Construction of Eigenvarieties in Small Cohomological Dimensions for Semi-Simple, Simply Connected Groups

Richard Hill

Received: June 23, 2007
Revised: September 27, 2007

Communicated by Don Blasius

Abstract. We study low order terms of Emerton's spectral sequence for simply connected, simple groups. As a result, for real rank 1 groups, we show that Emerton's method for constructing eigenvarieties is successful in cohomological dimension 1. For real rank 2 groups, we show that a slight modification of Emerton's method allows one to construct eigenvarieties in cohomological dimension 2.

2000 Mathematics Subject Classification: 11F33

Throughout this paper we shall use the following standard notation:

- k is an algebraic number field, fixed throughout.
- $\mathfrak{p}$, $\mathfrak{q}$ denote finite primes of k, and $k_\mathfrak{p}$, $k_\mathfrak{q}$ the corresponding local fields.
- $k_\infty = k \otimes_\mathbb{Q} \mathbb{R}$ is the product of the archimedean completions of k.
- $\mathbb{A}$ is the adèle ring of k.
- $\mathbb{A}_f$ is the ring of finite adèles of k.
- For a finite set S of places of k, we let

$$k_S = \prod_{v \in S} k_v, \qquad \mathbb{A}^S = {\prod_{v \notin S}}' k_v.$$

1 Introduction and Statements of Results

1.1 Interpolation of classical automorphic representations

Let $\mathbb{G}$ be a connected, algebraically simply connected, semi-simple group over a number field k. We fix once and for all a maximal compact subgroup $K_\infty \subset \mathbb{G}(k_\infty)$. Our assumptions on $\mathbb{G}$ imply that K_∞ is connected in the archimedean topology. This paper is concerned with the cohomology of the following "Shimura manifolds":

$$Y(K_f) = \mathbb{G}(k)\backslash\mathbb{G}(\mathbb{A})/K_\infty K_f,$$

where K_f is a compact open subgroup of $\mathbb{G}(\mathbb{A}_f)$. Let W be an irreducible finite dimensional algebraic representation of $\mathbb{G}$ over a field extension E/k. Such a representation gives rise to a local system $\mathcal{V}_W$ on $Y(K_f)$. We shall refer to the cohomology groups of this local system as the "classical cohomology groups":

$$H^\bullet_{\text{class.}}(K_f, W) := H^\bullet(Y(K_f), \mathcal{V}_W).$$

It is convenient to consider the direct limit over all levels K_f of these cohomology groups:

$$H^\bullet_{\text{class.}}(\mathbb{G}, W) = \varinjlim_{K_f} H^\bullet_{\text{class.}}(K_f, W).$$

There is a smooth action of $\mathbb{G}(\mathbb{A}_f)$ on $H^\bullet_{\text{class.}}(\mathbb{G}, W)$. Since W is a representation over a field E of characteristic zero, we may recover the finite level cohomology groups as spaces of K_f-invariants:

$$H^\bullet_{\text{class.}}(K_f, W) = H^\bullet_{\text{class.}}(\mathbb{G}, W)^{K_f}.$$

It has become clear that only a very restricted class of smooth representations of $\mathbb{G}(\mathbb{A}_f)$ may occur as subquotients of the classical cohomology $H^n_{\text{class.}}(\mathbb{G}, W)$. For example, in the case $E = \mathbb{C}$, Ramanujan's Conjecture (Deligne's Theorem) gives an archimedian bound on the eigenvalues of the Hecke operators. We shall be concerned here with the case that E is an extension of a non-archimedean completion of k.

Fix once and for all a finite prime $\mathfrak{p}$ of k over which $\mathbb{G}$ is quasi-split. Fix a Borel subgroup $\mathbb{B}$ of $\mathbb{G} \times_k k_\mathfrak{p}$ and a maximal torus $\mathbb{T} \subset \mathbb{B}$. We let E be a finite extension of $k_\mathfrak{p}$, large enough so that $\mathbb{G}$ splits over E. It follows that the irreducible algebraic representations of $\mathbb{G}$ over E are absolutely irreducible (§24.5 of [8]). By the highest weight theorem (§24.3 of [8]), an irreducible representation W of $\mathbb{G}$ over E is determined by its highest weight ψ_W, which is an algebraic character $\psi_W : \mathbb{T} \times_{k_\mathfrak{p}} E \to \mathrm{GL}_1/E$.

By a *tame level* we shall mean a compact open subgroup $K^\mathfrak{p} \subset \mathbb{G}(\mathbb{A}_f^\mathfrak{p})$. Fix a tame level $K^\mathfrak{p}$, and consider the spaces of $K^\mathfrak{p}$-invariants:

$$H^\bullet_{\text{class.}}(K^\mathfrak{p}, W) = H^\bullet_{\text{class.}}(\mathbb{G}, W)^{K^\mathfrak{p}}.$$

The group $\mathbb{G}(k_{\mathfrak{p}})$ acts smoothly on $H^{\bullet}_{\text{class.}}(K^{\mathfrak{p}}, W)$. We also have commuting actions of the level $K^{\mathfrak{p}}$ Hecke algebra:

$$\mathcal{H}(K^{\mathfrak{p}}) := \Big\{ f : K^{\mathfrak{p}} \backslash \mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})/K^{\mathfrak{p}} \to E : f \text{ has compact support} \Big\}.$$

In order to describe the representations of $\mathcal{H}(K^{\mathfrak{p}})$, recall the tensor product decomposition:

$$\mathcal{H}(K^{\mathfrak{p}}) = \mathcal{H}(K^{\mathfrak{p}})^{\text{ramified}} \otimes \mathcal{H}(K^{\mathfrak{p}})^{\text{sph}}, \tag{1}$$

where $\mathcal{H}(K^{\mathfrak{p}})^{\text{sph}}$ is commutative but infinitely generated, and $\mathcal{H}(K^{\mathfrak{p}})^{\text{ramified}}$ is non-commutative but finitely generated. Consequently the irreducible representations of $\mathcal{H}(K^{\mathfrak{p}})$ are finite-dimensional.

Let $\mathfrak{q} \neq \mathfrak{p}$ be a finite prime of k. We shall say that $\mathfrak{q}$ is unramified in $K^{\mathfrak{p}}$ if

(a) $\mathbb{G}$ is quasi-split over $k_{\mathfrak{q}}$, and splits over an unramified extension of $k_{\mathfrak{q}}$, and

(b) $K^{\mathfrak{p}} \cap \mathbb{G}(k_{\mathfrak{q}})$ is a hyper-special maximal compact subgroup of $\mathbb{G}(k_{\mathfrak{q}})$ (see [38]).

Let S be the set of finite primes $\mathfrak{q} \neq \mathfrak{p}$, which are ramified in $K^{\mathfrak{p}}$. This is a finite set, and we have

$$K^{\mathfrak{p}} = K_S \times \prod_{\mathfrak{q} \text{ unramified}} K_{\mathfrak{q}}, \qquad K_S = K^{\mathfrak{p}} \cap \mathbb{G}(k_S), \quad K_{\mathfrak{q}} = K^{\mathfrak{p}} \cap \mathbb{G}(k_{\mathfrak{q}}).$$

This gives the tensor product decomposition (1), where we take

$$\mathcal{H}(K^{\mathfrak{p}})^{\text{ramified}} = \mathcal{H}(K_S), \qquad \mathcal{H}(K^{\mathfrak{p}})^{\text{sph}} = \bigotimes_{\mathfrak{q} \text{ unramified}}{}^{'} \mathcal{H}(K_{\mathfrak{p}}).$$

For each unramified prime $\mathfrak{q}$, the Satake isomorphism (Theorem 4.1 of [12]) shows that $\mathcal{H}(K_{\mathfrak{q}})$ is finitely generated and commutative. Hence the irreducible representations of $\mathcal{H}(K^{\mathfrak{p}})^{\text{sph}}$ over $\bar{E}$ are 1-dimensional, and may be identified with elements of $(\text{Spec } \mathcal{H}(K^{\mathfrak{p}})^{\text{sph}})(\bar{E})$. Since the global Hecke algebra is infinitely generated, Spec $\mathcal{H}(K^{\mathfrak{p}})^{\text{sph}}$ is an infinite dimensional space. One might expect that the representations which occur as subquotients of $H^{\bullet}_{\text{class.}}(K^{\mathfrak{p}}, W)$ are evenly spread around this space. There is an increasing body of evidence [1, 2, 3, 10, 11, 13, 14, 15, 18, 21, 22] that this is not the case, and that in fact these representations are contained in a finite dimensional subset of Spec $\mathcal{H}(K^{\mathfrak{p}})^{\text{sph}}$, independent of W.

More precisely, let π be an irreducible representation of $\mathbb{G}(k_{\mathfrak{p}}) \times \mathcal{H}(K^{\mathfrak{p}})$, which occurs as a subquotient of $H^{n}_{\text{class.}}(K^{\mathfrak{p}}, W) \otimes_E \bar{E}$. We may decompose π as a tensor product:

$$\pi = \pi_{\mathfrak{p}} \otimes \pi^{\text{ramified}} \otimes \pi^{\text{sph}},$$

where π^{sph} is a character of $\mathcal{H}(K^{\mathfrak{p}})^{\text{sph}}$; π^{ramified} is an irreducible representation of $\mathcal{H}(K^{\mathfrak{p}})^{\text{ramified}}$ and $\pi_{\mathfrak{p}}$ is an irreducible smooth representation of $\mathbb{G}(k_{\mathfrak{p}})$. We can say very little about the pair (W,π) in this generality, so we shall make another restriction. We shall write $\mathrm{Jacq}_{\mathbb{B}}(\pi_{\mathfrak{p}})$ for the Jacquet module of $\pi_{\mathfrak{p}}$, with respect to $\mathbb{B}(k_{\mathfrak{p}})$. The Jacquet module is a smooth, finite dimensional representation of $\mathbb{T}(k_{\mathfrak{p}})$. It seems possible to say something about those pairs (π, W) for which $\pi_{\mathfrak{p}}$ has non-zero Jacquet module. Such representations $\pi_{\mathfrak{p}}$ are also said to have *finite slope*. Classically for $\mathrm{GL}_2/\mathbb{Q}$, representations of finite slope correspond to Hecke eigenforms for which the eigenvalue of $U_{\mathfrak{p}}$ is non-zero. By Frobenius reciprocity, such a $\pi_{\mathfrak{p}}$ is a submodule of a smoothly induced representation $\mathrm{ind}_{\mathbb{B}(k_{\mathfrak{p}})}^{\mathbb{G}(k_{\mathfrak{p}})}\theta$, where $\theta : \mathbb{T}(k_{\mathfrak{p}}) \to \bar{E}^{\times}$ is a smooth character. In order to combine the highest weight ψ_W, which is an algebraic character of $\mathbb{T}$, and the smooth character θ of $\mathbb{T}(k_{\mathfrak{p}})$, we introduce the following rigid analytic space (see [32] for background in rigid analytic geometry):

$$\hat{T}(A) = \mathrm{Hom}_{k_{\mathfrak{p}}-\mathrm{loc.an.}}(\mathbb{T}(k_{\mathfrak{p}}), A^{\times}), \qquad \begin{array}{c} A \text{ a commutative} \\ \text{Banach algebra over } E. \end{array}$$

Emerton defined the *classical point* corresponding to π to be the pair

$$(\theta\psi_W, \pi^{\mathrm{sph}}) \in \left(\hat{T} \times \mathrm{Spec}\ \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}\right)(\bar{E}).$$

We let $E(n, K^{\mathfrak{p}})_{\mathrm{class.}}$ denote the set of all classical points. Emerton defined the *eigenvariety* $E(n, K^{\mathfrak{p}})$ to be the rigid analytic Zariski closure of $E(n, K^{\mathfrak{p}})_{\mathrm{class.}}$ in $\hat{T} \times \mathrm{Spec}\ \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$.

Concretely, this means that for every unramified prime $\mathfrak{q}$ and each generator $T_{\mathfrak{q}}^i$ for the Hecke algebra $\mathcal{H}(K_{\mathfrak{q}})$, there is a holomorphic function $t_{\mathfrak{q}}^i$ on $E(n, K^{\mathfrak{p}})$ such that for every representation π in $H^n_{\mathrm{class.}}(K^{\mathfrak{p}}, W) \otimes \bar{E}$ of finite slope at $\mathfrak{p}$, the action of $T_{\mathfrak{q}}^i$ on π is by scalar multiplication by $t_{\mathfrak{q}}^i(x)$, where x is the corresponding classical point.

One also obtains a description of the action of the ramified part of the Hecke algebra. This description is different, since irreducible representations of $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ramified}}$ are finite dimensional rather than 1-dimensional. Instead one finds that there is a coherent sheaf $\mathcal{M}$ of $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ramified}}$-modules over $E(n, K^{\mathfrak{p}})$, such that, roughly speaking, the action of $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ramified}}$ on the fibre of a classical point describes the action of $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ramified}}$ on the corresponding part of the classical cohomology. A precise statement is given in Theorem 1 below.

Emerton introduced a criterion (Definition 1 below), according to which the Eigencurve $E(n, K^{\mathfrak{p}})$ is finite dimensional. More precisely, he was able to prove that the projection $E(n, K^{\mathfrak{p}}) \to \hat{T}$ is finite. If we let $\mathfrak{t}$ denote the Lie algebra of $\mathbb{T}(\bar{E})$, then there is a map given by differentiation at the identity element:

$$\hat{T} \to \check{\mathfrak{t}},$$

where $\check{\mathfrak{t}}$ is the dual space of $\mathfrak{t}$. It is worth noting that the image in $\check{\mathfrak{t}}$ of a classical point depends only on the highest weight ψ_W, since smooth characters have zero derivative. Emerton also proved, assuming his criterion, that the projection

$E(n, K^{\mathfrak{p}}) \to \check{\mathfrak{t}}$ has discrete fibres. As a result, one knows that the dimension of the eigencurve is at most the absolute rank of $\mathbb{G}$.
The purpose of this paper is to investigate Emerton's criterion for connected, simply connected, simple groups. Specifically, we show that Emerton's criterion holds for all such groups in dimension $n = 1$. Emerton's criterion typically fails in dimension $n = 2$. However we prove a weaker form of the criterion for $n = 2$, and we show that the weaker criterion is sufficient for most purposes.

1.2 Emerton's Criterion

Let p be the rational prime below $\mathfrak{p}$. In [18] Emerton introduced the following p-adic Banach spaces:

$$\tilde{H}^\bullet(K^{\mathfrak{p}}, \mathbb{Q}_p) = \left(\varprojlim_{s} \varinjlim_{K_{\mathfrak{p}}} H^\bullet(Y(K_{\mathfrak{p}} K^{\mathfrak{p}}), \mathbb{Z}/p^s) \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

For convenience, we also consider the direct limits of these spaces over all tame levels $K^{\mathfrak{p}}$:

$$\tilde{H}^\bullet(\mathbb{G}, \mathbb{Q}_p) = \varinjlim_{K^{\mathfrak{p}}} \tilde{H}^\bullet(K^{\mathfrak{p}}, \mathbb{Q}_p).$$

We have the following actions on these spaces:

- The group $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$ acts smoothly on $\tilde{H}^\bullet(\mathbb{G}, \mathbb{Q}_p)$; the subspace $\tilde{H}^\bullet(K^{\mathfrak{p}}, \mathbb{Q}_p)$ may be recovered as the $K^{\mathfrak{p}}$-invariants:
$$\tilde{H}^\bullet(K^{\mathfrak{p}}, \mathbb{Q}_p) = \tilde{H}^\bullet(\mathbb{G}, \mathbb{Q}_p)^{K^{\mathfrak{p}}}.$$
- The Hecke algebra $\mathcal{H}(K^{\mathfrak{p}})$ acts on $\tilde{H}^\bullet(K^{\mathfrak{p}}, \mathbb{Q}_p) \otimes E$.
- The group $\mathbb{G}(k_{\mathfrak{p}})$ acts continuously, but not usually smoothly on the Banach space $\tilde{H}^\bullet(K^{\mathfrak{p}}, \mathbb{Q}_{\mathfrak{p}})$. This is an admissible continuous representation of $\mathbb{G}(k_{\mathfrak{p}})$ in the sense of [33] (or [16], Definition 7.2.1).
- Recall that we have fixed a finite extension $E/k_{\mathfrak{p}}$, over which $\mathbb{G}$ splits. We let
$$\tilde{H}^\bullet(K^{\mathfrak{p}}, E) = \tilde{H}^\bullet(K^{\mathfrak{p}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} E.$$
The group $\mathbb{G}(k_{\mathfrak{p}})$ is a $\mathfrak{p}$-adic analytic group. Hence, we may define the subspace of $k_{\mathfrak{p}}$-locally analytic vectors in $\tilde{H}^\bullet(K^{\mathfrak{p}}, E)$ (see [16]):
$$\tilde{H}^\bullet(K^{\mathfrak{p}}, E)_{\text{loc.an.}}.$$
This subspace is $\mathbb{G}(k_{\mathfrak{p}})$-invariant, and is an admissible locally analytic representation of $\mathbb{G}(k_{\mathfrak{p}})$ (in the sense of [16], Definition 7.2.7). The Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ also acts on the subspace $\tilde{H}^\bullet(K^{\mathfrak{p}}, E)_{\text{loc.an.}}$.

For an irreducible algebraic representation W of $\mathbb{G}$ over E, we shall write $\check{W}$ be the contragredient representation. Emerton showed (Theorem 2.2.11 of [18]) that there is a spectral sequence:

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathfrak{g}}(\check{W}, \tilde{H}^q(K^{\mathfrak{p}}, E)_{\text{loc.an.}}) \implies H^{p+q}_{\text{class.}}(K^{\mathfrak{p}}, W). \tag{2}$$

Taking the direct limit over the tame levels $K^{\mathfrak{p}}$, there is also a spectral sequence (Theorem 0.5 of [18]):

$$\mathrm{Ext}^p_{\mathfrak{g}}(\check{W}, \tilde{H}^q(\mathbb{G}, E)_{\text{loc.an.}}) \implies H^{p+q}_{\text{class.}}(\mathbb{G}, W). \tag{3}$$

In particular, there is an edge map

$$H^n_{\text{class.}}(\mathbb{G}, W) \to \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^n(\mathbb{G}, E)_{\text{loc.an.}}). \tag{4}$$

DEFINITION 1. We shall say that $\mathbb{G}$ satisfies *Emerton's criterion in dimension* n if the following holds:

For every W, the edge map (4) is an isomorphism.

This is equivalent to the edge maps from $H^n_{\text{class.}}(K^{\mathfrak{p}}, W)$ to $\mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^n(K^{\mathfrak{p}}, E)_{\text{loc.an.}})$ being isomorphisms for every W and every tame level $K^{\mathfrak{p}}$.

THEOREM 1 (Theorem 0.7 of [18]). *Suppose Emerton's criterion holds for* $\mathbb{G}$ *in dimension* n*. Then we have:*

1. *Projection onto the first factor induces a finite map* $E(n, K^{\mathfrak{p}}) \to \hat{T}$.
2. *The map* $E(n, K^{\mathfrak{p}}) \to \check{\mathfrak{t}}$ *has discrete fibres.*
3. *If* (χ, λ) *is a point of the Eigencurve such that* χ *is locally algebraic and of non-critical slope (in the sense of [17], Definition 4.4.3), then* (χ, λ) *is a classical point.*
4. *There is a coherent sheaf* $\mathcal{M}$ *of* $\mathcal{H}(K^{\mathfrak{p}})^{\text{ramified}}$*-modules over* $E(n, K^{\mathfrak{p}})$ *with the following property. For any classical point* $(\theta\psi_W, \lambda) \in E(n, K^{\mathfrak{p}})$ *of non-critical slope, the fibre of* $\mathcal{M}$ *over the point* $(\theta\psi_W, \lambda)$ *is isomorphic (as a* $\mathcal{H}(K^{\mathfrak{p}})^{\text{ramified}}$*-module) to the dual of the* $(\theta\psi_W, \lambda)$*-eigenspace of the Jacquet module of* $H^n_{\text{class.}}(K^{\mathfrak{p}}, \check{W})$.

In fact Emerton proved this theorem for all reductive groups $\mathbb{G}/k$. He verified his criterion in the case $\mathbb{G} = \mathrm{GL}_2/\mathbb{Q}$, $n = 1$. He also pointed out that the criterion always holds for $n = 0$, since the edge map at $(0, 0)$ for any first quadrant $E_2^{\bullet,\bullet}$ spectral sequence is an isomorphism. Of course the cohomology of $\mathbb{G}$ is usually uninteresting in dimension 0, but his argument can be applied in the case where the derived subgroup of $\mathbb{G}$ has real rank zero. This is the case, for example, when $\mathbb{G}$ is a torus, or the multiplicative group of a definite quaternion algebra.

1.3 Our Main Results

For our main results, $\mathbb{G}$ is connected, simple and algebraically simply connected. We shall also assume that $\mathbb{G}(k_\infty)$ is not compact. We do not need to assume that $\mathbb{G}$ is absolutely simple. We shall prove the following.

Theorem 2. *Emerton's criterion holds in dimension* 1.

For cohomological dimensions 2 and higher, Emerton's criterion is quite rare. We shall instead use the following criterion.

Definition 2. We shall say that $\mathbb{G}$ satisfies the *weak Emerton criterion* in dimension n if

(a) for every non-trivial irreducible W, the edge map (4) is an isomorphism, and

(b) for the trivial representation W, the edge map (4) is injective, and its cokernel is a finite dimensional trivial representation of $\mathbb{G}(\mathbb{A}_f)$.

By simple modifications to Emerton's proof of Theorem 1, we shall prove the following in §4.

Theorem 3. *If the weak Emerton criterion holds for $\mathbb{G}$ in dimension n, then*

1. *Projection onto the first factor induces a finite map $E(n, K^{\mathfrak{p}}) \to \hat{T}$.*

2. *The map $E(n, K^{\mathfrak{p}}) \to \check{\mathfrak{t}}$ has discrete fibres.*

3. *If (χ, λ) is a point of the Eigencurve such that χ is locally algebraic and of non-critical slope, then either (χ, λ) is a classical point or (χ, λ) is the trivial representation of $\mathbb{T}(k_{\mathfrak{p}}) \times \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$.*

In order to state our next theorems, we recall the definition of the congruence kernel. As before, $\mathbb{G}/k$ is simple, connected and simply connected and $\mathbb{G}(k_\infty)$ is not compact. By a *congruence subgroup* of $\mathbb{G}(k)$, we shall mean a subgroup of the form

$$\Gamma(K_f) = \mathbb{G}(k) \cap (\mathbb{G}(k_\infty) \times K_f),$$

where $K_f \subset \mathbb{G}(\mathbb{A}_f)$ is compact and open. Any two congruence subgroups are commensurable.

An *arithmetic subgroup* is a subgroup of $\mathbb{G}(k)$, which is commensurable with a congruence subgroup. In particular, every congruence subgroup is arithmetic. The *congruence subgroup problem* (see the survey articles [30, 31]) is the problem of determining the difference between arithmetic subgroups and congruence subgroups. In particular, one could naively ask whether every arithmetic subgroup of $\mathbb{G}$ is a congruence subgroup. In order to study this question more precisely, Serre introduced two completions of $\mathbb{G}(k)$:

$$\hat{\mathbb{G}}(k) = \varprojlim_{K_f} \mathbb{G}(k)/\Gamma(K_f),$$

$$\tilde{\mathbb{G}}(k) = \varprojlim_{\Gamma \text{ arithmetic}} \mathbb{G}(k)/\Gamma.$$

There is a continuous surjective group homomorphism $\tilde{\mathbb{G}}(k) \to \hat{\mathbb{G}}(k)$. The *congruence kernel* $\mathrm{Cong}(\mathbb{G})$ is defined to be the kernel of this map. Recall the following:

Theorem 4 (Strong Approximation Theorem [23, 24, 25, 28, 29]). *Suppose $\mathbb{G}/k$ is connected, simple, and algebraically simply connected. Let S be a set of places of k, such that $\mathbb{G}(k_S)$ is not compact. Then $\mathbb{G}(k)\mathbb{G}(k_S)$ is dense in $\mathbb{G}(\mathbb{A})$.*

Under our assumptions on $\mathbb{G}$, the strong approximation theorem implies that $\hat{\mathbb{G}}(k) = \mathbb{G}(\mathbb{A}_f)$, and we have the following extension of topological groups:

$$1 \to \mathrm{Cong}(\mathbb{G}) \to \tilde{\mathbb{G}}(k) \to \mathbb{G}(\mathbb{A}_f) \to 1.$$

By the *real rank* of $\mathbb{G}$, we shall mean the sum

$$m = \sum_{\nu|\infty} \mathrm{rank}_{k_\nu} \mathbb{G}.$$

It follows from the non-compactness of $\mathbb{G}(k_\infty)$, that the real rank of $\mathbb{G}$ is at least 1. Serre [37] has conjectured that for $\mathbb{G}$ simple, simply connected and of real rank at least 2, the congruence kernel is finite; for real rank 1 groups he conjectured that the congruence kernel is infinite. These conjectures have been proved in many cases and there are no proven counterexamples (see the surveys [30, 31]).
Our next result is the following.

Theorem 5. *If the congruence kernel of $\mathbb{G}$ is finite then the weak Emerton criterion holds in dimension 2.*

Theorems 2 and 5 follow from our main auxiliary results:

Theorem 6. *Let $\mathbb{G}$ be as described above. Then $\tilde{H}^0(\mathbb{G}, E) = E$, with the trivial action of $\mathbb{G}(\mathbb{A}_f)$.*

Theorem 7. *Let $\mathbb{G}$ be as described above. Then*

$$\tilde{H}^1(\mathbb{G}, E) = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}), E)_{\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})-\mathrm{smooth}},$$

where $\mathrm{Cong}(\mathbb{G})$ denotes the congruence kernel of $\mathbb{G}$.

The reduction of Theorem 2 to Theorem 6 is given in §2, and the reduction of Theorem 5 to Theorem 7 is given in §3. Theorem 6 is proved in §6 and Theorem 7 is proved in §8.
Before going on, we point out that in some cases these cohomology spaces are uninteresting. In the case $E = \mathbb{C}$, the cohomology groups are related,

via generalizations of the Eichler–Shimura isomorphism, to certain spaces of automorphic forms. More precisely, Franke [19] has shown that

$$H^{\bullet}_{\mathrm{class.}}(K_f, W) = H^{\bullet}_{\mathrm{rel.Lie}}(\mathfrak{g}, K_\infty, W \otimes \mathcal{A}(K^f)),$$

where $\mathcal{A}(K^f)$ is the space of automorphic forms $\phi : \mathbb{G}(k)\backslash\mathbb{G}(\mathbb{A})/K_\infty K_f \to \mathbb{C}$. The right hand side is relative Lie algebra cohomology (see for example [9]). Since the constant functions form a subspace of $\mathcal{A}(K^f)$, we have a $(\mathfrak{g}, K_\infty)$-submodule $W \subset W \otimes \mathcal{A}(K^f)$. This gives us a map:

$$H^{n}_{\mathrm{rel.Lie}}(\mathfrak{g}, K_\infty, W) \to H^{n}_{\mathrm{class.}}(\mathbb{G}, W). \tag{5}$$

We shall say that the cohomology of $\mathbb{G}$ is *given by constants in dimension n* if the map (5) is surjective. For example the cohomology of $\mathrm{SL}_2/\mathbb{Q}$ is given by constants in dimensions 0 and 2, although (5) is only bijective in dimension 0. On the other hand, if $\mathbb{G}(k)\backslash\mathbb{G}(\mathbb{A})$ is compact then (5) is injective.

It is known that the cohomology of $\mathbb{G}$ is given by constants in dimensions $n < m$ and in dimensions $n > d - m$, where d is the common dimension of the spaces $Y(K_f)$ and m is the real rank of $\mathbb{G}$. One shows this by proving that the relative Lie algebra cohomology of any other irreducible $(\mathfrak{g}, K_\infty)$-subquotient of $W \otimes \mathcal{A}(K^f)$ vanishes in such dimensions (see for example Corollary II.8.4 of [9]).

If the cohomology is given by constants in dimension n, then $H^{n}_{\mathrm{class.}}(\mathbb{G}, W)$ is a finite dimensional vector space, equipped with the trivial action of $\mathbb{G}(\mathbb{A}_f)$. From the point of view of this paper, cohomology groups given by constants are uninteresting. Thus Theorem 2 is interesting only for groups of real rank 1, whereas Theorem 5 is interesting, roughly speaking, for groups of real rank 2.

In fact we can often do a little better than Theorem 3. We shall prove the following in §5:

THEOREM 8. *Let $\mathbb{G}/k$ be connected, semi-simple and algebraically simply connected and assume that the weak Emerton criterion holds in dimension n. Assume also that at least one of the following two conditions holds:*

- *(a) $H^{p}_{\mathrm{class.}}(\mathbb{G}, \mathbb{C})$ is given by constants in dimensions $p < n$ and $H^{n+1}_{\mathrm{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$; or*
- *(b) $\mathbb{G}(k)$ is cocompact in $\mathbb{G}(\mathbb{A})$.*

Then all conclusions of Theorem 1 hold for the eigenvariety $E(n, K^{\mathfrak{p}})$.

The theorem is valid, for example, in the following cases where Emerton's criterion fails:

- $\mathrm{SL}_3/\mathbb{Q}$ in dimension 2;
- $\mathrm{Sp}_4/\mathbb{Q}$ in dimension 2;

- Spin groups of quadratic forms over $\mathbb{Q}$ of signature $(2, l)$ with $l \geq 3$ in dimension 2;
- Special unitary groups $\mathrm{SU}(2, l)$ with $l \geq 3$ in dimension 2;
- SL_2/k, where k is a real quadratic field, in dimension 2.

Our results generalize easily to simply connected, semi-simple groups as follows. Suppose $\mathbb{G}/k$ is a direct sum of simply connected simple groups $\mathbb{G}_i/k$. Assume also that the tame level $K^{\mathfrak{p}}$ decomposes as a direct sum of tame levels $K_i^{\mathfrak{p}}$ in $\mathbb{G}_i(\mathbb{A}_f^{\mathfrak{p}})$. By the Künneth formula, we have a decomposition of the sets of classical points:

$$E(n, K^{\mathfrak{p}})_{\text{class.}} = \bigcup_{n_1+\cdots+n_s=n} \prod_{i=1}^{s} E(n_i, K_i^{\mathfrak{p}})_{\text{class.}}.$$

1.4 SOME HISTORY

Coleman and Mazur constructed the first "eigencurve" in [15]. In our current notation, they constructed the H^1-eigencurve for $\mathrm{GL}_2/\mathbb{Q}$. In fact they showed that the points of their eigencurve parametrize overconvergent eigenforms. Their arguments were based on earlier work of Hida [20] and Coleman [14] on families of modular forms. Similar results were subsequently obtained by Buzzard [10] for the groups GL_1/k, and for the multiplicative group of a definite quaternion algebra over $\mathbb{Q}$, and later more generally for totally definite quaternion algebras over totally real fields in [11]. Kassaei [21] treated the case that $\mathbb{G}$ is a form of GL_2/k, where k is totally real and $\mathbb{G}$ is split at exactly one archimedean place. Kissin and Lei in [22] treated the case $\mathbb{G} = \mathrm{GL}_2/k$ for a totally real field k, in dimension $n = [k : \mathbb{Q}]$.
Ash and Stevens [2, 3] obtained similar results for $\mathrm{GL}_n/\mathbb{Q}$ by quite different methods. More recently, Chenevier [13] constructed eigenvarieties for any twisted form of $\mathrm{GL}_n/\mathbb{Q}$ which is compact at infinity. Emerton's construction is apparently much more general, as his criterion is formulated for any reductive group over a number field. However, it seems to be quite rare for his criterion to hold. One might expect the weak criterion to hold more generally; in particular one might optimistically ask the following:

Question. For $\mathbb{G}/k$ connected, simple, algebraically simply connected and of real rank m, does the weak Emerton criterion always hold in dimension m?

Acknowledgements. The author benefited greatly from taking part in a study group on Emerton's work, organized by Kevin Buzzard. The author would like to thank all the participants in the London number theory seminar for many useful discussions. The author is also indebted to Prof. F. E. A. Johnson and Dr. Frank Neumann for their help with certain calculations, and to the anonymous referee for some helpful suggestions.

2 Proof of Theorem 2

Let $\mathbb{G}/k$ be simple, algebraically simply connected, and assume that $\mathbb{G}(k_\infty)$ is not compact. We shall prove in §6 that $\tilde{H}^0(\mathbb{G}, E) = E$, with the trivial action of $\mathbb{G}(\mathbb{A}_f)$. As a consequence of this, the terms $E_2^{p,0}$ in Emerton's spectral sequence (3) are Lie-algebra cohomology groups of finite dimensional representations:

$$E_2^{p,0} = H^p_{\mathrm{Lie}}(\mathfrak{g}, W).$$

Such cohomology groups are completely understood. We recall some relevant results:

Theorem 9 (Theorem 7.8.9 of [39]). *Let $\mathfrak{g}$ be a semi-simple Lie algebra over a field of characteristic zero, and let W be a finite-dimensional representation of $\mathfrak{g}$, which does not contain the trivial representation. Then we have for all $n \geq 0$,*

$$H^n_{\mathrm{Lie}}(\mathfrak{g}, W) = 0.$$

Theorem 10 (Whitehead's first lemma (Corollary 7.8.10 of [39])). *Let $\mathfrak{g}$ be a semi-simple Lie algebra over a field of characteristic zero, and let W be a finite-dimensional representation of $\mathfrak{g}$. Then we have*

$$H^1_{\mathrm{Lie}}(\mathfrak{g}, W) = 0.$$

Theorem 11 (Whitehead's second lemma (Corollary 7.8.12 of [39])). *Let $\mathfrak{g}$ be a semi-simple Lie algebra over a field of characteristic zero, and let W be a finite-dimensional representation of $\mathfrak{g}$. Then we have*

$$H^2_{\mathrm{Lie}}(\mathfrak{g}, W) = 0.$$

We shall use these results to verify Emerton's criterion in dimension 1, thus proving Theorem 2. We must verify that the edge map 4 is an isomorphism for $n = 1$ and for every irreducible algebraic representation W of $\mathbb{G}$. The small terms of the spectral sequence are:

$$E_2^{\bullet,\bullet} \quad : \quad \begin{array}{llll} \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^1(\mathbb{G}, E)) & & & \\ & H^0_{\mathrm{Lie}}(\mathfrak{g}, W) & H^1_{\mathrm{Lie}}(\mathfrak{g}, W) & H^2_{\mathrm{Lie}}(\mathfrak{g}, W) \end{array}$$

We therefore have an exact sequence:

$$0 \to H^1_{\mathrm{Lie}}(\mathfrak{g}, W) \to H^1_{\mathrm{class.}}(\mathbb{G}, W) \to \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^1(\mathbb{G}, E)) \to H^2_{\mathrm{Lie}}(\mathfrak{g}, W).$$

By Theorems 10 and 11 we know that the first and last terms are zero. Therefore the edge map is an isomorphism.

□

3 Proof of Theorem 5

Let $\mathbb{G}/k$ be connected, simple and simply connected, and assume that $\mathbb{G}(k_\infty)$ is not compact. In §8 we shall prove the isomorphism

$$\tilde{H}^1(\mathbb{G},\mathbb{Q}_p) = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)_{\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})-\mathrm{smooth}}.$$

As a consequence, we have:

Corollary 1. *If the congruence kernel of $\mathbb{G}$ is finite then $\tilde{H}^1(\mathbb{G},\mathbb{Q}_p)=0$.*

In this context, it is worth noting that the following may be proved by a similar method.

Theorem 12. *If the congruence kernel of $\mathbb{G}$ is finite then $\tilde{H}^{d-1}(\mathbb{G},\mathbb{Q}_p)=0$, where d is the dimension of the symmetric space $\mathbb{G}(k_\infty)/K_\infty$.*

We shall use the corollary to verify the weak Emerton criterion in dimension 2. Suppose first that W is a non-trivial irreducible algebraic representation of $\mathbb{G}$. We must show that the edge map (4) is an isomorphism. By Theorem 9 we know that the bottom row of the spectral sequence is zero, and by the corollary we know that the first row is zero. The small terms of the spectral sequence are as follows:

$$E_2^{\bullet,\bullet} \quad : \quad \begin{array}{cccc} \mathrm{Hom}_{\mathfrak{g}}(\check{W},\tilde{H}^2(\mathbb{G},E)_{\mathrm{loc.an.}}) & & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{array}$$

Hence in this case the edge map is an isomorphism.

In the case that W is the trivial representation, we must only verify that the edge map is injective and that its cokernel is a finite dimensional trivial representation of $\mathbb{G}(\mathbb{A}_f)$. We still know in this case that the first row of the spectral sequence is zero. For the bottom row, Theorems 10 and 11 tell us that the spectral sequence is as follows:

$$E_2^{\bullet,\bullet} \quad : \quad \begin{array}{cccc} \mathrm{Hom}_{\mathfrak{g}}(E,\tilde{H}^2(\mathbb{G},E)_{\mathrm{loc.an.}}) & & & \\ 0 & 0 & 0 & \\ E & 0 & 0 & H^3_{\mathrm{Lie}}(\mathfrak{g},E) \end{array}$$

It follows that we have an exact sequence

$$0 \to H^2_{\mathrm{class.}}(\mathbb{G},E) \to \mathrm{Hom}_{\mathfrak{g}}(E,\tilde{H}^2(\mathbb{G},E)_{\mathrm{loc.an.}}) \to H^3_{\mathrm{Lie}}(\mathfrak{g},E). \tag{6}$$

The action of $\mathbb{G}(\mathbb{A}_f)$ on $H^3_{\mathrm{Lie}}(\mathfrak{g},E)$ is trivial, since this action is defined by the (trivial) action on $\tilde{H}^0(\mathbb{G},E)=E$.

□

Remark. It is interesting to calculate the cokernel of the edge map in (6). In fact it is known that for any simple Lie algebra $\mathfrak{g}$ over a field E of characteristic zero, $H^3_{\mathrm{Lie}}(\mathfrak{g}, E) = E$. We therefore have by the Künneth formula:

$$H^3_{\mathrm{Lie}}(\mathfrak{g}, E) = E^d,$$

where d is the number of simple factors of $\mathbb{G} \times_k \bar{k}$. In particular, this is never zero. The exact sequence (6) can be continued for another term as follows:

$$0 \to H^2_{\mathrm{class.}}(\mathbb{G}, E) \to \tilde{H}^2(\mathbb{G}, E)^{\mathfrak{g}}_{\mathrm{loc.an.}} \to H^3_{\mathrm{Lie}}(\mathfrak{g}, E) \to H^3_{\mathrm{class.}}(\mathbb{G}, E)^{\mathbb{G}(\mathbb{A}_f)}.$$

In order to calculate the last term, we first choose an embedding of E in $\mathbb{C}$, and tensor with $\mathbb{C}$. There is a map

$$H^3_{\mathrm{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) \to H^3_{\mathrm{class.}}(\mathbb{G}, \mathbb{C})^{\mathbb{G}(\mathbb{A}_f)}.$$

If the k-rank of $\mathbb{G}$ is zero, then this map is an isomorphism. In other cases, it is often surjective, although the author does not know how to prove this statement in general. The groups $H^\bullet_{\mathrm{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C})$ are the cohomology groups of compact symmetric spaces (see §I.1.6 of [9]) and are completely understood. In particular, it is often the case that $H^3_{\mathrm{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$. This implies that the edge map in (6) often has a non-trivial cokernel.

4 Proof of Theorem 3

Theorem 3 is a variation on Theorem 1. In order to prove it, we recall some of the intermediate steps in Emerton's proof of Theorem 1.

In [17], Emerton introduced a new kind of Jacquet functor, $\mathrm{Jacq}_{\mathbb{B}}$, from the category of essentially admissible (in the sense of Definition 6.4.9 of [16]) locally analytic representations of $\mathbb{G}(k_\mathfrak{p})$ to the category of essentially admissible locally analytic representations of $\mathbb{T}(k_\mathfrak{p})$. This functor is left exact, and its restriction to the full subcategory of smooth representations is exact. Indeed, its restriction to smooth representations is the usual Jacquet functor of coinvariants.

Applying the Jacquet functor to the space $\tilde{H}^n(K^\mathfrak{p}, E)_{\mathrm{loc.an.}}$, one obtains an essentially admissible locally analytic representation of $\mathbb{T}(k_\mathfrak{p})$. On the other hand, the category of essentially admissible locally analytic representations of $\mathbb{T}(k_\mathfrak{p})$ is anti-equivalent to the category of coherent rigid analytic sheaves on $\hat{T}$ (Proposition 2.3.2 of [18]). We therefore have a coherent sheaf $\mathcal{E}$ on $\hat{T}$. Since the action of $\mathcal{H}(K^\mathfrak{p})$ on $\tilde{H}^n(K^\mathfrak{p}, E)_{\mathrm{loc.an.}}$ commutes with that of $\mathbb{G}(k_\mathfrak{p})$, it follows that $\mathcal{H}(K^\mathfrak{p})$ acts on $\mathcal{E}$. Let $\mathcal{A}$ be the image of $\mathcal{H}(K^\mathfrak{p})^{\mathrm{sph}}$ in the sheaf of endomorphisms of $\mathcal{E}$. Thus $\mathcal{A}$ is a coherent sheaf of commutative rings on $\hat{T}$. Writing Spec $\mathcal{A}$ for the relative spec of $\mathcal{A}$ over $\hat{T}$, we have a Zariski-closed embedding Spec $\mathcal{A} \to \hat{T} \times \mathrm{Spec}\, \mathcal{H}(K^\mathfrak{p})^{\mathrm{sph}}$. Since $\mathcal{A}$ acts as endomorphisms of $\mathcal{E}$, we may localize $\mathcal{E}$ to a coherent sheaf $\mathcal{M}$ on Spec $\mathcal{A}$.

Theorem 1 may be deduced from the following two results.

THEOREM 13 (2.3.3 of [18]). *(i) The natural projection* Spec $\mathcal{A} \to \hat{T}$ *is a finite morphism.*

(ii) The map Spec $\mathcal{A} \to \check{\mathfrak{t}}$ *has discrete fibres.*

(iii) The fibre of $\mathcal{M}$ *over a point* (χ, λ) *of* $\hat{T} \times \mathrm{Spec}\ \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$ *is dual to the* $(\mathbb{T}(k_{\mathfrak{p}}) = \chi, \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}} = \lambda)$*-eigenspace of* $\mathrm{Jacq}_{\mathbb{B}}(\tilde{H}^n(K^{\mathfrak{p}}, E)_{\mathrm{loc.an.}})$*. In particular, the point* (χ, λ) *lies in* Spec $\mathcal{A}$ *if and only if this eigenspace is non-zero.*

For any representation V of $\mathbb{G}(k_{\mathfrak{p}})$ over E, we shall write $V_{W-\mathrm{loc.alg.}}$ for the subspace of W-locally algebraic vectors in V. Note that under Emerton's criterion, we have

$$H^n_{\mathrm{class.}}(K^{\mathfrak{p}}, W) \otimes \check{W} = \tilde{H}^n(K^{\mathfrak{p}}, E)_{\check{W}-\mathrm{loc.alg.}}. \tag{7}$$

Hence $H^n_{\mathrm{class.}}(K^{\mathfrak{p}}, W) \otimes \check{W}$ is a closed subspace of $\tilde{H}^n(K^{\mathfrak{p}}, E)_{\mathrm{loc.an.}}$. By left-exactness of $\mathrm{Jacq}_{\mathbb{B}}$ we have an injective map

$$\mathrm{Jacq}_{\mathbb{B}}(H^n_{\mathrm{class.}}(K^{\mathfrak{p}}, W) \otimes \check{W}) \to \mathrm{Jacq}_{\mathbb{B}}(\tilde{H}^n(K^{\mathfrak{p}}, E)_{\mathrm{loc.an.}})$$

There are actions of $\mathbb{T}(k_{\mathfrak{p}})$ and $\mathcal{H}(K^{\mathfrak{p}})$ on these spaces, so we may restrict this map to eigenspaces:

$$\mathrm{Jacq}_{\mathbb{B}}(H^n_{\mathrm{class.}}(K^{\mathfrak{p}}, W) \otimes \check{W})^{(\chi,\lambda)} \to \mathrm{Jacq}_{\mathbb{B}}(\tilde{H}^n(K^{\mathfrak{p}}, E)_{\mathrm{loc.an.}})^{(\chi,\lambda)},$$
$$(\chi, \lambda) \in \hat{T} \times \mathrm{Spec}\ \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}.$$

The next result tells us that this restriction is often an isomorphism.

THEOREM 14 (Theorem 4.4.5 of [17]). *Let* V *be an admissible continuous representation of* $\mathbb{G}(k_{\mathfrak{p}})$ *on a Banach space. If* $\chi := \theta\psi_W \in \hat{T}(\bar{E})$ *is of non-critical slope, then the closed embedding*

$$\mathrm{Jacq}_{\mathbb{B}}(V_{W-\mathrm{loc.alg.}}) \to \mathrm{Jacq}_{\mathbb{B}}(V_{\mathrm{loc.an.}})$$

induces an isomorphism on χ*-eigenspaces.*

We recall Theorem 3.

THEOREM. *If the weak Emerton criterion holds for* $\mathbb{G}$ *in dimension* n*, then*

1. *Projection onto the first factor induces a finite map* $E(n, K^{\mathfrak{p}}) \to \hat{T}$.
2. *The map* $E(n, K^{\mathfrak{p}}) \to \check{\mathfrak{t}}$ *has discrete fibres.*
3. *If* (χ, λ) *is a point of the Eigencurve such that* χ *is locally algebraic and of non-critical slope, then either* (χ, λ) *is a classical point or* (χ, λ) *is the trivial representation of* $\mathbb{T}(k_{\mathfrak{p}}) \times \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$.

Proof. To prove the first two parts of the theorem, it is sufficient to show that $E(K^{\mathfrak{p}}, n)$ is a closed subspace of Spec $\mathcal{A}$. Since $E(n, K^{\mathfrak{p}})$ is defined to be the closure of the set of classical points, it suffices to show that each classical point is in Spec $\mathcal{A}$.

Suppose π is a representation appearing in $H^n_{\text{class.}}(K^{\mathfrak{p}}, W)$ and let $(\theta\psi_W, \lambda)$ be the corresponding classical point. This means that the (θ, λ)-eigenspace in the $\mathrm{Jacq}_{\mathbb{B}}(\pi)$ is non-zero. By exactness of the Jacquet functor on smooth representations, it follows that the (θ, λ) eigenspace in the Jacquet module of $H^n_{\text{class.}}(K^{\mathfrak{p}}, W)$ is non-zero. Hence by Proposition 4.3.6 of [17], the $(\theta\psi_W, \lambda)$-eigenspace in the Jacquet module of $H^n_{\text{class.}}(K^{\mathfrak{p}}, W) \otimes \check{W}$ is non-zero. By left-exactness of the Jacquet functor, it follows that the $(\theta\psi_W, \lambda)$ eigenspace in the Jacquet module of $\tilde{H}^n(\mathbb{G}, E)_{\text{loc.an.}}$ is non-zero. Hence by Theorem 13 (iii) it follows that the classical point is in Spec $\mathcal{A}$.

If $(\theta\psi, \lambda)$ is of non-critical slope then Theorem 14 shows that the converse also holds. □

5 Proof of Theorem 8

We first recall the statement:

Theorem. *Let $\mathbb{G}/k$ be connected, semi-simple and algebraically simply connected and assume that the weak Emerton criterion holds in dimension n. Assume also, that at least one of the following two conditions holds:*

(a) $H^p_{\text{class.}}(\mathbb{G}, \mathbb{C})$ is given by constants in dimensions $p < n$ and $H^{n+1}_{\text{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$; or

(b) $\mathbb{G}(k)$ is cocompact in $\mathbb{G}(\mathbb{A})$.

Then all the conclusions of Theorem 1 hold for the eigenvariety $E(n, K^{\mathfrak{p}})$.

Proof. To prove the theorem, we shall find a continuous admissible Banach space representation V, such that for every irreducible algebraic representation W, there is an isomorphism of smooth $\mathbb{G}(\mathbb{A}_f)$-modules

$$H^n_{\text{class.}}(\mathbb{G}, W) \cong \mathrm{Hom}_{\mathfrak{g}}(\check{W}, V_{\text{loc.an.}}). \tag{8}$$

Recall that by the weak Emerton criterion, we have an exact sequence of smooth $\mathbb{G}(\mathbb{A}_f)$-modules

$$0 \to H^n_{\text{class.}}(\mathbb{G}, E) \to \tilde{H}^n(\mathbb{G}, E)^{\mathfrak{g}}_{\text{loc.an.}} \to E^r \to 0, \qquad r \geq 0. \tag{9}$$

It follows, either from Lemma 1 or from Lemma 2 below, that all such sequences split. We therefore have a subspace $E^r \subset \tilde{H}^n(\mathbb{G}, E)$, on which $\mathbb{G}(\mathbb{A}_f)$ acts trivially. We define V to be the quotient, so that there is an exact sequence of admissible continuous representations of $\mathbb{G}(\mathbb{A}_f)$ on E-Banach spaces.

$$0 \to E^r \to \tilde{H}^n(\mathbb{G}, E) \to V \to 0. \tag{10}$$

Taking $\mathfrak{g}$-invariants of (10) and applying Whitehead's first lemma (Theorem 10), we have an exact sequence:

$$0 \to E^r \to \tilde{H}^n(\mathbb{G}, E)^{\mathfrak{g}}_{\text{loc.an.}} \to V^{\mathfrak{g}}_{\text{loc.an.}} \to 0. \tag{11}$$

On the other hand, E^r is a direct summand of $\tilde{H}^n(\mathbb{G}, E)^{\mathfrak{g}}_{\text{loc.an.}}$, so this sequence also splits. Comparing (9) and (11), we obtain

$$H^n_{\text{class.}}(\mathbb{G}, E) = V^{\mathfrak{g}}_{\text{loc.an.}} = \mathrm{Hom}_{\mathfrak{g}}(E, V_{\text{loc.an.}}).$$

This verifies (8) in the case that W is the trivial representation.
Now taking W to be a non-trivial irreducible representation, and applying $\mathrm{Hom}_{\mathfrak{g}}(\check{W}, -_{\text{loc.an.}})$ to (10), we obtain a long exact sequence:

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^n(\mathbb{G}, E)_{\text{loc.an.}}) \to \mathrm{Hom}_{\mathfrak{g}}(\check{W}, V_{\text{loc.an.}}) \to \mathrm{Ext}^1_{\mathfrak{g}}(\check{W}, E^r).$$

By Whitehead's first lemma, the final term above is zero. Hence, by the weak Emerton criterion, we have:

$$H^2_{\text{class.}}(\mathbb{G}, W) = \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^n(\mathbb{G}, E)_{\text{loc.an.}}) = \mathrm{Hom}_{\mathfrak{g}}(\check{W}, V_{\text{loc.an.}}).$$

$\square$

LEMMA 1. *Assume that $H^q_{\text{class.}}(\mathbb{G}, \mathbb{C})$ is given by constants in dimensions $q < n$ and $H^{n+1}_{\text{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$. Then*

$$\mathrm{Ext}^1_{\mathbb{G}(\mathbb{A}_f)}(E, H^n_{\text{class.}}(\mathbb{G}, E)) = 0,$$

where the Ext*-group is calculated from the category of smooth representations of $\mathbb{G}(\mathbb{A}_f)$ over E.*

Proof. Since we are dealing with smooth representations, the topology of E plays no role, so it is sufficient to prove that

$$\mathrm{Ext}^1_{\mathbb{G}(\mathbb{A}_f)}(\mathbb{C}, H^n_{\text{class.}}(\mathbb{G}, \mathbb{C})) = 0,$$

To prove this, it is sufficient to show that for every sufficiently large finite set S of finite primes of k, we have

$$\mathrm{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C}, H^n_{\text{class.}}(\mathbb{G}, \mathbb{C})) = 0.$$

For this, we shall use the spectral sequence of Borel (§3.9 of [7]; see also §2 of [6]):

$$\mathrm{Ext}^p_{\mathbb{G}(k_S)}(E, H^q_{\text{class.}}(\mathbb{G}, \mathbb{C})) \implies H^{p+q}_{S-\text{class.}}(\mathbb{G}, \mathbb{C}),$$

where $H^\bullet_{S-\text{class.}}(\mathbb{G}, -)$ denotes the direct limit over all S-congruence subgroups:

$$H^\bullet_{S-\text{class.}}(\mathbb{G}, -) = \varinjlim_{K^S} H^\bullet_{\text{Group}}(\Gamma_S(K^S), -),$$

$$\Gamma_S(K^S) = \mathbb{G}(k) \cap \left(\mathbb{G}(k_{\infty \cup S}) \times K^S \right).$$

By Proposition X.4.7 of [9], we know that

$$\operatorname{Ext}^p_{\mathbb{G}(k_S)}(\mathbb{C}, \mathbb{C}) = 0, \qquad p \geq 1.$$

Since $H^q_{\text{class.}}(\mathbb{G}, \mathbb{C})$ is a trivial representation of $\mathbb{G}(\mathbb{A}_f)$ in dimensions $q < n$, it follows from Borel's spectral sequence that $\operatorname{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C}, H^n_{\text{class.}}(\mathbb{G}, \mathbb{C}))$ injects into $H^{n+1}_{S-\text{class.}}(\mathbb{G}, \mathbb{C})$. On the other hand, it is shown in Theorem 1 of [6], that for S sufficiently large, $H^{n+1}_{S-\text{class.}}(\mathbb{G}, \mathbb{C})$ is isomorphic to $H^{n+1}_{\text{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C})$. Under the hypothesis that $H^{n+1}_{\text{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$, it follows that for S sufficiently large, $\operatorname{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C}, H^n_{\text{class.}}(\mathbb{G}, \mathbb{C})) = 0$. □

Lemma 2. *Assume that* $\mathbb{G}(k)$ *is cocompact in* $\mathbb{G}(\mathbb{A})$. *Then*

$$\operatorname{Ext}^1_{\mathbb{G}(\mathbb{A}_f)}(E, H^n_{\text{class.}}(\mathbb{G}, E)) = 0,$$

where the Ext*-group is calculated from the category of smooth representations of* $\mathbb{G}(\mathbb{A}_f)$ *over* E.

(The argument in fact shows that $\operatorname{Ext}^p_{\mathbb{G}(\mathbb{A}_f)}(E, H^q_{\text{class.}}(\mathbb{G}, E)) = 0$ for all $p > 0$.)

Proof. As in the proof of the previous lemma, we shall show that for S sufficiently large,

$$\operatorname{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C}, H^n_{\text{class.}}(\mathbb{G}, \mathbb{C})) = 0.$$

Recall that we have a decomposition:

$$L^2(\mathbb{G}(k) \backslash \mathbb{G}(\mathbb{A})) = \widehat{\bigoplus_{\pi}} m(\pi) \cdot \pi,$$

with finite multiplicities $m(\pi)$ and automorphic representations π. Here the $\hat{\oplus}$ denotes a Hilbert space direct sum. We shall write $\pi = \pi_\infty \otimes \pi_f$, where π_∞ is an irreducible unitary representation of $\mathbb{G}(k_\infty)$ and π_f is a smooth irreducible unitary representation of $\mathbb{G}(\mathbb{A}_f)$. This decomposition may by used to calculate the classical cohomology (Theorem VII.6.1 of [9]):

$$H^\bullet_{\text{class.}}(\mathbb{G}, \mathbb{C}) = \bigoplus_{\pi} m(\pi) \cdot H^\bullet_{\text{rel.Lie}}(\mathfrak{g}, K_\infty, \pi_\infty) \otimes \pi_f.$$

It is therefore sufficient to show that for each automorphic representation π, we have (for S sufficiently large) $\operatorname{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C}, \pi_f) = 0$. The smooth representation π_f decomposes as a tensor product of representations of $\mathbb{G}(k_\mathfrak{q})$ for $\mathfrak{q} \in S$, together with a representation of $\mathbb{G}(\mathbb{A}_f^S)$:

$$\pi_f = \left(\bigotimes_{\mathfrak{q} \in S} \pi_\mathfrak{q} \right) \otimes \pi_f^S.$$

This gives a decomposition of the cohomology:

$$\mathrm{Ext}^{\bullet}_{\mathbb{G}(k_S)}(\mathbb{C},\pi_f) = \left(\bigotimes_{\mathfrak{q}\in S} \mathrm{Ext}^{\bullet}_{\mathbb{G}(k_{\mathfrak{q}})}(\mathbb{C},\pi_{\mathfrak{q}})\right)\otimes \pi_f^S. \tag{12}$$

There are two cases to consider.
Case 1. Suppose π is the trivial representation, consisting of the constant functions on $\mathbb{G}(k)\backslash\mathbb{G}(\mathbb{A})$. Then by Proposition X.4.7 of [9], we have

$$\mathrm{Ext}^n_{\mathbb{G}(k_{\mathfrak{q}})}(\mathbb{C},\mathbb{C}) = 0, \qquad n\geq 1.$$

This implies by (12) that $\mathrm{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C},\mathbb{C})=0$.
Case 2. Suppose π is non-trivial, and hence contains no non-zero constant functions. If $\mathfrak{q}$ is a prime for which no factor of $\mathbb{G}(k_{\mathfrak{q}})$ is compact, then it follows from the strong approximation theorem that the local representation $\pi_{\mathfrak{q}}$ is non-trivial. This implies that

$$\mathrm{Ext}^0_{\mathbb{G}(k_{\mathfrak{q}})}(\mathbb{C},\pi_{\mathfrak{q}}) = \mathrm{Hom}_{\mathbb{G}(k_{\mathfrak{q}})}(\mathbb{C},\pi_{\mathfrak{q}})=0.$$

If S contains at least two such primes, then we have by (12)

$$\mathrm{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C},\pi_f)=0.$$

□

Remark. At first sight, it might appear that $\mathrm{Ext}^1_{\mathbb{G}(\mathbb{A}_f)}(\mathbb{C},H^n_{\mathrm{class.}}(\mathbb{G},\mathbb{C}))$ should always be zero; however this is not the case. For example, if $\mathbb{G}=\mathrm{SL}_2/\mathbb{Q}$ then

$$\mathrm{Ext}^1_{\mathrm{SL}_2(\mathbb{A}_f)}(\mathbb{C},H^1_{\mathrm{class.}}(\mathrm{SL}_2/\mathbb{Q},\mathbb{C}))=\mathbb{C}.$$

This may be verified using the spectral sequence of Borel cited above, together with the fact that $H^2_{\mathrm{rel.Lie}}(\mathfrak{sl}_2,\mathrm{SO}(2),\mathbb{C})=\mathbb{C}$.

6 PROOF OF THEOREM 6

We assume in this section that $\mathbb{G}/k$ is connected, simple and algebraically simply connected, and that $\mathbb{G}(k_\infty)$ is not compact.

PROPOSITION 1. *As topological spaces, we have* $Y(K_f)=\Gamma(K_f)\backslash\mathbb{G}(k_\infty)/K_\infty$.

Proof. By the strong approximation theorem (Theorem 4), $\mathbb{G}(k)\mathbb{G}(k_\infty)$ is a dense subgroup of $\mathbb{G}(\mathbb{A})$. Since $\mathbb{G}(k_\infty)K_f$ is open in $\mathbb{G}(\mathbb{A})$, this implies that $\mathbb{G}(k)\mathbb{G}(k_\infty)K_f$ is a dense, open subgroup of $\mathbb{G}(\mathbb{A})$. Since open subgroups are closed it follows that

$$\mathbb{G}(k)\mathbb{G}(k_\infty)K_f=\mathbb{G}(\mathbb{A}).$$

Quotienting out on the left by $\mathbb{G}(k)$, we have (as coset spaces):

$$(\mathbb{G}(k)\cap\mathbb{G}(k_\infty)K_f)\backslash(\mathbb{G}(k_\infty)K_f)=\mathbb{G}(k)\backslash\mathbb{G}(\mathbb{A}).$$

Substituting the definition of $\Gamma(K_f)$, we have:

$$\Gamma(K_f)\backslash\mathbb{G}(k_\infty)K_f = \mathbb{G}(k)\backslash\mathbb{G}(\mathbb{A}).$$

Quotienting out on the right by $K_\infty K_f$, we get:

$$\Gamma(K_f)\backslash\mathbb{G}(k_\infty)/K_\infty = Y(K_f).$$

□

In particular, this implies:

Corollary 2. *$Y(K_f)$ is connected.*

Proof. $\mathbb{G}(k_\infty)$ is connected. □

If K_f is sufficiently small then the group $\Gamma(K_f)$ is torsion-free. We shall assume that this is the case. Hence $Y(K_f)$ is a manifold. Its universal cover is $\mathbb{G}(\mathbb{R})/K$, and its fundamental group is $\Gamma(K_f)$.

Corollary 3. *If $\Gamma(K_f)$ is torsion-free then $H^\bullet(Y(K_f), -) = H^\bullet_{\mathrm{Group}}(\Gamma(K_f), -)$.*

Proof. This follows because $\Gamma(K_f)$ is the fundamental group of $Y(K_f)$, and the universal cover $\mathbb{G}(k_\infty)/K_\infty$ is contractible. See for example [36]. □

Corollary 4. *Let $\mathbb{G}/k$ be connected, simple, simply connected and assume $\mathbb{G}(k_\infty)$ is not compact. Then as $\mathbb{G}(\mathbb{A}_f)$-modules,*

$$\tilde{H}^0(\mathbb{G}, E)_{\mathrm{loc.an.}} = \tilde{H}^0(\mathbb{G}, E) = E.$$

Proof. Since every $Y(K_f)$ is connected, we have a canonical isomorphism:

$$H^0(Y(K_\mathfrak{p}K^\mathfrak{p}), \mathbb{Z}/p^s) = \mathbb{Z}/p^s.$$

Furthermore, the pull-back maps

$$H^0(Y(K_\mathfrak{p}K^\mathfrak{p}), \mathbb{Z}/p^s) \to H^0(Y(K'_\mathfrak{p}K^\mathfrak{p}), \mathbb{Z}/p^s) \qquad (K'_\mathfrak{p} \subset K_\mathfrak{p})$$

are all the identity on $\mathbb{Z}/p^s$. It follows that

$$\varinjlim_{K_\mathfrak{p}} H^0(Y(K^\mathfrak{p}K_\mathfrak{p}), \mathbb{Z}/p^s) = \mathbb{Z}/p^s.$$

Since the pull-back maps are all the identity, it follows that the action of $\mathbb{G}(k_\mathfrak{p})$ on this group is trivial. Taking the projective limit over s and tensoring with E we find that

$$\tilde{H}^0(K^\mathfrak{p}, E) = E.$$

The action of $\mathbb{G}(k_\mathfrak{p})$ is clearly still trivial, and hence every vector is locally analytic. The groups $\tilde{H}^0(K^\mathfrak{p}, E)$ for varying tame level $K^\mathfrak{p}$ form a direct system with respect to the pullback maps. These pullback maps are all the identity on E. Taking the direct limit over the tame levels, we obtain:

$$\tilde{H}^0(\mathbb{G}, E) = E.$$

Since the pullback maps are all the identity, it follows that the action of $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$ on $\tilde{H}^0(\mathbb{G}, E)$ is trivial. $\square$

7 Some Cohomology Theories

In this section we introduce some notation and recall some results, which will be needed in the proof of Theorem 7. It is worth mentioning that the theorem is much easier to prove in the case that $\mathbb{G}$ has finite congruence kernel. In that case one quite easily shows that $\tilde{H}^1 = 0$ by truncating the proof given in §8 shortly after the end of "step 1" of the proof. Furthermore, our main application (Theorem 5) requires only this easier case.

7.1 Discrete cohomology

Let G be a profinite group acting on an abelian group A. We say that the action is *smooth* if every element of A has open stabilizer in G. For a smooth G-module A, we define $H^\bullet_{\text{disc.}}(G, A)$ to be the cohomology of the complex of smooth cochains on G with values in A. Due to compactness, cochains take only finitely many values, so we have

$$H^\bullet_{\text{disc.}}(G, A) = \lim_{\overrightarrow{U}} H^\bullet_{\text{Group}}(G/U, A^U).$$

Here the limit is taken over the open normal subgroups U of G, and the cohomology groups on the right hand side are those of finite groups.

Theorem 15 (Hochschild–Serre spectral sequence (§2.6b of [35])). *Let G be a profinite group and A a discrete G-module on which G acts smoothly. Let H be a closed, normal subgroup. Then there is a spectral sequence:*

$$H^p_{\text{disc.}}(G/H, H^q_{\text{disc.}}(H, A)) \implies H^{p+q}_{\text{disc.}}(G, A).$$

For calculations with adèle groups, we need the following result on countable products of groups.

Proposition 2 (see §2.2 of [35]). *Let*

$$G = \prod_{i \in \mathbb{N}} G_i$$

be a countable product of profinite groups and let A be a discrete G-module. For any finite subset $S \subset \mathbb{N}$ we let

$$G_S = \prod_{i \in S} G_i.$$

Then

$$H^n_{\mathrm{disc.}}(G,A) = \varinjlim_{S} H^n_{\mathrm{disc.}}(G_S,A).$$

Here the limit is taken over all finite subsets with respect to the inflation maps.

Corollary 5. *Let G and A be as in the previous proposition, and assume that the action of G on A is trivial. Assume also that for a fixed n, we have:*

$$H^r_{\mathrm{disc.}}(G_i,A) = 0, \quad r = 1,\dots,n-1,\ i \in \mathbb{N}.$$

Then

$$H^n_{\mathrm{disc.}}(G,A) = \bigoplus_{i\in\mathbb{N}} H^n_{\mathrm{disc.}}(G_i,A).$$

Proof. Let $S \subset \mathbb{N}$ be a finite set and let $i \notin S$. We have a direct sum decomposition

$$G_{S\cup\{i\}} = G_S \oplus G_i.$$

Regarding this as a (trivial) group extension, we have a spectral sequence:

$$H^p_{\mathrm{disc.}}(G_S, H^q(G_i,A)) \implies H^{p+q}_{\mathrm{disc.}}(G_{S\cup\{i\}},A).$$

since the sum is direct, it follows that all the maps in the spectral sequence are zero, and we have

$$H^n_{\mathrm{disc.}}(G_{S\cup\{i\}},A) = \bigoplus_{r=0}^{n} H^{n-r}_{\mathrm{disc.}}(G_S, H^r_{\mathrm{disc.}}(G_i,A)).$$

By our hypothesis, most of these terms vanish, and we are left with:

$$H^n_{\mathrm{disc.}}(G_{S\cup\{i\}},A) = H^n_{\mathrm{disc.}}(G_S,A) \oplus H^n_{\mathrm{disc.}}(G_i,A).$$

By induction on the size of S, we deduce that

$$H^n_{\mathrm{disc.}}(G_S,A) = \bigoplus_{i\in S} H^n_{\mathrm{disc.}}(G_i,A).$$

The result follows from the previous proposition. □

7.2 Continuous cohomology

Again suppose that G is a profinite group, acting on an abelian topological group A. We call A a continuous G-module if the map $G\times A \to A$ is continuous. For a continuous G-module A, we define the continuous cohomology groups $H^\bullet_{\mathrm{cts}}(G,A)$ to be the cohomology of the complex of continuous cochains. If the topology on A is actually discrete then continuous cochains are in fact smooth, so we have

$$H^\bullet_{\mathrm{cts}}(G,A) = H^\bullet_{\mathrm{disc.}}(G,A).$$

7.3 Derived functors of inverse limit

Let **Ab** be the category of abelian groups. By a projective system in **Ab**, we shall mean a collection of objects A_s ($s \in \mathbb{N}$) and morphisms $\phi : A_{s+1} \to A_s$. We shall write $\mathbf{Ab}^{\mathbb{N}}$ for the category of projective systems in **Ab**. There is a functor

$$\varprojlim_s : \mathbf{Ab}^{\mathbb{N}} \to \mathbf{Ab}.$$

This functor is left-exact. It has right derived functors

$$\left(\varprojlim_s\right)^{\bullet} : \mathbf{Ab}^{\mathbb{N}} \to \mathbf{Ab}.$$

It turns out that $\left(\varprojlim_s\right)^n$ is zero for $n \geq 2$. The first derived functor has the following simple description due to Eilenberg. We define a homomorphism

$$\Delta : \prod_s A_s \to \prod_s A_s, \qquad \big(\Delta(a_\bullet)\big)_s = a_s - \phi(a_{s+1}).$$

With this notation we have

$$\varprojlim_s A_s = \ker \Delta.$$

Eilenberg showed that

$$\varprojlim_s{}^1 A_s = \operatorname{coker} \Delta.$$

A projective system A_s is said to satisfy the *Mittag–Leffler condition* if for every $s \in \mathbb{N}$ there is a $t \geq s$ such that for every $u \geq t$ the image of A_u in A_s is equal to the image of A_t in A_s.

Proposition 3 (Proposition 3.5.7 of [39]). *If A_s satisfies the Mittag–Leffler condition then $\varprojlim_s{}^1 A_s = 0$.*

This immediately implies:

Corollary 6 (Exercise 3.5.2 of [39]). *If A_s is a projective system of finite abelian groups then $\varprojlim_s{}^1 A_s = 0$.*

We shall use the derived functor $\varprojlim_s{}^1$ to pass between discrete and continuous cohomology:

Theorem 16 (Eilenberg–Moore Sequence (Theorem 2.3.4 of [27])). *Let G be a profinite group and A a projective limit of finite discrete continuous G-modules*

$$A = \varprojlim_s A_s.$$

Then there is an exact sequence:

$$0 \to \varprojlim_s{}^1 H^{n-1}_{\mathrm{disc.}}(G, A_s) \to H^n_{\mathrm{cts}}(G, A) \to \varprojlim_s H^n_{\mathrm{disc.}}(G, A_s) \to 0.$$

7.4 Stable Cohomology

For a continuous representation V of $\mathbb{G}(k_\mathfrak{p})$ over E, we shall write V_{st} for the set of smooth vectors. The functor $V \mapsto V_{\mathrm{st}}$ is left exact from the category of continuous admissible representations of $\mathbb{G}(k_\mathfrak{p})$ (in the sense of [33]) to the category of smooth representations. We shall write $H^\bullet_{\mathrm{st}}(\mathbb{G}(k_\mathfrak{p}), -)$ for the right-derived functors. This is called "stable cohomology" by Emerton (Definition 1.1.5 of [18]). It turns out that stable cohomology may be expressed in terms of continuous group cohomology as follows (Proposition 1.1.6 of [18]):

$$H^\bullet_{\mathrm{st}}(\mathbb{G}(k_\mathfrak{p}), V) = \varinjlim_{K_\mathfrak{p}} H^\bullet_{\mathrm{cts}}(K_\mathfrak{p}, V).$$

There is an alternative description of these derived functors which we shall also use. Let $V_{\mathrm{loc.an.}}$ denote the subspace of locally analytic vectors in V. There is an action of the Lie algebra $\mathfrak{g}$ on $V_{\mathrm{loc.an.}}$. Stable cohomology may be expressed in terms of Lie algebra cohomology as follows (Theorem 1.1.13 of [18]):

$$H^\bullet_{\mathrm{st}}(\mathbb{G}(k_\mathfrak{p}), V) = H^\bullet_{\mathrm{Lie}}(\mathfrak{g}, V_{\mathrm{loc.an.}}). \tag{13}$$

8 Proof of Theorem 7

In this section, we shall assume that $\mathbb{G}/k$ is connected, simply connected and simple, and that $\mathbb{G}(k_\infty)$ is not connected. We regard the vector space $\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Q}_p)$ as a p-adic Banach space with the supremum norm:

$$||\phi|| = \sup_{x \in \mathrm{Cong}(\mathbb{G})} |\phi(x)|_p.$$

We regard $\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}/p^s)$ as a discrete abelian group. The group $\mathbb{G}(\mathbb{A}_f)$ acts on these spaces as follows:

$$(g\phi)(x) = \phi(g^{-1}xg), \quad g \in \mathbb{G}(\mathbb{A}_f),\ x \in \mathrm{Cong}(\mathbb{G}).$$

Lemma 3. *The action of* $\mathbb{G}(\mathbb{A}_f)$ *on* $\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}/p^s)$ *is smooth.*

Proof. One may prove this directly; however it is implicit in the Hochschild–Serre spectral sequence. It is sufficient to show that the action of some open subgroup is smooth. Let K_f be a compact open subgroup of $\mathbb{G}(\mathbb{A}_f)$, and write write $\tilde{K}_f$ for the preimage of K_f in $\tilde{\mathbb{G}}(k)$. We therefore have an extension of profinite groups:

$$1 \to \mathrm{Cong}(\mathbb{G}) \to \tilde{K}_f \to K_f \to 1.$$

We shall regard $\mathbb{Z}/p^s$ as a trivial, and hence smooth, $\tilde{K}_f$-module. It follows that each $H^q_{\mathrm{disc.}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}/p^s)$ is a smooth K_f-module. On the other hand we have

$$\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}/p^s) = H^1_{\mathrm{disc.}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}/p^s).$$

□

LEMMA 4. *The action of* $\mathbb{G}(\mathbb{A}_f)$ *on the* p*-adic Banach space* $\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)$ *is continuous.*

Proof. It is sufficient to prove this for the open submodule $\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}_p)$. We have, as topological $\mathbb{G}(\mathbb{A}_f)$-modules:

$$\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}_p)=\varprojlim_{s}\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}/p^s).$$

Continuity follows from the previous Lemma. □

We shall say that a vector $v\in\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)$ is $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$-smooth if its stabilizer in $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$ is open. The set of such vectors will be written

$$\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)_{\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})-\mathrm{smooth}}.$$

THEOREM. *Assume* $\mathbb{G}/k$ *is connected, simple and simply connected, and that* $\mathbb{G}(k_\infty)$ *is not compact. Then we have an isomorphism of* $\mathbb{G}(\mathbb{A}_f)$*-modules:*

$$\tilde{H}^1(\mathbb{G},\mathbb{Q}_p)=\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)_{\mathbb{G}(\mathbb{A}^{\mathfrak{p}})-\mathrm{smooth}}.$$

Proof. Choose a level K_f small enough so that $\Gamma(K_f)$ is torsion-free. By Corollary 3, we have:

$$H^1(Y(K_f),\mathbb{Z}/p^s)=H^1_{\mathrm{Group}}(\Gamma(K_f),\mathbb{Z}/p^s).$$

Elements of $H^1_{\mathrm{Group}}(\Gamma(K_f),\mathbb{Z}/p^s)$ are group homomorphisms $\Gamma(K_f)\to\mathbb{Z}/p^s$. Let $\tilde{K}_f$ be the preimage of K_f in $\tilde{\mathbb{G}}(k)$; this is equal to the profinite completion of $\Gamma(K_f)$. It follows that homomorphisms $\Gamma(K_f)\to\mathbb{Z}/p^s$ correspond bijectively to continuous homomorphisms $\tilde{K}_f\to\mathbb{Z}/p^s$. We therefore have:

$$H^1(Y(K_f),\mathbb{Z}/p^s)=H^1_{\mathrm{disc.}}(\tilde{K}_f,\mathbb{Z}/p^s).$$

We have an extension of profinite groups:

$$1\to\mathrm{Cong}(\mathbb{G})\to\tilde{K}_f\to K_f\to 1.$$

This gives rise to a Hochschild–Serre spectral sequence (Theorem 15):

$$H^p_{\mathrm{disc.}}(K_f,H^q_{\mathrm{disc.}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}/p^s))\implies H^{p+q}_{\mathrm{disc.}}(\tilde{K}_f,\mathbb{Z}/p^s).$$

From this we have an inflation-restriction sequence containing the following terms:

$$\begin{aligned}0\to H^1_{\mathrm{disc.}}(K_f,\mathbb{Z}/p^s)\to H^1(Y(K_f),\mathbb{Z}/p^s)\to\\ \to H^1_{\mathrm{disc.}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}/p^s)^{K_f}\to H^2_{\mathrm{disc.}}(K_f,\mathbb{Z}/p^s)\end{aligned}\tag{14}$$

The proof of the theorem consists of applying the functors $\varinjlim_{K_{\mathfrak{p}}}$, $\varprojlim_{s}$, $-\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and $\varinjlim_{K^{\mathfrak{p}}}$ to the sequence (14).

Step 1. We first substitute $K_f = K_{\mathfrak{p}} K^{\mathfrak{p}}$, and apply the functor $\varinjlim_{K_{\mathfrak{p}}}$ to (14). We have by the Künneth formula:

$$\varinjlim_{K_{\mathfrak{p}}} H^1_{\mathrm{disc.}}(K_{\mathfrak{p}} K^{\mathfrak{p}}, \mathbb{Z}/p^s) = H^1_{\mathrm{disc.}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s),$$

$$\varinjlim_{K_{\mathfrak{p}}} H^2_{\mathrm{disc.}}(K_{\mathfrak{p}} K^{\mathfrak{p}}, \mathbb{Z}/p^s) = H^2_{\mathrm{disc.}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s).$$

By Lemma 3 we have:

$$\varinjlim_{K_{\mathfrak{p}}} H^1_{\mathrm{disc.}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K_{\mathfrak{p}} K^{\mathfrak{p}}} = H^1_{\mathrm{disc.}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K^{\mathfrak{p}}}.$$

Since the functor $\varinjlim_{K_{\mathfrak{p}}}$ is exact, the sequence remains exact:

$$\begin{aligned} 0 \to H^1_{\mathrm{disc.}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s) \to \varinjlim_{K_{\mathfrak{p}}} H^1(Y(K^{\mathfrak{p}} K_{\mathfrak{p}}), \mathbb{Z}/p^s) \to \\ \to H^1_{\mathrm{disc.}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K^{\mathfrak{p}}} \to H^2_{\mathrm{disc.}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s). \end{aligned} \tag{15}$$

Interlude. Before going on, we make some restrictions on the tame level $K^{\mathfrak{p}}$, and investigate the first and last terms in the sequence (15).
We shall assume that the tame level $K^{\mathfrak{p}}$ is a product of local factors:

$$K^{\mathfrak{p}} = \prod_{\mathfrak{q} \neq \mathfrak{p}} K_{\mathfrak{q}},$$

where each $K_{\mathfrak{q}}$ is a compact open subgroup of $\mathbb{G}(k_{\mathfrak{q}})$. Consider the following sets of finite primes of k:

$$S = \{\mathfrak{q} : \mathfrak{q} | p \text{ and } \mathfrak{q} \neq \mathfrak{p}\},$$

$$T = \{\mathfrak{q} : \mathfrak{q} \not| p \text{ and } K_{\mathfrak{q}} \neq [K_{\mathfrak{q}}, K_{\mathfrak{q}}]\}.$$

Both these sets are finite. We shall also assume from now on that for each prime $\mathfrak{q} \in T$, the group $K_{\mathfrak{q}}$ is chosen small enough so that it is a pro-q group, where q is the rational prime below $\mathfrak{q}$. In particular, for each $\mathfrak{q} \in T$ we have for $n \geq 1$,

$$H^n_{\mathrm{disc.}}(K_{\mathfrak{q}}, \mathbb{Z}/p^s) = 0. \tag{16}$$

We have a decomposition of $K^{\mathfrak{p}}$:

$$K^{\mathfrak{p}} = K_S \times K_T \times K^{S \cup T \cup \{\mathfrak{p}\}}, \tag{17}$$

where we are using the notation:

$$K_S = \prod_{\mathfrak{q} \in S} K_{\mathfrak{q}}, \quad K^S = \prod_{\mathfrak{q} \notin S} K_{\mathfrak{q}}.$$

By the Künneth formula and (16), (17), we have:

$$H^{\bullet}_{\mathrm{disc.}}(K^{\mathfrak{p}},\mathbb{Z}/p^s) = H^{\bullet}_{\mathrm{disc.}}(K_S K^{S\cup T\cup\{\mathfrak{p}\}},\mathbb{Z}/p^s). \tag{18}$$

By assumption, the group $K^{S\cup T\cup\{\mathfrak{p}\}}$ is perfect, so we have

$$H^1_{\mathrm{disc.}}(K^{S\cup T\cup\{\mathfrak{p}\}},\mathbb{Z}/p^s) = 0. \tag{19}$$

Again by the Künneth formula together with (18), (19), we have:

$$H^1_{\mathrm{disc.}}(K^{\mathfrak{p}},\mathbb{Z}/p^s) = H^1_{\mathrm{disc.}}(K_S,\mathbb{Z}/p^s). \tag{20}$$

$$H^2_{\mathrm{disc.}}(K^{\mathfrak{p}},\mathbb{Z}/p^s) = H^2_{\mathrm{disc.}}(K_S,\mathbb{Z}/p^s) \oplus H^2_{\mathrm{disc.}}(K^{S\cup T\cup\{\mathfrak{p}\}},\mathbb{Z}/p^s). \tag{21}$$

For each prime $\mathfrak{q} \notin S\cup T\cup\{\mathfrak{p}\}$, there is an open normal pro-q subgroup $L_{\mathfrak{q}} \subset K_{\mathfrak{q}}$. We shall write $G(\mathfrak{q})$ for the (finite) quotient group. We therefore have a Hochschild–Serre spectral sequence:

$$H^p_{\mathrm{Group}}(G(\mathfrak{q}), H^{\mathfrak{q}}_{\mathrm{disc.}}(L_{\mathfrak{q}},\mathbb{Z}/p^s)) \implies H^{p+q}_{\mathrm{disc.}}(K_{\mathfrak{q}},\mathbb{Z}/p^s).$$

This spectral sequence degenerates: for $n \ge 1$ we have

$$H^n(L_{\mathfrak{q}},\mathbb{Z}/p^s) = 0.$$

Hence,

$$H^{\bullet}_{\mathrm{disc.}}(K_{\mathfrak{q}},\mathbb{Z}/p^s) = H^{\bullet}_{\mathrm{Group}}(G(\mathfrak{q}),\mathbb{Z}/p^s), \qquad \mathfrak{q}\notin S\cup T\cup\{\mathfrak{p}\}. \tag{22}$$

Since $G(\mathfrak{q})$ is a finite perfect group, it has a universal central extension. We shall write $\pi_1(G(\mathfrak{q}))$ for the kernel of this extension, i.e. the Schur multiplier of $G(\mathfrak{q})$. By (22) we have:

$$H^2_{\mathrm{disc.}}(K_{\mathfrak{q}},\mathbb{Z}/p^s) = \mathrm{Hom}_{\mathrm{Group}}(\pi_1(G(\mathfrak{q})),\mathbb{Z}/p^s). \tag{23}$$

By Corollary 5 and (23) we have:

$$H^2_{\mathrm{disc.}}(K^{S\cup T\cup\{\mathfrak{p}\}},\mathbb{Z}/p^s) = \bigoplus_{\mathfrak{q}\notin S\cup T\cup\{\mathfrak{p}\}} \mathrm{Hom}_{\mathrm{Group}}(\pi_1(G(\mathfrak{q})),\mathbb{Z}/p^s). \tag{24}$$

From (21) and (24) we have:

$$H^2_{\mathrm{disc.}}(K^{\mathfrak{p}},\mathbb{Z}/p^s) = H^2(K_S,\mathbb{Z}/p^s) \oplus \mathrm{Hom}_{\mathrm{cts}}(\pi_1^{S\cup T\cup\{\mathfrak{p}\}},\mathbb{Z}/p^s), \tag{25}$$

where we are using the notation

$$\pi_1^{S\cup T\cup\{\mathfrak{p}\}} = \prod_{\mathfrak{q}\notin S\cup T\cup\{\mathfrak{p}\}} \pi_1(G(\mathfrak{q})).$$

The only property of $\pi_1^{S\cup T\cup\{\mathfrak{p}\}}$ which we require, is that it is a product of finite groups, not depending on s.

Step 2. We are now ready to apply the functor $\varprojlim_s$ to the sequence (15). To keep track of the exactness, we splice the sequence (15) into two:

$$0 \to H^1_{\text{disc.}}(K_S, \mathbb{Z}/p^s) \to \varinjlim_{K_\mathfrak{p}} H^1(Y(K^\mathfrak{p} K_\mathfrak{p}), \mathbb{Z}/p^s) \to A(s) \to 0, \tag{26}$$

$$0 \to A(s) \to H^1_{\text{disc.}}(\text{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K^\mathfrak{p}} \to H^2_{\text{disc.}}(K^\mathfrak{p}, \mathbb{Z}/p^s). \tag{27}$$

Step 2a. Applying the functor $\varprojlim_s$ to (26), we have a long exact sequence:

$$\begin{aligned} 0 \to \varprojlim_s H^1_{\text{disc.}}(K_S, \mathbb{Z}/p^s) \to \varprojlim_s \varinjlim_{K_\mathfrak{p}} H^1(Y(K^\mathfrak{p} K_\mathfrak{p}), \mathbb{Z}/p^s) \to \\ \to \varprojlim_s A(s) \to {\varprojlim_s}^1 H^1_{\text{disc.}}(K_S, \mathbb{Z}/p^s). \end{aligned} \tag{28}$$

In order to calculate the individual terms in (28), we shall use the Eilenberg–Moore sequence (see Theorem 16):

$$0 \to {\varprojlim_s}^1 H^{n-1}_{\text{disc.}}(K_S, \mathbb{Z}/p^s) \to H^n_{\text{cts}}(K_S, \mathbb{Z}_p) \to \varprojlim_s H^n_{\text{disc.}}(K_S, \mathbb{Z}/p^s) \to 0. \tag{29}$$

Taking $n = 1$ in (29) we have

$$\varprojlim_s H^1_{\text{disc.}}(K_S, \mathbb{Z}/p^s) = H^1_{\text{cts}}(K_S, \mathbb{Z}_p).$$

Since $[K_S, K_S]$ is open in K_S, it follows that:

$$\varprojlim_s H^1_{\text{disc.}}(K_S, \mathbb{Z}/p^s) = 0. \tag{30}$$

Also, since the groups $H^1_{\text{disc.}}(K_S, \mathbb{Z}/p^s)$ are all finite, it follows by Corollary 6 that

$${\varprojlim_s}^1 H^1_{\text{cts}}(K_S, \mathbb{Z}/p^s) = 0. \tag{31}$$

Substituting (30) and (31) into (28), we get

$$\varprojlim_s \varinjlim_{K_\mathfrak{p}} H^1(Y(K^\mathfrak{p} K_\mathfrak{p}), \mathbb{Z}/p^s) = \varprojlim_s A(s). \tag{32}$$

Step 2b. Applying the left-exact functor $\varprojlim_s$ to (27) and substituting (32) we obtain the following exact sequence:

$$\begin{aligned} 0 \to \varprojlim_s \varinjlim_{K_\mathfrak{p}} H^1(Y(K^\mathfrak{p} K_\mathfrak{p}), \mathbb{Z}/p^s) \to \varprojlim_s \left(H^1_{\text{disc.}}(\text{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K^\mathfrak{p}} \right) \\ \to \varprojlim_s H^2_{\text{disc.}}(K^\mathfrak{p}, \mathbb{Z}/p^s). \end{aligned} \tag{33}$$

We shall investigate the second and third terms in this sequence further. The functors $\varprojlim_s$ and $-^{K^{\mathfrak{p}}}$ commute, so we have

$$\varprojlim_s \left(H^1_{\text{disc.}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K^{\mathfrak{p}}}\right) = \left(\varprojlim_s H^1_{\text{disc.}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}/p^s)\right)^{K^{\mathfrak{p}}}. \tag{34}$$

Again by the Eilenberg–Moore sequence (29) we have by (34):

$$\varprojlim_s \left(H^1_{\text{disc.}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K^{\mathfrak{p}}}\right) = H^1_{\text{cts}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}_p)^{K^{\mathfrak{p}}}. \tag{35}$$

To calculate the third term in (33) we shall use (25). This shows that

$$\varprojlim_s H^2_{\text{disc.}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s) = \varprojlim_s H^2_{\text{disc.}}(K_S, \mathbb{Z}/p^s) \oplus \varprojlim_s \mathrm{Hom}_{\text{cts}}(\pi_1^{S\cup T\cup\{\mathfrak{p}\}}, \mathbb{Z}/p^s). \tag{36}$$

Since $\pi_1^{S\cup T\cup\{\mathfrak{p}\}}$ is a product of finite groups, it follows that

$$\varprojlim_s \mathrm{Hom}_{\text{cts}}(\pi_1^{S\cup T\cup\mathfrak{p}}, \mathbb{Z}/p^s) = 0.$$

Substituting this into (36), we obtain:

$$\varprojlim_s H^2_{\text{disc.}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s) = \varprojlim_s H^2_{\text{disc.}}(K_S, \mathbb{Z}/p^s). \tag{37}$$

Substituting (31) into the Eilenberg–Moore sequence (29), we have:

$$\varprojlim_s H^2_{\text{cts}}(K_S, \mathbb{Z}/p^s) = H^2_{\text{cts}}(K_S, \mathbb{Z}_p). \tag{38}$$

Substituting (38) into (37) we have:

$$\varprojlim_s H^2_{\text{cts}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s) = H^2_{\text{cts}}(K_S, \mathbb{Z}_p).$$

The sequence (33) is therefore

$$0 \to \varprojlim_s \varinjlim_{K_{\mathfrak{p}}} H^1(Y(K^{\mathfrak{p}}K_{\mathfrak{p}}), \mathbb{Z}/p^s) \to H^1_{\text{cts}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}_p)^{K^{\mathfrak{p}}} \to H^2_{\text{cts}}(K_S, \mathbb{Z}_p). \tag{39}$$

Step 3. We next apply the exact functor $-\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ to (39). Note that since $K^{\mathfrak{p}}$ and $\mathrm{Cong}(\mathbb{G})$ are compact, we have

$$C^\bullet_{\text{cts}}(K^{\mathfrak{p}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = C^\bullet_{\text{cts}}(K^{\mathfrak{p}}, \mathbb{Q}_p),$$

$$C^\bullet_{\text{cts}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = C^\bullet_{\text{cts}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Q}_p).$$

Furthermore, since $\mathbb{Q}_p$ is flat over $\mathbb{Z}_p$, we have

$$H^\bullet_{\text{cts}}(K^{\mathfrak{p}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^\bullet_{\text{cts}}(K^{\mathfrak{p}}, \mathbb{Q}_p),$$

$$H^{\bullet}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = H^{\bullet}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p).$$

Since $H^1_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}_p)$ is torsion-free, we have

$$\left(H^1_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}_p)^{K^{\mathfrak{p}}}\right)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = H^1_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)^{K^{\mathfrak{p}}}.$$

Again, since $\mathbb{Q}_p$ is flat over $\mathbb{Z}_p$, we have an exact sequence:

$$0\to \tilde{H}^1(K^{\mathfrak{p}},\mathbb{Q}_p)\to H^1_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)^{K^{\mathfrak{p}}}\to H^2_{\mathrm{cts}}(K_S,\mathbb{Q}_p). \tag{40}$$

Step 4. Applying the exact functor $\varinjlim_{K^{\mathfrak{p}}}$ to (40), we have an exact sequence

$$0\to \tilde{H}^1(\mathbb{G},\mathbb{Q}_p)\to H^1_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)_{\mathbb{G}(\mathbb{A}^{\mathfrak{p}})-\mathrm{smooth}}\to H^2_{\mathrm{st}}(\mathbb{G}(k_S),\mathbb{Q}_p). \tag{41}$$

As $\mathbb{G}(k_S)$ is a $\mathbb{Q}_p$-analytic group, the stable cohomology may be expressed in terms of Lie algebra cohomology (using (13)):

$$H^2_{\mathrm{st}}(\mathbb{G}(k_S),\mathbb{Q}_p) = H^2_{\mathrm{Lie}}(\mathfrak{g}\otimes_k k_S,\mathbb{Q}_p),$$

where we are regarding $\mathfrak{g}\otimes_k k_S$ as a Lie algebra over $\mathbb{Q}_p$. By Whitehead's second lemma (Theorem 11) we have

$$H^2_{\mathrm{st}}(\mathbb{G}(k_S),\mathbb{Q}_p)=0.$$

Hence

$$\tilde{H}^1(\mathbb{G},\mathbb{Q}_p) = H^1_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)_{\mathbb{G}(\mathbb{A}^{\mathfrak{p}})-\mathrm{smooth}}.$$

□

9 Some Examples

9.1 $\mathrm{SL}_2/\mathbb{Q}$

Let $\mathbb{G}=\mathrm{SL}_2/\mathbb{Q}$. Since $\mathfrak{g}$ is 3-dimensional, the spectral sequence has non-zero terms only in columns 0 to 3. Since arithmetic subgroups have virtual cohomological dimension 1, it follows that $\tilde{H}^n=0$ for $n>1$. Taking W to be the trivial representation, the E_2 sheet of the spectral sequence is as follows:

$$E_2^{\bullet,\bullet}\quad:\quad\begin{array}{cccc} \tilde{H}^1(\mathbb{G},E)^{\mathfrak{g}}_{\mathrm{loc.an.}} & E & 0 & 0\\ E & 0 & 0 & E\end{array}$$

The connection map $E_2^{1,1}\to E_2^{3,0}$ is an isomorphism, and the spectral sequence stabilizes at E_3 as follows:

$$E_3^{\bullet,\bullet}\quad:\quad\begin{array}{cccc} \tilde{H}^1(\mathbb{G},E)^{\mathfrak{g}}_{\mathrm{loc.an.}} & 0 & 0 & 0\\ E & 0 & 0 & 0\end{array}$$

9.2 $\mathrm{SL}_1(D)$ for an indefinite quaternion algebra D

Let k be a totally real field and let D be a quaternion algebra over k, which is indefinite at exactly one real place of k. We shall consider the group $\mathbb{G}(-) = \mathrm{SL}_1(D \otimes_k -)$ over k. Arithmetic subgroups of $\mathbb{G}$ have virtual cohomological dimension 2, so we have classical cohomology groups in dimensions 0, 1 and 2. In dimensions 0 and 2 these are given by constants, and are 1-dimensional. On the other hand it is easy to show that $\tilde{H}^2(\mathbb{G}, \mathbb{Q}_p) = 0$. The E_2 sheet of the spectral sequence is as follows:

$$E_2^{\bullet,\bullet} \quad : \quad \begin{array}{cccc} \tilde{H}^1(\mathbb{G},E)^{\mathfrak{g}}_{\mathrm{loc.an.}} & E^2 & 0 & 0 \\ E & 0 & 0 & E \end{array}$$

The connection map $E_2^{1,1} \to E_2^{3,0}$ is surjective, and the spectral sequence stabilizes at E_3 as follows:

$$E_3^{\bullet,\bullet} \quad : \quad \begin{array}{cccc} \tilde{H}^1(\mathbb{G},E)^{\mathfrak{g}}_{\mathrm{loc.an.}} & E & 0 & 0 \\ E & 0 & 0 & 0 \end{array}$$

9.3 SL_2/k for k real quadratic

Let k be a real quadratic field and consider the group $\mathbb{G} = \mathrm{SL}_2/k$. The non-zero classical cohomology groups are the following:

$$\begin{array}{rcl} H^0_{\mathrm{class.}}(\mathbb{G},W) & = & W^{\mathbb{G}}, \\ H^2_{\mathrm{class.}}(\mathbb{G},W) & & \text{infinite dimensional.} \end{array}$$

It is known in this case (see [37]) that the congruence kernel of $\mathbb{G}$ is trivial. We therefore have $\tilde{H}^1(\mathbb{G},E) = 0$, and we can also show that $\tilde{H}^3(\mathbb{G},E) = 0$. Therefore the weak Emerton criterion holds in dimension 2. We also have $H^3(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$. Therefore we may apply Theorem 8 to the eigenvariety $E(2, K^{\mathfrak{p}})$. The E_2-sheet of the spectral sequence is as follows:

$$E_2^{\bullet,\bullet} \quad : \quad \begin{array}{cccc} \tilde{H}^2(\mathbb{G},E)^{\mathfrak{g}}_{\mathrm{loc.an.}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E & 0 & 0 & E \end{array}$$

The map $\tilde{H}^2(\mathbb{G},E)^{\mathfrak{g}}_{\mathrm{loc.an.}} \to E$ in the E_3-sheet is surjective, and the spectral sequence stabilizes at the E_4-sheet:

$$E_4^{\bullet,\bullet} \quad : \quad \begin{array}{cccc} H^2_{\mathrm{class.}}(\mathbb{G},E) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 \end{array}$$

9.4 $\mathrm{SL}_3/\mathbb{Q}$

Arithmetic subgroups of $\mathrm{SL}_3(\mathbb{Q})$ have virtual cohomological dimension 4, as the symmetric space is 5-dimensional. We have the following non-zero classical cohomology groups:

$$\begin{array}{rcl} H^0_{\text{class.}}(\mathbb{G}, W) & = & W^{\mathbb{G}}, \\ H^2_{\text{class.}}(\mathbb{G}, W) & = & \text{infinite dimensional}, \\ H^3_{\text{class.}}(\mathbb{G}, W) & = & \text{infinite dimensional}. \end{array}$$

It was shown in [4] that the congruence kernel is trivial. Hence the weak Emerton criterion holds in dimension 2, and in fact the only non-zero Banach space representations are:

$$\begin{array}{rcl} \tilde{H}^0(\mathbb{G}, E) & = & E, \\ \tilde{H}^2(\mathbb{G}, E) & = & \text{infinite dimensional}, \\ \tilde{H}^3(\mathbb{G}, E) & = & \text{infinite dimensional}. \end{array}$$

Furthermore, $H^3_{\text{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$. We may therefore apply Theorem 8 to the eigenvariety $E(2, K^{\mathfrak{p}})$. One can use Poincaré duality to construct an eigenvariety interpolating $H^3_{\text{class.}}$.
The author has not been able to calculate all of the terms of the spectral sequence. However the E_2-sheet is as follows:

$$E_2^{\bullet,\bullet}: \quad \begin{array}{ccccccccc} \tilde{H}^3(\mathbb{G}, E)^{\mathfrak{g}}_{\text{loc.an.}} & ? & ? & ? & ? & ? & ? & 0 & 0 \\ \tilde{H}^2(\mathbb{G}, E)^{\mathfrak{g}}_{\text{loc.an.}} & \mathrm{Ext}^1_{\mathfrak{g}}(E, \tilde{H}^2(\mathbb{G}, E)_{\text{loc.an.}}) & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ E & 0 & 0 & E & 0 & E & 0 & 0 & E \end{array}$$

This is stable by the E_5-sheet, and most things are known:

$$E_5^{\bullet,\bullet} \quad : \quad \begin{array}{ccccccccc} ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ H^2_{\text{class.}}(\mathbb{G}, E) & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

9.5 $\mathrm{Sp}_4/\mathbb{Q}$

Arithmetic subgroups of $\mathrm{Sp}_4(\mathbb{Q})$ have cohomological dimension 5, as the symmetric space is 6-dimensional. It was shown in [5] that the congruence kernel is trivial. Furthermore $H^3(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$. We may therefore apply Theorem 8 to give a construction of the H^2-eigencurve. By Poincaré duality, it is also possible to construct a reasonable H^4-eigenvariety.

9.6 Spin$(2,l)$ $(l \geq 3)$

Let L be a $\mathbb{Z}$-lattice equipped with a quadratic form of signature $(2,l)$ with $l \geq 3$. We let $\mathbb{G}/\mathbb{Q}$ be the corresponding Spin group. This has real rank 2, and the corresponding symmetric space has dimension $2l$. The congruence kernel was shown to be trivial for such groups by Kneser [26]. Hence $\mathbb{G}$ satisfies the weak Emerton criterion in dimension 2. It turns out that $H^3(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$, so we may apply Theorem 8 to $E(2, K^{\mathfrak{p}})$.

References

[1] A. Ash, D. Pollack, and G. Stevens. Rigidity of p-adic cohomology classes of congruence subgroups of $\mathrm{GL}(n,\mathbb{Z})$. *preprint*, 2007.

[2] A. Ash and G. Stevens. p-adic deformations of cohomology classes of subgroups of $\mathrm{GL}(n,\mathbf{Z})$. *Collect. Math.*, 48 (1–2) 1–30, 1997. Journées Arithmétiques (Barcelona, 1995).

[3] A. Ash and G. Stevens. p-adic deformations of cohomology classes of subgroups of $\mathrm{GL}(N,\mathbb{Z})$: the non-ordinary case. *preprint*, April 2000.

[4] H. Bass, M. Lazard, and J.-P. Serre. Sous-groupes d'indice fini dans $\mathbf{SL}(n, \mathbf{Z})$. *Bull. Amer. Math. Soc.*, 70:385–392, 1964.

[5] H. Bass, J. Milnor, and J.-P. Serre. Solution of the congruence subgroup problem for SL_n $(n \geq 3)$ and Sp_{2n} $(n \geq 2)$. *Publications mathématiques de l'I.H.É.S.*, 33:59–137, 1967.

[6] D. Blasius, J. Franke, and F. Grunewald. Cohomology of S-arithmetic subgroups in the number field case. *Invent. Math.*, 116(1-3):75–93, 1994.

[7] A. Borel. Cohomologie de sous-groupes discrets et représentations de groupes semi-simples. In *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974)*, pages 73–112. Astérisque, No. 32–33. Soc. Math. France, Paris, 1976.

[8] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1991.

[9] A. Borel and N. R. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, vol. 67 of *Mathematical surveys and monographs.* American Mathematical Society, Providence, R.I., 2nd edition, 2000.

[10] K. Buzzard. On p-adic families of automorphic forms. In *Modular curves and abelian varieties*, volume 224 of *Progr. Math.*, pages 23–44. Birkhäuser, Basel, 2004.

[11] K. Buzzard. Eigenvarieties. *preprint*, 2006.

[12] P. Cartier. Representations of p-adic groups: a survey. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 111–155. Amer. Math. Soc., Providence, R.I., 1979.

[13] G. Chenevier. Familles p-adiques de formes automorphes pour GL_n. *J. Reine Angew. Math.*, 570:143–217, 2004.

[14] R. F. Coleman. p-adic Banach spaces and families of modular forms. *Invent. Math.*, 127(3):417–479, 1997.

[15] R. Coleman and B. Mazur. The eigencurve. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 1–113. Cambridge Univ. Press, Cambridge, 1998.

[16] M. Emerton. *Locally analytic vectors in representations of locally p-adic analytic groups.* preprint, 2004.

[17] M. Emerton. Jacquet modules of locally analytic representations of p-adic reductive groups. I. Construction and first properties. *Ann. Sci. École Norm. Sup. (4)*, 39(5):775–839, 2006.

[18] M. Emerton. On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms. *Invent. Math.*, 164(1):1–84, 2006.

[19] J. Franke. Harmonic analysis in weighted L_2-spaces. *Ann. Sci. École Norm. Sup. (4)*, 31(2):181–279, 1998.

[20] H. Hida. *Elementary theory of L-functions and Eisenstein series*, volume 26 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 1993.

[21] P. L. Kassaei. $\mathcal{P}$-adic modular forms over Shimura curves over totally real fields. *Compos. Math.*, 140(2):359–395, 2004.

[22] M. Kisin and K. F. Lai. Overconvergent Hilbert modular forms. *Amer. J. Math.*, 127(4):735–783, 2005.

[23] M. Kneser. Starke Approximation in algebraischen Gruppen. I. *J. Reine Angew. Math.*, 218:190–203, 1965.

[24] M. Kneser. Hasse principle for H^1 of simply connected groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 159–163. Amer. Math. Soc., Providence, R.I., 1966.

[25] M. Kneser. Strong approximation. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 187–196. Amer. Math. Soc., Providence, R.I., 1966.

[26] M. Kneser. Normalteiler ganzzahliger Spingruppen. *J. Reine Angew. Math.*, 311/312:191–214, 1979.

[27] J. Neukirch, A. Schmidt, and K. Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2000.

[28] V. P. Platonov. The problem of strong approximation and the Kneser-Tits hypothesis for algebraic groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 33:1211–1219, 1969.

[29] G. Prasad. Strong approximation for semi-simple groups over function fields. *Ann. of Math. (2)*, 105(3):553–572, 1977.

[30] M. S. Raghunathan. The congruence subgroup problem. *Proc. Indian Acad. Sci. Math. Sci.*, 114(4):299–308, 2004.

[31] A. S. Rapinchuk. Congruence subgroup problem for algebraic groups: old and new. *Astérisque*, (209):11, 73–84, 1992. Journées Arithmétiques, 1991 (Geneva).

[32] P. Schneider. Basic notions of rigid analytic geometry. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 369–378. Cambridge Univ. Press, Cambridge, 1998.

[33] P. Schneider and J. Teitelbaum. Banach space representations and Iwasawa theory. *Israel J. Math.*, 127:359–380, 2002.

[34] P. Schneider and J. Teitelbaum. Locally analytic distributions and p-adic representation theory, with applications to GL_2. *J. Amer. Math. Soc.*, 15(2):443–468 (electronic), 2002.

[35] J.-P. Serre. *Cohomologie galoisienne*, volume 5 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, fifth edition, 1994.

[36] J.-P. Serre. Cohomologie des groupes discrets. In *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970)*, pages 77–169. Ann. of Math. Studies, No. 70. Princeton Univ. Press, Princeton, N.J., 1971.

[37] J.-P. Serre. Le problème des groupes de congruence pour SL_2. *Ann. of Math. (2)*, 92:489–527, 1970.

[38] J. Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 29–69. Amer. Math. Soc., Providence, R.I., 1979.

[39] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

Richard Hill
University College London,
London WC1E 6BT, UK.
rih@math.ucl.ac.uk