On the Structure of Calabi-Yau Categories with a Cluster Tilting Subcategory

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Abstract. We prove that for $d \geq 2$, an algebraic d-Calabi-Yau triangulated category endowed with a d-cluster tilting subcategory is the stable category of a DG category which is perfectly $(d+1)$-Calabi-Yau and carries a non degenerate t-structure whose heart has enough projectives.

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Contents

1. Introduction

In this article, we propose a description of a class of Calabi-Yau categories using the formalism of DG-categories and the notion of 'stabilization', as used for the description of triangulated orbit categories in section 7 of [21]. For $d \geq 2$, let $\mathcal{C}$ be an algebraic d-Calabi-Yau triangulated category endowed with a d-cluster tilting subcategory $\mathcal{T}$, *cf.* [23] [18] [19], see also [3] [13] [14]. Such categories occur for example,

- in the representation-theoretic approach to Fomin-Zelevinsky's cluster algebras [12], *cf.* [6] [9] [15] and the references given there,
- in the study of Cohen-Macaulay modules over certain isolated singularities, *cf.* [17] [23] [16], and the study of non commutative crepant resolutions [36], *cf.* [17].

From $\mathcal{C}$ and $\mathcal{T}$ we construct an exact dg category $\mathcal{B}$, which is perfectly $(d+1)$-Calabi-Yau, and a non-degenerate aisle $\mathcal{U}$, *cf.* [25], in $\mathrm{H}^0(\mathcal{B})$ whose heart has enough projectives. We prove, in theorem 7.1, how to recover the category $\mathcal{C}$ from $\mathcal{B}$ and $\mathcal{U}$ using a general procedure of stabilization defined in section 7. This extends previous results of [24] to a more general framework.
It follows from [30] that for $d = 2$, up to derived equivalence, the category $\mathcal{B}$ only depends on $\mathcal{C}$ (with its enhancement) and not on the choice of $\mathcal{T}$. In the appendix, we show how to naturally extend a t-structure, *cf.* [2], on the compact objects of a triangulated category to the whole category.

Example Let k be a field, A a finite-dimensional hereditary k-algebra and $\mathcal{C} = \mathcal{C}_A$ the cluster category of A, see [7] [8], i.e. the quotient of the bounded derived category of finitely generated modules over A by the functor $F = \tau^{-1}[1]$, where τ denotes the AR-translation and [1] denotes the shift functor.
Then $\mathcal{B}$ is given by the dg algebra, see section 7 of [21],

$$B = A \oplus (DA)[-3]$$

and theorem 7.1 reduces to the equivalence

$$\mathcal{D}^b(\mathcal{B})/\mathsf{per}(\mathcal{B}) \xrightarrow{\sim} \mathcal{C}_A$$

of section 7.1 of [21].

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3. Preliminaries

Let k be a field. Let $\mathcal{E}$ be a k-linear Frobenius category with split idempotents. Suppose that its stable category $\mathcal{C} = \underline{\mathcal{E}}$, with suspension functor S, has finite-dimensional Hom-spaces and admits a Serre functor Σ, see [4]. Let $d \geq 2$ be an integer. We suppose that $\mathcal{C}$ is Calabi-Yau of CY-dimension d, *i.e.* [27] there is an isomorphism of triangle functors

$$S^d \xrightarrow{\sim} \Sigma .$$

We fix such an isomorphism once and for all. See section 4 of [23] for several examples of the above situation.
For $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$, we put

$$\mathrm{Ext}^n(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X, S^n Y) .$$

We suppose that $\mathcal{C}$ is endowed with a d-cluster tilting subcategory $\mathcal{T} \subset \mathcal{C}$, *i.e.*

a) $\mathcal{T}$ is a k-linear subcategory,
b) $\mathcal{T}$ is functorially finite in $\mathcal{C}$, i.e. the functors $\mathrm{Hom}_{\mathcal{C}}(?, X)|\mathcal{T}$ and $\mathrm{Hom}_{\mathcal{C}}(X, ?)|\mathcal{T}$ are finitely generated for all $X \in \mathcal{C}$,
c) we have $\mathrm{Ext}^i(T, T') = 0$ for all $T, T' \in \mathcal{T}$ and all $0 < i < d$ and
d) if $X \in \mathcal{C}$ satisfies $\mathrm{Ext}^i(T, X) = 0$ for all $0 < i < d$ and all $T \in \mathcal{T}$, then T belongs to $\mathcal{T}$.

Let $\mathcal{M} \subset \mathcal{E}$ be the preimage of $\mathcal{T}$ under the projection functor. In particular, $\mathcal{M}$ contains the subcategory $\mathcal{P}$ of the projective-injective objects in $\mathcal{M}$. Note that $\mathcal{T}$ equals the quotient $\underline{\mathcal{M}}$ of $\mathcal{M}$ by the ideal of morphisms factoring through a projective-injective.
We dispose of the following commutative square:

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{T} \arrow[r, hook] & \underline{\mathcal{E}} = \mathcal{C} . \end{tikzcd}$$

We use the notations of [20]. In particular, for an additive category $\mathcal{A}$, we denote by $\mathcal{C}(\mathcal{A})$ (resp. $\mathcal{C}^-(\mathcal{A})$, $\mathcal{C}^b(\mathcal{A})$, ...) the category of unbounded (resp. right bounded, resp. bounded, ...) complexes over $\mathcal{A}$ and by $\mathcal{H}(\mathcal{A})$ (resp. $\mathcal{H}^-(\mathcal{A})$, $\mathcal{H}^b(\mathcal{A})$, ...) its quotient modulo the ideal of nullhomotopic morphisms. By [26], *cf.* also [31], the projection functor $\mathcal{E} \to \underline{\mathcal{E}}$ extends to a canonical triangle functor $\mathcal{H}^b(\mathcal{E})/\mathcal{H}^b(\mathcal{P}) \to \underline{\mathcal{E}}$. This induces a triangle functor $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \underline{\mathcal{E}}$. It is shown in [30] that this functor is a localization functor. Moreover, the projection functor $\mathcal{H}^b(\mathcal{M}) \to \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ induces an equivalence from the subcategory $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$ of bounded $\mathcal{E}$-acyclic complexes with components in $\mathcal{M}$ onto its kernel. Thus, we have a short exact sequence of triangulated categories

$$0 \longrightarrow \mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{C} \longrightarrow 0 .$$

Let $\mathcal{B}$ be the dg (=differential graded) subcategory of the category $\mathcal{C}^b(\mathcal{M})_{dg}$ of bounded complexes over $\mathcal{M}$ whose objects are the $\mathcal{E}$-acyclic complexes. We denote by $G : \mathcal{H}^-(\mathcal{M}) \to \mathcal{D}(\mathcal{B}^{op})^{op}$ the functor which takes a right bounded complex X over $\mathcal{M}$ to the dg module

$$B \mapsto \mathrm{Hom}^\bullet_{\mathcal{M}}(X, B) ,$$

where B is in $\mathcal{B}$.

Remark 3.1. By construction, the functor G restricted to $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$ establishes an equivalence

$$G : \mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \xrightarrow{\sim} \operatorname{per}(\mathcal{B}^{op})^{op} .$$

Recall that if P is a right bounded complex of projectives and A is an acyclic complex, then each morphism from P to A is nullhomotopic. In particular, the complex $\operatorname{Hom}^{\bullet}_{\mathcal{M}}(P, A)$ is nullhomotopic for each P in $\mathcal{H}^-(\mathcal{P})$. Thus G takes $\mathcal{H}^-(P)$ to zero, and induces a well defined functor (still denoted by G)

$$G : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{D}(\mathcal{B}^{op})^{op} .$$

4. EMBEDDING

PROPOSITION 4.1. *The functor G is fully faithful.*

For the proof, we need a number of lemmas.
It is well-known that the category $\mathcal{H}^-(\mathcal{E})$ admits a semiorthogonal decomposition, *cf.* [5], formed by $\mathcal{H}^-(\mathcal{P})$ and its right orthogonal $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{E})$, the full subcategory of the right bounded $\mathcal{E}$-acyclic complexes. For X in $\mathcal{H}^-(\mathcal{E})$, we write

$$\mathbf{p}X \to X \to \mathbf{a}_p X \to S\mathbf{p}X$$

for the corresponding triangle, where $\mathbf{p}X$ is in $\mathcal{H}^-(\mathcal{P})$ and $\mathbf{a}_pX$ is in $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{E})$. If X lies in $\mathcal{H}^-(\mathcal{M})$, then clearly $\mathbf{a}_pX$ lies in $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$ so that we have an induced semiorthogonal decomposition of $\mathcal{H}^-(\mathcal{M})$.

LEMMA 4.1. *The functor $\Upsilon : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$ which takes X to $\mathbf{a}_pX$ is fully faithful.*

Proof. By the semiorthogonal decomposition of $\mathcal{H}^-(\mathcal{M})$, the functor $X \mapsto \mathbf{a}_pX$ induces a right adjoint of the localization functor

$$\mathcal{H}^-(\mathcal{M}) \longrightarrow \mathcal{H}^-(\mathcal{M})/\mathcal{H}^-(\mathcal{P})$$

and an equivalence of the quotient category with the right orthogonal $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$.

$$\begin{tikzcd}
& \mathcal{H}^-(\mathcal{P}) \arrow[d, hook, shift right] & \\
& \mathcal{H}^-(\mathcal{M}) \arrow[u, shift right] \arrow[d, shift right] & \mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M}) = \mathcal{H}(\mathcal{P})^{\perp} \arrow[l, hook'] \\
\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[r, hook] & \mathcal{H}^-(\mathcal{M})/\mathcal{H}^-(\mathcal{P}) \arrow[u, hook', shift right] \arrow[ur, dashed, "\sim"] &
\end{tikzcd} .$$

Moreover, it is easy to see that the canonical functor

$$\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{H}^-(\mathcal{M})/\mathcal{H}^-(\mathcal{P})$$

is fully faithful so that we obtain a fully faithful functor

$$\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$$

taking X to $\mathbf{a}_p X$. $\surd$

Remark 4.1. Since the functor G is triangulated and takes $\mathcal{H}^-(\mathcal{P})$ to zero, for X in $\mathcal{H}^b(\mathcal{M})$, the adjunction morphism $X \to \mathbf{a}_p X$ yields an isomorphism

$$G(X) \xrightarrow{\sim} G(\mathbf{a}_p X) = G(\Upsilon X)\,.$$

Let $\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ be the full subcategory of the derived category $\mathcal{D}(\mathcal{M})$ formed by the right bounded complexes whose homology modules lie in the subcategory $\operatorname{Mod}\underline{\mathcal{M}}$ of $\operatorname{Mod}\mathcal{M}$. The Yoneda functor $\mathcal{M} \to \operatorname{Mod}\mathcal{M}$, $M \mapsto M^\wedge$, induces a full embedding

$$\Psi : \mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M}) \hookrightarrow \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})\,.$$

We write $\mathcal{V}$ for its essential image. Under Ψ, the category $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$ is identified with $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})$. Let $\Phi : \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M}) \to \mathcal{D}(\mathcal{B}^{op})^{op}$ be the functor which takes X to the dg module

$$B \mapsto \operatorname{Hom}^\bullet(X_c, \Psi(B))\,,$$

where B is in $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$ and X_c is a cofibrant replacement of X for the projective model structure on $\mathcal{C}(\mathcal{M})$. Since for each right bounded complex M with components in $\mathcal{M}$, the complex $M^\wedge$ is cofibrant in $\mathcal{C}(\mathcal{M})$, it is clear that the functor $G : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{D}(\mathcal{B}^{op})^{op}$ is isomorphic to the composition $\Phi \circ \Psi \circ \Upsilon$. We dispose of the following commutative diagram

$$\begin{tikzcd}
\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[r, hook, "\Upsilon"] & \mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M}) \arrow[rr, "\Psi"] \arrow[dr, "\sim"] & & \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M}) \arrow[r, "\Phi"] & \mathcal{D}(\mathcal{B}^{op})^{op} \\
& & \mathcal{V} \arrow[ur, hook] \arrow[urr, dashed] & & \\
\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \arrow[uu, hook] \arrow[r, equal] & \mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \arrow[uu, hook] \arrow[rr, "\sim"] & & \operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M}) \arrow[uu, hook] \arrow[ul, hook] \arrow[r, "\sim"] & \operatorname{per}(\mathcal{B}^{op})^{op} \arrow[uu, hook]
\end{tikzcd}$$

Lemma 4.2. *Let Y be an object of $\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$.*

a) *Y lies in $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ iff $\mathrm{H}^p(Y)$ is a finitely presented $\underline{\mathcal{M}}$-module for all $p \in \mathbb{Z}$ and vanishes for all but finitely many p.*

b) *Y lies in $\mathcal{V}$ iff $\mathrm{H}^p(Y)$ is a finitely presented $\underline{\mathcal{M}}$-module for all $p \in \mathbb{Z}$ and vanishes for all $p \gg 0$.*

Proof. a) Clearly the condition is necessary. For the converse, suppose first that Y is a finitely presented $\underline{\mathcal{M}}$-module. Then, as an $\mathcal{M}$-module, Y admits a resolution of length $d+1$ by finitely generated projective modules by theorem 5.4 b) of [23]. It follows that Y belongs to $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})$. Since $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ is triangulated, it also contains all shifts of finitely presented $\underline{\mathcal{M}}$-modules and all extensions of shifts. This proves the converse.

b) Clearly the condition is necessary. For the converse, we can suppose without loss of generality that $Y^n = 0$, for all $n \geq 1$ and that Y^n belongs to $\mathrm{proj}\,\mathcal{M}$, for $n \leq 0$. We now construct a sequence

$$\cdots \to P_n \to \cdots \to P_1 \to P_0$$

of complexes of finitely generated projective $\mathcal{M}$-modules such that P_n is quasi-isomorphic to $\tau_{\geq -n} Y$ for each n and that, for each $p \in \mathbb{Z}$, the sequence of $\mathcal{M}$-modules P_n^p becomes stationary. By our assumptions, we have $\tau_{\geq 0} Y \xrightarrow{\sim} \mathrm{H}^0(Y)$. Since $\mathrm{H}^0(Y)$ belongs to $\mathrm{mod}\,\underline{\mathcal{M}}$, we know by theorem 5.4 c) of [23] that it belongs to $\mathrm{per}(\mathcal{M})$ as an $\mathcal{M}$-module. We define P_0 to be a finite resolution of $\mathrm{H}^0(Y)$ by finitely generated $\mathcal{M}$-modules. For the induction step, consider the following truncation triangle associated with Y

$$S^{i+1}\mathrm{H}^{-i-1}(Y) \to \tau_{\geq -i-1} Y \to \tau_{\geq -i} Y \to S^{i+2}\mathrm{H}^{-i-1}(Y)\,,$$

for $i \geq 0$. By the induction hypothesis, we have constructed $P_0, \ldots, P_i$ and we dispose of a quasi-isomorphism $P_i \xrightarrow{\sim} \tau_{\geq -i} Y$. Let Q_{i+1} be a finite resolution of $S^{i+2}\mathrm{H}^{-i-1}(Y)$ by finitely presented projective $\mathcal{M}$-modules. We dispose of a morphism $f_i : P_i \to Q_{i+1}$ and we define P_{i+1} as the cylinder of f_i. We define P as the limit of the P_i in the category of complexes. We remark that Y is quasi-isomorphic to P and that P belongs to $\mathcal{V}$. This proves the converse. $\surd$

Let X be in $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$.

Remark 4.2. Lemma 4.2 shows that the natural t-structure of $\mathcal{D}(\mathcal{M})$ restricts to a t-structure on $\mathcal{V}$. This allows us to express $\Psi(X)$ as

$$\Psi(X) \xrightarrow{\sim} \operatorname*{holim}_i \tau_{\geq -i} \Psi(X)\,,$$

where $\tau_{\geq -i}\Psi(X)$ is in $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$.

LEMMA 4.3. *We dispose of the following isomorphism*

$$\Phi(\Psi(X)) = \Phi(\operatorname*{holim}_i \tau_{\geq -i}\Psi(X)) \xrightarrow{\sim} \operatorname*{holim}_i \Phi(\tau_{\geq -i}\Psi(X))\,.$$

Proof. It is enough to show that the canonical morphism induces a quasi-isomorphism when evaluated at any object B of $\mathcal{B}$. We have

$$\Phi(\operatorname*{holim}_i \tau_{\geq -i}\Psi(X))(B) = \mathrm{Hom}^\bullet(\operatorname*{holim}_i \tau_{\geq -i}\Psi(X), B)\,,$$

but since B is a bounded complex, for each $n \in \mathbf{Z}$, the sequence

$$i \mapsto \mathrm{Hom}^n(\tau_{\geq -i}\Psi(X), B)$$

stabilizes as i goes to infinity. This implies that

$$\mathrm{Hom}^\bullet(\operatorname*{holim}_i \tau_{\geq -i}\Psi(X), B) \xleftarrow{\sim} \operatorname*{holim}_i \Phi(\tau_{\geq -i}\Psi(X))(B)\,.$$

$\surd$

LEMMA 4.4. *The functor Φ restricted to the category $\mathcal{V}$ is fully faithful.*

Proof. Let X, Y be in $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$. The following are canonically isomorphic :

$$
\begin{aligned}
& \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi\Psi X, \Phi\Psi Y) \\
& \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi\Psi Y, \Phi\Psi X) \\
(4.1) \qquad & \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\operatorname{hocolim}_i \Phi\tau_{\geq -i}\Psi Y, \operatorname{hocolim}_j \Phi\tau_{\geq -j}\Psi X) \\
& \operatorname{holim}_i \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi\tau_{\geq -i}\Psi Y, \operatorname{hocolim}_j \Phi\tau_{\geq -j}\Psi X) \\
(4.2) \qquad & \operatorname{holim}_i \operatorname{hocolim}_j \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi\tau_{\geq -i}\Psi Y, \Phi\tau_{\geq -j}\Psi X) \\
& \operatorname{holim}_i \operatorname{hocolim}_j \mathrm{Hom}_{\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(\tau_{\geq -j}\Psi X, \tau_{\geq -i}\Psi Y) \\
(4.3) \qquad & \operatorname{holim}_i \mathrm{Hom}_{\mathcal{V}}(\operatorname{holim}_j \tau_{\geq -j}\Psi X, \tau_{\geq -i}\Psi Y) \\
& \mathrm{Hom}_{\mathcal{V}}(\Psi(X), \Psi(Y)) \,.
\end{aligned}
$$

Here (4.1) is by the lemma 4.3 seen in $\mathcal{D}(\mathcal{B}^{op})$, (4.2) is by the fact that $\Phi\tau_{\geq -i}\Psi Y$ is compact and (4.3) is by the fact that $\tau_{\geq -i}\Psi Y$ is bounded. $\surd$

It is clear now that lemmas 4.1, 4.3 and 4.4 imply the proposition 4.1.

5. Determination of the image of G

Let $L_\rho : \mathcal{D}^-(\underline{\mathcal{M}}) \to \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ be the restriction functor induced by the projection functor $\mathcal{M} \to \underline{\mathcal{M}}$. L_ρ admits a left adjoint $L : \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M}) \to \mathcal{D}^-(\underline{\mathcal{M}})$ which takes Y to $Y \otimes^{\mathbb{L}}_{\mathcal{M}} \underline{\mathcal{M}}$. Let $\mathcal{B}^-$ be the dg subcategory of $\mathcal{C}^-(\mathrm{Mod}\,\mathcal{M})_{dg}$ formed by the objects of $\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ that are in the essential image of the restriction of Ψ to $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$. Let $\mathcal{B}'$ be the DG quotient, *cf.* [11], of $\mathcal{B}^-$ by its quasi-isomorphisms. It is clear that the dg categories $\mathcal{B}'$ and $\mathcal{B}$ are quasi-equivalent, *cf.* [22], and that the natural dg functor $\mathcal{M} \to \mathcal{C}^-(\mathrm{Mod}\,\mathcal{M})_{dg}$ factors through $\mathcal{B}^-$. Let $R' : \mathcal{D}(\mathcal{B}^{op})^{op} \to \mathcal{D}(\underline{\mathcal{M}}^{op})^{op}$ be the restriction functor induced by the dg functor $\underline{\mathcal{M}} \to \mathcal{B}'$. Let $\Phi' : \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M}) \to \mathcal{D}(\mathcal{B}'^{op})^{op}$ be the functor which takes X to the dg module

$$B' \mapsto \mathrm{Hom}^\bullet(X_c, B')\,,$$

where B' is in $\mathcal{B}'$ and X_c is a cofibrant replacement of X for the projective model structure on $\mathcal{C}(\mathrm{Mod}\,\mathcal{M})$. Finally let $\Gamma : \mathcal{D}(\underline{\mathcal{M}}) \to \mathcal{D}(\underline{\mathcal{M}}^{op})^{op}$ be the functor that sends Y to

$$M \mapsto \mathrm{Hom}^\bullet(Y_c, \underline{\mathcal{M}}(?, M))\,,$$

where Y_c is a cofibrant replacement of Y for the projective model structure on $\mathcal{C}(\mathrm{Mod}\,\underline{\mathcal{M}})$ and M is in $\underline{\mathcal{M}}$.

We dispose of the following diagram :

$$
\begin{tikzcd}
 & & & \mathcal{D}(\mathcal{B}^{op})^{op} & \mathcal{B} \arrow[d] \\
\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[r, hook, "\Upsilon"] & \mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M}) \arrow[r, "\Psi"] & \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M}) \arrow[r, "\Phi'"] \arrow[ur, "\Phi"] \arrow[d, "L"'] & \mathcal{D}(\mathcal{B}'^{op})^{op} \arrow[u, "\sim"'] \arrow[d, "R'"] & \mathcal{B}' \\
 & & \mathcal{D}^-(\underline{\mathcal{M}}) \arrow[r, "\Gamma"'] & \mathcal{D}(\underline{\mathcal{M}}^{op})^{op} & \underline{\mathcal{M}} \arrow[u]
\end{tikzcd}
$$

LEMMA 5.1. *The following square*

$$
\begin{tikzcd}
\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M}) \arrow[r, "\Phi'"] \arrow[d, "L"'] & \mathcal{D}(\mathcal{B}'^{op})^{op} \arrow[d, "R'"] & \mathcal{B}' \\
\mathcal{D}^-(\underline{\mathcal{M}}) \arrow[r, "\Gamma"'] & \mathcal{D}(\underline{\mathcal{M}}^{op})^{op} & \underline{\mathcal{M}} \arrow[u]
\end{tikzcd}
$$

is commutative.

Proof. By definition $(R' \circ \Phi')(X)(M)$ equals $\mathrm{Hom}^\bullet(X_c, \underline{\mathcal{M}}(?, \mathcal{M}))$. Since $\underline{\mathcal{M}}(?, M)$ identifies with $L_\rho M^\wedge$ and by adjunction, we have

$$\mathrm{Hom}^\bullet(X_c, \underline{\mathcal{M}}(?, M)) \xrightarrow{\sim} \mathrm{Hom}^\bullet(X_c, L_\rho M^\wedge) \xrightarrow{\sim} \mathrm{Hom}^\bullet((LX)_c, \underline{\mathcal{M}}(?, M)) \, ,$$

where the last member equals $(\Gamma \circ L)(X)(M)$. $\surd$

LEMMA 5.2. *The functor L reflects isomorphisms.*

Proof. Since L is a triangulated functor, it is enough to show that if $L(Y) = 0$, then $Y = 0$. Let Y be in $\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ such that $L(Y) = 0$. We can suppose, without loss of generality, that $\mathrm{H}^p(Y) = 0$ for all $p > 0$. Let us show that $\mathrm{H}^0(Y) = 0$. Indeed, since $\mathrm{H}^0(Y)$ is an $\underline{\mathcal{M}}$-module, we have $\mathrm{H}^0(Y) \cong L^0\mathrm{H}^0(Y)$, where $L^0 : \mathrm{Mod}\,\mathcal{M} \to \mathrm{Mod}\,\underline{\mathcal{M}}$ is the left adjoint of the inclusion $\mathrm{Mod}\,\underline{\mathcal{M}} \to \mathrm{Mod}\,\mathcal{M}$. Since $\mathrm{H}^p(Y)$ vanishes in degrees $p > 0$, we have

$$L^0\mathrm{H}^0(Y) = \mathrm{H}^0(LY) \, .$$

By induction, one concludes that $\mathrm{H}^p(Y) = 0$ for all $p \leq 0$. $\surd$

PROPOSITION 5.1. *An object Y of $\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ lies in the essential image of the functor $\Psi \circ \Upsilon : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ iff $\tau_{\geq -n}Y$ is in $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$, for all $n \in \mathbb{Z}$ and $L(Y)$ belongs to $\mathrm{per}(\underline{\mathcal{M}})$.*

Proof. Let X be in $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$. By lemma 4.2 a), $\tau_{\geq -n}\Psi\Upsilon(X)$ is in $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$, for all $n \in \mathbb{Z}$. Since X is a bounded complex, there exists an $s \ll 0$ such that for all $m < s$ the m-components of $\Upsilon(X)$ are in $\mathcal{P}$, which implies that $L\Psi\Upsilon(X)$ belongs to $\mathrm{per}(\underline{\mathcal{M}})$.
Conversely, suppose that Y is an object of $\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ which satisfies the conditions. By lemma 4.2, Y belongs to $\mathcal{V}$. Thus we have $Y = \Psi(Y')$ for some Y'

in $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$. We now consider Y' as an object of $\mathcal{H}^-(\mathcal{M})$ and also write Ψ for the functor $\mathcal{H}^-(\mathcal{M}) \to \mathcal{D}^-(\mathcal{M})$ induced by the Yoneda functor. We can express Y' as

$$Y' \xleftarrow{\sim} \operatorname{hocolim}_i \sigma_{\geq -i} Y' ,$$

where the $\sigma_{\geq -i}$ are the naive truncations. By our assumptions on Y', $\sigma_{\geq -i}Y'$ belongs to $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$, for all $i \in \mathbb{Z}$. The functors Ψ and L clearly commute with the naive truncations $\sigma_{\geq -i}$ and so we have

$$L(Y) = L(\Psi Y') \xleftarrow{\sim} \operatorname{hocolim}_i L(\sigma_{\geq -i}\Psi Y') \xrightarrow{\sim} \operatorname{hocolim}_i \sigma_{\geq -i} L(\Psi Y') .$$

By our hypotheses, $L(Y)$ belongs to $\mathrm{per}(\underline{\mathcal{M}})$ and so there exists an $m \gg 0$ such that

$$L(Y) = L(\Psi Y') \xleftarrow{\sim} \sigma_{\geq -m} L(\Psi Y') = L(\sigma_{\geq -m}\Psi Y') .$$

By lemma 5.2, the inclusion

$$\Psi(\sigma_{\geq -m} Y)' = \sigma_{\geq -m}\Psi Y' \longrightarrow \Psi(Y') = Y$$

is an isomorphism. But since $\sigma_{\geq -m}Y'$ belongs to $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$, Y identifies with $\Psi(\sigma_{\geq -m}Y')$. $\surd$

Remark 5.1. It is clear that if X belongs to $\mathrm{per}(\underline{\mathcal{M}})$, then $\Gamma(X)$ belongs to $\mathrm{per}(\underline{\mathcal{M}}^{op})^{op}$. We also have the following partial converse.

Lemma 5.3. *Let X be in $\mathcal{D}^-_{\mathrm{mod}\,\underline{\mathcal{M}}}(\underline{\mathcal{M}})$ such that $\Gamma(X)$ belongs to $\mathrm{per}(\underline{\mathcal{M}}^{op})^{op}$. Then X is in $\mathrm{per}(\underline{\mathcal{M}})$.*

Proof. By lemma 4.2 b) we can suppose, without loss of generality, that X is a right bounded complex with finitely generated projective components. Applying Γ, we get a perfect complex $\Gamma(X)$. In particular $\Gamma(X)$ is homotopic to zero in high degrees. But since Γ is an equivalence

$$\mathrm{proj}\,\underline{\mathcal{M}} \xrightarrow{\sim} (\mathrm{proj}\,\underline{\mathcal{M}}^{op})^{op} ,$$

it follows that X is already homotopic to zero in high degrees. $\surd$

Remark 5.2. The natural right aisle on $\mathcal{D}(\mathcal{M})$ is the full subcategory of the objects X such that $\mathsf{H}^n(X) = 0$ for all $n < 0$. The associated truncation functor $\tau_{\geq 0}$ takes $\mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ to itself. Therefore, the natural right aisle on $\mathcal{D}(\mathcal{M})$ restricts to a natural right aisle $\mathcal{U}^{op}$ on $\mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M})$.

Definition 5.1. *Let $\mathcal{U}$ be the natural left aisle in $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}$ associated with $\mathcal{U}^{op}$.*

Lemma 5.4. *The natural left aisle $\mathcal{U}$ on $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op} \xrightarrow{\sim} \mathrm{per}(\mathcal{B}^{op})$ satisfies the conditions of proposition A.1 b).*

Proof. Clearly the natural left aisle $\mathcal{U}$ in $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}$ is non-degenerate. We need to show that for each $C \in \mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}$, there is an integer N such that $\mathrm{Hom}(C, S^N U) = 0$ for each $U \in \mathcal{U}$. We dispose of the following isomorphism

$$\mathrm{Hom}_{\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}}(C, S^N\mathcal{U}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(S^{-N}\mathcal{U}^{op}, C) ,$$

where $\mathcal{U}^{op}$ denotes the natural right aisle on $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$. Since by theorem 5.4 c) of [23] an $\underline{\mathcal{M}}$-module admits a projective resolution of length $d+1$ as an $\mathcal{M}$-module and C is a bounded complex, we conclude that for $N \gg 0$

$$\mathrm{Hom}_{\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(S^{-N}\mathcal{U}^{op}, C) = 0\,.$$

This proves the lemma. $\surd$

We denote by $\tau_{\leq n}$ and $\tau_{\geq n}$, $n \in \mathbb{Z}$, the associated truncation functors on $\mathcal{D}(\mathcal{B}^{op})^{op}$.

Lemma 5.5. *The functor $\Phi : \mathcal{D}^{-}_{\underline{\mathcal{M}}}(\mathcal{M}) \to \mathcal{D}(\mathcal{B}^{op})^{op}$ restricted to the category $\mathcal{V}$ is exact with respect to the given t-structures.*

Proof. We first prove that $\Phi(\mathcal{V}_{\leq 0}) \subset \mathcal{D}(\mathcal{B}^{op})^{op}_{\leq 0}$. Let X be in $\mathcal{V}_{\leq 0}$. We need to show that $\Phi(X)$ belongs to $\mathcal{D}(\mathcal{B}^{op})^{op}_{\leq 0}$. The following have the same classes of objects :

$$\mathcal{D}(\mathcal{B}^{op})^{op}_{\leq 0}$$

$$\mathcal{D}(\mathcal{B}^{op})_{>0}$$

$$(\mathrm{per}(\mathcal{B}^{op})_{\leq 0})^{\perp} \tag{5.1}$$

$$^{\perp}(\mathrm{per}(\mathcal{B}^{op})^{op})_{>0}\,, \tag{5.2}$$

where in (5.1) we consider the right orthogonal in $\mathcal{D}(\mathcal{B}^{op})$ and in (5.2) we consider the left orthogonal in $\mathcal{D}(\mathcal{B}^{op})^{op}$. These isomorphisms show us that $\Phi(X)$ belongs to $\mathcal{D}(\mathcal{B}^{op})^{op}_{\leq 0}$ iff

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi(X), \Phi(P)) = 0\,,$$

for all $P \in \mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})_{>0}$. Now, by lemma 4.4 the functor Φ is fully faithful and so

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi(X), \Phi(P)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(X, P)\,.$$

Since X belongs to $\mathcal{V}_{\leq 0}$ and P belongs to $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})_{>0}$, we conclude that

$$\mathrm{Hom}_{\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(X, P) = 0\,,$$

which implies that $\Phi(X) \in \mathcal{D}(\mathcal{B}^{op})^{op}_{\leq 0}$. Let us now consider X in $\mathcal{V}$. We dispose of the truncation triangle

$$\tau_{\leq 0}X \to X \to \tau_{>0}X \to S\tau_{\leq 0}X\,.$$

The functor Φ is triangulated and so we dispose of the triangle

$$\Phi\tau_{\leq 0}X \to X \to \Phi\tau_{>0}X \to S\Phi\tau_{\leq 0}X\,,$$

where $\Phi\tau_{\leq 0}X$ belongs to $\mathcal{D}(\mathcal{B}^{op})^{op}_{\leq 0}$. Since Φ induces an equivalence between $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ and $\mathrm{per}(\mathcal{B}^{op})^{op}$ and $\mathrm{Hom}(P, \tau_{>0}X) = 0$, for all P in $\mathcal{V}_{\leq 0}$, we conclude that $\Phi\tau_{>0}X$ belongs to $\mathcal{D}(\mathcal{B}^{op})^{op}_{>0}$. This implies the lemma. $\surd$

Definition 5.2. *Let $\mathcal{D}(\mathcal{B}^{op})^{op}_{f}$ denote the full triangulated subcategory of $\mathcal{D}(\mathcal{B}^{op})^{op}$ formed by the objects Y such that $\tau_{\geq -n}Y$ is in $\mathrm{per}(\mathcal{B}^{op})^{op}$, for all $n \in \mathbb{Z}$, and $R(Y)$ belongs to $\mathrm{per}(\underline{\mathcal{M}}^{op})^{op}$.*

Proposition 5.2. *An object Y of $\mathcal{D}(\mathcal{B}^{op})^{op}$ lies in the essential image of the functor $G : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{D}(\mathcal{B}^{op})^{op}$ iff it belongs to $\mathcal{D}(\mathcal{B}^{op})^{op}_f$.*

Proof. Let X be in $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$. It is clear that the $\tau_{\geq -n}G(X)$ are in $\mathrm{per}(\mathcal{B}^{op})^{op}$ for all $n \in \mathbb{Z}$. By proposition 5.1 we know that $L\Psi\Upsilon(X)$ belongs to $\mathrm{per}(\underline{\mathcal{M}})$. By lemma 5.1 and remark 5.1 we conclude that $RG(X)$ belongs to $\mathrm{per}(\underline{\mathcal{M}}^{op})^{op}$. Let now Y be in $\mathcal{D}(\mathcal{B}^{op})^{op}_f$. We can express it, by the dual of lemma A.2 as the homotopy limit of the following diagram

$$\cdots \to \tau_{\geq -n-1}Y \to \tau_{\geq -n}Y \to \tau_{\geq -n+1}Y \to \cdots ,$$

where $\tau_{\geq -n}Y$ belongs to $\mathrm{per}(\mathcal{B}^{op})^{op}$, for all $n \in \mathbb{Z}$. But since Φ induces an equivalence between $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ and $\mathrm{per}(\mathcal{B}^{op})^{op}$, this last diagram corresponds to a diagram

$$\cdots \to M_{-n-1} \to M_{-n} \to M_{-n+1} \to \cdots$$

in $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$. Let $p \in \mathbb{Z}$. The relations among the truncation functors imply that the image of the above diagram under each homology functor H^p, $p \in \mathbb{Z}$, is stationary as n goes to $+\infty$. This implies that

$$\mathrm{H}^p \operatorname*{holim}_n M_{-n} \xrightarrow{\sim} \lim_n \mathrm{H}^p M_{-n} \cong \mathrm{H}^p M_j ,$$

for all $j < p$. We dispose of the following commutative diagram

$$\begin{tikzcd} \operatorname*{holim}_n M_{-n} \arrow[r] \arrow[d] & \operatorname*{holim}_n \tau_{\geq -i} M_{-n} \cong M_{-i} \\ \tau_{\geq -i} \operatorname*{holim}_n M_{-n} \arrow[ur, "\sim"] & \end{tikzcd}$$

which implies that

$$\tau_{\geq -i} \operatorname*{holim}_n M_{-n} \xrightarrow{\sim} M_{-i} ,$$

for all $i \in \mathbb{Z}$. Since $\operatorname*{holim}_n M_{-n}$ belongs to $\mathcal{V}$, lemma 4.3 allows us to conclude that $\Phi(\operatorname*{holim}_n M_{-n}) \cong Y$. We now show that $\operatorname*{holim}_n M_{-n}$ satisfies the conditions of proposition 5.1. We know that $\tau_{\geq -i} \operatorname*{holim}_n M_{-n}$ belongs to $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$, for all $i \in \mathbb{Z}$. By lemma 5.1 $(\Gamma \circ L)(\operatorname*{holim}_n M_{-n})$ identifies with $R(Y)$, which is in $\mathrm{per}(\underline{\mathcal{M}}^{op})^{op}$. Since $\operatorname*{holim}_n M_{-n}$ belongs to $\mathcal{V}$, its homologies lie in $\mathrm{mod}\,\underline{\mathcal{M}}$ and so we are in the conditions of lemma 5.1, which implies that $L(\operatorname*{holim}_n M_{-n})$ belongs to $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$. This finishes the proof. $\surd$

6. Alternative description

In this section, we present another characterization of the image of G, which was identified as $\mathcal{D}(\mathcal{B}^{op})^{op}_f$ in proposition 5.2. Let M denote an object of $\mathcal{M}$ and also the naturally associated complex in $\mathcal{H}^b(\mathcal{M})$. Since the category $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ is generated by the objects $M \in \mathcal{M}$ and the functor G is fully faithful, we remark that $\mathcal{D}(\mathcal{B}^{op})^{op}_f$ equals the triangulated subcategory of $\mathcal{D}(\mathcal{B}^{op})^{op}$ generated by the objects $G(M)$, $M \in \mathcal{M}$. The rest of this section is concerned with the problem of characterizing the objects $G(M)$, $M \in \mathcal{M}$. We denote by P_M the projective $\underline{\mathcal{M}}$-module $\underline{\mathcal{M}}(?, M)$ associated with $M \in \mathcal{M}$ and by X_M the image of M under $\Psi \circ \Upsilon$.

Lemma 6.1. *We dispose of the following isomorphism*

$$\mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(X_M, Y) \xleftarrow{\sim} \mathrm{Hom}_{\mathrm{mod}\,\underline{\mathcal{M}}}(P_M, \mathrm{H}^0(Y))\,,$$

for all $Y \in \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$.

Proof. Clearly X_M belongs to $\mathcal{D}_{\underline{\mathcal{M}}}(\mathcal{M})_{\leq 0}$ and is of the form

$$\cdots \to P_n^\wedge \to \cdots \to P_1^\wedge \to P_0^\wedge \to M^\wedge \to 0\,,$$

where $P_n \in \mathcal{P}$, $n \geq 0$. Now Yoneda's lemma and the fact that $\mathrm{H}^m(Y)(P_n) = 0$, for all $m \in \mathbb{Z}$, $n \geq 0$, imply the lemma. $\surd$

Remark 6.1. Since the functor Φ restricted to $\mathcal{V}$ is fully faithful and exact, we have

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(G(M), \Phi(Y)) \xleftarrow{\sim} \mathrm{Hom}_{\mathrm{per}(\mathcal{B}^{op})^{op}}(\Phi(P_M), \mathrm{H}^0(\Phi(Y)))\,,$$

for all $Y \in \mathcal{V}$.

We now characterize the objects $G(M) = \Phi(X_M)$, $M \in \mathcal{M}$, in the triangulated category $\mathcal{D}(\mathcal{B}^{op})$. More precisely, we give a description of the functor

$$R_M := \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(?, \Phi(X_M)) : \mathcal{D}(\mathcal{B}^{op})^{op} \to \mathrm{Mod}\, k$$

using an idea of M. Van den Bergh, *cf.* lemma 2.13 of [10]. Consider the following functor

$$F_M := \mathrm{Hom}_{\mathrm{per}(\mathcal{B}^{op})}(\mathrm{H}^0(?), \Phi(P_M)) : \mathrm{per}(\mathcal{B}^{op})^{op} \to \mathrm{mod}\, k\,.$$

Remark 6.2. Remark 6.1 shows that the functor R_M when restricted to $\mathrm{per}(\mathcal{B}^{op})$ coincides with F_M.

Let DF_M be the composition of F_M with the duality functor $D = \mathrm{Hom}(?, k)$. Note that DF_M is homological.

Lemma 6.2. *We dispose of the following isomorphism of functors on* $\mathrm{per}(\mathcal{B}^{op})$

$$DF_M \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi(X_M), ?[d+1])\,.$$

Proof. The following functors are canonically isomorphic to $DF\Phi$:

$$\begin{aligned}
& D\mathrm{Hom}_{\mathrm{per}(\mathcal{B}^{op})}(\mathrm{H}^0\Phi(?), \Phi(P_M)) \\
(6.1)\quad & D\mathrm{Hom}_{\mathrm{per}(\mathcal{B}^{op})}(\Phi\mathrm{H}^0(?), \Phi(P_M)) \\
(6.2)\quad & D\mathrm{Hom}_{\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(P_M, \mathrm{H}^0(?)) \\
(6.3)\quad & D\mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(X_M, ?) \\
(6.4)\quad & \mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(?[-d-1], X_M) \\
(6.5)\quad & \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi(?)[-d-1], \Phi(X_M)) \\
(6.6)\quad & \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi(X_M), \Phi(?)[d+1])
\end{aligned}$$

Step (6.1) follows from the fact that Φ is exact. Step (6.2) follows from the fact that Φ is fully faithful and we are considering the opposite category. Step (6.3) is a consequence of lemma 6.1. Step (6.4) follows from the $(d+1)$-Calabi-Yau property and remark 4.2. Step (6.5) is a consequence of Φ being fully faithful and step (6.6) is a consequence of working in the opposite category. Since the functor Φ^{op} establish an equivalence between $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}$ and $\mathrm{per}(\mathcal{B}^{op})$ the lemma is proven. $\surd$

Now, since the category $\mathrm{Mod}\, k$ is cocomplete, we can consider the left Kan extension, *cf.* [28], E_M of DF_M along the inclusion $\mathrm{per}(\mathcal{B}^{op}) \hookrightarrow \mathcal{D}(\mathcal{B}^{op})$. We dispose of the following commutative square :

$$\begin{tikzcd} \mathrm{per}(\mathcal{B}^{op}) \arrow[r, "DF_M"] \arrow[d, hook] & \mathrm{mod}\, k \arrow[d, hook] \\ \mathcal{D}(\mathcal{B}^{op}) \arrow[r, dashed, "E_M"'] & \mathrm{Mod}\, k\,. \end{tikzcd}$$

For each X of $\mathcal{D}(\mathcal{B}^{op})$, the comma-category of morphisms $P \to X$ from a perfect object P to X is filtered. Therefore, the functor E_M is homological. Moreover, it preserves coproducts and so DE_M is cohomological and transforms coproducts into products. Since $\mathcal{D}(\mathcal{B}^{op})$ is a compactly generated triangulated category, the Brown representability theorem, *cf.* [29], implies that there is a $Z_M \in \mathcal{D}(\mathcal{B}^{op})$ such that

$$DE_M \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(?, Z_M)\,.$$

Remark 6.3. Since the duality functor D establishes and anti-equivalence in $\mathrm{mod}\, k$, the functor DE_M restricted to $\mathrm{per}(\mathcal{B}^{op})$ is isomorphic to F_M.

Theorem 6.1. *We dispose of an isomorphism*

$$G(M) \xrightarrow{\sim} Z_M\,.$$

Proof. We now construct a morphism of functors from R_M to DE_M. Since R_M is representable, by Yoneda's lemma it is enough to construct an element in $DE_M(\Phi(X_M))$. Let $\mathcal{C}$ be the category $\mathrm{per}(\mathcal{B}^{op}) \downarrow \Phi(X_M)$, whose objects are the morphisms $Y' \to \Phi(X_M)$ and let $\mathcal{C}'$ be the category $X_M \downarrow \mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$, whose objects are the morphisms $X_M \to X'$. The following are canonically isomorphic :

$$DE_M(\Phi(X_M))$$

$$D\,\mathrm{colim}_{\mathcal{C}}\mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi(X_M), Y'[d+1]) \tag{6.7}$$

$$D\,\mathrm{colim}_{\mathcal{C}'}\mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(X'[-d-1], X_M) \tag{6.8}$$

$$\mathrm{D}\,\mathrm{colim}_{i}\,\mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}((\tau_{\geq -i}X_M)[-d-1], X_M) \tag{6.9}$$

$$\lim_{i}\,\mathrm{DHom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}((\tau_{\geq -i}X_M)[-d-1], X_M)$$

$$\lim_{i}\,\mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(X_M, \tau_{\geq -i}X_M) \tag{6.10}$$

Step (6.7) is a consequence of the definition of the left Kan extension and lemma 6.2. Step (6.8) is obtained by considering the opposite category. Step (6.9) follows from the fact that the system $(\tau_{\geq -i}X_M)_{i\in\mathbb{Z}}$ forms a cofinal system for the index system of the colimit. Step (6.10) follows from the $(d+1)$-Calabi-Yau property. Now, the image of the identity by the canonical morphism

$$\mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(X_M, X_M) \longrightarrow \lim_i \mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(X_M, \tau_{\geq -i}X_M)\,,$$

gives us an element of $(DE_M)(\Phi(X_M))$ and so a morphism of functors from R_M to DE_M. We remark that this morphism is an isomorphism when evaluated at the objects of $\mathrm{per}(\mathcal{B}^{op})$. Since both functors R_M and DE_M are cohomological, transform coproducts into products and $\mathcal{D}(\mathcal{B}^{op})$ is compactly generated, we conclude that we dispose of an isomorphism

$$G(M) \xrightarrow{\sim} Z_M\,.$$

$\surd$

7. The main theorem

Consider the following commutative square as in section 3:

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{T} \arrow[r, hook] & \underline{\mathcal{E}} = \mathcal{C}\,. \end{tikzcd}$$

In the previous sections we have constructed, from the above data, a dg category $\mathcal{B}$ and a left aisle $\mathcal{U} \subset \mathrm{H}^0(\mathcal{B})$, see [25], satisfying the following conditions :

- $\mathcal{B}$ is an exact dg category over k such that $\mathrm{H}^0(\mathcal{B})$ has finite-dimensional Hom-spaces and is Calabi-Yau of CY-dimension $d+1$,
- $\mathcal{U} \subset \mathrm{H}^0(\mathcal{B})$ is a non-degenerate left aisle such that :
  - for all $B \in \mathcal{B}$, there is an integer N such that $\mathrm{Hom}_{\mathrm{H}^0(\mathcal{B})}(B, S^N U) = 0$ for each $U \in \mathcal{U}$,
  - the heart $\mathcal{H}$ of the t-structure on $\mathrm{H}^0(\mathcal{B})$ associated with $\mathcal{U}$ has enough projectives.

Let now $\mathcal{A}$ be a dg category and $\mathcal{W} \subset \mathrm{H}^0(\mathcal{A})$ a left aisle satisfying the above conditions. We can consider the following general construction : Let $\mathcal{Q}$ denote the category of projectives of the heart $\mathcal{H}$ of the t-structure on $\mathrm{H}^0(\mathcal{A})$ associated with $\mathcal{W}$. We claim that the following inclusion

$$\mathcal{Q} \hookrightarrow \mathcal{H} \hookrightarrow \mathrm{H}^0(\mathcal{A})\,,$$

lifts to a morphism $\mathcal{Q} \xrightarrow{j} \mathcal{A}$ in the homotopy category of small dg categories, *cf.* [20] [32] [33] [34] [35]. Indeed, recall the following argument from section 7 of [22]: Let $\tilde{\mathcal{Q}}$ be the full dg subcategory of $\mathcal{A}$ whose objects are the same as those of $\mathcal{Q}$. Let $\tau_{\leq 0}\tilde{\mathcal{Q}}$ denote the dg category obtained from $\tilde{\mathcal{Q}}$ by applying the truncation functor $\tau_{\leq 0}$ of complexes to each Hom-space. We dispose of the following diagram in the category of small dg categories

$$\begin{tikzcd} & \tilde{\mathcal{Q}} \arrow[r, hook] & \mathcal{A} \\ & \tau_{\leq 0}\tilde{\mathcal{Q}} \arrow[u] \arrow[d] & \\ \mathcal{Q} \arrow[r, equal] & \mathrm{H}^0(\tilde{\mathcal{Q}}) & . \end{tikzcd}$$

Let X, Y be objects of $\mathcal{Q}$. Since X and Y belong to the heart of a t-structure in $\mathrm{H}^0(\mathcal{A})$, we have

$$\mathrm{Hom}_{\mathrm{H}^0(\mathcal{A})}(X, Y[-n]) = 0\,,$$

for $n \geq 1$. The dg category $\mathcal{A}$ is exact, which implies that

$$\mathrm{H}^{-n}\mathrm{Hom}^{\bullet}_{\tilde{\mathcal{Q}}}(X,Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{H}^0(\mathcal{A})}(X, Y[-n]) = 0\,,$$

for $n \geq 1$. This shows that the dg functor $\tau_{\leq 0}\tilde{\mathcal{Q}} \rightarrow \mathrm{H}^0(\tilde{\mathcal{Q}})$ is a quasi-equivalence and so we dispose of a morphism $\mathcal{Q} \xrightarrow{j} \mathcal{A}$ in the homotopy category of small dg categories. We dispose of a triangle functor $j^* : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{Q})$ given by restriction. By proposition A.1, the left aisle $\mathcal{W} \subset \mathrm{H}^0(\mathcal{A})$ admits a smallest extension to a left aisle $\mathcal{D}(\mathcal{A}^{op})^{op}_{\leq 0}$ on $\mathcal{D}(\mathcal{A}^{op})^{op}$. Let $\mathcal{D}(\mathcal{A}^{op})^{op}_{f}$ denote the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{op})^{op}$ formed by the objects Y such that $\tau_{\geq -n}Y$ is in $\mathrm{per}(\mathcal{A}^{op})^{op}$, for all $n \in \mathbb{Z}$, and $j^*(Y)$ belongs to $\mathrm{per}(\mathcal{Q}^{op})^{op}$.

Definition 7.1. *The stable category of $\mathcal{A}$ with respect to $\mathcal{W}$ is the triangle quotient*

$$\mathrm{stab}(\mathcal{A}, \mathcal{W}) = \mathcal{D}(\mathcal{A}^{op})^{op}_{f}/\mathrm{per}(\mathcal{A}^{op})^{op}\,.$$

We are now able to formulate the main theorem. Let $\mathcal{B}$ be the dg category and $\mathcal{U} \subset \mathrm{H}^0(\mathcal{B})$ the left aisle constructed in sections 1 to 5.

Theorem 7.1. *The functor G induces an equivalence of categories*

$$\tilde{G} : \mathcal{C} \xrightarrow{\sim} \mathrm{stab}(\mathcal{B}, \mathcal{U})\,.$$

Proof. We dispose of the following commutative diagram :

$$\begin{tikzcd} \mathcal{C} \arrow[r, dashed, "\tilde{G}", "\sim"'] & \mathrm{stab}(\mathcal{B},\mathcal{U}) \\ \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[u] \arrow[r, "G", "\sim"'] & \mathcal{D}(\mathcal{B}^{op})^{op}_{f} \arrow[u] \\ \mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \arrow[u] \arrow[r, "\sim"'] & \mathrm{per}(\mathcal{B}^{op})^{op}\,. \arrow[u] \end{tikzcd}$$

The functor G is an equivalence since it is fully faithful by proposition 4.1 and essentially surjective by proposition 5.2. Since we dispose of an equivalence $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \xrightarrow{\sim} \mathrm{per}(\mathcal{B}^{op})^{op}$ by construction of $\mathcal{B}$ and the columns of the above diagram are short exact sequences of triangulated categories, the theorem is proved. $\surd$

Appendix A. Extension of t-structures

Let $\mathcal{T}$ be a compactly generated triangulated category with suspension functor S. We denote by $\mathcal{T}_c$ the full triangulated sub-category of $\mathcal{T}$ formed by the compact objects, see [29]. We use the terminology of [25]. Let $\mathcal{U} \subseteq \mathcal{T}_c$ be a left aisle on $\mathcal{T}_c$, i.e. a full additive subcategory $\mathcal{U}$ of $\mathcal{T}_c$ which satisfies:

a) $S\mathcal{U} \subset \mathcal{U}$,

b) $\mathcal{U}$ is stable under extensions, i.e. for each triangle

$$X \to Y \to Z \to SX$$

of $\mathcal{T}_c$, we have $Y \in \mathcal{U}$ whenever $X, Z \in \mathcal{U}$ and

c) the inclusion functor $\mathcal{U} \hookrightarrow \mathcal{T}_c$ admits a right adjoint.

As shown in [25], the concept of aisle is equivalent to that of t-structure.

Proposition A.1. a) *The left aisle $\mathcal{U}$ admits a smallest extension to a left aisle $\mathcal{T}_{\leq 0}$ on $\mathcal{T}$.*

b) *If $\mathcal{U} \subseteq \mathcal{T}_c$ is non-degenerate (*i.e., *$f : X \to Y$ is invertible iff $\mathrm{H}^p(f)$ is invertible for all $p \in \mathbb{Z}$) and for each $X \in \mathcal{T}_c$, there is an integer N such that $\mathrm{Hom}(X, S^N U) = 0$ for each $U \in \mathcal{U}$, then $\mathcal{T}_{\leq 0}$ is also non-degenerate.*

Proof. a) Let $\mathcal{T}_{\leq 0}$ be the smallest full subcategory of $\mathcal{T}$ that contains $\mathcal{U}$ and is stable under infinite sums and extensions. It is clear that $\mathcal{T}_{\leq 0}$ is stable by S since $\mathcal{U}$ is. We need to show that the inclusion functor $\mathcal{T}_{\leq 0} \hookrightarrow \mathcal{T}$ admits a right adjoint. For completeness, we include the following proof, which is a variant of the 'small object argument', *cf.* also [1]. We dispose of the following recursive procedure. Let $X = X_0$ be an object in $\mathcal{T}$. For the initial step consider all morphisms from any object P in $\mathcal{U}$ to X_0. This forms a set I_0 since $\mathcal{T}$ is compactly generated and so we dispose of the following triangle

$$\coprod_{f \in I_0} P \longrightarrow X_0 \longrightarrow X_1 \rightsquigarrow \coprod_{f \in I_0} P\,.$$

For the induction step consider the above construction with X_n, $n \geq 1$, in the place of X_{n-1} and I_n in the place of I_{n-1}. We dispose of the following diagram

$$\begin{tikzcd}
X = X_0 \arrow[r] & X_1 \arrow[r] \arrow[dl, squiggly] & X_2 \arrow[r] \arrow[dl, squiggly] & X_3 \arrow[r] \arrow[dl, squiggly] & \cdots \arrow[r] & X' \\
\coprod_{f \in I_0} P \arrow[u] & \coprod_{f \in I_1} P \arrow[u] & \coprod_{f \in I_2} P \arrow[u] & \coprod_{f \in I_3} P \arrow[u] & &
\end{tikzcd},$$

where X' denotes the homotopy colimit of the diagram $(X_i)_{i\in\mathbb{Z}}$. Consider now the following triangle

$$S^{-1}X' \to X'' \to X \to X'\,,$$

where the morphism $X \to X'$ is the transfinite composition in our diagram. Let P be in $\mathcal{U}$. We remark that since P is compact, $\mathrm{Hom}_{\mathcal{T}}(P, X') = 0$. This also implies, by construction of $\mathcal{T}_{\leq 0}$, that $\mathrm{Hom}_{\mathcal{T}}(R, X') = 0$, for all R in $\mathcal{T}_{\leq 0}$. The long exact sequence obtained by applying the functor $\mathrm{Hom}_{\mathcal{T}}(R, ?)$ to the triangle above shows that

$$\mathrm{Hom}(R, X'') \xrightarrow{\sim} \mathrm{Hom}(R, X)\,.$$

Let X''_{n-1}, $n \geq 1$, be an object as in the following triangle

$$X = X_0 \to X_n \to X''_{n-1} \to S(X)\,.$$

A recursive application of the octahedron axiom implies that X''_{n-1} belongs to $S(\mathcal{T}_{\leq 0})$, for all $n \geq 1$. We dispose of the isomorphism

$$\underset{n}{\mathrm{hocolim}}\, X''_{n-1} \xrightarrow{\sim} S(X'')\,.$$

Since $\underset{n}{\mathrm{hocolim}}\, X''_{n-1}$ belongs to $S(\mathcal{T}_{\leq 0})$, we conclude that X'' belongs to $\mathcal{T}_{\leq 0}$. This shows that the functor that sends X to X'' is the right adjoint of the inclusion functor $\mathcal{T}_{\leq 0} \hookrightarrow \mathcal{T}$. This proves that $\mathcal{T}_{\leq 0}$ is a left aisle on $\mathcal{T}$. We now show that the t-structure associated to $\mathcal{T}_{\leq 0}$, *cf.* [25], extends, from $\mathcal{T}_c$ to $\mathcal{T}$, the one associated with $\mathcal{U}$. Let X be in $\mathcal{T}_c$. We dispose of the following truncation triangle associated with $\mathcal{U}$

$$X_{\mathcal{U}} \to X \to X^{\mathcal{U}^{\perp}} \to SX_{\mathcal{U}}\,.$$

Clearly $X_{\mathcal{U}}$ belongs to $\mathcal{T}_{\leq 0}$. We remark that $\mathcal{U}^{\perp} = \mathcal{T}_{\leq 0}^{\perp}$, and so $X^{\mathcal{U}^{\perp}}$ belongs to $\mathcal{T}_{>0} := \mathcal{T}_{\leq 0}^{\perp}$.
We now show that $\mathcal{T}_{\leq 0}$ is the smallest extension of the left aisle $\mathcal{U}$. Let $\mathcal{V}$ be an aisle containing $\mathcal{U}$. The inclusion functor $\mathcal{V} \hookrightarrow \mathcal{T}$ commutes with sums, because it admits a right adjoint. Since $\mathcal{V}$ is stable under extensions and suspensions, it contains $\mathcal{T}_{\leq 0}$.

b) Let X be in $\mathcal{T}$. We need to show that $X = 0$ iff $\mathrm{H}^p(X) = 0$ for all $p \in \mathbb{Z}$. Clearly the condition is necessary. For the converse, suppose that $\mathrm{H}^p(X) = 0$ for all $p \in \mathbb{Z}$. Let n be an integer. Consider the following truncation triangle

$$\mathrm{H}^{n+1}(X) \to \tau_{>n}X \to \tau_{>n+1}X \to S\mathrm{H}^{n+1}(X)\,.$$

Since $\mathrm{H}^{n+1}(X) = 0$ we conclude that

$$\tau_{>n}X \in \bigcap_{m\in\mathbb{Z}} \mathcal{T}_{>m}\,,$$

for all $n \in \mathbb{Z}$. Now, let C be a compact object of $\mathcal{T}$. We know that there is a $k \in \mathbb{Z}$ such that $C \in \mathcal{T}_{\leq k}$. This implies that

$$\mathrm{Hom}_{\mathcal{T}}(C, \tau_{>n}X) = 0$$

for all $n \in \mathbb{Z}$, since $\tau_{>n}X$ belongs to $(\mathcal{T}_{\leq k})^{\perp}$. The category $\mathcal{T}$ is compactly generated and so we conclude that $\tau_{>n}X = 0$, for all $n \in \mathbb{Z}$. The following truncation triangle

$$\tau_{\leq n}X \to X \to \tau_{>n}X \to S\tau_{\leq n}X \,,$$

implies that $\tau_{\leq n}X$ is isomorphic to X for all $n \in \mathbb{Z}$. This can be rephrased as saying that

$$X \in \bigcap_{n\in\mathbb{N}} \mathcal{T}_{\leq -n} \,.$$

Now by our hypothesis there is an integer N such that

$$\mathrm{Hom}_{\mathcal{T}}(C, \mathcal{U}_{\leq -N}) = 0 \,.$$

Since C is compact and by construction of $\mathcal{T}_{\leq -N}$, we have

$$\mathrm{Hom}_{\mathcal{T}}(C, \mathcal{T}_{\leq -N}) = 0 \,.$$

This implies that $\mathrm{Hom}_{\mathcal{T}}(C, X) = 0$, for all compact objects C of $\mathcal{T}$. Since $\mathcal{T}$ is compactly generated, we conclude that $X = 0$. This proves the converse. $\surd$

Lemma A.1. *Let $(Y_p)_{p\in\mathbb{Z}}$ be in $\mathcal{T}$. We dispose of the following isomorphism*

$$\mathrm{H}^n\,(\coprod_p Y_p) \xleftarrow{\sim} \coprod_p \mathrm{H}^n(Y_p)\,,$$

for all $n \in \mathbb{Z}$.

Proof. By definition $\mathrm{H}^n := \tau_{\geq n}\,\tau_{\leq n}\,, n \in \mathbb{Z}$. Since $\tau_{\geq n}$ admits a right adjoint, it is enough to show that $\tau_{\leq n}$ commute with infinite sums. We consider the following triangle

$$\coprod_p \tau_{\leq n}Y_p \to \coprod_p Y_p \to \coprod_p \tau_{>n}Y_p \to S(\coprod_p \tau_{\leq n}Y_p)\,.$$

Here $\coprod_p \tau_{\leq n}Y_p$ belongs to $\mathcal{T}_{\leq n}$ since $\mathcal{T}_{\leq n}$ is stable under infinite sums. Let P be an object of $S^n\mathcal{U}$. Since P is compact, we have

$$\mathrm{Hom}_{\mathcal{T}}(P, \coprod_p \tau_{>n}Y_p) \xleftarrow{\sim} \coprod_p \mathrm{Hom}_{\mathcal{T}}(P, \tau_{>n}Y_p) = 0\,.$$

Since $\mathcal{T}_{\leq n}$ is generated by $S^n\mathcal{U}$, $\coprod_i \tau_{>n}Y_p$ belongs to $\mathcal{T}_{>n}$. Since the truncation triangle of $\coprod_p Y_p$ is unique, this implies the following isomorphism

$$\coprod_p \tau_{\leq n}Y_p \xrightarrow{\sim} \tau_{\leq n}(\coprod_p Y_p)\,.$$

This proves the lemma. $\surd$

Proposition A.2. *Let X be an object of $\mathcal{T}$. Suppose that we are in the conditions of proposition A.1 b). We dispose of the following isomorphism*

$$\operatorname*{hocolim}_i \tau_{\leq i}X \xrightarrow{\sim} X\,.$$

Proof. We need only show that

$$\mathrm{H}^n(\operatorname*{hocolim}_i \tau_{\leq i}X) \xrightarrow{\sim} \mathrm{H}^n(X)\,,$$

for all $n \in \mathbb{Z}$. We dispose of the following triangle, *cf.* [29],

$$\coprod_p \tau_{\leq p}X \to \coprod_q \tau_{\leq q}X \to \operatorname*{hocolim}_i \tau_{\leq i}X \to S(\coprod_p \tau_{\leq p}X)\,.$$

Since the functor H^n is homological, for all $n \in \mathbb{Z}$ and it commutes with infinite sums by lemma A.1, we obtain a long exact sequence

$$\cdots \to \coprod_p \mathrm{H}^n(\tau_{\leq p}X) \to \coprod_q \mathrm{H}^n(\tau_{\leq q}X) \to \mathrm{H}^n\,(\operatorname*{hocolim}_i \tau_{\leq i}X) \to$$
$$\to \coprod_p \mathrm{H}^n\, S(\tau_{\leq p}X) \to \coprod_q \mathrm{H}^n\, S(\tau_{\leq q}X) \to \cdots$$

We remark that the morphism $\coprod_p \mathrm{H}^n\, S(\tau_{\leq p}X) \to \coprod_q \mathrm{H}^n\, S(\tau_{\leq q}X)$ is a split monomorphism and so we obtain

$$\mathrm{H}^n(X) = \operatorname*{colim}_i \mathrm{H}^n(\tau_{\leq i}X) \xrightarrow{\sim} \mathrm{H}^n(\operatorname*{hocolim}_i \tau_{\leq i}X)\,.$$

$\surd$

References

[1] L. Alonso, A. Jeremias, M. Souto, *Construction of t-structures and equivalences of derived categories*, Trans. Amer. Math. Soc., no. 355, 2003, 2523–2543.

[2] A. Beilinson, J. Bernstein, P. Deligne, *Analyse et topologie sur les espaces singuliers*, Astérisque, vol. 100, Soc. Math. France, 1982.

[3] R. Bocklandt, *Graded Calabi Yau Algebras of dimension* 3, preprint math. RT/0603558.

[4] A. Bondal, M. Kapranov, *Representable functors, Serre functors, and reconstructions* (Russian), Izv. Akad. Nauk SSSR Ser. Math. 53 (1989), no. 6, 1183–1205, 1337; translation in Math. USSR-Izv. 35 (1990), no. 3, 519–541.

[5] A. Bondal, D. Orlov, *Semiorthogonal decomposition for algebraic varieties*, preprint MPIM 95/15 (1995), preprint math. AG/9506012.

[6] A. Buan, R. Marsh, I. Reiten, G. Todorov, *Clusters and seeds in acyclic cluster algebras*, preprint, math. RT/0510359.

[7] A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, *Tilting theory and cluster combinatorics*, Advances in Mathematics, no. 204, 2006, 572–618.

[8] P. Caldero, F. Chapoton, R. Schiffler, *Quivers with relations arising from (A_n case)*, Trans. Amer. Math. Soc., no. 358, 2006, 1347–1364.

[9] P. Caldero, B. Keller, *From triangulated categories to cluster algebras II*, preprint math. RT/0510251.

[10] D.Christensen, B. Keller, A. Neeman, *Failure of Brown representability in derived categories*, Topology 40 (2001), no. 6, 1339–1361.

[11] V. Drinfeld, *DG quotients of DG categories*, J. Alg. 272 (2004), 643–691.

[12] S. Fomin, A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529.

[13] V. Ginzburg, *Calabi-Yau algebras*, preprint, math. AG/0612139.

[14] V. Ginzburg, *Lectures in Noncommutative geometry*, preprint, math. AG/0506603.

[15] C. Geiß, B. Leclerc, J. Schröer, *Rigid modules over preprojective algebras*, Inv. Math., no. 165, 2006, 589–632.

[16] O. Iyama, Y. Yoshino, *Mutations in triangulated categories and rigid Cohen-Macaulay modules*, preprint in preparation.

[17] O. Iyama, I. Reiten, *Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras*, preprint, math. RT/0605136.

[18] O. Iyama, *Higher dimensional Auslander-Reiten theory on maximal orthogonal subcategories*, preprint, math. RT/0411631.

[19] O. Iyama, *Maximal orthogonal subcategories of triangulated categories satisfying Serre duality*, Oberwolfach report 6, 2005.

[20] B. Keller, *On differential graded categories*, preprint math. KT/0601185.

[21] B. Keller, *On triangulated orbit categories*, Doc. Math. 10 (2005), 551–581.

[22] B. Keller, *Deriving DG categories*, Ann. Scient. Ec. Norm. Sup. 27 (1994), 63–102.

[23] B. Keller, I. Reiten, *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, preprint, math. RT/0512471.

[24] B. Keller, I. Reiten, *Acyclic Calabi-Yau categories are cluster categories*, Oberwolfach report 23, (2006), 27–29.

[25] B. Keller, D. Vossieck, *Aisles in derived categories*, Bull. Soc. Math. Belg. Ser. A 40 (1988), no. 2, 239–253.

[26] B. Keller, D. Vossieck, *Sous les catégories derivées*, C. R. Acad. Sci. Paris Ser. I Math 305 (1987) no. 6, 225–228.

[27] M. Kontsevich, *Triangulated categories and geometry*, Course at the École Normale Supérieure, Paris, Notes taken by J. Bellaïche, J.-F. Dat, I. Marin, G. Racinet and H. Randriambololona, 1998.

[28] S. Maclane, *Categories for the working mathematician.* Second edition, Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998.

[29] A. Neeman, *Triangulated Categories*, Annals of Mathematics Studies, Vol. 148, Princeton University Press, Princeton, NJ, 2001.

[30] Y. Palu, *Catégories triangulées et algèbres amassées*, Ph. D. thesis in preparation.

[31] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. (2) 43 (1991), no. 1, 37–48.

[32] G. Tabuada, *A new Quillen model for the Morita homotopy theory of DG categories*, arxiv:math/0701205.

[33] G. Tabuada, *Addendum to "Invariants additifs de dg-catégories"*, Int. Math. Res. Notices, (2006), Art. ID 75853.

[34] G. Tabuada, *Invariants additifs de dg-catégories*, Int. Math. Res. Notices 53 (2005), 3309–3339.

[35] G. Tabuada, *Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories*, C. R. Math. Acad. Sci. Paris 340 (2005), no. 1, 15–19.
[36] M. Van den Bergh, *Non-commutative crepant resolutions*, The Legacy of Niels Hendrik Abel, Springer, (2002), 749–770.

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