Rings of Integers of Type $K(\pi, 1)$

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Abstract. We investigate the Galois group $G_S(p)$ of the maximal p-extension unramified outside a finite set S of primes of a number field in the (tame) case, when no prime dividing p is in S. We show that the cohomology of $G_S(p)$ is 'often' isomorphic to the étale cohomology of the scheme $Spec(\mathcal{O}_k \smallsetminus S)$, in particular, $G_S(p)$ is of cohomological dimension 2 then.

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1 Introduction

We call a connected locally noetherian scheme Y a '$K(\pi,1)$' for a prime number p if the higher homotopy groups of the p-completion $Y_{et}^{(p)}$ of its étale homotopy type Y_{et} vanish. In this paper we consider the case of an arithmetic curve, where the $K(\pi,1)$-property is linked with open questions in the theory of Galois groups with restricted ramification of number fields:

Let k be a number field, S a finite set of nonarchimedean primes of k and p a prime number. For simplicity, we assume that p is odd or that k is totally imaginary. By a p-extension we understand a Galois extension whose Galois group is a (pro-) p-group. Let $k_S(p)$ denote the maximal p-extension of k unramified outside S and put $G_S(p) = Gal(k_S(p)|k)$. A systematic study of this group had been started by Šafarevič, and was continued by Koch, Kuz'min, Wingberg and many others; see [NSW], VIII, §7 for basic properties of $G_S(p)$. In geometric terms (and omitting the base point), we have

$$G_S(p) \cong \pi_1\big((Spec(\mathcal{O}_k) \smallsetminus S)_{et}^{(p)}\big).$$

As is well known to the experts, if S contains the set S_p of primes dividing p, then $Spec(\mathcal{O}_k) \smallsetminus S$ is a $K(\pi,1)$ for p (see Proposition 2.3 below). In particular, if $S \supset S_p$, then $G_S(p)$ is of cohomological dimension less or equal to 2.

The group $G_S(p)$ is most mysterious in the *tame* case, i.e. when $S \cap S_p = \varnothing$. In this case, examples when $Spec(\mathcal{O}_k) \smallsetminus S$ is *not* a $K(\pi, 1)$ are easily constructed. On the contrary, until recently not a single $K(\pi, 1)$-example was known. The following properties of the group $G_S(p)$ were known so far:

- $G_S(p)$ is a 'fab-group', i.e. U^{ab} is finite for each open subgroup $U \subset G$.
- $G_S(p)$ can be infinite (Golod-Šafarevič).
- $G_S(p)$ is a finitely presented pro-p-group (Koch).

A conjecture of Fontaine and Mazur [FM] asserts that $G_S(p)$ has no infinite p-adic analytic quotients.

In 2005, Labute considered the case $k = \mathbb{Q}$ and found finite sets S of prime numbers (called strictly circular sets) with $p \notin S$ such that $G_S(p)$ has cohomological dimension 2. In [S1] the author showed that, in the examples given by Labute, $Spec(\mathbb{Z}) \smallsetminus S$ is a $K(\pi, 1)$ for p.
The objective of this paper is a systematic study of the $K(\pi, 1)$-property. Our focus is on the tame case, where we conjecture that rings of integers of type $K(\pi, 1)$ are cofinal in the following sense:

Conjecture 1. *Let k be a number field and let p be a prime number. Assume that $p \neq 2$ or that k is totally imaginary. Let S be a finite set of primes of k with $S \cap S_p = \varnothing$. Let, in addition, a set T of primes of Dirichlet density $\delta(T) = 1$ be given. Then there exists a finite subset $T_1 \subset T$ such that $Spec(\mathcal{O}_k) \smallsetminus (S \cup T_1)$ is a $K(\pi, 1)$ for p.*

Of course we may assume that $T \cap S_p = \varnothing$ in the conjecture. Our main result is the following

Theorem 1. *Conjecture 1 is true if the number field k does not contain a primitive p-th root of unity and the class number of k is prime to p.*

Explicit examples of rings of integers of type $K(\pi, 1)$ can be found in [La], [S1] (for $k = \mathbb{Q}$) and in [Vo] (for k imaginary quadratic).

The $K(\pi, 1)$-property has strong consequences. We write $X = Spec(\mathcal{O}_k)$ and assume in all results that $p \neq 2$ or that k is totally imaginary. Primes $\mathfrak{p} \in S \smallsetminus S_p$ with $\mu_p \not\subset k_\mathfrak{p}$ are redundant in the sense that removing these primes from S does not change $(X \smallsetminus S)^{(p)}_{et}$, see section 4. In the tame case, we may therefore restrict our considerations to sets of primes whose norms are congruent to 1 modulo p. These are the results.

Theorem 2. *Let S be a finite non-empty set of primes of k whose norms are congruent to 1 modulo p. If $X \smallsetminus S$ is a $K(\pi, 1)$ for p and $G_S(p) \neq 1$, then the following hold.*

(i) *$cd\, G_S(p) = 2$, $scd\, G_S(p) = 3$.*

(ii) *$G_S(p)$ is a duality group.*

The dualizing module D of $G_S(p)$ is given by $D = tor_p C_S(k_S(p))$, i.e. it is the subgroup of p-torsion elements in the S-idèle class group of $k_S(p)$.

Remarks: 1. If $X \smallsetminus S$ is a $K(\pi, 1)$ for p and $G_S(p) = 1$, then k is imaginary quadratic, $\#S = 1$ and $p = 2$ or 3. See Proposition 7.4 for a more precise statement.

2. We have a natural exact sequence

$$0 \to \mu_{p^\infty}(k_S(p)) \to \bigoplus_{\mathfrak{p} \in S} \operatorname{Ind}\,^{G_\mathfrak{p}(k_S(p)|k)}_{Gal(k_S(p)|k)}\ \mu_{p^\infty}(k_S(p)_\mathfrak{p}) \to$$
$$\operatorname{tor}_p C_S(k_S(p)) \to \mathcal{O}^\times_{k_S(p),S} \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0\ ,$$

where $\mathcal{O}^\times_{k_S(p),S}$ is the group of S-units of $k_S(p)$ and $\mu_{p^\infty}(K)$ denotes the group of all p-power roots of unity in a field K. Note that $\mu_{p^\infty}(k_S(p))$ is finite, while, by Theorem 3 below, for $\mathfrak{p} \in S$ the field $k_S(p)_\mathfrak{p}$ contains all p-power roots of unity.

3. In the wild case $S \supset S_p$, where $X \smallsetminus S$ is always a $K(\pi, 1)$ for p, $G_S(p)$ is of cohomological dimension 1 or 2. The strict cohomological dimension is conjecturally equal to 2 (=Leopoldt's conjecture for each finite subextension of k in $k_S(p)$). In the wild case, $G_S(p)$ is often, but not always a duality group, cf. [NSW] Prop. 10.7.13.

Allowing ramification at a prime $\mathfrak{p}$ does not mean that the ramification is realized globally. Therefore it is a natural and interesting question how far we get locally at the primes in S when going up to $k_S(p)$. See [NSW] X, §3 for results in the wild case. In the tame case, we have the following

Theorem 3. *Let S be a finite non-empty set of primes of k whose norms are congruent to 1 modulo p. If $X \smallsetminus S$ is a $K(\pi, 1)$ for p and $G_S(p) \neq 1$, then*

$$k_S(p)_\mathfrak{p} = k_\mathfrak{p}(p)$$

for all primes $\mathfrak{p} \in S$, i.e. $k_S(p)$ realizes the maximal p-extension of the local field $k_\mathfrak{p}$.

Remark: Under the assumptions of the theorem, let $\mathfrak{q} \notin S$. Then either $\mathfrak{q}$ splits completely in $k_S(p)$, or $k_S(p)$ realizes the maximal unramified p-extension $k^{nr}_\mathfrak{q}(p)$. We do not know whether the completely split case actually occurs.

The next result addresses the question of enlarging the set S without destroying the $K(\pi, 1)$-property.

Theorem 4. *Let S' be a finite non-empty set of primes of k whose norms are congruent to 1 modulo p and let $S \subset S'$ be a nonempty subset. Assume that $X \smallsetminus S$ is a $K(\pi, 1)$ for p and that $G_S(p) \neq 1$. If each $\mathfrak{q} \in S' \smallsetminus S$ does not split completely in $k_S(p)$, then $X \smallsetminus S'$ is a $K(\pi, 1)$ for p. Furthermore, in this* *case, the arithmetic form of Riemann's existence theorem holds: the natural homomorphism*

$$\mathop{\ast}_{\mathfrak{p}\in S'\setminus S(k_S(p))} T_{\mathfrak{p}}(k_{S'}(p)|k_S(p)) \longrightarrow Gal(k_{S'}(p)|k_S(p))$$

is an isomorphism, i.e. $Gal(k_{S'}(p)|k_S(p))$ *is the free pro-p product of a bundle of inertia groups.*

Finally, we deduce a statement on universal norms of unit groups.

Theorem 5. *Let* S *be a finite non-empty set of primes of* k *whose norms are congruent to* 1 *modulo* p. *Assume that* $X \smallsetminus S$ *is a* $K(\pi,1)$ *for* p *and that* $G_S(p) \neq 1$. *Then*

$$\varprojlim_{K\subset k_S(p)} \mathcal{O}_K^\times \otimes \mathbb{Z}_p \;=\; 0 \;=\; \varprojlim_{K\subset k_S(p)} \mathcal{O}_{K,S}^\times \otimes \mathbb{Z}_p,$$

where K *runs through all finite subextensions of* k *in* $k_S(p)$, $\mathcal{O}_K^\times$ *and* $\mathcal{O}_{K,S}^\times$ *are the groups of units and* S*-units of* K, *respectively, and the transition maps are the norm maps.*

The structure of this paper is as follows. First we give the necessary definitions and make some calculations of étale cohomology groups for which we couldn't find an appropriate reference. In section 4, we deal with the first obstruction against the $K(\pi,1)$-property, the h^2-defect. Then we recall Labute's results on mild pro-p-groups, which we use in the proof of Theorem 1 given in section 6. In the last three sections we prove Theorems 2–5.

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2 First observations

We tacitly assume all schemes to be connected and omit base points from the notation. Let Y be a locally noetherian scheme and let p be a prime number. We denote by $Y_{et}^{(p)}$ the p-completion of the étale homotopy type of Y, see [AM], [Fr]. By $\widetilde{Y}(p)$ we denote the universal pro-p-covering of Y. The projection $\widetilde{Y}(p) \to Y$ is Galois with group $\pi_1^{et}(Y)(p) = \pi_1(Y_{et}^{(p)})$, cf. [AM], (3.7). Any discrete p-torsion $\pi_1^{et}(Y)(p)$-module M defines a locally constant sheaf on Y_{et}, which we denote by the same letter. The Hochschild-Serre spectral sequence defines natural homomorphisms

$$\phi_{M,i} : H^i(\pi_1^{et}(Y)(p), M) \longrightarrow H^i_{et}(Y,M),\ i \geq 0.$$

Since $H^1_{et}(\widetilde{Y}(p), M) = 0$, the map $\phi_{M,i}$ is an isomorphism for $i = 0$ and 1, and is injective for $i = 2$. For a pro-p-group G we denote by $K(G,1)$ the associated Eilenberg-MacLane space ([AM], (2.6)).

PROPOSITION 2.1. *The following conditions are equivalent:*

(i) *The classifying map* $Y_{et}^{(p)} \longrightarrow K(\pi_1^{et}(Y)(p),1)$ *is a weak equivalence.*

(ii) $\pi_i(Y_{et}^{(p)}) = 0$ *for all* $i \geq 2$.

(iii) $H^i_{et}(\widetilde{Y}(p), \mathbb{Z}/p\mathbb{Z}) = 0$ *for all* $i \geq 1$.

(iv) $\phi_{\mathbb{Z}/p\mathbb{Z},i} : H^i(\pi_1^{et}(Y)(p), \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^i_{et}(Y, \mathbb{Z}/p\mathbb{Z})$ *is an isomorphism for all* $i \geq 0$.

(v) $\phi_{M,i} : H^i(\pi_1^{et}(Y)(p), M) \longrightarrow H^i_{et}(Y, M)$ *is an isomorphism for all* $i \geq 0$ *and any discrete* p*-torsion* $\pi_1^{et}(Y)(p)$*-module* M.

Proof. The equivalences (i)⇔(ii)⇔(v) are the content of [AM], (4.3), (4.4). The equivalence (iii)⇔(iv) follows in a straightforward manner from the Hochschild-Serre spectral sequence. The implication (v)⇒(iv) is trivial.
Assume that (iv) holds. As $\pi_1^{et}(Y)(p)$ is a pro-p-group, any finite p-primary $\pi_1^{et}(Y)(p)$-module M has a composition series with graded pieces isomorphic to $\mathbb{Z}/p\mathbb{Z}$ with trivial $\pi_1^{et}(Y)(p)$-action ([NSW], Corollary 1.7.4). Hence, if M is finite, the five-lemma implies that $\phi_{M,i}$ is an isomorphism for all i. An arbitrary discrete p-primary $\pi_1^{et}(Y)(p)$-module is the filtered inductive limit of finite p-primary $\pi_1^{et}(Y)(p)$-modules. Since group cohomology ([NSW], Proposition 1.5.1) and étale cohomology ([AGV], VII, 3.3) commute with filtered inductive limits, $\phi_{M,i}$ is an isomorphism for all i and all discrete p-torsion $\pi_1^{et}(Y)(p)$-modules M. This implies (v) and completes the proof. □

DEFINITION. *We say that* Y *is a '*$K(\pi,1)$ *for* p*' if the equivalent conditions of Proposition 2.1 are satisfied.*

Now let k be a number field, S a finite set of nonarchimedean primes of k and p a prime number. We put $X = Spec(\mathcal{O}_k)$. The following observation is straightforward.

COROLLARY 2.2. *Let* k' *be a finite subextension of* k *in* $k_S(p)$ *and let* $X' = Spec(\mathcal{O}_{k'})$, $S' = S(k')$. *Then the following are equivalent.*

(i) $X \smallsetminus S$ *is a* $K(\pi,1)$ *for* p,

(ii) $X' \smallsetminus S'$ *is a* $K(\pi,1)$ *for* p.

Proof. Both schemes have the same universal pro-p-covering. □

We denote by S_p and S_∞ the set of primes of k dividing p and the set of archimedean primes of k, respectively. For a set S of primes (which may contain archimedean places), let $k_S(p)$ be the maximal p-extension of k unramified outside S and $G_S(p) = Gal(k_S(p)|k)$. For a finite set S of nonarchimedean primes of k we have the identification

$$\pi_1^{et}((X \smallsetminus S)_{et}^{(p)}) = G_{S\cup S_\infty}(p).$$

If p is odd or k is totally imaginary, then $G_S(p) = G_{S\cup S_\infty}(p)$. The following proposition is given for sake of completeness. It deals with the 'wild' case $S \supset S_p$, and is well known.

Proposition 2.3. *If S contains S_p, then $X \smallsetminus S$ is a $K(\pi,1)$ for p.*

Proof. We verify condition (v) of Proposition 2.1. Let $k_{S\cup S_\infty}$ be the maximal extension of k unramified outside $S \cup S_\infty$ and put $G_{S\cup S_\infty} = Gal(k_{S\cup S_\infty}|k)$. For any p-primary discrete $G_{S\cup S_\infty}(p)$-module M the homomorphism $\phi_{M,i}$ factors as

$$H^i(G_{S\cup S_\infty}(p), M) \to H^i(G_{S\cup S_\infty}, M) \to H^i_{et}(X \smallsetminus S, M).$$

By [NSW], Cor. 10.4.8, the left map is an isomorphism. That also the right map is an isomorphism follows in a straightforward manner by using the Kummer sequence, the Principal Ideal Theorem and known properties of the Brauer group, see for example [Zi], Prop. 3.3.1. or [Mi], II Prop. 2.9. $\square$

Remark: If $p = 2$ and k has real places it is useful to work with the modified étale site defined by T. Zink [Zi], which takes the real archimedean places of k into account. Proposition 2.3 holds true also for the modified étale site, see [S2], Thm. 6.

3 Calculation of étale cohomology groups

As a basis of our investigations, we need the calculation of the étale cohomology groups of open subschemes of $Spec(\mathcal{O}_k)$ with values in the constant sheaf $\mathbb{Z}/p\mathbb{Z}$. Let p be a fixed prime number. All cohomology groups are taken with respect to the constant sheaf $\mathbb{Z}/p\mathbb{Z}$, which we omit from the notation. Furthermore, we use the notation

$$h^i(-) = \dim_{\mathbb{F}_p} H^i_{et}(-) \quad (= \dim_{\mathbb{F}_p} H^i_{et}(-, \mathbb{Z}/p\mathbb{Z})\ )$$

for the occurring cohomology groups. We start with some well-known local computations.

Proposition 3.1. *Let k be a nonarchimedean local field of characteristic zero and residue characteristic ℓ. Let $X = Spec(\mathcal{O}_k)$ and let x be the closed point of X. Then the étale local cohomology groups $H^i_x(X)$ vanish for $i \le 1$ and $i \ge 4$, and*

$$h^2_x(X) = \begin{cases} \delta, & \text{if } \ell \neq p, \\ \delta + [k:\mathbb{Q}_p], & \text{if } \ell = p, \end{cases}$$

where $\delta = 1$ if $\mu_p \subset k$ and zero otherwise. Furthermore, $h_x^3(X) = \delta$. In particular, we have the Euler-Poincaré characteristic formula

$$\sum_{i=0}^{3}(-1)^i h_x^i(X) = \begin{cases} 0, & \text{if } \ell \neq p, \\ [k:\mathbb{Q}_p], & \text{if } \ell = p. \end{cases}$$

Proof. As X is henselian, we have isomorphisms $H^i_{et}(X) \cong H^i_{et}(x)$ for all i, and therefore

$$h^i(X) = \begin{cases} 1 & \text{for } i = 0,1, \\ 0 & \text{for } i \geq 2. \end{cases}$$

Furthermore, $X \smallsetminus \{x\} = \mathit{Spec}(k)$, hence $H^i_{et}(X \smallsetminus \{x\}) \cong H^i(k)$. The local duality theorem (cf. [NSW], Theorem 7.2.15) shows $h^2(X \smallsetminus \{x\}) = \delta$, and by [NSW], Corollary 7.3.9, we have

$$h^1(X \smallsetminus \{x\}) = \begin{cases} 1+\delta & \text{if } \ell \neq p, \\ 1+\delta+[k:\mathbb{Q}_p] & \text{if } \ell = p. \end{cases}$$

Furthermore, the natural homomorphism $H^1_{et}(X) \to H^1_{et}(X \smallsetminus \{x\})$ is injective. Therefore the result of the proposition follows from the exact excision sequence

$$\cdots \to H^i_x(X) \to H^i_{et}(X) \to H^i_{et}(X \smallsetminus \{x\}) \to H^{i+1}_x(X) \to \cdots .$$

□

Now let k be a number field, S a finite set of nonarchimedean primes of k and $X = \mathit{Spec}(\mathcal{O}_k)$. We assume for simplicity that p is odd or that k is totally imaginary, so that we can ignore the archimedean places of k for cohomological considerations. We introduce the following notation

r_1	the number of real places of k
r_2	the number of complex places of k
r	$= r_1 + r_2$, the number of archimedean places of k
S_p	the set of places of k dividing p
δ	equal to 1 if $\mu_p \subset k$ and zero otherwise
$\delta_\mathfrak{p}$	equal to 1 if $\mu_p \subset k_\mathfrak{p}$ and zero otherwise
$\mathit{Cl}(k)$	the ideal class group of k
$\mathit{Cl}_S(k)$	the S-ideal class group of k
h_k	$= \#\mathit{Cl}(k)$, the class number of k
${}_nA$	$= \ker(A \xrightarrow{\cdot n} A)$, where A is an abelian group and $n \in \mathbb{N}$
A/n	$= \operatorname{coker}(A \xrightarrow{\cdot n} A)$, where A is an abelian group and $n \in \mathbb{N}$.

PROPOSITION 3.2. *Assume that $p \neq 2$ or that k is totally imaginary. Then $H^i_{et}(X \smallsetminus S) = 0$ for $i \geq 4$, and*

$$\chi(X \smallsetminus S) := \sum_{i=0}^{3}(-1)^i h^i(X \smallsetminus S) = r - \sum_{\mathfrak{p} \in S \cap S_p} [k_\mathfrak{p} : \mathbb{Q}_p] .$$

In particular,

$$\chi(X \smallsetminus S) = \begin{cases} r, & \text{if } S \cap S_p = \varnothing\,, \\ -r_2, & \text{if } S \supset S_p\,. \end{cases}$$

Proof. The assertion for $S = S_p$ is well known, see [Mi], II Theorem 2.13 (a). Consider the exact excision sequence

$$\cdots \to \bigoplus_{\mathfrak{p}\in S} H^i_{\mathfrak{p}}(X_{\mathfrak{p}}) \to H^i_{et}(X) \to H^i_{et}(X \smallsetminus S) \to \bigoplus_{\mathfrak{p}\in S} H^{i+1}_{\mathfrak{p}}(X_{\mathfrak{p}}) \to \cdots\ ,$$

where $X_{\mathfrak{p}} = \mathit{Spec}(\mathcal{O}_{k,\mathfrak{p}})$ is the spectrum of the completion of $\mathcal{O}_k$ at $\mathfrak{p}$. Using this excision sequence for $S = S_p$, Proposition 3.1 implies the result for $S = \varnothing$, noting that $\sum_{\mathfrak{p}\in S_p}[k_{\mathfrak{p}} : \mathbb{Q}_p] - r_2 = [k : \mathbb{Q}] - r_2 = r$. The result for arbitrary S follows from the case $S = \varnothing$, the above excision sequence and from Proposition 3.1. □

In order to give formulae for the individual cohomology groups, we consider the Kummer group (cf. [NSW], VIII, §6)

$$V_S(k) := \{a \in k^\times \mid a \in k_{\mathfrak{p}}^{\times p} \text{ for } \mathfrak{p} \in S \text{ and } a \in U_{\mathfrak{p}} k_{\mathfrak{p}}^{\times p} \text{ for } \mathfrak{p} \notin S\}/k^{\times p},$$

where $U_{\mathfrak{p}}$ denotes the unit group of the local field $k_{\mathfrak{p}}$ (convention: $U_{\mathfrak{p}} = k_{\mathfrak{p}}^\times$ if $\mathfrak{p}$ is archimedean).[1] $V_S(k)$ is a finite group. More precisely, we have the following

Proposition 3.3. *There exists a natural exact sequence*

$$0 \longrightarrow \mathcal{O}_k^\times/p \longrightarrow V_\varnothing(k) \longrightarrow {}_p\mathit{Cl}(k) \longrightarrow 0\,.$$

In particular,

$$\dim_{\mathbb{F}_p} V_\varnothing(k) = \dim_{\mathbb{F}_p} {}_p\mathit{Cl}(k) + \dim_{\mathbb{F}_p} \mathcal{O}_k^\times/p = \dim_{\mathbb{F}_p} {}_p\mathit{Cl}(k) + r - 1 + \delta.$$

If S is arbitrary and $\mathfrak{p} \notin S$ is an additional prime of k, then we have a natural exact sequence

$$0 \longrightarrow V_{S\cup\{\mathfrak{p}\}}(k) \xrightarrow{\ \phi\ } V_S(k) \longrightarrow U_{\mathfrak{p}} k_{\mathfrak{p}}^{\times p}/k_{\mathfrak{p}}^{\times p}.$$

For $\mathfrak{p} \notin S_p$, we have $\dim_{\mathbb{F}_p} \operatorname{coker}(\phi) \le \delta_{\mathfrak{p}}$.

Proof. Sending an $a \in V_\varnothing(k)$ to the class in $\mathit{Cl}(k)$ of the fractional ideal $\mathfrak{a}$ with $(a) = \mathfrak{a}^p$ yields a surjective homomorphism $V_\varnothing(k) \to {}_p\mathit{Cl}(k)$ with kernel $\mathcal{O}_k^\times/p$. This, together with Dirichlet's Unit Theorem, shows the first statement. The second exact sequence follows immediately from the definitions. There are natural isomorphisms

$$U_{\mathfrak{p}} k_{\mathfrak{p}}^{\times p}/k_{\mathfrak{p}}^{\times p} \cong U_{\mathfrak{p}}/U_{\mathfrak{p}} \cap k_{\mathfrak{p}}^{\times p} = U_{\mathfrak{p}}/U_{\mathfrak{p}}^p.$$

For $\mathfrak{p} \notin S_p$ we have $\dim_{\mathbb{F}_p} U_{\mathfrak{p}}/U_{\mathfrak{p}}^p = \delta_{\mathfrak{p}}$, showing the last statement. □

[1] In terms of flat cohomology, we have $V_S(k) = \ker\left(H^1_{fl}(X \smallsetminus S, \mu_p) \to \bigoplus_{\mathfrak{p}\in S} H^1(k_{\mathfrak{p}}, \mu_p)\right)$.

The étale cohomology groups of $X \smallsetminus S$ have the following dimensions.

Theorem 3.4. *Let S be a finite set of nonarchimedean primes of k. Assume $p \neq 2$ or that k is totally imaginary. Then $H^i_{et}(X \smallsetminus S) = 0$ for $i \geq 4$ and*

$$\begin{aligned} h^0(X \smallsetminus S) &= 1\,, \\ h^1(X \smallsetminus S) &= 1 + \sum_{\mathfrak{p}\in S} \delta_\mathfrak{p} - \delta + \dim_{\mathbb{F}_p} V_S + \sum_{\mathfrak{p}\in S\cap S_p} [k_\mathfrak{p} : \mathbb{Q}_p] - r, \\ h^2(X \smallsetminus S) &= \sum_{\mathfrak{p}\in S} \delta_\mathfrak{p} - \delta + \dim_{\mathbb{F}_p} V_S + \theta\,, \\ h^3(X \smallsetminus S) &= \theta\,. \end{aligned}$$

Here θ is equal to 1 if $\delta = 1$ and $S = \varnothing$, and zero in all other cases.

Proof. The statement on h^0 is trivial and the vanishing of H^i for $i \geq 4$ was already part of Proposition 3.2. Artin-Verdier duality (see [Ma], 2.4 or [Mi], Theorem 3.1) shows

$$H^3_{et}(X)^\vee \cong \mathrm{Hom}_X(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m) = \mu_p(k).$$

Consider the exact excision sequence

$$\bigoplus_{\mathfrak{p}\in S} H^3_\mathfrak{p}(X_\mathfrak{p}) \xrightarrow{\alpha} H^3_{et}(X) \xrightarrow{\beta} H^3_{et}(X \smallsetminus S) \to \bigoplus_{\mathfrak{p}\in S} H^4_\mathfrak{p}(X_\mathfrak{p}).$$

By Proposition 3.1, the right hand group is zero, hence β is surjective. By the local duality theorem (see [Ma], 2.4, [Mi], II Corollary 1.10), the dual map to α is the natural inclusion

$$\mu_p(k) \to \bigoplus_{\mathfrak{p}\in S} \mu_p(k_\mathfrak{p}),$$

which is injective, unless $\delta = 1$ and $S = \varnothing$. Therefore $h^3(X \smallsetminus S) = 1$ if $\delta = 1$ and $S = \varnothing$, and zero otherwise. Using the isomorphism $H^1(G_S(p)) \xrightarrow{\sim} H^1_{et}(X \smallsetminus S)$, the statement on h^1 follows from the corresponding formula for the first cohomology of $G_S(p)$, see [NSW], Theorem 8.7.11. Finally, the result for h^2 follows by using the Euler-Poincaré characteristic formula in Proposition 3.2. □

Corollary 3.5. *Assume that $\delta = 0$ or $S \neq \varnothing$. Then $X \smallsetminus S$ is a $K(\pi,1)$ for p if and only if the following conditions* (i) *and* (ii) *are satisfied.*

(i) $\phi_2\colon\ H^2(G_S(p)) \hookrightarrow H^2_{et}(X \smallsetminus S)$ *is an isomorphism,*

(ii) $cd\, G_S(p) \leq 2$.

Proof. The given conditions are obviously necessary. Furthermore, ϕ_0 and ϕ_1 are isomorphisms and $H^i_{et}(X \smallsetminus S) = 0$ for $i \geq 3$ by Theorem 3.4. Therefore (i) and (ii) imply that ϕ_i is an isomorphism for all i. Hence condition (iv) of Proposition 2.1 is satisfied for $X \smallsetminus S$ and p. □

Let F be a locally constant sheaf on $(X \smallsetminus S)_{et}$. For each prime $\mathfrak{p}$ the composite map $\mathcal{O}_{k,S} \to k \to k_\mathfrak{p}$ induces natural maps $H^i_{et}(X \smallsetminus S, F) \to H^i(k_\mathfrak{p}, F)$ for all $i \geq 0$.

DEFINITION. *For any locally constant sheaf F on $(X \smallsetminus S)_{et}$ we put*

$$Ш^i(k,S,F) := \ker\Big(H^i_{et}(X \smallsetminus S, F) \longrightarrow \prod_{\mathfrak{p}\in S} H^i(k_\mathfrak{p}, F)\Big).$$

Assume a prime number p is fixed. Then we write $Ш^i(k,S) := Ш^i(k,S,\mathbb{Z}/p\mathbb{Z})$ and, following historical notation, we put $Б_S(k) := V_S(k)^\vee$, where $\vee$ denotes the Pontryagin dual.

The next theorem is sharper than [NSW], Thm. 8.7.4, as the group $Ш^2(G_S)$, which was considered there, is a subgroup of $Ш^2(k,S)$. If $p = 2$ and k has real places, then Theorem 3.6 remains true after replacing étale cohomology with its modified version.

THEOREM 3.6. *Assume $p \neq 2$ or that k is totally imaginary. Then there exists a natural isomorphism*

$$Ш^2(k,S) \xrightarrow{\sim} Б_S(k).$$

Proof. The proof of [NSW], Thm. 8.7.4 can be adapted to show also the stronger statement here. However, we decided to take the short cut by using flat duality. For any prime $\mathfrak{p}$ of k one easily computes the local cohomology groups for the flat topology as $H^1_{fl,\mathfrak{p}}(X,\mu_p) = 0$ and $H^2_{fl,\mathfrak{p}}(X,\mu_p) \cong k_\mathfrak{p}^\times / U_\mathfrak{p} k_\mathfrak{p}^{\times p}$. Therefore excision and Kummer theory imply an exact sequence

$$0 \to H^1_{fl}(X,\mu_p) \to k^\times / k^{\times p} \to \bigoplus_{\mathfrak{p}} k_\mathfrak{p}^\times / U_\mathfrak{p} k_\mathfrak{p}^{\times p}.$$

As $H^1_{fl}(X^h_\mathfrak{p},\mu_p) \cong U_\mathfrak{p}/p$, we obtain the exact sequence

$$(*) \qquad 0 \to V_S(k) \to H^1_{fl}(X,\mu_p) \to \bigoplus_{\mathfrak{p}\in S} H^1_{fl}(X^h_\mathfrak{p},\mu_p).$$

By excision, and noting $H^3_\mathfrak{p}(X,\mathbb{Z}/p\mathbb{Z}) \cong H^2(k_\mathfrak{p},\mathbb{Z}/p\mathbb{Z})$, we have an exact sequence

$$(**) \qquad \bigoplus_{\mathfrak{p}\in S} H^2_\mathfrak{p}(X,\mathbb{Z}/p\mathbb{Z}) \to H^2_{et}(X,\mathbb{Z}/p\mathbb{Z}) \to Ш^2(k,S) \to 0.$$

Comparing sequences $(*)$ and $(**)$ via local and global flat duality, we obtain the asserted isomorphism. □

We provide the following lemma for further use.

Lemma 3.7. *Let $K \subset k_S(p)$ be an extension of k inside $k_S(p)$ and let $(X \smallsetminus S)_K$ be the normalization of $X \smallsetminus S$ in K. If $\delta = 0$, or $S \neq \varnothing$ or $K|k$ is infinite, then*

$$H^3_{et}((X \smallsetminus S)_K) = 0.$$

Proof. We denote the normalization of $X \smallsetminus S$ in any algebraic extension field k' of k by $(X \smallsetminus S)_{k'}$. Étale cohomology commutes with inverse limits of schemes if the transition maps are affine (see [AGV], VII, 5.8). Therefore

$$H^3((X \smallsetminus S)_K) = \varinjlim_{k' \subset K} H^3((X \smallsetminus S)_{k'}),$$

where k' runs through all finite subextensions of k in K. If $\delta = 0$ or $S \neq \varnothing$, then, by Theorem 3.4, $H^3_{et}((X \smallsetminus S)_{k'}) = 0$ for all k' and the limit is obviously zero. Assume $\delta = 1$ and $S = \varnothing$. Then, by Artin-Verdier duality,

$$H^3_{et}(X_{k'}) \cong \mu_p(k')^\vee.$$

For $k' \subset k'' \subset K$, the transition map

$$H^3_{et}(X_{k'}) \to H^3_{et}(X_{k''})$$

is the dual of the norm map $N_{k''|k'}\colon\ \mu_p(k'') \to \mu_p(k')$, hence the zero map if $k' \neq k''$. As $K|k$ is infinite, the limit vanishes. $\square$

4 Removing the h^2-defect

We start by extending the notions introduced before to infinite sets of primes S. Let k be a number field and S a set of nonarchimedean primes of k. We set $X = \mathit{Spec}(\mathcal{O}_k)$ and

$$X \smallsetminus S = \mathit{Spec}(\mathcal{O}_{k,S}),$$

which makes sense also if S is infinite. Let F be a sheaf on $X \smallsetminus S$ which comes by restriction from $X \smallsetminus T$ for some finite subset $T \subset S$. As each open subscheme of X is affine, we have

$$H^i_{et}(X \smallsetminus S, F) \cong \varinjlim_{\substack{T \subset S' \subset S \\ S' \text{ finite}}} H^i_{et}(X \smallsetminus S', F)$$

for all $i \geq 0$.

We fix a prime number p and put the running assumption that k is totally imaginary if $p = 2$. Hence we may ignore archimedean primes for cohomological considerations. The notion of being a $K(\pi,1)$ for p extends in an obvious manner to the case when S is infinite. Also the isomorphism

$$\text{Ш}^2(k,S) \xrightarrow{\sim} \text{Б}_S(k)$$

generalizes to infinite S by passing to the limit over all finite subsets $S' \subset S$. In particular, $Ш^2(k,S)$ is finite.

For the remainder of this paper, we assume that $S \cap S_p = \varnothing$. We also keep the running assumption $p \neq 2$ or k is totally imaginary.

> *For shorter notation, we drop p wherever possible. We write G_S instead of $G_S(p)$, k_S for $k_S(p)$, and so on. Unless mentioned otherwise, all cohomology groups are taken with values in $\mathbb{Z}/p\mathbb{Z}$. We keep this notational convention for the rest of this paper.*

If $\mathfrak{p} \nmid p$ is a prime with $\mu_p \not\subset k_\mathfrak{p}$, then every p-extension of the local field $k_\mathfrak{p}$ is unramified (see [NSW], Proposition 7.5.1). Therefore primes $\mathfrak{p} \notin S_p$ with $N(\mathfrak{p}) \not\equiv 1 \bmod p$ cannot ramify in a p-extension. Removing all these redundant primes from S, we obtain a subset $S_{\min} \subset S$ which has the property that $G_S = G_{S_{\min}}$. Moreover, we have the

Lemma 4.1. *The natural map*

$$(X \smallsetminus S)^{(p)}_{et} \longrightarrow (X \smallsetminus S_{\min})^{(p)}_{et}$$

is a weak homotopy equivalence.

Proof. By [AM], (4.3), it suffices to show that for every discrete p-primary G_S-module M the natural maps $H^i_{et}(X \smallsetminus S_{\min}, M) \to H^i_{et}(X \smallsetminus S, M)$ are isomorphisms for all i. By the same argument, as in the proof of Proposition 2.1, (iv)⇒(v), we may suppose that $M = \mathbb{Z}/p\mathbb{Z}$. Using the excision sequence, it therefore suffices to show that the group $H^i_\mathfrak{p}(X \smallsetminus S_{\min}, \mathbb{Z}/p\mathbb{Z})$ vanishes for all $\mathfrak{p} \in S \smallsetminus S_{\min}$. This follows from Proposition 3.1. □

Therefore we can replace S by $S_{\min}$ and make the following notational convention for the rest of this paper.

> *The word 'prime' means nonarchimedean prime with norm $\equiv 1 \bmod p$.*

At this point it is useful to redefine the notion of Dirichlet density.

Definition. *Let S be a set of primes of k (of norm $\equiv 1 \bmod p$). The p-density $\Delta^p(S)$ is defined by*

$$\Delta^p(S) = \delta_{k(\mu_p)}\big(S(k(\mu_p))\big),$$

where $S(k(\mu_p))$ is the set of prolongations of primes in S to $k(\mu_p)$ and $\delta_{k(\mu_p)}$ denotes the Dirichlet density on the level $k(\mu_p)$. In other words,

$$\Delta^p(S) = d \cdot \delta_k(S), \text{ where } d = [k(\mu_p) : k].$$

The set of all primes (of norm $\equiv 1 \bmod p$) has p-density equal to 1.

Proposition 4.2. *Let S be a set of primes of p-density $\Delta^p(S)=1$. Then there exists a finite subset $T\subset S$ with $Б_T(k)=0$. In particular, $Б_S(k)=0=$ Ш$^2(k,S)$.*

Proof. By the Hasse principle for the module μ_p, see [NSW], Thm. 9.1.3 (ii), and Kummer theory, the natural map

$$k^\times/k^{\times p}\longrightarrow\prod_{\mathfrak{p}\in S}k_\mathfrak{p}^\times/k_\mathfrak{p}^{\times p}$$

is injective, hence $V_S(k)=0$. Furthermore $V_\varnothing(k)$ is finite. Choosing to each nonzero element α of $V_\varnothing(k)$ a prime $\mathfrak{p}\in S$ with $\alpha\notin k_\mathfrak{p}^{\times p}$, we obtain a finite subset $T\subset S$ with $V_T(k)=0$. □

Theorem 4.3. *Let k be a number field and let S be a set of primes of k of p-density $\Delta^p(S)=1$. Then $X\smallsetminus S$ is a $K(\pi,1)$ for p.*

Proof. Let $T\subset S$ be a finite subset. By [NSW], Thm. 9.2.2 (ii), the natural map

$$H^1_{et}(X\smallsetminus(S\cup S_p))\longrightarrow\prod_{\mathfrak{p}\in T\cup S_p}H^1(k_\mathfrak{p})$$

is surjective. A class in $H^1_{et}(X\smallsetminus(S\cup S_p))$ which maps to zero in $H^1(k_\mathfrak{p})$ for all $\mathfrak{p}\mid p$ is contained in $H^1_{et}(X\smallsetminus S)$. Therefore, also the map

$$H^1_{et}(X\smallsetminus S)\longrightarrow\prod_{\mathfrak{p}\in T}H^1(k_\mathfrak{p})$$

is surjective. Hence the maximal elementary abelian extension of k in k_S realizes the maximal elementary abelian extension of $k_\mathfrak{p}$ in $k_\mathfrak{p}(p)$ for all $\mathfrak{p}\in S$. Applying the same argument to each finite subextension of k in k_S, we conclude that k_S realizes $k_\mathfrak{p}(p)$ for all $\mathfrak{p}\in S$. In particular,

$$\prod_{\mathfrak{p}\in S(k_S)}H^2((k_S)_\mathfrak{p})=0.$$

Furthermore, by Proposition 4.2, Ш$^2(K,S(K))=0$ for all finite subextensions K of k in k_S. We obtain

$$H^2_{et}((X\smallsetminus S)_{k_S})=0.$$

As there is no cohomology in dimension greater or equal 3, condition (iii) of Proposition 2.1 is satisfied. □

In order to proceed, we make the following definitions.

Definition. *Let S be a finite set of primes (of norm $\equiv 1 \bmod p$).*

(i) *We say that S is p-large if $Б_S(k,p)=0$.*

(ii) *We put*

$$\delta^2_S(k) = h^2(X \smallsetminus S) - h^2(G_S)$$

and call this number the h^2-defect of S (with respect to p).

(iii) *We denote by k_S^{el} the maximal elementary abelian p-extension of k inside k_S.*

If S is p-large, then $\text{Ш}^2(k,S)=0$, and so, for any set $T \supset S$, the natural maps $H^2(G_S) \to H^2(G_T)$ and $H^2_{et}(X \smallsetminus S) \to H^2_{et}(X \smallsetminus T)$ are injective.

Lemma 4.4. *Let S be p-large and let $\mathfrak{p}$ be a prime (of norm $\equiv 1 \bmod p$) which does not split completely in $k_S^{el}|k$. Then*

$$\delta^2_{S\cup\{\mathfrak{p}\}}(k) \le \delta^2_S(k).$$

Furthermore, the natural map $H^2(G_{S\cup\{\mathfrak{p}\}}) \longrightarrow H^2(k_\mathfrak{p})$ is surjective.

Proof. Put $S' = S \cup \{\mathfrak{p}\}$. By Theorem 3.4, the extension $k_{S'}^{el}|k$ is ramified at $\mathfrak{p}$. Therefore $k_{S'}^{el}$ realizes the maximal elementary abelian p-extension $k_\mathfrak{p}^{el}$ of the local field $k_\mathfrak{p}$, i.e. the map

$$H^1(G_{S'}(p)) \longrightarrow H^1(k_\mathfrak{p})$$

is surjective. As the cup-product $H^1(k_\mathfrak{p}) \times H^1(k_\mathfrak{p}) \to H^2(k_\mathfrak{p})$ is surjective, the natural map

$$H^2(G_{S'}) \longrightarrow H^2(k_\mathfrak{p})$$

is also surjective. The statement of the lemma now follows from the commutative and exact diagram

$$\begin{tikzcd}
0 \arrow[r] & H^2(G_S) \arrow[r] \arrow[d, hook] & H^2_{et}(X \smallsetminus S) \arrow[d, hook] \\
0 \arrow[r] & H^2(G_{S'}) \arrow[r] & H^2_{et}(X \smallsetminus S') \arrow[d] \\
 & & H^2(k_\mathfrak{p}).
\end{tikzcd}$$

□

Lemma 4.5. *Let S be p-large and let $\mathfrak{p}$ be a prime. Let T be a set of primes of p-density $\Delta^p(T)=1$. Then there exists a prime $\mathfrak{p}' \in T$ such that*

(i) *$\mathfrak{p}'$ does not split completely in $k_S^{el}|k$.*

(ii) *$\mathfrak{p}$ does not split completely in $k_{S\cup\{\mathfrak{p}'\}}^{el}|k$.*

In particular, $\delta^2_{S\cup\{\mathfrak{p},\mathfrak{p}'\}}(k) \le \delta^2_S(k)$.

Proof. If $\mathfrak{p}$ does not split completely in $k_S^{el}|k$, then condition (ii) is void. By assumption, $\varDelta^p(T)=\delta_{k(\mu_p)}(T(k(\mu_p)))=1$. By Čebotarev's density theorem, we can find a prime $\mathfrak{P}'\in T(k(\mu_p))$ which does not split completely in $k_S^{el}(\mu_p)|k(\mu_p)$. Then $\mathfrak{p}'=\mathfrak{P}'|_k$ satisfies (i). Therefore we may assume that $\mathfrak{p}$ splits completely in $k_S^{el}|k$. By class field theory, there exists an $s\in k^\times$ with

(a) $v_\mathfrak{p}(s)\equiv 1 \bmod p$,
(b) $v_\mathfrak{q}(s)\equiv 0 \bmod p$ for all $\mathfrak{q}\notin S$, $\mathfrak{q}\neq\mathfrak{p}$, and
(c) $s\in k_\mathfrak{q}^{\times p}$ for all $\mathfrak{q}\in S$.

Since S is p-large, s is well-defined as an element in $k^\times/k^{\times p}$. Now consider the extensions $k(\mu_p,\sqrt[p]{s})$ and $k_{S\cup\{\mathfrak{p}\}}^{el}(\mu_p)$ of $k(\mu_p)$. The first one might be contained in the second (only if $\zeta_p\in k$) but this does not matter. Using Čebotarev's density theorem, we find $\mathfrak{P}'\in T(k(\mu_p))$ such that $Frob_{\mathfrak{P}'}$ is non-zero in $Gal(k(\mu_p,\sqrt[p]{s})|k(\mu_p))$ and non-zero in $Gal(k_{S\cup\{\mathfrak{p}\}}^{el}(\mu_p)|k(\mu_p))$. We put $\mathfrak{p}'=\mathfrak{P}'|_k$. Then $\mathfrak{p}'$ does not split completely in $k_S^{el}|k$ and $s\notin k_{\mathfrak{p}'}^{\times p}=k(\mu_p)_{\mathfrak{P}'}^{\times p}$. We claim that $\mathfrak{p}$ does not split completely in $k_{S\cup\{\mathfrak{p}'\}}^{el}|k$: Otherwise there would exist a $t\in k^\times$ satisfying conditions (a) – (c) above and with $t\in k_{\mathfrak{p}'}^{\times p}$. Since $s/t\in Б_S(k)=0$, we obtain $s/t\in k^{\times p}$. Hence $s\in k_{\mathfrak{p}'}^{\times p}$ giving a contradiction. Hence condition (i) and (ii) are satisfied. $\square$

Lemma 4.6. *Let S be a finite set of primes and let T be a set of primes of p-density $\varDelta^p(T)=1$. Then there exists a finite subset $T_1\subset T$ such that $S\cup T_1$ is p-large and such that the natural inclusion*

$$H^2(G_{S\cup T_1}(k))\hookrightarrow H^2_{et}(X\smallsetminus(S\cup T_1))$$

is an isomorphism.

Proof. We first move finitely many primes from T to S to achieve that S is p-large. If $\delta_S^2(k)$ is zero, we are ready. Otherwise, consider the commutative diagram

$$\begin{array}{ccc} H^2(G_S) & \hookrightarrow & H^2(G_{S\cup T}) \\ \big\downarrow & & \big\downarrow\wr \\ H^2_{et}(X\smallsetminus S) & \hookrightarrow & H^2_{et}(X\smallsetminus(S\cup T)) \end{array}$$

in which the right hand isomorphism follows from Theorem 4.3. Let $x\in H^2_{et}(X\smallsetminus S)$ but $x\notin H^2(G_S)$. Then there exists a finite subset $T_0\subset T$ such that $x\in H^2(G_{S\cup T_0})$. Let $T_0=\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$. We choose $\mathfrak{p}_1',\dots,\mathfrak{p}_n'\in T$ according to Lemma 4.5 and put $T_1=\{\mathfrak{p}_1,\dots,\mathfrak{p}_n,\mathfrak{p}_1',\dots,\mathfrak{p}_n'\}$. Then the natural map

$$H^2(G_{S\cup T_1})\xrightarrow{\phi}\prod_{i=1}^n H^2(k_{\mathfrak{p}_i})\times\prod_{i=1}^n H^2(k_{\mathfrak{p}_i'})$$

is surjective. We have $H^2(G_S)\subset\ker(\phi)$ and also $x\in\ker(\phi)$. Hence $\delta^2_{S\cup T_1}(k)<\delta_S^2(k)$. Iterating this process, we obtain a set with trivial h^2-defect. $\square$

5 Review of mild pro-p-groups

In the following we recall definitions and results from J. Labute's paper [La]. Only interested in our application, we are slightly less general than Labute.

Let R be a principal ideal domain and let L be the free R-Lie algebra over $\xi_1, \ldots, \xi_n$, $n \geq 1$. We view L as graded algebra where the degree of ξ_i is 1. Let $\rho_1, \ldots, \rho_m$, $m \geq 1$, be homogeneous elements in L with ρ_i of degree h_i and let $\mathfrak{r} = (\rho_1, \ldots, \rho_m)$ be the ideal of L generated by $\rho_1, \ldots, \rho_m$. Let $\mathfrak{g} = L/\mathfrak{r}$ and $U_\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$. Then $M = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a $U_\mathfrak{g}$-module via the adjoint representation.

Definition. *The sequence $\rho_1, \ldots, \rho_m$ is called* strongly free *if $U_\mathfrak{g}$ is a free R-module and $M = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is the free $U_\mathfrak{g}$-module on the images of $\rho_1, \ldots, \rho_m$ in M.*

Let us consider the special case when $R = k[\pi]$ is the polynomial ring in one variable π over a field k. Then $\bar{L} = L/\pi$ is a free k-Lie algebra and the next theorem shows that we can detect strong freeness by reduction. We denote the image in $\bar{L}$ of an element $\rho \in L$ by $\bar{\rho}$.

Theorem 5.1. ([La], Th. 3.10) *The sequence $\rho_1, \ldots, \rho_m$ in L is strongly free if and only if the sequence $\bar{\rho}_1, \ldots, \bar{\rho}_m$ is strongly free in $\bar{L}$.*

Over fields, we have the following criterion for strong freeness. Let $R = k$ be a field, X a finite set and $S \subset X$ a subset. Let $L(X)$ be the free Lie algebra over X and let $\mathfrak{a}$ be the ideal of $L(X)$ generated by the elements $\xi \in X \smallsetminus S$. Put

$$T = \{\ [\xi, \xi'] \mid \xi \in X \smallsetminus S,\ \xi' \in S\} \subset \mathfrak{a}.$$

Theorem 5.2. ([La], Th. 3.3, Cor. 3.5) *If $\rho_1, \ldots, \rho_m$ are homogeneous elements of $\mathfrak{a}$ which lie in the k-span of T modulo $[\mathfrak{a}, \mathfrak{a}]$ and which are linearly independent over k modulo $[\mathfrak{a}, \mathfrak{a}]$ then the sequence $\rho_1, \ldots, \rho_m$ is strongly free in L.*

Let p be an odd prime number and let G be a pro-p-group. We consider the descending p-central series $(G_n)_{n \geq 1}$, which is defined recursively by

$$G_1 = G,\ G_{n+1} = G_n^p[G, G_n].$$

The quotients $\mathrm{gr}_n(G) = G_n/G_{n+1}$, denoted additively, are $\mathbb{F}_p$-vector spaces. The graded vector space

$$\mathrm{gr}(G) = \bigoplus_{n \geq 1} \mathrm{gr}_n(G)$$

has a Lie algebra structure over the polynomial ring $\mathbb{F}_p[\pi]$, where multiplication by π is induced by $x \mapsto x^p$ and the bracket operation for homogeneous elements is induced by the commutator operation in G, see [NSW], III, §8. For $g \in G$, $g \neq 1$, let the natural number $h(g)$ be defined by

$$g \in G_{h(G)},\ g \notin G_{h(G)+1}.$$

This definition makes sense because $\bigcap_n G_n = \{1\}$, see [NSW], Prop. 3.8.2. The image $\omega(g)$ of g in $\mathrm{gr}_{h(g)}(G)$ is called the *initial form* of g.

Let F be the free pro-p-group over elements $x_1,\dots,x_n$, $n \geq 1$. Then $h(x_i) = 1$, $i = 1,\dots,n$, and

$$L = \mathrm{gr}(F)$$

is the free $\mathbb{F}_p[\pi]$-Lie algebra over $\xi_1,\dots,\xi_n$, where $\xi_i = \omega(x_i)$, $i = 1,\dots,n$, see [Lz]. Let $r_1,\dots,r_m$, $m \geq 1$, be a sequence of elements in $F_2 = F^p[F,F] \subset F$ and let $R = (r_1,\dots,r_m)_F$ be the closed normal subgroup of F generated by $r_1,\dots,r_m$. Put $\rho_i = \omega(r_i) \in L$.

Definition. *A pro-p-group G is called* mild *if there exists a presentation*

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

with F a free pro-p-group on generators $x_1,\dots,x_n$ and $R = (r_1,\dots,r_m)_F$ such that the associated sequence $\rho_1,\dots,\rho_m$ is strongly free in $L = gr(F)$.

Essential for our application is the following property of mild pro-p-groups.

Theorem 5.3. ([La], Th. 1.2(c)) *If G is a mild pro-p-group, then $cd\, G = 2$.*

Now let G be a finitely presented pro-p-group and let

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

be a minimal presentation, i.e. F is the free pro-p-group on generators $x_1,\dots,x_n$ where $n = \dim_{\mathbb{F}_p} H^1(G)$ and $R = (r_1,\dots,r_m)_F$ with $m = \dim_{\mathbb{F}_p} H^2(G)$, cf. [NSW], (3.9.5). Then the images $\xi_i = \omega(x_i)$, $i = 1,\dots,n$, of $x_1,\dots,x_n$ are a basis of the $\mathbb{F}_p$-vector space $F/F_2 = H_1(F) = H_1(G) = G/G_2$. For $y \in F_n$ and $a \in \mathbb{Z}_p$ the class of y^a modulo F_{n+1} only depends on the residue class $\bar a \in \mathbb{F}_p$ of a. Every element $r \in R \subset F_2$ has a representation

$$r \equiv \prod_{j=1}^{n} (x_j^p)^{a_j} \cdot \prod_{1\leq k<l\leq n} [x_k, x_l]^{a_{kl}} \mod F_3,$$

where $a_j, a_{k,l} \in \mathbb{F}_p$. These coefficients are uniquely defined and can be calculated as follows. As F is free, we have an isomorphism $H_2(G) = R_G^{ab}/p$. Let $\bar r \in H_2(G)$ be the image of r and let $\chi_1,\dots,\chi_n \in H^1(G)$ be the dual $\mathbb{F}_p$-basis of $\xi_1,\dots,\xi_n$.

Theorem 5.4. $a_{kl} = -\bar r(\chi_k \cup \chi_l)$ *for $k < l$.*

For a proof see [NSW], Prop. 3.9.13, which also gives a description of the a_j using the Bockstein operator.

Using the results above, we obtain a criterion for mildness.

THEOREM 5.5. *Let G be a finitely presented pro-p-group. Assume there exists a basis $\chi_1, \ldots, \chi_n$ of $H^1(G)$, a basis $\bar{r}_1, \ldots, \bar{r}_m$ of $H_2(G)$ and a natural number a, $1 \le a < n$, such that the following conditions are satisfied*

(i) $\bar{r}_i(\chi_k \cup \chi_l) = 0$ *for* $a < k < l \le n$ *and* $i = 1, \ldots, m$.

(ii) *The* $m \times a(n-a)$ *matrix*

$$\Big(\bar{r}_i(\chi_k \cup \chi_l)\Big)_{i,(k,l)}, \ 1 \le i \le m, \ 1 \le k \le a < l \le n$$

has rank m.

Then G is a mild pro-p-group.

Proof. Let $\xi_1, \ldots, \xi_n \in H_1(G)$ be the dual basis of $\chi_1, \ldots, \chi_n$. We choose a minimal presentation

$$(*) \qquad 1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

and generators $x_1, \ldots, x_n \in F$ mapping to $\xi_1, \ldots, \xi_n \in H_1(F) = H_1(G)$. Then we choose elements $r_1, \ldots, r_m \in R$ mapping to $\bar{r}_1, \ldots, \bar{r}_m \in R_G^{ab}/p = H_2(G)$. Let $X = \{\xi_1, \ldots, \xi_n\}$. Then $L = \mathrm{gr}(F)$ is the free $\mathbb{F}_p[\pi]$-Lie algebra over X and $\bar{L} = L/\pi$ is the free $\mathbb{F}_p$-Lie algebra over X. In order to show that G is mild, we have to show that the initial forms $\rho_1, \ldots, \rho_m$ of $r_1, \ldots, r_m$ are a strongly free sequence in L. By Theorem 5.1 it suffices to show that $\bar{\rho}_1, \ldots, \bar{\rho}_m \subset \bar{L}$ are a strongly free sequence. By condition (ii) and Theorem 5.4, we have $\bar{\rho}_1, \ldots, \bar{\rho}_m \in \mathrm{gr}_2(\bar{L}) = F_2/F_3F^p$.
Now we apply Theorem 5.2 with $S = \{\xi_{a+1}, \ldots, \xi_n\} \subset X$. In the notation of this theorem, $\mathfrak{a}$ is the ideal generated by $\xi_1, \ldots, \xi_a$ in $\bar{L}$ and

$$T = \{[\xi_i, \xi_j] \mid 1 \le i \le a, \ a+1 \le j \le n\}.$$

By condition (i) and Theorem 5.4, we have $\bar{\rho}_i$ in the $\mathbb{F}_p$-span H of T modulo $[\mathfrak{a}, \mathfrak{a}]$. The elements of T are a basis of H and the coefficient matrix of $\bar{\rho}_1, \ldots, \bar{\rho}_m$ is up to sign the transpose of the matrix written in condition (ii). Hence $\bar{\rho}_1, \ldots, \bar{\rho}_m$ are linearly independent and, by Theorem 5.2, a strongly free sequence. This concludes the proof. □

6 EXISTENCE OF $K(\pi,1)$'S

Let k be a number field and let p be a prime number with $\mu_p \not\subset k$ and assume that $Cl(k)(p) = 0$. The exact sequence

$$0 \longrightarrow \mathcal{O}_k^\times \longrightarrow k^\times \xrightarrow{(v_\mathfrak{q})_\mathfrak{q}} \bigoplus_\mathfrak{q} \mathbb{Z} \longrightarrow Cl(k) \longrightarrow 0$$

implies the exactness of

$$0 \longrightarrow \mathcal{O}_k^\times/p \longrightarrow k^\times/k^{\times p} \longrightarrow \bigoplus_\mathfrak{q} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0.$$

Let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ be a finite set of primes of norm $\equiv 1 \bmod p$. We choose for $i = 1, \ldots, m$ elements $s_i \in k^\times/k^{\times p}$ with $v_{\mathfrak{p}_i}(s_i) \equiv 1 \bmod p$ and $v_{\mathfrak{q}}(s_i) \equiv 0 \bmod p$ for all primes $\mathfrak{q} \neq \mathfrak{p}_i$ of k. Let furthermore, $e_1, \ldots, e_r$, $r = r_1 + r_2 - 1$, be a basis of $\mathcal{O}_k^\times/p$.

Consider the field

$$K = k(\mu_p, \sqrt[p]{s_1}, \ldots, \sqrt[p]{s_m}, \sqrt[p]{e_1}, \ldots, \sqrt[p]{e_r}).$$

An inspection of the ramification behaviour shows that $Gal(K|k(\mu_p))$ has the Galois group $(\mathbb{Z}/p\mathbb{Z})^{m+r}$: Indeed, $k(\mu_p, \sqrt[p]{e_1}, \ldots, \sqrt[p]{e_r})|k(\mu_p)$ is unramified outside S_p and has Galois group $(\mathbb{Z}/p\mathbb{Z})^r$ by Kummer theory. Adjoining $\sqrt[p]{s_i}$, $i = 1, \ldots, m$, yields a cyclic extension of degree p which is unramified outside $S_p \cup \{\mathfrak{p}_i\}$ and ramified at $\mathfrak{p}_i$.
Since $\mu_p \not\subset k$, the extensions $k_S^{el}(\mu_p)|k(\mu_p)$ and $K|k(\mu_p)$ lie in different eigenspaces for the action of $Gal(k(\mu_p)|k)$. Therefore $Kk_S^{el}|k(\mu_p)$ has Galois group $(\mathbb{Z}/p\mathbb{Z})^{m+r+n}$, with $n = \dim_{\mathbb{F}_p} Gal(k_S^{el}|k) = \dim_{\mathbb{F}_p} H^1(G_S)$.

Assume now that we are given

- a set of primes T of k with $T \cap S = \varnothing$ and with p-density $\Delta_k^p(T) = 1$,
- a nonzero element $F \in Gal(k_S^{el}|k) = Gal(k_S^{el}(\mu_p)|k(\mu_p))$,
- to each $\mathfrak{p}_i$, $i = 1, \ldots, m$, a condition C_i which says “split” or “inert”.

By Čebotarev's density theorem applied to the extension $Kk_S^{el}(\mu_p)|k(\mu_p)$, we find a prime $\mathfrak{P} \in T(Kk_S^{el}(\mu_p))$ such that

- the image of $Frob_{\mathfrak{P}}$ in $Gal(k(\mu_p, \sqrt[p]{e_1}, \ldots, \sqrt[p]{e_r})|k(\mu_p))$ is trivial,
- the image of $Frob_{\mathfrak{P}}$ in $Gal(k(\mu_p, \sqrt[p]{s_i})|k(\mu_p))$ is trivial if C_i is “split” and nontrivial otherwise, and
- the image of $Frob_{\mathfrak{P}}$ in $Gal(k_S^{el}(\mu_p)|k(\mu_p))$ is equal to F.

Let $\mathfrak{p} \in T$ be the restriction of $\mathfrak{P}$ to k. Then the natural map $\mathcal{O}_k^\times/p \to k_{\mathfrak{p}}^\times/k_{\mathfrak{p}}^{\times p}$ is the zero map. Since ${}_pCl(k) = 0$, we obtain $\mathcal{O}_k^\times/p \xrightarrow{\sim} V_\varnothing(k) = V_{\{\mathfrak{p}\}}(k)$. By Theorem 3.4, $k_{\{\mathfrak{p}\}}^{el}|k$ is cyclic of order p and $\mathfrak{p}$ is ramified in this extension. Recall that $H^1_{nr}(G_{\mathfrak{p}})$ is defined as the exact annihilator of the inertia group $T_{\mathfrak{p}}(k_{\mathfrak{p}}^{el}|k_{\mathfrak{p}}) \subset H_1(G_{\mathfrak{p}})$ in the natural pairing

$$H_1(G_{\mathfrak{p}}) \times H^1(G_{\mathfrak{p}}) \longrightarrow \mathbb{F}_p.$$

Dually, $T_{\mathfrak{p}}(k_{\mathfrak{p}}^{el}|k_{\mathfrak{p}})$ is the exact annihilator of $H^1_{nr}(G_{\mathfrak{p}})$. The equation $T_{\mathfrak{p}}(k_{\{\mathfrak{p}\}}^{el}|k) = Gal(k_{\{\mathfrak{p}\}}^{el}|k)$ yields an isomorphism

$$H^1(G_{\{\mathfrak{p}\}}) \xrightarrow{\sim} H^1(G_{\mathfrak{p}})/H^1_{nr}(G_{\mathfrak{p}}).$$

By class field theory, $\mathfrak{p}_i$ splits in $k_{\{\mathfrak{p}\}}^{el}|k$ if and only if there exists an element $s_i' \in k^\times/k^{\times p}$ with $v_{\mathfrak{p}_i}(s_i') \equiv 1 \bmod p$, $v_{\mathfrak{q}}(s_i') \equiv 0 \bmod p$ for all $\mathfrak{q} \neq \mathfrak{p}_i$ and $s_i' \in k_{\mathfrak{p}}^{\times p}$. Then s_i'/s_i lies in $\mathcal{O}_k^\times/p$, and therefore $s_i \in k_{\mathfrak{p}}^{\times p}$. Hence $\mathfrak{p}_i$ splits in $k^{el}_{\{\mathfrak{p}\}}|k$ if and only if s_i is a p-th power in $k_\mathfrak{p}$. On the other hand, by our choice of $\mathfrak{P}$, s_i is a p-th power in $k_\mathfrak{p}$ if and only if C_i is "split". Therefore the following holds:

- the natural map $\mathcal{O}_k^\times/p \to k_\mathfrak{p}^\times/k_\mathfrak{p}^{\times p}$ is the zero map,
- $Frob_\mathfrak{p} = F \in Gal(k_S^{el}|k)$,
- $k^{el}_{\{\mathfrak{p}\}}|k$ is cyclic of order p,
- each $\mathfrak{p}_i$, $i = 1, \ldots, m$, satisfies condition C_i in $k^{el}_{\{\mathfrak{p}\}}|k$.

Now assume that $\text{Б}_{S\smallsetminus\{\mathfrak{q}\}}(k) = 0$ for all $\mathfrak{q} \in S$, in particular, S is p-large. Then all $\mathfrak{p}_i \in S$ ramify in $k_S^{el}|k$ and the 1-dimensional subspaces $T_{\mathfrak{p}_i}(k_S^{el}|k)$, $i = 1, \ldots, m$, in $H_1(G_S)$ are pairwise different and generate $H_1(G_S)$. Furthermore assume that $\delta_S^2(k) = 0$. As $\mathfrak{p}$ does not split completely in $k_S^{el}|k$, Lemma 4.4 implies $\delta^2_{S\cup\{\mathfrak{p}\}}(k) = 0$. Since $\mu_p \not\subset k$ and by Theorem 3.4, the natural maps $H^2(G_S) \to \prod_{\mathfrak{q}\in S} H^2(G_\mathfrak{q})$ and $H^2(G_{S\cup\{\mathfrak{p}\}}) \to \prod_{\mathfrak{q}\in S\cup\{\mathfrak{p}\}} H^2(G_\mathfrak{q})$ are isomorphisms. We denote the $\mathfrak{q}$-component of a global cohomology class α by $\alpha_\mathfrak{q}$.

Next we fix a primitive p-th root of unity in $k(\mu_p)$ and to each $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ a prolongation to $k(\mu_p)$. After this choice we have identifications $\mu_p(k_\mathfrak{p}) = \mu_p((Kk_S^{el})_\mathfrak{P}) \cong \mathbb{F}_p$ and $\mu_p(k_{\mathfrak{p}_i}) \cong \mathbb{F}_p$, $i = 1, \ldots, m$. In particular, we have an isomorphism $H^2(G_\mathfrak{p}) = H^2(G_\mathfrak{p}, \mu_p) = \mathbb{F}_p$, and similarly for the $\mathfrak{p}_i$. Via these isomorphisms we consider the $\mathfrak{q}$-component $\alpha_\mathfrak{q}$ of a class $\alpha \in H^2(G_{S\cup\{\mathfrak{p}\}})$ as an element in $\mathbb{F}_p$. Let

$$\pi_\mathfrak{p} \in H^1(G_\mathfrak{p})/H^1_{nr}(G_\mathfrak{p}) = H^1(G_\mathfrak{p}, \mu_p)/H^1_{nr}(G_\mathfrak{p}, \mu_p) = k_\mathfrak{p}^\times/U_\mathfrak{p} k_\mathfrak{p}^{\times p}$$

be the image of a uniformizer and let $\chi_\mathfrak{p} \in H^1(G_{\{\mathfrak{p}\}})$ be the unique pre-image. We denote the image of $\chi_\mathfrak{p}$ in $H^1(G_{S\cup\{\mathfrak{p}\}})$ by the same letter. Thus $\chi_\mathfrak{p}$ maps to $\pi_\mathfrak{p}$ under the natural map $H^1(G_{S\cup\{\mathfrak{p}\}}) \to H^1(G_\mathfrak{p})/H^1_{nr}(G_\mathfrak{p})$. Consider the exact pairing

$$H^1_{nr}(G_\mathfrak{p}) \times H^1(G_\mathfrak{p})/H^1_{nr}(G_\mathfrak{p}) \to H^2(G_\mathfrak{p}) = \mathbb{F}_p,$$

which is induced by local Tate duality, see [NSW], Thm. 7.2.15. Let $\delta : k_\mathfrak{p}^\times/k_\mathfrak{p}^{\times p} \xrightarrow{\sim} H^1(G_\mathfrak{p})$ be the boundary isomorphism of the Kummer sequence and let $rec : k_\mathfrak{p}^\times/k_\mathfrak{p}^{\times p} \xrightarrow{\sim} H_1(G_\mathfrak{p})$ be the mod-p reciprocity map. Put $\varphi = rec \circ \delta^{-1}$. Then the image of $\chi_\mathfrak{p}$ under the composition

$$H^1(G_S) \longrightarrow H^1(G_\mathfrak{p}) \xrightarrow{\phi} H_1(G_\mathfrak{p}) \longrightarrow H_1(G_S)$$

is $Frob_\mathfrak{p}$, the Frobenius automorphism of the unramified prime $\mathfrak{p}$ in $k_S^{el}|k$. By [NSW], Prop. 7.2.13 [2]), the diagram

$$\begin{array}{ccccc}
H^1(G_\mathfrak{p}) & \times & H^1(G_\mathfrak{p}) & \xrightarrow{\cup} & H^2(G_\mathfrak{p}) \\
\Big\| & & \wr\big\downarrow\varphi & & \wr\big\downarrow inv \\
H^1(G_\mathfrak{p}) & \times & H_1(G_\mathfrak{p}) & \xrightarrow{can} & \mathbb{F}_p
\end{array}$$

[2]This proposition contains a sign error, see the errata file on the author's homepage.

commutes. We obtain for any $\chi \in H^1(G_S) \subset H^1(G_{S\cup\{\mathfrak{p}\}})$ the following formula for the $\mathfrak{p}$-component of $\chi \cup \chi_\mathfrak{p} \in H^2(G_{S\cup\{\mathfrak{p}\}})$:

$$(\chi \cup \chi_\mathfrak{p})_\mathfrak{p} = \chi(\mathit{Frob}_\mathfrak{p}).$$

The image of $\chi_\mathfrak{p}$ in $H^1(G_{\mathfrak{p}_i})$ obviously lies in the subgroup $H^1_{nr}(G_{\mathfrak{p}_i})$. By the same argument, noting that the cup-product is anti-symmetric, we obtain the equality

$$(\chi \cup \chi_\mathfrak{p})_{\mathfrak{p}_i} = -\chi_\mathfrak{p}(\mathit{Frob}_{\mathfrak{p}_i}),$$

for any $\chi \in H^1(G_S)$ mapping to $\pi_{\mathfrak{p}_i} \in H^1(G_{\mathfrak{p}_i})/H^1_{nr}(G_{\mathfrak{p}_i})$, where $\mathit{Frob}_{\mathfrak{p}_i}$ is the Frobenius automorphism of the unramified prime $\mathfrak{p}_i$ in $k^{el}_{\{\mathfrak{p}\}}|k$. As $\chi_\mathfrak{p}$ is unramified at $\mathfrak{p}_i$, the element $(\chi \cup \chi_\mathfrak{p})_{\mathfrak{p}_i}$ depends only on the image of χ in the one-dimensional $\mathbb{F}_p$-vector space $H^1(G_{\mathfrak{p}_i})/H^1_{nr}(G_{\mathfrak{p}_i})$. Since $\mathfrak{p}_i$ ramifies in $k^{el}_S|k$, the map $H^1(G_S) \to \mathbb{F}_p$, $\chi \mapsto (\chi \cup \chi_\mathfrak{p})_{\mathfrak{p}_i}$ is the linear form associated to an element $t_i \in T_{\mathfrak{p}_i}(k^{el}_S|k) \subset H_1(G_S)$.

Summing up and using the notation and choices above, we obtain the

LEMMA 6.1. *Let k be a number field and let p be a prime number with $\mu_p \not\subset k$ and $Cl(k)(p) = 0$. Let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ be a finite p-large set of primes and assume $\delta^2_S(k) = 0$. Let for $i = 1, \ldots, m$ elements $\varepsilon_i \in \{0,1\}$ and for $i = 1, \ldots, n$ elements $d_i \in \mathbb{F}_p$ be given, where not all d_i are zero. Let $\chi_1, \ldots, \chi_n$ be a basis of $H^1(G_S)$. Furthermore, let T be a set of primes of p-density $\Delta_p(T) = 1$ and with $T \cap S = \varnothing$.*
Then there exists a prime $\mathfrak{p} \in T$ such that the following conditions hold with respect to the identifications $H^2(G_{\mathfrak{p}_i}) = \mathbb{F}_p$, $i = 1, \ldots, m$, and $H^2(G_\mathfrak{p}) = \mathbb{F}_p$.

- *$\mathfrak{p}$ does not split completely in $k^{el}_S|k$,*
- *$k^{el}_{\{\mathfrak{p}\}}|k$ is cyclic of order p,*
- *$\chi_1, \ldots, \chi_n, \chi_\mathfrak{p}$ is a basis of $H^1(G_{S\cup\{\mathfrak{p}\}})$,*
- *$(\chi_i \cup \chi_\mathfrak{p})_\mathfrak{p} = d_i$ for $i = 1, \ldots, n$,*
- *For $i = 1, \ldots, m$ we have $c_i = 0$ if and only if $\varepsilon_i = 0$, where $c_i \in T_{\mathfrak{p}_i}(k^{el}_S|k) \subset H_1(G_S)$ represents the map $H^1(G_S) \to \mathbb{F}_p$, $\chi \mapsto (\chi \cup \chi_\mathfrak{p})_{\mathfrak{p}_i}$.*

Now we are able to prove the following result, which is unessentially sharper than Theorem 1 of the introduction.

THEOREM 6.2. *Let k be a number field and let p be a prime number such that*

$$\mu_p \not\subset k \quad \text{and} \quad Cl(k)(p) = 0.$$

Let S be a finite set of primes of k and let T be a set of primes of p-density $\Delta^p(T) = 1$. Then there exists a finite subset $T_1 \subset T$ such that $\mathit{Spec}(\mathcal{O}_k) \smallsetminus (S \cup T_1)$ is a $K(\pi,1)$ for p.

Proof. We may suppose that $T \cap S = \varnothing$. After moving finitely many primes of T to S, we may assume that the following conditions hold:

- $Б_{S\smallsetminus\{\mathfrak{p}\}}(k)=0$ for all $\mathfrak{p}\in S$,
- $\delta_S^2(k)=0$.

Now let $S=\{\mathfrak{p}_1,\dots,\mathfrak{p}_m\}$. Then $m=h^2(G_S)$. Let $n=m-r=h^1(G_S)$. We will achieve the $K(\pi,1)$-situation by adding m further primes to S.
We choose any basis $\chi_1,\dots,\chi_n$ of $H^1(G_S)$. Let $t_1,\dots,t_m$ be generators of the inertia groups $T_{\mathfrak{p}_i}(k_S^{el}|k)\subset H_1(G_S)$. Now we add a prime $\mathfrak{p}_{m+1}$ in the following way:
Let $i_1\in\{1,\dots,n\}$ be an index such that $\chi_{i_1}(t_1)\neq 0$, and let $i_1'\in\{1,\dots,n\}$, $i_1'\neq i_1$, be any other index. Now, according to Lemma 6.1, we put the conditions

$$\varepsilon_1=1 \text{ and } \varepsilon_i=0 \text{ for } i\in\{2,\dots,m\},$$

$$d_{i_1'}=1 \text{ and } d_i=0 \text{ for } i\in\{1,\dots,n\},\ i\neq i_1'$$

to choose a prime $\mathfrak{p}_{m+1}\in T$ such that for $i=1,\dots,n$

$$(\chi_i\cup\chi_{\mathfrak{p}_{m+1}})_{\mathfrak{p}_1}=\lambda_1\chi_i(t_1),\lambda_1\in\mathbb{F}_p^\times,\ (\chi_i\cup\chi_{\mathfrak{p}_{m+1}})_{\mathfrak{p}_j}=0,\ j=2,\dots,m$$

$$\text{and } (\chi_i\cup\chi_{\mathfrak{p}_{m+1}})_{\mathfrak{p}_{m+1}}=d_i.$$

Then in the matrix

$$\Big((\chi_i\cup\chi_{\mathfrak{p}_{m+1}})_{\mathfrak{p}_j}\Big)_{\substack{i=1,\dots,n\\ j=1,\dots,m+1}}$$

the i_1-line has entry $\neq 0$ at $(i_1,1)$ and all other entries zero, while the i_1'-line has some entry at $(i_1',1)$, the entry 1 at $(i_1',m+1)$ and all other entries zero.
In order to proceed, we put $\chi_{n+1}=\chi_{\mathfrak{p}_{m+1}}$ and choose an index $i_2\in\{1,\dots,n\}$ with $\chi_{i_2}(t_2)\neq 0$ and any $i_2'\in\{1,\dots,n\}$ with $i_2'\neq i_2$. We choose conditions as before, completed by $\varepsilon_{m+1}=0$ and $d_{n+1}=0$. Then we choose $\mathfrak{p}_{m+2}$ according to Lemma 6.1 and such that in the matrix

$$\Big((\chi_i\cup\chi_{\mathfrak{p}_{m+2}})_{\mathfrak{p}_j}\Big)_{\substack{i=1,\dots,n\\ j=1,\dots,m+2}}$$

the i_2-line has entry $\neq 0$ at $(i_2,2)$ and all other entries zero, while the i_2'-line has some entry at $(i_2',2)$, the entry 1 at $(i_2',m+2)$ and all other entries zero.
In addition, our choice implies

$$(\chi_{\mathfrak{p}_{m+1}}\cup\chi_{\mathfrak{p}_{m+2}})_{\mathfrak{p}_{m+1}}=0=(\chi_{\mathfrak{p}_{m+1}}\cup\chi_{\mathfrak{p}_{m+2}})_{\mathfrak{p}_{m+2}}$$

As $\chi_{\mathfrak{p}_{m+1}}$ and $\chi_{\mathfrak{p}_{m+2}}$ are unramified at $\mathfrak{p}_1,\dots,\mathfrak{p}_m$ by construction, we have furthermore $(\chi_{\mathfrak{p}_{m+1}}\cup\chi_{\mathfrak{p}_{m+2}})_{\mathfrak{p}_i}=0$ for $i=1,\dots,m$.

Now we proceed to construct $\mathfrak{p}_{m+3},\dots,\mathfrak{p}_{2m}$ in a similar way, and apply Theorem 5.5 with $a=m$. For each j, the j-th of the m-steps in the construction produced the two lines $((i_j,j),-)$ and $((i_j',j),-)$ in the $nm\times 2m$-matrix

$$\Big((\chi_i\cup\chi_{\mathfrak{p}_j})_{\mathfrak{p}_k}\Big)_{\substack{i=1,\dots,n,j=m+1,\dots,2m\\ k=1,\dots,2m}}$$

According to our choices these $2m$ lines are linearly independent, hence the matrix has rank $2m$. Putting $T_1=\{\mathfrak{p}_{m+1},\ldots,\mathfrak{p}_{2m}\}$, we conclude by Theorem 5.5 that $G_{S\cup T_1}$ is a mild pro-p-group. Hence $cd\, G_{S\cup T_1}=2$ by Theorem 5.3. By Lemma 4.4, we didn't produce new h^2-defect during our construction, hence $\delta^2_{S\cup T_1}(k)=0$. As the étale cohomology is trivial in dimension ≥ 3, we conclude that the homomorphisms

$$\phi_i : H^i(G_{S\cup T_1},\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^i_{et}(Spec(\mathcal{O}_k)\smallsetminus(S\cup T_1),\mathbb{Z}/p\mathbb{Z})$$

are isomorphisms for all $i\geq 0$. Hence condition (v) of Proposition 2.1 is satisfied. $\square$

7 Consequences of the $K(\pi,1)$-property

In this section we assume that S is finite and we exclude the case $S=\varnothing$ from our considerations. Keeping all conventions made before, we assume

$p\neq 2$ or k is totally imaginary and S is a non-empty finite set of nonarchimedean primes $\mathfrak{p}$ with norm $N(\mathfrak{p})\equiv 1 \bmod p$.

Lemma 7.1. *G_S is a fab-group, i.e. the abelianization U^{ab} of every open subgroup U of G_S is finite.*

Proof. Let $U\subset G_S$ be an open subgroup. The abelianization U^{ab} of U is a finitely generated abelian pro-p-group. If U^{ab} were infinite, it would have a quotient isomorphic to $\mathbb{Z}_p$, which by Galois theory corresponds to a $\mathbb{Z}_p$-extension K_∞ of the number field $K=k_S^U$ inside k_S. By [NSW], Theorem 10.3.20 (ii), a $\mathbb{Z}_p$-extension of a number field is ramified at at least one prime dividing p. This contradicts $K_\infty\subset k_S$ and we conclude that U^{ab} is finite. $\square$

The group theoretical structure of the local Galois groups is well known.

Proposition 7.2. *Let $\mathfrak{p}\in S$. Then $Gal(k_\mathfrak{p}(p)|k_\mathfrak{p})$ is the pro-p-group on two generators σ,τ subject to the relation $\sigma\tau\sigma^{-1}=\tau^q$. The element τ is a generator of the inertia group, σ is a Frobenius lift and $q=N(\mathfrak{p})$.*

Proof. This follows from [NSW], Thm. 7.5.2 by passing to the maximal pro-p-factor group. $\square$

We obtain the following corollary.

Corollary 7.3. *Assume that G_S is infinite. Then, for each $\mathfrak{p}\in S$, the decomposition group $G_\mathfrak{p}$ of $\mathfrak{p}$ in G_S has infinite index.*

Proof. The decomposition group $G_\mathfrak{p}$ is a quotient of the local Galois group $Gal(k_\mathfrak{p}(p)|k_\mathfrak{p})$. If $G_\mathfrak{p}\subset G_S$ would have finite index, it would be an infinite fab-group by Lemma 7.1. By Proposition 7.2, each infinite quotient of $Gal(k_\mathfrak{p}(p)|k_\mathfrak{p})$ has a surjection to $\mathbb{Z}_p$ and is therefore not a fab-group. This contradiction shows that $G_\mathfrak{p}$ has infinite index in G_S. $\square$

The next proposition classifies the degenerate $K(\pi,1)$-case.

Proposition 7.4. *$X \smallsetminus S$ is a $K(\pi,1)$ and $G_S = 1$ if and only if $S = \{\mathfrak{p}\}$ consists of a single prime and one of the following cases occurs.*

(a) *$p = 2$, $k \neq \mathbb{Q}(\sqrt{-1})$ is imaginary quadratic, $2 \nmid h_k$ and $N(\mathfrak{p}) \not\equiv 1 \bmod 4$,*

(b) *$p = 2$, $k = \mathbb{Q}(\sqrt{-1})$ and $N(\mathfrak{p}) \not\equiv 1 \bmod 8$,*

(c) *$p = 3$, $k = (\mathbb{Q}\sqrt{-3})$ and $N(\mathfrak{p}) \not\equiv 1 \bmod 9$.*

Proof. Assume $G_S = 1$ and that $X \smallsetminus S$ is a $K(\pi,1)$. Then $H^i_{et}(X \smallsetminus S) = 0$ for all $i \geq 1$. In particular, $p \nmid h_k$. By Theorem 3.4, $h^2(X \smallsetminus S) = 0$ implies $\delta = 1$, $\#S = 1$ and $V_S = 0$. Then, using $h^1(X \smallsetminus S) = 0$, we obtain $r = 1$. As $\delta = 1$, the following possibilities remain

(a) $p = 2$, $k \neq \mathbb{Q}(\sqrt{-1})$ is imaginary quadratic and $2 \nmid h_k$,

(b) $p = 2$, $k = \mathbb{Q}(\sqrt{-1})$,

(c) $p = 3$, $k = (\mathbb{Q}\sqrt{-3})$.

In all cases, Proposition 3.3 yields an isomorphism $\mathcal{O}_k^\times/p \xrightarrow{\sim} V_\varnothing$. The second exact sequence of Proposition 3.3 and the isomorphism $U_\mathfrak{p}/p \cong U_\mathfrak{p} k_\mathfrak{p}^{\times p}/k_\mathfrak{p}^{\times p}$ imply

$$0 = V_S = \ker\big(\mathcal{O}_k^\times/p \to U_\mathfrak{p}/p\big).$$

Note that $\mathcal{O}_k^\times/p$ is one-dimensional. In case (a), the unit -1 is a generator of $\mathcal{O}_k^\times/2$ which must not be a square in $U_\mathfrak{p}$, implying $N(\mathfrak{p}) \not\equiv 1 \bmod 4$. In case (b), $\sqrt{-1}$ is a generator, and in case (c), a generator is given by $\zeta_3 = \frac{1}{2}(-1+\sqrt{-3})$. The assertions in the cases (b) and (c) follow similarly. Conversely, assume we are in case (a), (b) or (c). Then we can reverse the given arguments and obtain $h^i(X \smallsetminus S) = 0$ for all $i \geq 1$. $\square$

Theorem 7.5. *Assume $G_S \neq 1$ and that $X \smallsetminus S$ is a $K(\pi,1)$. Then the following hold.*

(i) *$cd\, G_S = 2$, $scd\, G_S = 3$.*

(ii) *G_S is a duality group (of dimension 2).*

Proof. By Lemma 7.1 and Corollary 3.5, G_S is a fab-group and $cd\ G_S \leq 2$. Now the assertions follow in a purely group-theoretical way:
As $G_S \neq 1$ and G_S^{ab} is finite, G_S is not free, and we obtain $cd\, G_S = 2$. By [NSW], Proposition 3.3.3, it follows that $scd\, G_S \in \{2,3\}$. Assume $scd\, G = 2$. We consider the G_S-module

$$D_2(\mathbb{Z}) := \varinjlim_{U} U^{ab},$$

where the limit runs over all open normal subgroups $U \triangleleft G_S$ and for $V \subset U$ the transition map is the transfer Ver: $U^{ab} \to V^{ab}$, i.e. the dual of the corestriction map cor: $H^2(V,\mathbb{Z}) \to H^2(U,\mathbb{Z})$ (see [NSW], I, §5). By [NSW], Theorem 3.6.4 (iv), we obtain $G_S^{ab} = D_2(\mathbb{Z})^{G_S}$. On the other hand, U^{ab} is finite for all U and the group theoretical version of the Principal Ideal Theorem (see [Ne], VI, Theorem 7.6) implies $D_2(\mathbb{Z}) = 0$. Hence $G_S^{ab} = 0$ which implies $G_S = 1$ producing a contradiction. Hence $scd\, G_S = 3$.
It remains to show that G_S is a duality group. By [NSW], Theorem 3.4.6, it suffices to show that the terms

$$D_i(G_S,\mathbb{Z}/p\mathbb{Z}) := \varinjlim_{U} H^i(U,\mathbb{Z}/p\mathbb{Z})^\vee$$

are zero for $i = 0, 1$. Here U runs through the open subgroups of G_S, $\vee$ denotes the Pontryagin dual and the transition maps are the duals of the corestriction maps. For $i = 0$, and $V \subsetneqq U$, the transition map

$$\mathrm{cor}^\vee : \mathbb{Z}/p\mathbb{Z} = H^0(V,\mathbb{Z}/p\mathbb{Z})^\vee \to H^0(U,\mathbb{Z}/p\mathbb{Z})^\vee = \mathbb{Z}/p\mathbb{Z}$$

is multiplication by $(U : V)$, hence zero. Therefore $D_0(G_S,\mathbb{Z}/p\mathbb{Z}) = 0$, as G_S is infinite. Furthermore,

$$D_1(G_S,\mathbb{Z}/p\mathbb{Z}) = \varinjlim_{U} U^{ab}/p = 0$$

by the Principal Ideal Theorem. This finishes the proof. □

In order to proceed, we introduce some notation in order to deal with the case of infinite extensions. For a (possibly infinite) algebraic extension K of k we denote by $S(K)$ the set of prolongations of primes in S to K. The set $S(K)$ carries a profinite topology in a natural way. Now assume that $M|K|k$ is a tower of Galois extensions. We denote the inertia group of a prime $\mathfrak{p} \in S(K)$ in the extension $M|K$ by $T_\mathfrak{p}(M|K)$. For $i \geq 0$ we write

$$\sideset{}{'}\bigoplus_{\mathfrak{p}\in S(K)} H^i(T_\mathfrak{p}(M|K),\mathbb{Z}/p\mathbb{Z}) \stackrel{df}{=} \varinjlim_{k'\subset K} \bigoplus_{\mathfrak{p}\in S(k')} H^i(T_\mathfrak{p}(M|k'),\mathbb{Z}/p\mathbb{Z}),$$

where the limit on the right hand side runs through all finite subextensions k' of k in K. The $Gal(K|k)$-module $\bigoplus'_{\mathfrak{p}\in S(K)} H^i(T_\mathfrak{p}(M|K),\mathbb{Z}/p\mathbb{Z})$ is the maximal discrete submodule of the product $\prod_{\mathfrak{p}\in S(K)} H^i(T_\mathfrak{p}(M|K),\mathbb{Z}/p\mathbb{Z})$.

Whenever we deal with local terms associated to the elements of $S(K)$ (e.g. étale cohomology groups) we use restricted sums, which are, in the same manner as above, defined as the inductive limit over the similar terms associated to all finite subextensions of k in K.

A natural question is how far we get locally at the primes in S when going up to k_S.

Proposition 7.6. *Assume that* $X \smallsetminus S$ *is a* $K(\pi,1)$ *and that* $G_S \neq 1$*. Then* k_S *realizes the maximal unramified* p*-extension of* $k_\mathfrak{p}$ *for all* $\mathfrak{p} \in S$*, i.e.*

$$k_\mathfrak{p}^{nr}(p) \subset (k_S)_\mathfrak{p} \quad \textit{for all } \mathfrak{p} \in S.$$

If $\mathfrak{p} \in S$ *ramifies in* k_S*, then* $(k_S)_\mathfrak{p} = k_\mathfrak{p}(p)$*, i.e.* k_S *realizes the maximal* p*-extension of* $k_\mathfrak{p}$*.*

Proof. For an integral normal scheme Y we write Y_L for the normalization of Y in an algebraic extension L of its function field. Then $(X \smallsetminus S)_{k_S}$ is the universal pro-p covering of $X \smallsetminus S$. We consider the following part of the excision sequence for the pair $(X_{k_S}, (X \smallsetminus S)_{k_S})$

$$H^2_{et}((X \smallsetminus S)_{k_S}) \to \bigoplus_{\mathfrak{p}\in S(k_S)}{}' H^3_\mathfrak{p}((X_{k_S})_\mathfrak{p}) \to H^3_{et}(X_{k_S}).$$

As G_S is infinite, Lemma 3.7 implies $H^3_{et}(X_{k_S}) = 0$. By condition (iii) of Proposition 2.1 we have $H^2_{et}((X \smallsetminus S)_{k_S}) = 0$. Hence $H^3_\mathfrak{p}((X_{k_S})_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in S(k_S)$. As $H^i_{et}((X_{k_S})_\mathfrak{p}) = 0$ for $i \geq 2$, we obtain

$$H^3_\mathfrak{p}((X_{k_S})_\mathfrak{p}) \cong H^2((k_S)_\mathfrak{p}),$$

where the group on the right hand side is Galois cohomology with values in $\mathbb{Z}/p\mathbb{Z}$. As $\mu_p \subset k_\mathfrak{p}$ by assumption, the vanishing of $H^2((k_S)_\mathfrak{p})$ implies $p^\infty \mid [(k_S)_\mathfrak{p} : k_\mathfrak{p}]$. In other words, the decomposition group $G_\mathfrak{p}(k_S|k)$ of each $\mathfrak{p} \in S$ is infinite. As a subgroup of G_S, it has cohomological dimension ≤ 2. Furthermore, $G_\mathfrak{p}(k_S|k)$ is a factor group of the local Galois group $Gal(k_\mathfrak{p}(p)|k_\mathfrak{p})$, which, by Proposition 7.2, has only three quotients of cohomological dimension less or equal to 2: itself, the trivial group and the Galois group of the maximal unramified p-extension of $k_\mathfrak{p}$. Hence $k_\mathfrak{p}^{nr}(p) \subseteq (k_S)_\mathfrak{p}$ and $(k_S)_\mathfrak{p} = k_\mathfrak{p}(p)$ if $\mathfrak{p}$ ramifies in k_S. □

In order to deduce Theorem 3, it remains to show that each $\mathfrak{p} \in S$ ramifies in k_S. The following lemma provides a first step.

Lemma 7.7. *Let* $\mathfrak{p} \in S$ *be a prime and let* $S' = S \smallsetminus \{\mathfrak{p}\}$*. Assume that the natural injection* $V_S \hookrightarrow V_{S'}$ *is an isomorphism. Then* $\mathfrak{p}$ *ramifies in* k_S*.*

Proof. Since the map $H^1(G_S) \to H^1_{et}(X \smallsetminus S)$ is an isomorphism, Theorem 3.4 implies

$$\dim_{\mathbb{F}_p} H^1(G_S) = 1 + \#S - \delta + \dim_{\mathbb{F}_p} V_S - r,$$

and the same formula holds with S replaced by S'. Hence

$$\dim_{\mathbb{F}_p} H^1(G_S) = \dim_{\mathbb{F}_p} H^1(G_{S'}) + 1\,.$$

In particular, $G_{S'}$ is a proper quotient of G_S and therefore $\mathfrak{p}$ ramifies in k_S. □

Corollary 7.8. *Assume that $X \smallsetminus S$ is a $K(\pi,1)$ and that $G_S \neq 1$. Let $\mathfrak{p} \in S$ be a prime and let $S' = S \smallsetminus \{\mathfrak{p}\}$. Assume that $V_{S'} = 0$. Then $(k_S)_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$.*

Remark: If $V_\varnothing = 0$, then the given criterion applies to any set S and each $\mathfrak{p} \in S$. This was used in [S1] for $k = \mathbb{Q}$ and in [Vo] for imaginary quadratic number fields. If the unit rank of k is non-zero, then $V_\varnothing \neq 0$ and the criterion applies only to sufficiently large sets S.

8 Enlarging the set of primes

Next we consider the problem of enlarging the set S.

Proposition 8.1. *Let $S \subset S'$ be finite sets of primes of norm congruent to 1 modulo p. Assume that $X \smallsetminus S$ is a $K(\pi,1)$ and that $G_S \neq 1$. Further assume that each $\mathfrak{q} \in S' \smallsetminus S$ does not split completely in k_S. Then the following holds.*

(i) *$X \smallsetminus S'$ is a $K(\pi,1)$.*

(ii) *$(k_{S'})_{\mathfrak{q}} = k_{\mathfrak{q}}(p)$ for all $\mathfrak{q} \in S' \smallsetminus S$.*

Furthermore, $H^i(Gal(k_{S'}|k_S)) = 0$ for $i \geq 2$. For $i = 1$ we have a natural isomorphism

$$H^1(Gal(k_{S'}|k_S)) \cong \bigoplus_{\mathfrak{p} \in S' \smallsetminus S(k_S))}{}' H^1(T_{\mathfrak{p}}(k_{S'}|k_S)),$$

In particular, $Gal(k_{S'}|k_S)$ is a free pro-p-group.

Proof. Let $\mathfrak{q} \in S' \smallsetminus S$. Since $\mathfrak{q}$ does not split completely in k_S and since $cd\, G_S = 2$, the decomposition group of $\mathfrak{q}$ in $k_S|k$ is a non-trivial and torsion-free quotient of $\mathbb{Z}_p \cong G(k_{\mathfrak{q}}^{nr}(p)|k_{\mathfrak{q}})$. Therefore $(k_S)_{\mathfrak{q}}$ is the maximal unramified p-extension of $k_{\mathfrak{q}}$. We denote the normalization of an integral normal scheme Y in an algebraic extension L of its function field by Y_L. Then $(X \smallsetminus S)_{k_S}$ is the universal pro-p covering of $X \smallsetminus S$. We consider the étale excision sequence for the pair $((X \smallsetminus S)_{k_S}, (X \smallsetminus S')_{k_S})$. By assumption, $X \smallsetminus S$ is a $K(\pi,1)$, hence $H^i_{et}((X \smallsetminus S)_{k_S}) = 0$ for $i \geq 1$ by condition (iii) of Proposition 2.1. This implies isomorphisms

$$H^i_{et}\big((X \smallsetminus S')_{k_S}\big) \xrightarrow{\sim} \bigoplus_{\mathfrak{p} \in S' \smallsetminus S(k_S)}{}' H^{i+1}_{\mathfrak{p}}\big(((X \smallsetminus S)_{k_S})_{\mathfrak{p}}\big)$$

for $i \geq 1$. As k_S realizes the maximal unramified p-extension of $k_{\mathfrak{q}}$ for all $\mathfrak{q} \in S' \smallsetminus S$, the schemes $((X \smallsetminus S)_{k_S})_{\mathfrak{p}}$, $\mathfrak{p} \in S' \smallsetminus S(k_S)$ have trivial cohomology with values in $\mathbb{Z}/p\mathbb{Z}$ and we obtain isomorphisms

$$H^i((k_S)_{\mathfrak{p}}) \xrightarrow{\sim} H^{i+1}_{\mathfrak{p}}\big(((X \smallsetminus S)_{k_S})_{\mathfrak{p}}\big)$$

for $i \geq 1$. These groups vanish for $i \geq 2$. This implies

$$H^i_{et}((X \smallsetminus S')_{k_S}) = 0$$

for $i \geq 2$. The scheme $(X \smallsetminus S')_{k_{S'}}$ is the universal pro-p covering of $(X \smallsetminus S')_{k_S}$. The Hochschild-Serre spectral sequence yields an inclusion

$$H^2(Gal(k_{S'}|k_S)) \hookrightarrow H^2_{et}((X \smallsetminus S')_{k_S}) = 0.$$

Hence $Gal(k_{S'}|k_S)$ is a free pro-p-group and

$$H^1(Gal(k_{S'}|k_S)) \xrightarrow{\sim} H^1_{et}((X \smallsetminus S')_{k_S}) \cong \bigoplus_{\mathfrak{p} \in S' \smallsetminus S(k_S)}{}' H^1((k_S)_\mathfrak{p}).$$

This shows that each $\mathfrak{p} \in S' \smallsetminus S(k_S)$ ramifies in $k_{S'}|k_S$, and since the Galois group is free, $k_{S'}$ realizes the maximal p-extension of $(k_S)_\mathfrak{p}$. In particular,

$$H^1(T_\mathfrak{p}(k_{S'}|k_S)) \cong H^1((k_S)_\mathfrak{p})$$

for all $\mathfrak{p} \in S' \smallsetminus S(k_S)$. Using that $Gal(k_{S'}|k_S)$ is free, the Hochschild-Serre spectral sequence induces an isomorphism

$$0 = H^2_{et}((X \smallsetminus S')_{k_S}) \xrightarrow{\sim} H^2_{et}((X \smallsetminus S')_{k_{S'}})^{Gal(k_{S'}|k_S)}.$$

Hence $H^2_{et}((X \smallsetminus S')_{k_{S'}}) = 0$, since $Gal(k_{S'}|k_S)$ is a pro-p-group. Condition (iii) of Proposition 2.1 implies that $X \smallsetminus S'$ is a $K(\pi, 1)$. □

Corollary 8.2. *Assume that $X \smallsetminus S$ is a $K(\pi,1)$, and let $S \subset S'$ be a finite set of primes of norm $\equiv 1 \bmod p$. Assume that each $\mathfrak{q} \in S' \smallsetminus S$ does not split completely in k_S. Then the arithmetic form of Riemann's existence theorem holds, i.e. the natural homomorphism*

$$\mathop{\ast}_{\mathfrak{p} \in S' \backslash S(k_S)} T_\mathfrak{p}(k_{S'}|k_S) \longrightarrow Gal(k_{S'}|k_S)$$

is an isomorphism. Here $T_\mathfrak{p}$ is the inertia group and $\ast$ denotes the free pro-p-product of a bundle of pro-p-groups, cf. [NSW], Ch. IV, §3.

Proof. By Proposition 8.1 and by the calculation of the cohomology of a free product ([NSW], 4.3.10 and 4.1.4), ϕ is a homomorphism between free pro-p-groups which induces an isomorphism on mod p cohomology. Therefore ϕ is an isomorphism. □

9 Proof of Theorems 3 and 5

Theorem 9.1. *Assume that $X \smallsetminus S$ is a $K(\pi,1)$ and $G_S \neq 1$. Then k_S realizes the maximal p-extension $k_\mathfrak{p}(p)$ of the local field $k_\mathfrak{p}$ for all $\mathfrak{p} \in S$.*

Proof. The decomposition groups of primes in S have infinite index by Corollary 7.3. By Corollary 2.2, we may replace k by a finite subextension in k_S, and therefore assume that $\#S \geq 2$.
By Proposition 7.6, it suffices to show that each $\mathfrak{p} \in S$ ramifies in k_S. Let $\mathfrak{p} \in S$ be a prime which does not ramify in k_S and put $S' = S \smallsetminus \{\mathfrak{p}\}$. By Lemma 7.7, the natural injection $\phi\colon\ V_S \hookrightarrow V_{S'}$ is not an isomorphism. By Proposition 3.3, the cokernel of ϕ is one-dimensional. By Theorem 3.4, we obtain

$$h^2(X \smallsetminus S') = h^2(X \smallsetminus S).$$

As $G_S = G_{S'}$, we have $cd\, G_{S'} = 2$ and

$$h^2(G_S) = h^2(G_{S'}) \leq h^2(X \smallsetminus S') = h^2(X \smallsetminus S).$$

As $X \smallsetminus S$ is a $K(\pi, 1)$, equality holds. Therefore the injection $H^2(G_{S'}) \hookrightarrow H^2_{et}(X \smallsetminus S')$ is an isomorphism. By Corollary 3.5, $X \smallsetminus S'$ is a $K(\pi, 1)$. By Proposition 7.6, $\mathfrak{p}$ does not split completely in $k_{S'} = k_S$. By Proposition 8.1, k_S realizes the maximal p-extension of $k_\mathfrak{p}$. This yields a contradiction. $\square$

Now we are in the position to show Theorem 5.

Proof of Theorem 5. We have $H^2_{et}((X \smallsetminus S)_{k_S}) = 0$ by condition (iii) of Proposition 2.1. By Theorem 9.1, the local cohomology groups $H^2_\mathfrak{p}((X_{k_S})_\mathfrak{p})$ vanish for all $\mathfrak{p} \in S(k_S)$. Therefore the excision sequence yields $H^2_{et}(X_{k_S}) = 0$. By the flat duality theorem of Artin-Mazur ([Mi], III Corollary 3.2) we have $H^2_{et}(X_K)^\vee \cong H^1_{fl}(X_K, \mu_p)$ for each finite subextension K of k in k_S. Hence

$$\varprojlim_{K \subset k_S} H^1_{fl}(X_K, \mu_p) = 0.$$

The flat Kummer sequence $0 \to \mu_p \to \mathbb{G}_m \xrightarrow{\cdot p} \mathbb{G}_m \to 0$ implies compatible exact sequences

$$0 \to \mathcal{O}_K^\times/p \to H^1_{fl}(X_K, \mu_p) \to {}_pH^1_{fl}(X_K, \mathbb{G}_m)$$

for all K. We obtain

$$\varprojlim_{K \subset k_S} \mathcal{O}_K^\times/p = 0\,.$$

The topological Nakayama-Lemma (see [NSW], Corollary 5.2.8) for the compact $\mathbb{Z}_p$-module $\varprojlim \mathcal{O}_K^\times \otimes \mathbb{Z}_p$ therefore implies

$$\varprojlim_{K \subset k_S} \mathcal{O}_K^\times \otimes \mathbb{Z}_p = 0\,.$$

Tensoring the exact sequences (cf. [NSW], Lemma 10.3.11)

$$0 \to \mathcal{O}_K^\times \to \mathcal{O}_{K,S}^\times \to \bigoplus_{\mathfrak{p} \in S(K)} (K_\mathfrak{p}^\times/U_\mathfrak{p}) \to Cl(K) \to Cl_S(K) \to 0$$

by (the flat $\mathbb{Z}$-algebra) $\mathbb{Z}_p$, we obtain exact sequences of finitely generated, hence compact, $\mathbb{Z}_p$-modules. The field k_S admits no unramified p-extensions. Therefore class field theory implies $\varprojlim_K Cl(K)(p) = 0$, where K runs through all finite subextensions of k in k_S. Thus, passing to the projective limit over K, we obtain the exact sequence

$$0 \to \varprojlim_{K \subset k_S} \mathcal{O}_K^\times \otimes \mathbb{Z}_p \to \varprojlim_{K \subset k_S} \mathcal{O}_{K,S}^\times \otimes \mathbb{Z}_p \to \varprojlim_{K \subset k_S} \bigoplus_{\mathfrak{p} \in S(K)} (K_\mathfrak{p}^\times / U_\mathfrak{p}) \otimes \mathbb{Z}_p \to 0.$$

As k_S realizes the maximal unramified p-extension of $k_\mathfrak{p}$ for all $\mathfrak{p} \in S$, local class field theory implies the vanishing of the right hand limit. Therefore the result for the S-units follows from the corresponding result for the units. □

We have proven all assertions but the statement on the dualizing module in Theorem 2. In [S1], Th. 5.2 we showed this statement under the assumption that k_S realizes the maximal p-extension $k_\mathfrak{p}(p)$ of $k_\mathfrak{p}$ for all $\mathfrak{p} \in S$. This assumption has been shown above, hence the result follows.

Added in proof. As pointed out by K. Wingberg, the proof of Proposition 8.1 does not use that S and S' are disjoint from S_p. Hence Theorem 1 also holds if $S \cap S_p \neq \varnothing$, i.e. the result extends to the mixed case.

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