Eigenvalue Clusters of the Landau Hamiltonian in the Exterior of a Compact Domain

Alexander Pushnitski and Grigori Rozenblum

Received: August 30, 2007
Revised: October 20, 2007

Communicated by Heinz Siedentop

Abstract. We consider the Schrödinger operator with a constant magnetic field in the exterior of a compact domain on the plane. The spectrum of this operator consists of clusters of eigenvalues around the Landau levels. We discuss the rate of accumulation of eigenvalues in a fixed cluster.

2000 Mathematics Subject Classification: Primary 35P20; Secondary 35Q40.
Keywords and Phrases: Schrödinger operator, magnetic field, spectral asymptotics, exterior problem

1 Introduction

1.1 Preliminaries

The Landau Hamiltonian describes a charged particle confined to a plane in a constant magnetic field. The Landau Hamiltonian is one of the earliest explicitly solvable quantum mechanical models. Its spectrum consists of the Landau levels,[1] infinitely degenerate eigenvalues placed at the points of an arithmetic progression.

In [7], the Landau Hamiltonian was considered in the exterior of a compact obstacle. Introducing the obstacle produces clusters of eigenvalues of finite multiplicity around the Landau levels. Various asymptotics (high energy, semi-classical) of these eigenvalue clusters were studied in [7]. In this paper we focus

[1] It is a little known fact that this was worked out by Fock two years before Landau; see [4, 9].

on a different aspect of the spectral analysis of this model: for a fixed eigenvalue cluster, we consider the rate of accumulation of eigenvalues in this cluster to the Landau level. We describe this rate of accumulation rather precisely in terms of the logarithmic capacity of the obstacle.
Our construction is motivated by the recent progress in the study of the Landau Hamiltonian on the whole plane perturbed by a compactly supported or fast decaying electric or magnetic field, see [15, 13, 3, 16]. In particular, we use some operator theoretic constructions from [15] and [13] and some concrete analysis (related to logarithmic capacity) from [3].

1.2 The Landau Hamiltonian

We will write $x = (x_1, x_2) \in \mathbb{R}^2$ and identify $\mathbb{R}^2$ with $\mathbb{C}$ in the standard way, setting $z = x_1 + ix_2 \in \mathbb{C}$. The Lebesgue measure in $\mathbb{R}^2$ will be denoted by dx and in $\mathbb{C}$ by $dm(z)$. The derivatives with respect to x_1, x_2 are denoted by $\partial_k = \partial_{x_k}$; we set, as usual, $\bar\partial = (\partial_1 + i\partial_2)/2$, $\partial = (\partial_1 - i\partial_2)/2$.
We denote by $B > 0$ the magnitude of the constant magnetic field in $\mathbb{R}^2$. We choose the gauge $A(x) = (A_1(x), A_2(x)) = (-\frac{1}{2}Bx_2, \frac{1}{2}Bx_1)$ for the magnetic vector potential associated with this field. The magnetic Hamiltonian on the whole plane is defined as

$$X_0 = -(\nabla - iA)^2 \quad \text{in } L^2(\mathbb{R}^2). \tag{1}$$

More precisely, for $u \in C_0^\infty(\mathbb{R}^2)$ we set

$$\|u\|^2_{H^1_A} = \int_{\mathbb{R}^2} |i\nabla u(x) + A(x)u(x)|^2 \, dx \tag{2}$$

and define X_0 as the selfadjoint operator which corresponds to the closure of the quadratic form $\|u\|^2_{H^1_A}$, $u \in C_0^\infty(\mathbb{R}^2)$.
It is well known (see [4, 9] or [10]) that the spectrum of X_0 consists of the eigenvalues $\Lambda_q = (2q+1)B$, $q = 0, 1, \dots$, of infinite multiplicity. In particular, we have

$$\|u\|^2_{H^1_A} \geq B\|u\|^2_{L^2}, \quad u \in C_0^\infty(\mathbb{R}^2). \tag{3}$$

We will denote by $\mathcal{L}_q$ the eigenspace of X_0 corresponding to Λ_q and by P_q the operator of orthogonal projection onto $\mathcal{L}_q$ in $L^2(\mathbb{R}^2)$. Later on, we will need an explicit description of $\mathcal{L}_q$; this will be discussed in section 4.2.
Let $\Omega \subset \mathbb{R}^2$ be an open set. In order to define the magnetic Hamiltonian in Ω, it is convenient to consider the associated quadratic form. Following [12], we denote by $H^1_A(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{H^1_A}$. The quadratic form $\|u\|^2_{H^1_A}$ is closed in $L^2(\Omega)$ and (by (3)) positively definite. This form defines a self-adjoint operator in $\mathbb{R}^2$ which we denote by $X(\Omega)$. If Ω is bounded by a smooth curve, then the usual computations show that this definition of $X(\Omega)$ corresponds to setting the Dirichlet boundary condition on $\partial\Omega$. The operator X_0 corresponds to taking $\Omega = \mathbb{R}^2$ in the above definitions.

Note that for a bounded Ω, the norm in $H^1_A(\Omega)$ is equivalent to the standard Sobolev norm $H^1(\Omega)$; in particular, in this case the embedding $H^1_A(\Omega) \subset L^2(\Omega)$ is compact.

1.3 Main results

Let $K \subset \mathbb{R}^2$ be a compact set and K^c its complement. Our main results concern the spectrum of the operator $X(K^c)$. First we state a preliminary result which gives a general description of the spectrum of $X(K^c)$. This result is already known (see [7]) but as part of our construction, we provide a simple proof in Section 1.4.

PROPOSITION 1.1. *Let $K \subset \mathbb{R}^2$ be a compact set. Then*

$$\sigma_{ess}(X(K^c)) = \sigma_{ess}(X_0) = \cup_{q=0}^{\infty}\{\Lambda_q\}, \quad \Lambda_q = (2q+1)B.$$

Moreover, for all q and all $\lambda \in (\Lambda_{q-1}, \Lambda_q)$ the number of eigenvalues of $X(K^c)$ in (λ, Λ_q) is finite.

In other words, the last statement means that the eigenvalues of $X(K^c)$ can accumulate to the Landau levels only from above.
For all $q \geq 0$, we enumerate the eigenvalues of $X(K^c)$ in $(\Lambda_q, \Lambda_{q+1})$:

$$\lambda_1^q \geq \lambda_2^q \geq \dots$$

Proposition 1.1 ensures that $\lambda_n^q \to \Lambda_q$ as $n \to \infty$. Below we describe the rate of this convergence. Roughly speaking, we will see that for large n,

$$\frac{a^n}{n!} \leq \lambda_n^q - \Lambda_q \leq \frac{b^n}{n!} \tag{4}$$

with some a, b depending on K. In order to discuss the dependence of a, b on the domain K, let us introduce the following notation:

$$\begin{aligned} \Delta_q(K) &= \limsup_{n\to\infty}[n!(\lambda_n^q - \Lambda_q)]^{1/n}, \\ \delta_q(K) &= \liminf_{n\to\infty}[n!(\lambda_n^q - \Lambda_q)]^{1/n}. \end{aligned} \tag{5}$$

The estimates for these spectral characteristics will be given in terms of the logarithmic capacity of K, which is denoted by $\mathrm{Cap}\,(K)$. Below we recall the definition and basic properties of logarithmic capacity; for the details, see e.g. [11] or [18]. For a measure $\mu \geq 0$ in $\mathbb{R}^2$, its logarithmic energy is defined by

$$I(\mu) = \int\int \log\frac{1}{|z-t|}d\mu(z)d\mu(t).$$

For a compact set $K \subset \mathbb{R}^2$, its logarithmic capacity is defined by

$$\mathrm{Cap}\,(K) = e^{-V(K)},$$
$$V(K) = \inf\{I(\mu) \mid \mu \geq 0 \text{ is a measure in } \mathbb{R}^2, \quad \operatorname{supp}\mu \subset K, \quad \mu(K) = 1\}.$$

The logarithmic capacity of compact sets in $\mathbb{R}^2$ has the following properties:
(i) if $K_1 \subset K_2$ then Cap $K_1 \leq$ Cap K_2;
(ii) the logarithmic capacity of a disc of radius r is r;
(iii) if $K_2 = \{\alpha x \mid x \in K_1\}$, $\alpha > 0$, then Cap $K_2 = \alpha\,$Cap K_1;
(iv) Cap K coincides with the logarithmic capacity of the polynomial convex hull $\mathrm{Pc}\,(K)$ of K (=the complement of the unbounded connected component of K^c).
(v) Continuity of capacity: if $K_\varepsilon = \{x \in \mathbb{R}^2 \mid \mathrm{dist}\,(x, K) \leq \varepsilon\}$, then Cap $K_\varepsilon \to$ Cap K as $\varepsilon \to +0$.
To extend the notion of capacity to arbitrary Borel sets E, one defines the inner and outer capacities

$$\mathrm{Cap}_i(E) = \sup\{\mathrm{Cap}\,(K) \mid K \subset E, \quad K \text{ compact}\}$$
$$\mathrm{Cap}_o(E) = \inf\{\mathrm{Cap}_i(U) \mid E \subset U, \quad U \text{ open}\}.$$

Then every Borel set E is capacitable in the sense that $\mathrm{Cap}_i(E) = \mathrm{Cap}_o(E)$ and one can simply write $\mathrm{Cap}\,(E)$ for this common value. We will also need another version of inner capacity, which we denote by $\mathrm{Cap}_-(K)$ and define by

$$\sup\{\mathrm{Cap}\ S \mid \ S \subset K \text{ is a compact set with a Lipschitz boundary}\}.$$

Theorem 1.2. *Let $K \subset \mathbb{R}^2$ be a compact set; then for all $q \geq 0$ one has*

$$\Delta_q(K) \leq \frac{B}{2}(\mathrm{Cap}\,(K))^2,$$
$$\delta_q(K) \geq \frac{B}{2}(\mathrm{Cap}_-(\mathrm{Pc}\,(K)))^2.$$

In particular, if K is a compact set with a Lipschitz boundary, then $\Delta_q(K) = \delta_q(K) = \frac{B}{2}(\mathrm{Cap}\,(K))^2$ for all $q \geq 0$.

The lower bound in the above theorem is strictly positive if and only if $\mathrm{Pc}\,(K)$ has a non-empty interior. In particular, for such compacts the number of eigenvalues λ_1^q, λ_2^q, $\dots$ is infinite for each q. However, even for some compacts without interior points, lower spectral bounds can be obtained. In particular, this can be done for the compact K being a smooth (not necessarily closed) curve.

Theorem 1.3. *Let $K \subset \mathbb{R}^2$ be a C^∞ smooth simple curve of a finite length. Then for all $q \geq 0$, one has*

$$\Delta_q(K) = \delta_q(K) = \frac{B}{2}(\mathrm{Cap}\,(K))^2.$$

Remark. 1. One can prove that

$$\text{if } \mathrm{Cap}\,(K) = 0, \text{ then } C_0^\infty(K^c) \text{ is dense in } H_A^1(\mathbb{R}^2). \tag{6}$$

It follows that for K of zero capacity, $H^1_A(K^c) = H^1_A(\mathbb{R}^2)$ and therefore $X(K^c) = X_0$. Thus, for such K the spectrum of $X(K^c)$ consists of Landau levels Λ_q.

The statement (6) seems to be well known to the experts in the field although it is difficult to pinpoint the exact reference. One can use the argument of [1], Theorem 9.9.1; this argument applies to the usual H^1 Sobolev norm, but it is very easy to modify it for the norm H^1_A. In this theorem the Bessel capacity rather than the logarithmic capacity is used; however, the Bessel capacity of a compact set vanishes if and only if its logarithmic capacity vanishes. In order to prove the latter fact (again, well known to experts) one has to combine Theorem 2.2.7 in [1] and Sect.II.4 in [11].

2. We do not know whether it possible for Λ_q to remain eigenvalues of $X(K^c)$ of infinite multiplicity if $\mathrm{Cap}\,(K) > 0$.

3. Following the proof of Theorem 1.2 and using the results of [3], it is easy to show that for $q = 0$, the lower bound in this theorem can be replaced by the following one:

$$\delta_0(K) \geq \frac{B}{2}(\mathrm{Cap}\,(K_-))^2,$$
$$K_- = \{z \in \mathbb{C} \mid \limsup_{r\to+0} \frac{\log m(Pc(K) \cap D_r(z))}{\log r} < \infty\},$$

where $D_r(z) = \{\zeta \in \mathbb{C} \mid |\zeta - z| \leq r\}$, and $m(\cdot)$ is the Lebesgue measure.

4. Analysing the proof of Theorem 1.3, it is easy to see that if we are only interested in its statement for finitely many q, it suffices to require some finite smoothness of the curve K.

1.4 Outline of the proof

Let us write $L^2(\mathbb{R}^2) = L^2(K^c)\oplus L^2(K)$. (If the Lebesgue measure of K vanishes then, of course, $L^2(K) = \{0\}$.) With respect to this decomposition, let us define

$$R(K^c) = X(K^c)^{-1} \oplus 0 \text{ in } L^2(\mathbb{R}^2) = L^2(K^c) \oplus L^2(K). \tag{7}$$

Clearly, for any $\lambda \neq 0$ we have

$$\lambda \in \sigma(X(K^c)) \Leftrightarrow \lambda^{-1} \in \sigma(R(K^c)) \tag{8}$$

with the same multiplicity. Thus, it suffices for our purposes to study the spectrum of the operator $R(K^c)$.
First note that in the "free" case $K = \emptyset$ we have $R(\mathbb{R}^2) = X_0^{-1}$ and the spectrum of X_0^{-1} consists of the eigenvalues Λ_q^{-1} of infinite multiplicity and their point of accumulation, zero.

Next, it turns out (see section 3) that

$$R(K^c) = X_0^{-1} - W, \quad \text{where } W \geq 0 \text{ is compact.} \tag{9}$$

Thus, the Weyl's theorem on the invariance of the essential spectrum under compact perturbations ensures that $\sigma_{ess}(R(K^c)) = \sigma_{ess}(X_0^{-1})$. Moreover, a simple operator theoretic argument (see e.g. [2, Theorem 9.4.7]) shows that the eigenvalues of $R(K^c)$ do not accumulate to the inverse Landau levels Λ_q^{-1} from above. Thus, the spectrum of $R(K^c)$ consists of zero and the eigenvalue clusters $\{(\lambda_1^q)^{-1}, (\lambda_2^q)^{-1}, \dots\}$ with the eigenvalues in the q'th cluster accumulating to Λ_q^{-1}. In section 2.3 we show that the rate of accumulation of $(\lambda_n^q)^{-1}$ to Λ_q^{-1} can be described in terms of the spectral asymptotics of the Toeplitz type operator $P_q W P_q$; here W is defined by (9) and P_q is the projection onto $\mathcal{L}_q = \operatorname{Ker}(X_0 - \Lambda_q) = \operatorname{Ker}(X_0^{-1} - \Lambda_q^{-1})$.
The spectrum of $P_q W P_q$ is studied in sections 4 and 5, using the results of [3]. The fact that the logarithmic capacity of the domain appears in this problem probably deserves some explanation. In [3], the spectral asymptotics of $P_q W P_q$ was related to the asymptotics of the singular numbers of the embedding of the Segal-Bargmann space $\mathcal{F}^2$ (see section 4.2 below) into an L^2 space with the weight related to W. Following the technique of [14], in [3] the analysis of this asymptotics is then reduced to the analysis of the sequence of polynomials of a complex variable, orthogonal with respect to the relevant weight. After this, the results of [18] ensure that the asymptotics of these polynomials is determined by the logarithmic capacity of the support of the weight.

1.5 Acknowledgements

A large part of this work was completed during the authors' stay at the Isaac Newton Institute for Mathematical Sciences (Cambridge, UK) in the framework of the programme "Spectral Theory and Partial Differential Equations". It is a pleasure to thank the Institute and the organisers of the programme for providing this opportunity. The authors are also grateful to Uzy Smilansky for useful discussions and to Nikolai Filonov for helpful remarks on the text of the manuscript.

2 Some abstract results

Here we collect some general operator theoretic statements that are used in the proof. The statements themselves, with the exception of the last one, are almost obvious, but spelling them out explicitly helps explain the main ideas of our construction.

2.1 Quadratic forms

Our arguments can be stated most succinctly if we are allowed to deal with quadratic forms whose domains are not necessarily dense in the Hilbert space.

Here is the corresponding abstract framework; related constructions appeared before in the literature; see e.g. [17].

Let a be a closed positive definite quadratic form in a Hilbert space $\mathcal{H}$ with the domain $d[a]$. Let the closure of $d[a]$ in $\mathcal{H}$ be $\mathcal{H}_a$. Then the form a defines a self-adjoint operator A in $\mathcal{H}_a$. Let $J_a : \mathcal{H}_a \to \mathcal{H}$ be the natural embedding operator; its adjoint $J_a^* : \mathcal{H} \to \mathcal{H}_a$ acts as the orthogonal projection onto the subspace $\mathcal{H}_a$ of $\mathcal{H}$. The operator $J_a A^{-1} J_a^*$ in $\mathcal{H}$ can be considered as the direct sum

$$J_a A^{-1} J_a^* = A^{-1} \oplus 0 \text{ in the decomposition } \mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_a^\perp;$$

here we have in mind (7). Now let b be another closed positive definite form in $\mathcal{H}$ and let B, $d[b]$, $\mathcal{H}_b$, J_b be the corresponding objects constructed for this form.

Proposition 2.1. *Suppose that $d[b] \subset d[a]$ and $b[x,y] = a[x,y]$ for all $x, y \in d[b]$. Then:*
(i) $J_b B^{-1} J_b^* \le J_a A^{-1} J_a^*$ on $\mathcal{H}$;
(ii) if $x \in d[b] \cap \mathrm{Dom}\,(A)$, then $x \in \mathrm{Dom}\,(B)$, $Bx = Ax$, and $J_b B^{-1} J_b^* Ax = J_a A^{-1} J_a^* Ax$.

Proof. It suffices to consider the case $\mathcal{H}_a = \mathcal{H}$.
(i) The hypothesis implies

$$b[x,x] = a[J_b x, J_b x] \quad \text{for all } x \in d[b].$$

This can be recast as $\|B^{1/2}x\| = \|A^{1/2}J_b x\|$, $x \in d[b]$. It follows that the operator $A^{1/2} J_b B^{-1/2}$ is an isometry on $\mathcal{H}_b$ and therefore $A^{1/2} J_b B^{-1/2} J_b^*$ is a contraction on $\mathcal{H}$. By conjugation, we get that $\|J_b B^{-1/2} J_b^* A^{1/2} z\| \le \|z\|$ for all $z \in d[a]$. The last statement is equivalent to $\|J_b B^{-1/2} J_b^* u\| \le \|A^{-1/2} u\|$ for all $u \in \mathcal{H}$, and so $J_b B^{-1} J_b^* \le A^{-1}$ as required.
(ii) Let $y \in d[b]$; then

$$b[x,y] = a[x,y] = (Ax, y),$$

and so $x \in \mathrm{Dom}\,(B)$ and $Bx = Ax$. Next, $J_a A^{-1} J_a^* Ax = A^{-1}Ax = x$, and

$$J_b B^{-1} J_b^* Ax = J_b B^{-1} J_b^* Bx = J_b B^{-1} Bx = J_b x = x,$$

which proves the required statement. □

2.2 Shift in enumeration

The asymptotics of the type discussed in Theorems 1.2 and 1.3 is independent of a shift in the enumeration of eigenvalues. This is a consequence of the following elementary fact. Let $b_1 \ge b_2 \ge \dots$ be a sequence of positive numbers such that $\limsup_{n\to\infty} [n! b_n]^{1/n} < \infty$. Then for all $\ell \in \mathbb{Z}$,

$$\lim_{n\to\infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} [n! b_{n+\ell}]^{1/n} = \lim_{n\to\infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} [n! b_n]^{1/n}. \tag{10}$$

2.3 Accumulation of eigenvalues

Having in mind (9), let us consider the following general situation. Let T be a self-adjoint operator and let Λ be an isolated eigenvalue of T of infinite multiplicity with the corresponding eigenprojection P_Λ. Let $\tau > 0$ be such that

$$((\Lambda - 2\tau, \Lambda + 2\tau) \setminus \{\Lambda\}) \cap \sigma(T) = \emptyset.$$

Next, let $W \geq 0$ be a compact operator; consider the spectrum of $T - W$. The Weyl's theorem on the invariance of the essential spectrum under compact perturbations ensures that

$$((\Lambda - 2\tau, \Lambda + 2\tau) \setminus \{\Lambda\}) \cap \sigma_{ess}(T - W) = \emptyset.$$

Moreover, a simple argument (see e.g. [2, Theorem 9.4.7]) shows that the eigenvalues of $T - W$ do not accumulate to Λ from above (i.e. $(\Lambda, \Lambda + \epsilon) \cap \sigma(T - W) = \emptyset$ for some $\epsilon > 0$).
We will need a description of the eigenvalues of $T - W$ below Λ in terms of the eigenvalues of the Toeplitz operator $P_\Lambda W P_\Lambda$. Let $\mu_1 \geq \mu_2 \geq \cdots$ be the eigenvalues of $P_\Lambda W P_\Lambda$; in order to exclude degenerate cases, let us assume that this operator has infinite rank. Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of $T - W$ in the interval $(\Lambda - \tau, \Lambda)$.

Proposition 2.2. *Under the above assumptions, for any $\epsilon > 0$ there exists $\ell \in \mathbb{Z}$ such that for all sufficiently large n, one has*

$$(1 - \epsilon)\mu_{n+\ell} \leq \Lambda - \lambda_n \leq (1 + \epsilon)\mu_{n-\ell}.$$

The proof borrows its key element from [8, Lemma 1.1]. An alternative proof can be found in [15, Proposition 4.1].

Proof. 1. We denote $S = T - W$ and $Q_\Lambda = I - P_\Lambda$ and consider the operators

$$R_\pm = \epsilon P_\Lambda W P_\Lambda + \frac{1}{\epsilon} Q_\Lambda W Q_\Lambda \pm (P_\Lambda W Q_\Lambda + Q_\Lambda W P_\Lambda).$$

and

$$S_\pm = P_\Lambda(T - (1 \pm \epsilon)W)P_\Lambda + Q_\Lambda(T - (1 \pm \frac{1}{\epsilon})W)Q_\Lambda.$$

We have

$$S = S_+ + R_- = S_- - R_+.$$

2. Since W is compact, the operators $R_\pm$ are also compact. Since $R_\pm$ can be represented as

$$R_\pm = (\sqrt{\epsilon} P_\Lambda \pm \frac{1}{\sqrt{\epsilon}} Q_\Lambda) W (\sqrt{\epsilon} P_\Lambda \pm \frac{1}{\sqrt{\epsilon}} Q_\Lambda)$$

and $W \geq 0$, we see that $R_\pm \geq 0$.

3. Let us discuss the spectrum of $S_\pm$ in $(\Lambda-\tau,\Lambda)$. Clearly, the spectrum of $P_\Lambda(T-(1\pm\epsilon)W)P_\Lambda = \Lambda P_\Lambda - (1\pm\epsilon)P_\Lambda W P_\Lambda$ consists of the eigenvalues $\Lambda-(1\pm\epsilon)\mu_n$. Next, since by assumption, $T\mid_{\operatorname{Ran} Q_\Lambda}$ has no spectrum in $(\Lambda-2\tau,\Lambda+2\tau)$ and W is compact, we see that $Q_\Lambda(T-(1\pm\frac{1}{\epsilon})W)Q_\Lambda\mid_{\operatorname{Ran} Q_\Lambda}$ has only finitely many eigenvalues in the interval $(\Lambda-\tau,\Lambda+\tau)$. Since the operators $P_\Lambda(T-(1\pm\epsilon)W)P_\Lambda$ and $Q_\Lambda(T-(1\pm\frac{1}{\epsilon})W)Q_\Lambda$ act in orthogonal subspaces of our Hilbert space, the spectrum of $S_\pm$ is the union of the spectra of these operators.

So we arrive at the following conclusion. Let $\nu_1^\pm \le \nu_2^\pm \le \cdots$ denote the eigenvalues of $S_\pm$ in $(\Lambda-\tau,\Lambda)$. Then

$$\nu_n^+ = \Lambda-(1+\epsilon)\mu_{n-i}, \quad \nu_n^- = \Lambda-(1-\epsilon)\mu_{n-j}, \tag{11}$$

for some integers i,j and all sufficiently large n.

4. Let us prove that $\lambda_n \le \nu_{n+k}^-$ for some integer k and all sufficiently large n. Denote $\delta=(\lambda_1-\Lambda+\tau)/2$ and let us write $R_+ = R_+^{(1)}+R_+^{(2)}$, where $0\le R_+^{(1)}\le \delta I$ and $\operatorname{rank} R_-^{(2)}<\infty$. Denote by $N_S(\alpha,\beta)$ the number of eigenvalues of S in the interval (α,β). Writing $S=S_- - R_+^{(1)} - R_+^{(2)}$, we get for any $\lambda\in(\lambda_1,\Lambda)$:

$$\begin{aligned} N_S(\Lambda-\tau,\lambda) &= N_S(\lambda_1-2\delta,\lambda) \\ &\ge N_{S_- - R_+^{(1)}}(\lambda_1-2\delta,\lambda) - \operatorname{rank} R_-^{(2)} \ge N_{S_-}(\lambda_1-\delta,\lambda) - \operatorname{rank} R_-^{(2)}. \end{aligned}$$

The second inequality above follows from $\sigma(R_+^{(1)}) \subset [0,\delta]$ (see [2, Lemma 9.4.3]). These inequalities for the eigenvalue counting functions can be rewritten as $\lambda_n \le \nu_{n+k}^-$ with some integer k.

In the same way, one proves that $\lambda_n \ge \nu_{n-k}^+$ for large n and some integer k. Taken together with (11), this yields the required result. $\square$

3 Preliminaries and reduction to Toeplitz operators

Let $K\subset\mathbb{R}^2$ be a compact set; we return to the discussion of the spectrum of $X(K^c)$ and start with some general remarks.

First we would like to point out that the spectral asymptotics that we are interested in is independent of the "holes" in the domain K:

$$\delta_q(K)=\delta_q(Pc(K)), \quad \Delta_q(K)=\Delta_q(Pc(K)). \tag{12}$$

Indeed, let us write $K^c=\Omega\cup\Sigma$, where Ω is the unbounded connected component of K^c and Ω and Σ are disjoint. With respect to the direct sum decomposition $L^2(K^c)=L^2(\Omega)\oplus L^2(\Sigma)$, we have $X(K^c)=X(\Omega)\oplus X(\Sigma)$. By the compactness of the embedding $H_A^1(\Sigma)\subset L^2(\Sigma)$, the operator $X(\Sigma)$ has a compact resolvent. Thus, on any bounded interval of the real line the spectrum of $X(K^c)$ differs from the spectrum of $X(\Omega)$ by at most finitely many eigenvalues. By (10), this yields (12).

Next, we apply the abstract reasoning of section 2.1 to the quadratic form $a[u] = \|u\|^2_{H^1_A(K^c)}$ with domain $d[a] = H^1_A(K^c)$, considering $L^2(\mathbb{R}^2)$ as the main Hilbert space $\mathcal{H}$. We consider the operator $R(K^c)$ (see (7)) and write $R(K^c) = X_0^{-1} - W$. Proposition 2.2 suggests that in order to find the rate of accumulation of the eigenvalues of $R(K^c)$ to Λ_q^{-1}, one should study the spectrum of the Toeplitz type operators $P_q W P_q$. This is done in the next section. Denote by $\mu_1^q \geq \mu_2^q \geq \dots$ the eigenvalues of $P_q W P_q$. We will prove

Proposition 3.1. *Let $K \subset \mathbb{R}^2$ be a compact set and $q \geq 0$. Then*

$$\limsup_{n\to\infty} (n!\mu_n^q)^{1/n} \leq \frac{B}{2}(\mathrm{Cap}\,(K))^2,$$
$$\liminf_{n\to\infty} (n!\mu_n^q)^{1/n} \geq \frac{B}{2}(\mathrm{Cap}\,_-(K))^2.$$

If K is a C^∞ smooth curve, then one has

$$\lim_{n\to\infty} (n!\mu_n^q)^{1/n} = \frac{B}{2}(\mathrm{Cap}\,(K))^2.$$

Now we can prove our main statements.

Proof of Theorem 1.1 and Theorem 1.2. Combining Proposition 3.1, Proposition 2.2 and (12), we get the estimates for the quantities

$$\lim_{n\to\infty} \sup[n!(\Lambda_q^{-1} - (\lambda_n^q)^{-1})]^{1/n} \leq \frac{B}{2}(\mathrm{Cap}\,(K))^2,$$
$$\lim_{n\to\infty} \inf[n!(\Lambda_q^{-1} - (\lambda_n^q)^{-1})]^{1/n} \geq \frac{B}{2}(\mathrm{Cap}\,_-(Pc(K))^2$$

for any compact K. If K is a C^∞ smooth curve, we get

$$\lim_{n\to\infty} [n!(\Lambda_q^{-1} - (\lambda_n^q)^{-1})]^{1/n} = \frac{B}{2}(\mathrm{Cap}\,(K))^2.$$

An elementary argument shows that

$$\lim_{n\to\infty} \begin{Bmatrix}\sup\\ \inf\end{Bmatrix} [n!(\Lambda_q^{-1} - (\lambda_n^q)^{-1})]^{1/n} = \lim_{n\to\infty} \begin{Bmatrix}\sup\\ \inf\end{Bmatrix} [n!(\lambda_n^q - \Lambda_q)]^{1/n}.$$

This yields the required statements. $\square$

Proof of (9). Let D be a disc such that $K \subset D$. By Proposition 2.1(i), we get

$$D^c \subset K^c \subset \mathbb{R}^2 \Rightarrow R(D^c) \leq R(K^c) \leq X_0^{-1}$$

and so

$$0 \leq X_0^{-1} - R(K^c) \leq X_0^{-1} - R(D^c). \tag{13}$$

Thus, $W = X_0^{-1} - R(K^c)$ is non-negative; let us address compactness.

It is well known that if $0 \le V_1 \le V_2$ are self-adjoint operators and V_2 is compact, then V_1 is also compact. Thus, by (13), in order to prove the compactness of W, it suffices to check that $X_0^{-1} - R(D^c)$ is compact.
Let $\Gamma = \partial D$. Employing the same arguments as in the proof of (12), we see that $X(\Gamma^c)^{-1} - R(D^c)$ is the inverse of the magnetic operator on the disc and hence a compact operator. Thus, it suffices to prove that the difference

$$X_0^{-1} - X(\Gamma^c)^{-1} = (X_0^{-1} - R(D^c)) - (X(\Gamma^c)^{-1} - R(D^c))$$

is compact.
Let us compute the quadratic form of this difference. Let $f, g \in L^2(\mathbb{R}^2)$, $X_0^{-1}f = u$, $X(\Gamma^c)^{-1}g = v$. We have

$$((X_0^{-1} - X(\Gamma^c)^{-1})f, g) = (u, X(\Gamma^c)v) - (X_0 u, v).$$

Integrating by parts and noting that $v \in \mathrm{Dom}\,(X(\Gamma^c))$ vanishes on Γ, we get

$$(u, X(\Gamma^c)v) - (X_0 u, v) = \int_\Gamma (n_A v(s)^+ + n_A v(s)^-) u(s) ds \tag{14}$$

where $n_A v(s) = (\nabla - iA(s))v \cdot \mathbf{n}(s)$, $\mathbf{n}(s)$ is the exterior normal to Γ at the point s and the superscripts $+$ and $-$ indicate that the limits of the functions are taken on the circle Γ by approaching it from the outside or inside.
Take a smooth cut-off function $\omega \in C_0^\infty(\mathbb{R}^2)$ such that $\omega(x) = 1$ in the neighborhood of D. Then we can replace u, v by $u_1 = \omega u$, $v_1 = \omega v$ in the r.h.s. of (14). By the local elliptic regularity we have $u_1 \in H^2(\mathbb{R}^2)$, $v_1 \in H^2(\Gamma^c)$, and the corresponding Sobolev norms of u_1, v_1 can be estimated via the L^2-norms of f, g. Now it remains to notice that the trace mapping $u_1 \mapsto u_1|_\Gamma$ is compact as considered from $H^2(\mathbb{R}^2)$ to $L^2(\Gamma)$, and the mappings $v_1 \mapsto (n_A v_1)^\pm$ are compact as considered from $H^2(\Gamma^c)$ to $L^2(\Gamma)$. It follows that the difference $X_0^{-1} - X(\Gamma^c)^{-1}$ is compact, as required. □

4 The spectrum of Toeplitz operators

4.1 Restriction operators and the associated Toeplitz operators

Let μ be a finite measure in $\mathbb{R}^2$ with a compact support. Consider the restriction operator

$$\gamma_0 : C_0^\infty(\mathbb{R}^2) \ni u \mapsto u\,|_{supp(\mu)} \in L^2(\mu).$$

We are interested in two special cases, namely when μ is the restriction of the Lebesgue measure to a set with Lipschitz boundary and when μ is the arc length measure on a simple smooth curve. In both cases γ_0 can be extended by continuity to a bounded and compact operator $\gamma : H_A^1(\mathbb{R}^2) \to L^2(\mu)$.
Next, let $J : H_A^1(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be the embedding operator, $J : u \mapsto u$. Then the adjoint $J^* : L^2(\mathbb{R}^2) \to H_A^1(\mathbb{R}^2)$ acts as $J^* : u \mapsto {X_0}^{-1/2}u$.

For $q \geq 0$, consider the operators $T_q(\mu)$ in $L^2(\mathbb{R}^2)$ defined by the quadratic form

$$(T_q(\mu)u, u)_{L^2(\mathbb{R}^2)} = \int_{\operatorname{supp}\mu} |(P_q u)(x)|^2 d\mu(x), \quad u \in L^2(\mathbb{R}^2).$$

This operator can be represented as

$$T_q(\mu) = (\gamma J^* {X_0}^{1/2} P_q)^* (\gamma J^* {X_0}^{1/2} P_q) = \Lambda_q (\gamma J^* P_q)^* (\gamma J^* P_q).$$

Since γ is compact by assumption, the operator $T_q(\mu)$ is also compact.
Fix $q \geq 0$; let $s_1^q \geq s_2^q \geq \dots$ be the eigenvalues of $T_q(\mu)$ in $L^2(\mathbb{R}^2)$.

Proposition 4.1. *(i) Let μ be the restriction of the Lebesgue measure onto a bounded domain $K \subset \mathbb{R}^2$ with a Lipschitz boundary. Then*

$$\lim_{n\to\infty} (n! s_n^q)^{1/n} = \frac{B}{2} (\operatorname{Cap}(K))^2.$$

(ii) Let μ be the arc length measure of a C^∞ smooth simple curve of a finite length. Then

$$\lim_{n\to\infty} (n! s_n^q)^{1/n} = \frac{B}{2} (\operatorname{Cap}(\operatorname{supp}\mu))^2.$$

Before proving this proposition, we need to recall the description of the subspaces $\mathcal{L}_q$.

4.2 The structure of subspaces $\mathcal{L}_q$

Denote $\Psi(z) = \frac{1}{4} B|z|^2$. Let us define the creation and annihilation operators (first introduced in this context by Fock [5])

$$\mathfrak{Q} = -2ie^{-\Psi} \overline{\partial} e^{\Psi} = -2i\overline{\partial} - \frac{B}{2} iz$$
$$\mathfrak{Q}^* = -2ie^{\Psi} \partial e^{-\Psi} = -2i\partial + \frac{B}{2} i\overline{z}.$$

The Landau Hamiltonian can be expressed as

$$X_0 = \mathfrak{Q}^*\mathfrak{Q} + B = \mathfrak{Q}\mathfrak{Q}^* - B. \tag{15}$$

The spectrum and spectral subspaces of X_0 can be described in the following way. The equation $(X_0 - B)u = 0$ is equivalent to

$$\mathfrak{Q}u = -2ie^{-\Psi} \bar{\partial}(e^{\Psi} u) = 0.$$

This means that $f = e^{\Psi} u$ is an entire analytic function such that $e^{-\Psi} f \in L^2(\mathbb{C})$. The space of such functions f is called Fock or Segal-Bargmann space $\mathcal{F}^2$ (see [6] for an extensive discussion). So $\mathcal{L}_0 = e^{-\Psi}\mathcal{F}^2$. Further eigenspaces $\mathcal{L}_q$, $q = 1, 2, \dots$, are obtained as $\mathcal{L}_q = (\mathfrak{Q}^*)^q \mathcal{L}_0$. The operators $\mathfrak{Q}^*, \mathfrak{Q}$ act between the subspaces $\mathcal{L}_q$ as

$$\mathfrak{Q}^* : \mathcal{L}_q \mapsto \mathcal{L}_{q+1}, \quad \mathfrak{Q} : \mathcal{L}_q \mapsto \mathcal{L}_{q-1}, \quad \mathfrak{Q} : \mathcal{L}_0 \mapsto \{0\}, \tag{16}$$

and are, up to constant factors, isometries on $\mathcal{L}_q$. In particular, the substitution

$$\mathcal{L}_q \ni u = C_q^{-1}(\mathfrak{Q}^*)^q e^{-\Psi} f, \quad f \in \mathcal{F}^2, \quad C_q = \sqrt{q!(2B)^q} \tag{17}$$

gives a unitary equivalence of spaces $\mathcal{L}_q$ and $\mathcal{F}^2$.

4.3 Proof of Proposition 4.1

(i) The proof is given in [3, Lemma 3.1] for $q = 0$ and [3, Lemma 3.2] for $q \geq 0$.
(ii) For $q = 0$ the result again follows from Lemma 3.1 in [3]. Although the reasoning there concerns the operators $T_q(v) = (vP_q)^*(vP_q)$ where the function v is separated from zero on a compact, it goes through for $T_q(\mu)$. Only notational changes are required; one simply has to replace the measure $v(z)dm(z)$ by $d\mu(z)$.
For $q \geq 1$ below we apply the reduction to the lowest Landau level similar to the proof of Lemma 3.2 in [3].
Denote $d\tilde{\mu}(z) = e^{-\Psi(z)}d\mu(z)$. Applying the unitary equivalence (17), we get for $u \in \mathcal{L}_q$

$$(T_q(\mu)u, u)_{L^2(\mathbb{R}^2)} = C_q^{-2}\|(2\partial - B\overline{z})^q f\|^2_{L^2(\tilde{\mu})}. \tag{18}$$

In particular, for $q = 0$

$$(T_0(\mu)u, u)_{L^2(\mathbb{R}^2)} = C_0^{-2}\|f\|^2_{L^2(\tilde{\mu})}. \tag{19}$$

Below we separately prove the upper and lower bound for the quadratic form (18).
1. *Upper bound.* Consider the open δ-neighborhood $U_\delta \subset \mathbb{C}^1$ of the curve Γ. As it follows from the Cauchy integral formula, for some constant $C_1(q, \delta)$, the inequality

$$\|\partial^k f\|^2_{L^2(\tilde{\mu})} \leq C_1(q, \delta) \int_{U_\delta} |f(z)|^2 dm(z).$$

holds for all functions $f \in \mathcal{F}^2$. Thus, we have the estimate

$$\|(2\partial - B\overline{z})^q f\|^2_{L^2(\tilde{\mu})} \leq C_2(q, \delta) \int_{U_\delta} |f(z)|^2 dm(z).$$

Using (18), (19), we arrive at the estimate

$$T_q(\mu) \leq CT_0(\chi_{U_\delta}(x)dx), \tag{20}$$

where χ_{U_δ} is the characteristic function of the set U_δ. Now we can again apply the estimate of [3, Lemma 3.1] to the eigenvalues $s_1(\delta) \geq s_2(\delta) \geq \dots$ of $T_0(\chi_{U_\delta}(x)dx)$. This estimate together with (20) yields

$$\lim_{n\to\infty}(n!s_n)^{1/n} \leq \lim_{n\to\infty}(n!s_n(\delta))^{1/n} \leq \frac{B}{2}(\mathrm{Cap}\,(U_\delta))^2.$$

Finally, by the continuity of capacity, $\mathrm{Cap}\,(U_\delta) \to \mathrm{Cap}\,(\Gamma)$ as $\delta \to 0$, and this proves the upper bound.
2. *Lower bound.* The lower bound for the spectrum of $T_q(\mu)$ requires a little more work. Let $\sigma : [0, s] \to \mathbb{C}$ be the parameterization of Γ by the arc length. Since f is analytic, we have

$$(\partial f)(\sigma(t)) = \rho(t) \frac{d}{dt} f(\sigma(t)) \tag{21}$$

with some smooth factor $\rho(t)$, $|\rho(t)| = 1$.
Next, due to the compactness of the embedding $H^1(0, s) \subset L^2(0, s)$, for any $\beta > 0$ there exists a subspace of $H^1(0, s)$ of a finite codimension such that for any u in this subspace,

$$\int_0^s |u(t)|^2 dt \le \beta^2 \int_0^s |u'(t)|^2 dt. \tag{22}$$

It follows from (21) and (22) that for any $\beta > 0$ there exists a subspace of $\mathcal{F}^2$ of a finite codimension such that for any f in this subspace

$$\int_\Gamma |f(z)|^2 d\tilde{\mu}(z) \le \beta^2 \int_\Gamma |\partial f(z)|^2 d\tilde{\mu}(z).$$

Arguing by induction, we obtain that for any $\beta > 0$ there exists a subspace $N = N(\beta, q) \subset \mathcal{F}^2$ of finite codimension such that for all $f \in N(\beta, q)$

$$\int_\Gamma |\partial^k f(z)|^2 d\tilde{\mu}(z) \le \beta^2 \int_\Gamma |\partial^q f(z)|^2 d\tilde{\mu}(z), \quad \forall k = 0, 1, \dots, q-1. \tag{23}$$

Using (23) and choosing β sufficiently small, we can estimate the form (18) from below as follows:

$$\begin{aligned} \|(2\partial - B\overline{z})^q f\|^2_{L^2(\tilde{\mu})} &\ge (\|(2\partial)^q f\|_{L^2(\tilde{\mu})} - \sum_{k=0}^{q-1} C_{q,k} \|\partial^k f\|_{L^2(\tilde{\mu})})^2 \\ &\ge \|(2\partial)^q f\|^2_{L^2(\tilde{\mu})} (1 - \sum_{k=0}^{q-1} C_{q,k}\beta)^2 = C_1 \|\partial^q f\|^2_{L^2(\tilde{\mu})} \ge C_2 \|f\|^2_{L^2(\tilde{\mu})} \end{aligned}$$

for all $f \in N(\beta, q)$. Using (18) and (19), we arrive at the lower bound

$$T_q(\mu) \ge C T_0(\mu) + F$$

where F is a finite rank operator. For the eigenvalues of $T_0(\mu)$ the required lower estimates are already obtained by reference to [3, Lemma 3.1]; this completes the proof of the lower bound. □

5 PROOF OF PROPOSITION 3.1

We will prove separately upper and lower bounds.

5.1 PROOF OF THE UPPER BOUND

1. Let $U \subset \mathbb{R}^2$ be an open bounded set with a Lipschitz boundary, $K \subset U$, and let $\omega \in C_0^\infty(\mathbb{R}^2)$ be such that $\omega|_K = 1$ and $\omega|_{U^c} = 0$. Denote $\tilde{\omega} = 1 - \omega$. Note that for any $\psi \in \mathcal{H}$, the function $\tilde{\omega} P_q \psi$ belongs both to $\mathrm{Dom}\,(X_0)$ and to the form domain of $X(K^c)$. Thus, by Proposition 2.1(ii) (with $A = X_0$ and $B = X(K^c)$), we have $W X_0 \tilde{\omega} P_q \psi = 0$. Thus, we have

$$
\begin{aligned}
(W P_q \psi, P_q \psi) &= \frac{1}{\Lambda_q^2}(W X_0 P_q \psi, X_0 P_q \psi) \\
&= \frac{1}{\Lambda_q^2}(W X_0(\omega + \tilde{\omega}) P_q \psi, X_0(\omega + \tilde{\omega}) P_q \psi) = \frac{1}{\Lambda_q^2}(W X_0 \omega P_q \psi, X_0 \omega P_q \psi).
\end{aligned}
$$

Since $W = X_0^{-1} - R(K^c) \le X_0^{-1}$, we have

$$
(W X_0 \omega P_q \psi, X_0 \omega P_q \psi) \le (X_0^{-1} X_0 \omega P_q \psi, X_0 \omega P_q \psi) = \|\omega P_q \psi\|^2_{H_A^1}.
$$

Using (15), we get

$$
\begin{aligned}
\|\omega P_q \psi\|^2_{H_A^1} &= \|\mathfrak{Q}^* \omega P_q \psi\|^2 - B\|\omega P_q \psi\|^2 \le \|\mathfrak{Q}^* \omega P_q \psi\|^2 \\
&= \|\omega \mathfrak{Q}^* P_q \psi - 2i(\partial \omega) P_q \psi\|^2 \le 2\|\mathfrak{Q}^* P_q \psi\|^2_{L^2(U)} + C_1 \|P_q \psi\|^2_{L^2(U)}.
\end{aligned}
$$

2. Due to the compactness of the embedding $H_0^1(U) \subset L^2(U)$, for any $\beta > 0$ there exists a subspace of $H_0^1(U)$ of a finite codimension such that for all elements u of this subspace,

$$
\int_U |u(x)|^2 dx \le \beta^2 \int_U |\nabla u(x)|^2 dx = \beta^2 \int_U |2\partial u(x)|^2 dx.
$$

Taking β sufficiently small, we obtain

$$
\begin{aligned}
\|\mathfrak{Q}^* u\|_{L^2(U)} &\ge \|2\partial u\|_{L^2(U)} - \frac{B}{2}\|\overline{z} u\|_{L^2(U)} \ge \|2\partial u\|_{L^2(U)} - \frac{B}{2}\sup_U |z| \|u\|_{L^2(U)} \\
&\ge (1 - \frac{B}{2}\beta \sup_U |z|)\|2\partial u\|_{L^2(U)} \ge \frac{1}{2}\|2\partial u\|_{L^2(U)} \ge \frac{1}{2\beta}\|u\|_{L^2(U)}
\end{aligned}
$$

for all u in our subspace. It follows that on a subspace of $\psi \in L^2(\mathbb{R}^2)$ of a finite codimension,

$$
(W P_q \psi, P_q \psi)_{L^2(\mathbb{R}^2)} \le 2\|\mathfrak{Q}^* P_q \psi\|^2_{L^2(U)} + 4\beta^2 \|\mathfrak{Q}^* P_q \psi\|^2_{L^2(U)} \le C\|P_{q+1} \psi\|^2_{L^2(U)};
$$

the last inequality holds true by (16).
Thus, we have

$$
P_q W P_q \le C_3 P_{q+1} \chi_U P_{q+1} + F,
$$

where χ_U is the characteristic function of U, and F is a finite rank operator.

3. From Proposition 4.1 we get

$$\limsup_{n\to\infty}(n!\mu_n)^{1/n} \leq \frac{1}{2}B(\mathrm{Cap}\, U)^2.$$

Since U can be chosen such that Cap U is arbitrarily close to Cap K, by the continuity property of capacity, we get the required upper bound.

5.2 Proof of the lower bound

1. Let γ, J, μ be as in section 4.1. Consider the quadratic form in $L^2(\mathbb{R}^2)$

$$\|u\|^2_{H^1_A(\mathbb{R}^2)} + \int_{\mathrm{supp}\,\mu} |u(x)|^2 d\mu(x) = \|X_0^{1/2}u\|^2_{L^2(\mathbb{R}^2)} + \|\gamma J^* X_0^{1/2} u\|^2_{L^2(\mathbb{R}^2)}$$

defined for $u \in H^1_A(\mathbb{R}^2)$. This form is closed and positively defined on $L^2(\mathbb{R}^2)$. Denote by $\tilde{X}$ the corresponding self-adjoint operator in $L^2(\mathbb{R}^2)$. We have

$$\tilde{X} = X_0 + X_0^{1/2} J\gamma^*\gamma J^* X_0^{1/2} = X_0^{1/2}(I + J\gamma^*\gamma J^*)X_0^{1/2}$$

and therefore

$$X_0^{-1} - \tilde{X}^{-1} = X_0^{-1/2}[J\gamma^*\gamma J^*(I + J\gamma^*\gamma J^*)^{-1}]X_0^{-1/2}.$$

Since γ is compact by assumption, we have $J\gamma^*\gamma J^* \leq I$ on a subspace of a finite codimension. Thus,

$$X_0^{-1} - \tilde{X}^{-1} \geq \frac{1}{2} X_0{}^{-1/2} J\gamma^*\gamma J^* X_0^{-1/2}$$

on a subspace of finite codimension, and so

$$P_q(X_0^{-1} - \tilde{X}^{-1})P_q \geq \frac{1}{2\Lambda_q}(\gamma J^* P_q)^*(\gamma J^* P_q) + F \tag{24}$$

where F is a finite rank operator.
2. Now let $K \subset \mathbb{R}^2$ be a compact with a non-empty interior. Let $K_1 \subset K$ be a set with a Lipschitz boundary. Let μ be the restriction of the Lebesgue measure on K_1. By Proposition 2.1(i), we have $X_0^{-1} \geq \tilde{X}^{-1} \geq R(K^c)$. It follows that

$$P_q(X_0^{-1} - R(K^c))P_q \geq P_q(X_0^{-1} - \tilde{X}^{-1})P_q. \tag{25}$$

From here, using (24) and Proposition 4.1(i), we get the required lower bound in the first part of Proposition 3.1. Finally, consider the case of K being a smooth curve. Let μ be the arc measure of the curve. Then, again by (24) and (25), and applying Proposition 4.1(ii), we get the second part of Proposition 3.1.

References

[1] Adams, D. R., Hedberg, L. I., *Function Spaces and Potential Theory* Springer, 1995.

[2] Birman, M. Sh., Solomjak, M. Z., *Spectral theory of selfadjoint operators in Hilbert space.* D. Reidel Publishing Co., 1987.

[3] Filonov, N., Pushnitski, A., *Spectral asymptotics of Pauli operators and orthogonal polynomials in complex domains.* Comm. Math. Phys. 264, 759–772 (2006).

[4] Fock, V., *Bemerkung zur Quantelung des harmonischen Oszillators im Magnetfeld,* Z. Phys 47, 446–448 (1928).

[5] Fock, V., *Konfigurationsraum und zweite Quantelung,* Z. Phys. 75, 622–647 (1932).

[6] Folland, G., *Harmonic Analysis in Phase Space,* Ann. Math. Studies, 122, Princeton Univ. Press, Princeton, 1989.

[7] Hornberger, K., Smilansky, U., *Magnetic edge states,* Physics Reports, 367, 249–385 (2002).

[8] Iwatsuka, A., Tamura, H., *Asymptotic distribution of negative eigenvalues for two-dimensional Pauli operators with nonconstant magnetic fields.* Ann. Inst. Fourier (Grenoble) 48, no. 2, 479–515 (1998).

[9] Landau, L., *Diamagnetismus der Metalle.* Z. Phys. 64, 629–637 (1930).

[10] Landau, L., Lifshitz, E., *Quantum Mechanics (Non-relativistic theory).* 3rd edition, Pergamon Press, Oxford, 1977.

[11] Landkof, N. S., *Foundations of modern potential theory.* Springer, NY, 1972.

[12] Lieb, E. H., Loss, M., *Analysis.* American Mathematical Society, 1997.

[13] Melgaard, M., Rozenblum, G., *Eigenvalue asymptotics for weakly perturbed Dirac and Schrödinger operators with constant magnetic fields of full rank.* Comm. Partial Differential Equations 28, 697–736 (2003).

[14] Parfënov, O. G. *The widths of some classes of entire functions*, Mat. Sb. 190 (1999), no. 4, 87–94; translation in Sb. Math. 190, no. 3-4, 561–568 (1999).

[15] Raikov, G., Warzel, S., *Quasi-classical versus non-classical spectral asymptotics for magnetic Schroödinger operators with decreasing electric potentials.* Rev. Math. Phys. 14, 1051–1072 (2002).

[16] Rozenblum, G., Tashchiyan G., *On the spectrum of the perturbed Landau Hamiltonian.* To appear in Comm. Partial Differential Equations.

[17] Simon, B., *A canonical decomposition for quadratic forms with applications to monotone convergence theorems.* J. Funct. Anal. 28, no. 3, 377–385 (1978).

[18] Stahl, H., Totik, V., *General orthogonal polynomials*, Cambridge Univ. Press, Cambridge, 1992.

Alexander Pushnitski
Department of Mathematics
King's College London
Strand London WC2R 2LS
U.K.
alexander.pushnitski@kcl.ac.uk

Grigori Rozenblum
Department of Mathematical Sciences
Chalmers University of Technology
and University of Gothenburg
Chalmers Tvärgatan, 3, S-412 96
Gothenburg, Sweden
grigori@math.chalmers.se