Hirzebruch-Mumford Proportionality and Locally Symmetric Varieties of Orthogonal Type

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Abstract. For many classical moduli spaces of orthogonal type there are results about the Kodaira dimension. But nothing is known in the case of dimension greater than 19. In this paper we obtain the first results in this direction. In particular the modular variety defined by the orthogonal group of the even unimodular lattice of signature $(2, 8m + 2)$ is of general type if $m \geq 5$.

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1 Modular varieties of orthogonal type

Let L be an integral indefinite lattice of signature $(2, n)$ and $(\, , \,)$ the associated bilinear form. By $\mathcal{D}_L$ we denote a connected component of the homogeneous type IV complex domain of dimension n

$$\mathcal{D}_L = \{[w] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (w, w) = 0, \ (w, \overline{w}) > 0\}^+.$$

$\mathrm{O}^+(L)$ is the index 2 subgroup of the integral orthogonal group $\mathrm{O}(L)$ that leaves $\mathcal{D}_L$ invariant. Any subgroup Γ of $\mathrm{O}^+(L)$ of finite index determines a modular variety

$$\mathcal{F}_L(\Gamma) = \Gamma \setminus \mathcal{D}_L.$$

By [BB] this is a quasi-projective variety.

For some special lattices L and subgroups $\Gamma < \mathrm{O}^+(L)$ one obtains in this way the moduli spaces of polarised abelian or Kummer surfaces ($n = 3$, see [GH]), the moduli space of Enriques surfaces ($n = 10$, see [BHPV]), and the moduli spaces of polarised or lattice-polarised K3 surfaces ($0 < n \leq 19$, see [Nik1, Dol]). Other interesting modular varieties of orthogonal type include the period domains of irreducible symplectic manifolds: see [GHS3].

It is natural to ask about the birational type of $\mathcal{F}_L(\Gamma)$. For many classical moduli spaces of orthogonal type there are results about the Kodaira dimension, but nothing is known in the case of dimension greater than 19. In this paper we obtain the first results in this direction. We determine the Kodaira dimension of many quasi-projective varieties associated with two series of even lattices.

To explain what these varieties are, we first introduce the *stable orthogonal group* $\widetilde{\mathrm{O}}(L)$ of a nondegenerate even lattice L. This is defined (see [Nik2] for more details) to be the subgroup of $\mathrm{O}(L)$ which acts trivially on the discriminant group $A_L = L^\vee/L$, where $L^\vee$ is the dual lattice. If $\Gamma < \mathrm{O}(L)$ then we write $\widetilde{\Gamma} = \Gamma \cap \widetilde{\mathrm{O}}(L)$. Note that if L is unimodular then $\widetilde{\mathrm{O}}(L) = \mathrm{O}(L)$.

The first series of varieties we want to study, which we call the modular varieties of *unimodular type*, is

$$\mathcal{F}_{II}^{(m)} = \mathrm{O}^+(II_{2,8m+2})\backslash\mathcal{D}_{II_{2,8m+2}}. \tag{1}$$

$\mathcal{F}_{II}^{(m)}$ is of dimension $8m + 2$ and arises from the even unimodular lattice of signature $(2, 8m + 2)$

$$II_{2,8m+2} = 2U \oplus mE_8(-1),$$

where U denotes the hyperbolic plane and $E_8(-1)$ is the negative definite lattice associated to the root system E_8. The variety $\mathcal{F}_{II}^{(2)}$ is the moduli space of elliptically fibred K3 surfaces with a section (see e.g. [CM, Section 2]). The case $m = 3$ is of particular interest: it arises in the context of the fake Monster Lie algebra [B1].

The second series, which we call the modular varieties of *K3 type*, is

$$\mathcal{F}_{2d}^{(m)} = \widetilde{\mathrm{O}}^+(L_{2d}^{(m)})\backslash\mathcal{D}_{L_{2d}^{(m)}}. \tag{2}$$

$\mathcal{F}_{2d}^{(m)}$ is of dimension $8m + 3$ and arises from the lattice

$$L_{2d}^{(m)} = 2U \oplus mE_8(-1) \oplus \langle -2d\rangle,$$

where $\langle -2d\rangle$ denotes a lattice generated by a vector of square $-2d$.

The first three members of the series $\mathcal{F}_{2d}^{(m)}$ have interpretations as moduli spaces. $\mathcal{F}_{2d}^{(2)}$ is the moduli space of polarised $K3$ surfaces of degree $2d$. For $m = 1$ the 11-dimensional variety $\mathcal{F}_{2d}^{(1)}$ is the moduli space of lattice-polarised K3 surfaces, where the polarisation is defined by the hyperbolic lattice $\langle 2d\rangle \oplus E_8(-1)$ (see [Nik1, Dol]). For $m = 0$ and d prime the 3-fold $\mathcal{F}_{2d}^{(0)}$ is the moduli space of polarised Kummer surfaces (see [GH]).

Theorem 1.1 *The modular varieties of unimodular and* K3 *type are varieties of general type if m and d are sufficiently large. More precisely:*

(i) If $m \geq 5$ then the modular varieties $\mathcal{F}_{II}^{(m)}$ and $\mathcal{F}_{2d}^{(m)}$ (for any $d \geq 1$) are of general type.

(ii) For $m = 4$ the varieties $\mathcal{F}_{2d}^{(4)}$ are of general type if $d \geq 3$ and $d \neq 4$.

(iii) For $m = 3$ the varieties $\mathcal{F}_{2d}^{(3)}$ are of general type if $d \geq 1346$.

(iv) For $m = 1$ the varieties $\mathcal{F}_{2d}^{(1)}$ are of general type if $d \geq 1537488$.

Remark. The methods of this paper are also applicable if $m = 2$. Using them, one can show that the moduli space $\mathcal{F}_{2d}^{(2)}$ of polarised K3 surfaces of degree $2d$ is of general type if $d \geq 231000$. This case was studied in [GHS2], where, using a different method involving special pull-backs of the Borcherds automorphic form Φ_{12} on the domain $\mathcal{D}_{II_{2,26}}$, we proved that $\mathcal{F}_{2d}^{(2)}$ is of general type if $d > 61$ or $d = 46$, 50, 54, 57, 58, 60.
The methods of [GHS2] do not appear to be applicable in the other cases studied here. Instead, the proof of Theorem 1.1 depends on the existence of a good toroidal compactification of $\mathcal{F}_L(\Gamma)$, which was proved in [GHS2], and on the exact formula for the Hirzebruch-Mumford volume of the orthogonal group found in [GHS1].
We shall construct pluricanonical forms on a suitable compactification of the modular variety $\mathcal{F}_L(\Gamma)$ by means of modular forms. Let $\Gamma < \mathrm{O}^+(L)$ be a subgroup of finite index, which naturally acts on the affine cone $\mathcal{D}_L^\bullet$ over $\mathcal{D}_L$. In what follows we assume that $\dim \mathcal{D}_L \geq 3$.

Definition 1.2 *A modular form of weight k and character $\chi\colon \Gamma \to \mathbb{C}^*$ with respect to the group Γ is a holomorphic function*

$$F : \mathcal{D}_L^\bullet \to \mathbb{C}$$

which has the two properties

$$\begin{aligned} F(tz) &= t^{-k}F(z) \quad \forall\, t \in \mathbb{C}^*, \\ F(g(z)) &= \chi(g)F(z) \quad \forall\, g \in \Gamma. \end{aligned}$$

The space of modular forms is denoted by $M_k(\Gamma, \chi)$. The space of cusp forms, i.e. modular forms vanishing on the boundary of the Baily–Borel compactification of $\Gamma\backslash\mathcal{D}_L$, is denoted by $S_k(\Gamma, \chi)$. We can reformulate the definition of modular forms in geometric terms. Let $F \in M_{kn}(\Gamma, \det^k)$ be a modular form, where n is the dimension of $\mathcal{D}_L$. Then

$$F(dZ)^k \in H^0(\mathcal{F}_L(\Gamma)^\circ, \Omega^{\otimes k}),$$

where dZ is a holomorphic volume form on $\mathcal{D}_L$, Ω is the sheaf of germs of canonical n-forms on $\mathcal{F}_L(\Gamma)$ and $\mathcal{F}_L(\Gamma)^\circ$ is the open smooth part of $\mathcal{F}_L(\Gamma)$ such that the projection $\pi : \mathcal{D}_L \to \Gamma\backslash\mathcal{D}_L$ is unramified over $\mathcal{F}_L(\Gamma)^\circ$.

The main question in the proof of Theorem 1.1 is how to extend the form $F(dZ)^k$ to $\mathcal{F}_L(\Gamma)$ and to a suitable toroidal compactification $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$. There are three possible kinds of obstruction to this, which we call (as in [GHS2]) elliptic, reflective and cusp obstructions. Elliptic obstructions arise if $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$ has non-canonical singularities arising from fixed loci of the action of the group Γ. Reflective obstructions arise because the projection π is branched along divisors whose general point is smooth in $\mathcal{F}_L(\Gamma)$. Cusp obstructions arise when we extend the form from $\mathcal{F}_L(\Gamma)$ to $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$.
The problem of elliptic obstructions was solved for $n \geq 9$ in [GHS2].

Theorem 1.3 *([GHS2, Theorem 2.1]) Let L be a lattice of signature $(2, n)$ with $n \geq 9$, and let $\Gamma < \mathrm{O}^+(L)$ be a subgroup of finite index. Then there exists a toroidal compactification $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$ of $\mathcal{F}_L(\Gamma) = \Gamma\backslash\mathcal{D}_L$ such that $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$ has canonical singularities and there are no branch divisors in the boundary. The branch divisors in $\mathcal{F}_L(\Gamma)$ arise from the fixed divisors of reflections.*

Reflective obstructions, that is branch divisors, are a special problem related to the orthogonal group. They do not appear in the case of moduli spaces of polarised abelian varieties of dimension greater than 2, where the modular group is the symplectic group. There are no quasi-reflections in the symplectic group even for $g = 3$.
The branch divisor is defined by special reflective vectors in the lattice L. This description is given in §2. To estimate the reflective obstructions we use the Hirzebruch-Mumford proportionality principle and the exact formula for the Hirzebruch-Mumford volume of the orthogonal group found in [GHS1]. We do the numerical estimation in §4.
We treat the cusp obstructions in §3, using special cusp forms of low weight (the lifting of Jacobi forms) constructed in [G2] and the low-weight cusp form trick (see [G2] and [GHS2]).

2 The branch divisors

To estimate the obstruction to extending pluricanonical forms to a smooth projective model of $\mathcal{F}_L(\widetilde{\mathrm{O}}^+(L))$ we have to determine the branch divisors of the projection

$$\pi\colon \mathcal{D}_L \to \mathcal{F}_L(\widetilde{\mathrm{O}}^+(L)) = \widetilde{\mathrm{O}}^+(L) \setminus \mathcal{D}_L. \tag{3}$$

According to [GHS2, Corollary 2.13] these divisors are defined by reflections $\sigma_r \in \mathrm{O}^+(L)$, where

$$\sigma_r(l) = l - \frac{2(l,r)}{(r,r)}r,$$

coming from vectors $r \in L$ with $r^2 < 0$ that are *stably reflective*: by this we mean that r is primitive and σ_r or $-\sigma_r$ is in $\widetilde{\mathrm{O}}^+(L)$. By a (k)-*vector* for $k \in \mathbb{Z}$ we mean a primitive vector r with $r^2 = k$.

Let D be the exponent of the finite abelian group A_L and let the *divisor* $\operatorname{div}(r)$ of $r \in L$ be the positive generator of the ideal (l, L). We note that $r^* = r/\operatorname{div}(r)$ is a primitive vector in $L^\vee$. In [GHS2, Propositions 3.1–3.2] we proved the following.

Lemma 2.1 *Let L be an even integral lattice of signature $(2,n)$. If $\sigma_r \in \widetilde{\mathrm{O}}^+(L)$ then $r^2 = -2$. If $-\sigma_r \in \widetilde{\mathrm{O}}^+(L)$, then $r^2 = -2D$ and $\operatorname{div}(r) = D \equiv 1$ mod 2 or $r^2 = -D$ and $\operatorname{div}(r) = D$ or $D/2$.*

We need also the following well-known property of the stable orthogonal group.

Lemma 2.2 *For any sublattice M of an even lattice L the group $\widetilde{\mathrm{O}}(M)$ can be considered as a subgroup of $\widetilde{\mathrm{O}}(L)$.*

Proof. Let $M^\perp$ be the orthogonal complement of M in L. We have as usual

$$M \oplus M^\perp \subset L \subset L^\vee \subset M^\vee \oplus (M^\perp)^\vee.$$

We can extend $g \in \widetilde{\mathrm{O}}(M)$ to $M \oplus M^\perp$ by putting $g|_{M^\perp} \equiv \mathrm{id}$. It is clear that $g \in \widetilde{\mathrm{O}}(M \oplus M^\perp)$. For any $l^\vee \in L^\vee$ we have $g(l^\vee) \in l^\vee + (M \oplus M^\perp)$. In particular, $g(l) \in L$ for any $l \in L$ and $g \in \widetilde{\mathrm{O}}(L)$. $\square$

We can describe the components of the branch locus in terms of homogeneous domains. For r a stably reflective vector in L we put

$$H_r = \{[w] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (w, r) = 0\},$$

and let $\mathcal{N}$ be the union of all hyperplane sections $H_r \cap \mathcal{D}_L$ over all stably reflective vectors r.

Proposition 2.3 *Let $r \in L$ be a stably reflective vector: suppose that r and L do not satisfy $D = 4$, $r^2 = -4$, $\operatorname{div}(r) = 2$. Let K_r be the orthogonal complement of r in L. Then the associated component $\pi(H_r \cap \mathcal{D}_L)$ of the branch locus $\mathcal{N}$ is of the form $\widetilde{\mathrm{O}}^+(K_r)\backslash\mathcal{D}_{K_r}$.*

Proof. We have $H_r \cap \mathcal{D}_L = \mathbb{P}(K_r) \cap \mathcal{D}_L = \mathcal{D}_{K_r}$. Let

$$\Gamma_{K_r} = \{\varphi \in \widetilde{\mathrm{O}}^+(L) \mid \varphi(K_r) = K_r\}. \qquad (4)$$

Γ_{K_r} maps to a subgroup of $\mathrm{O}^+(K_r)$. The inclusion of $\widetilde{\mathrm{O}}(K_r)$ in $\widetilde{\mathrm{O}}(L)$ (Lemma 2.2) preserves the spinor norm (see [GHS1, §3.1]), because K_r has signature $(2, n-1)$ and so $\widetilde{\mathrm{O}}^+(K_r)$ becomes a subgroup of $\widetilde{\mathrm{O}}^+(L)$.
Therefore the image of Γ_{K_r} contains $\widetilde{\mathrm{O}}^+(K_r)$ for any r. Now we prove that this image coincides with $\widetilde{\mathrm{O}}^+(K_r)$ for all r, except perhaps if $D = 4$, $r^2 = -4$ and $\operatorname{div}(r) = 2$.

Let us consider the inclusions

$$\langle r\rangle \oplus K_r \subset L \subset L^\vee \subset \langle r\rangle^\vee \oplus K_r^\vee.$$

By standard arguments (see [GHS2, Proposition 3.6]) we see that

$$|\det K_r| = \frac{|\det L|\cdot|r^2|}{\operatorname{div}(r)^2} \quad\text{and}\quad [L:\langle r\rangle\oplus K_r] = \frac{|r^2|}{\operatorname{div}(r)} = 1 \ \text{ or } \ 2.$$

If the index is 1, then it is clear that the image of Γ_{K_r} is $\widetilde{\mathrm{O}}^+(K_r)$. Let us assume that the index is equal to 2. In this case the lattice $\langle r\rangle^\vee$ is generated by $r^\vee = -r/(r,r) = r^*/2$, where $r^* = r/\operatorname{div}(r)$ is a primitive vector in $L^\vee$. In particular $r^\vee$ represents a non-trivial class in $\langle r\rangle^\vee \oplus K_r^\vee$ modulo $L^\vee$. Let us take $k^\vee \in K_r^\vee$ such that $k^\vee \notin L^\vee$. Then $k^\vee + r^\vee \in L^\vee$ and

$$\varphi(k^\vee) - k^\vee \equiv r^\vee - \varphi(r^\vee) \mod L.$$

We note that if $\varphi\in\Gamma_{K_r}$ then $\varphi(r) = \pm r$. Hence

$$\varphi(k^\vee) - k^\vee \equiv \begin{cases} 0 \mod L & \text{if } \varphi(r) = r \\ r^* \mod L & \text{if } \varphi(r) = -r \end{cases}$$

Since $\varphi(r^*) \equiv r^* \mod L$, we cannot have $\varphi(r) = -r$ unless $\operatorname{div}(r) = 1$ or 2. Therefore we have proved that $\varphi(k^\vee) \equiv k^\vee \mod K_r$ $(K_r = K_r^\vee \cap L)$, except possibly if $D = 4$, $r^2 = -4$, $\operatorname{div}(r) = 2$. □

The group $\widetilde{\mathrm{O}}^+(L)$ acts on $\mathcal{N}$. We need to estimate the number of components of $\widetilde{\mathrm{O}}^+(L)\setminus\mathcal{N}$. This will enable us to estimate the reflective obstructions to extending pluricanonical forms which arise from these branch loci.

For the even unimodular lattice $II_{2,8m+2}$ any primitive vector r has $\operatorname{div}(r) = 1$. Consequently r is stably reflective if and only if $r^2 = -2$.

For $L_{2d}^{(m)}$ the reflections and the corresponding branch divisors arise in two different ways, according to Lemma 2.1. We shall classify the orbits of such vectors.

Proposition 2.4 *Suppose d is a positive integer.*

(i) *Any two (-2)-vectors in the lattice $II_{2,8m+2}$ are equivalent modulo $\mathrm{O}^+(II_{2,8m+2})$, and the orthogonal complement of a (-2)-vector r is isometric to*

$$K_{II}^{(m)} = U \oplus mE_8(-1) \oplus \langle 2\rangle.$$

(ii) *There is one $\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})$-orbit of (-2)-vectors r in $L_{2d}^{(m)}$ with $\operatorname{div}(r) = 1$. If $d \equiv 1 \bmod 4$ then there is a second orbit of (-2)-vectors, with $\operatorname{div}(r) = 2$. The orthogonal complement of a (-2)-vector r in $L_{2d}^{(m)}$ is isometric to*

$$K_{2d}^{(m)} = U \oplus mE_8(-1) \oplus \langle 2\rangle \oplus \langle -2d\rangle,$$

if $\operatorname{div}(r) = 1$, *and to*

$$N_{2d}^{(m)} = U \oplus mE_8(-1) \oplus \begin{pmatrix} 1 & 2 \\ \frac{1-d}{2} & 1 \end{pmatrix},$$

if $\operatorname{div}(r) = 2$.

(iii) *The orthogonal complement of a $(-2d)$-vector r in $L_{2d}^{(m)}$ is isometric to*

$$II_{2,8m+2} = 2U \oplus mE_8(-1)$$

if $\operatorname{div}(r) = 2d$, *and to*

$$K_2^{(m)} = U \oplus mE_8(-1) \oplus \langle 2\rangle \oplus \langle -2\rangle \quad \text{or} \quad T_{2,8m+2} = U \oplus U(2) \oplus mE_8(-1)$$

if $\operatorname{div}(r) = d$.

(iv) *Suppose $d > 1$. The number of $\widetilde{\mathrm{O}}(L_{2d}^{(m)})$-orbits of $(-2d)$-vectors with* $\operatorname{div}(r) = 2d$ *is* $2^{\rho(d)}$. *The number of $\widetilde{\mathrm{O}}(L_{2d}^{(m)})$-orbits of $(-2d)$-vectors with* $\operatorname{div}(r) = d$ *is*

$$\begin{cases} 2^{\rho(d)} & \text{if } d \text{ is odd or } d \equiv 4 \mod 8; \\ 2^{\rho(d)+1} & \text{if } d \equiv 0 \mod 8; \\ 2^{\rho(d)-1} & \text{if } d \equiv 2 \mod 4. \end{cases}$$

Here $\rho(d)$ is the number of prime divisors of d.

Proof. If the lattice L contains two hyperbolic planes then according to the well-known result of Eichler (see [E, §10]) the $\widetilde{\mathrm{O}}^+(L)$-orbit of a primitive vector $l \in L$ is completely defined by two invariants: by its length (l,l) and by its image $l^* + L$ in the discriminant group A_L, where $l^* = l/\operatorname{div}(l)$.
i) If u is a primitive vector of an even unimodular lattice $II_{2,8m+2}$ then $\operatorname{div}(u) = 1$ and there is only one $\mathrm{O}(II_{2,8m+2})$-orbit of (-2)-vectors. Therefore we can take r to be a (-2)-vector in U, and the form of the orthogonal complement is obvious.
ii) In the lattice $L_{2d}^{(m)}$ we fix a generator h of its $\langle -2d\rangle$-part. Then for any $r \in L_{2d}^{(m)}$ we can write $r = u + xh$, where $u \in II_{2,8m+2}$ and $x \in \mathbb{Z}$. It is clear that $\operatorname{div}(r)$ divides r^2. If $f | \operatorname{div}(r)$, where $f = 2$, d or $2d$, then the vector u is also divisible by f. Therefore the (-2)-vectors form two possible orbits of vectors with divisor equal to 1 or 2. If $r^2 = -2$ and $\operatorname{div}(r) = 2$ then $u = 2u_0$ with $u_0 \in 2U \oplus mE_8(-1)$ and we see that in this case $d \equiv 1 \mod 4$. This gives us two different orbits for such d. In both cases we can find a (-2)-vector r in the sublattice $U \oplus \langle -2d\rangle$. Elementary calculation gives us the orthogonal complement of r.
iii) This was proved in [GHS2, Proposition 3.6] for $m = 2$. For general m the proof is the same.

iv) To find the number of orbits of $(-2d)$-vectors we have to consider two cases.
a) Let $\operatorname{div}(r) = 2d$. Then $r = 2du + xh$ and $r^* \equiv (x/2d)h \mod L$, where $u \in II_{2,8m+2}$ and x is modulo $2d$. Moreover $(r,r) = 4d^2(u,u) - x^2 2d = -2d$. Thus $x^2 \equiv 1 \mod 4d$. This congruence has $2^{\rho(d)}$ solutions modulo $2d$. For any such $x \mod 2d$ we can find a vector u in $2U \oplus mE_8(-1)$ with $(u,u) = (x^2-1)/2d$. Then $r = 2du + xh$ is primitive (because u is not divisible by any divisor of x) and $(r,r) = -2d$.
b) Let $\operatorname{div}(r) = d$. Then $r = du + xh$, where u is primitive, $r^* \equiv (x/d)h \mod L$ and x is modulo d. We have $(r^*, r^*) \equiv -2x^2/d \mod 2\mathbb{Z}$ and $x^2 \equiv 1 \mod d$. For any solution modulo d we can find as above $u \in 2U \oplus mE_8(-1)$ such that $r = du + xh$ is primitive and $(r,r) = -2d$. It is easy to see that the number of solutions $\{x \mod d \,|\, x^2 \equiv 1 \mod d\}$ is as stated. □

Remark. To calculate the number of the branch divisors arising from vectors r with $r^2 = -2d$ one has to divide the corresponding number of orbits found in Proposition 2.4(iv) by 2 if $d > 2$. This is because $\pm r$ determine different orbits but the same branch divisor. For $d = 2$ the proof shows that there is one divisor for each orbit given in Proposition 2.4(iv).

3 Modular forms of low weight

In this section we let $L = 2U \oplus L_0$ be an even lattice of signature $(2,n)$ with two hyperbolic planes. We choose a primitive isotropic vector c_1 in L. This vector determines a 0-dimensional cusp and a tube realisation of the domain $\mathcal{D}_L$. The tube domain (see the definition of $\mathcal{H}(L_1)$ below) is a complexification of the positive cone of the hyperbolic lattice $L_1 = c_1^\perp/c_1$. If $\operatorname{div}(c_1) = 1$ we call this cusp *standard* (as above, by [E] there is only one standard cusp). In this case $L_1 = U \oplus L_0$. In [GHS2, §4] we proved that any 1-dimensional boundary component of $\widetilde{\mathrm{O}}^+(L) \setminus \mathcal{D}_L$ contains the standard 0-dimensional cusp if every isotropic (with respect to the discriminant form: see [Nik2, §1.3]) subgroup of A_L is cyclic.
Let us fix a 1-dimensional cusp by choosing two copies of U in L. (One has to add to c_1 a primitive isotropic vector $c_2 \in L_1$ with $\operatorname{div}(c_2) = 1$). Then $L = U \oplus L_1 = U \oplus (U \oplus L_0)$ and the construction of the tube domain may be written down simply in coordinates. We have

$$\mathcal{H}(L_1) = \mathcal{H}_n = \{Z = (z_n, \dots, z_1) \in \mathbb{H}_1 \times \mathbb{C}^{n-2} \times \mathbb{H}_1;\ (\operatorname{Im} Z, \operatorname{Im} Z)_{L_1} > 0\},$$

where $Z \in L_1 \otimes \mathbb{C}$ and $(z_{n-1}, \dots, z_2) \in L_0 \otimes \mathbb{C}$. (We represent Z as a column vector.) An isomorphism between $\mathcal{H}_n$ and $\mathcal{D}_L$ is given by

$$\begin{array}{rcl} p\colon \mathcal{H}_n & \longrightarrow & \mathcal{D}_L \\ Z = (z_n, \dots, z_1) & \longmapsto & \big(-\frac{1}{2}(Z,Z)_{L_1} : z_n : \dots : z_1 : 1\big). \end{array} \tag{5}$$

The action of $\mathrm{O}^+(L \otimes \mathbb{R})$ on $\mathcal{H}_n$ is given by the usual fractional linear transformations. A calculation shows that the Jacobian of the transformation of $\mathcal{H}_n$

defined by $g \in \mathrm{O}^+(L \otimes \mathbb{R})$ is equal to $\det(g) j(g,Z)^{-n}$, where $j(g,Z)$ is the last ($(n+2)$-nd) coordinate of $g\big(p(Z)\big) \in \mathcal{D}_L$. Using this we define the automorphic factor

$$\begin{array}{rcl} J\colon\ \mathrm{O}^+(L \otimes \mathbb{R}) \times \mathcal{H}_{n+2} & \to & \mathbb{C}^* \\ (g,Z) & \mapsto & (\det g)^{-1} \cdot j(g,Z)^n. \end{array}$$

The connection with pluricanonical forms is the following. Consider the form

$$dZ = dz_1 \wedge \cdots \wedge dz_n \in \Omega^n(\mathcal{H}_n).$$

$F(dZ)^k$ is a Γ-invariant k-fold pluricanonical form on $\mathcal{H}_n$, for Γ a subgroup of finite index of $\mathrm{O}^+(L)$, if $F\big(g(Z)\big) = J(g,Z)^k F(Z)$ for any $g \in \Gamma$; in other words if $F \in M_{nk}(\Gamma, \det^k)$ (see Definition 1.2). To prove Theorem 1.1 we need cusp forms of weight smaller than the dimension of the corresponding modular variety.

Proposition 3.1 *For unimodular type, cusp forms of weight* $12 + 4m$ *exist: that is*

$$\dim S_{12+4m}(\mathrm{O}^+(II_{2,8m+2})) > 0.$$

For K3 *type we have the bounds*

$$\begin{array}{rcl} \dim S_{11+4m}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) & > & 0 \ \ \textit{if}\ \ d > 1; \\ \dim S_{10+4m}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) & > & 0 \ \ \textit{if}\ \ d \geq 1; \\ \dim S_{7+4m}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) & > & 0 \ \ \textit{if}\ \ d \geq 4; \\ \dim S_{6+4m}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) & > & 0 \ \ \textit{if}\ \ d = 3 \ \ \textit{or}\ \ d \geq 5; \\ \dim S_{5+4m}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) & > & 0 \ \ \textit{if}\ \ d = 5 \ \ \textit{or}\ \ d \geq 7; \\ \dim S_{2+4m}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) & > & 0 \ \ \textit{if}\ \ d > 180. \end{array}$$

Proof. For any $F(Z) \in M_k(\widetilde{\mathrm{O}}^+(L))$ we can consider its Fourier-Jacobi expansion at the 1-dimensional cusp fixed above

$$F(Z) = f_0(z_1) + \sum_{m \geq 1} f_m(z_1; z_2, \dots z_{n-1}) \exp(2\pi i m z_n).$$

A lifting construction of modular forms $F(Z) \in M_k(\widetilde{\mathrm{O}}^+(L))$ with trivial character by means of the first Fourier–Jacobi coefficient is given in [G1], [G2]. We note that $f_1(z_1; z_2, \dots, z_{n-1}) \in J_{k,1}(L_0)$, where $J_{k,1}(L_0)$ is the space of the Jacobi forms of weight k and index 1. A more general construction of the additive lifting was given in [B2] but for our purpose the construction of [G2] is sufficient.

The dimension of $J_{k,1}(L_0)$ depends only on the discriminant form and the rank of L_0 (see [G2, Lemma 2.4]). In particular, for the special cases of $L = II_{2,8m+2}$ and $L = L_{2d}^{(m)}$ we have

$$J_{k+4m,1}^{\mathrm{cusp}}(mE_8(-1)) \cong S_k(\mathrm{SL}_2(\mathbb{Z}))$$

and

$$J_{k+4m,1}^{\mathrm{cusp}}(mE_8(-1) \oplus \langle -2d \rangle) \cong J_{k,d}^{\mathrm{cusp}},$$

where $J_{k,d}^{\mathrm{cusp}}$ is the space of the usual Jacobi cusp forms in two variables of weight k and index d (see [EZ]) and $S_k(\mathrm{SL}_2(\mathbb{Z}))$ is the space of weight k cusp forms for $\mathrm{SL}_2(\mathbb{Z})$.

The lifting of a Jacobi cusp form of index one is a cusp form of the same weight with respect to $\mathrm{O}^+(II_{2,8m+2})$ or $\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})$ with trivial character. The fact that we get a cusp form was proved in [G2] for maximal lattices, i.e., if d is square-free. In [GHS2, §4] we extended this to all lattices L for which the isotropic subgroups of the discriminant A_L are all cyclic, which is true in all cases considered here.

To prove the unimodular type case of Proposition 3.1 we can take the Jacobi form corresponding to the cusp form $\Delta_{12}(\tau)$. Using the Jacobi lifting construction we obtain a cusp form of weight $12 + 4m$ with respect to $\mathrm{O}^+(II_{2,8m+2})$.

For the K3 type case we need the dimension formula for the space of Jacobi cusp forms $J_{k,d}^{\mathrm{cusp}}$ (see [EZ]). For a positive integer l one sets

$$\{l\}_{12} = \begin{cases} \lfloor \frac{l}{12} \rfloor & \text{if } l \not\equiv 2 \bmod 12 \\ \lfloor \frac{l}{12} \rfloor - 1 & \text{if } l \equiv 2 \bmod 12. \end{cases}$$

Then if $k > 2$ is even

$$\dim J_{k,d}^{\mathrm{cusp}} = \sum_{j=0}^{d} \left(\{k + 2j\}_{12} - \left\lfloor \frac{j^2}{4d} \right\rfloor \right),$$

and if k is odd

$$\dim J_{k,d}^{\mathrm{cusp}} = \sum_{j=1}^{d-1} \left(\{k - 1 + 2j\}_{12} - \left\lfloor \frac{j^2}{4d} \right\rfloor \right).$$

This gives the bounds claimed. For $k = 2$, using the results of [SZ] one can also calculate $\dim J_{2,d}^{\mathrm{cusp}}$: there is an extra term, $\lceil \sigma_0(d)/2 \rceil$, where $\sigma_0(d)$ denotes the number of divisors of d. This gives $\dim J_{2,d}^{\mathrm{cusp}} > 0$ if $d > 180$ and for some smaller values of d. □

4 Kodaira dimension results

In this section we prove Theorem 1.1. We first explain the geometric background. Let $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$ be a toroidal compactification as in Theorem 1.3. In particular all singularities are canonical and there is no ramification divisor which is contained in the boundary. Then the canonical divisor (as a $\mathbb{Q}$-divisor) is given by $K_{\mathcal{F}_L(\Gamma)^{\rm tor}} = nM - V - D$ where M is the line bundle of modular forms of weight 1, n is the dimension of $\mathcal{F}_L(\Gamma)$, V is the branch locus (which is given by reflections) and D is the boundary. Hence in order to construct k-fold pluricanonical forms we must find modular forms of weight kn which vanish of order k along the branch divisor and the boundary. This also suffices since $\mathcal{F}_L(\Gamma)^{\rm tor}$ has canonical singularities.

Our strategy is the following. For $\Gamma \subseteq \widetilde{\rm O}^+(L)$ we choose a cusp form $F_a \in S_a(\Gamma)$ of low weight a, i.e. a strictly less than the dimension. Then we consider elements $F \in F_a^k M_{k(n-a)}(\Gamma, \det^k)$: for simplicity we assume that k is even. Such an F vanishes to order at least k on the boundary of any toroidal compactification. Hence if dZ is the volume element on $\mathcal{D}_L$ defined in §3 it follows that $F(dZ)^k$ extends as a k-fold pluricanonical form to the general point of every boundary component of $\mathcal{F}_L(\Gamma)^{\rm tor}$. Since we have chosen the toroidal compactification so that all singularities are canonical and that there is no ramification divisor which is contained in the boundary the only obstructions to extending $F(dZ)^k$ to a smooth projective model are the reflective obstructions, coming from the ramification divisor of the quotient map $\pi\colon \mathcal{D}_L \to \mathcal{F}_L(\Gamma)$ studied in §2.

Let $\mathcal{D}_K$ be an irreducible component of this ramification divisor. Recall from Proposition 2.3 that $\mathcal{D}_K = \mathbb{P}(K \otimes \mathbb{C}) \cap \mathcal{D}_L$ where $K = K_r$ is the orthogonal complement of a stably reflective vector r. For the lattices chosen in Theorem 1.1 all irreducible components of the ramification divisor are given in Proposition 2.4.

Proposition 4.1 *We assume that k is even and that the dimension $n \geq 9$. For $\Gamma \subseteq \widetilde{\rm O}^+(L)$, the obstruction to extending forms $F(dZ)^k$ where $F \in F_a^k M_{k(n-a)}(\Gamma)$ to $\mathcal{F}_L(\Gamma)^{\rm tor}$ lies in the space*

$$B = \bigoplus_K B(K) = \bigoplus_K \bigoplus_{\nu=0}^{k/2-1} M_{k(n-a)+2\nu}(\Gamma \cap \widetilde{\rm O}^+(K)),$$

where the direct sum is taken over all irreducible components $\mathcal{D}_K$ of the ramification divisor of the quotient map $\pi\colon \mathcal{D}_L \to \mathcal{F}(\Gamma)$.

Proof. Let $\sigma \in \Gamma$ be plus or minus a reflection whose fixed point locus is $\mathcal{D}_K$. We can extend the differential form provided that F vanishes of order k along every irreducible component $\mathcal{D}_K$ of the ramification divisor.

If F_a vanishes along $\mathcal{D}_K$ then K gives no restriction on the second factor of the modular form F.

Now let $\{w = 0\}$ be a local equation for $\mathcal{D}_K$. Then $\sigma^*(w) = -w$ (this is independent of whether σ or $-\sigma$ is the reflection). For every modular form $F \in M_k(\Gamma)$ of even weight we have $F\big(\sigma(z)\big) = F(z)$. This implies that if $F(z) \equiv 0$ on $\mathcal{D}_K$, then F vanishes to even order on $\mathcal{D}_K$.

We denote by $M_{2b}(\Gamma)(-\nu\mathcal{D}_K)$ the space of modular forms of weight $2b$ which vanish of order at least ν along $\mathcal{D}_K$. Since the weight is even we have $M_{2b}(\Gamma)(-\mathcal{D}_K) = M_{2b}(\Gamma)(-2\mathcal{D}_K)$. For $F \in M_{2b}(\Gamma)(-2\nu\mathcal{D}_K)$ we consider $(F/w^{2\nu})$ as a function on $\mathcal{D}_K$. From the definition of modular form (Definition 1.2) it follows that this function is holomorphic, $\Gamma\cap\Gamma_K$-invariant (see equation (4)) and homogeneous of degree $2b+2\nu$. Thus $(F/w^{2\nu})|_{\mathcal{D}_K} \in M_{2(b+\nu)}(\Gamma\cap\Gamma_K)$. In Proposition 2.3 we saw that, Γ_K contains $\widetilde{\mathrm{O}}^+(K)$ as subgroup of $\widetilde{\mathrm{O}}^+(L)$(with equality in almost all cases), so we may replace $\Gamma\cap\Gamma_K$ by $\Gamma\cap\widetilde{\mathrm{O}}^+(K)$. In this way we obtain an exact sequence

$$0 \to M_{2b}(\Gamma)(-(2+2\nu)\mathcal{D}_K) \to M_{2b}(\Gamma)(-2\nu\mathcal{D}_K) \to M_{2(b+\nu)}(\Gamma\cap\widetilde{\mathrm{O}}^+(K)),$$

where the last map is given by $F \mapsto F/w^{2\nu}$. This gives the result. □

Now we proceed with the proof of Theorem 1.1.

Let L be a lattice of signature $(2,n)$ and $\Gamma < \widetilde{\mathrm{O}}^+(L)$: recall that k is even. According to Proposition 4.1 we can find pluricanonical differential forms on $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$ if

$$C_B(\Gamma) = \dim M_{k(n-a)}(\Gamma) - \sum_K \dim B(K) > 0, \tag{6}$$

where summation is taken over all irreducible components of the ramification divisor (see the remark at the end of §2). It now remains to estimate the dimension of $B(K)$ for each of the finitely many components of the ramification locus in the cases we are interested in, namely $\Gamma = \mathrm{O}^+(II_{2,8m+2})$ and $\Gamma = \widetilde{\mathrm{O}}^+(L_{2d}^{(m)})$.

According to the Hirzebruch-Mumford proportionality principle

$$\dim M_k(\Gamma) = \frac{2}{n!}\,\mathrm{vol}_{HM}(\Gamma)k^n + O(k^{n-1}).$$

The exact formula for the Hirzebruch-Mumford volume vol_{HM} for any indefinite orthogonal group was obtained in [GHS1]. It depends mainly on the determinant and on the local densities of the lattice L. Here we simply quote the estimates of the dimensions of certain spaces of cusp forms.

The case of $II_{2,8m+2}$ is easier because the branch divisor has only one irreducible component defined by any (-2)-vector r. According to Proposition 2.4 the orthogonal complement K_r is $K_{II}^{(m)}$. This lattice differs from the lattice $L_2^{(m)}$, whose Hirzebruch-Mumford volume was calculated in [GHS1, §3.5], only by one copy of the hyperbolic plane. Therefore

$$\mathrm{vol}_{HM}\,\widetilde{\mathrm{O}}^+(L_2^{(m)}) = (B_{8m+4}/(8m+4))\,\mathrm{vol}_{HM}\,\widetilde{\mathrm{O}}^+(K_{II}^{(m)}),$$

and hence, for even k,

$$\dim M_k(\widetilde{\mathrm{O}}^+(K_{II}^{(m)})) = \frac{2^{1-4m}}{(8m+1)!}\cdot\frac{B_2\ldots B_{8m+2}}{(8m+2)!!}\,k^{8m+1} + O(k^{8m}),$$

where the B_i are Bernoulli numbers. Assume that $m \geq 3$. Let us take a cusp form

$$F \in S_{4m+12}(\mathrm{O}^+(II_{2,8m+2}))$$

from Proposition 3.1. In this case the dimension of the obstruction space B of Proposition 4.1 for the pluricanonical forms of order $k = 2k_1$ is given by

$$\sum_{\nu=0}^{k_1-1} \dim M_{(4m-10)k+2\nu}(\widetilde{\mathrm{O}}^+(K_{II}^{(m)})) =$$
$$\frac{2^{4m+2}}{(8m+2)!} \cdot \frac{B_2 \dots B_{8m+2}}{(8m+2)!!}\big((1+\frac{1}{4m-10})^{8m+2}-1\big)((4m-10)k_1)^{8m+2}$$
$$+O(k^{8m+1})$$

In [GHS1, §3.3] we computed the leading term of the dimension of the space of modular forms for $\mathrm{O}^+(II_{2,8m+2})$. Comparing these two we see that the constant $C_B(\mathrm{O}^+(II_{2,8m+2}))$ in the obstruction inequality (6) is positive if and only if

$$\frac{B_{4m+2}}{4m+2} > \big(1+\frac{1}{4m-10}\big)^{8m+2} - 1. \tag{7}$$

Moreover $\mathcal{F}_{II}^{(m)}$ is of general type if $C_B(\mathrm{O}^+(II_{2,8m+2})) > 0$. From Stirling's formula

$$5\sqrt{\pi n}(\frac{n}{\pi e})^{2n} > |B_{2n}| > 4\sqrt{\pi n}(\frac{n}{\pi e})^{2n}. \tag{8}$$

Using this estimate we easily obtain that (7) holds if $m \geq 5$. Therefore we have proved Theorem 1.1 for the lattice $II_{2,8m+2}$.

Next we consider the lattice $L_{2d}^{(m)}$ of K3 type. For this lattice the branch divisor of $\mathcal{F}_{2d}^{(m)}$ is calculated in Proposition 2.4. It contains one or two (if $d \equiv 1 \bmod 4$) components defined by (-2)-vectors and some number of components defined by $(-2d)$-vectors. To estimate the obstruction constant $C_B(\Gamma)$ in (6) we use the dimension formulae for the space of modular forms with respect to the group $\widetilde{\mathrm{O}}^+(M)$, where M is one of the following lattices from Proposition 2.4: $L_{2d}^{(m)}$ (the main group); $K_{2d}^{(m)}$ and $N_{2d}^{(m)}$ (the (-2)-obstruction); $M_{2,8m+2}$, $K_2^{(m)}$ and $T_{2,8m+2}$ (the $(-2d)$-obstruction). The corresponding dimension formulae were found in [GHS1] (see §§3.5, 3.6.1–3.6.2, 3.3 and 3.4). The branch divisor of $(-2d)$-type appears only if $d > 1$. We note that

$$\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(T_{2,8m+2})) > \mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(K_2^{(m)})). \tag{9}$$

Therefore in order to estimate $C_B(\Gamma)$ we can assume that all $(-2d)$-divisors defined by stably reflective $(-2d)$-vectors r with $\mathrm{div}(r) = d$ (see Proposition 2.4) are of the type $T_{2,8m+2}$.

We put $k = 2k_1$, $w = n - a$ and $n = 8m+3$. For the obstruction constant in (6) we obtain

$$C_B(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) > \dim M_{2k_1w}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) - B_{(-2)} - B_{(-2d)} \tag{10}$$

where

$$B_{(-2)} = \dim B(K_{2d}^{(m)}) + \dim B(N_{2d}^{(m)}),$$

$$B_{(-2d)} = 2^{\rho(d)-1}(\dim B(M_{2,8m+2}) + 2^{h_d} \dim B(T_{2,8m+2}))$$

and $B(K)$ is the obstruction space from Proposition 4.1. By h_d we denote the sum $\delta_{0,d(8)} - \delta_{2,d(4)}$, where $d(n)$ is $d \bmod n$ and δ is the Kronecker delta (see Proposition 2.4 and the remark following it).

For any lattice considered above

$$\begin{aligned} B(K) &= \sum_{\nu=0}^{k_1-1} \dim M_{2(k_1 w+\nu)}(\widetilde{\mathrm{O}}^+(K)) \\ &= \frac{2^{8m+3}}{(8m+3)!} E_w(8m+3) \mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(K))(k_1 w)^{8m+3} + O(k_1^{8m+2}) \end{aligned}$$

where $E_w(8m+3) = (1+\frac{1}{w})^{8m+3}$.

All terms in (10) contain a common factor. First

$$\dim M_{2k_1 w}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) = C_{m,d}^{k_1,w} \left| \frac{B_{8m+4}}{B_{4m+2}} \right| \sqrt{d} + O(k_1^{8m+2}), \tag{11}$$

where

$$C_{m,d}^{k_1,w} = \frac{2^{4m+1+\delta_{1,d}}}{(8m+3)!} \frac{|B_2 \ldots B_{8m+2}|}{(8m+2)!!} \frac{|B_{4m+2}|}{4m+2} d^{4m+\frac{3}{2}} \prod_{p|d} (1-p^{-(4m+2)})(k_1 w)^{8m+3}.$$

We note that $2^{4m+1}\frac{B_{4m+2}}{4m+2} = \pi^{-(4m+2)}\Gamma(4m+2)\zeta(4m+2)$.

From [GHS1, (16)] it follows that

$$\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(K_{2d}^{(m)})) = 2^{\delta_{1,d}-\delta_{4,d(8)}} \frac{B_2 \ldots B_{8m+2}}{(8m+2)!!} d^{4m+\frac{3}{2}} \pi^{-(4m+2)} \Gamma(4m+2) L(4m+2, \left(\frac{4d}{*}\right)).$$

We can use the formula for the volume of $N_{2d}^{(m)}$ in the following form:

$$\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(N_{2d}^{(m)})) = 2^{1+\delta_{1,d}-(8m+4)} d^{4m+\frac{3}{2}} \frac{B_2 \ldots B_{8m+2}}{(8m+2)!!} \pi^{-(4m+2)} \Gamma(4m+2) L(4m+2, \left(\frac{d}{*}\right))$$

(see [GHS1, 3.6.2]). It follows that

$$B_{(-2)} = C_{m,d}^{k_1,w} E_w(8m+3)(2^{8m+3-\delta_{4,d(8)}} P_K(4m+2) + P_N(4m+2)) + O(k_1^{8m+2})$$

where

$$P_K(n) = (1-2^{-n})^{\delta_{0,d(2)}} \frac{L(n, \left(\frac{4d}{*}\right))}{L(n,\chi_{0,4d})} \prod_{p|d} \frac{1-p^{-n}}{1+p^{-n}}$$

and

$$P_N(n) = \frac{L(n, \left(\frac{d}{*}\right))}{L(n,\chi_{0,d})} \prod_{p|d} \frac{1-p^{-n}}{1+p^{-n}}.$$

Here $\chi_{0,f}$ denotes the principal Dirichlet character modulo f.
We note that $|P_K(n)| < 1$ and $|P_N(n)| < 1$ for any d. We conclude that

$$B_{(-2)} < C_{m,d}^{k_1,w} E_w(8m+3) b_{(-2)}$$

where $b_{(-2)} = 2^{8m+3} - 1$.
The $(-2d)$-contribution is calculated according to [GHS1, 3.3–3.4]. We note that $\widetilde{\mathrm{O}}^+(T_{2,8m+2})$ is a subgroup of $\widetilde{\mathrm{O}}^+(M_{2,8m+2})$. We obtain

$$B_{(-2d)} \le C_{m,d}^{k_1,w} E_w(8m+3) b_{(-2d)}$$

where for $d > 2$

$$b_{(-2d)} = \frac{2^{\rho(d)}}{d} \left(\frac{4}{d}\right)^{4m+\frac{1}{2}} 4(2^{h_d}(1+2^{-(4m+2)} - 2^{-(8m+3)}) + 2^{-(8m+3)}).$$

As a result we see that that the obstruction constant $C_B(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)}))$ is positive if

$$\beta_{m,d}^{(w)} = \left|\frac{B_{4m+2}}{B_{8m+4}}\right| E_w(8m+3)(b_{(-2)} + b_{(-2d)}) < \sqrt{d}.$$

Using (8) we get

$$\left|\frac{B_{4m+2}}{B_{8m+4}}\right| < \frac{5}{4\sqrt{2}} \left(\frac{\pi e}{2m+1}\right)^{4m+2} \frac{1}{2^{8m+4}}.$$

For $m \ge 5$ we choose a cusp form F_a of weight $a = 4m+10$, i.e. we take $w = 4m-7$ in Proposition 4.1. Such a cusp form exists for all $d \ge 1$ by Proposition 3.1. Using the fact that $\beta_{(-2d)} \le \beta_{(-4)}$ for any $d \ge 2$ and the value $b_{(-4)} = 2^{4m+\frac{5}{2}}$, we see that

$$\beta_{m,d}^{(4m-7)} < (1+\frac{1}{4m-7})^{8m+3} \frac{5}{8\sqrt{2}} \left(\frac{\pi e}{2m+1}\right)^{4m+2} \frac{2^{8m+3} + 2^{4m+\frac{5}{2}} + 1}{2^{8m+3}},$$

which is smaller than 1 if $m \ge 5$. This proves Theorem 1.1 for $m \ge 5$.
For $m = 4$ there exists a cusp form F_a of weight $4m+6$ if $d \ne 1, 2, 4$, i.e. we take $w = 4m-3$. To see that $\beta_{4,d}^{(13)} < \sqrt{d}$ we need check this only for $d = 3$ because $b_{(-2d)} < b_{(-6)}$ for $d > 3$. One can do it by direct calculation.

For $m \leq 3$ we choose F_a of weight $4m+2$, i.e. we take $w = 4m+1$. Such a cusp form exists if $d > 180$ according to Proposition 3.1. For such d we see that $\beta_{(-2d)} < 1$. Then the obstruction constant $C_B(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)}))$ is positive if

$$\left|\frac{B_{4m+2}}{B_{8m+2}}\right|(1+\frac{1}{4m+1})^{8m+3}(2^{8m+3}+2) < \sqrt{d}.$$

This inequality gives us the bound on d in Theorem 1.1.
This completes the proof of Theorem 1.1.

In the proof of Theorem 1.1 above we have seen that the (-2)-part of the branch divisor forms the most important reflective obstruction to the extension of the $\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})$-invariant differential forms to a smooth compact model of $\mathcal{F}_{2d}^{(m)}$. Let us consider the double covering $\mathcal{SF}_{2d}^{(m)}$ of $\mathcal{F}_{2d}^{(m)}$ for $d > 1$ determined by the special orthogonal group:

$$\mathcal{SF}_{2d}^{(m)} = \widetilde{\mathrm{SO}}^+(L_{2d}^{(m)}) \setminus \mathcal{D}_{L_{2d}^{(m)}} \to \mathcal{F}_{2d}^{(m)}.$$

Here the branch divisor does not contain the (-2)-part. Theorem 4.2 below shows that there are only five exceptional varieties $\mathcal{SF}_{2d}^{(m)}$ with $m > 0$ and $d > 1$ that are possibly not of general type.
The variety $\mathcal{SF}_{2d}^{(2)}$ can be interpreted as the moduli space of K3 surfaces of degree $2d$ with spin structure: see [GHS2, §5]. The three-fold $\mathcal{SF}_{2d}^{(0)}$ is the moduli space of $(1,t)$-polarised abelian surfaces.

Theorem 4.2 *The variety $\mathcal{SF}_{2d}^{(m)}$ is of general type for any $d > 1$ if $m \geq 3$. If $m = 2$ then $\mathcal{SF}_{2d}^{(2)}$ is of general type if $d \geq 3$. If $m = 1$ then $\mathcal{SF}_{2d}^{(1)}$ is of general type if $d = 5$ or $d \geq 7$.*

Proof. The case $m = 2$ is [GHS2, Theorem 5.1], and the result for $m \geq 5$ is immediate from Theorem 1.1. For $m = 1$, 3 and 4 we can prove more than what follows from Theorem 1.1.
The branch divisor of $\mathcal{SF}_{2d}^{(m)}$ is defined by the reflections in vectors $r \in L_{2d}^{(m)}$ such that $-\sigma_r \in \widetilde{\mathrm{SO}}^+(L_{2d}^{(m)})$, because the rank of $L_{2d}^{(m)}$ is odd. Therefore $r^2 = -2d$, by Proposition 2.1.
If $F \in M_{2k+1}(\widetilde{\mathrm{SO}}^+(L_{2d}^{(m)}))$ is a modular form (note that the character det is trivial), $d > 1$ and $z \in \mathcal{D}^\bullet_{L_{2d}^{(m)}}$ is such that $(z, r) = 0$, then

$$F(z) = F(-\sigma_r(z)) = F(-z) = (-1)^{2k+1}F(z).$$

Therefore any modular form of odd weight for $\widetilde{\mathrm{SO}}^+(L_{2d}^{(m)})$ vanishes on the branch divisor.

To apply the low-weight cusp form trick used in the proof of Theorem 1.1 one needs a cusp form of weight smaller than $\dim \mathcal{SF}_{2d}^{(m)} = 8m + 3$. By Proposition 3.1 there exists a cusp form $F_{11+4m} \in S_{11+4m}(\widetilde{\mathrm{SO}}^+(L_{2d}^{(m)}))$. For $m \geq 3$ we have that $11 + 4m < 8m + 3$. Therefore the differential forms $F_{11+4m}^k F_{(4m-8)k}(dZ)^k$, for arbitrary $F_{(4m-8)k} \in M_{(4m-8)k}(\widetilde{\mathrm{SO}}^+(L_{2d}^{(m)}))$, extend to the toroidal compactification of $\mathcal{SF}_{2d}^{(m)}$ constructed in Theorem 1.3. This proves the cases $m \geq 3$ of the theorem.

For the case $m = 1$ we use a cusp form of weight 9 with respect to $\widetilde{\mathrm{SO}}^+(L_{2d}^{(1)})$ constructed in Proposition 3.1. □

We can obtain some information also for some of the remaining cases.

Proposition 4.3 *The spaces $\mathcal{SF}_8^{(1)}$ and $\mathcal{SF}_{12}^{(1)}$ have non-negative Kodaira dimension.*

Proof. By Proposition 3.1 there are cusp forms of weight 11 for $\widetilde{\mathrm{SO}}^+(L_8^{(1)})$ and $\widetilde{\mathrm{SO}}^+(L_{12}^{(1)})$. The weight of these forms is equal to the dimension. By the well-known observation of Freitag [F, Hilfssatz 2.1, Kap. III] these cusp forms determine canonical differential forms on the 11-dimensional varieties $\mathcal{SF}_8^{(1)}$ and $\mathcal{SF}_{12}^{(1)}$. □

These varieties may perhaps have intermediate Kodaira dimension as it seems possible that a reflective modular form of canonical weight exists for $L_8^{(1)}$ and $L_{12}^{(1)}$.

In [GHS2] we used pull-backs of the Borcherds modular form Φ_{12} on $\mathcal{D}_{II_{2,26}}$ to show that many moduli spaces of K3 surfaces are of general type. We can also use Borcherds products to prove results in the opposite direction.

Theorem 4.4 *The Kodaira dimension of $\mathcal{F}_{II}^{(m)}$ is $-\infty$ for $m = 0$, 1 and 2.*

Proof. For $m = 0$ we can see immediately that the quotient is rational: a straightforward calculation gives that $\mathcal{F}_{II}^{(0)} = \Gamma\backslash\mathbb{H}_1 \times \mathbb{H}_1$ where $\mathbb{H}_1$ is the usual upper half plane and Γ is the degree 2 extension of $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ by the involution which interchanges the two factors. Compactifying this, we obtain the projective plane $\mathbb{P}_2$.

For $m = 1$, 2 we argue differently. There are modular forms similar to Φ_{12} for the even unimodular lattices $II_{2,10}$ and $II_{2,18}$. They are Borcherds products Φ_{252} and Φ_{127} of weights 252 and 127 respectively, defined by the automorphic functions

$$\Delta(\tau)^{-1}(\tau)E_4(\tau)^2 = q^{-1} + 504 + q(\dots)$$

and

$$\Delta(\tau)^{-1}(\tau)E_4(\tau) = q^{-1} + 254 + q(\dots),$$

where $q = \exp(2\pi i\tau)$ and $\Delta(\tau)$ and $E_4(\tau)$ are the Ramanujan delta function and the Eisenstein series of weight 4 (see [B1]). The divisors of Φ_{252} and Φ_{127} coincide with the branch divisors of $\mathcal{F}_{II}^{(1)}$ and $\mathcal{F}_{II}^{(2)}$ defined by the (-2)-vectors. Moreover Φ_{252} and Φ_{127} each vanishes with order one along the respective divisor. Therefore if $F_{10k}(dZ)^k$ (or $F_{18k}(dZ)^k$) defines a pluricanonical differential form on a smooth model of a toroidal compactification of $\mathcal{F}_{II}^{(1)}$ or $\mathcal{F}_{II}^{(2)}$, then F_{10k} (or F_{18k}) is divisible by Φ_{252}^k (or Φ_{127}^k), since F_{10k} or F_{18k} must vanish to order at least k along the branch divisor. This is not possible, because the quotient would be a holomorphic modular form of negative weight. $\square$

We have already remarked that the space $\mathcal{F}_{II}^{(2)}$ is the moduli space of K3 surfaces with an elliptic fibration with a section. Using the Weierstrass equations it is then clear that this moduli space is unirational (such K3 surfaces are parametrised by a linear system in the weighted projective space $\mathbb{P}(4, 6, 1, 1)$). In fact it is even rational: see [Le].

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