Geometric Methods for Cohomological Invariants

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Abstract. We explain how to exploit Rost's theory of Chow groups with coefficients to carry some computations of cohomological invariants. In particular, we use the idea of the "stratification method" introduced by Vezzosi.

We recover a number of known results, with very different proofs. We obtain some new information on spin groups.

Keywords and Phrases: cohomological invariants, algebraic cycles.

§1. Introduction

In what follows, k is a base field, and all other fields considered in this paper will be assumed to contain k. We fix a prime number p which is different from $char(k)$, and for any field K/k, we write $H^i(K)$ for $H^i(Gal(K_s/K), \mathbf{Z}/p(i))$. Here K_s is a separable closure of K, and $\mathbf{Z}/p(i)$ is the i-th Tate twist of $\mathbf{Z}/p$, that is $\mathbf{Z}/p \otimes \boldsymbol{\mu}_p^{\otimes i}$. (See the end of this introduction for more information on our choice of coefficients.)

1.1. Cohomological invariants

Given a functor $A(-)$ from fields over k to the category of pointed sets, the (mod p) *cohomological invariants* of A are all transformations of functors

$$a : A(-) \to H^*(-).$$

Typical examples for A include

$A(K)$ = isomorphism classes of nondegenerate quadratic forms over K,
$A(K)$ = isomorphism classes of octonions algebras over K,
etc...

For more examples and a very good introduction to the subject, see [5].
We are mostly interested in the situation where $A(K)$ is the set of isomorphism classes over K of some type of "algebraic object", especially when all such objects become isomorphic over algebraically closed fields. In this case, if we write X_K for the base point in $A(K)$, then we obtain a group scheme G over k by setting $G(K) = Aut_K(X_K)$, assuming Aut_K is appropriately defined. Moreover, by associating to each object $Y_K \in A(K)$ the variety of isomorphisms (in a suitable sense) from Y_K to the base point X_K, we obtain a $1:1$ correspondence between $A(K)$ and $H^1(K, G)$. Recall that this is the (pointed) set of all G-torsors over K, i.e. varieties acted on by G which become isomorphic to G with its translation action on itself when the scalars are extended to the algebraic closure $\bar{K}$.
For example, in the case of quadratic forms, resp. of octonion algebras, we have $A(K) = H^1(K, \mathbf{O}_n)$, resp. $A(K) = H^1(K, G_2)$.
In more geometric terms, a G-torsor over K is a principal G-bundle over $\mathrm{Spec}(K)$. In this way we see that cohomological invariants

$$H^1(-, G) \to H^*(-)$$

are analogous to characteristic classes in topology. We shall call them the mod p cohomological invariants of G. They form a graded $H^*(k)$-algebra which we denote by $Inv(G)$ (the prime p being implicit).

1.2. Versal torsors and classifying spaces

A simple way of constructing a G-torsor is to start with a G-principal bundle $E \to X$ and take the fibre over a point $\mathrm{Spec}(K) \to X$. As it turns out, there always exist bundles $\pi : E \to X$ with the particularly nice feature that *any* G-torsor is obtained, up to isomorphism, as a fibre of π. In this case, the generic torsor T_κ over $\kappa = k(X)$ is called *versal*. We shall also say that π is a versal bundle.
A strong result of Rost (presented in [5]) asserts that an invariant $a \in Inv(G)$ is entirely determined by its value $a(T_\kappa)$ on a versal torsor. It follows that $Inv(G) \subset H^*(\kappa)$, and more precisely one can show that

$$Inv(G) \subset A^0(X, H^*).$$

The right hand side refers to the cohomology classes in $H^*(\kappa)$ which are *unramified* at all divisors of X (see *loc. cit.*). (The notation comes from Rost's theory of Chow groups with coefficients, see below and section 2.)
There are two well-known constructions of versal bundles. One can embed G in a "special" group S (eg $S = \mathbf{GL}_n$, $S = \mathbf{SL}_n$ or $\mathbf{Sp}_n$) and take $E = S$, $X = S/G$. Alternatively, one can pick a representation V of G such that the action is free on a nonempty open subset $U \subset V$, and take $E = U$, $X = U/G$. In either situation, there are favorable cases when we actually have

$$Inv(G) = A^0(X, H^*).$$

For versal bundles of the first type, this happens when S is simply connected (see Merkurjev [8]). For versal bundles of the second type, this happens when the complement of U has codimension ≥ 2 ([5], letter by Totaro).
In this paper we shall restrict our attention exclusively to the second type of versal torsors, for reasons which we shall explain in a second. We shall view U/G as an approximation to the *classifying space* of G (see [13]). It will not cause any confusion, hopefully, to call this variety BG (see §2.3).

1.3. Rost's Chow groups with coefficients

The group $A^0(X, H^*)$, for any variety X, is the first in a sequence of groups $A^n(X, H^*)$, Rost's *Chow groups with coefficients*. They even form a (bigraded) ring when X is smooth. The term "Chow groups" is used since the usual mod p Chow groups of X may be recovered as $CH^n X \otimes_{\mathbf{Z}} \mathbf{Z}/p = A^n(X, H^0)$.
Rost's groups have extremely good properties: homotopy invariance, long exact sequence associated to an open subset, and so on. There is even a spectral sequence for fibrations.
Our aim in this paper, very briefly, is to use these geometric properties in order to get at $A^0(BG, H^*) = Inv(G)$.
To achieve this, we shall be led to introduce the equivariant Chow groups $A^n_G(X, H^*)$, defined when X is acted on by the algebraic group G. The definition of these is in perfect analogy with the case of equivariant, ordinary Chow groups as in [3], [13], where "Borel constructions" are used. When $X = \mathrm{Spec}(k)$, we have $A^n_G(\mathrm{Spec}(k), H^*) = A^n(BG, H^*)$ where BG is a classifying space "of the second type" as above.
The key observation for us will be that $A^0_G(V, H^*) = Inv(G)$ when V is a representation of G (thus a variety which is equivariantly homotopic to $\mathrm{Spec}(k)$, so to speak). This allows us to cut out V into smaller pieces, do some geometry, and eventually implement the "stratification method", which was first introduced by Vezzosi in [14] in the context of ordinary Chow groups.

1.4. Results

A large number of the results that we shall obtain in this paper are already known, although we provide a completely different approach for these. Occasionally we refine the results and sometimes we even get something new.
In this introduction we may quote the following theorem (3.2.1 in the text):

Theorem – *Let $p = 2$ and $n \geq 2$. There are exact sequences:*

$$0 \longrightarrow Inv(\mathbf{O}_n) \longrightarrow Inv(\mathbf{Z}/2 \times \mathbf{O}_{n-1}) \xrightarrow{r} Inv(\mathbf{O}_{n-2})$$

$$0 \longrightarrow Inv(\mathbf{SO}_{2n}) \longrightarrow Inv(\mathbf{Z}/2 \times \mathbf{SO}_{2n-1}) \xrightarrow{r} Inv(\mathbf{SO}_{2n-2})$$

$$0 \longrightarrow Inv(\mathbf{Spin}_{2n}) \longrightarrow Inv(\mathbf{Z}/2 \times \mathbf{Spin}_{2n-1}) \xrightarrow{r} Inv(\mathbf{Spin}_{2n-2})$$

Moreover, the image of the map r contains the image of the restriction map $Inv(\mathbf{O}_n) \to Inv(\mathbf{O}_{n-2})$, resp. $Inv(\mathbf{SO}_{2n}) \to Inv(\mathbf{SO}_{2n-2})$, resp. $Inv(\mathbf{Spin}_{2n}) \to Inv(\mathbf{Spin}_{2n-2})$.

In each of the three cases the existence of the map r was not known, as far as I am aware, while the other half of the exact sequence is described by Garibaldi in [4]. Of course the computation of $Inv(\mathbf{O}_n)$ and $Inv(\mathbf{SO}_n)$ has been completed (see [5] again), and indeed we shall derive it from the theorem. This affords a new construction of an invariant first defined by Serre (and written originally b_1). On the other hand the invariants of $\mathbf{Spin}_n$ are not known in general: they have been computed for $n \leq 12$ only ([4]), so our theorem might be of some help.

In any case, our emphasis in this paper is with methods rather than specific results, and the point that we are trying to make is that the stratification method is a powerful one. It provides us with a place to start when trying to tackle the computation of $Inv(G)$, whatever G may be. It is much more mechanical than any other approach that I am aware of.

Still, the reader will find in what follows a number of examples of computations of $Inv(G)$ for various G's: products of copies of $\boldsymbol{\mu}_p$ in 2.2.3, the group $\mathbf{Spin}_7$ in 4.2.2, the wreath product $\mathbf{G}_m \wr \mathbf{Z}/2$ in 4.2.1, the dihedral group D_8 in 5.2.3, etc.

1.5. Organization of the paper

We start by presenting Rost's definition of Chow groups with coefficients in section 2. We also indicate how to construct the equivariant Chow groups, and we mention a number of basic tools such as the Künneth formula.

In section 3 we introduce the stratification method, and prove the above results on orthogonal groups. We compute $Inv(\mathbf{O}_n)$ and $Inv(\mathbf{SO}_n)$ completely.

In section 4 we explain how one could use the stratification method with projective representations rather than ordinary ones. We recover a result (corollary 4.1.3) which was proved and exploited fruitfully in [4].

We proceed to introduce the Bloch & Ogus spectral sequence in section 5. As we have mentioned already, the stratification method has been used by many authors (including, Vezzosi, Vistoli and myself) to compute CH^*BG and H^*BG, and to study the cycle map $CH^*BG \to H^*BG$. It turns out that the spectral sequence shows, roughly speaking, that cohomological invariants measure the failure of this cycle map to be an isomorphism (in small degrees this is strictly true). This was the motivation to try and extend the stratification method to cohomological invariants.

We conclude in section 6 with some easy remarks on invariants with values in other cycle modules, particularly algebraic K-theory.

Notations & Conventions. We insist on the assumptions that we have made at the beginning of this introduction: p is a fixed prime, k is a fixed base field, $p \neq char(k)$, and $H^*(K)$ means mod p Galois cohomology as defined more precisely above. Our particular choice of twisting for the coefficients has been dictated by the desire to obtain a "cycle module" in the sense described in section 2, and at the same time keep things as simple as possible. There are more general cycle modules, including Galois cohomology simply with some

more general coefficients (see [11]). In most applications, we have $p = 2$ anyway, in which case the twisting does not interfere.
We will say that G is an algebraic group to indicate that G is a smooth, affine group scheme over k [in alternative terminology: G is a linear algebraic group over $\bar{k}$ defined over k]. Subgroups will always be assumed to be closed and smooth. We shall encounter many times the stabilizer of an element under an action of G: in each case, it will be trivially true that the scheme-theoretic stablizer is closed and smooth. [Alternatively: subgroups are always defined over k. So will all stabilizers we encounter.]
We write $Inv(G)$ for the mod p cohomological invariants of G, even though the letter p does not appear in the notation. We caution that in [4], these would be called the invariants of G with values in $\boldsymbol{\mu}_p$ (for $p = 2$ there is no difference).

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§2. Rost's Chow groups with coefficients

2.1. Definitions

In [11], Rost defines a *cycle module* to be a functor $M : F \mapsto M(F)$ from fields containing k to graded abelian groups, equipped with structural data. Quoting Rost, these divide into "the even ones", namely restriction and corestriction maps, and "the odd ones": it is required that $M(F)$ should have the structure of a module over $K_*(F)$ (Milnor's K-theory of F), and there should exist residue maps for discrete valuations. The fundamental example for us is $M(F) = H^*(F)$ (mod p Galois cohomology as defined in the introduction). Another example is Milnor's K-theory itself, $M(F) = K_*(F)$.
Given a cycle module M, Rost defines for any variety X over k the Chow groups with coefficients in M, written $A_i(X, M)$ ($i \geq 0$). These are bigraded; for the cycle modules above, we have the summands $A_i(X, H^j)$ and $A_i(X, K_j)$. Moreover, the classical and mod p Chow groups can be recovered as $CH_iX = A_i(X, K_0)$ and $CH_iX \otimes_{\mathbf{Z}} \mathbf{Z}/p = A_i(X, H^0)$.
When X is of pure dimension n, we shall reindex the Chow groups by putting $A^i(X, H^j) = A_{n-i}(X, H^j)$. Our interest in the theory of Chow groups with coefficients stems from the concrete description of $A^0(X, H^j)$ when X is irreducible: it is the subgroup of $H^j(k(X))$ comprised of those cohomology classes in degree j which are unramified at all divisors of X. Hence, when X is a classifying space for an algebraic group G (as in 1.2), we have $A^0(X, H^j) = Inv^j(G)$.
Rost's Chow groups have all the properties of ordinary Chow groups, and more: in fact they were designed to be more flexible than ordinary Chow groups, particularly in fibred situations. For the time being, we shall be content to list the following list of properties; we will introduce more as we go along. Here X will denote an equidimensional variety.

1. When X is smooth, $A^*(X, H^*)$ is a graded-commutative ring.

2. A map $f : X \to Y$ induces $f^* : A^*(Y, H^*) \to A^*(X, H^*)$ whenever Y is smooth or f is flat.

3. A proper map $f : X \to Y$ induces $f_* : A_*(X, H^*) \to A_*(Y, H^*)$.

4. There is a projection formula: $f_*(xf^*(y)) = f_*(x)y$ (here $f : X \to Y$ is proper, and X and Y are smooth).

5. Let $i : C \to X$ be the inclusion of a closed subvariety, and let $j : U \to X$ denote the inclusion of the open complement of C. Then there is a long exact sequence:

$$\cdots \longrightarrow A_m(C, H^n) \xrightarrow{i_*} A_m(X, H^n) \xrightarrow{j^*}$$
$$A_m(U, H^n) \xrightarrow{r} A_{m-1}(C, H^{n-1}) \longrightarrow \cdots$$

   Moreover the residue map r, or connecting homomorphism, satisfies

$$r(xj^*(y)) = r(x)i^*(y).$$

6. If $\pi : E \to X$ is the projection map of a vector bundle or an affine bundle, then π^* is an isomorphism.

There are particular cases when we can say more about the exact sequence in property (5). When $n = 0$ for instance, we recover the usual localisation sequence:

$$CH_mC \otimes_{\mathbf{Z}} \mathbf{Z}/p \longrightarrow CH_mX \otimes_{\mathbf{Z}} \mathbf{Z}/p \longrightarrow CH_mU \otimes_{\mathbf{Z}} \mathbf{Z}/p \longrightarrow 0.$$

Also, when X is equidimensional and C has codimension ≥ 1 (for example when X is irreducible and U is nonempty), then the following portion of the sequence is exact:

$$0 \longrightarrow A^0(X, H^n) \xrightarrow{j^*} A^0(U, H^n).$$

Example 2.1.1. Let us compute $A^0(\mathbf{G}_m, H^*)$, where $\mathbf{G}_m = \mathbf{GL}_1$ is the punctured affine line. The inclusion of the origin in $\mathbf{A}^1$ yields the exact sequence:

$$0 \longrightarrow A^0(\mathbf{A}^1, H^*) \longrightarrow A^0(\mathbf{G}_m, H^*) \longrightarrow$$
$$A^0(\mathrm{Spec}(k), H^{*-1}) \longrightarrow A^1(\mathbf{A}^1, H^{*-1}) .$$

Now, $\mathbf{A}^1$ can be seen as a (trivial) vector bundle over Spec(k), so $A^i(\mathbf{A}^1, H^*) = A^i(\mathrm{Spec}(k), H^*)$. From the definitions, we see directly that $A^i(\mathrm{Spec}(k), H^*) = 0$ when $i > 0$, and $A^0(\mathrm{Spec}(k), H^*) = H^*(k)$.
So we have the exact sequence of $H^*(k)$-modules:

$$0 \longrightarrow H^*(k) \longrightarrow A^0(\mathbf{G}_m, H^*) \longrightarrow H^{*-1}(k) \longrightarrow 0.$$

It follows that $A^0(\mathbf{G}_m, H^*)$ is a free $H^*(k)$-module on two generators, one in degree 0, the other in degree 1. In fact, if we write $k(\mathbf{G}_m) = k(t)$, it is not difficult to see that the element $(t) \in H^1(k(t)) = k(t)^\times/(k(t)^\times)^p$ can be taken as the generator in degree 1. For this one only has to unwind the definition of r as a residue map: see [5], chap. II and III. Essentially the reason is that the divisor of the function t on $\mathbf{A}^1$ is the origin.
In any case, we shall use the letter t for this generator. When k is algebraically closed, so that the ring $H^*(k)$ can be seen as the field $\mathbf{F}_p$ concentrated in degree 0, then we can write $A^0(\mathbf{G}_m, H^*) = \mathbf{F}_p[t]/(t^2)$, an exterior algebra.

Remark 2.1.2 – Let X be smooth, and suppose that X has a k-rational point $\mathrm{Spec}(k) \to X$. The induced map on Chow groups gives a splitting for the map $H^*(k) \to A^0(X, H^*)$ coming from the projection $X \to \mathrm{Spec}(k)$. It follows that we can write

$$A^0(X, H^*) = H^*(k) \oplus A^0(X, H^*)_{norm}.$$

The elements in the second summand are said to be *normalised*. Note that the splitting may *a priori* depend on the choice of a k-rational "base point" for X. There is an analogous notion for cohomological invariants. Namely, an invariant $n \in Inv(G)$ is called normalised when it is 0 on the trivial torsor; an invariant $c \in Inv(G)$ is called constant when there is an $\alpha \in H^*(k)$ such that for any torsor T over a field K, the value $c(T)$ is the image of α under the natural map $H^*(k) \to H^*(K)$. Any invariant $a \in Inv(G)$ may be written $a = n + c$ with n normalised and c constant, and we may write

$$Inv(G) = H^*(k) \oplus Inv(G)_{norm}$$

where the second summand consists of normalised elements.
Whenever X is a classifying space of the types considered in the introduction, there is a canonical choice of a point $\mathrm{Spec}(k) \to X$ such that the pullback of the versal torsor over X is the trivial torsor. It follows that $A^0(X, H^*)_{norm} = Inv(G)_{norm}$.
Let us give an application. The group $\boldsymbol{\mu}_p$ acts freely on $\mathbf{G}_m$, and the quotient is again $\mathbf{G}_m$. Since $\mathbf{G}_m$ is special, it follows from the introduction that the torsor $\mathbf{G}_m \to \mathbf{G}_m$ is versal, so $Inv(\boldsymbol{\mu}_p) \subset A^0(\mathbf{G}_m, H^*)$. It is not immediate that this inclusion is an equality, as $\mathbf{G}_m$ is not a classifying space: $\mathbf{G}_m$ is not simply-connected, and the origin has only codimension 1 in $\mathbf{A}^1$, so we are in neither of the two favorable cases.
To show that it is indeed an equality, we remark that $A^0(\mathbf{G}_m, H^*)_{norm}$ is a free $H^*(k)$-module on one generator of degree 1, from the previous example. As a result, if we can find a non-zero, normalised invariant of $\boldsymbol{\mu}_p$ in degree 1, then the map of $H^*(k)$-modules $Inv(\boldsymbol{\mu}_p)_{norm} \to A^0(\mathbf{G}_m, H^*)_{norm}$ will certainly be surjective. Such an invariant is easy to find. Indeed, take the identity $H^1(K, \boldsymbol{\mu}_p) \to H^1(K)$. (Should you try to compute the mod p invariants of $\boldsymbol{\mu}_\ell$, where ℓ is a prime different from p, you would find no normalized invariants at all. It is an easy exercise to prove this using the techniques to be introduced in this paper).

2.2. Spectral sequences and Künneth formula

Apart from Milnor K-theory and Galois cohomology, there is one extra type of cycle module which we shall consider in this paper. In the presence of a map $f : X \to Y$, and having picked a primary cycle module M, there is for each $n \geq 0$ a new cycle module written $A_n[f; M]$, see [11],§7. These are functors from fields κ "over Y", that is fields with a map $\mathrm{Spec}(\kappa) \to Y$, to graded abelian groups – which turns out to be enough to define $A_*(Y, A_n[f; M])$. Quite simply, if we write $X_\kappa = \mathrm{Spec}(\kappa) \times_Y X$, the definition is:

$$A_n[f; M](\kappa) = A_n(X_\kappa, M).$$

When f and M are understood, we shall write $\mathcal{A}_n$ for $A_n[f; M]$. Also, when f is flat, so that the fibres all have the same dimension d, we may change the grading to follow the codimension, thus defining $\mathcal{A}^n = \mathcal{A}_{d-n}$.

Theorem 2.2.1 – *Let $f : X \to Y$ be a flat map, and suppose that Y is equidimensional. Let $\mathcal{A}_n = A_n[f; M]$ for a cycle module M. Then there exists a convergent spectral sequence:*

$$A^r(Y, \mathcal{A}^s) \Rightarrow A^{r+s}(X, M).$$

See [11],§8, for a proof.

Corollary 2.2.2 – *Let X be a scheme (over k), and write X_κ for the scheme obtained by extending the scalars to the field κ. Assume that $A^0(X, H^*)$ is a free $H^*(k)$-module of finite rank, and that*

$$A^0(X_\kappa, H^*) = A^0(X, H^*) \otimes_{H^*(k)} H^*(\kappa).$$

Then there is a Künneth isomorphism:

$$A^0(X \times Y, H^*) = A^0(X, H^*) \otimes_{H^*(k)} A^0(Y, H^*)$$

for any equidimensional Y.

The hypotheses of this corollary should be compared with [5], 16.5.

Proof. Let $f : X \times Y \to Y$ be the projection. By the theorem, we know that $A^0(X \times Y, H^*) = A^0(Y, \mathcal{A}^0)$.
However, for any field $\mathrm{Spec}(\kappa) \to Y$, the fibre of f above $\mathrm{Spec}(k)$ is X_κ as in the statement of the corollary. Our hypothesis is then that $\mathcal{A}^0(\kappa) = A^0(X_\kappa, H^*)$ is a direct sum of copies of $H^*(\kappa)$, and more precisely that $\mathcal{A}^0$ splits up as the direct sum of several copies of the cycle module H^*. The result follows. $\square$

As an illustration, we have the following proposition.

Proposition 2.2.3 – *There are invariants $t^{(i)} \in Inv((\boldsymbol{\mu}_p)^n)$ for $1 \leq i \leq n$, each of degree 1, such that $Inv((\boldsymbol{\mu}_p)^n)$ is a free $H^*(k)$-module on the products $t^{(i_1)}t^{(i_2)} \cdots t^{(i_k)}$, for all sequences $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.*
When k is algebraically closed, $Inv((\mathbf{Z}/p)^n)$ is an exterior algebra $\Lambda(t^{(1)}, \ldots, t^{(n)})$ over $\mathbf{F}_p$.

Proof. The case $n = 1$ has been dealt with in remark 2.1.2. The general case follows from corollary 2.2.2. □

2.3. Equivariant Chow groups

The good properties of Chow groups with coefficients will allow us to define the equivariant Chow groups of a variety acted on by an algebraic group. This will be in strict analogy with [3] and [13], to which we will refer for details.
So let G be a linear algebraic group over k. There exists a representation V of G such that the action is free outside of a closed subset S; moreover, we can make our choices so that the codimension of S is arbitrarily large (*loc. cit.*). Put $U = V - S$.
Let X be any equidimensional scheme over k with an action of G. Define

$$X_G = \frac{U \times X}{G}.$$

We shall restrict our attention to those pairs (X, U) for which X_G is a scheme rather than just an algebraic space. Lemma 9 and Proposition 23 in [3] provide simple conditions on X under which an appropriate U may be found. This will be amply sufficient for the examples that we need to study in this paper.[1]
The notation X_G hides the dependence on U because, as it turns out, the Chow groups $A^i(X_G, H^*)$ do not depend on the choice of V or S, as long as the codimension of S is large enough (depending on i). One may prove this using the "double fibration argument" as in [13]. We write $A^i_G(X, H^*)$ for this group (or the limit taken over all good pairs (V, S), if you want).
A map $f : X \to Y$ induces a map $f_G : X_G \to Y_G$ (the notation X_G will mean that a choice of V and S has been made). It follows that the equivariant Chow groups with coefficients $A^*_G(-, H^*)$ are functorial, and indeed they have all the properties listed in §2.1. The proof of this uses that if f is flat, proper, an open immersion, a vector bundle projection, etc, then f_G is respectively flat, proper, an open immsersion, or a vector bundle projection (see [3]). It may be useful to spell out that a G-invariant open subset U in an equidimensional G-variety X, whose complement C has codimension c, gives rise to a long exact sequence:

$$\cdots \to A^m_G(C, H^n) \to A^{m+c}_G(X, H^n) \to A^{m+c}_G(U, H^n) \to A^{m+1}_G(C, H^{n-1}) \to \cdots .$$

We shall refer to it as the equivariant long exact sequence associated to U.

Example 2.3.1. We shall write BG for $\mathrm{Spec}(k)_G$. It follows from the results above and from the introduction that

$$A^0_G(\mathrm{Spec}(k), H^*) = A^0(BG, H^*) = Inv(G).$$

[1] I am grateful to the referee for pointing out this technical difficulty. I also agree with him or her that it would be desirable to extend Rost's theory of Chow groups with coefficients to algebraic spaces. This is certainly not the place to do so.

Example 2.3.2. Suppose that the action of G on X is free, and that the quotient X/G exists as a scheme. Then there is a natural map $X_G \to X/G$. Moreover X_G is an open subset in $(V \times X)/G$, which in turn is (the total space of) a vector bundle over X/G. The complement of X_G in $(V \times X)/G$ can be taken to have an arbitrarily large codimension (namely, it is that of S). In a given degree i, we may thus choose V and S appropriately and obtain:

$$A^i_G(X, H^*) = A^i(X_G, H^*) = A^i((V \times X)/G, H^*) = A^i(X/G, H^*).$$

(The second equality coming from the long exact sequence as in §2.1, property (5).)
Thus when the action is free, the equivariant Chow groups are just the Chow groups of the quotient.

Notations 2.3.3 – As we have already done in this section, we shall write X_G to signify that we have chosen V and S with the codimension of S large enough for the computation at hand. Thus we can and will write indifferently $A^*_G(X, H^*)$ or $A^*(X_G, H^*)$.
Moreover, we shall write EG for $U = V - S$, with the same convention. The quotient EG/G is written BG, and we write $A^*(BG, H^*)$ much more often than $A^*_G(\mathrm{Spec}(k), H^*)$.

Remark 2.3.4 – Whenever we have a theory at hand which has the properties listed in §2.1, we can define the equivariant analog exactly as above. Apart from Chow groups with coefficients, we shall have to consider at one point the étale cohomology of schemes, namely the groups $H^i_{\mathrm{et}}(X, \mathbf{Z}/p(i))$, which we shall write simply $H^i_{\mathrm{et}}(X)$, the coefficients being understood. In fact we shall only encounter the expression $H^i_{\mathrm{et}}(BG)$.
When $k = \mathbf{C}$, we have $H^*_{\mathrm{et}}(BG) = H^*(BG, \mathbf{F}_p)$, where on the right hand side we use topological cohomology and a model for the classifying space BG in the classical, topological sense of the word. To see this, note first that $H^*_{\mathrm{et}}((V - S)/G) = H^*((V - S)/G, \mathbf{F}_p)$ since we are using finite coefficients. Moreover, we can arrange V and S so that $V - S$ is a Stiefel variety (see [13]), and this provides sufficiently many V's and S's (that is, the codimension of S can be arbitrarily large, with $V - S$ a Stiefel variety). As a result, we can restrict attention to these particular pairs (V, S) when forming the limit $H^*_{\mathrm{et}}(BG) = \lim H^*((V - S)/G, \mathbf{F}_p)$ without changing the result, which is then clearly $H^*(E/G, \mathbf{F}_p)$ where E is the infinite Stiefel variety. This space E is contractible, so E/G is a topological model for BG.

§3. The stratification method

3.1. The idea of the method

Let G be a linear algebraic group. The stratification method is a procedure to compute $A^0(BG, H^*) = Inv(G)$. We shall not try to present it as a mechanical algorithm, but rather as a heuristic recipe. This being said, one of the virtues

of the method is that it is closer to a systematic treatment of the question than any other approach that the author is aware of.
The stratification method rests on two very elementary facts.

1. If K is a closed subgroup of G, then the equivariant Chow groups of the variety G/K have an easy description: indeed there is an isomorphism of varieties:
$$(G/K)_G = \frac{EG \times G/K}{G} = \frac{EG}{K} = BK.$$
Thus $A^*_G(G/K, H^*) = A^*(BK, H^*)$.

2. Suppose that V is a representation of G. Then it is a G-equivariant vector bundle over a point, and therefore $V_G \to \mathrm{Spec}(k)_G = BG$ is a vector bundle, too. As a result, $A^*_G(V, H^*) = A^*(BG, H^*)$.

Things are put together in the following way. One starts with a well-chosen representation V, and then cuts V into smaller pieces. These smaller pieces would ideally be orbits or families of orbits parametrized in a simple manner. Using (1), and hoping that the stabilizer groups K showing up are easy enough to understand, one computes the Chow groups of the small pieces. Applying then repeatedly the equivariant long exact sequence from §2.3, one gets at $A^0_G(V, H^*)$. By (2), this is $A^0(BG, H^*)$.
Any particular application of the method will require the use of *ad hoc* geometric arguments. Before giving an example however, we propose to add a couple of lemmas to our toolkit.

Lemma 3.1.1 – *Suppose that G and K are algebraic groups acting on the left on X, and suppose that the actions commute. Assume further that the action of K alone gives a K-principal bundle $X \to X/K = Y$. Then*
$$A^*_G(Y, H^*) = A^*_{G\times K}(X, H^*).$$

Proof. This is simply an elaboration on example 2.3.2. □

We shall have very often the occasion of using this lemma in the following guise:

Lemma 3.1.2 – *Suppose that G contains a subgroup G' which is an extension*
$$1 \to N \to G' \to K \to 1.$$
Let $G \times K$ act on G/N by $(\sigma, \tau)\cdot[g] = [\sigma g \tau^{-1}]$. Assume moreover that K acts on a variety X, and that there is a K-principal bundle $G/N \times X \to Y$.
Then
$$A^*_G(Y, H^*) = A^*_{G'}(X, H^*).$$
If the product $N \times K$ is direct, we have $X_N \times K = BN \times X_K$.

Here the variety Y is an example of what we call "a family of orbits (of G) parametrized in a simple manner". Note that when K is the trivial group and $X = \mathrm{Spec}(k)$, we recover property (1) above.

Proof. By lemma 3.1.1, we have

$$A^*_G(Y, H^*) = A^*_{G\times K}(G/N \times X, H^*).$$

Now let E_1 be an open subset in a representation of G, on which the action is free, and suppose that the codimension of the complement of E_1 is large enough. Similarly, pick E_2 for K; then $E_1 \times E_2$ can be chosen for the group $G\times K$. We regard E_1 as a trivial K-space and E_2 as a trivial G-space. Finally, G' has natural maps to both G and K, and we combine these to see $E_1 \times E_2$ as a G'-space, with non-trivial action on each factor.
We write:

$$\begin{array}{rcl} G/N \times X_G \times K & = & (E_1 \times E_2 \times \frac{G}{N} \times X)/G \times K \\ & = & (\frac{E_1\times G/N}{G} \times E_2 \times X)/K \end{array}.$$

By property (1) above, we may identify $\frac{E_1\times G/N}{G}$ with $E_1/N = BN$. Moreover we have arranged things so that, under this identification, the action of K on $\frac{E_1\times G/N}{G}$ translates into its natural action on E_1/N (that is, the action induced from that which G' possesses as a subgroup of G normalising N). We proceed:

$$\begin{array}{rcl} G/N \times X_G \times K & = & (E_1/N \times E_2 \times X)/K \\ & = & (E_1 \times E_2 \times X)/G' \end{array}.$$

When the product is direct, the action of K on E_1/N is trivial. This concludes the proof. □

As an example, we may apply this to actions of G on disjoint unions:

LEMMA 3.1.3 – *Suppose that G acts on Y and that Y is the disjoint union of varieties $Y_1 \coprod \dots \coprod Y_n$. Let $X = Y_1$, and suppose that X contains a point in each orbit, that is, suppose that the natural map*

$$f : G \times X \to Y$$

is surjective. Let K be the subgroup of G leaving X invariant. Then there is an identification:

$$A^0_G(Y, H^*) = A^0_K(X, H^*).$$

Proof. This is immediate from the previous lemma, with N the trivial group. □

3.2. THE EXAMPLE OF $\mathbf{O}_n$, $\mathbf{SO}_n$, AND $\mathbf{Spin}_n$

For this example we take $p = 2$ (so that $char(k) \neq 2$).
Let q be a non-degenerate quadratic form on a k-vector space V of dimension n. Assume moreover that q has maximal Witt index. Then its automorphism group, which we will denote by $\mathbf{O}_n$, is split. Similarly one has the groups $\mathbf{SO}_n$ and $\mathbf{Spin}_n$, which are also split.

Theorem 3.2.1 – *Let $p = 2$ and $n \geq 2$. There are exact sequences:*

$$0 \longrightarrow Inv(\mathbf{O}_n) \longrightarrow Inv(\mathbf{Z}/2 \times \mathbf{O}_{n-1}) \xrightarrow{r} Inv(\mathbf{O}_{n-2})$$

$$0 \longrightarrow Inv(\mathbf{SO}_{2n}) \longrightarrow Inv(\mathbf{Z}/2 \times \mathbf{SO}_{2n-1}) \xrightarrow{r} Inv(\mathbf{SO}_{2n-2})$$

$$0 \longrightarrow Inv(\mathbf{Spin}_{2n}) \longrightarrow Inv(\mathbf{Z}/2 \times \mathbf{Spin}_{2n-1}) \xrightarrow{r} Inv(\mathbf{Spin}_{2n-2})$$

Moreover, the image of the map r contains the image of the restriction map $Inv(\mathbf{O}_n) \to Inv(\mathbf{O}_{n-2})$, resp. $Inv(\mathbf{SO}_{2n}) \to Inv(\mathbf{SO}_{2n-2})$, resp. $Inv(\mathbf{Spin}_{2n}) \to Inv(\mathbf{Spin}_{2n-2})$.

Proof. Step 1. Let G_n denote either of $\mathbf{O}_n$, $\mathbf{SO}_n$, or $\mathbf{Spin}_n$. Then G_n has the canonical representation V. Let V' denote $V - \{0\}$. We write U for the open subset in V' on which q is non-zero, and we write C for its complement.
The codimension of $\{0\}$ in V is n. Consider the long exact sequence:

$$0 \to A^0_{G_n}(V, H^*) \to A^0_{G_n}(V', H^*) \to$$
$$0 = A^{1-n}_{G_n}(\{0\}, H^{*-1}) \to A^1_{G_n}(V, H^{*-1}) \to A^1_{G_n}(V', H^{*-1}).$$

It follows that $A^0_{G_n}(V, H^*) = A^0_{G_n}(V', H^*) = Inv(G_n)$. Moreover the last map above is surjective when $* = 1$ (cf §2.1), so that $A^1_{G_n}(V, H^0) = A^1_{G_n}(V', H^0) = CH^1 BG_n \otimes_{\mathbf{Z}} \mathbf{Z}/2$.
Turning to the equivariant long exact sequence associated to the open set U in V', we finally get:

$$0 \longrightarrow Inv(G_n) \longrightarrow A^0_{G_n}(U, H^*) \longrightarrow A^0_{G_n}(C, H^{*-1}). \qquad (\dagger)$$

When $* = 1$ we have:

$$0 \to Inv^1(G_n) \to A^0_{G_n}(U, H^1) \to$$
$$A^0(C, H^0) \to CH^1 BG_n \otimes_{\mathbf{Z}} \mathbf{Z}/2 \to CH^1_{G_n} U \otimes_{\mathbf{Z}} \mathbf{Z}/2 \to 0. \quad (\dagger\dagger)$$

Step 2. Let $Q = q^{-1}(1)$. We shall use the following result.

Lemma 3.2.2 – *The action of G_n on Q is transitive. Moreover the stabilizer of a k-rational point is isomorphic to G_{n-1}, and we get an isomorphism of k-varieties $Q = G_n/G_{n-1}$.*
The action of G_n on C is transitive. Moreover the stabilizer of a k-rational point is isomorphic to a semi-direct product $H \rtimes G_{n-2}$, where H is an algebraic group isomorphic to affine space as a variety, and we get an isomorphism of k-varieties $C = G_n/H \rtimes G_{n-2}$. Finally, the map $B(H \rtimes G_{n-2}) \to BG_{n-2}$ is an affine bundle.

For a proof, see [10].
From this we can at once identify the last term in the exact sequence ($\dagger$). Indeed, if $S = H \rtimes G_{n-2}$ as in the lemma, then $A^0_{G_n}(C, H^*) = A^0(BS, H^*)$ as explained at the beginning of this §. However, since $BS \to BG_{n-2}$ is an affine

bundle, we draw from §2.1, property (6), that $A^0(BS, H^*) = A^0(BG_{n-2}, H^*) = Inv(G_{n-2})$.

Step 3. We turn to the term $A^0_{G_n}(U, H^*)$. Extend the action of G_n on Q to an action on $Q \times \mathbf{G}_m$ which is trivial on the second factor. The group $\mathbf{Z}/2 = \{1, \tau\}$ also acts on $Q \times \mathbf{G}_m$ by $\tau(x, t) = (-x, -t)$. The two actions commute. Finally, there is a map $Q \times \mathbf{G}_m \to U$ defined by $(x, t) \mapsto tx$. It is G_n-equivariant, and a $\mathbf{Z}/2$-principal bundle.

We wish to apply lemma 3.1.2 with $N = G_{n-1}$, $K = \mathbf{Z}/2$, and $G' = N \times K$. This is of course the time when we need to assume that n is even if G_n is $\mathbf{SO}_n$ or $\mathbf{Spin}_n$. Then the element $-Id$ generates the copy of $\mathbf{Z}/2$ that we need. We conclude:

$$A^0_{G_n}(U, H^*) = A^0(BG_{n-1} \times (\mathbf{G}_m)_{\mathbf{Z}/2}, H^*).$$

We may easily compute $A^0((\mathbf{G}_m)_{\mathbf{Z}/2}, H^*)$. Indeed, using that the action is free, with quotient $\mathbf{G}_m$, this is $A^0(\mathbf{G}_m, H^*)$ from example 2.3.2. As observed in remark 2.1.2, this happens to equal $A^0(B\mathbf{Z}/2, H^*) = Inv(\mathbf{Z}/2)$.

Now, by the Künneth formula (corollary 2.2.2), which we may use as $Inv(\mathbf{Z}/2)$ is a free $H^*(k)$-module, we draw

$$A^0(BG_{n-1} \times (\mathbf{G}_m)_{\mathbf{Z}/2}, H^*) = Inv(G_{n-1}) \otimes_{H^*(k)} Inv(\mathbf{Z}/2) = Inv(\mathbf{Z}/2 \times G_{n-1}).$$

Step 4. We need to prove the last sentence in the theorem. We claim that there is an $x \in Inv(\mathbf{Z}/2 \times G_{n-1})$ such that $r(x) = 1 \in Inv(G_{n-2})$. Granted this, the theorem follows from the formula $r(xj^*(y)) = r(x)i^*(y) = i^*(y)$ (see §2.1), applied here with i^* denoting the restriction $Inv(G_n) \to Inv(G_{n-2})$ and j^* denoting the restriction $Inv(G_n) \to Inv(\mathbf{Z}/2 \times G_{n-1})$.

To prove the claim we start by computing $CH^1_{G_n} U$ (in the rest of this proof we shall write simply CH for the mod 2 Chow groups). As above this is $CH^1 BG_{n-1} \times (\mathbf{G}_m)_{\mathbf{Z}/2}$. Arguing as in example 2.3.2, we see that we may replace $(\mathbf{G}_m)_{\mathbf{Z}/2}$ by $\mathbf{G}_m/\mathbf{Z}/2 = \mathbf{G}_m$, as far as computing the Chow groups goes. Now, $\mathbf{G}_m$ being an open set in affine space, it has a trivial Chow ring and there is a Künneth formula, so:

$$CH^1 BG_{n-1} \times (\mathbf{G}_m)_{\mathbf{Z}/2} = CH^1 BG_{n-1}.$$

Therefore the last map in (††) is the restriction map $CH^1 BG_n \to CH^1 BG_{n-1}$. When $G_n = \mathbf{Spin}_n$, which is simply-connected, we have $CH^1 BG_n = 0$. In [10], the reader will find a proof that $CH^1 B\mathbf{SO}_{2n} = 0$ also and that $CH^1 B\mathbf{O}_n \to CH^1 B\mathbf{O}_{n-1}$ is an isomorphism. The claim follows from the exact sequence (††). □

3.3. End of the computation for $\mathbf{O}_n$

The main novelty in theorem 3.2.1 is with the spin groups (specifically, with the second half of the exact sequence presented, as the first half was obtained by Garibaldi in [4]). The invariants of $\mathbf{O}_n$ and $\mathbf{SO}_n$ are completely known, see

[5]. In this section and the next, however, we show how to recover these results from theorem 3.2.1. In the case of $\mathbf{O}_n$ this is so close to the proof given in *loc. cit.*that we shall omit most of the details. In the case of $\mathbf{SO}_n$ on the other hand, our method is considerably different, so we have thought it worthwhile to present it. We continue with $p = 2$.
A major role will be played by *Stiefel-Whitney classes*. Recall that $H^1(K, \mathbf{O}_n)$ is the set of isomorphism classes of non-degenerate quadratic forms on K^n. Given such a form q, the classes $w_i(q) \in H^*(K)$, for $i \geq 0$, have been defined by Milnor [9] (originally they were defined in mod 2 Milnor K-theory). They are 0 for $i > n$ and if $w_t(q) = \sum w_i(q)t^i$, then one has

$$w_t(q \oplus q') = w_t(q)w_t(q').$$

Each w_i can be seen as a cohomological invariant in $Inv(\mathbf{O}_n)$. In fact one has

Proposition 3.3.1 – *The $H^*(k)$-module $Inv(\mathbf{O}_n)$ is free with a basis given by the classes w_i, for $0 \leq i \leq n$. Moreover, the following multiplicative formula holds:*

$$\begin{array}{rcll} w_1 w_{i-1} & = & w_i & (i \text{ odd}), \\ & = & (-1)w_{i-1} & (i \text{ even}). \end{array}$$

In the formula, (-1) stands for the image of $-1 \in k$ in $H^1(k) = k^\times/(k^\times)^2$.

Sketch proof. From theorem 3.2.1, we have an injection

$$0 \to Inv(\mathbf{O}_n) \to Inv(\mathbf{Z}/2 \times \mathbf{O}_{n-1})$$

where $\mathbf{Z}/2 \times \mathbf{O}_{n-1}$ is seen as a subgroup of $\mathbf{O}_n$, the nonzero element in the copy of $\mathbf{Z}/2$ corresponding to the $-Id$ matrix. An immediate induction then shows that $Inv(\mathbf{O}_n)$ injects into $Inv((\mathbf{Z}/2)^n)$, the invariants of the elementary abelian 2-group comprised of the diagonal matrices in $\mathbf{O}_n$ with ± 1 as entries. (This could be seen as a consequence of the surjection $H^1(K, (\mathbf{Z}/2)^n) \to H^1(K, \mathbf{O}_n)$, which in turns expresses the fact that quadratic forms may be diagonalized).
We have computed $Inv(\mathbf{Z}/2)$ in remark 2.1.2, and we may apply corollary 2.2.2 to obtain $Inv((\mathbf{Z}/2)^n)$. The symmetric group S_n acts on this ring, and the image of $Inv(\mathbf{O}_n)$ must certainly lie in the subring fixed by this action. The latter is easily seen to be a free $H^*(k)$-module on the images of the Stiefel-Whitney classes.
The multiplicative formula is obtained by direct computation. $\square$

3.4. End of the computation for $\mathbf{SO}_n$

Since $\mathbf{SO}_n$ is a subgroup of $\mathbf{O}_n$, the restriction map $Inv(\mathbf{O}_n) \to Inv(\mathbf{SO}_n)$ defines Stiefel-Whitney classes for $\mathbf{SO}_n$.
Now, as we have defined it, the group $\mathbf{SO}_n$ has the following interpretation for its torsors: $H^1(K, \mathbf{SO}_n)$ is the set of isomorphism classes of non-degenerate quadratic forms on K^n whose discriminant is 1. Recall that the discriminant

is the determinant of any matrix representing the bilinear form associated to q, viewed as an element of $K^\times/(K^\times)^2$. As it turns out, the discriminant of q is precisely $w_1(q)$ when seen as a class in the additive group $H^1(K)$. So we have $w_1 = 0$ in $Inv(\mathbf{SO}_n)$. From the formula $w_1 w_{2i} = w_{2i+1}$, we have in fact $w_{2i+1} = 0$ in $Inv(\mathbf{SO}_n)$.

Proposition 3.4.1 – *When $n = 2m+1$ is odd, $Inv(\mathbf{SO}_n)$ is a free $H^*(k)$-module with basis $\{w_0, w_2, w_4, \ldots, w_{2m}\}$.*
When $n = 2m$ is even, $Inv(\mathbf{SO}_n)$ is a free $H^*(k)$-module with basis

$$\{w_0, w_2, w_4, \ldots, w_{2m-2}, b^{(1)}_{n-1}\},$$

where $b^{(1)}_{n-1}$ is an invariant of degree $n-1$.

The invariant $b^{(1)}_{n-1}$ was introduced by Serre [5]. It was originally denoted by b_1, but we try to keep lowerscripts for the degree whenever possible.

Proof. The odd case is easy, for we have an isomorphism

$$\mathbf{Z}/2 \times \mathbf{SO}_{2m+1} = \mathbf{O}_{2m+1}.$$

From the Künneth formula (corollary 2.2.2), it follows that $Inv(\mathbf{SO}_{2m+1})$ is the quotient of $Inv(\mathbf{O}_{2m+1})$ by the ideal generated by w_1. From the formula in proposition 3.3.1, this is the submodule generated by the odd Stiefel-Whitney classes. The result follows.
Turning to the even case, we use the exact sequence from theorem 3.2.1:

$$0 \longrightarrow Inv(\mathbf{SO}_{2m}) \longrightarrow Inv(\mathbf{Z}/2 \times \mathbf{SO}_{2m-1}) \xrightarrow{r} Inv(\mathbf{SO}_{2m-2})$$

We proceed by induction, assuming the result for $n-2$.
The group $\mathbf{Z}/2 \times \mathbf{SO}_{2m-1}$ is seen as a subgroup of $\mathbf{SO}_{2m}$, were again the copy of $\mathbf{Z}/2$ is identified with $\{\pm Id\}$. However, there is also a canonical isomorphism of $\mathbf{Z}/2 \times \mathbf{SO}_{2m-1}$ with $\mathbf{O}_{2m-1}$, as above. The corresponding injective map $\mathbf{O}_{2m-1} \to \mathbf{SO}_{2m}$ thus obtained induces the map $H^1(K, \mathbf{O}_{2m-1}) \to H^1(K, \mathbf{SO}_{2m})$ which sends a quadratic form q to $\tilde{q} = (q \otimes \det(q)) \oplus \det(q)$. Here $\det(q)$ is the 1-dimensional quadratic form with corresponding 1×1 matrix given by the discriminant of q.
We have $w_{2i+1}(\tilde{q}) = 0$ as explained above. To compute the even Stiefel-Whitney classes, we note that $w_i(\tilde{q}) = w_i((q \oplus \langle 1\rangle) \otimes \det(q)) = w_i(q \otimes \det(q))$, and we recall that if we factorise formally

$$\sum w_i(q) t^i = \prod (1 + a_i t^i),$$

then

$$\sum w_i(q \otimes \det(q)) t^i = \prod (1 + (a_i + w_1(q)) t^i).$$

It follows that $w_{2i}(\tilde{q}) = w_{2i}(q) + w_1(q) R_i(q)$, and the remainder $w_1(q)R_i(q)$ can thus be written as a linear combination over $H^*(k)$ of the classes $w_j(q)$ with $j < 2i$. Moreover $w_{2m}(\tilde{q}) = 0$.

Now, write w_i, resp w'_i, resp w''_i, for the Stiefel-Whitney classes in $Inv(\mathbf{SO}_{2m})$, resp $Inv(\mathbf{O}_{2m-1})$, resp $Inv(\mathbf{SO}_{2m-2})$, and regard $Inv(\mathbf{SO}_{2m})$ as a subring of $Inv(\mathbf{O}_{2m-1})$. Thus we have $w_{2i+1} = 0$ and $w_{2i} = w'_{2i} + w'_1 R_i$.

Let M, resp N, denote the $H^*(k)$-submodule of $Inv(\mathbf{O}_{2m-1})$ generated by the classes w'_{2i+1} for $0 \leq i \leq m-2$, resp by the classes w_{2i} for $0 \leq i \leq m-1$ together with $b^{(1)}_{n-1} = w'_{2m+1} + (-1)w'_1 R_m$. Then M and N are free modules, and $Inv(\mathbf{O}_{2m-1}) = M \oplus N$. We claim (i) that the residue map r is 0 on N (ie $r(b^{(1)}_{n-1}) = 0$), and (ii) that r maps M injectively onto a free submodule in $Inv(\mathbf{SO}_{2m-2})$. This will complete the induction step, as we will have $Inv(\mathbf{SO}_{2m}) = \ker(r) = N$.

Both parts of the claim are proved by the same computation. We first note from theorem 3.2.1 that $r(w'_1) = 1$, and we compute:

$$\begin{array}{rcl} r(w'_{2i+1}) & = & r(w'_1 w'_{2i}) \\ & = & r(w'_1 w_{2i}) - r((w'_1)^2 R'_i) \\ & = & w''_{2i} - r((-1) w'_1 R_i) \\ & = & w''_{2i} - (-1) r(w'_1 R_i) \end{array} .$$

Take $i = m-1$ to obtain (i) (since $w''_{2m-2} = 0$). Property (ii) is clear.

It remains to start the induction. The group $\mathbf{SO}_2$ is a torus, so $Inv(\mathbf{SO}_2) = H^*(k)$. One gets the result for $\mathbf{SO}_4$ from this and theorem 3.2.1. This is very similar to the induction step (but easier), and will be left to the reader. $\square$

§4. Projective variants

4.1. Projective bundles

Consider the n-th projective space $\mathbf{P}^n$. It has an open subset isomorphic to $\mathbf{A}^n$, and from the long exact sequence of §2.1, (5), we draw at once $A^0(\mathbf{P}^n, H^*) = H^*(k)$.

Now, if $V \to X$ is a vector bundle, we may form the associated projective bundle $\pi : \mathbf{P}(V) \to X$. The fibre of π over $\mathrm{Spec}(\kappa) \to X$ is $\mathbf{P}^{n-1}_\kappa$, where n is the rank of V. Thus we see that the induced cycle module $\mathcal{A}^0 = A^0[\pi, H^*]$ on X is isomorphic to H^*, and from theorem 2.2.1 we have:

Lemma 4.1.1 – *Let $\pi : \mathbf{P}(V) \to X$ denote the projective bundle associated to the vector bundle $V \to X$. Then*

$$A^0(\mathbf{P}(V), H^*) = A^0(X, H^*).$$

Remark 4.1.2 – It should be kept in mind that the above argument, which rests on the spectral sequence of theorem 2.2.1, has been used for the sake of concision only. The result is a consequence of the following general statement:

$$A^*(\mathbf{P}(V), H^*) = A^*(X, H^*)[\zeta]/(\zeta^n)$$

with $\zeta \in A^1(\mathbf{P}(V), H^0)$. This is perfectly analogous to the usual statement for ordinary Chow groups, and is no harder to prove. We shall have no use for the complete statement in the sequel, however, and therefore we omit the lengthy argument.

Corollary 4.1.3 – *Let V be a representation of the linear algebraic group G. Form the projective representation $\mathbf{P}(V)$, and assume that there is a k-rational point in $\mathbf{P}(V)$ whose orbit is open, and isomorphic to G/S. Then there is an injection*

$$0 \to Inv(G) \to Inv(S).$$

Proof. We have a vector bundle $V_G \to BG$ and the associated projective bundle is $\mathbf{P}(V)_G \to BG$. From the lemma,

$$Inv(G) = A^0(BG, H^*) = A^0(\mathbf{P}(V)_G, H^*) = A^0_G(\mathbf{P}(V), H^*).$$

We may view G/S as an open subset in $\mathbf{P}(V)$, and from the equivariant long exact sequence we have

$$0 \to Inv(G) \to A^0_G(G/S, H^*).$$

As noted in §3.1, we have $A^0_G(G/S, H^*) = Inv(S)$. □

4.2. Applications

Example 4.2.1. As a very simple illustration, we may compute the invariants of a wreath product $\mathbf{G}_m \wr \mathbf{Z}/2$, that is a semi-direct product $\mathbf{G}_m^2 \rtimes \mathbf{Z}/2$ where $\mathbf{Z}/2$ permutes the two copies of the multiplicative group.
Indeed, this group may be seen as the normalizer of a maximal torus in $\mathbf{GL}_2$, and thus it has a canonical representation W of dimension 2. The space $\mathbf{P}(W)$ is just two orbits, and that of $[1,1]$ is open with stabilizer $\mathbf{G}_m \times \mathbf{Z}/2$. From corollary 4.1.3 (and corollary 2.2.2), we know that the restriction map from $Inv(\mathbf{G}_m^2 \rtimes \mathbf{Z}/2)$ to $Inv(\mathbf{Z}/2)$ is injective. It is also certainly surjective, since the inclusion $\mathbf{Z}/2 \to \mathbf{G}_m^2 \rtimes \mathbf{Z}/2$ is a section for the projection $\mathbf{G}_m^2 \rtimes \mathbf{Z}/2 \to \mathbf{Z}/2$. Finally, we conclude that $Inv(\mathbf{G}_m^2 \rtimes \mathbf{Z}/2) = Inv(\mathbf{Z}/2)$ (regardless of the choice of p).

Example 4.2.2. In [4], Garibaldi gives many examples of applications of corollary 4.1.3. It is possible to recover a good number of his results using the techniques of this paper. Let us illustrates this with $\mathbf{Spin}_7$ at $p = 2$.
If Δ denotes the spin representation of $\mathbf{Spin}_7$, one can show that there exists an open orbit U in $\mathbf{P}(\Delta)$ with stabilizer $G_2 \times \mathbf{Z}/2$ (G_2 is the split group of that type). Hence an injection

$$0 \to Inv(\mathbf{Spin}_7) \to Inv(G_2 \times \mathbf{Z}/2).$$

Garibaldi computes the image of this, creating invariants of $\mathbf{Spin}_7$ by restricting invariants of $\mathbf{Spin}_8$, and exploiting the fact that further restriction from $Inv(\mathbf{Spin}_7)$ to $Inv(\mathbf{Z}/2)$ is zero, since it factors through the invariants of a maximal torus.

Our version of this computation is to study the complement of U in $\mathbf{P}(\Delta)$. It consists of a single orbit with stabilizer K, and there is an affine bundle map $BK \to B(\mathbf{SL}_3 \times \mathbf{Z}/2)$. Therefore we have an exact sequence

$$0 \to Inv^d(\mathbf{Spin_7}) \to Inv^d(G_2 \times \mathbf{Z}/2) \to Inv^{d-1}(\mathbf{SL}_3 \times \mathbf{Z}/2).$$

This seems to be typical of the stratification method when projective representations are used: one can obtain some results very rapidly, but they are not always as accurate as one may wish.
Instead, if we proceed exactly as in the proof of theorem 3.2.1, we obtain:

$$0 \to Inv^d(\mathbf{Spin_7}) \to Inv^d(G_2 \times \mathbf{Z}/2) \to Inv^{d-1}(\mathbf{SL}_3) \to 0.$$

Since $\mathbf{SL}_3$ is special, $Inv(\mathbf{SL}_3) = H^*(k)$. From [5] we have that $Inv(G_2)$ is a free $H^*(k)$-module on two generators of degree 0 and 3. By corollary 2.2.2, $Inv(G_2 \times \mathbf{Z}/2)$ is a free $H^*(k)$-modules with 4 generators of degree 0, 1, 3 and 4. It follows that $Inv(\mathbf{Spin_7})$ is a free $H^*(k)$-module on 3 generators of degree 0, 3 and 4.

§5. The Bloch & Ogus spectral sequence

5.1. The spectral sequence

In this section, we assume that k is algebraically closed. The prime p being fixed as always, we shall write $H^i_{\text{et}}(X)$ for the étale cohomology group $H^i_{\text{et}}(X, \mathbf{Z}/p)$.

Theorem 5.1.1 (Rost, Bloch, Ogus) – *Let $k = \bar{k}$, and let X be equidimensional. Then there is a spectral sequence*

$$E_2^{r,s} = A^r(X, H^{s-r}) \Rightarrow H^{r+s}_{\text{et}}(X).$$

In particular the E_2 page is zero under the first diagonal, and the resulting map

$$A^n(X, H^0) = CH^n X \otimes_{\mathbf{Z}} \mathbf{Z}/p \to H^n_{\text{et}}(X)$$

is the usual cycle map.

A word of explanation on the authorship of the spectral sequence. To start with, there is the well-known coniveau spectral sequence, which converges to $H^n_{\text{et}}(X)$ and for which there is a description of the E_1 term. In [2] Bloch and Ogus prove that the E_2 term can be identified with $H^r_{Zar}(X, \mathcal{H}^s)$, where $\mathcal{H}^s$ is the sheafification of $U \mapsto H^s_{\text{et}}(U)$. They deduce that the groups are 0 under the diagonal, and prove the statement about the cycle map. More than 20 years later, in [11], corollary 6.5, Rost proves that the E_2 term can also be described using his Chow groups with coefficients as in the theorem. The sequence seems to be usually refered to as the Bloch & Ogus spectral sequence.
Given an algebraic group G, we may take a model for BG to play the role of X, and obtain:

COROLLARY 5.1.2 – *There is a map*

$$H^*_{\mathrm{et}}(BG) \to Inv(G)$$

which vanishes (when $* > 0$*) on the image of the cycle map*

$$CH^*BG \otimes_{\mathbf{Z}} \mathbf{Z}/p \to H^{2*}_{\mathrm{et}}(BG).$$

EXAMPLE 5.1.3. The Stiefel-Whitney classes as in 3.3 can be defined as the images of the elements with the same name in the cohomology of $B\mathbf{O}_n$. The corollary explains why they square to zero in $Inv(\mathbf{O}_n)$, since they square to Chern classes in cohomology, and these come from the Chow ring (see [13] for details).

REMARK 5.1.4 – When G is a finite group, viewed as a 0-dimensional algebraic group, there is a natural map

$$H^*(G, \mathbf{F}_p) \to H^*_{\mathrm{et}}(BG),$$

where $H^*(G, \mathbf{F}_p)$ is the usual cohomology of G as a discrete group. Indeed, there is a Galois covering $EG \to BG$ with group G, so we may see this map as coming from the corresponding Hoschild-Serre spectral sequence. Since we assume in this section that k is algebraically closed, it follows easily that the map is an isomorphism.
Composing this with the map from the previous corollary, we obtain the homomorphism

$$H^*(G, \mathbf{F}_p) \to Inv(G)$$

which was considered in [5]. Of course the direct definition given in *loc. cit.* is much preferable.

COROLLARY 5.1.5 (TO THEOREM 5.1.1) – *There are natural isomorphisms:*

$$Inv^1(G) = H^1_{\mathrm{et}}(BG)$$

and

$$Inv^2(G) = \frac{H^2_{\mathrm{et}}(BG)}{CH^1BG \otimes_{\mathbf{Z}} \mathbf{Z}/p}.$$

The denominator in the second isomorphism is really the image of $CH^1BG \otimes_{\mathbf{Z}} \mathbf{Z}/p$ in étale cohomology via the cycle map. When $k = \mathbf{C}$, the cycle map is injective in degree 1 and in fact $CH^1BG = H^2(BG, \mathbf{Z})$ (topological cohomology), see [13].

COROLLARY 5.1.6 (TO THEOREM 5.1.1) – *Any class x in the kernel of the cycle map*

$$CH^2BG \otimes_{\mathbf{Z}} \mathbf{Z}/p \to H^4_{\mathrm{et}}(BG)$$

determines an invariant $r_x \in Inv^3(BG)$, which is well-defined up to the image of $H^3_{\mathrm{et}}(BG) \to Inv^3(BG)$. If x is nonzero, neither is r_x.

We think of r_x as a simplified version of the Rost invariant.

Proof. Our assumption implies that the class x, viewed as an element of bidegree $(2,2)$ on the E_2 page of the spectral sequence under discussion, must be hit by a differential. Let $r_x \in E_2^{0,3} = Inv^3(G)$ be such that $d_2(r_x) = x$. This element is well-defined up to the kernel of d_2. Since further differentials d_r for $r > 2$ are zero on $E_r^{0,3}$, we see that r_x is defined up to elements which survive to the E_∞ page. These are, by definition, the elements in the image of $H^3_{\mathrm{et}}(BG) \to Inv^3(BG)$. $\square$

5.2. Applications

For simplicity, we shall take $k = \mathbf{C}$, the complex numbers, in the applications. In this case according to remark 2.3.4, the étale cohomology of BG as above coincides with the topological cohomology of a topological classifying space (i.e. a quotient EG/G of a contractible space EG endowed with a free G-action).
We start with a proposition which should be compared with statements 31.15 and 31.20 in [7], for which there is no proof available as far as I am aware.

Proposition 5.2.1 – *Over the complex numbers, there are isomorphisms*

$$Inv^1(G) = Hom(\pi_0(G), \mathbf{Z}/p)$$

and

$$Inv^2(G) = p\text{-torsion in } H^3(BG, \mathbf{Z}).$$

In particular, if G is connected then $Inv^1(G) = 0$, and if G is 1-connected then $Inv^2(G) = 0$.

Proof. From corollary 5.1.5, we have $Inv^1(G) = H^1(BG, \mathbf{F}_p)$, and of course this is $Hom(\pi_1(BG), \mathbf{Z}/p)$. However $\pi_1(BG) = \pi_0(G)$.
As noted above, the cycle map is injective in degree 1 over the complex numbers, and $CH^1BG = H^2(BG, \mathbf{Z})$. From corollary 5.1.5 again, we see that

$$Inv^2(G) = \frac{H^2(BG, \mathbf{F}_p)}{H^2(BG, \mathbf{Z}) \otimes \mathbf{Z}/p},$$

and it is elementary to show that this maps injectively onto the p-torsion in $H^3(BG, \mathbf{Z})$ via the Bockstein.
The statement about connected groups is trivial. When G is 1-connected, it is automatically 2-connected, since any real Lie group has $\pi_2(G) = 0$. Thus BG is 3-connected and we draw $H^3(BG, \mathbf{Z}) = 0$ from Hurewicz's theorem. $\square$

Example 5.2.2. Consider the case of the exceptional group G_2, for $k = \mathbf{C}$ and $p = 2$. It is reductive and 1-connected, so by the proposition we have $Inv^i(G_2) = 0$ for $i = 1, 2$. Moreover, the Chow ring of G_2 has been computed over the complex numbers, see [6]. It turns out that the map $CH^2BG_2 \otimes_{\mathbf{Z}} \mathbf{Z}/2 \to H^4(BG_2, \mathbf{F}_2)$ has exactly one nonzero element. From corollary 5.1.6,

we know that there is a nonzero invariant $e_3 \in Inv^3(G_2)$. It is uniquely defined as $H^3(BG_2, \mathbf{F}_2) = 0$.
It is proved in [5] that for any field k of characteristic $\neq 2$, $Inv(G_2)$ is in fact a free $H^*(k)$-module on the generators 1 and e_3.

Example 5.2.3. We take $k = \mathbf{C}$, $p = 2$, and $G = D_8 = \mathbf{Z}/4 \rtimes \mathbf{Z}/2$, the dihedral group. We shall completely compute $Inv(G)$ by showing first that things reduce to corollary 5.1.5.
Let t be a generator of $\mathbf{Z}/4$ in G, and let τ be the second generator, so that $\tau(t) = \tau t \tau^{-1} = t^{-1}$. We let G act on $V = \mathbf{A}^2$ via

$$t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

and

$$\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding map $G \to \mathbf{GL}_2$ is an embedding, so G acts freely outside of a finite number of lines. Let U denote the open complement. We have an injection

$$0 \to A^0_G(V, H^*) = Inv(G) \to A^0_G(U, H^*) = A^0(U/G, H^0),$$

by example 2.3.2. Now, U/G is a variety of dimension 2 over an algebraically closed field, so $H^*(k(U)) = 0$ for $* > 2$. It follows that $Inv(G)$ is concentrated in degrees ≤ 2.
Since we are working over $\mathbf{C}$, we have $H^*_{\text{et}}(BG) = H^*(BG, \mathbf{F}_2) = H^*(G, \mathbf{F}_2)$. The cohomology of G is well-known, see for example [1]:

$$H^*(G, \mathbf{F}_2) = \mathbf{F}_2[x_1, y_1, w_2]/(x_1 y_1 = 0).$$

More precisely, G can be presented as an "extraspecial group", ie as a central extension

$$0 \to \mathbf{Z}/2 \to G \to E \to 0$$

where $E \approx \mathbf{Z}/2 \times \mathbf{Z}/2$. If we choose x_1 and y_1 so that $H^*(E, \mathbf{F}_2) = \mathbf{F}_2[x_1, y_1]$, then we can pullback these classes to the cohomology of G where they will give the classes with the same name in the description above. (Besides, the cohomology class of the extention is $x_1 y_1$.)
We have immediately, by corollary 5.1.5, that $Inv^1(G) = \mathbf{F}_2 \cdot x_1 \oplus \mathbf{F}_2 \cdot y_1$. In degree 2, we note that $H^2(G, \mathbf{F}_2)$ is generated additively by the classes x_1^2, y_1^2, and w_2. The Chow ring of BE is $CH^*BE \otimes_{\mathbf{Z}} \mathbf{Z}/2 = \mathbf{F}_2[x_1^2, y_1^2]$, from which we know that the classes x_1^2 and y_1^2 in the cohomology of BG certainly come from the Chow ring of BG.
On the other hand, $Sq^1 w_2 = w_2(x_1 + y_1) \neq 0$ (*loc. cit.*), while the Steenrod operation Sq^1 is zero on classes coming from the integral cohomology, and *a fortiori* it is zero on the classes coming from the Chow ring.

As a result, we have finally $Inv^2(G) = \mathbf{F}_2 \cdot w_2$.
Over a general field, we could reach a similar conclusion by studying the geometric situation a bit more carefully. The variety U/G, for example, can be shown to be the open subset in $\mathbf{A}^2$ obtained by removing the axis $Y = 0$ and the two parabolae $X \pm 2Y^2 = 0$. Alternatively, you might want to use that D_8 is a 2-Sylow in S_4, and exploit the double coset formula as in [5], chap. V.

§6. Other cycle modules

We shall conclude with a few simple remarks on other possible cycle modules. Given any cycle module M as in §2, we may define the invariants $Inv(G, M)$ of G with values in M to be the natural transformations of functors

$$H^1(-, G) \to M(-).$$

It is straightforward to establish the inclusion

$$Inv(G, M) \subset A^0(BG, M)$$

for any BG which is the base of a versal G-principal bundle. Indeed, the arguments in [5] hold *verbatim* (and in fact in *loc. cit.* the reader will find a similar inclusion even for invariants with values in the Witt ring, even though the Witt ring satisfies weaker properties than cycle modules do.)
We may also define the equivariant Chow groups $A^*_G(X, M)$ for varieties X acted on by G, exactly as we have done for $M = H^*$. When this is done, we may take BG to be $\mathrm{Spec}(k)_G$ (as in the rest of this paper), and we have an equality

$$Inv(G, M) = A^0(BG, M) = A^0_G(\mathrm{Spec}(k), M)$$

by Totaro's arguments as in [5].
The techniques we have used for Galois cohomology may be used for any cycle module. Let us illustrate this for $M = K_*$, the algebraic K-theory of fields (Milnor or Quillen, it will not affect the sequel). In this case we have $A^*(X, K_0) = CH^*X$, and the parallel with our previous computations becomes even more obvious.
Arguing as in example 2.1.1, we obtain that $A^0(\mathbf{G}_m, K_*)$ is a free K_*-module on two generators, one in dimension 0, the other in dimension 1. If p is a prime number, we may use this to compute the invariants of $\boldsymbol{\mu}_p$.
The group $\boldsymbol{\mu}_p$ has a 1-dimensional representation V, and $\mathbf{G}_m$ sits in V as a $\boldsymbol{\mu}_p$-invariant open subset whose complement is a point $\mathrm{Spec}(k)$. We obtain the exact sequence (where $G = \boldsymbol{\mu}_p$):

$$0 \to A^0_G(V, K_*) \to A^0_G(\mathbf{G}_m, K_*) \to A^0_G(\mathrm{Spec}(k), K_{*-1}) \to A^1_G(V, K_{*-1}).$$

The action on $\mathbf{G}_m$ is free with quotient $\mathbf{G}_m$, so as in example 2.3.2 we draw $A^0_G(\mathbf{G}_m, K_*) = A^0(\mathbf{G}_m, K_*)$. Thus we may rewrite the exact sequence:

$$0 \longrightarrow Inv(G, K_*) \longrightarrow A^0(\mathbf{G}_m, K_*) \longrightarrow Inv(G, K_{*-1}),$$

and for $* = 1$ it is worth writing the extra term:

$$0 \to Inv(G, K_1) \to A^0(\mathbf{G}_m, K_1) \to Inv(G, K_0) \xrightarrow{s_*} CH^1BG \to 0.$$

Let us explain the notation s_* and what this map looks like. We call s the zero section $s : BG \to V_G$ of the vector bundle $\pi : V_G \to BG$, and s_* is the induced pushforward map. We know that π^* is an isomorphism, and that there is a projection formula $s_*(\pi^*(x)y) = xs_*(y)$. If we use π^* as an identification, it follows that $s_*(x) = c_1(V)x$ where $c_1(V) = s_*(1)$ is the first *Chern class* of V. Finally the map $s_* : Inv(G, K_0) \to CH^1BG$ is simply the surjective map $\mathbf{Z} \to CH^1BG$ sending 1 to $c_1(V)$. Now, CH^1BG is p-torsion by a transfer argument (or see [13], example 13.1, which shows that CH^1BG may well be 0 depending on k). In any case, s_* has a kernel isomorphic to $\mathbf{Z}$.
We conclude that the 1-dimensional generator for $A^0(\mathbf{G}_m, K_*)$ must map to a generator for this kernel. As a result, $Inv(G, K_*)$ is reduced to K_*, i.e. *the group* $\boldsymbol{\mu}_p$ *has no nonconstant invariants in algebraic K-theory at all.*
For $p = 2$ for example, assuming that $char(k) \neq 2$, we may use the surjection $H^1(K, (\mathbf{Z}/2)^n) \to H^1(K, \mathbf{O}_n)$ for fields containing k to deduce that $Inv(\mathbf{O}_n, K_*)$ injects into $Inv((\mathbf{Z}/2)^n, K_*)$. Thus, after an obvious Künneth argument, we see that $\mathbf{O}_n$ has no nonconstant invariants in algebraic K-theory, either. In particular, there is no natural way of lifting the Siefel-Whitney classes to integral Milnor K-theory.

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