12 *(1) The center $\mathfrak{Z}(E)$ of a W^*-correspondence E over M is a W^*-correspondence over the center $\mathfrak{Z}(M)$ of M.*

(2) Let σ be a faithful normal representation of M on the Hilbert space $\mathcal{E}$, and for $\xi \in E$, define $\Phi(\xi) := L_\xi$ where L_ξ maps $\mathcal{E}$ to $E \otimes \mathcal{E}$ via the formula $L_\xi(h) = \xi \otimes h$. Then the pair (σ, Φ) defines an isomorphism of $\mathfrak{Z}(E)$ onto $\mathfrak{Z}(E^\sigma)$ in the sense of Definition 2.2. (Here, $\mathfrak{Z}(E)$ is a correspondence over $\mathfrak{Z}(M)$ and $\mathfrak{Z}(E^\sigma)$ is a correspondence over $\mathfrak{Z}(\sigma(M)') = \mathfrak{Z}(\sigma(M)) = \sigma(\mathfrak{Z}(M))$).

(3) Given a faithful representation σ of M on the Hilbert space $\mathcal{E}$ and $\gamma \in \mathbb{D}(E^\sigma)$, then γ lies in the center of E^σ if and only if the representation $\sigma \times \gamma^*$ maps $H^\infty(E)$ into $\sigma(M)$.

PROOF. It is clear that $\mathfrak{Z}(E)$ is a bimodule over $\mathfrak{Z}(M)$ and, to prove (1), we need only show that the inner product of two elements in $\mathfrak{Z}(E)$ lies in $\mathfrak{Z}(M)$.

For $a \in M$, $\xi_1, \xi_2 \in \mathfrak{Z}(E)$ we have

$$a\langle \xi_1, \xi_2 \rangle = \langle \xi_1 a^*, \xi_2 \rangle = \langle a^* \xi_1, \xi_2 \rangle = \langle \xi_1, a\xi_2 \rangle = \langle \xi_1, \xi_2 a \rangle = \langle \xi_1, \xi_2 \rangle a.$$

Hence the inner product lies in the center of M, proving (1). We fix a faithful representation σ of M on $\mathcal{E}$. For $\xi \in \mathfrak{Z}(E)$, $a \in M$ and $h \in \mathcal{E}$ we have $L_\xi \sigma(a)h = \xi \otimes_\sigma \sigma(a)h = \xi a \otimes h = a\xi \otimes h = (a \otimes I)L_\xi h$. Hence, $L_\xi \in E^\sigma$. Given $b \in \sigma(M)'$ and $h \in \mathcal{E}$ we have $L_\xi bh = \xi \otimes bh = (I_E \otimes b)L_\xi h$. Thus L_ξ lies in $\mathfrak{Z}(E^\sigma)$.
For $\xi \in \mathfrak{Z}(E)$, $a, b \in \mathfrak{Z}(M)$, and $h \in \mathcal{E}$, $L_{a\xi b}h = a\xi b \otimes h = \xi ab \otimes h = \xi \otimes \sigma(a)\sigma(b)h = (I \otimes \sigma(a))L_\xi \sigma(b)h$ hence

$$\Phi(a\xi b) = \sigma(a)\Phi(\xi)\sigma(b).$$

For $\xi_1, \xi_2 \in \mathfrak{Z}(E)$ we have $L_{\xi_1}^* L_{\xi_2} = \sigma(\langle \xi_1, \xi_2 \rangle)$. Therefore the pair (σ, Φ) is an isomorphism of $\mathfrak{Z}(E)$ into $\mathfrak{Z}(E^\sigma)$.
To prove that the map Φ is onto, fix an $\eta \in \mathfrak{Z}(E^\sigma)$. Then, η is a map from $\mathcal{E}$ to $E \otimes_\sigma \mathcal{E}$ satisfying

$$\eta\sigma(a) = (a \otimes I)\eta \tag{23}$$

and

$$\eta b = (I \otimes b)\eta, \tag{24}$$

for $a \in M$ and $b \in \sigma(M)'$. Define the map $\psi : E \to B(\mathcal{E})$ by $\psi(\zeta) = \eta^* L_\zeta$ and note that for $b \in \sigma(M)'$ and $h \in \mathcal{E}$, $\eta^* L_\zeta bh = \eta^*(\zeta \otimes bh) = \eta^*(I \otimes b)L_\zeta h$. Using (24) the latter is equal to $b\eta^* L_\zeta h$. Hence $\psi(\zeta)$ lies in $\sigma(M)$. Also $\psi(\zeta a) = \psi(\zeta)\sigma(a)$ for all $a \in M$ and it then follows from the self duality of E that there is an $\xi \in E$ with $\langle \xi, \zeta \rangle = \sigma^{-1}(\psi(\zeta))$. Thus, for all $\zeta \in E$, $L_\xi^* L_\zeta = \sigma(\langle \xi, \zeta \rangle) = \eta^* L_\zeta$ and we conclude that $\eta = L_\xi$.
It follows from (23) that, for all $a \in M$, $L_{\xi a} = \eta\sigma(a) = (a \otimes I)\eta = L_{a\xi}$, showing that ξ lies in $\mathfrak{Z}(E)$.
Finally, to prove (3), fix an $\eta \in \mathbb{D}(E^\sigma)$ and write (T, σ) for the covariant pair associated with $\sigma \times \eta^*$ (so that, $\tilde{T} = \eta^*$). Then the representation maps $H^\infty(E)$ into $\sigma(M)$ if and only if, for each $\xi \in E$, $T(\xi) \in \sigma(M)$. This holds iff, for all $b \in \sigma(M)'$, $\xi \in E$ and $h \in \mathcal{E}$, $\tilde{T}(I_\mathcal{E} \otimes b)(\xi \otimes h) = T(\xi)bh = bT(\xi)h = b\tilde{T}(\xi \otimes h)$; that is, if and only if $\tilde{T}(I_\mathcal{E} \otimes b) = b\tilde{T}$ for every $b \in \sigma(M)'$. But the last statement says that η lies in the center of E^σ. $\square$
The following example may help to show that the center of a correspondence is much less "inert" than the center of a von Neumann algebra.

Example 4.13 *Let M be a von Neumann algebra and let α be an endomorphism of M. Then we obtain a W^*-correspondence over M, denoted ${}_\alpha M$, by taking M with its usual right action and inner product give by the formula, $\langle \xi, \eta \rangle = \xi^* \eta$ and by letting α implement the left action. Then an element ξ in ${}_\alpha M$ lies in the center of ${}_\alpha M$ if and only if ξ intertwines α and the identity endomorphism; i.e., $\xi \in \mathfrak{Z}({}_\alpha M)$ if and only if $\alpha(a)\xi = \xi a$ for all $a \in M$. $\mathfrak{Z}({}_\alpha M)$ is a much studied object in the literature and the preceding lemma spells out some of its important elementary properties.*

Our goal now is to develop the properties of Möbius transformations of $\mathbb{D}(E^\sigma)$ and to identify those that implement automorphisms of $H^\infty(E)$. To this end, fix a faithful representation σ of M on a Hilbert space $\mathcal{E}$. Set $N = \sigma(M)'$, write $K = \mathcal{E} \oplus (E \otimes_\sigma \mathcal{E})$, and define the (necessarily faithful) representation ρ of N on K by the formula

$$\rho(S) = \begin{pmatrix} S & 0 \\ 0 & I \otimes S \end{pmatrix}, \ \ S \in N.$$

For $\gamma \in \mathbb{D}(E^\sigma)$ we set $\Delta_\gamma := (I_{\mathcal{E}} - \gamma^*\gamma)^{1/2}$ - an element in $B(\mathcal{E})$ - and $\Delta_{\gamma*} := (I_{E\otimes\mathcal{E}} - \gamma\gamma^*)^{1/2}$ - an element in $B(E \otimes \mathcal{E})$. When γ is understood, then we shall simply write Δ for Δ_γ and Δ_* for Δ_{γ^*}. Given $\gamma \in \mathbb{D}(E^\sigma)$ we define the map g_γ on $\mathbb{D}(E^\sigma)^*$ by the formula,

$$g_\gamma(z^*) = \Delta_\gamma(I - z^*\gamma)^{-1}(\gamma^* - z^*)\Delta_{\gamma^*}^{-1}, \tag{25}$$

$z \in \mathbb{D}(E^\sigma)$. Then g_γ is a biholomorphic automorphism of $\mathbb{D}(E^\sigma)^*$ that maps 0 to γ^* and γ^* to 0. Further, $g_\gamma^2 = id$, and every biholomorphic map g of $\mathbb{D}(E^\sigma)^*$ is of the form

$$g = w \circ g_\gamma$$

where w is an isometry on $(E^\sigma)^*$ and $\gamma^* = w^{-1}g(0)$ [20]. When γ lies in the center of E^σ, we see that g_γ maps the center onto itself and it follows that every biholomorphic automorphism of the open unit ball of $(E^\sigma)^*$ that preserves the center is of the form

$$g = w \circ g_\gamma$$

where γ lies in the center and w is an isometry on $(E^\sigma)^*$ that preserves the center.

If $z \in \mathbb{D}(E^\sigma)$, then the series $\sum_{n=0}^\infty (z^*\gamma)^n$ converges in norm to the operator in N, $(I - z^*\gamma)^{-1} = \sum_{n=0}^\infty (z^*\gamma)^n$. One easily calculates, then, that

$$g_\gamma(z^*) = \Delta\gamma^*\Delta_*^{-1} - \Delta(I - z^*\gamma)^{-1}z^*\Delta_*.$$

Recall that the equation $U(z \otimes h) = z(h)$ defines a Hilbert space isomorphism $U : E^\sigma \otimes \mathcal{E} \to E \otimes \mathcal{E}$ [31, p. 369]. Consequently, as maps on $\mathcal{E}$, $UL_z = z$ and $z^* = L_z^*U^*$. Thus we may write

$$g_\gamma(z^*) = \Delta\gamma^*\Delta_*^{-1} - \Delta(I - L_z^*U^*\gamma)^{-1}L_z^*U^*\Delta_*.$$

We write $K_1 = E \otimes_\sigma \mathcal{E}$ for the second summand in $K = \mathcal{E} \oplus (E \otimes_\sigma \mathcal{E})$ and we let q_1 denote the projection from K onto K_1. Likewise, we set $K_2 = \mathcal{E}$ with projection q_2. Corresponding to the direct sum decomposition, we define V by the formula

$$V := \begin{pmatrix} \Delta\gamma^*\Delta_*^{-1} & -\Delta \\ U^*\Delta_* & U^*\gamma \end{pmatrix} : \begin{pmatrix} K_1 \\ \mathcal{E} \end{pmatrix} \to \begin{pmatrix} K_2 \\ E^\sigma \otimes \mathcal{E} \end{pmatrix}. \tag{26}$$

If we calculate VV^*, we find that the off diagonal terms vanish and the terms on the diagonal are $\Delta\gamma^*\Delta_*^{-2}\gamma\Delta+\Delta^2$ and $U^*(\Delta_*^2+\gamma\gamma^*)U$. Since $\Delta_*^2+\gamma\gamma^* = I_{E\otimes\mathcal{E}}$, the latter expression is $U^*U = I_{E^\sigma\otimes\mathcal{E}} = q_2$. For the first expression, we note that $\gamma^*\Delta_*^{-2}\gamma = \gamma^*(I-\gamma\gamma^*)^{-1}\gamma = (I-\gamma^*\gamma)^{-1}-1$ and $\Delta\gamma^*\Delta_*^{-2}\gamma\Delta+\Delta^2 = \Delta((I-\gamma^*\gamma)^{-1}-I)\Delta+\Delta^2 = I_{\mathcal{E}}$. This shows that V is a coisometry. Similar computations show that it is, in fact, a unitary operator. Thus V is a transfer operator.

We want to apply Theorem 3.6 to obtain an element $X\in H^\infty(E)$ with $\widehat{X}(\eta^*) = g_\gamma(\eta^*)$, for $\eta\in\mathbb{D}(E^\sigma)$. To do this, we first let F be the correspondence E^σ and then F^ρ is a correspondence over $\rho(N)'$. In order to apply Theorem 3.6 we let M, in that theorem, be the von Neumann algebra $\rho(N)'$ and let σ there be the identity representation of $\rho(N)'$ on K (so that $\mathcal{E}$ there is K). E in that theorem will be F^ρ and N there (the commutant of $\sigma(M)$) will be $\rho(N)$. The representation τ of N then will be the map ρ^{-1} of $\rho(N)$ on $\mathcal{E}$ (so that $\mathcal{E}$ will play the role of H there). Also, q_1 will be as above. We set $A=\Delta\gamma^*\Delta_*^{-1}$, $B=-\Delta$, $C=U^*\Delta_*$ and $D=U^*\gamma$. These A,B,C and D give rise to the matricial operator V of equation (26). In order to show that the assumptions of Theorem 3.6 are satisfied, we have to show that these operators (A,B,C and D) all have the required intertwining properties. (Note that we have already checked that V is a unitary operator).

The required intertwining properties are:

(a) $A=\Delta\gamma^*\Delta_*^{-1}$ lies in $q_2\rho(N)'q_1$.

(b) $B=-\Delta$ lies in N'.

(c) For every $S\in N$, $U^*\Delta_*(I_E\otimes S) = (S\otimes I_{\mathcal{E}})U^*\Delta_*$ on $E\otimes\mathcal{E}$.

(d) For every $S\in N$, $U^*\gamma S = (I_E\otimes S)U^*\gamma$ on $\mathcal{E}$.

Indeed, recall that γ lies in the center of E^σ and, thus, for $S\in N$, $\gamma S = (I\otimes S)\gamma$. Therefore Δ commutes with N and Δ_* commutes with $I\otimes S$ for $S\in N$. This implies (a) and (b). Recall that, for $h\in\mathcal{E}$, $U^*\gamma h = \gamma\otimes h$ and, thus, $U^*\gamma Sh = \gamma\otimes Sh = (I\otimes S)(\gamma\otimes h) = (I\otimes S)U^*\gamma h$ proving (d). For (c), it suffices to note that $U(S\otimes I)U^* = I\otimes S$ and Δ_* commutes with $I\otimes S$ for all $S\in N$.

We can now apply Theorem 3.6. Since F^ρ plays the role of E in that theorem and the identity representation of $\rho(N)'$, id, plays the role of σ, E^σ in that theorem is replaced by $(F^\rho)^{id}$ which, by the duality theorem [31, Theorem 3.6] is isomorphic to $F=E^\sigma$. We therefore conclude:

Lemma 4.14 *For every $\gamma\in\mathbb{D}(\mathfrak{Z}(E^\sigma))$, there is an X in $H^\infty(F^\rho)$ with $\|X\|\le 1$ such that, for all $z\in\mathbb{D}(E^\sigma)$, $\widehat{X}(z^*) = g_\gamma(z^*)$.*

Note that $g_\gamma(z^*)$ is an operator from $E\otimes\mathcal{E}$ into $\mathcal{E}$ and can be viewed as an operator in $B(K)$ which is where the values of X, as an element of $H^\infty(F^\rho)$, lie.

We can now use [31, Theorem 5.3] to prove the following.

Corollary 4.15 *Fix $\gamma \in \mathbb{D}(\mathfrak{Z}(E^\sigma))$ as above. Then, for every $z_1, z_2, \ldots, z_k$ in $\mathbb{D}(E^\sigma)$, the map on $M_k(\sigma(M)')$ defined by the $k \times k$ matrix*

$$((id - \theta_{g_\gamma(z_i^*)^*, g_\gamma(z_j^*)^*}) \circ (id - \theta_{z_i, z_j})^{-1})$$

is completely positive.

Proof. Applying [31, Theorem 5.3] to X of Lemma 4.14, we get the complete positivity of the map defined by the matrix

$$((I - Ad(g_\gamma(z_i^*), g_\gamma(z_j^*))) \circ (id - \theta_{z_i, z_j})^{-1}).$$

But note that, for every $b \in \sigma(M)'$, $Ad(g_\gamma(z_i^*), g_\gamma(z_j^*))(\rho(b)) = g_\gamma(z_i^*)\rho(b)g_\gamma(z_j^*)^* = \langle g_\gamma(z_i^*)^*, bg_\gamma(z_j^*)^* \rangle = \theta_{g_\gamma(z_i^*)^*, g_\gamma(z_j^*)^*}(b)$. $\square$

Corollary 4.16 *Let $Z : \mathbb{D}(E^\sigma)^* \to B(\mathcal{E})$ be a Schur class operator function and let γ be in $\mathbb{D}(\mathfrak{Z}(E^\sigma))$. Then the function $Z_\gamma : \mathbb{D}((E^\sigma)^*) \to B(\mathcal{E})$ defined by*

$$Z_\gamma(\eta^*) = Z(g_\gamma(\eta^*))$$

is also a Schur class operator function.

Proof. For every η_i, η_j in $\mathbb{D}(E^\sigma)$ we have $(id - Ad(Z(g_\gamma(\eta_i^*)), Z(g_\gamma(\eta_j^*)))) \circ (id - \theta_{\eta_i, \eta_j})^{-1} = ((id - Ad(Z(g_\gamma(\eta_i^*)), Z(g_\gamma(\eta_j^*)))) \circ (id - \theta_{g_\gamma(\eta_i^*)^*, g_\gamma(\eta_j^*)^*})^{-1}) \circ (id - \theta_{g_\gamma(\eta_i^*)^*, g_\gamma(\eta_j^*)^*}) \circ (id - \theta_{\eta_i, \eta_j})^{-1})$. Hence the map associated with Z_γ is a composition of two completely positive maps and is, therefore, completely positive. $\square$

For the statement of the next lemma, recall from [31, end of Section 2] that every $X \in H^\infty(E)$ has a "Fourier series" expansion given by a sequence of "Fourier coefficient operators" $\{\mathbb{E}_j\}$. (In [31] we wrote $\{\Phi_j\}$ for this sequence). Each map $\mathbb{E}_j$ is completely contractive, w^*-continuous and $\mathbb{E}_j(T_{\xi_1} T_{\xi_2} \cdots T_{\xi_k}) = T_{\xi_1} T_{\xi_2} \cdots T_{\xi_k}$ if $j = k$ and is zero otherwise. The Cesaro means of the "Fourier series" of X converge to X in the w^*-topology.

Lemma 4.17 *Let σ be a normal, faithful, representation of M on a Hilbert space H and let $K(\sigma)$ denote the kernel of the map $X \to \widehat{X}$ defined in equation (16).*

(i) $K(\sigma) \subseteq \{X \in H^\infty(E) \mid \mathbb{E}_0(X) = \mathbb{E}_1(X) = 0\}$.

(ii) If, for every $k \in \mathbb{N}$, $\vee\{(\eta^{\otimes k})(H) \mid \eta \in \mathbb{D}(E^\sigma)\} = E^{\otimes k} \otimes H$, then $K(\sigma) = \{0\}$.

(iii) Every completely isometric automorphism α of $H^\infty(E)$ that is a w^*-homeomorphism and is implemented by a biholomorphic map of $\mathbb{D}(E^\sigma)$ in the sense of (22) leaves $K(\sigma)$ invariant. In particular, $K(\sigma)$ is invariant under the action of the gauge group and, thus, under the maps $\mathbb{E}_k$, $k \geq 0$.

Proof. Write C_1 for $\{X \in H^\infty(E) \mid \mathbb{E}_0(X) = \mathbb{E}_1(X) = 0\}$. Then for every $X \in H^\infty(E)$, $X = \mathbb{E}_0(X) + \mathbb{E}_1(X) + X_1$ where $X_1 \in C_1$. Note that for every $\eta \in \mathbb{D}(E^\sigma)$, every $0 < t \leq 1$ and every $k \geq 0$, $\mathbb{E}_k(X)((t\eta)^*) = t^k \mathbb{E}(X)(\eta^*)$. Thus, for $X \in K(\sigma)$, $0 = X((t\eta)^*) = \mathbb{E}_0(X)(\eta^*) + t\mathbb{E}_1(X)(\eta^*) + t^2 S$ where S is some bounded operator on H. Since this holds for every $0 < t \leq 1$, we have (by differentiation) $\mathbb{E}_0(X) = 0$ and $\mathbb{E}_1(X)(\eta^*) = 0$ for all $\eta \in \mathbb{D}(E^\sigma)$. Write $\mathbb{E}_1(X) = T_\xi$ (for some $\xi \in E$). Then, for all $h \in H$ and $\eta \in \mathbb{D}(E^\sigma)$, $0 = \mathbb{E}_1(X)(\eta^*)h = \eta^*(\xi \otimes h)$. Since $\vee\{\eta(H) \mid \eta \in \mathbb{D}(E^\sigma)\} = E \otimes H$ ([31, Lemma 3.5]), we find that $\xi \otimes h = 0$ for all $h \in H$. Since E is faithful, this implies that $\xi = 0$, completing the proof of (i).
We can also write $0 = X((t\eta)^*) = \mathbb{E}_0(X)(\eta^*) + t\mathbb{E}_1(X)(\eta^*) + \cdots + t^k \mathbb{E}_k(X)(\eta^*) + t^{k+1} S$ and conclude that $\mathbb{E}_j(X)(\eta^*) = 0$ for all $j \leq k$. We can then continue as above but to be able to conclude that $\mathbb{E}_k(X) = 0$ we need the condition in part (ii) (to replace the use of [31, Lemma 3.5] in the argument above).
To prove (iii), note that the invariance of $K(\sigma)$ under an automorphism α as in (iii) follows from (22). The invariance under the gauge group (and under $\mathbb{E}_k$) is then immediate. $\square$
The following proposition is obvious if $K(\sigma) = \{0\}$. But, in fact, it holds for every faithful, normal representation σ. The argument uses an idea from [17, Proof of Theorem 4.11].

Proposition 4.18 *Let σ be a faithful, normal representation of M and let α, β be two homomorphisms of $H^\infty(E)$ into itself such that β is completely isometric, surjective and a w^*-homeomorphism, while α is completely contractive and w^*-continuous. Suppose they satisfy the equation*

$$\widehat{\alpha(X)}(\eta^*) = \widehat{\beta(X)}(\eta^*)$$

for all $X \in H^\infty(E)$ and $\eta \in \mathbb{D}(E^\sigma)$. Then $\alpha = \beta$.

Proof. It is clearly enough to assume $\beta = id$ and $\widehat{\alpha(X)}(\eta^*) = \widehat{X}(\eta^*)$. Note that α, viewed as a representation of $H^\infty(E)$ on $\mathcal{F}(E) \otimes_\sigma H$ (whose restriction to $\varphi_\infty(M)$ is $\varphi_\infty(\cdot) \otimes I_H$), can be written as $(\varphi_\infty(\cdot) \otimes I_H) \times \zeta^*$ for some ζ in the closed unit ball of the $\varphi_\infty(\cdot) \otimes I_H$-dual of E. Thus, for $k \in \mathcal{F}(E) \otimes_\sigma H$, $\alpha(T_\xi)k = (\zeta^*)(\xi \otimes k)$ and $\|\alpha(T_\xi)k\| \leq \|\xi \otimes k\| = \|T_\xi k\|$.
Fix $h \in H$ viewed as the zeroth summand of $\mathcal{F}(E) \otimes_\sigma H$. Then for every $\xi \in E$,

$$\|\alpha(T_\xi)h\| \leq \|T_\xi h\|.$$

By construction $\alpha(T_\xi) - T_\xi \in K(\sigma)$. But also, by Lemma 4.17(i), for every $X \in K(\sigma)$, Xh is orthogonal to $T_\xi h$. Thus

$$\|\alpha(T_\xi)h\|^2 = \|(\alpha(T_\xi) - T_\xi)h\|^2 + \|T_\xi h\|^2 \geq \|T_\xi h\|^2.$$

We conclude that for every $h \in H$, $(\alpha(T_\xi) - T_\xi)h = 0$. It follows that $\alpha(T_\xi) = T_\xi$ for all $\xi \in E$. Since α is a w^*-continuous homomorphism, $\alpha(X) = X$ for all $X \in H^\infty(E)$. $\square$

The following lemma will prove very useful when we deal with a representation σ for which $K(\sigma) \neq \{0\}$. It relates the σ-dual with the π-dual where π is the representation defined in the proof of Lemma 3.8 (for which $K(\pi) = \{0\}$).

LEMMA 4.19 *Let σ be a faithful representation of M on H and π be the representation $\varphi_\infty \otimes I_H$ of M on $K := \mathcal{F}(E) \otimes H$. Let $\psi : \sigma(M)' \to (\varphi_\infty(M) \otimes I_H)'$ be defined by $\psi(b) = I_E \otimes b$ and let $\Psi : E^\sigma \to E^\pi$ be defined by $\Psi(\eta) = I_{\mathcal{F}(E)} \otimes \eta$. Then we have the following.*

(1) The pair (ψ, Ψ) is an isomorphism of E^σ into (not necessarily onto) E^π satisfying
$$\Psi(\eta)P_H = P_{E\otimes H}\Psi(\eta) = \eta \ , \ \eta \in E^\sigma$$
where P_H is the projection from K to H (viewed as a subspace) and $P_{E\otimes H}$ is the projection of $E \otimes K$ onto $E \otimes H$.

(2) For every $X \in H^\infty(E)$ and $\zeta \in E^\pi$ that satisfies $\zeta P_H = P_{E\otimes H}\zeta$, we have $\zeta|H \in E^\sigma$ and the restriction of $\widehat{X}(\zeta^*)$ to H (viewed as a summand of $\mathcal{F}(E) \otimes H = H \oplus E \otimes H \oplus \cdots$) is $\widehat{X}((\zeta|H)^*)$.

(3) There is an isomorphism Φ of $\mathfrak{Z}(E^\sigma)$ onto $\mathfrak{Z}(E^\pi)$ satisfying
$$\Phi(\gamma)P_H = P_{E\otimes H}\Phi(\gamma) = \gamma \ , \ \gamma \in \mathfrak{Z}(E^\sigma).$$

(4) For $\eta \in E^\sigma$ and $\gamma \in \mathfrak{Z}(E^\sigma)$,
$$g_{\Phi(\gamma)}(\Psi(\eta)^*)P_{E\otimes H} = P_H g_{\Phi(\gamma)}(\Psi(\eta)^*) = g_\gamma(\eta^*).$$

PROOF. It is clear that ψ is indeed an isomorphism into $(\varphi_\infty(M) \otimes I_H)'$. Note that it follows from the intertwining property of $\eta \in E^\sigma$ that $\Psi(\eta)$ is a well defined bounded operator. To show that Ψ maps E^σ into E^π, fix $\eta \in E^\sigma$, $\theta \otimes h \in \mathcal{F}(E) \otimes H$ and $a \in M$ and compute $(I_{\mathcal{F}(E)} \otimes \eta)\pi(a)(\theta \otimes h) = (I_{\mathcal{F}(E)} \otimes \eta)(\varphi_\infty(a)\theta \otimes h) = \varphi_\infty(a)\theta \otimes \eta(h)$, where we view $\mathcal{F}(E) \otimes E$ as the subspace of $\mathcal{F}(E)$ consisting of all the positive tensor powers of E. But the last expression is equal to $(\varphi_\infty(a) \otimes I_H)(I_{\mathcal{F}(E)} \otimes \eta)(\theta \otimes h)$, showing that $\Psi(\eta) \in E^\pi$.
To show that the map is a bimodule map, fix $\eta \in E^\sigma$, $b, c \in \sigma(M)'$ and $\theta \otimes h \in \mathcal{F}(E) \otimes H$. Then $\Psi(c\eta b)(\theta \otimes h) = \theta \otimes (c\eta b)h = \theta \otimes (I_E \otimes c)\eta bh = \psi(c)(\theta \otimes \eta bh) = \psi(c)\Psi(\eta)(\theta \otimes bh) = \psi(c)\Psi(\eta)\psi(b)(\theta \otimes h)$, proving that the image of Ψ lies in E^π. Regarding the inner product, we have: $\langle \Psi(\eta_1), \Psi(\eta_2)\rangle = \Psi(\eta_1)^*\Psi(\eta_2) = (I_{\mathcal{F}(E)} \otimes \eta_1)^*(I_{\mathcal{F}(E)} \otimes \eta_2) = (I_{\mathcal{F}(E)} \otimes \eta_1^*\eta_2) = \psi(\langle \eta_1, \eta_2\rangle)$ for all $\eta_1, \eta_2 \in E^\sigma$. Thus (ψ, Ψ) is an isomorphism of E^σ into E^π. The proof of the equation $\Psi(\eta)P_H = P_{E\otimes H}\Psi(\eta) = \eta$ for $\eta \in E^\sigma$ is easy. This proves (1).
To prove (2), let $\zeta \in E^\pi$ satisfy $\zeta P_H = P_{E\otimes H}\zeta$ and fix $a \in M$ and $h \in H$. Then $(\zeta|H)\sigma(a)h = \zeta(\varphi_\infty(a) \otimes I_H)h = (\varphi_E(a) \otimes I_K)P_{E\otimes H}\zeta h = (\varphi_E(a) \otimes I_H)(\zeta|H)h$. Thus, $\zeta|H \in E^\sigma$. To prove that $\widehat{X}((\zeta|H)^*) = \widehat{X}(\zeta^*)|H$, let, first, consider $X = \varphi_\infty(a)$ for $a \in M$. Then $\widehat{X}(\zeta^*) = \varphi_\infty(a) \otimes I_H$ and $\widehat{X}((\eta|H)^*) = \sigma(a)$ and (2)

holds in this case. Take $X = T_\xi$ for some $\xi \in E$. Then, for $h \in H \subseteq \mathcal{F}(E) \otimes H$, $\widehat{X}(\zeta^*)h = \zeta^*(\xi \otimes h) = (\zeta|H)^*(\xi \otimes h) = \widehat{X}((\zeta|H)^*)h$. In particular, we see that H is invariant for all $\widehat{X}(\zeta^*)$ where X runs over a set of generators. Thus, H is invariant under $\widehat{X}(\zeta^*)$ for all $X \in H^\infty(E)$ and (2) holds for all X's in a w^*-dense subalgebra of $H^\infty(E)$. Since the map $X \mapsto \widehat{X}(\zeta^*)$ is w^*-continuous, we are done.

To prove (3), recall from Lemma 4.12 (2) that both $\mathfrak{Z}(E^\sigma)$ and $\mathfrak{Z}(E^\pi)$ are isomorphic to $\mathfrak{Z}(E)$. Combining these two isomorphisms, we get Φ. More precisely, every $\eta \in \mathfrak{Z}(E^\sigma)$ is equal to L_ξ for some $\xi \in \mathfrak{Z}(E)$ (that is, $\eta(h) = \xi \otimes h$, $h \in H$). Then we set $\Phi(\eta)k = \xi \otimes k$ for $k \in K = \mathcal{F}(E) \otimes H$. The equation $\Phi(\gamma)P_H = P_{E\otimes H}\Phi(\gamma) = \gamma$, $\gamma \in \mathfrak{Z}(E^\sigma)$ follows easily.

Part (4) follows from (1) and (3). □

Fix $X \in H^\infty(E)$ with $\|X\| \leq 1$, let $\pi = \varphi_\infty \otimes I_H$, as in Lemma 3.8, and let γ be an element of $\mathbb{D}(\mathfrak{Z}(E^\pi))$. Then if $\widehat{X}$ is the Schur class operator function on $\mathbb{D}((E^\pi)^*)$ determined by X then by Corollary 4.16, $\widehat{X} \circ g_\gamma$ also is a Schur class operator function on $\mathbb{D}((E^\pi)^*)$. By Corollary 3.9 there is an element $\alpha_\gamma(X)$ in $H^\infty(E)$, whose norm does not exceed 1, such that $\widehat{\alpha_\gamma(X)} = \widehat{X} \circ g_\gamma$. Further, by Lemma 3.8, this element is uniquely defined. We can, of course, extend this to a map, α_γ, from $H^\infty(E)$ to itself such that, for $X \in H^\infty(E)$ and $\eta \in \mathbb{D}((E^\pi)^*)$,

$$\widehat{\alpha_\gamma(X)}(\eta^*) = \widehat{X}(g_\gamma(\eta^*)). \tag{27}$$

Lemma 4.20 *Let σ and π be as in Lemma 4.19. Then:*

(i) For every $\gamma \in \mathbb{D}(\mathfrak{Z}(E^\pi))$, α_γ, defined by equation (27) is an automorphism of the algebra $H^\infty(E)$ that is completely isometric and is a homeomorphism with respect to the ultraweak topology.

(ii) For every $\gamma \in \mathbb{D}(\mathfrak{Z}(E^\sigma))$ let α_γ be defined to be $\alpha_{\Phi(\gamma)}$ (with Φ as in Lemma 4.19). Then, for every $X \in H^\infty(E)$ and $\eta \in E^\sigma$,

$$\widehat{\alpha_\gamma(X)}(\eta^*) = \widehat{X}(g_\gamma(\eta^*)). \tag{28}$$

Proof. We first prove (i). Linearity and multiplicativity of α_γ are easy to check. Since $g_\gamma^2 = id$, α_γ is invertible (with $\alpha_\gamma^{-1} = \alpha_\gamma$). So it is an automorphism. Since α_γ maps the closed unit ball of $H^\infty(E)$ into itself (as does the inverse map), α_γ is isometric. It is, in fact, completely isometric. To see this, consider, for $n \in \mathbb{N}$, the algebra $H^\infty(M_n(E))$, associated with the W^*-correspondence $M_n(E)$ over the von Neumann algebra $M_n(M)$. The corresponding Fock space is $M_n(\mathcal{F}(E))$ and the algebra can be identified with $M_n(H^\infty(E))$. The representation σ of M gives rise to a representation σ_n of $M_n(M)$ on $H^{(n)} = \mathbb{C}^n \otimes H$ (with $\sigma_n(M_n(M))' = I_{\mathbb{C}^n} \otimes \sigma(M)' \cong \sigma(M)')$. One can check that $E^\sigma \cong (M_n(E))^{\sigma_n}$. For $\gamma \in \mathfrak{Z}(E^\sigma)$, write γ' for the corresponding element of $\mathfrak{Z}(M_n(E^\sigma))$. Then $\alpha_{\gamma'}$ acts on $M_n(H^\infty(E))$ by applying α_γ to each

entry. Since we know that $\alpha_{\gamma'}$ is an isometry, it follows that α_γ is a complete isometry.
It is left to show that α_γ is continuous with respect to the ultraweak topology. For this, let $\{X_\beta\}$ be a net in the closed unit ball of $H^\infty(E)$ that converges ultraweakly to X. Since evaluating at η^* (for η in the open unit ball) amounts to applying a ultraweakly continuous representation , we have, for every such η, $\widehat{X_\beta}(\eta^*) \to \widehat{X}(\eta^*)$ in the weak operator topology. Since this holds for $g_\gamma(\eta^*)$ in place of η, we see that, for every η in the open unit ball of E^σ,

$$\widehat{\alpha_\gamma(X_\beta)}(\eta^*) \to \widehat{\alpha_\gamma(X)}(\eta^*).$$

Using Lemma 3.8, we find that $\alpha_\gamma(X_\beta) \to \alpha_\gamma(X)$ in the ultraweak topology. This proves (i).
Part (ii) of the lemma results from the following computation

$$\widehat{\alpha_\gamma(X)}(\eta^*) = \widehat{\alpha_{\Phi(\gamma)}(X)}(\Psi(\eta)^*)|H = \widehat{X}(g_{\Phi(\gamma)}(\Psi(\eta)^*))|H$$

$$= \widehat{X}(g_{\Phi(\gamma)}(\Psi(\eta)^*)|E \otimes H) = \widehat{X}(g_\gamma(\eta)^*),$$

where we used equation (27) and Lemma 4.19. $\square$
Note that we needed to use the representation π in order to define, for every $X \in H^\infty(E)$, the element $\alpha_\gamma(X)$ in $H^\infty(E)$ satisfying (27). That is, we used the fact that $K(\pi) = 0$. Once we defined it, it may be more convenient to work with the original representation σ (which can be chosen to be an arbitrary faithful representation) and invoke (28). Note that, using Proposition 4.18, we see that there is only one automorphism that satisfies (28).

Theorem 4.21 *Let E be a W^*-correspondence over M and let σ be a faithful normal representation of M on a Hilbert space H. Let α be an isometric automorphism of $H^\infty(E)$ and assume that $g : \mathbb{D}(E^\sigma)^* \to \mathbb{D}(E^\sigma)^*$ is a biholomorphic automorphism of $\mathbb{D}(E^\sigma)^*$ such that*

$$\widehat{\alpha(X)}(\eta^*) = \widehat{X}(g(\eta^*)),$$

for all $X \in H^\infty(E)$ and all $\eta \in E^\sigma$. Then:

(i) $g(\mathbb{D}\mathfrak{Z}((E^\sigma)^*)) \subseteq \mathbb{D}\mathfrak{Z}((E^\sigma)^*)$.

(ii) There is a $\gamma \in \mathbb{D}\mathfrak{Z}((E^\sigma))$ and a unitary operator u in $\mathcal{L}(E)$ such that $u(\mathfrak{Z}(E)) = \mathfrak{Z}(E)$ and such that

$$g(\eta^*) = g_\gamma(\eta^*) \circ (u \otimes I_{\mathcal{E}})$$

(as a map from $E \otimes_\sigma H$ to H).

(iii) With u as in (ii), there is an automorphism α_u of $H^\infty(E)$ such that $\alpha_u(T_\xi) = T_{u\xi}$ for every $\xi \in E$.

(iv) With u and γ as in (ii),

$$\alpha = \alpha_\gamma \circ \alpha_u$$

where α_γ is the automorphism defined in equation (27) (and satisfies (28)).

(v) For every $\eta_1, \eta_2, \ldots, \eta_k$ in the open unit ball of E^σ, the map defined by the $k \times k$ matrix

$$((id - \theta_{g(\eta_i^*)^*, g(\eta_j^*)^*}) \circ (id - \theta_{\eta_i,\eta_j})^{-1})$$

is completely positive.

Proof. Note first that, since α is an isometric automorphism, it maps $\varphi_\infty(M)$ onto itself.

Suppose η lies in $\mathbb{D}(\mathfrak{Z}(E^\sigma)^*)$. Then, by part (3) of Lemma 4.12, $\widehat{X}(\eta^*) \in \sigma(M)$ for every $X \in H^\infty(E)$. But then, for every X, $\widehat{X}(g(\eta^*))$ lies in $\sigma(M)$, showing that $g(\eta^*) \in \mathfrak{Z}(E^\sigma)$. This proves (i).

The discussion following Lemma 4.12 shows that we can write $g = w \circ g_\gamma$ for some γ in $\mathbb{D}\mathfrak{Z}((E^\sigma))$ and an isometry w on $(E^\sigma)^*$ that preserves the center. Let α_γ be the automorphism described in Lemma 4.20(ii) and write $\beta = \alpha_\gamma^{-1} \circ \alpha$. Then it follows that

$$\widehat{\beta(X)}(\eta^*) = \widehat{X}(w\eta^*)$$

for $X \in H^\infty(E)$ and $\eta \in \mathbb{D}(E^\sigma)$.

For $\eta = 0$ and $Y \in H^\infty(E)$ we have $\widehat{Y}(0) = \sigma(\mathbb{E}_0(Y))$ where $\mathbb{E}_0$ is the conditional expectation of $H^\infty(E)$ onto M (where M is viewed as the "zeroth term"). Thus, $\sigma(\mathbb{E}_0(\beta(X))) = \widehat{\beta(X)}(0) = \widehat{X}(0) = \sigma(\mathbb{E}_0(X))$ for every $X \in H^\infty(E)$. Since σ is faithful, $\mathbb{E}_0(\beta(X)) = \mathbb{E}_0(X)$. Thus, for every $\xi \in E$, $\mathbb{E}_0(\beta(T_\xi)) = 0$ and we can write

$$\beta(T_\xi) = T_\theta + Y \tag{29}$$

where Y lies in $(T_E)^2 H^\infty(E)$. Write C for $(T_E)^2 H^\infty(E)$. Since (29) holds for all $\xi \in E$, $\beta(C) \subseteq C$. We can apply the same arguments to β^{-1}, in place of β, and find that $\beta^{-1}(C) \subseteq C$. Applying β^{-1} to (29), we find that

$$\beta^{-1}(T_\theta) = T_\xi + Z \tag{30}$$

for some $Z \in C$.

Arguing as in the proof of Proposition 4.18, we find that, for every $h \in H$, $\|\beta(T_\xi)h\| \le \|T_\xi h\|$ and $\|\beta(T_\xi)\|^2 = \|Yh\|^2 + \|T_\theta h\|^2 \ge \|T_\theta h\|^2$. Thus $\|T_\xi h\| \ge \|T_\theta h\|$. Applying the same arguments to β^{-1} (using (30) in place of (29)) we find that $\|T_\theta h\| \ge \|T_\xi h\|$ and, thus, $\|T_\xi h\| = \|T_\theta h\|$ and, consequently, $Yh = 0$ for all $h \in H$. Thus $Y = 0$ and $\beta(T_\xi) = T_\theta$. Since β is isometric, $\|T_\xi\| = \|T_\theta\|$. It follows that $\|\xi\| = \|\theta\|$. If we write $\theta = u\xi$ (and recall that then $\beta(T_\xi) = T_{u\xi}$) then u is a linear isometry. We also have, for $a \in M$, $T_{u(\xi a)} = \beta(T_{\xi a}) = \beta(T_\xi a) = \beta(T_\xi)a = T_{u(\xi)}a = T_{u(\xi)a}$. Hence u is an isometric

(right) module map and, therefore, u lies in $\mathcal{L}(E)$. Since β is an automorphism, u is a unitary operator. We also have $\beta(T_\xi) = T_{u\xi}$, so $\beta = \alpha_u$ (in the notation of (iii)). This proves (iii) and (iv).
Recall that $\widehat{\beta(X)}(\eta^*) = \widehat{X}(w\eta^*)$ and set $X = T_\xi$ to get $\widehat{T_{u\xi}}(\eta^*) = \widehat{\beta(T_\xi)}(\eta^*) = \widehat{T_\xi}(w\eta^*)$. Hence $\eta^* L_{u\xi} = (w\eta^*)L_\xi$. Applying this to $h \in \mathcal{E}$ we get $\eta^*(u\xi \otimes h) = (w\eta^*)(\xi \otimes h)$. Hence $w\eta^* = \eta^* \circ (u \otimes I)$, proving $g(\eta^*) = g_\gamma(\eta^*) \circ (u \otimes I_{\mathcal{E}})$.
To prove (ii) we need only to show that u preserves the center of E. So fix $\xi \in \mathfrak{Z}(E)$. By Lemma 4.12, L_ξ^* lies in the center of $(E^\sigma)^*$. Thus wL_ξ^* lies in $\mathfrak{Z}((E^\sigma)^*)$. But $wL_\xi^* = L_\xi^* \circ (u \otimes I) = L_{u^*\xi}$. Thus $L_{u^*\xi}$ lies in $\mathfrak{Z}((E^\tau)^*)$. Using Lemma 4.12 again we get $u^*\xi \in \mathfrak{Z}(E)$. This shows that $u^*\mathfrak{Z}(E) \subseteq \mathfrak{Z}(E)$ and, applying the same argument to β^{-1}, we complete the proof of (ii).
To prove (v), fix $b \in \sigma(M)'$ and η_i, η_j in $\mathbb{D}(E^\sigma)$ and compute $\langle g(\eta_i^*), b \cdot g(\eta_j^*)\rangle = g(\eta_i^*)(I_E \otimes b)g(\eta_j^*)^* = g_\gamma(\eta_i^*)(u \otimes I_{\mathcal{E}})(I_E \otimes b)(u^* \otimes I_{\mathcal{E}})g_\gamma(\eta_j)^* = g_\gamma(\eta_i^*)(I_E \otimes b)g_\gamma(\eta_j^*)^* = \langle g_\gamma(\eta_i^*), b \cdot g_\gamma(\eta_j^*)\rangle$. Thus (v) follows from Corollary 4.15. $\square$
Combining Theorem 4.21 with Theorem 4.9, we get the following.

Theorem 4.22 *Let E be a faithful W^*-correspondence over M where $\mathfrak{Z}(M)$ is atomic. Let α be an automorphism of $H^\infty(E)$ that is completely isometric and a w^*-homeomorphism and leaves $\varphi_\infty(M)$ elementwise fixed and let σ be a faithful representation of M.*
Then there is a $\gamma \in \mathbb{D}\mathfrak{Z}((E^\sigma))$ and a unitary operator u in $\mathcal{L}(E)$, satisfying $u(\mathfrak{Z}(E)) = \mathfrak{Z}(E)$, such that

$$\alpha = \alpha_\gamma \circ \alpha_u,$$

where α_γ is the automorphism defined in Lemma 4.20 and $\alpha_u(T_\xi) = T_{u\xi}$ for every $\xi \in E$.
In particular, if $\mathfrak{Z}(E) = \{0\}$, every such automorphism is α_u for some unitary operator $u \in \mathcal{L}(E)$.

Theorem 4.22 provides another perspective on the results from [26, 27]. The analytic crossed products discussed there are of the form $H^\infty(E)$, where E is the correspondence ${}_\alpha M$ associated with a von Neumann algebra M and an automorphism α that is properly outer. This means that $\mathfrak{Z}(E) = \{0\}$. Theorem 4.22 implies that all automorphisms of $H^\infty(E)$ are given by automorphisms of M.

5 Examples : Graph Algebras

In this section we consider some examples that come from directed graphs. We shall assume for simplicity that our graphs have finitely many vertices and edges. We write $\mathcal{Q}$ both for the graph and for its set of edges. The space of vertices will be denoted V. We shall write s and r for the source and range maps on $\mathcal{Q}$, mapping $\mathcal{Q}$ to V, and we shall think of an edge e in $\mathcal{Q}$ as "pointing" from $s(e)$ to $r(e)$. For simplicity, we shall also assume that r is surjective, i.e., we shall assume that $\mathcal{Q}$ is *without sources.* Write $\mathcal{Q}^*$ for the set of all finite paths in $\mathcal{Q}$, i.e., the path category generated by $\mathcal{Q}$. An element in $\mathcal{Q}$ will be written $\alpha = e_1 e_2 \cdots e_k$, where $s(e_i) = r(e_{i+1})$. We set $s(\alpha) = s(e_k)$, $r(\alpha) = r(e_1)$, and $|\alpha| = k$, the length of α. We will also view vertex $v \in V$ as a "path of length 0", and we extend r and s to V simply by setting $r(v) = s(v) = v$.

Let M be $C(V)$, the set of complex-valued functions on V. Of course, M is a finite dimensional commutative von Neumann algebra. Likewise, we let E be $C(\mathcal{Q})$, the set of complex-valued functions on $\mathcal{Q}$. Then we define an M-bimodule structure on E as follows: for $f \in E$, $\psi \in M$ and $e \in \mathcal{Q}$,

$$(f\psi)(e) := f(e)\psi(s(e)),$$

and

$$(\psi f)(e) := \psi(r(e))f(e).$$

Note that the "no sources" assumption implies that the left action of M is faithful. An M-valued inner product on E will be given by the formula

$$\langle f, g\rangle(v) = \sum_{s(e)=v} \overline{f(e)} g(e),$$

for $f, g \in E$ and $v \in V$. With these operations, E becomes a W^*-correspondence over M. The algebra $H^\infty(E)$ in this case will be written $H^\infty(\mathcal{Q})$. In the literature, $H^\infty(\mathcal{Q})$ is sometimes denoted $\mathcal{L}_\mathcal{Q}$. It is the ultraweak closure of the tensor algebra $\mathcal{T}_+(E(\mathcal{Q}))$ acting on the Fock space of $\mathcal{F}(E(\mathcal{Q}))$. For $e \in \mathcal{Q}$, let δ_e be the δ-function at e, i.e., $\delta_e(e') = 1$ if $e = e'$ and is zero otherwise. Then T_{δ_e} is a partial isometry that we denote by S_e. Also, for $v \in V$, P_v is defined to be $\varphi_\infty(\delta_v)$. Then each P_v is a projection and it is an easy matter to see that the families $\{S_e : e \in \mathcal{Q}\}$ and $\{P_v : v \in V\}$ form a *Cuntz-Toeplitz family* in the sense that the following conditions are satisfied:

(i) $P_v P_u = 0$ if $u \neq v$,

(ii) $S_e^* S_f = 0$ if $e \neq f$

(iii) $S_e^* S_e = P_{s(e)}$ and

(iv) $\sum_{r(e)=v} S_e S_e^* \leq P_v$ for all $v \in V$.

In fact, these particular families yield a faithful representation of the Cuntz-Toeplitz algebra $\mathcal{T}(E(\mathcal{Q}))$ [19]. The algebra $\mathcal{T}_+(E(\mathcal{Q}))$ is the norm-closed (unstarred) algebra that they generate inside $\mathcal{T}(E(\mathcal{Q}))$ and $H^\infty(\mathcal{Q})$ is the ultraweak closure of $\mathcal{T}_+(E(\mathcal{Q}))$. The algebra $\mathcal{T}_+(E(\mathcal{Q}))$ was first defined and studied in [25], providing examples of the theory developed in [28]. It was called a quiver algebra there because in pure algebra, graphs of the form $\mathcal{Q}$ are called quivers. (Hence the notation we use here.) The properties of quiver algebras were further developed in [29]. In [23], the focus was on $H^\infty(\mathcal{Q})$ and the authors called this algebra a free semigroupoid algebras. Both algebras are often represented as algebras of operators on $l_2(\mathcal{Q}^*)$, and it will be helpful to understand how

from the perspective of this note. Let H_0 be a Hilbert space whose dimension equals the number of vertices, let $\{e_v |\ v \in V\}$ be a fixed orthonormal basis for H_0 and let σ_0 be the diagonal representation of $M = C(V)$ on H_0. Then $l_2(\mathcal{Q}^*)$ is isomorphic to $\mathcal{F}(E(\mathcal{Q}))\otimes_{\sigma_0} H_0$ where the isomorphism maps an element ξ_α of the standard orthonormal basis of $l_2(\mathcal{Q}^*)$ to $\delta_\alpha \otimes e_{s(e)}$ (where, for $\alpha = e_1 \cdots e_k$, $\delta_\alpha = \delta_{e_1} \otimes \cdots \otimes \delta_{e_k} \in E^{\otimes k}$). The partial isometries S_e can then be viewed as the shift operators $S_e \xi_\alpha = \xi_{e\alpha}$. Thus, the representations of $\mathcal{T}_+(E(\mathcal{Q}))$ and $H^\infty(\mathcal{Q})$ on $l_2(\mathcal{Q}^*)$ are just the representations induced by σ_0.
Quite generally, a completely contractive covariant representation of $E(\mathcal{Q})$ on a Hilbert space H is given by a representation σ of $M = C(V)$ on H and by a contractive map $\tilde{T} : E \otimes_\sigma H \to H$ satisfying equation (2). The representation σ is given by the projections $Q_v = \sigma(\delta_v)$ whose sum is I. Also, from $\tilde{T}$ we may define maps $T(e) \in B(H)$ by the equation $T(e)h = \tilde{T}(\delta_e \otimes h)$ and it is easy to check that $\tilde{T}\tilde{T}^* = \sum_e T(e)T(e)^*$ and $T(e) = Q_{r(e)}T(e)Q_{s(e)}$. Thus to every completely contractive representation of the quiver algebra $\mathcal{T}_+(E(\mathcal{Q}))$ we associate a family $\{T(e)|e \in \mathcal{Q}\}$ of maps on H that satisfy $\sum_e T(e)T(e)^* \leq I$ and $T(e) = Q_{r(e)}T(e)Q_{s(e)}$. Conversely, every such family defines a representation, written $\sigma \times T$ (or $\sigma \times \tilde{T}$), satisfying $(\sigma \times T)(S_e) = T(e)$ and $(\sigma \times T)(P_v) = Q_v$.
We fix σ to be σ_0 and write H in place of H_0. So that, in this case, each projection Q_v is one dimensional (with range equal to $\mathbb{C}e_v$). Then obviously $\sigma(M)' = \sigma(M)$. To describe the σ-dual of E, write $\mathcal{Q}^{-1}$ for the directed graph obtained from $\mathcal{Q}$ by reversing all arrows, so that $s(e^{-1}) = r(e)$ and $r(e^{-1}) = s(e)$. Sometimes $\mathcal{Q}^{-1}$ is denoted $\mathcal{Q}^{op}$ and is called the opposite graph. Note that the Hilbert space $E\otimes_\sigma H_0$ is spanned by the orthonormal basis $\{\delta_e \otimes e_{s(\alpha)}\}$.
Fix $\eta \in E^\sigma$ and note that its covariance property implies that, for every $e \in \mathcal{Q}$, $\eta^*(\delta_e \otimes e_{s(e)}) = \eta^*(\delta_{r(e)}\delta_e \otimes e_{s(e)}) = Q_{r(e)}\eta^*(\delta_e \otimes e_{s(e)}) = \overline{\eta(e^{-1})}e_{r(e)}$ for some $\overline{\eta(e^{-1})} \in \mathbb{C}$. The reason for the "strange" way of writing that scalar is that we can view η as an element of $E(\mathcal{Q}^{-1})$ and the correspondence structure on E^σ, as described in Proposition 2.13, fits the correspondence structure of $E(\mathcal{Q}^{-1})$. Consequently, we can identify the two and write

$$E^\sigma = E(\mathcal{Q}^{-1}).$$

(See Example 4.3 in [31] for a description of the structure of the dual correspondence for more general representations σ). It will also be convenient to write η matricially with respect to the orthonormal bases $\{\delta_v \mid v \in V\}$ of H_0 and $\{\delta_e \otimes e_{s(e)}\}_{e\in\mathcal{Q}}$ of $E \otimes H_0$ as

$$(\eta)_{e,r(e)} = \eta(e^{-1}). \tag{31}$$

Suppose $\eta \in \mathbb{D}(E^\sigma)$. For every $X \in H^\infty(\mathcal{Q})$, we have defined $X(\eta^*)$ as an element of $B(H)$ in Remark 2.14. For the generators of $H^\infty(\mathcal{Q})$, the definition yields the equations,

$$\widehat{P_v}(\eta^*) = \theta_{v,v}\ ,\ v \in V \tag{32}$$

and

$$\widehat{S_e}(\eta^*) = \overline{\eta(e^{-1})}\theta_{r(e),s(e)}\ ,\ e \in \mathcal{Q} \tag{33}$$

where $\theta_{v,w}$ is the partial isometry operator on H that maps e_w to e_v and vanishes on $(e_w)^{\perp}$. For a general $X \in H^\infty(\mathcal{Q})$, $\widehat{X}(\eta^*)$ is obtained by using the linearity, multiplicativity and w^*-continuity of the map $X \mapsto \widehat{X}(\eta^*)$.
The proof of the next lemma is straightforward and is omitted.

Lemma 5.1 *The centers of the correspondences $E(\mathcal{Q})$ and $E(\mathcal{Q}^{-1})$ are given by the formulae*

$$\mathfrak{Z}(E(\mathcal{Q})) = span\{\delta_e \mid s(e) = r(e)\}$$

and

$$\mathfrak{Z}(E(\mathcal{Q}^{-1})) = span\{\delta_{e^{-1}} \mid s(e) = r(e)\}.$$

The following proposition is immediate from Theorem 4.22.

Proposition 5.2 *If there is no $e \in \mathcal{Q}$ with $s(e) = r(e)$, then every automorphism α of $H^\infty(\mathcal{Q})$ that is completely isometric, w^*-homeomorphic and leaves $\varphi_\infty(C(V))$ elementwise fixed (that is, does not permute the vertices) is of the form α_u for some unitary $u \in \mathcal{L}(E(\mathcal{Q}))$. That is,*

$$\alpha(S_e) = \sum_{s(f)=s(e)} u_{f,e} S_f$$

where the scalars $u_{f,e}$ are given by $u_{f,e} = (u(\delta_e))(f)$. (Note that this is zero if $s(f) \neq s(e)$, since $u(\delta_e) = u(\delta_e \delta_{s(e)}) = u(\delta_e)\delta_{s(e)}$).

We note, as we did at the beginning of Section 4, that the assumptions made on the automorphism can be weakened using arguments of [22] but we shall not elaborate on this here.

Example 5.3 *Let $\mathcal{Q}$ be an n-cycle (for $n > 1$) ; that is $V = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{Q} = \{e_1, \ldots, e_n\}$ where e_i is the arrow from v_1 to v_{i+1} (or to v_1 when $i = n$). Then, for every α as in Proposition 5.2, there are $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $|\lambda_i| = 1$, such that $\alpha(S_{e_i}) = \lambda_i S_{e_i}$ for all i.*

The rest of this section will be devoted to the study of the following example, which is very simple, yet provides a full array of the structures we have been studying.

Example 5.4 *Let the vertex set of the graph have two elements: $V = \{v, w\}$. Suppose the edge set consists of three elements $\mathcal{Q} = \{e, f, g\}$, where e is the arrow from v to w, so $s(e) = v$, $r(e) = w$; f is an arrow from w to v; and g is a loop based at w, $s(g) = r(g) = w$.*

Then by Lemma 5.1, $\mathfrak{Z}(E(\mathcal{Q})) = \mathbb{C}\delta_g$. We know from Theorem 4.22 that every automorphism α is the composition of an automorphism, written α_u associated with a unitary in $\mathcal{L}(E(\mathcal{Q}))$ that maps δ_g into $\lambda_3\delta_g$ (with $|\lambda_3| = 1$) and an automorphism associated with a “Möbius transformation".

As noted in Proposition 5.2, $(u(\delta_{e'}))(f') = 0$ unless $s(e') = s(f')$, so that $u(\delta_e) \in \mathbb{C}\delta_e$ and $u(\delta_f) \in span\{\delta_f, \delta_g\}$. Since u^* is unitary, we have that $u(\delta_f) = \lambda_f \delta_f$. Thus

$$\alpha_u(S_e) = \lambda_e S_e, \;\; \alpha_u(S_f) = \lambda_f S_f \tag{34}$$

and

$$\alpha_u(S_g) = \lambda_g S_g$$

for $\lambda_e, \lambda_f, \lambda_g$ with absolute value 1.
It is left to analyze the Möbius transformations and the corresponding automorphisms. Since the center of E^σ are scalar multiples of $\delta_{g^{-1}}$, the Möbius transformations are associated with scalars $\lambda \in \mathbb{D}$ (in fact, with $\lambda\delta_{g^{-1}}$) and will be denoted τ_λ, $\lambda \in \mathbb{D}$. We have

$$\tau_\lambda(\eta^*) = \Delta_\lambda(I - \eta^*(\lambda\delta_{g^{-1}}))^{-1}(\bar{\lambda}\delta_{g^{-1}} - \eta^*)\Delta_{\lambda*}^{-1} \tag{35}$$

where $\Delta_\lambda = (I_H - (\lambda\delta_{g^{-1}})^*(\lambda\delta_{g^{-1}}))^{1/2}$ and $\Delta_{\lambda*} = (I_{E\otimes H} - (\lambda\delta_{g^{-1}})(\lambda\delta_{g^{-1}})^*)^{1/2}$. It will be convenient to write $\tau_\lambda(\eta^*)$ matricially as a map from $E \otimes H$, with the ordered orthonormal basis $\{\delta_e \otimes \delta_v, \delta_f \otimes \delta_w, \delta_g \otimes \delta_w\}$, to H, with the ordered orthonormal basis $\{\delta_v, \delta_w\}$. Using the formula (31), we see that

$$\eta = \begin{pmatrix} 0 & \eta(e^{-1}) \\ \eta(f^{-1}) & 0 \\ 0 & \eta(g^{-1}) \end{pmatrix}$$

and

$$\lambda\delta_{g^{-1}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \lambda \end{pmatrix}.$$

The computation of the expression in (35) yields

$$\tau_\lambda(\eta^*) = \begin{pmatrix} 0 & -\overline{\eta(f^{-1})} & 0 \\ \frac{-\overline{\eta(e^{-1})}(1-|\lambda|^2)^{1/2}}{1-\lambda\overline{\eta(g^{-1})}} & 0 & \frac{\bar{\lambda}-\overline{\eta(g^{-1})}}{1-\lambda\overline{\eta(g^{-1})}} \end{pmatrix}.$$

Thus

$$\overline{\tau_\lambda(\eta^*)^*(e^{-1})} = \frac{-\overline{\eta(e^{-1})}(1-|\lambda|^2)^{1/2}}{1-\lambda\overline{\eta(g^{-1})}} = -\overline{\eta(e^{-1})}(1-|\lambda|^2)^{1/2}\sum_{k=0}^{\infty}(\lambda\overline{\eta(g^{-1})})^k,$$

$$\overline{\tau_\lambda(\eta^*)^*(f^{-1})} = -\overline{\eta(f^{-1})},$$

and

$$\overline{\tau_\lambda(\eta^*)^*(g^{-1})} = \frac{\bar{\lambda}-\overline{\eta(g^{-1})}}{1-\lambda\overline{\eta(g^{-1})}} = (\bar{\lambda}-\overline{\eta(g^{-1})})\sum_{k=0}^{\infty}(\lambda\overline{\eta(g^{-1})})^k.$$

This suggests setting

$$T(e) = -(1-|\lambda|^2)^{1/2}\sum_{k=0}^{\infty}(\lambda S_g)^k S_e,$$

$$T(f) = -S_f$$

and

$$T(g) = -(\bar{\lambda}P_w - S_g)\sum_{k=0}^{\infty}(\lambda S_g)^k.$$

Using (32), (33) and the fact that the map $X \mapsto \widehat{X}(\eta^*)$ is a continuous homomorphism, we get

$$\widehat{T(e)}(\eta^*) = \overline{\tau_\lambda(\eta^*)^*(e^{-1})}\theta_{w,v}$$

,

$$\widehat{T(f)}(\eta^*) = \overline{\tau_\lambda(\eta^*)^*(f^{-1})}\theta_{v,w}$$

and

$$\widehat{T(g)}(\eta^*) = \overline{\tau_\lambda(\eta^*)^*(g^{-1})}\theta_{w,w}.$$

Using Theorem 4.9, Theorem 4.22, Equation (34) and Theorem 4.18, we conclude the following.

Theorem 5.5 *(1) For every $\lambda \in \mathbb{D}$, there is a unique automorphism α_λ of $H^\infty(\mathcal{Q})$ such that, for every $e' \in \{e, f, g\}$, $\alpha_\lambda(S_{e'}) - T(e') \in K(\sigma)$.*

(2) Every completely isometric, w^*-homeomorphic automorphism α of $H^\infty(\mathcal{Q})$ can be written

$$\alpha = \alpha_u \circ \alpha_\lambda$$

where $\lambda \in \mathbb{D}$ and $\alpha_u(S_{e'}) = \lambda_{e'}S_{e'}$ for every $e' \in \{e, f, g\}$ (where λ_e, λ_f and λ_g are complex numbers of absolute value 1).

Proof. The only thing that we need to clarify here is that, in part (2), we do not have to require that α fixes P_v and P_w. Indeed, assume that α satisfies $\alpha(P_v) = P_w$ and $\alpha(P_w) = P_v$. Then $\alpha(S_e) = P_v\alpha(S_e)P_w$ and, thus, $\mathbb{E}_0(\alpha(S_e)) = 0$ and $\mathbb{E}_1(\alpha(S_e)) \in \mathbb{C}S_f$. Similarly, we get $\mathbb{E}_0(\alpha(S_f)) = \mathbb{E}_1(\alpha(S_g)) = 0$, $\mathbb{E}_1(\alpha(S_f)) \in \mathbb{C}S_e$ and $\mathbb{E}_0(\alpha(S_g)) \in \mathbb{C}P_v$. Thus, S_g is not in the range of α, contradicting the surjectivity of α. $\Box$

Finally, we note the following.

Proposition 5.6 *In this example, $K(\sigma)$ is the ideal generated by the commutator $[S_g, S_eS_f]$.*

Proof. Since we shall not use this result, we only sketch the idea of the proof. It follows from Lemma 4.17 that it suffices to analyze $\mathbb{E}_k(K(\sigma))$ for a given k. Since $K(\sigma)$ is an ideal, it suffices to consider $P_{v'}\mathbb{E}_k(K(\sigma))P_{v''}$ for fixed $v', v'' \in \{v, w\}$. Evaluating an element of $P_{v'}\mathbb{E}_k(K(\sigma))P_{v''}$ in η^* yields a polynomial in three the variables $z_1 = \overline{\eta(e^{-1})}, z_2 = \overline{\eta(f^{-1})}$ and $z_3 = \overline{\eta(f^{-1})}$. This polynomial is defined on a small enough neighborhood of 0 and, from the definition of $K(\sigma)$, it vanishes there. It follows that its coefficients are all 0. This shows that an element in $P_{v'}\mathbb{E}_k(K(\sigma))P_{v''}$ is a linear combination of sums of the form $\sum a_i S_{\alpha_i}$ (for some paths α_i) where $\sum a_i = 0$ and for every i, j, the paths α_i and α_j satisfy $s(\alpha_i) = s(\alpha_j) = v''$, $r(\alpha_i) = r(\alpha_j) = v'$ and both paths contain the same edges (with the same multiplicities) but in a different order. A moment's reflection shows that this can happen only if the two paths are identical except that, at some points, one path travels along g and then along ef while the other path "chooses" to travel first along ef and then along g. This shows that the element in $P_{v'}\mathbb{E}_k(K(\sigma))P_{v''}$ lies in the ideal generated by $[S_g, S_e S_f]$. $\square$

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