Dynamical Symmetries in Supersymmetric Matrix[1]

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Abstract. We reveal a dynamical SU(2) symmetry in the asymptotic description of supersymmetric matrix models. We also consider a recursive approach for determining the ground state, and point out some additional properties of the model(s).

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1 Introduction

Supersymmetric matrix models derive from superstring theory which ultimately aims at a quantum field theoretic model of all known forces, including gravity. Some of the basic mathematical properties of supersymmetric matrix models are still open and pose a challenge to mathematics.

One of the key properties of supersymmetric matrix models - often assumed for granted in physics, but difficult to prove mathematically - is the existence of a ground state. I.e., the self-adjoint and nonnegative Hamiltonian operator $H = H^* \geq 0$ specifying the supersymmetric matrix model under consideration is assumed to have an eigenvalue at 0, the bottom of its spectrum. Since its spectrum is purely essential and covers the entire positive half axis, $\sigma(H) = [0, \infty)$ (see [3, 11]), the existence of zero-energy eigenstates, i.e., the non-triviality $\mathrm{Ker}(H) \neq 0$ of the zero-energy subspace, is not a consequence of standard methods of regular perturbation theory.

The Hamiltonian H acts on (an appropriate dense domain in) the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^{d(N^2-1)}) \otimes \mathcal{F}$ of square-integrable functions of coordinate variables

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$x \equiv (x_{tA})_{t\in\mathcal{D},A\in\mathcal{N}}$ with values in the fermion Fock space $\mathcal{F}$ spanned by vectors of the form $\lambda^\dagger_{\alpha_1,A_1}\cdots\lambda^\dagger_{\alpha_k,A_k}|0\rangle$, where $\{\lambda^\dagger_{\alpha A}\}_{\alpha\in\mathcal{S},A\in\mathcal{N}}$ are standard fermion creation operators, and $|0\rangle$ is the vacuum vector. The index ranges are denoted by $\mathcal{D} := \{1,2,\ldots,d\}$, $\mathcal{N} := \{1,2,\ldots,N^2-1\}$, and $\mathcal{S} := \{1,2,\ldots,d-1\}$. Note that $\dim_{\mathbb{C}}\mathcal{F} = 2^{(d-1)(N^2-1)} < \infty$.

On $\mathcal{H}$, the Hamiltonian assumes the form

$$H \;=\; -\Delta_x\otimes\mathbf{1} + V(x)\otimes\mathbf{1} + H_F, \tag{1}$$

where Δ_x is the Laplacian on $\mathbb{R}^{d(N^2-1)}$. The potential V is a homogenous, quartic polynomial in the coordinates x,

$$V(x) \;=\; \frac{1}{2} f_{ABC}\, f_{AB'C'}\, x_{sB}\, x_{tC}\, x_{sB'}\, x_{tC'}, \tag{2}$$

with $(f_{ABC})_{A,B,C\in\mathcal{N}}$ being real, antisymmetric structure constants of $\mathrm{SU}(N)$ and using Einstein's summation convention (i.e., repeated indices are summed over). The operator H_F, the fermionic part of the Hamiltonian, is linear in x, but quadratic in the fermion creation operators $\lambda^\dagger_{\alpha A}$ and their adjoints $\lambda_{\alpha A} = (\lambda^\dagger_{\alpha A})^*$, the fermion annihilation operators,

$$\begin{aligned} H_F \;=\; f_{ABC}\Big\{ & -2i\Big(\sum_{t=1}^{d-2} x_{t,C}\,\Gamma^t_{\alpha,\beta}\Big)\,\lambda^\dagger_{\alpha A}\,\lambda_{\beta B} \\ & + (x_{d-1,C}+ix_{d,C})\,\lambda^\dagger_{\alpha A}\,\lambda^\dagger_{\alpha B} \;-\; (x_{d-1,C}-ix_{d,C})\,\lambda_{\alpha A}\,\lambda_{\alpha B}\Big\}, \end{aligned} \tag{3}$$

where $(\Gamma^t)_{t=1}^{d-2}$ are purely imaginary, antisymmetric $(d-1)\times(d-1)$ matrices that represent the Clifford algebra $\{\Gamma^s,\Gamma^t\} = 2\delta_{st}\cdot\mathbf{1}_{(d-1)\times(d-1)}$, with $s,t\in\{1,2,\ldots,d-2\}$ and $d\in\{2,3,5,9\}$.

The Hamiltonian H commutes with the generators $\{J_A\}_{A\in\mathcal{N}}$ of $\mathrm{su}(N)$, where $J_A = \frac{-i}{2} f_{ABC}(2x_{sB}\partial_{sC} + \lambda^\dagger_{\alpha B}\lambda_{\alpha C} + \lambda_{\alpha C}\lambda^\dagger_{\alpha B})$ and $\partial_{sC} := \frac{\partial}{\partial x_{sC}}$, and the ground state sought for is required to be $\mathrm{SU}(N)$-invariant. That is, the spectral analysis of H is carried out on the subspace $\mathcal{H}_0 = \bigcap_{A\in\mathcal{N}}\mathrm{Ker}(J_A) \subseteq \mathcal{H}$. On $\mathcal{H}_0$, the Hamiltonian H arises as the restriction of the square of supercharges $(Q_\beta)_{\beta\in\mathcal{S}}$. These supercharges are selfadjoint (matrix-valued, first-order partial differential) operators on $\mathcal{H}$, which we don't describe here in detail, but we note that the Hamiltonian $H\restriction_{\mathcal{H}_0} = Q_\beta^2\restriction_{\mathcal{H}_0} \geq 0$, is manifestly nonnegative on $\mathcal{H}_0$.

Two main difficulties arise in the analysis of H:

(a) The quartic potential V has many vanishing directions. E.g., given $\vec{e} = (e_A)_{A\in\mathcal{N}} \in \mathbb{R}^{N^2-1}$ and denoting $\vec{x}_t = (x_{tA})_{A\in\mathcal{N}}$, the potential $V(x)$ vanishes on all hyperplanes $M(\vec{e}) = \{x \mid \forall t\in\mathcal{D}:\ \vec{x}_t\in\mathbb{R}\vec{e}\}$. So, even though the potential V grows to $\lim_{\eta\to\infty} V(\eta x) = \infty$, for almost all $x\in\mathbb{R}^{d(N^2-1)}$, this growth at infinity is not confining enough for H to have purely discrete spectrum, as shown in [3].

(b) The fermionic part H_F of the Hamiltonian is indefinite, so it doesn't contribute an obviously confining term to $-\Delta+V(x)$. Yet, their sum H restricted to $\mathcal{H}_0$ is nonnegative and is expected to yield a zero eigenvalue at the bottom of its spectrum, for $d=9$. In contrast, if $d=2$ and $N=2$ then zero is not an eigenvalue of H, as was shown in [5].

A lot of effort was put into the question of existence of zero-energy states in these SU(N)-invariant supersymmetric matrix models given by $H\upharpoonright_{\mathcal{H}_0}$. The original formulation uses Clifford variables $\{\Theta_{\hat\alpha,A}\}_{\hat\alpha\in\widehat{\mathcal{S}},A\in\mathcal{N}}$, $\{\Theta_{\hat\alpha,A},\Theta_{\hat\beta,B}\}=\delta_{\hat\alpha\hat\beta}\delta_{AB}$ rather than fermion creation and annihilation operators employed here, where $\widehat{\mathcal{S}}:=\{1,2,\ldots,2(d-1)\}$ and the relation between Clifford variables and the fermion creation and annihilation operators is the standard one, $\Theta_{\alpha,A}:=\frac{1}{\sqrt2}\big(\lambda^\dagger_{\alpha A}+\lambda_{\alpha A}\big)$ and $\Theta_{\alpha+d-1,A}:=\frac{-i}{\sqrt2}\big(\lambda^\dagger_{\alpha A}-\lambda_{\alpha A}\big)$, for all $\alpha\in\mathcal{S}$ and $A\in\mathcal{N}$. In terms of these Clifford variables, the Hamiltonian reads $\widetilde H=[-\Delta_x+V(x)]\otimes\mathbf{1}+\widetilde H_F$, where

$$\widetilde H_F \;=\; i\,f_{ABC}\,x_{tC}\,\gamma^t_{\hat\alpha,\hat\beta}\,\Theta_{\hat\alpha,A}\,\Theta_{\hat\beta,B}, \tag{4}$$

and $(\gamma^t)_{t\in\mathcal{D}}$ are real, symmetric $2(d-1)\times2(d-1)$ matrices given by

$$\gamma^t \;:=\; \begin{pmatrix} 0 & i\Gamma^t \\ -i\Gamma^t & 0\end{pmatrix}, \qquad \gamma^8 \;:=\; \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0\end{pmatrix}, \qquad \gamma^9 \;:=\; \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1}\end{pmatrix}, \tag{5}$$

with $t=1,2,\ldots,d-2$ [6]. The matrices $(\gamma^t)_{t\in\mathcal{D}}$ form a real representation of the Clifford algebra $\{\gamma^s,\gamma^t\}=2\delta_{st}\cdot\mathbf{1}$, with $s,t\in\mathcal{D}$, of minimal dimension, provided $d=2,3,5,9$.

The reason for recalling the form $\widetilde H$ of the Hamiltonian is that $\widetilde H$ is manifestly Spin(d)-invariant. The fermion creation and annihilation operators leading to (3) correspond to the particular choice (5) of $(\gamma^t)_{t\in\mathcal{D}}$. In attempts to construct a ground state explicitly [9], fermion creation and annihilation operators are used rather than Clifford variables. This is so, mainly because they provide the Hilbert space on which the Hamiltonian acts irreducibly from the very beginning. Namely, the creation and annihilation operators, $\lambda^\dagger_{\alpha A},\lambda_{\alpha A}$, with $\alpha\in\mathcal{S}$ and $a\in\mathcal{N}$, form a representation of the canonical anticommutation relations (CAR): $\{\lambda^\dagger_{\alpha A},\lambda^\dagger_{\beta B}\}=\{\lambda_{\alpha A},\lambda_{\beta B}\}=0$ and $\{\lambda_{\alpha A},\lambda^\dagger_{\beta B}\}=\delta_{\alpha\beta}\delta_{AB}$, where the anticommutator is $\{a,b\}:=ab+ba$. The CAR have an explicit representation as linear operators on the fermion Fock space

$$\mathcal{F} \;=\; \bigoplus_{k=0}^{(d-1)(N^2-1)} \operatorname{span}\big\{\lambda^\dagger_{\alpha_1,A_1}\cdots\lambda^\dagger_{\alpha_k,A_k}|0\rangle \;\big|\; \alpha_j\in\mathcal{S},\ A_j\in\mathcal{N}\big\}, \tag{6}$$

which is a complex Hilbert space of dimension $2^{(d-1)(N^2-1)}$. The vectors $\big\{\lambda^\dagger_{\alpha_1,A_1}\cdots\lambda^\dagger_{\alpha_k,A_k}|0\rangle \;\big|\; \alpha_j\in\mathcal{S},\ A_j\in\mathcal{N}\big\}\subseteq\mathcal{F}$ form an orthonormal basis; $|0\rangle$ is called vacuum vector. The Hilbert space $\mathcal{H}$ can be viewed as a direct integral

$$\mathcal{H} \;=\; \int^{\oplus}_{\mathbb{R}^{d(N^2-1)}} \mathcal{F}\,dx \;=\; L^2\big(\mathbb{R}^{d(N^2-1)};\,\mathcal{F}\big), \tag{7}$$

whose elements are linear combinations of the form

$$\Psi(x) = \sum_{k=0}^{(d-1)(N^2-1)} \Psi_k(x), \tag{8}$$

$$\Psi_k(x) = \sum_{\alpha_1,\ldots,\alpha_k\in\mathcal{S}} \sum_{A_1,\ldots,A_k\in\mathcal{N}} \psi^{(k)}_{\alpha_1,A_1\ldots,\alpha_k,A_k}(x)\, \lambda^\dagger_{\alpha_1,A_1}\cdots\lambda^\dagger_{\alpha_k,A_k}|0\rangle. \tag{9}$$

While the fermion creation and annihilation operators above (and in attempts to explicitly construct a ground state as in [9]) are chosen *independently of* x, the asymptotic form of the ground state wave function was determined with the help of *space-dependent* fermions [7, 6, 5]. The analysis of the asymptotic form $H^\infty\Psi = 0$ of the solutions of $H\Psi = 0$ leads for $N = 2$ and $d = 2, 3, 5$ to absence of a zero-energy states, as proved in [5], since these solutions are not square-integrable at $r \to \infty$, where $r > 0$ is introduced in (12), below. On the other hand, for $d = 9$, this asymptotic form of the wave function is square-integrable at infinity, and it is believed that for $d = 9$, the supersymmetric matrix models do possess zero enery eigenstates, for all $N \in \mathbb{N}$. This belief is supported by the recent existence proof for a related model [4].
Main Results: In this paper we study the asymptotic Hamiltonian H^∞ described in detail in (16)–(18). The asympotic Hamiltonian $H^\infty = H^\infty_B + H^\infty_F$ splits into a bosonic part H^∞_B and a fermionic part H^∞_F, similar to the full Hamiltonian H.
The bosonic Hamiltonian H^∞_B is a sum of harmonic oscillators and we first focus our attention on the ground state subspace of H^∞_B with corresponding ground state energy $2(d-1)$. This leads us to study the spectral properties of $2(d-1) + H^\infty_F$. We derive a dynamical SU(2) symmetry in (39) and observe the formation of 'Cooper pairs' [e.g., in the ground state of $2(d-1) + H^\infty_F$ computed in (46) and (48)] that arise in the SO(d)-breaking formulation when diagonalizing certain ingredients of the fermionic part of the Hamiltonian.
Thereafter, we transform the zero energy equation on Fock space into a system of graded equations (52) obtained by its natural grading derived from the fermion number. We show that this system of equations can be solved by a recursive insertion (58) of solutions, provided a certain invertibility condition on the graded Hamiltonians hold, which is known to hold true for the first recursion step (54). We finally observe a sum rule for the graded equations and apply this to the asymptotic ground state of $s_d + H^\infty_F$ (62)–(64).
To ease the reading, we carry out our analysis first in the case $N = 2$, i.e., for the asymptotic SU(2) theory. In the last section we note that several features extend to the non-asymptotic case and/or to general $N \geq 2$. We mostly restrict the dimension d to the most interesting case $d = 9$.

2 Asymptotic form of the Hamiltonian

The bosonic configuration space is a set of $d = 2, 3, 5$, or 9 traceless hermitian matrices $\{X_s\}_{s=1}^d$ corresponding to the Lie algebra su(N) of the gauge group

SU(N). Given selfadjoint generators $\vec{T} \equiv (T_A)_{A\in\mathcal{N}}$ of su(N) with $[T_A, T_B] = i f_{ABC} T_C$, the coordinates x derive from expanding $X_t = x_{tA} T_A = \vec{x}_t \cdot \vec{T}$ in these generators.

For simplicity, we start by taking $N = 2$ and $2\vec{T}$ to be the Pauli matrices $\vec{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$, so $f_{ABC} = \epsilon_{ABC}$. The potential, which for general $N \geq 2$ is given by

$$V(x) \;:=\; -\frac{1}{2} \sum_{s,t\in\mathcal{D}} \operatorname{Tr}\big([X_s, X_t]^2\big), \tag{10}$$

assumes the form

$$V(x) \;=\; \frac{1}{2} \sum_{s,t\in\mathcal{D}} \big(\vec{x}_s \wedge \vec{x}_t\big)^2, \tag{11}$$

where $(\vec{x}\wedge\vec{y})_A = \epsilon_{ABC}\, x_B\, y_C$, as usual. Observe from (11) that $V(x)$ vanishes if $x \in M := \bigcup_{\vec{e}\in\mathbb{R}^3} M(\vec{e})$, recalling that $M(\vec{e})$ are the hyperplanes $M(\vec{e}) = \{x \mid \forall t \in \mathcal{D} :\ \vec{x}_t \in \mathbb{R}\vec{e}\}$. We remark that, for $N = 2$, the condition $x \in M$ is even equivalent to $V(x) = 0$, while a necessary condition for the vanishing of the potential is more complicated for $N > 2$. We coordinatize $x \in \mathbb{R}^{3d}$ by (see, e.g., [6, 5])

$$\vec{x}_t \;=\; r\, E_t\, \vec{e} \;+\; r^{-1/2}\, \vec{y}_t, \tag{12}$$

for $t = 1, \ldots, d$, where $\vec{e} \in \mathbb{R}^3$ and $E = (E_1, \ldots, E_d) \in \mathbb{R}^d$ are unit vectors, $r > 0$, and $\vec{y}_t \in \mathbb{R}^3$ are perpendicular to both E and $\vec{e}$ in the sense that $E_s \vec{y}_s = \vec{e}\cdot\vec{y}_t = 0$, for all $t = 1, \ldots, d$. They derive from $x \in \mathbb{R}^{3d} \setminus \{0\}$ by the requirement that the euclidean length of the projection $x^{\|}$ of x along $\mathbb{R}\cdot E \otimes \vec{e}$ be maximal. Indeed, $\vec{e}$ and E are normalized eigenvectors of $(x_{tA}x_{tB})_{A,B=1}^{3}$ and $(x_{sA}x_{tA})_{s,t=1}^{d}$, respectively, corresponding to the largest eigenvalue $r^2 > 0$ which, in turn, is the square of the length $r = |x^{\|}| = \langle E \otimes \vec{e}, x\rangle = E_t x_{tA} e_A$ of $x^{\|}$. The component $x^{\perp} = x - x^{\|}$ perpendicular to $E\otimes\vec{e}$ then yields $\vec{y}_t = r^{1/2}\vec{x}_t^{\perp}$. Writing E as $E(\widetilde{E}, \theta, \varphi) = \big(\cos\theta\, \widetilde{E},\ \sin\theta\, \cos\varphi,\ \sin\theta\, \sin\varphi\big)$ in spherical coordinates, the coordinate vectors $\vec{x}_t$ assume the form

$$\vec{x}_t \;=\; r\, \cos\theta\, \widetilde{E}_t\, \vec{e} \;+\; r^{-1/2}\, \vec{y}_t, \tag{13}$$

for $t = 1, \ldots, d-2$, and

$$\vec{x}_{d-1} + i\vec{x}_d \;=\; r\, \sin\theta\, e^{i\varphi}\, \vec{e} \;+\; r^{-1/2}\, (\vec{y}_{d-1} + i\vec{y}_d), \tag{14}$$

where $\vec{e} \in \mathbb{R}^3$ and $\widetilde{E} = (\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{d-2}) \in \mathbb{R}^{d-2}$ are unit vectors, $\theta \in (-\pi/2, \pi/2)$, $\varphi \in (0, 2\pi)$, and $r > 0$.

To derive the asymptotic form of the Hamiltonian (cf. [5, 9]), we substitute (13)–(14) (and the corresponding differentials) into H, divide by r, and obtain

$$\frac{1}{r}\, H \quad \to \quad H^{\infty} \tag{15}$$

as the resulting limit, as $r \to \infty$. Note that, while the difference of H/r and H^{∞} is of lower order in r, the limit $r \to \infty$ is formal, as this difference is an unbounded (differential) operator. Moreover, it ignores the question whether the coordinate change (13)–(14) defines a global diffeomorphism. The asymptotic Hamiltonian H^∞ in (15) is of the form

$$H^\infty \; = \; H_B^\infty \; + \; H_F^\infty, \tag{16}$$

where the bosonic part

$$H_B^\infty \; := \; -(1 - E_s E_t)\,(1 - e_A e_B)\,\frac{\partial^2}{\partial y_{sA}\,\partial y_{tB}} \; + \; y_{sA}\,y_{sA} \tag{17}$$

is a sum of harmonic oscillators in the variables $\vec{y}_t$ in all $s_d = 2(d-1)$ spatial directions perpendicular to both E and $\vec{e}_t$, with ground state energy equal to $s_d = 2(d-1) = 2, 4, 8, 16$, for $d = 2, 3, 5, 9$, respectively, and ground state eigenvector $\Phi_0(\vec{y}_1, \ldots, \vec{y}_d) = \exp[-(1 - E_s E_t)\,(1 - e_A e_B)\,y_{sA}\,y_{tB}]$.
The fermionic part H_F^∞ of the asymptotic Hamiltonian H^∞ in (16) results from (3) by inserting (13)–(14), with $\vec{y}_t = 0$ and $r = 1$,

$$\begin{aligned} H_F^\infty \;&=\; 2\,\cos\theta\,(-ie_C\epsilon_{ABC})\,\Gamma_{\alpha\beta}\,\lambda_{\alpha A}^\dagger\,\lambda_{\beta B} \\ &\quad + \;\sin\theta\,e^{i\varphi}\,(e_C\epsilon_{ABC})\,\lambda_{\alpha A}^\dagger\lambda_{\alpha B}^\dagger + \;\sin\theta\,e^{-i\varphi}\,(e_C\epsilon_{ABC})\,\lambda_{\alpha B}\lambda_{\alpha A}, \end{aligned} \tag{18}$$

with $\Gamma_{\alpha\beta} := \sum_{t=1}^{d-2} \widetilde{E}_t \Gamma_{\alpha\beta}^t$.
We henceforth assume the unit vectors $\widetilde{E} \in \mathbb{R}^{d-2}$, $\vec{e} \in \mathbb{R}^3$ and $\theta \in (-\pi/2, \pi/2)$, $\varphi \in (0, 2\pi)$ to be fixed. Our goal is to find an explicit unitary transformation on Fock space $\mathcal{F}$ which brings H_F^∞ into normal (i.e., particle number preserving) form. It is a well-known fact (see, e.g., [2, 1]) that, since H_F^∞ is quadratic in the creation and annihilation operators, such a unitary exists and is of the form $\lambda_k^\dagger \mapsto u_{k,\ell}\lambda_\ell^\dagger + v_{k,\ell}\lambda_\ell^\dagger$, i.e., linear. While the existence of such a unitary follows from the diagonalizability of self-adjoint matrices, the determination of that unitary in an *explicit and managable form* can be difficult and depends on the special properties of the model (here: H_F^∞) under consideration.
Note that both $(M_{AB} := -ie_C\epsilon_{ABC})_{A,B=1,2,3}$ and $(\Gamma_{\alpha\beta})_{\alpha,\beta=1}^{d-1}$ are imaginary, antisymmetric, and thus self-adjoint matrices.
Since $M\vec{v} = i\vec{e} \wedge \vec{v}$, an orthonormal basis $\{\vec{e}, \vec{n}_+, \vec{n}_-\} \subseteq \mathbb{C}^3$ of eigenvectors, $M\vec{e} = 0$,

$$M\vec{n}_\pm \; = \; i\vec{e} \wedge \vec{n}_\pm \; = \; \pm\vec{n}_\pm \tag{19}$$

is given by the usual orthonormal *dreibein*: $\vec{e} \perp \vec{n}_+ \perp \vec{n}_-$. We choose $\vec{n}_\pm = \vec{n}_\pm(\vec{e})$ to depend continuously on $\vec{e}$ and to obey $\overline{\vec{n}_\pm} = \vec{n}_\mp$. Hence

$$-ie_C\epsilon_{ABC} \; = \; (\vec{n}_+)_A\,\overline{(\vec{n}_+)_B} \; - \; (\vec{n}_-)_A\,\overline{(\vec{n}_-)_B} \; = \; (\vec{n}_+)_A\,(\vec{n}_-)_B \; - \; (\vec{n}_-)_A\,(\vec{n}_+)_B. \tag{20}$$

Similary, for $d = 3, 5, 9$, we observe that, due to $\Gamma^2 = \widetilde{E}^2 \cdot \mathbf{1} = \mathbf{1}$ and $\mathrm{Tr}\,\Gamma = 0$, there is an orthonormal basis $\{\tilde{e}_{\sigma j} |\; \sigma = \pm, j = 1, \ldots, (d-1)/2\} \subseteq \mathbb{C}^{d-1}$ of eigenvectors of Γ such that

$$\Gamma\tilde{e}_{\pm j} \; = \; \pm\tilde{e}_{\pm j}, \tag{21}$$

for all $j = 1, \ldots, (d-1)/2$. Again we choose $\widetilde{E} \mapsto \tilde{e}_{\pm j}$ continuous and $\overline{\tilde{e}_{\pm j}} = \tilde{e}_{\mp j}$. So,

$$\Gamma_{\alpha\beta} \;=\; \sum_j \Big[(\tilde{e}_{+j})_\alpha \, (\tilde{e}_{-j})_\beta \;-\; (\tilde{e}_{-j})_\alpha \, (\tilde{e}_{+j})_\beta \Big], \tag{22}$$

where the summation ranges over $j = 1, \ldots, (d-1)/2$.
Using the orthonormal (eigen)vectors $\tilde{e}_{\pm j}$, $\vec{n}_\pm$, and $\vec{n}_0 := \vec{e}$, we define space-dependent fermion creation operators (for $d = 3, 5, 9$)

$$\lambda^\dagger_{\sigma j \tau} \;:=\; (\tilde{e}_{\sigma j})_\alpha \, (\vec{n}_\tau)_A \, \lambda^\dagger_{\alpha A}, \tag{23}$$

where $\sigma \in \{+,-\}$, $j = 1, \ldots, (d-1)/2$, and $\tau \in \{+,-,0\}$. Note that the matrix U defined by $U_{\sigma j\tau, \alpha A} := (\tilde{e}_{\sigma j})_\alpha \, (\vec{n}_\tau)_A$ is unitary and, hence, $\lambda^\dagger_{\alpha A} \mapsto \lambda^\dagger_{\sigma j \tau}$ is implemented by a unitary conjugation on (the operators on) Fock space $\mathcal{F}$. I.e., $\lambda_{\sigma j \tau}|0\rangle = 0$ and $\lambda^\dagger_{\sigma j \tau}, \lambda_{\sigma j \tau}$ fulfill the CAR.
Using the new creation operators $\lambda^\dagger_{\sigma j \tau}$, we introduce

$$A_j^\dagger \;\;:=\;\; i e^{i\varphi} \, \lambda^\dagger_{+j+} \, \lambda^\dagger_{-j-}, \tag{24}$$

$$B_j^\dagger \;\;:=\;\; i e^{-i\varphi} \, \lambda^\dagger_{-j+} \, \lambda^\dagger_{+j-}, \tag{25}$$

and $A_j := (A_j^\dagger)^*$ and $B_j := (B_j^\dagger)^*$, for $j = 1, \ldots, (d-1)/2$, which may be considered *(Cooper) pair creation and annihilation operators*. Note that these operators obey commutation relations somewhat reminiscent to the canonical ones, namely

$$[A_j^\dagger \,,\, A_k^\dagger] \;=\; [B_j^\dagger \,,\, B_k^\dagger] \;=\; [A_j^\dagger \,,\, B_k^\dagger] \;=\; [A_j \,,\, B_k^\dagger] \;=\; 0, \tag{26}$$

$$[A_j \,,\, A_k^\dagger] \;=\; \delta_{kj}\big(N_j^{(A)} - 1\big) \;:=\; \delta_{kj}\big(\lambda^\dagger_{+j+}\,\lambda_{+j+} \,+\, \lambda^\dagger_{-j-}\,\lambda_{-j-} \,-\, 1\big), \tag{27}$$

$$[B_j \,,\, B_k^\dagger] \;=\; \delta_{kj}\big(N_j^{(B)} - 1\big) \;:=\; \delta_{kj}\big(\lambda^\dagger_{-j+}\,\lambda_{-j+} \,+\, \lambda^\dagger_{+j-}\,\lambda_{+j-} \,-\, 1\big) \tag{28}$$

The asymptotic Hamiltonian H^∞, when acting on the ground state of H_B^∞, can be written as

$$s_d \,+\, H_F^\infty \;=\; H_0^\infty \,+\, H_+^\infty \,+\, H_-^\infty, \tag{29}$$

where

$$H_0^\infty \;\;:=\;\; s_d \;+\; 2\cos\theta \sum_j (N_j^{(A)} - N_j^{(B)}), \tag{30}$$

$$H_+^\infty \;\;:=\;\; 2\sin\theta \sum_j (A_j^\dagger + B_j^\dagger), \tag{31}$$

$$H_-^\infty \;\;:=\;\; 2\sin\theta \sum_j (A_j + B_j). \tag{32}$$

We remark that the degrees of freedom defined by the parallel fermions $\lambda^\dagger_{\pm j 0} = (\tilde{e}_{\pm j})_\alpha e_A \lambda_{\alpha A}$ do not appear in H_F^∞ and can be dropped, henceforth.

For the d=2 case we instead of (23) define $\lambda_{\pm}^{\dagger} := (\vec{n}_{\pm})_A \lambda_A^{\dagger}$, and the corresponding expressions for the asymptotic Hamiltonian in (29) are simply

$$H_0^\infty = 2, \quad H_+^\infty = 2C^\dagger, \quad H_-^\infty = 2C, \quad C^\dagger := ie^{i\varphi}\lambda_+^\dagger \lambda_-^\dagger. \tag{33}$$

3 Dynamical Symmetry

For definiteness, we restrict our attention in this and the following sections to the most interesting case: $d = 9$. Denoting

$$J_+ \otimes \mathbf{1} := A^\dagger := \textstyle\sum_j A_j^\dagger, \quad \mathbf{1} \otimes J_+ := B^\dagger := \textstyle\sum_j B_j^\dagger, \tag{34}$$

$$J_- \otimes \mathbf{1} := A := \textstyle\sum_j A_j, \quad \mathbf{1} \otimes J_- := B := \textstyle\sum_j B_j, \tag{35}$$

$$J_3 \otimes \mathbf{1} := \tfrac{1}{2}(N^{(A)} - 4) := \tfrac{1}{2}\big(\textstyle\sum_j N_j^{(A)} - 4\big), \tag{36}$$

$$\mathbf{1} \otimes J_3 := \tfrac{1}{2}(N^{(B)} - 4) := \tfrac{1}{2}\big(\textstyle\sum_j N_j^{(B)} - 4\big), \tag{37}$$

with

$$[J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm, \quad J_\pm = J_1 \pm iJ_2, \tag{38}$$

Eqs. (29)–(32) can be written as

$$4 + \frac{1}{4}H_F^\infty = (2 + \cos\theta\, J_3 + \sin\theta\, J_1) \otimes \mathbf{1} + \mathbf{1} \otimes (2 - \cos\theta\, J_3 + \sin\theta\, J_1), \tag{39}$$

thus exhibiting the dynamical symmetry mentioned above. (We recall that a dynamical symmetry refers to the situation that the Hamiltonian, being one of the generators of a symmetry Lie group, has nontrivial commutation relations with the other symmetry generators rather than commuting with them.)

The relevant SU(2) representations are the tensor product of four spin $\frac{1}{2}$ representations, i.e., direct sums of two singlets [note that both $(A_1A_3 + A_2A_4 - A_1A_4 - A_2A_3)|0\rangle$ and $(A_1A_2 + A_3A_4 - A_1A_4 - A_2A_3)|0\rangle$ are annihilated by $A^\dagger$, A , and $\frac{1}{2}(N^{(A)} - 4)$], three spin 1 representations, and (most importantly, as providing the zero-energy state of H) one spin 2 representation acting irreducibly on the space spanned by the orthonormal states

$$|0\rangle, \quad \frac{1}{2}A^\dagger|0\rangle, \quad \frac{1}{\sqrt{24}}(A^\dagger)^2|0\rangle, \quad \frac{1}{12}(A^\dagger)^3|0\rangle, \quad \frac{1}{4!}(A^\dagger)^4|0\rangle. \tag{40}$$

Restricting to that space (correspondingly for the $B^\dagger$'s), we can write

$$J_+ \;=\; \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}, \quad J_- \;=\; \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{41}$$

$$J_3 \;=\; \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}. \tag{42}$$

Since the spectrum of $\sin\theta J_1 \pm \cos\theta J_3$ is the same as that of J_3, the spectrum of $4+\frac{1}{4}H_F^\infty$ clearly consists of all integers between zero and eight,

$$\sigma\Big(4+\frac{1}{4}H_F^\infty\Big) \;=\; \{0,1,2,\ldots,8\}. \tag{43}$$

Its unique zero-energy state Ψ is most easily obtained by solving individually, for each $A_j^\dagger$ resp. $B_j^\dagger$ degree of freedom,

$$\Big(1\pm\cos\theta\;\sigma_3^{(j)}+\sin\theta\;\sigma_1^{(j)}\Big)\,\Psi \;=\; e^{\mp\frac{1}{2}\theta i\sigma_2^{(j)}}\big(1\pm\sigma_3^{(j)}\big)e^{\pm\frac{1}{2}\theta i\sigma_2^{(j)}}\,\Psi \;\stackrel{!}{=}\; 0, \tag{44}$$

where we identify

$$2\,J_k \;=\; \sigma_k\otimes 1\otimes 1\otimes 1 \;+\ldots+\; 1\otimes 1\otimes 1\otimes\sigma_k \;\equiv\; \sum_{j=1}^{4}\sigma_k^{(j)}. \tag{45}$$

In our notation $\sigma_3^{(j)}|0\rangle=-|0\rangle$ and $\sigma_3^{(j)}A_j^\dagger|0\rangle=+A_j^\dagger|0\rangle$, and we easily find the solution to (44) as

$$\Psi \;=\; \Big(\prod_j e^{-\frac{\theta}{2}i\sigma_2^{(j)}}\Big)\Big(\prod_j e^{\frac{\theta}{2}i\sigma_2^{(j)}}B_j^\dagger\Big)|0\rangle \;=\; \frac{1}{4!}\,e^{-\theta i(J_2\otimes 1-1\otimes J_2)}\,(B^\dagger)^4|0\rangle. \tag{46}$$

Using the nilpotency of $A_j^\dagger$ and $B_j^\dagger$ for

$$e^{\alpha(A_j^\dagger-A_j)}|0\rangle=\cos\alpha\,e^{\tan\alpha\;A_j^\dagger}|0\rangle \text{ and } e^{-\alpha(B_j^\dagger-B_j)}B_j^\dagger|0\rangle=\sin\alpha\,e^{\cot\alpha\;B_j^\dagger}|0\rangle, \tag{47}$$

the ground state can also be written as

$$\begin{aligned}\Psi \;&=\; \qquad\qquad (48)\\ &=\; \frac{1}{16}\,e^{-4i\varphi}\,(\sin\theta)^{-4}\prod_j\big\{\big(\sin\theta-(1-\cos\theta)A_j^\dagger\big)\big(\sin\theta-(1+\cos\theta)B_j^\dagger\big)\big\}|0\rangle\\ &=\; \frac{1}{16}\,e^{-4i\varphi}\,(\sin\theta)^4\;\exp\big[-\frac{1-\cos\theta}{\sin\theta}A^\dagger\;-\;\frac{1+\cos\theta}{\sin\theta}B\big]|0\rangle \;\sim\; e^{-C_\theta}|0\rangle,\end{aligned}$$

with $C_\theta := \frac{1-\cos\theta}{\sin\theta}(J_+ \otimes 1) + \frac{1+\cos\theta}{\sin\theta}(1 \otimes J_+)$. Alternatively, one can solve the 2×2 matrix eigenvector equations resulting from (44),

$$\begin{aligned} \big(1 + \cos\theta\,(N_j^{(A)} - 1) + \sin\theta\,(A_j^\dagger + A_j)\big)\Psi &= 0, & (49)\\ \big(1 - \cos\theta\,(N_j^{(B)} - 1) + \sin\theta\,(B_j^\dagger + B_j)\big)\Psi &= 0, & (50) \end{aligned}$$

to obtain (48).
For d=2 the asymptotic ground state is easily found from (33),

$$\Psi \;=\; \tfrac{1}{\sqrt{2}} e^{-C^\dagger}|0\rangle \;=\; \tfrac{1}{\sqrt{2}}(1 - C^\dagger)|0\rangle. \tag{51}$$

An interesting feature of the form (46) for the ground state is that it expresses it as a spin-rotation by an angle θ applied to some reference state $(B^\dagger)^4|0\rangle$ (which itself also varies in the first $d-2$ directions in space according to (13), (23), (21)).

4 Graded chain of Hamiltonians

We henceforth drop the superscript "∞" and write $H_0 = H_0^\infty$, $H_+ = H_+^\infty$, and $H_- = H_-^\infty$. Consider the grade- resp. fermion number-ordered equations

$$\begin{array}{ccccccc} & & H_0\Psi_0 & + & H_-\Psi_2 & = & 0,\\ H_+\Psi_0 & + & H_0\Psi_2 & + & H_-\Psi_4 & = & 0,\\ H_+\Psi_2 & + & H_0\Psi_4 & + & H_-\Psi_8 & = & 0,\\ & & \vdots & & & & \\ H_+\Psi_{12} & + & H_0\Psi_{14} & + & H_-\Psi_{16} & = & 0,\\ H_+\Psi_{14} & + & H_0\Psi_{16} & & & = & 0, \end{array} \tag{52}$$

which we obtain by writing $\Psi = \sum_{n=0}^{16} \Psi_n$, requiring $(N^{(A)} + N^{(B)})\Psi_n \;=\; n\,\Psi_n$ and the ground state equation

$$(16 + H_F^\infty)\Psi \;=\; (H_0 + H_+ + H_-)(\Psi_0 + \Psi_2 + \ldots + \Psi_{16}) \;\stackrel{!}{=}\; 0. \tag{53}$$

(Recall that we have dropped the eight non-dynamical parallel fermions $\lambda^\dagger_{\pm j0} = (\tilde{e}_{\pm j})_\alpha e_A \lambda_{\alpha A}$.)
The following method to construct the ground state we believe to be relevant also for the fully interacting, non-asymptotic theory. We use the first equation in (52) to express Ψ_0 in terms of Ψ_2,

$$\Psi_0 \;=\; -H_0^{-1}\, H_-\, \Psi_2. \tag{54}$$

H_0 is certainly invertible on the zero-fermion subspace, even in the full theory (cf. [9]). Inserting (54), the second equation in (52) can be written as

$$H_2\Psi_2 + H_-\Psi_4 \;=\; 0, \quad \text{with} \quad H_2 \;:=\; H_0 - H_+\, H_0^{-1}\, H_-, \tag{55}$$

which yields

$$\Psi_2 \;=\; -H_2^{-1} H_- \Psi_4, \tag{56}$$

provided H_2 is invertible on $H_-\Psi_4$, resp. the two-fermion sector of Fock space. Continuing in this manner, denoting

$$\widehat{\mathscr{H}_{2k}} \;:=\; \operatorname{span}\left\{(A^\dagger)^m (B^\dagger)^n |0\rangle\right\}_{m,n=0,1,2,3,4,\; m+n=k} \tag{57}$$

for the considered $2k$-fermion subspace, we find that if we assume invertibility of H_{2k} on $\widehat{\mathscr{H}_{2k}}$ we can form

$$H_{2(k+1)} \;:=\; H_0 - H_+ H_{2k}^{-1} H_- \tag{58}$$

on $\widehat{\mathscr{H}_{2(k+1)}}$ and solve for Ψ_{2k} in terms of $\Psi_{2(k+1)}$. The final equation for Ψ_{16} is $H_{16}\Psi_{16} = 0$.
For concreteness, denote an orthonormal basis of $\widehat{\mathscr{H}} = \oplus_k \widehat{\mathscr{H}_{2k}}$ by $|k,l\rangle := |k\rangle \otimes |l\rangle$, where, as in (40),

$$|k\rangle := \frac{1}{k!\sqrt{\binom{4}{k}}} J_+^k |0\rangle. \tag{59}$$

Then, e.g., $H_+ H_0^{-1} H_-$ acts on $\widehat{\mathscr{H}}$ 'tridiagonally' according to

$$\begin{aligned}
\frac{1}{\sin^2\theta} H_+ H_0^{-1} H_- |k,l\rangle \;=&\\
&\left(\frac{k(5-k)}{4+(k-l-1)\cos\theta} + \frac{l(5-l)}{4+(k-l+1)\cos\theta}\right)|k,l\rangle\\
&+ \frac{\sqrt{l(5-l)(k+1)(4-k)}}{4+(k-l+1)\cos\theta}|k+1,l-1\rangle\\
&+ \frac{\sqrt{k(5-k)(l+1)(4-l)}}{4+(k-l-1)\cos\theta}|k-1,l+1\rangle.
\end{aligned} \tag{60}$$

Calculating the spectra of H_{2k} on $\widehat{\mathscr{H}_{2k}}$ (e.g., with the help of a computer) one can verify the invertibility of all H_{2k} on $\widehat{\mathscr{H}_{2k}}$ for $k < 8$, while H_{16} is identically zero on $\widehat{\mathscr{H}_{16}}$. Hence, one can also start with the state $\Psi_{16} \sim A^4 B^4 |0\rangle$ (with correct normalization in θ) and generate the lower grade parts of the full asymptotic ground state Ψ using the relations (54), (56), etc.
We finish this section by noting a simple consequence of the graded form (52) of the ground state equation $H\Psi = 0$ (for general d and N). Taking the inner product of the grade $2k$-equation with Ψ_{2k} yields

$$\begin{aligned}
\langle \Psi_{2k}, H_-\Psi_{2(k+1)}\rangle \;&=\; -\langle H_0\rangle_{2k} \;-\; \langle \Psi_{2k}, H_+\Psi_{2(k-1)}\rangle\\
&=\; -\langle H_0\rangle_{2k} \;-\; \overline{\langle \Psi_{2(k-1)}, H_-\Psi_{2k}\rangle},
\end{aligned} \tag{61}$$

where $\langle H_0\rangle_{2k} := \langle\Psi_{2k}, H_0\Psi_{2k}\rangle$. The first equation reads $\langle\Psi_0, H_-\Psi_2\rangle = -\langle H_0\rangle_0$ which is real. The second then becomes $\langle\Psi_2, H_-\Psi_4\rangle = -\langle H_0\rangle_2 + \langle H_0\rangle_0$, and so on, so that in the last step one obtains

$$\sum_{k=0}^{\Lambda}(-1)^k\langle H_0\rangle_{2k} \;=\; 0, \tag{62}$$

where Λ is the total number of fermions in the relevant Fock space.
It is instructive to verify (62) for the asymptotic $N=2$ case studied above, since there all relevant terms can be calculated explicitly. Using the basis (59) and the notation $\alpha := 1-\cos\theta$, $\beta := 1+\cos\theta$, we find

$$\Psi \sim e^{-C_\theta}|0\rangle \;=\; \sum_k \frac{(-1)^k\sqrt{\binom{4}{k}}}{(\sin\theta)^k}\,\alpha^k\,|k\rangle \;\otimes\; \sum_l \frac{(-1)^l\sqrt{\binom{4}{l}}}{(\sin\theta)^l}\,\beta^l\,|l\rangle. \tag{63}$$

Hence,

$$\langle\Psi_{2n}, H_0\Psi_{2n}\rangle \;=\; \frac{1}{64}(\sin\theta)^{8-2n}\sum_{k+l=n}\binom{4}{k}\binom{4}{l}\big(4+(k-l)\cos\theta\big)\,\alpha^{2k}\,\beta^{2l}. \tag{64}$$

5 General SU(N)

We now compute the ground state energy of

$$\widetilde{H}_F \;=\; i\,f_{ABC}\,x_{tC}\,\gamma^t_{\hat\alpha,\hat\beta}\,\Theta_{\hat\alpha,A}\,\Theta_{\hat\beta,B}, \tag{65}$$

for general $N\geq 2$ and in regions of the configuration space where the potential V is zero (see eqs. (2), (3), and recalling that f_{ABC} denote the structure constants of SU(N) in an orthonormal basis). By (10), the vanishing of the potential V means that all X_s are commuting, hence can be written $X_s = UD_sU^\dagger$ where U is unitary and independent of s and the D_s are diagonal. If we look into a particular direction (which corresponds to fixing $\vec{e}$ in the SU(2) case) and choose a basis $\{T_A\}$ accordingly, we may write $X_s = D_s = x_{sA}T_A = x_{s\tilde k}T_{\tilde k}$ and $x_{sa}=0$, where $\tilde k = 1,\ldots,N-1$ are indices in the Cartan subalgebra and $a,b = N,\ldots,N^2-1$ denote the remaining indices.
Denoting the eigenvalues of X_t by μ^t_k, i.e., $X_t = \mathrm{diag}(\mu^t_1,\ldots,\mu^t_N)$, the eigenvectors $\{e_{kl}\}_{k\neq l}$ of $M^t_{ab} := -if_{abC}x_{tC} = -if_{ab\tilde k}x_{t\tilde k}$ satisfy (cf. e.g. [8])

$$M^t e_{kl} \;=\; (\mu^t_k-\mu^t_l)e_{kl} \;=:\; \mu^t_{kl}e_{kl}, \quad (e^a_{kl})^* \;=\; e^a_{lk}. \tag{66}$$

The crucial observation is that these eigen*vectors* are independent of t. Now,

$$\widetilde{H}_F \;=\; -\gamma^t_{\hat\alpha,\hat\beta}\,M^t_{ab}\,\Theta_{\hat\alpha,A}\,\Theta_{\hat\beta,B}, \;=\; W_{\hat\alpha a,\hat\beta b}\,\Theta_{\hat\alpha,a}\,\Theta_{\hat\beta,b}, \tag{67}$$

where $W := -\sum_t\gamma^t\otimes M^t$. From the above observations we have the ansatz $E_{\mu kl} := v_\mu\otimes e_{kl}$ for the eigenvectors of W, yielding

$$W\,E_{\mu kl} \;=\; -\sum_t \gamma^t\,v\otimes M^t\,e_{kl} \;=\; \gamma(k,l)\,v_\mu\otimes e_{kl}, \tag{68}$$

where $\gamma(k,l) := -\sum_t \mu_{kl}^t \gamma^t$ squares to $\sum_t (\mu_{kl}^t)^2$. Denoting by $v_\mu = v_{\pm jkl}$ the corresponding 16 eigenvectors of $\gamma(k,l)$, we find

$$W\, E_{\pm jkl} \;=\; \pm\sqrt{\sum_t (\mu_{kl}^t)^2}\; E_{\pm jkl}, \tag{69}$$

and $\widetilde{H}_F$ therefore has

$$E_0 \;:=\; -16 \sum_{k<l} \sqrt{\sum_{t=1}^{9} (\mu_k^t - \mu_l^t)^2} \tag{70}$$

as its lowest eigenvalue.

This agrees with the following two previously known cases: [8], where only X_9 is assumed to have large eigenvalues so that $E_0 \to -16 \sum_{k<l} |\mu_k^9 - \mu_l^9|$; as well as the SU(2)-case studied above and in [7], where (13) with, e.g., $e_A = \delta_{A3}$ gives $E_0 = -16r$. Note also [10], where the eigenvalues of $\widetilde{H}_F$ are stated, with the SU(N) symmetry fixed.

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