Motivic Tubular Neighborhoods

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Abstract. We construct motivic versions of the classical tubular neighborhood and the punctured tubular neighborhood, and give applications to the construction of tangential base-points for mixed Tate motives, algebraic gluing of curves with boundary components, and limit motives.

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1. Introduction

Let $i : A \to B$ be a closed embedding of finite CW complexes. One useful fact is that A admits a cofinal system of neighborhoods T in B with $A \to T$ a deformation retract. This is often used in the case where B is a differentiable manifold, showing for example that A has the homotopy type of the differentiable manifold T. This situation occurs in algebraic geometry, for instance in the case of the inclusion of the special fiber in a degeneration of smooth varieties $\mathcal{X} \to C$ over the complex numbers.

To some extent, one has been able to mimic this construction in purely algebraic terms. The rigidity theorems of Gillet-Thomason [14], extended by Gabber (details appearing a paper of Fujiwara [13]) indicated that, at least through the eyes of torsion étale sheaves, the topological tubular neighborhood can be replaced by the Hensel neighborhood. However, basic examples of non-torsion phenomena, even in the étale topology, show that the Hensel neighborhood cannot always be thought of as a tubular neighborhood, perhaps the simplest example being the sheaf $\mathbb{G}_m$.[1]

Our object in this paper is to construct an algebraic version of the tubular neighborhood which has the basic properties of the topological construction, at least for a reasonably large class of cohomology theories. It turns out that

[1]If $\mathcal{O}$ is a local ring with residue field k and maximal ideal $\mathfrak{m}$, the surjection $\mathbb{G}_m(\mathcal{O}) \to \mathbb{G}_m(k)$ has kernel $(1+\mathfrak{m})^\times$, which is in general non-zero, even for $\mathcal{O}$ Hensel

a "homotopy invariant" version of the Hensel neighborhood does the job, at least for theories which are themselves homotopy invariant. If one requires in addition that the given cohomology theory has a Mayer-Vietoris property for the Nisnevich topology, then one also has an algebraic version of the punctured tubular neighborhood. We extend these constructions to the case of a (reduced) strict normal crossing subscheme by a Mayer-Vietoris procedure, giving us the tubular neighborhood and punctured tubular neighborhood of a normal crossing subscheme of a smooth k-scheme.

Morel and Voevodsky have constructed an algebro-geometric version of homotopy theory, in the setting of presheaves of spaces or spectra on the category of smooth varieties over a reasonable base scheme B; we concentrate on the $\mathbb{A}^1$-homotopy category of spectra, $\mathcal{SH}_{\mathbb{A}^1}(B)$. For a map $f: X \to Y$, they construct a pair of adjoint functors

$$Rf_*: \mathcal{SH}_{\mathbb{A}^1}(X) \to \mathcal{SH}_{\mathbb{A}^1}(Y)$$
$$Lf^*: \mathcal{SH}_{\mathbb{A}^1}(Y) \to \mathcal{SH}_{\mathbb{A}^1}(X).$$

If we have a closed immersion $i: W \to X$ with open complement $j: U \to X$, then one has the functor

$$Li^*Rj_*: \mathcal{SH}_{\mathbb{A}^1}(U) \to \mathcal{SH}_{\mathbb{A}^1}(W)$$

One of our main results is that, in case W is a strict normal crossing subscheme of a smooth X, the restriction of Li^*Rj_*E to a Zariski presheaf on W can be viewed as the evaluation of E on the punctured tubular neighborhood of W in X.

Consider a morphism $p: X \to \mathbb{A}^1$ and take $i: W \to X$ to be the inclusion of $p^{-1}(0)$. Following earlier constructions of Spitzweck [43], Ayoub has constructed a "unipotent specialization functor" in the motivic setting, essentially (in the case of a semi-stable degeneration) by evaluating Li^*Rj_*E on a cosimplicial version of the appropriate path space on $\mathbb{G}_m$ with base-point 1. Applying the same idea to our tubular neighborhood construction gives a model for this specialization functor, again only as a Zariski presheaf on $p^{-1}(0)$.

Ayoub has also defined a motivic monodromy operator and monodromy sequence involving the unipotent specialization functor and the functor Li^*Rj_*, for theories with $\mathbb{Q}$-coefficients that satisfy a certain additional condition (see definition 9.2.2). We give a model for this construction by combining our punctured tubular neighborhood with a $\mathbb{Q}$-linear version of the $\mathbb{G}_m$-path space mentioned above. We conclude with an application of our constructions to the moduli spaces of smooth curves and a construction of a specialization functor for category of mixed Tate motives, which in some cases yields a purely algebraic construction of tangential base-points. Of course, the construction of Ayoub, when restricted to the triangulated category of Tate motives, also gives such a specialization functor, but we hope that the explicit nature of our construction will be useful for applications.

We have left to another paper the task of checking the compatibilities of our constructions with others via the appropriate realization functor. As we have already mentioned, our punctured tubular neighborhood construction is comparable with the motivic version of the functor Li^*Rj_* for the situation of a normal crossing scheme $i : D \to X$ with open complement $j : X \setminus D \to X$; this should imply that it is a model for the analogous functor after realization. Similarly, our limit cohomology construction should transform after realization to the appropriate version of the sheaf of vanishing cycles, at least in the case of a semi-stable degeneration, and should be comparable with the constructions of Rappaport-Zink [37] as well as the limit mixed Hodge structure of Katz [22] and Steenbrink [44]. Our specialization functor for Tate motives should be compatible with the Betti, étale and Hodge realizations; similarly, realization functors applied to our limit motive should yield for example the limit mixed Hodge structure. We hope that our rather explicit construction of the limiting motive will be useful in giving a geometric view to the limit mixed Hodge structure of a semi-stable degeneration but we have not attempted an investigation of these issues in this paper.

My interest in this topic began as a result of several discussions on limit motives with Spencer Bloch and Hélène Esnault, whom I would like to thank for their encouragement and advice. I would also like to thank Hélène Esnault for clarifying the role of the weight filtration leading to the exactness of Clemens-Schmidt monodromy sequence (see Remark 9.3.6). An earlier version of our constructions used an analytic (i.e. formal power series) neighborhood instead of the Hensel version now employed; I am grateful to Fabien Morel for suggesting this improvement. Finally, I want to thank Joseph Ayoub for explaining his construction of the nearby cycles functor; his comments suggested to us the use of the cosimplicial path space in our construction of limit cohomology. In addition, Ayoub noticed a serious error in our first attempt at constructing the monodromy sequence; the method used in this version is following his suggestions and comments. Finally, we would like to thank the referee for giving unusually thorough and detailed comments and suggestions, which have substantially improved this paper. In particular, the material in sections 7 comparing our construction with the categorical ones of Morel-Voevodsky, as well as the comparison with Ayoub's specialization functor and monodromy sequence in section 8.3 and section 9 was added following the suggestion of the referee, who also supplied the main ideas for the proofs.

2. Model structures and other preliminaries

2.1. Presheaves of simplicial sets. We recall some facts on the model structures in categories of simplicial sets, spectra, associated presheaf categories and certain localizations. For details, we refer the reader to [17] and[19].

For a small category I and category $\mathcal{C}$, we will denote the category of functors from I to $\mathcal{C}$ by $\mathcal{C}^I$.

We let **Ord** denote the category with objects the finite ordered sets $[n] := \{0, \ldots, n\}$ (with the standard ordering) and morphisms the order-preserving maps of sets. For a category $\mathcal{C}$, the functor categories $\mathcal{C}^{\mathbf{Ord}}$, $\mathcal{C}^{\mathbf{Ord}^{\mathrm{op}}}$ are the categories of *cosimplicial objects of* $\mathcal{C}$, resp. *simplicial objects of* $\mathcal{C}$. For $\mathcal{C} = \mathbf{Sets}$, we have the category of simplicial sets, $\mathbf{Spc} := \mathbf{Sets}^{\mathbf{Ord}^{\mathrm{op}}}$, and similarly for $\mathcal{C}$ the category of pointed sets, $\mathbf{Sets}_*$, the category of pointed simplicial sets $\mathbf{Spc}_* := \mathbf{Sets}_*^{\mathbf{Ord}^{\mathrm{op}}}$.

We give $\mathbf{Spc}$ and $\mathbf{Spc}_*$ the standard model structures: cofibrations are (pointed) monomorphisms, weak equivalences are weak homotopy equivalences on the geometric realization, and fibrations are detemined by the right lifting property (RLP) with respect to trivial cofibrations; the fibrations are then exactly the Kan fibrations. We let $|A|$ denote the geometric realization, and $[A, B]$ the homotopy classes of (pointed) maps $|A| \to |B|$.

For an essentially small category $\mathcal{C}$, we let $\mathbf{Spc}(\mathcal{C})$ be the category of presheaves of simplicial sets on $\mathcal{C}$. We give $\mathbf{Spc}(\mathcal{C})$ the so-called injective model structure, that is, the cofibrations and weak equivalences are the pointwise ones, and the fibrations are determined by the RLP with respect to trivial cofibrations. We let $\mathcal{H}\mathbf{Spc}(\mathcal{C})$ denote the associated homotopy category (see [17] for details on these model structures for $\mathbf{Spc}$ and $\mathbf{Spc}(\mathcal{C})$).

2.2. PRESHEAVES OF SPECTRA. Let $\mathbf{Spt}$ denote the category of spectra. To fix ideas, a spectrum will be a sequence of pointed simplicial sets $E_0, E_1, \ldots$ together with maps of pointed simplicial sets $\epsilon_n : S^1 \wedge E_n \to E_{n+1}$. Maps of spectra are maps of the underlying simplicial sets which are compatible with the attaching maps ϵ_n. The stable homotopy groups $\pi_n^s(E)$ are defined by

$$\pi_n^s(E) := \lim_{m\to\infty} [S^{m+n}, E_m].$$

The category $\mathbf{Spt}$ has the following model structure: Cofibrations are maps $f : E \to F$ such that $E_0 \to F_0$ is a cofibration, and for each $n \geq 0$, the map

$$E_{n+1} \coprod_{S^1\wedge E_n} S^1 \wedge F_n \to F_{n+1}$$

is a cofibration. Weak equivalences are the stable weak equivalences, i.e., maps $f : E \to F$ which induce an isomorphism on π_n^s for all n. Fibrations are characterized by having the RLP with respect to trivial cofibrations. We write $\mathcal{SH}$ for the homotopy category of $\mathbf{Spt}$.

For $X \in \mathbf{Spc}_*$, we have the suspension spectrum $\Sigma^\infty X := (X, \Sigma X, \Sigma^2 X, \ldots)$ with the identity bonding maps. Dually, for a spectrum $E := (E_0, E_1, \ldots)$ we have the 0-space $\Omega^\infty E := \lim_n \Omega^n E_n$. These operations form a Quillen pair of adjoint functors $(\Sigma^\infty, \Omega^\infty)$ between $\mathbf{Spc}_*$ and $\mathbf{Spt}$, and thus induce adjoint functors on the homotopy categories.

Let $\mathcal{C}$ be a category. A functor $E : \mathcal{C}^{\mathrm{op}} \to \mathbf{Spt}$ is called a *presheaf of spectra* on $\mathcal{C}$.

We use the following model structure on the category of presheaves of spectra (see [19]): Cofibrations and weak equivalences are given pointwise, and fibrations are characterized by having the RLP with respect to trivial cofibrations. We denote this model category by $\mathbf{Spt}(\mathcal{C})$, and the associated homotopy category by $\mathcal{H}\mathbf{Spt}(\mathcal{C})$.

As a particular example, we have the model category of simplicial spectra $\mathbf{Spt}^{\mathbf{Ord}^{\mathrm{op}}} = \mathbf{Spt}(\mathbf{Ord})$. We have the *total spectrum functor*

$$\mathrm{Tot} : \mathbf{Spt}(\mathbf{Ord}) \to \mathbf{Spt}$$

which preserves weak equivalences. The adjoint pair $(\Sigma^\infty, \Omega^\infty)$ extend pointwise to define a Quillen pair on the presheaf categories and an adjoint pair on the homotopy categories.

Let B be a noetherian separated scheme of finite Krull dimension. We let $\mathbf{Sm}/B$ denote the category of smooth B-schemes of finite type over B. We often write $\mathbf{Spc}(B)$ and $\mathcal{H}\mathbf{Spc}(B)$ for $\mathbf{Spc}(\mathbf{Sm}/B)$ and $\mathcal{H}\mathbf{Spc}(\mathbf{Sm}/B)$, and write $\mathbf{Spt}(B)$ and $\mathcal{H}\mathbf{Spt}(B)$ for $\mathbf{Spt}(\mathbf{Sm}/B)$ and $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}/B)$.

For $Y \in \mathbf{Sm}/B$, a subscheme $U \subset Y$ of the form $Y \setminus \cup_\alpha F_\alpha$, with $\{F_\alpha\}$ a possibly infinite set of closed subsets of Y, is called *essentially smooth over* B; the category of essentially smooth B-schemes is denoted $\mathbf{Sm}^{\mathrm{ess}}$.

2.3. Local model structure. If the category $\mathcal{C}$ has a topology, there is often another model structure on $\mathbf{Spc}(\mathcal{C})$ or $\mathbf{Spt}(\mathcal{C})$ which takes this into account. We consider the case of the small Nisnevich site X_{Nis} on a scheme X (assumed to be noetherian, separated and of finite Krull dimension), and the big Nisnevich sites $\mathbf{Sm}/B_{\mathrm{Nis}}$ or $\mathbf{Sch}/B_{\mathrm{Nis}}$, as well as the Zariski versions X_{Zar}, $\mathbf{Sm}/B_{\mathrm{Zar}}$, etc. We describe the Nisnevich version for spectra below; the definitions and results for the Zariski topology and for spaces are exactly parallel.

Definition 2.3.1. A map $f : E \to F$ of presheaves of spectra on X_{Nis} is a *local weak equivalence* if the induced map on the Nisnevich sheaf of stable homotopy groups $f_* : \pi^s_m(E)_{\mathrm{Nis}} \to \pi^s_m(F)_{\mathrm{Nis}}$ is an isomorphism of sheaves for all m. A map $f : E \to F$ of presheaves of spectra on $\mathbf{Sm}/B_{\mathrm{Nis}}$ or $\mathbf{Sch}/B_{\mathrm{Nis}}$ is a local weak equivalence if the restriction of f to X_{Nis} is a local weak equivalence for all $X \in \mathbf{Sm}/B$ or $X \in \mathbf{Sch}/B$. □

The Nisnevich local model structure on the category of presheaves of spectra on X_{Nis} has cofibrations given pointwise, weak equivalences the local weak equivalences and fibrations are characterized by having the RLP with respect to trivial cofibrations. We write $\mathbf{Spt}(X_{\mathrm{Nis}})$ for this model category, and $\mathcal{H}\mathbf{Spt}(X_{\mathrm{Nis}})$ for the associated homotopy category. The Nisnevich local model categories $\mathbf{Spt}(\mathbf{Sm}/B_{\mathrm{Nis}})$ and $\mathbf{Spt}(\mathbf{Sch}/B_{\mathrm{Nis}})$, with homotopy categories $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}/B_{\mathrm{Nis}})$ and $\mathcal{H}\mathbf{Spt}(\mathbf{Sch}/B_{\mathrm{Nis}})$, are defined similarly. A similar localization gives model categories of presheaves of spaces $\mathbf{Spc}(X_{\mathrm{Nis}})$, $\mathbf{Spc}(X_{\mathrm{Zar}})$, $\mathbf{Spc}(\mathbf{Sm}/B_{\mathrm{Nis}})$, etc., and homotopy categories $\mathcal{H}\mathbf{Spc}(X_{\mathrm{Nis}})$, $\mathcal{H}\mathbf{Spc}(X_{\mathrm{Zar}})$, $\mathcal{H}\mathbf{Spc}(\mathbf{Sm}/B_{\mathrm{Nis}})$, etc. We also have the adjoint pair $(\Sigma^\infty, \Omega^\infty)$ in this setting. For details, we refer the reader to [19].

Remark 2.3.2. Let E be in $\mathbf{Spt}(\mathbf{Sm}/B_{\mathrm{Nis}})$, and let

$$\begin{tikzcd} W' \arrow[r, "i'"] \arrow[d] & Y' \arrow[d, "f"] \\ W \arrow[r, "i"'] & X \end{tikzcd}$$

be an *elementary Nisnevich square*, i.e., f is étale, $i : W \to X$ is a closed immersion, the square is cartesian, and $W' \to W$ is an isomorphism, with X and X' in $\mathbf{Sm}/B$ (see [34, Definition 1.3, pg. 96]) .
If E is fibrant in $\mathbf{Spt}(\mathbf{Sm}/B_{\mathrm{Nis}})$ then E transforms each elementary Nisnevich square to a homotopy cartesian square in $\mathbf{Spt}$. Conversely, suppose that E transforms each elementary Nisnevich square to a homotopy cartesian square in $\mathbf{Spt}$. Then E is *quasi-fibrant*, i.e., for all $Y \in \mathbf{Sm}/B$, the canonical map $E(Y) \to E_{fib}(Y)$, where E_{fib} is the fibrant model of E, is a weak equivalence. See [19] for details.
If we define an elementary Zariski square as above, with $X' \to X$ an open immersion, the same holds in the model category $\mathbf{Spt}(\mathbf{Sm}/B_{\mathrm{Zar}})$. More precisely, one can show (see e.g. [45]) that, if E transforms each elementary Zarisk square to a homotopy cartesian square in $\mathbf{Spt}$, then E *satisfies Mayer-Vietoris* for the Zariski topology: if $X \in \mathbf{Sm}/B$ is a union of Zariski open subschemes U and V, then the evident sequence

$$E(X) \to E(U) \oplus E(V) \to E(U \cap V)$$

is a homotopy fiber sequence in $\mathcal{SH}$. □

Remark 2.3.3. Let $\mathcal{C}$ be a small category with an initial object $\emptyset$ and admiting finite coproducts over $\emptyset$, denoted $X \amalg Y$. A functor $E : \mathcal{C}^{\mathrm{op}} \to \mathbf{Spt}$ is called *additive* if for each X, Y in $\mathcal{C}$, the canonical map

$$E(X \amalg Y) \to E(X) \oplus E(Y)$$

in $\mathcal{SH}$ is an isomorphism. It is easy to show that if $E \in \mathbf{Spt}(\mathbf{Sm}/B)$ satisfies Mayer-Vietoris for the Zariski topology, and $E(\emptyset) \cong 0$ in $\mathcal{SH}$, then E is additive. From now on, we will assume that *all* our presheaves of spectra E satisfy $E(\emptyset) \cong 0$ in $\mathcal{SH}$. □

2.4. $\mathbb{A}^1$-local structure. One can perform a Bousfield localization on $\mathbf{Spc}(\mathbf{Sm}/B_{\mathrm{Nis}})$ or $\mathbf{Spt}(\mathbf{Sm}/B_{\mathrm{Nis}})$ so that the maps $\Sigma^\infty X \times \mathbb{A}^1_+ \to \Sigma^\infty X_+$ induced by the projections $X \times \mathbb{A}^1 \to X$ become weak equivalences. We call the resulting model structure the *Nisnevich-local* $\mathbb{A}^1$*-model structure*, denoted $\mathbf{Spc}_{\mathbb{A}^1}(\mathbf{Sm}/B_{\mathrm{Nis}})$ or $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/B_{\mathrm{Nis}})$. One has the Zariski-local versions as well. We denote the homotopy categories for the Nisnevich version by $\mathcal{H}_{\mathbb{A}^1}(B)$ (for spaces) and $\mathcal{SH}_{\mathbb{A}^1}(B)$ (for spectra). For the Zariski versions, we indicate the topology in the notation. We also have the adjoint pair $(\Sigma^\infty, \Omega^\infty)$ in this setting. For details, see [30, 31, 34].

2.5. Additional notation. Given $W \in \mathbf{Sm}/S$, we have restriction functors

$$\mathbf{Spc}(S) \to \mathbf{Spc}(W_{\mathrm{Zar}})$$
$$\mathbf{Spt}(S) \to \mathbf{Spt}(W_{\mathrm{Zar}});$$

we write the restriction of some $E \in \mathbf{Spc}(S)$ to $\mathbf{Spc}(W_{\mathrm{Zar}})$ as $E(W_{\mathrm{Zar}})$. We use a similar notation for the restriction of E to $\mathbf{Spt}(W_{\mathrm{Zar}})$, or for restrictions to W_{Nis}. More generally, if $p : Y \to W$ is a morphism in $\mathbf{Sm}/S$, we write $E(Y/W_{\mathrm{Zar}})$ for the presheaf $U \mapsto E(Y \times_W U)$ on W_{Zar}.

For $Z \subset Y$ a closed subset, $Y \in \mathbf{Sm}/S$ and for $E \in \mathbf{Spc}(S)$ or $E \in \mathbf{Spt}(S)$, we write $E^Z(Y)$ for the homotopy fiber of the restriction map

$$E(Y) \to E(Y \setminus Z).$$

We define the presheaf $E^{Z_{\mathrm{Zar}}}(Y)$ by setting, for $U \subset Z$ a Zariski open subscheme with closed complement F,

$$E^{Z_{\mathrm{Zar}}}(Y)(U) := E^U(Y \setminus F).$$

A *co-presheaf* on a category $\mathcal{C}$ with values in $\mathcal{A}$ is just an $\mathcal{A}$-valued preheaf on $\mathcal{C}^{\mathrm{op}}$.

As usual, we let Δ^n denote the algebraic n-simplex

$$\Delta^n := \operatorname{Spec} \mathbb{Z}[t_0, \ldots, t_n]/\sum_i t_i - 1,$$

and Δ^* the cosimplicial scheme $n \mapsto \Delta^n$. For a scheme X, we have $\Delta^n_X := X \times \Delta^n$ and the cosimplicial scheme Δ^*_X.

Let B be a scheme as above. For $E \in \mathbf{Spc}(B)$ or in $\mathbf{Spt}(B)$, we say that E is *homotopy invariant* if for all $X \in \mathbf{Sm}/B$, the pull-back map $E(X) \to E(X \times \mathbb{A}^1)$ is a weak equivalence (resp., stable weak equivalence). We say that E *satisfies Nisnevich excision* if E transforms elementary Nisnevich squares to homotopy cartesian squares.

3. Tubular neighborhoods for smooth pairs

Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$. In this section, we construct the tubular neighborhood $\tau_\epsilon^{\hat{X}}(W)$ of W in X as a functor from W_{Zar} to cosimplicial pro-k-schemes. Given $E \in \mathbf{Spc}(k)$, we can evaluate E on $\tau_\epsilon^{\hat{X}}(W)$, yielding the presheaf of spaces $E(\tau_\epsilon^{\hat{X}}(W))$ on W_{Zar}, which is our main object of study.

3.1. The cosimplicial pro-scheme $\tau_\epsilon^{\hat{X}}(W)$. For a closed immersion $W \to T$ in $\mathbf{Sm}/k$, let T^W_{Nis} be the category of Nisnevich neighborhoods of W in T, i.e., objects are étale maps $p : T' \to T$ of finite type, together with a section $s : W \to T'$ to p over W. Morphisms are morphisms over T which respect the sections. Note that T^W_{Nis} is a left-filtering essentially small category.

Sending $(p : T' \to T, s : W \to T')$ to $T' \in \mathbf{Sm}/k$ defines the pro-object $\hat{T}^h_W$ of $\mathbf{Sm}/k$; the sections $s : W \to T'$ give rise to a map of the constant pro-scheme W to $\hat{T}^h_W$, denoted

$$\hat{i}_W : W \to \hat{T}^h_W.$$

Given a k-morphism $f : S \to T$, and closed immersions $i_V : V \to S$, $i_W : W \to T$ such that $f \circ i_V$ factors through i_W (by $\bar{f} : V \to W$), we have the pull-back functor

$$f^* : T^W_{\mathrm{Nis}} \to S^V_{\mathrm{Nis}},$$
$$f^*(T' \to T, s : W \to T') := (T' \times_T S, (s \circ \bar{f}, i_V)).$$

This gives us the map of pro-objects $f^h : \hat{S}^h_V \to \hat{T}^h_W$, so that sending $W \to T$ to $\hat{T}^h_W$ and f to f^h becomes a pseudo-functor.
We let $f^h : \hat{S}^h_V \to \hat{T}^h_W$ denote the induced map on pro-schemes. If f happens to be a Nisnevich neighborhood of $W \to X$ (so $\bar{f} : V \to W$ is an isomorphism) then $f^h : \hat{S}^h_V \to \hat{T}^h_W$ is clearly an isomorphism.

Remark 3.1.1. The pseudo-functor $(W \to T) \mapsto \hat{T}^h_W$ can be rectified to an honest functor by first replacing T^W_{Nis} with the cofinal subcategory $T^W_{\mathrm{Nis},0}$ of neighborhoods $T' \to T$, $s : W \to T'$ such that each connected component of T' has non-empty intersection with $s(W)$. One notes that $T^W_{\mathrm{Nis},0}$ has only identity automorphisms, so we replace $T^W_{\mathrm{Nis},0}$ with a choice of a full subcategory $T^W_{\mathrm{Nis},00}$ giving a set of representatives of the isomorphism classes in $T^W_{\mathrm{Nis},0}$. Given a map of pairs of closed immersions $f : (V \xrightarrow{i_V} S) \to (W \xrightarrow{i_W} T)$ as above, we modify the pull-back functor f^* defined above by passing to the connected component of $(s \circ \bar{f}, i_V)(V)$ in $T' \times_T S$. We thus have the honest functor $(W \to T) \mapsto T^W_{\mathrm{Nis},00}$ which yields an equivalent pro-object $\hat{T}^h_W$.
As pointed out by the referee, one can also achieve strict functoriality by rectifying the fiber product; in any case, we will use a strictly functorial version from now on without comment. □

For a closed immersion $i : W \to X$ in $\mathbf{Sm}/k$, set $\hat{\Delta}^n_{X,W} := (\widehat{\Delta^n_X})^h_{\Delta^n_W}$. The cosimplicial scheme

$$\Delta^*_X : \mathbf{Ord} \to \mathbf{Sm}/k$$
$$[n] \mapsto \Delta^n_X$$

thus gives rise to the cosimplicial pro-scheme

$$\hat{\Delta}^*_{X,W} : \mathbf{Ord} \to \mathbf{Pro\text{-}Sm}/k$$
$$[n] \mapsto \hat{\Delta}^n_{X,W}$$

The maps $\hat{i}_{\Delta^n_W} : \Delta^n_W \to (\widehat{\Delta^n_X})^h_{\Delta^n_W}$ give the closed immersion of cosimplicial pro-schemes

$$\hat{i}_W : \Delta^*_W \to \hat{\Delta}^*_{X,W}.$$

Also, the canonical maps $\pi_n : \hat{\Delta}^n_{X,W} \to \Delta^n_X$ define the map

$$\pi_{X,W} : \hat{\Delta}^*_{X,W} \to \Delta^*_X.$$

Let $(p : X' \to X, s : W \to X')$ be a Nisnevich neighborhood of (W, X). The map

$$p : \hat{\Delta}^n_{X',W} \to \hat{\Delta}^n_{X,W}$$

is an isomorphism respecting the closed immersions $\hat{i}_W$. Thus, sending a Zariski open subscheme $U \subset W$ with complement $F \subset W \subset X$ to $\hat{\Delta}^n_{X\setminus F,U}$ defines a co-presheaf $\hat{\Delta}^n_{\hat{X},W_{\mathrm{Zar}}}$ on W_{Zar} with values in pro-objects of $\mathbf{Sm}/k$; we write $\tau^{\hat{X}}_\epsilon(W)$ for the cosimplicial object

$$n \mapsto \hat{\Delta}^n_{\hat{X},W_{\mathrm{Zar}}}.$$

We use the $\hat{X}$ in the notation because the co-presheaf $\hat{\Delta}^n_{\hat{X},W_{\mathrm{Zar}}}$ is depends only on the Nisnevich neighborhood of W in X.
Let $\Delta^*_{W_{\mathrm{Zar}}}$ denote the co-presheaf on W_{Zar} defined by $U \mapsto \Delta^*_U$. The closed immersions $\hat{i}_U$ define the natural transformation

$$\hat{i}_W : \Delta^*_{W_{\mathrm{Zar}}} \to \tau^{\hat{X}}_\epsilon(W).$$

The maps $\pi_{X\setminus F,W\setminus F}$ for $F \subset W$ a Zariski closed subset define the map

$$\pi_{X,W} : \tau^{\hat{X}}_\epsilon(W) \to \Delta^*_{X|W_{\mathrm{Zar}}}$$

where $X|W_{\mathrm{Zar}}$ is the co-presheaf $W \setminus F \mapsto X \setminus F$ on W_{Zar}. We let

$$\bar{\pi}_{X,W} : \tau^{\hat{X}}_\epsilon(W) \to X|W_{\mathrm{Zar}} \tag{3.1.1}$$

denote the composition of $\pi_{X.W}$ with the projection $\Delta^*_{X|W_{\mathrm{Zar}}} \to X|W_{\mathrm{Zar}}$.

3.2. Evaluation on spaces. Let $i : W \to T$ be a closed immersion in $\mathbf{Sm}/k$. For $E \in \mathbf{Spc}(T)$, we have the space $E(\hat{T}^W_h)$, defined by

$$E(\hat{T}^h_W) := \operatorname*{colim}_{(p:T'\to T,s:W\to T')\in T^W_{\mathrm{Nis}}} E(T').$$

Given a Nisnevich neighborhood $(p : T' \to T, s : W \to T')$, we have the isomorphism

$$p^* : E(\hat{T}^h_W) \to E(\hat{T'}^h_{s(W)}).$$

Thus, for each open subscheme $j : U \to W$, we may evaluate E on the cosimplicial pro-scheme $\tau^{\hat{X}}_\epsilon(W)(U)$, giving us the presheaf of simplicial spectra $E(\tau^{\hat{X}}_\epsilon(W))$ on W_{Zar}:

$$E(\tau^{\hat{X}}_\epsilon(W))(U) := E(\tau^{\hat{X}}_\epsilon(W)(U)).$$

Now suppose that E is in $\mathbf{Spc}(k)$. The map $\hat{i}_W : \Delta^*_{W_{\mathrm{Zar}}} \to \tau^{\hat{X}}_\epsilon(W))$ gives us the map of presheaves on W_{Zar}

$$i^*_W : E(\tau^{\hat{X}}_\epsilon(W)) \to E(\Delta^*_{W_{\mathrm{Zar}}}).$$

Similarly, the map $\pi_{X,W}$ gives the map of presheaves on W_{Zar}

$$\pi^*_{X,W} : E(\Delta^*_{X|W_{\mathrm{Zar}}}) \to E(\tau^{\hat{X}}_\epsilon(W)).$$

The main result of this section is

THEOREM 3.2.1. *Let E be in $\mathbf{Spc}(k)$. Then the map $i_W^* : E(\tau_\epsilon^{\hat{X}}(W)) \to E(\Delta^*_{W_{\mathrm{Zar}}})$ is a weak equivalence for the Zariski-local model structure, i.e., for each point $w \in W$, the map $i^*_{W,w}$ on the stalks at w is a weak equivalence of the associated total space.*

3.3. PROOF OF THEOREM 3.2.1. The proof relies on two lemmas.

LEMMA 3.3.1. *Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$, giving the closed immersion $\mathbb{A}^1_W \to \mathbb{A}^1_X$. For $t \in \mathbb{A}^1(k)$, we have the section $i_t : W \to \mathbb{A}^1_W$, $i_t(w) := (w, t)$. Then for each $E \in \mathbf{Spc}(k)$, the maps*

$$i_0^*, i_1^* : E(\hat{\Delta}^*_{\mathbb{A}^1_X,\mathbb{A}^1_W}) \to E(\hat{\Delta}^*_{X,W})$$

are homotopic.

Proof. This is just an adaptation of the standard triangulation argument. For each order-preserving map $g = (g_1, g_2) : [m] \to [1] \times [n]$, let

$$T_g : \Delta^m \to \Delta^1 \times \Delta^n,$$

be the affine-linear extension of the map on the vertices

$$v_i \mapsto (v_{g_1(i)}, v_{g_2(i)}).$$

$\mathrm{id}_X \times T_g$ induces the map

$$\hat{T}_g : \hat{\Delta}^m_{X,W} \to (\widehat{\Delta^1 \times \Delta^n_X})^h_{\Delta^1 \times \Delta^n_W}$$

We note that the isomorphism $(t_0, t_1) \mapsto t_0$ of (Δ^1, v_1, v_0) with $(\mathbb{A}^1, 0, 1)$ induces an isomorphism of cosimplicial schemes

$$\hat{\Delta}^*_{\mathbb{A}^1_X,\mathbb{A}^1_W} \cong (\widehat{\Delta^1 \times \Delta^*_X})^h_{\Delta^1 \times \Delta^*_W}.$$

The maps

$$\hat{T}_g^* : E(\hat{\Delta}^n_{\mathbb{A}^1_X,\mathbb{A}^1_W}) \to E(\hat{\Delta}^m_{X,W})$$

induce a simplicial homotopy T between i_0^* and i_1^*. Indeed, we have the simplicial sets $\Delta[n] : \mathrm{Hom}_{\mathbf{Ord}}(-, [n])$. Let $(\Delta^1 \times \Delta^*)^{\Delta[1]}$ be the cosimplicial scheme

$$n \mapsto (\Delta^1 \times \Delta^n)^{\Delta[1]([n])} := \prod_{s \in \Delta[1]([n])} \Delta^1 \times \Delta^n$$

where the product is $\times_{\mathbb{Z}}$. The inclusions $\delta_0, \delta_1 : [0] \to [1]$ thus induce the maps of cosimplicial schemes

$$\delta_0^*, \delta_1^* : (\Delta^1 \times_k \Delta^*)^{\Delta[1]} \to \Delta^1 \times_k \Delta^*.$$

The maps T_g satisfy the identities necessary to define a map of cosimplicial schemes

$$T : \Delta^* \to (\Delta^1 \times \Delta^*)^{\Delta[1]}.$$

with $\delta_0^* \circ T = i_0$, $\delta_1^* \circ T = i_1$. Applying the functor ^{h}, we see that the maps $\hat{T}_g$ define the map of cosimplicial schemes

$$\hat{T} : \hat{\Delta}^*_{X,W} \to (\hat{\Delta}^*_{\mathbb{A}^1_X,\mathbb{A}^1_W})^{\Delta[1]},$$

with $\delta_0^* \circ \hat{T} = \hat{i}_0$, $\delta_1^* \circ \hat{T} = \hat{i}_1$; we then apply E. □

Lemma 3.3.2. *Take $W \in \mathbf{Sm}_k$. Let $X = \mathbb{A}^n_W$ and let $i : W \to X$ be the 0-section. Then $i^*_W : E(\hat{\Delta}^*_{X,W}) \to E(\Delta^*_W)$ is a homotopy equivalence.*

Proof. Let $p : X \to W$ be the projection, giving the map

$$\hat{p} : \hat{\Delta}^*_{X,W} \to \hat{\Delta}^*_{W,W} = \Delta^*_W$$

and $\hat{p}^* : E(\Delta^*_W) \to E(\hat{\Delta}^*_{X,W})$. Clearly $\hat{i}^*_W \circ \hat{p}^* = \mathrm{id}$, so it suffices to show that $\hat{p}^* \circ \hat{i}^*_W$ is homotopic to the identity.
For this, we use the multiplication map $\mu : \mathbb{A}^1 \times \mathbb{A}^n \to \mathbb{A}^n$,

$$\mu(t; x_1, \ldots, x_n) := (tx_1, \ldots, tx_n).$$

The map $\mu \times \mathrm{id}_{\Delta^*}$ induces the map

$$\hat{\mu} : (\widehat{\mathbb{A}^1 \times \mathbb{A}^n_W} \times \Delta^*)^h_{\mathbb{A}^1 \times 0_W \times \Delta^*} \to (\widehat{\mathbb{A}^n_W \times \Delta^*})^h_{0_W \times \Delta^*}$$

with $\hat{\mu} \circ \hat{i}_0 = \hat{i}_W \circ \hat{p}$ and $\hat{\mu} \circ \hat{i}_1 = \mathrm{id}$. Since $\hat{i}^*_0$ and $\hat{i}^*_1$ are homotopic by Lemma 3.3.1, the proof is complete. □

To complete the proof of Theorem 3.2.1, take a point $w \in W$. Then replacing X with a Zariski open neighborhood of w, we may assume there is a Nisnevich neigborhood $X' \to X$, $s : W \to X'$ of W in X such that $W \to X'$ is in turn a Nisnevich neighborhood of the zero-section $W \to \mathbb{A}^n_W$, $n = \mathrm{codim}_X W$. Since $E(\hat{\Delta}^n_{X,W})$ is thus weakly equivalent to $E(\hat{\Delta}^n_{\mathbb{A}^n_W, 0_W})$, the result follows from Lemma 3.3.2.

Corollary 3.3.3. *Suppose that $E \in \mathbf{Spc}(\mathbf{Sm}/k)$, resp. $E \in \mathbf{Spt}(\mathbf{Sm}/k)$ is homotopy invariant. Then for $i : W \to X$ a closed immersion, there is a natural isomorphism in $\mathcal{H}\mathbf{Spc}(W_{\mathrm{Zar}})$, resp. $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$*

$$E(\tau^{\hat{X}}_\epsilon(W)) \cong E(\tau^{\hat{N}_i}_\epsilon(0_W))$$

Here N_i is the normal bundle of the immersion i, and $0_W \subset N_i$ is the 0-section.

Proof. This follows directly from Theorem 3.2.1: Since E is homotopy invariant, the canonical map

$$E(T) \to E(\Delta^*_T)$$

is a weak equivalence for each $T \in \mathbf{Sm}/k$. The desired isomorphism in the respective homotopy category is constructed by composing the isomorphisms

$$\begin{aligned} E(\tau^{\hat{X}}_\epsilon(W)) &\xrightarrow{i^*_W} E(\Delta^*_{W_{\mathrm{Zar}}}) \leftarrow E(W_{\mathrm{Zar}}) \\ &= E(0_{W\mathrm{Zar}}) \to E(\Delta^*_{0_{W\mathrm{Zar}}}) \xleftarrow{i^*_{0_W}} E(\tau^{\hat{N}_i}_\epsilon(0_W)). \end{aligned}$$

□

4. Punctured tubular neighborhoods

Our real interest is not in the tubular neighborhood $\tau^{\hat{X}}_\epsilon(W)$, but in the punctured tubular neighborhood $\tau^{\hat{X}}_\epsilon(W)^0$. In this section, we define this object and discuss its basic properties.

4.1. Definition of the punctured neighborhood. Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$. We have the closed immersion of cosimplicial pro-schemes

$$\hat{i} : \Delta^*_W \to \hat{\Delta}^*_{X,W}$$

giving for each n the open complement $\hat{\Delta}^n_{X\setminus W} := \hat{\Delta}^n_{X,W} \setminus \Delta^n_W$. We may pass to the cofinal subcategory of Nisnevich neighborhoods of Δ^n_W in Δ^n_X,

$$(p : T \to \Delta^n_X, s : \Delta^n_W \to T)$$

for which the diagram

$$\begin{tikzcd} T \setminus s(\Delta^n_W) \arrow[r] \arrow[d] & T \arrow[d] \\ \Delta^n_X \setminus \Delta^n_W \arrow[r] & \Delta^n_X \end{tikzcd}$$

is cartesian, giving us the cosimplicial proscheme $n \mapsto \hat{\Delta}^n_{X\setminus W}$ and the map

$$\hat{j} : \hat{\Delta}^*_{X\setminus W} \to \hat{\Delta}^*_{X,W},$$

which defines the "open complement" $\hat{\Delta}^*_{X\setminus W}$ of Δ^*_W in $\hat{\Delta}^*_{X,W}$. Extending this construction to all open subschemes of X, we have the co-presheaf on W_{Zar},

$$U = W \setminus F \mapsto \hat{\Delta}^*_{(X\setminus F)\setminus U},$$

which we denote by $\tau^{\hat{X}}_\epsilon(W)^0$.

Let $\Delta^n_{(X\setminus W)_{\mathrm{Zar}}}$ be the constant co-presheaf on W_{Zar} with value $\Delta^n_{X\setminus W}$, giving the cosimplicial co-presheaf $\Delta^*_{(X\setminus W)_{\mathrm{Zar}}}$. The maps

$$\hat{j}_U : \hat{\Delta}^*_{(X\setminus F)\setminus U} \to \hat{\Delta}^*_{(X\setminus F),U}$$

define the map $\hat{j} : \tau^{\hat{X}}_\epsilon(W)^0 \to \tau^{\hat{X}}_\epsilon(W)$. The maps $\hat{\Delta}^*_{U\setminus W\cap U} \to \Delta^*_{X\setminus W}$ give us the map

$$\pi : \tau^{\hat{X}}_\epsilon(W)^0 \to \Delta^*_{X\setminus W}$$

where we view $\Delta^*_{X\setminus W}$ as the constant co-sheaf on W_{Zar}.

To give a really useful result on the presheaf $E(\tau^{\hat{X}}_\epsilon(W)^0)$, we will need to impose additional conditions on E. These are

(1) E is homotopy invariant
(2) E satisfies Nisnevich. excision

One important consequence of these properties is the purity theorem of Morel-Voevodsky:

Theorem 4.1.1 (Purity [34, theorem 2.23]). *Suppose $E \in \mathbf{Spt}(k)$ is homotopy invariant and satisfies Nisnevich excision. Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$ and $s : W \to N_i$ the 0-section of the normal bundle. Then there is an isomorphism in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$*

$$E^{W_{\mathrm{Zar}}}(X) \to E^{W_{\mathrm{Zar}}}(N_i)$$

□

Let $E(X|W_{\mathrm{Zar}})$ be the presheaf on W_{Zar}

$$W \setminus F \mapsto E(X \setminus F)$$

and $E(X \setminus W)$ the constant presheaf.
Let

$$\mathrm{res} : E(X|W_{\mathrm{Zar}}) \to E(\tau_\epsilon^{\hat{X}}(W))$$
$$\mathrm{res}^0 : E(X \setminus W) \to E(\tau_\epsilon^{\hat{X}}(W)^0)$$

be the pull-back by the natural maps $\tau_\epsilon^{\hat{X}}(W)(W \setminus F) \to X \setminus F$, $\tau_\epsilon^{\hat{X}}(W)^0 \to X \setminus W$. Let $E^{\Delta_W^*}(\tau_\epsilon^{\hat{X}}(W)) \in \mathbf{Spt}(W_{\mathrm{Zar}})$ be the homotopy fiber of the restriction map

$$\hat{j} : E(\tau_\epsilon^{\hat{X}}(W)) \to E(\tau_\epsilon^{\hat{X}}(W)^0).$$

The commutative diagram in $\mathbf{Spt}(W_{\mathrm{Zar}})$

$$\begin{tikzcd}
E(X|W_{\mathrm{Zar}}) \arrow[r, "j^*"] \arrow[d, "\mathrm{res}"'] & E(X \setminus W) \arrow[d, "\mathrm{res}^0"] \\
E(\tau_\epsilon^{\hat{X}}(W)) \arrow[r, "\hat{j}^*"'] & E(\tau_\epsilon^{\hat{X}}(W)^0)
\end{tikzcd}$$

induces the map of homotopy fiber sequences

$$\begin{tikzcd}
E^{W_{\mathrm{Zar}}}(X) \arrow[r] \arrow[d, "\psi"'] & E(X|W_{\mathrm{Zar}}) \arrow[r, "j^*"] \arrow[d, "\mathrm{res}"'] & E(X \setminus W) \arrow[d, "\mathrm{res}^0"] \\
E^{\Delta_W^*}(\tau_\epsilon^{\hat{X}}(W)) \arrow[r] & E(\tau_\epsilon^{\hat{X}}(W)) \arrow[r, "j^*"'] & E(\tau_\epsilon^{\hat{X}}(W)^0)
\end{tikzcd}$$

We can now state the main result for $E(\tau_\epsilon^{\hat{X}}(W)^0)$.

Theorem 4.1.2. *Suppose that $E \in \mathbf{Spt}(k)$ is homotopy invariant and satisfies Nisnevich excision. Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$. Then the map ψ is a Zariski local weak equivalence.*

Proof. Let $i_{\Delta^*} : \Delta_W^* \to \Delta_X^*$ be the immersion $\mathrm{id} \times i$. For $U = W \setminus F \subset W$, $\tau_\epsilon^{\hat{X}}(W)^0(U)$ is the cosimplicial pro-scheme with n-cosimplices

$$\tau_\epsilon^{\hat{X}}(W)^0(U)^n = \hat{\Delta}^n_{X\setminus F, U} \setminus \Delta^n_U$$

so by Nisnevich excision we have the natural isomorphism

$$\alpha : E^{\Delta^*_{W_{\mathrm{Zar}}}}(\Delta^*_{X|W_{\mathrm{Zar}}}) \to E^{\Delta_W^*}(\tau_\epsilon^{\hat{X}}(W)),$$

where $E^{\Delta^*_{W_{\mathrm{Zar}}}}(\Delta^*_{X|W_{\mathrm{Zar}}})(W\setminus F)$ is the total spectrum of the simplicial spectrum

$$n \mapsto E^{\Delta^n_{W\setminus F}}(\Delta^n_{X\setminus F}).$$

The homotopy invariance of E implies that the pull-back

$$E^{W\setminus F}(X \setminus F) \to E^{\Delta^n_{W\setminus F}}(\Delta^n_{X\setminus F})$$

is a weak equivalence for all n, so we have the weak equivalence

$$\beta : E^{W_{\mathrm{Zar}}}(X) \to E^{\Delta^*_{W_{\mathrm{Zar}}}}(\Delta^*_{X|W_{\mathrm{Zar}}}).$$

It follows from the construction that $\psi = \alpha\beta$, completing the proof. □

Corollary 4.1.3. *There is a distinguished triangle in* $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$

$$E^{W_{\mathrm{Zar}}}(X) \to E(W_{\mathrm{Zar}}) \to E(\tau_\epsilon^{\hat{X}}(W)^0)$$

Proof. By Theorem 3.2.1, the map $\hat{i}^* : E(\tau_\epsilon^{\hat{X}}(W)) \to E(\Delta^*_{W_{\mathrm{Zar}}})$ is a weak equivalence; using homotopy invariance again, the map

$$E(W_{\mathrm{Zar}}) \to E(\Delta^*_{W_{\mathrm{Zar}}})$$

is a weak equivalence. Combining this with Theorem 4.1.2 yields the result. □

For homotopy invariant $E \in \mathbf{Spt}(k)$, we let

$$\phi_E : E(\tau_\epsilon^{\hat{N}_i}(0_W)) \to E(\tau_\epsilon^{\hat{X}}(W)).$$

be the isomorphism in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$ given by corollary 3.3.3.

Corollary 4.1.4. *Suppose that $E \in \mathbf{Spt}(k)$ is homotopy invariant and satisfies Nisnevich excision. Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$ and let $N_i^0 = N_i \setminus 0_W$.*

(1) The restriction maps

$$\mathrm{res} : E(N_i/W_{\mathrm{Zar}}) \to E(\tau_\epsilon^{\hat{N}_i}(0_W))$$
$$\mathrm{res}^0 : E(N_i^0/W_{\mathrm{Zar}}) \to E(\tau_\epsilon^{\hat{N}_i}(0_W)^0)$$

are weak equivalences in $\mathbf{Spt}(W_{\mathrm{Zar}})$.

(2) There is a canonical isomorphism in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$

$$\phi_E^0 : E(\tau_\epsilon^{\hat{N}_i}(0_W)^0) \to E(\tau_\epsilon^{\hat{X}}(W)^0)$$

(3) Consider the diagram (in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$*):*

$$\begin{tikzcd}
E^{0_{W_{\mathrm{Zar}}}}(N_i) \arrow[r] \arrow[d, equal] & E(N_i/W_{\mathrm{Zar}}) \arrow[r] \arrow[d, "\mathrm{res}_E"] & E(N_i^0/W_{\mathrm{Zar}}) \arrow[d, "\mathrm{res}^0_E"] \\
E^{0_{W_{\mathrm{Zar}}}}(N_i) \arrow[r] \arrow[d, "\pi"'] & E(\tau_\epsilon^{\hat{N}_i}(0_W)) \arrow[r, "\hat{j_N}^*"] \arrow[d, "\phi_E"] & E(\tau_\epsilon^{\hat{N}_i}(0_W)^0) \arrow[d, "\phi^0_E"] \\
E^{W_{\mathrm{Zar}}}(X) \arrow[r] & E(\tau_\epsilon^{\hat{X}}(W)) \arrow[r, "\hat{j}^*"'] & E(\tau_\epsilon^{\hat{X}}(W)^0) \\
E^{W_{\mathrm{Zar}}}(X) \arrow[r] \arrow[u, equal] & E(X|W_{\mathrm{Zar}}) \arrow[r, "j^*"'] \arrow[u, "\mathrm{res}_E"'] & E(X \setminus W) \arrow[u, "\mathrm{res}^0_E"']
\end{tikzcd}$$

The first and last rows are the homotopy fiber sequences defining the presheaves $E^{0_{W_{\mathrm{Zar}}}}(N_i)$ and $E^{W_{\mathrm{Zar}}}(X)$, respectively, the second row and third rows are the *distinguished triangles of Theorem 4.1.2, and π is the Morel-Voevodsky purity isomorphism. Then this diagram commutes and each triple of vertical maps defines a map of distinguished triangles.*

Proof. It follows directly from the weak equivalence (in Theorem 4.1.2) of the homotopy fiber of

$$\hat{j}^* : E(\tau_\epsilon^{\hat{X}}(W)) \to E(\tau_\epsilon^{\hat{X}}(W)^0)$$

with $E^{W_{\mathrm{Zar}}}(X)$ that the triple $(\mathrm{id}, \mathrm{res}_E, \mathrm{res}_E^0)$ defines a map of distinguished triangles. The same holds for the map of the first row to the second row; we now verify that this latter map is an isomorphism of distinguished triangles. For this, let $s : W \to N_i$ be the zero-section. We have the isomorphism $i_W^* : E(\tau_\epsilon^{\hat{N}_i}(0_W)) \to E(W_{\mathrm{Zar}})$ defined as the diagram of weak equivalences

$$E(\tau_\epsilon^{\hat{N}_i}(0_W)) \xrightarrow{i^*_{\Delta^*_{W_{\mathrm{Zar}}}}} E(\Delta^*_{W_{\mathrm{Zar}}}) \xleftarrow{\iota_{0*}} E(W_{\mathrm{Zar}}).$$

From this, it is easy to check that the diagram

$$\begin{tikzcd}
E(N_i/W_{\mathrm{Zar}}) \arrow[r, "\mathrm{res}_E"] \arrow[dr, "s^*"'] & E(\tau_\epsilon^{\hat{N}_i}(0_W)) \arrow[d, "i_W^*"] \\
& E(W_{\mathrm{Zar}})
\end{tikzcd}$$

commutes in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$. As E is homotopy invariant, s^* is an isomorphism, hence res_E is an isomorphism as well. This completes the proof of (1).

The proof of (2) and (3) uses the standard deformation diagram. Let $\bar{\mu} : \bar{Y} \to X \times \mathbb{A}^1$ be the blow-up of $X \times \mathbb{A}^1$ along W, let $\bar{\mu}^{-1}[X \times 0]$ denote the proper transform, and let $\mu : Y \to X \times \mathbb{A}^1$ be the open subscheme $\bar{Y} \setminus \bar{\mu}^{-1}[X \times 0]$. Let $p : Y \to \mathbb{A}^1$ be $p_2 \circ \mu$. Then $p^{-1}(0) = N_i$, $p^{-1}(1) = X \times 1 = X$, and Y contains the proper transform $\bar{\mu}^{-1}[W \times \mathbb{A}^1]$, which is mapped isomorphically by μ to $W \times \mathbb{A}^1 \subset X \times \mathbb{A}^1$. We let $\tilde{i} : W \times \mathbb{A}^1 \to Y$ be the resulting closed immersion. The restriction of $\tilde{i}$ to $W \times 0$ is the zero-section $s : W \to N_i$ and the restriction of $\tilde{i}$ to $W \times 1$ is $i : W \to X$. The resulting diagram is

(4.1.1)
$$\begin{tikzcd}
W \arrow[r, "i_0"] \arrow[d, "s"'] & W \times \mathbb{A}^1 \arrow[d, "\tilde{i}"'] & W \arrow[l, "i_1"'] \arrow[d, "i"] \\
N_i \arrow[r, "i_0"] \arrow[d, "p_0"'] & Y \arrow[d, "p"'] & X \arrow[l, "i_1"'] \arrow[d, "p_1"] \\
0 \arrow[r, "i_0"'] & \mathbb{A}^1 & 1 \arrow[l, "i_1"]
\end{tikzcd}$$

Together with Theorem 4.1.2, diagram (4.1.1) gives us two maps of distinguished triangles:

$$\Big[E^{0_{W\times\mathbb{A}^1}\mathrm{Zar}}(Y) \to E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)) \to E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)^0)\Big]$$
$$\xrightarrow{i_1^*}$$
$$\Big[E^{W_{\mathrm{Zar}}}(X) \to E(\tau_\epsilon^{\hat{X}}(W)) \to E(\tau_\epsilon^{\hat{X}}(W)^0)\Big]$$

and

$$\Big[E^{0_{W\times\mathbb{A}^1}\mathrm{Zar}}(Y) \to E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)) \to E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)^0)\Big]$$
$$\xrightarrow{i_0^*}$$
$$\Big[E^{0_W\mathrm{Zar}}(N_i) \to E(\tau_\epsilon^{\hat{N}_i}(0_W)) \to E(\tau_\epsilon^{\hat{N}_i}(0_W)^0)\Big]$$

As above, we have the commutative diagram

$$\begin{tikzcd}
E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)) \arrow[r, "i_0^*"] \arrow[d, "i^*_{W\times\mathbb{A}^1}"'] & E(\tau_\epsilon^{\hat{N}_i}(0_W)) \arrow[d, "i_W^*"'] \\
E(W\times\mathbb{A}^1) \arrow[r, "i_0^*"'] & E(W).
\end{tikzcd}$$

As E is homotopy invariant, the maps i_W^*, $i^*_{W\times\mathbb{A}^1}$ and $i_0^*: E(W\times\mathbb{A}^1)\to E(W)$ are isomorphisms, hence

$$i_0^* : E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)) \to E(\tau_\epsilon^{\hat{N}_i}(0_W))$$

is an isomorphism. Similarly,

$$i_1^* : E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)) \to E(\tau_\epsilon^{\hat{X}}(W))$$

is an isomorphism. The proof of the Morel-Voevodsky purity theorem [34, Theorem 2.23] shows that

$$i_0^* : E^{0_{W\times\mathbb{A}^1}\mathrm{Zar}}(Y) \to E^{0_W\mathrm{Zar}}(N_i)$$
$$i_1^* : E^{0_{W\times\mathbb{A}^1}\mathrm{Zar}}(Y) \to E^{W_{\mathrm{Zar}}}(X)$$

are weak equivalences; the purity isomorphism π is by definition $i_1^*\circ(i_0^*)^{-1}$. Thus, both i_0^* and i_1^* define isomorphisms of distinguished triangles, and

$$i_1^*\circ(i_0^*)^{-1} : E(\tau_\epsilon^{\hat{N}_i}(0_W)) \to E(\tau_\epsilon^{\hat{X}}(W))$$

is the map ϕ_E. Defining ϕ_E^0 to be the isomorphism

$$i_1^*\circ(i_0^*)^{-1} : E(\tau_\epsilon^{\hat{N}_i}(0_W)^0) \to E(\tau_\epsilon^{\hat{X}}(W)^0)$$

proves both (2) and (3). □

Remarks 4.1.5. (1) It follows from the construction of ϕ_E and ϕ_E^0 that both of these maps are natural in E.

(2) The maps ϕ_E^0 are natural in the closed immersion $i : W \to X$ in the following sense: Suppose we have closed immersions $i_j : W_j \to X_j$, $j = 1, 2$ and a morphism $f : (W_1, X_1) \to (W_2, X_2)$ of pairs of immersions such that f restricts to a morphism $X_1 \setminus W_1 \to X_2 \setminus W_2$. Fix E and let ϕ_{jE}^0 be the map corresponding to the immersions i_j We have the evident maps

$$\iota : \tau_\epsilon^{\hat{X}_1}(W_1)^0 \to \tau_\epsilon^{\hat{X}_2}(W_2)^0 \qquad \eta : \tau_\epsilon^{\hat{N}_{i_1}}(W_1)^0 \to \tau_\epsilon^{\hat{N}_{i_2}}(W_2)^0$$

Then the diagram

$$\begin{tikzcd} E(\tau_\epsilon^{\hat{N}_{i_2}}(W)^0) \arrow[r, "\phi_{2E}"] \arrow[d, "\eta^*"'] & E(\tau_\epsilon^{\hat{Y}}(W)^0) \arrow[d, "\iota^*"] \\ E(\tau_\epsilon^{\hat{N}_{i_1}}(W)^0) \arrow[r, "\phi_{1E}"'] & E(\tau_\epsilon^{\hat{X}}(W)^0) \end{tikzcd}$$

commutes. Indeed, the map f induces a map of deformation diagrams.
□

5. The exponential map

If $i : M' \to M$ is a submanifold of a differentiable manifold M, there is a diffeomorphism exp of the normal bundle $N_{M'/M}$ of M' in M with the tubular neighborhood $\tau_\epsilon^M(M')$. In addition, exp restricts to a diffeomorphism $\exp^0$ of the punctured normal bundle $N_{M'/M} \setminus 0_{M'}$ with the punctured tubular neighborhood $\tau_\epsilon^M(M') \setminus M'$. Classically, this has been used to define the boundary map in the Gysin sequence for $M' \to M$, by using the restriction map $\exp^{0*} : H^*(M \setminus M') \to H^*(N_{M'/M} \setminus 0_{M'})$ followed by the Thom isomorphism $H^*(N_{M'/M} \setminus 0_{M'}) \cong H^{*-d}(M')$.
In this section, we use our punctured tubular neighborhood to construct a purely algebraic version of the classical exponential map, at least for the associated suspension spectra. We will use this in section 11 to define a purely algebraic version of the gluing of Riemann surfaces along boundary components.

5.1. Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$ with normal bundle $p : N_i \to W$. We have the map

$$\exp : N_i \to X$$

in $\mathbf{Spc}(k)$, defined as the composition $N_i \to W \to X$. We also have the Morel-Voevodsky purity isomorphism

$$\pi : \mathrm{Th}(N_i) \to X/(X \setminus W)$$

in $\mathcal{H}(k)$. In fact:

Lemma 5.1.1. *The diagram*

$$\begin{tikzcd} N_i \arrow[r, "q'"] \arrow[d, "\exp"'] & \mathrm{Th}(N_i) \arrow[d, "\pi"] \\ X \arrow[r, "q"'] & X/(X \setminus W) \end{tikzcd} \tag{5.1.1}$$

commutes in $\mathcal{H}(k)$.

Proof. As we have already seen, the construction of the purity isomorphism π relies on the deformation to the normal bundle; we retain the notation from the proof of corollary 4.1.4. We have the total space $Y \to \mathbb{A}^1$ of the deformation. The fiber Y_0 over $0 \in \mathbb{A}^1$ is canonically isomorphic to N_i and the fiber Y_1 over 1 is canonically isomorphic to X; the inclusions $W \times 0 \to Y_0$, $W \times 1 \to Y_1$ are isomorphic to the zero-section $s : W \to N_i$ and the original inclusion $i : W \to X$, respectively. The proper transform $\mu^{-1}[W \times \mathbb{A}^1]$ is isomorphic to $W \times \mathbb{A}^1$, giving the closed immersion $\iota : W \times \mathbb{A}^1 \to Y$. The diagram thus induces maps in $\mathbf{Spc}(k)$:

$$\bar{i}_0 : \mathrm{Th}(N_i) \to Y/(Y \setminus W \times \mathbb{A}^1)$$
$$\bar{i}_1 : X/X \setminus W \to Y/(Y \setminus W \times \mathbb{A}^1)$$

which are isomorphisms in $\mathcal{H}(k)$ (see [34, Thm. 2.23]); the purity isomorphism is by definition $\pi := \bar{i}_1^{-1} \circ \bar{i}_0$.
We have the commutative diagram in $\mathbf{Spc}(k)$:

$$\begin{tikzcd} W \arrow[r, "i_0"] \arrow[d, shift left, "s"] \arrow[rr, bend left, "\mathrm{id}"] & W \times \mathbb{A}^1 \arrow[r, shift left, "p_1"] \arrow[d, "\iota"] & W \arrow[l, shift left, "i_1"] \arrow[d, "i"] \\ N_i \arrow[u, shift left, "p"] \arrow[r] \arrow[d, "q'"'] & Y \arrow[d] & X \arrow[l] \arrow[d, "q"] \\ \mathrm{Th}(N_i) \arrow[r, "\bar{i}_0"'] & Y/(Y \setminus W \times \mathbb{A}^1) & X/X \setminus W \arrow[l, "\bar{i}_1"] \end{tikzcd}$$

from which the result follows directly. □

Remark 5.1.2. Since we have the homotopy cofiber sequences:

$$N_i \setminus 0_W \to N_i \to \mathrm{Th}(N_i) \to \Sigma(N_i \setminus 0_W)_+$$
$$X \setminus W \to X \to X/(X \setminus W) \to \Sigma(X \setminus W)_+$$

the diagram (5.1.1) induces a map

$$\Sigma(N_i \setminus 0_W)_+ \to \Sigma(X \setminus W)_+$$

in $\mathcal{H}(k)$, however, this map is not uniquely determined, hence is not canonical. □

5.2. The construction. In this section we define a canonical map

$$\exp^0 : \Sigma^\infty(N_i \setminus 0_W)_+ \to \Sigma^\infty(X \setminus W)_+$$

in $\mathcal{SH}_{\mathbb{A}^1}(k)$ which yields the map of distinguished triangles in $\mathcal{SH}_{\mathbb{A}^1}(k)$:

$$\begin{tikzcd}
\Sigma^\infty(N_i \setminus 0_W)_+ \arrow[r] \arrow[d, "\exp^0"'] & \Sigma^\infty N_{i+} \arrow[r] \arrow[d, "\exp"'] & \Sigma^\infty \mathrm{Th}(N_i) \arrow[d, "\pi"] \\
\Sigma^\infty(X \setminus W)_+ \arrow[r] & \Sigma^\infty X_+ \arrow[r] & \Sigma^\infty X/(X \setminus W)
\end{tikzcd}$$

To define $\exp^0$, we apply Corollary 4.1.4 with E a fibrant model of $\Sigma^\infty(X\setminus W)_+$. Denote the composition

$$E(X \setminus W) \xrightarrow{\mathrm{res}^0_E} E(\tau^{\hat{X}}_\epsilon(W)^0)(W) \xrightarrow{(\phi^0_E)^{-1}} E(\tau^{\hat{N}_i}_\epsilon(0_W)^0)(W) \xrightarrow{(\mathrm{res}^0_E)^{-1}} E(N^0_i)$$

by $\exp^{0*}_E$. Since E is fibrant, we have canonical isomorphisms

$$\begin{aligned}
\pi_0 E(N^0_i) &\cong \mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(\Sigma^\infty N^0_{i+}, E) \\
&\cong \mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(\Sigma^\infty N^0_{i+}, \Sigma^\infty(X \setminus W)_+)
\end{aligned}$$

$$\begin{aligned}
\pi_0 E(X \setminus W) &\cong \mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(\Sigma^\infty(X \setminus W)_+, E) \\
&\cong \mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(\Sigma^\infty(X \setminus W)_+, \Sigma^\infty(X \setminus W)_+)
\end{aligned}$$

so $\exp^{0*}_E$ induces the map

$$\mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(\Sigma^\infty(X \setminus W)_+, \Sigma^\infty(X \setminus W)_+) \xrightarrow{\exp^{0*}_E} \mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(\Sigma^\infty N^0_{i+}, \Sigma^\infty(X \setminus W)_+).$$

We set

$$\exp^0 := \exp^{0*}_E(\mathrm{id}).$$

To finish the construction, we show

Proposition 5.2.1. *The diagram, with rows the evident homotopy cofiber sequences,*

$$\begin{tikzcd}
\Sigma^\infty(N_i \setminus 0_W)_+ \arrow[r] \arrow[d, "\exp^0"'] & \Sigma^\infty N_{i+} \arrow[r] \arrow[d, "\exp"'] & \Sigma^\infty \mathrm{Th}(N_i) \arrow[r, "\partial"] \arrow[d, "\pi"] & \Sigma\Sigma^\infty(N_i \setminus 0_W)_+ \arrow[d, "\Sigma\exp^0"'] \\
\Sigma^\infty(X \setminus W)_+ \arrow[r] & \Sigma^\infty X_+ \arrow[r] & \Sigma^\infty X/(X \setminus W) \arrow[r, "\partial"'] & \Sigma\Sigma^\infty(X \setminus W)_+
\end{tikzcd}$$

commutes in $\mathcal{SH}_{\mathbb{A}^1}(k)$.

Proof. It suffices to show that, for all fibrant $E \in \mathbf{Spt}(k)$, the diagram formed by applying $\mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(-, E)$ to our diagram commutes. This latter diagram is the same as applying π_0 to the diagram

(5.2.1)
$$\begin{tikzcd}
E(N_i \setminus 0_W) & E(N_i) \arrow[l] & E^{0_W}(N_i) \arrow[l] & \Omega E(N_i \setminus 0_W)) \arrow[l, "\partial"'] \\
E(X \setminus W) \arrow[u, "\exp^{0*}"'] & E(X) \arrow[l] \arrow[u, "\exp^*"'] & E^W(X) \arrow[l] \arrow[u, "\pi^*"] & \Omega E(X \setminus W) \arrow[l, "\partial"] \arrow[u, "\Omega\exp^{0*}"']
\end{tikzcd}$$

where the rows are the evident homotopy fiber sequences. It follows by the definition of $\exp^0$ and exp that this diagram is just the "outside" of the diagram in Corollary 4.1.4(3), extended to make the distinguished triangles explicit. Thus the diagram (5.2.1) commutes, which finishes the proof. □

Remark 5.2.2. The exponential maps exp and $\exp^0$ are natural with respect to maps of closed immersions $f : (W' \xrightarrow{i'} X') \to (W \xrightarrow{i} X)$ satisfying the cartesian condition of remark 4.1.5(2). This follows from the naturality of the isomorphisms ϕ_E, ϕ^0_E described in Remark 4.1.5, and the functoriality of the (punctured) tubular neighborhood construction. □

6. Neighborhoods of normal crossing schemes

We extend our results to the case of a strict normal crossing divisor $W \subset X$ by using a Mayer-Vietoris construction.

6.1. Normal crossing schemes. Let D be a reduced effective Cartier divisor on a smooth k-scheme X with irreducible components $D_1,\ldots, D_m$. For each $I \subset \{1, \ldots, m\}$, we set

$$D_I := \cap_{i\in I} D_i$$

We let $i : D \to X$ the inclusion. For each $I \neq \emptyset$, we let $\iota_I : D_I \to D$, $i_I : D_I \to X$ be the inclusions; for $I \subset J \subset \{1, \ldots, m\}$ we have as well the inclusion $\iota_{I,J} : D_J \to D_I$.
Recall that D is a *strict normal crossing divisor* if for each I, D_I is smooth over k and $\mathrm{codim}_X D_I = |I|$.
We extend this notion a bit: We call a closed subscheme $D \subset X$ a *strict normal crossing subscheme* if X is in $\mathbf{Sm}/k$ and, locally on X, there is a smooth locally closed subscheme $Y \subset X$ containing D such that D is a strict normal crossing divisor on Y

6.2. The tubular neighborhood. Let $D \subset X$ be a strict normal crossing subscheme with irreducible components $D_1, \ldots, D_m$. For each $I \subset \{1, \ldots, m\}$, $I \neq \emptyset$, we have the tubular neighborhood co-presheaf $\tau^{\hat{X}}_\epsilon(D_I)$ on D_I. The various inclusions $\iota_{I,J}$ give us the maps of co-presheaves

$$\hat{\iota}_{I,J} : \iota_{I,J*}(\tau^{\hat{X}}_\epsilon(D_J)) \to \tau^{\hat{X}}_\epsilon(D_I);$$

pushing forward by the maps ι_I yields the diagram of co-presheaves on D_{Zar} (with values in cosimplicial pro-objects of $\mathbf{Sm}/k$)

$$I \mapsto \iota_{I*}(\tau_\epsilon^{\hat{X}}(D_I)) \tag{6.2.1}$$

indexed by the non-empty $I \subset \{1, \ldots, m\}$. We have as well the diagram of identity co-presheaves

$$I \mapsto \iota_{I*}(D_{I\mathrm{Zar}}) \tag{6.2.2}$$

as well as the diagram

$$I \mapsto \iota_{I*}(\Delta^*_{D_{I\mathrm{Zar}}}) \tag{6.2.3}$$

We denote these diagrams by $\tau_\epsilon^{\hat{X}}(D)$, $D_\bullet$ and $\Delta^*_{D_\bullet}$, respectively. The projections $p_I : \Delta^*_{D_{I\mathrm{Zar}}} \to D_{I\mathrm{Zar}}$ and the closed immersions $\hat{\iota}_{D_I} : \Delta^*_{D_{I\mathrm{Zar}}} \to \tau_\epsilon^{\hat{X}}(D_I)$ yield the natural transformations

$$D_\bullet \xleftarrow{p_\bullet} \Delta^*_{D_\bullet} \xrightarrow{\hat{\iota}_{D_\bullet}} \tau_\epsilon^{\hat{X}}(D).$$

Now take $E \in \mathbf{Spt}(k)$. Applying E to the diagram (6.2.1) yields the diagram of presheaves on D_{Zar}

$$I \mapsto \iota_{I*}(E(\tau_\epsilon^{\hat{X}}(D_I)))$$

Similarly, applying E to (6.2.2) and (6.2.3) yields the diagrams of presheaves on D_{Zar}

$$I \mapsto \iota_{I*}(E(D_{I\mathrm{Zar}}))$$

and

$$I \mapsto \iota_{I*}(E(\Delta^*_{D_{I\mathrm{Zar}}})).$$

Definition 6.2.1. For $D \subset X$ a strict normal crossing subscheme and $E \in \mathbf{Spt}(k)$, set

$$E(\tau_\epsilon^{\hat{X}}(D)) := \operatorname*{holim}_{I \neq \emptyset} \iota_{I*}(E(\tau_\epsilon^{\hat{X}}(D_I))).$$

Similarly, set

$$E(D_\bullet) := \operatorname*{holim}_{I \neq \emptyset} \iota_{I*}(E(D_I))$$
$$E(\Delta^*_{D_\bullet}) := \operatorname*{holim}_{I \neq \emptyset} \iota_{I*}(E(\Delta^*_{D_I}))$$

□

The natural transformations $\hat{\iota}_D$ and $p_\bullet$ yield the maps of presheaves on D_{Zar}

$$E(D_\bullet) \xrightarrow{p_\bullet^*} E(\Delta^*_{D_\bullet}) \xleftarrow{\hat{\iota}_D^*} E(\tau_\epsilon^{\hat{X}}(D)).$$

Proposition 6.2.2. *Suppose $E \in \mathbf{Spt}(\mathbf{Sm}/k)$ is homotopy invariant and satisfies Nisnevich excision. Then the maps $\hat{\iota}_D^*$ and $p_\bullet^*$ are Zariski-local weak equivalences.*

Proof. The maps p_I^* are pointwise weak equivalences by homotopy invariance. By Theorem 3.2.1, the maps $\hat{\iota}_{D_I}$ are Zariski-local weak equivalences. Since the homotopy limits are finite, the stalk of each homotopy limit is weakly equivalence to the homotopy limit of the stalks. By [8] this suffices to conclude that the map on the homotopy limits is a Zariski-local weak equivalence. □

Remark 6.2.3. One could also attempt a more direct definition of $\tau_\epsilon^{\hat{X}}(D)$ by just using our definition in the smooth case $i : W \to X$ and replacing the smooth W with the normal crossing scheme D, in other words, the co-presheaf on D_{Zar}

$$D \setminus F \mapsto \hat{\Delta}^*_{X\setminus F, D\setminus F}.$$

Labeling this choice $\tau_\epsilon^{\hat{X}}(D)_{\text{naive}}$, and considering $\tau_\epsilon^{\hat{X}}(D)_{\text{naive}}$ as a constant diagram, we have the evident map of diagrams

$$\phi : \tau_\epsilon^{\hat{X}}(D) \to \tau_\epsilon^{\hat{X}}(D)_{\text{naive}}$$

We were unable to determine if ϕ induces a weak equivalence after evaulation on $E \in \mathbf{Spt}(k)$, even assuming that E is homotopy invariant and satisfies Nisnevich excision. We were also unable to construct such an E for which ϕ fails to be a weak equivalence. □

6.3. The punctured tubular neighborhood. To define the punctured tubular neighborhood $\tau_\epsilon^{\hat{X}}(D)^0$, we proceed as follows: Fix a subset $I \subset \{1, \ldots, m\}$, $I \neq \emptyset$. Let $p : X' \to X$, $s : D_I \to X'$ be a Nisnevich neighborhood of D_I in X, and let $D_{X'} = p^{-1}(D)$. Sending $X' \to X$ to $\Delta^n_{D_{X'}}$ gives us the pro-scheme $\hat{\Delta}^n_{D\subset X, D_I}$, and the closed immersion $\hat{\Delta}^n_{D\subset X, D_I} \to \hat{\Delta}^n_{X, D_I}$. Varying n, we have the cosimplicial pro-scheme $\hat{\Delta}^*_{D\subset X, D_I}$, and the closed immersion $\hat{\Delta}^*_{D\subset X, D_I} \to \hat{\Delta}^*_{X, D_I}$.

Take a closed subset $F \subset D_I$, and let $U := D_I \setminus F$. As in the definition of the punctured tubular neighborhood of a smooth closed subscheme in section 4.1, we pass to the appropriate cofinal subcategory of Nisnevich neighborhoods to show that the open complements $\hat{\Delta}^n_{X\setminus F, U} \setminus \hat{\Delta}^n_{D\setminus F\subset X\setminus F, U}$ for varying n form a cosimplicial pro-scheme

$$n \mapsto \hat{\Delta}^n_{X\setminus F, U} \setminus \hat{\Delta}^n_{D\setminus F\subset X\setminus F, U}.$$

Similarly, we set

$$\tau_\epsilon^{\hat{X}}(D, D_I)^0(U) := \hat{\Delta}^*_{X\setminus F, U} \setminus \hat{\Delta}^*_{D\setminus F\subset X\setminus F, U}.$$

This forms the co-presheaf $\tau_\epsilon^{\hat{X}}(D, D_I)^0$ on $D_{I\text{Zar}}$. The open immersions

$$\hat{j}_I(U)^n : \hat{\Delta}^n_{X\setminus F, U} \setminus \hat{\Delta}^n_{D\setminus F\subset X\setminus F, U} \to \hat{\Delta}^n_{X\setminus F, U}$$

define the map

$$\hat{j}_I(U) : \tau_\epsilon^{\hat{X}}(D, D_I)^0(U) \to \tau_\epsilon^{\hat{X}}(D_I)(U),$$

giving the map of co-presheaves

$$\hat{j}_I : \tau_\epsilon^{\hat{X}}(D, D_I)^0 \to \tau_\epsilon^{\hat{X}}(D_I).$$

For $J \subset I$, we have the map $\hat{\iota}_{J,I} : \hat{\Delta}^*_{X,D_I} \to \hat{\Delta}^*_{X,D_J}$ and

$$\hat{\iota}_{J,I}^{-1}(\hat{\Delta}^*_{D\subset X,D_J}) = \hat{\Delta}^*_{D\subset X,D_I}.$$

Thus we have the map $\hat{\iota}^0_{J,I} : \tau^{\hat{X}}_\epsilon(D, D_I)^0 \to \tau^{\hat{X}}_\epsilon(D, D_J)^0$ and the diagram of co-presheaves on D_{Zar}

$$I \mapsto \iota_{I*}(\tau^{\hat{X}}_\epsilon(D, D_I)^0) \tag{6.3.1}$$

which we denote by $\tau^{\hat{X}}_\epsilon(D)^0$. The maps $\hat{j}_I$ define the map

$$\hat{j} : \tau^{\hat{X}}_\epsilon(D)^0 \to \tau^{\hat{X}}_\epsilon(D).$$

The projection maps $\pi_I : \tau^{\hat{X}}_\epsilon(D_I) \to X$ (where we consider X as the constant co-presheaf on $D_{I\mathrm{Zar}}$) restrict to maps $\pi^0_I : \tau^{\hat{X}}_\epsilon(D, D_I)^0 \to X \setminus D$, which in turn induce the map

$$\pi^0 : \tau^{\hat{X}}_\epsilon(D)^0 \to X \setminus D,$$

where we consider $X \setminus D$ the constant diagram of constant co-presheaves on D_{Zar}.

Definition 6.3.1. For $E \in \mathbf{Spt}(k)$, let $E(\tau^{\hat{X}}_\epsilon(D)^0)$ be the presheaf on D_{Zar},

$$E(\tau^{\hat{X}}_\epsilon(D)^0) := \operatorname*{holim}_{\emptyset\neq I\subset\{1,\ldots,m\}} \iota_{I*}E(\tau^{\hat{X}}_\epsilon(D, D_I)^0).$$

□

The map $\hat{j}$ defines the map of presheaves

$$\hat{j}^* : E(\tau^{\hat{X}}_\epsilon(D)) \to E(\tau^{\hat{X}}_\epsilon(D)^0).$$

We let $E^{D_{\mathrm{Zar}}}(\tau^{\hat{X}}_\epsilon(D))$ denote the homotopy fiber of $\hat{j}^*$. Via the commutative diagram

$$\begin{tikzcd} E(X \setminus F) \arrow[r, "j^*"] \arrow[d, "\pi^*"'] & E(X \setminus D) \arrow[d, "\pi^{0*}"] \\ E(\tau^{\hat{X}}_\epsilon(D))(D \setminus F) \arrow[r, "\hat{j}^*"'] & E(\tau^{\hat{X}}_\epsilon(D))^0(D \setminus F) \end{tikzcd}$$

we have the canonical map

$$\pi^*_D : E^{D_{\mathrm{Zar}}}(X) \to E^{D_{\mathrm{Zar}}}(\tau^{\hat{X}}_\epsilon(D)).$$

We want to show that the map π^*_D is a weak equivalence, assuming that E is homotopy invariant and satisfies Nisnevich excision. We first consider a simpler situation. We begin by noting the following

Lemma 6.3.2. *Let $\square^n_0$ denote the category of non-empty subsets of $\{1, \ldots, n\}$ with maps the inclusions, let $\mathcal{C}$ be a small category and let $F : \mathcal{C} \times \square^n_0 \to \mathbf{Spt}^{\mathbf{Ord}^{\mathrm{op}}}$ be a functor. Let $\mathrm{holim}_{\square^n_0} F : \mathcal{C} \to \mathbf{Spt}^{\mathbf{Ord}^{\mathrm{op}}}$ be the functor with* *value the simplicial spectrum* $m \mapsto \mathrm{holim}_{\square_0^n} F(i \times [m])$ *at* $i \in \mathcal{C}$. *There is a isomorphism*

$$\mathrm{Tot}(\operatorname*{holim}_{\square_0^n} F) \to \operatorname*{holim}_{\square_0^n} \mathrm{Tot}(F).$$

in $\mathcal{H}\mathbf{Spt}(\mathcal{C}^{\mathrm{op}})$.

Proof. Letting $\square^n$ be the category of all subsets of $\{1, \ldots, n\}$ (including the empty set), we may extend F to $F* : \square^n \to \mathbf{Spt}(\mathbf{Ord}^{\mathrm{op}})$ by $F*(\emptyset) = 0$. Similarly, given a functor $G : \square^n \to \mathbf{Spt}$, we may extend G to $G_\natural : \square_0^{n+1} \to \mathbf{Spt}$ by $G_\natural(I) = 0$, $G_\natural(I \cup \{n+1\}) = G(I)$ for $I \subset \{1, \ldots, n\}$. We define the *iterated homotopy fiber* of G, $\mathrm{fib}_n G \in \mathbf{Spt}$, by

$$\mathrm{hofib}_n(G) := \operatorname*{holim}_{\square_0^{n+1}} G_\natural.$$

One easily checks that for a map $g : A \to B$ of spectra, considered in the evident manner as a functor $g_1 : \square^1 \to \mathbf{Spt}$, we have $\mathrm{hofib} g = \mathrm{hofib}_1 g_1$. More generally, if we let $i_-, i_+ : \square^{n-1} \to \square^n$ be the inclusions

$$i_-(I) := I, \ i_+(I) := I \cup \{n\}$$

we have the evident natural transformation $\omega : i_- \to i_+$ and for $G : \square^n \to \mathbf{Spt}$ a functor, we have a natural isomorphism

$$\mathrm{hofib}(\mathrm{hofib}_{n-1} G \circ i_- \xrightarrow{\mathrm{hofib}_{n-1} G(\omega)} \mathrm{hofib}_{n-1} G \circ i_+) \cong \mathrm{hofib}_n G,$$

hence the name iterated homotopy fiber. Finally, one has the natural isomorphism

$$\mathrm{hofib}_n G* \cong \Omega \operatorname*{holim}_{\square_0^n} G$$

for $G : \square_0^n \to \mathbf{Spt}$.
Since Tot is compatible with suspension we may replace our original functor F with $\Sigma F \cong \Omega^{-1} F$; using induction on n, it suffices to show that there is a natural isomorphism in $\mathcal{H}\mathbf{Spt}(\mathcal{C}^{\mathrm{op}})$

$$\mathrm{Tot}(\mathrm{hofib} F) \to \mathrm{hofib}\mathrm{Tot}(F)$$

for $F : A \to B$ a map in $\mathbf{Spt}^{\mathcal{C} \times \mathbf{Ord}^{\mathrm{op}}}$.
For this, note that for $f : X \to Y$ a map of spectra, there is a natural weak equivalence

$$a(f) : \Sigma \mathrm{hofib} f \to \mathrm{hocofib} f$$

Since Tot commutes with suspension and preserves weak equivalences, it suffices to define a natural weak equivalence

$$\mathrm{Tot}(\mathrm{hocofib} f) \to \mathrm{hocofib}(\mathrm{Tot} f).$$

In fact, since Tot preserves cofiber squares and is compatible with the wedge action of pointed simplicial sets on $\mathbf{Spt}^{\mathbf{Ord}^{\mathrm{op}}}$ and $\mathbf{Spt}$, there is a natural isomorphism $\mathrm{Tot}(\mathrm{hocofib} f) \to \mathrm{hocofib}(\mathrm{Tot} f)$, completing the proof. $\square$

This lemma allows us to define a simplicial model for $E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))$, induced by the cosimplicial structure on the co-presheaves $\tau_\epsilon^{\hat{X}}(D_I)$ and $\tau_\epsilon^{\hat{X}}(D_I)^0$. In fact, let

$$E(\tau_\epsilon^{\hat{X}}(D))_n := \operatorname*{holim}_{I\neq\emptyset} \iota_{I*}E(\hat{\Delta}^n_{\hat{X},D_{I\mathrm{Zar}}})$$
$$E(\tau_\epsilon^{\hat{X}}(D)^0)_n := \operatorname*{holim}_{I\neq\emptyset} \iota_{I*}E(\hat{\Delta}^n_{\hat{X},D_{I\mathrm{Zar}}} \setminus \hat{\Delta}^n_{D\subset X,D_{I\mathrm{Zar}}})$$

and set

$$E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))_n := \mathrm{hofib}(\hat{j}_n^* : E(\tau_\epsilon^{\hat{X}}(D))_n \to E(\tau_\epsilon^{\hat{X}}(D)^0)_n).$$

It follows from lemma 6.3.2 that $E(\tau_\epsilon^{\hat{X}}(D))$, $E(\tau_\epsilon^{\hat{X}}(D)^0)$ and $E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))$ are isomorphic in the homotopy category to the total presheaves of spectra associated to the simplical presheaves

$$n \mapsto E(\tau_\epsilon^{\hat{X}}(D))_n$$
$$n \mapsto E(\tau_\epsilon^{\hat{X}}(D)^0)_n$$
$$n \mapsto E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))_n$$

respectively. The map π_D^* is defined by considering $E^{D_{\mathrm{Zar}}}(X)$ as a constant simplicial object. Let

$$\pi_{D,0}^* : E^{D_{\mathrm{Zar}}}(X) \to E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))_0$$

be the map of $E^{D_{\mathrm{Zar}}}(X)$ to the 0-simplices of $E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))$.

Proposition 6.3.3. *Suppose that E satisfies Nisnevich excision and D is a strict normal crossing subscheme of X. Then $\pi_{D,0}^*$ is a weak equivalence.*

Before we give the proof of this result, we prove two preliminary lemmas.

Lemma 6.3.4. *Let x be a point on a finite type k-scheme X, let $Y = \operatorname{Spec}\mathcal{O}_{X,x}$ and Z and W be closed subschemes of Y. Then $\hat{Y}_Z^h \times_Y W \cong \hat{W}^h_{Z\cap W}$.*

Proof. Since Y and W are local, the pro-schemes $\hat{Y}_Z^h$ and $\hat{W}_{Z\cap W}$ are represented by local Y-schemes. If $Y' \to Y, s : Z \to Y'$ is a Nisnevich neighborhood of Z in Y, and $i : Z\cap W \to W$ is the inclusion, then $Y' \times_Y W \to W$, $(s_{|Z\cap W}, i) : Z\cap W \to Y'\times_Y W$ is a Nisnevich neighborhood of $Z\cap W$ in W, giving us the W-morphism

$$f : \hat{W}^h_{Z\cap W} \to \hat{Y}^h_Z \times_Y W.$$

As W is local, we have a co-final family in the category of all finite type étale morphisms $W' \to W$ of the form $W' = \operatorname{Spec}(\mathcal{O}_W[T]/F)_G$, i.e., the localization of $\mathcal{O}_W[T]/F$ with respect to some $G \in \mathcal{O}[T]$, where $(\partial F/\partial T, F)$ is the unit ideal in $\mathcal{O}_W[T]_G$. Those $W' \to W$ of this form which give a Nisnevich neighborhood of $Z\cap W$ are those for which F contains a linear factor, modulo the ideal $I_{Z\cap W}$ of $Z\cap W$. Each such pair (F, G) lifts to a pair $(\tilde{F}, \tilde{G})$ of elements in $\mathcal{O}_Y[T]$

such that $\mathrm{Spec}\,(\mathcal{O}_Y[T]/\tilde{F})_{\tilde{G}} \to Y$ is étale, and such that the linear factor in F mod $I_{Z\cap W}$ lifts to a linear factor of $\tilde{F}$ mod I_Z. This easily implies that f is an isomorphism. $\square$

Let $i : W \to Y$ be a closed immersion of finite type k-schemes, $E \in \mathbf{Spt}(Y_{\mathrm{Zar}})$. Define the functor

$$i^! : \mathbf{Spt}(Y_{\mathrm{Zar}}) \to \mathbf{Spt}(W_{\mathrm{Zar}})$$

by

$$(i^! E)(W \setminus F) := \mathrm{hofib}(E(Y \setminus F) \to E(Y \setminus W))$$

for each $F \subset W$ closed.
For each $I \subset \{1, \ldots, m\}$, let $\iota_I : D_I \to D$ be the inclusion. For $J \subset I$, and $F \subset D$ closed, the diagram of restriction maps

$$\begin{tikzcd} E(D \setminus (D_I \cap F)) \arrow[r] \arrow[d] & E(D \setminus D_I) \arrow[d] \\ E(D \setminus (D_J \cap F)) \arrow[r] & E(D \setminus D_J) \end{tikzcd}$$

gives the map

$$\iota_{I*}\iota_I^! E \to \iota_{J*}\iota_J^! E$$

Lemma 6.3.5. *Suppose $E \in \mathbf{Spt}(D_{\mathrm{Zar}})$ is satisfies Zariski excision. Then the evident map*

$$\operatorname*{hocolim}_{I \in \square_0^{n\,\mathrm{op}}} \iota_{I*}\iota_I^! E \to E$$

is a pointwise weak equivalence.

Proof. Suppose temporarily that D is an arbitrary finite type k-scheme, written as a union of two closed subschemes: $D = D^1 \cup D^2$, and take an $E \in \mathbf{Spt}(D_{\mathrm{Zar}})$ which is additive. Let $D^{12} := D^1 \cap D^2$, with inclusions $\iota^j : D^j \to D$, $\iota^{12} : D^{12} \to D$, $\iota^{j,12} : D^{12} \to D^j$. We have the natural map

$$\mathrm{hocolim}\left[\begin{tikzcd} \iota_{12*}\iota^{!12}E \arrow[r, "\iota_*^{1,12}"] \arrow[d, "\iota_*^{2,12}"'] & \iota_*^1\iota^{1!}E \\ \iota_*^2\iota^{2!}E & \end{tikzcd}\right] \xrightarrow{\alpha} E$$

We first show that α is a pointwise weak equivalence. It suffices to show that α is a weak equivalence on global sections, equivalently, that the diagram

$$\begin{tikzcd} E^{D^{12}}(D) \arrow[r] \arrow[d] & E^{D^1}(D) \arrow[d] \\ E^{D^2}(D) \arrow[r] & E(D) \end{tikzcd}$$

is homotopy cocartesian.

The homotopy cofiber of $E^{D^1}(D) \to E(D)$ is homotopy equivalence to $E(D \setminus D^1)$ and the homotopy cofiber of $E^{D^{12}}(D) \to E^{D^2}(D)$ is homotopy equivalent to $E^{D^2 \setminus D^{12}}(D \setminus D^{12})$. Since

$$D \setminus D^{12} = D^1 \setminus D^{12} \amalg D^2 \setminus D^{12}$$

and E is additive, the map on the homotopy cofibers is a weak equivalence, as desired.

The proof of the lemma now follows easily by induction on the number m of irreducible components of $D = \cup_{i=1}^m D_i$. Indeed, write $D = D^1 \cup D^2$, with $D^1 = D_1$ and $D^2 = \cup_{i=2}^m D_i$. Note that the Zariski excision property is preserved by the functor $i^!$ and that a presheaf that satisfies Zariski excision is additive. By induction the maps

$$\operatorname*{hocolim}_{\emptyset \neq I \subset \{2,\ldots,m\}} \iota_{I*}\iota_I^! E \to \iota_*^2 \iota^{2!} E$$

$$\operatorname*{hocolim}_{\{1\} \subsetneq I \subset \{1,\ldots,m\}} \iota_{I*}\iota_I^! E \to \iota_*^{12} \iota^{12!} E$$

are pointwise weak equivalences. Thus the map

$$\operatorname*{hocolim}_{I \in \square_0^{n\mathrm{op}}} \iota_{I*}\iota_I^! E \to \operatorname{hocolim} \left[\begin{array}{ccc} \iota_{12*}\iota^{!12} E & \xrightarrow{\iota_*^{1,12}} & \iota_*^1 \iota^{1!} E \\ {\scriptstyle \iota_*^{2,12}} \downarrow & & \\ \iota_*^2 \iota^{2!} E & & \end{array} \right]$$

is a pointwise weak equivalence; combined with our previous computation, this proves the lemma. $\square$

Proof of proposition 6.3.3. Write D as a sum, $D = \sum_{i=1}^m D_i$ with each D_i smooth (but not necessarily irreducible), and with m minimal. We proceed by induction on m.

For $m = 1$, Nisnevich excision implies that the natural map

$$E^{D_{\mathrm{Zar}}}(X) \to E^{D_{\mathrm{Zar}}}(\hat{X}_D^h)$$

is a weak equivalence in $\mathbf{Spt}(D_{\mathrm{Zar}})$. Since D is smooth, the map $E^{D_{\mathrm{Zar}}}(\hat{X}_D^h) \to E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))_0$ is an isomorphism, which proves the result in this case.

By lemma 6.3.5 it suffices to show that $\iota_I^!(\pi_{D,0}^*)$ is a weak equivalence for all I. More generally, let $\iota_{I,J} : D_I \to D_J$ be the inclusion for $I \subset J$. If E satisfies Zariski excision on D_{Zar}, the same holds for $\iota_I^! E$ on $D_{I,\mathrm{Zar}}$ and there is a natural weak equivalence

$$\iota_{J,I}^!(\iota_I^! E) \to \iota_J^! E$$

Thus it suffices to show that $\iota_i^!(\pi_{D,0}^*)$ is a weak equivalence for all $i \in \{1, \ldots, m\}$, e.g., for $i = m$. In what follows, we will only apply $\iota_I^!$ to presheaves E which satisfy Zariski excision, which suffices for the proof.

We use the following notation: for $W \subset D_I$ a closed subset, we let $E^{W_{\mathrm{Zar}}}(X)$ denote the presheaf on D_I

$$E^{W_{\mathrm{Zar}}}(X)(D_I \setminus F) := E^{W \setminus F}(X \setminus F).$$

We use the same notation for the presheaf

$$D \setminus F \mapsto E^{W \setminus F}(X \setminus F)$$

on D_{Zar}, relying on the context to make the meaning clear.
Clearly $\iota_m^! \iota_{m*} E^{D_{m\mathrm{Zar}}}(X) \to E^{D_{m\mathrm{Zar}}}(X)$ is a weak equivalence and the map $E^{D_{m\mathrm{Zar}}}(X) \to E^{D_{\mathrm{Zar}}}(X)$ induces a weak equivalence $\iota_m^! E^{D_{m\mathrm{Zar}}}(X) \to \iota_m^! E^{D_{\mathrm{Zar}}}(X)$, so we need to show that

$$E^{D_{m\mathrm{Zar}}}(X) \to \iota_m^! E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))_0 = \iota_m^! (\operatorname*{holim}_{I \neq \emptyset} E^{\hat{X}_{D_I} \times_X D_{\mathrm{Zar}}}(\hat{X}_{D_I})$$

is a weak equivalence.
For this, we decompose the set of non-empty $I \subset \{1, \ldots, m\}$ into three sets:

1. $I = \{m\}$,
2. I with $m \notin I$,
3. I with $\{m\} \subsetneqq I$.

Let

$$\begin{aligned} E_1 &:= \iota_m^! E^{\hat{X}^h_{D_m} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_m}) \\ E_2 &:= \operatorname*{holim}_{m \notin I} \iota_m^! E^{\hat{X}^h_{D_I} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_I}) \\ E_3 &:= \operatorname*{holim}_{\{m\} \subsetneqq I} \iota_m^! E^{\hat{X}^h_{D_I} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_I}) \end{aligned}$$

We thus have the isomorphism

$$\iota_m^! \left(\operatorname*{holim}_{I \neq \emptyset} E^{\hat{X}^h_{D_I} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_I}) \right) \cong \operatorname{holim} \left[\begin{array}{ccc} & & E_1 \\ & & \downarrow \\ E_2 & \longrightarrow & E_3 \end{array} \right]$$

For I of type 2, lemma 6.3.4 says that the natural map

$$\hat{X}^h_{D_{I \cup \{m\}}} \times_X D_m \to \hat{X}^h_{D_I} \times_X D_m$$

is an isomorphism. Since the restriction map

$$\iota_m^! E^{\hat{X}^h_{D_I} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_I}) \to \iota_m^! E^{\hat{X}^h_{D_{I \cup \{m\}}} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_{I \cup \{m\}}})$$

identifies itself with the pull-back

$$E^{\hat{X}^h_{D_I} \times_X D_{m\mathrm{Zar}}}(\hat{X}^h_{D_I}) \to E^{\hat{X}^h_{D_{I \cup \{m\}}} \times_X D_{m\mathrm{Zar}}}(\hat{X}^h_{D_{I \cup \{m\}}})$$

the Nisnevich excision property of E implies that $E_2 \to E_3$ is a weak equivalence. Thus

$$E_1 \to \operatorname{holim} \left[\begin{array}{ccc} & & E_1 \\ & & \downarrow \\ E_2 & \longrightarrow & E_3 \end{array} \right]$$

is a weak equivalence, and

$$E^{D_m\mathrm{Zar}}(X) \to \iota_m^! E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))_0 = \iota_m^! \left(\operatorname*{holim}_{I \neq \emptyset} E^{\hat{X}^h_{D_I} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_I}) \right)$$

is identified with

$$E^{D_m\mathrm{Zar}}(X) \to \iota_m^! E^{\hat{X}^h_{D_m} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_m}) = E^{\hat{X}^h_{D_m} \times_X D_m\mathrm{Zar}}(\hat{X}^h_{D_m}),$$

which is a weak equivalence by Nisnevich excision. □

Proposition 6.3.6. *Suppose that E is homotopy invariant and satisfies Nisnevich excision, and D is a strict normal crossing subscheme of X. Then*

$$\pi_D^* : E^{D_{\mathrm{Zar}}}(X) \to E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))$$

is a weak equivalence in $\mathbf{Spt}(D_{\mathrm{Zar}})$.

Proof. Let $p_n : \Delta^n_D \to D$ be the projection. Applying Proposition 6.3.3 to the strict normal crossing subscheme Δ^n_D of Δ^n_X, the map

$$\pi_{\Delta^n_D,0} : p_{n*}E^{\Delta^n_{D\mathrm{Zar}}}(\Delta^n_X) \to p_{n*}E^{\Delta^n_{D\mathrm{Zar}}}(\tau_\epsilon^{\widehat{\Delta^n_X}}(\Delta^n_D))_0$$

is a weak equivalence for each n. Thus

$$\pi_{\Delta^*_D} : p_*E^{\Delta^*_{D\mathrm{Zar}}}(\Delta^*_X) \to E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))$$

is a weak equivalence. Indeed, $E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D)$ is a simplicial object with n-simplices $p_{n*}E^{\Delta^n_{D\mathrm{Zar}}}(\tau_\epsilon^{\widehat{\Delta^n_X}}(\Delta^n_D))_0$. Since E is homotopy invariant, the map

$$p^* : E^{D_{\mathrm{Zar}}}(X) \to p_*E^{\Delta^*_{D\mathrm{Zar}}}(\Delta^*_X)$$

is a weak equivalence, whence the result. □

We can now state and prove the main result for strict normal crossing schemes.

Theorem 6.3.7. *Let D be a strict normal crossing scheme on some $X \in \mathbf{Sm}/k$ and take $E \in \mathbf{Spt}(k)$ which is homotopy invariant and satisfies Nisnevich excision. Then there is a natural distinguished triangle in* $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$

$$E^{D_{\mathrm{Zar}}}(X) \xrightarrow{\alpha_D} E(D_\bullet) \xrightarrow{\beta_D} E(\tau_\epsilon^{\hat{X}}(D)^0)$$

Proof. By proposition 6.3.6, we have the weak equivalence

$$\pi_D^* : E^{D_{\mathrm{Zar}}}(X) \to E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D)).$$

By Proposition 6.2.2, we have the isomorphism

$$(p_\bullet^*)^{-1}\hat{\iota}_D^* : E(\tau_\epsilon^{\hat{X}}(D)) \to E(D_\bullet).$$

in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$. Since $E^D(\tau_\epsilon^{\hat{X}}(D))$ is by definition the homotopy fiber of the restriction map $E(\tau_\epsilon^{\hat{X}}(D)) \to E(\tau_\epsilon^{\hat{X}}(D)^0)$, the result is proved. $\square$

7. Comparison isomorphisms

We give a comparison of our tubular neighborhood construction with the categorical version Li^*Rj_* of Morel-Voevodsky.

7.1. Model structure and cross functors. Fix a noetherian separated scheme S of finite Krull dimension, and let $\mathbf{Sch}_S$ denote the category of finite type S-schemes (for our application, we will take $S = \operatorname{Spec} k$ for a field k). Morel-Voevodsky show how to make the catgory $\mathcal{SH}_{\mathbb{A}^1}(X)$ functorial in $X \in \mathbf{Sch}_S$, defining an adjoint pair of exact functors Lf^*, Rf_* for each morphism $f : Y \to X$ in $\mathbf{Sch}_S$. Roendigs shows in [39] how to achieve this on the model category level and in addition that this structure extends to give cross functors $(f_*, f^*, f^!, f_!)$ as defined by Voevodsky and investigated in detail by Ayoub [3]. We begin by describing the model structure used by Roendigs, which is different from the one we have used up to now, and recalling his main results.
For $B \in \mathbf{Sch}_S$, we denote by $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ the model structure on $\mathbf{Spc}_*(\mathbf{Sm}/B)$ described by Roendigs in [39]. To describe this model structure, we first recall the projective model structure $\mathbf{Spc}(\mathbf{Sm}/B)_{\mathrm{proj}}$ on $\mathbf{Spc}(\mathbf{Sm}/B)$. Here the weak equivalences and fibrations are the pointwise ones and the cofibrations are generated by the maps

$$Z \times \partial\Delta^n \to Z \times \Delta^n,$$

with $Z \in \mathbf{Sm}/B$. This induces a model structure $\mathbf{Spc}_*(\mathbf{Sm}/B)_{\mathrm{proj}}$ on $\mathbf{Spc}_*(\mathbf{Sm}/B)$ by forgetting/adjoing a base-point. One has a functorial cofibrant replacement $E^c \to E$ defined as in [34, Lemma 1.16].
The model structure $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ is defined by Bousfield localization: the cofibrations are the same as in $\mathbf{Spc}_*(\mathbf{Sm}/B)_{\mathrm{proj}}$. E is fibrant if $E(\emptyset)$ is contractible, E is a fibrant in $\mathbf{Spc}_*(\mathbf{Sm}/B)_{\mathrm{proj}}$, E transforms elementary Nisnevich squares to homotopy fiber squares and transforms $Z \times \mathbb{A}^1 \to Z$ to a weak equivalence. A map $A \to B$ is a weak equivalence if $\mathcal{H}om(B^c, E) \to \mathcal{H}om(A^c, E)$ is a weak equivalence for each fibrant E. The fibrations in $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ are determined by having the right lifting property with respect to trivial cofibrations.
Let $f : X \to Y$ be a morphism in $\mathbf{Sch}_S$. We have the functor

$$f_* : \mathbf{Spc}_*(\mathbf{Sm}/X) \to \mathbf{Spc}_*(\mathbf{Sm}/Y)$$

defined by pre-composition with the pull-back functor $- \times_Y X$, i.e.

$$f_*E(Y' \to Y) := E(Y' \times_Y X \to X).$$

f_* has the left adjoint f^* defined as the Kan extension, and characterized by $f^*(Y'_+) = Y' \times_Y X_+$ for $Y' \to Y \in \mathbf{Sm}/Y$. In case f is a smooth morphism, f^* is given by precomposition with the functor

$$f \circ - : \mathbf{Sm}/X \to \mathbf{Sm}/Y,$$

and thus has the left adjoint $f_\sharp$ characterized by

$$f_\sharp(Z \xrightarrow{p} X) = Z \xrightarrow{fp} Y$$

on the representable presheaves. We have

Proposition 7.1.1 (proposition 2.18 of [39]). *Let $f : X \to Y$ be a morphism in $\mathbf{Sch}_S$. Then (f^*, f_*) is a Quillen adjoint pair $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/X) \leftrightarrow \mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/Y)$. If f is smooth, then $(f_\sharp, f^*)$ is a Quillen adjoint pair $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/Y) \leftrightarrow \mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/X)$.*

For spectra, the projective model structure $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/B)_{\mathrm{proj}}$ on $\mathbf{Spt}(\mathbf{Sm}/B)$ is defined as follows: For $\phi : E \to F$ a morphism in $\mathbf{Spt}(\mathbf{Sm}/B)$, $\phi : E \to F$ is a cofibration if $\phi_0 : E_0 \to F_0$ is a cofibration in $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ and if for each $n \geq 1$, the map

$$\phi_n \cup \Sigma\phi_{n-1} : E_n \cup_{\Sigma E_{n-1}} \Sigma F_{n-1} \to F_n$$

is a cofibration in $\mathbf{Spc}_*(\mathbf{Sm}/B)_{\mathrm{proj}}$. Weak equivalences (resp. fibrations) are maps ϕ such that ϕ_n is a weak equivalence (resp. fibration) in $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ for all n. There is a functorial cofibrant replacement $E^c \to E$.
Now for the motivic model structure $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/B)$: The cofibrations are the same as in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/B)_{\mathrm{proj}}$. ϕ is a fibration if ϕ_n is a fibration in $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ for all n and the diagram

$$\begin{tikzcd} E_n \arrow[r] \arrow[d, "\phi_n"'] & \Omega E_{n+1} \arrow[d, "\Omega\phi_{n+1}"] \\ F_n \arrow[r] & \Omega F_{n+1} \end{tikzcd}$$

is homotopy cartesian in $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ for all n. There is a fibrant replacement functor $E \to E^f$; $\phi : E \to F$ is a weak equivalence if $\phi^f : E^f \to F^f$ is a weak equivalence in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/B)_{\mathrm{proj}}$.
Given $f : X \to Y$ in $\mathbf{Sch}_S$, define the functors $f_* : \mathbf{Spt}(\mathbf{Sm}/X) \to \mathbf{Spt}(\mathbf{Sm}/Y)$ and $f^* : \mathbf{Spt}(\mathbf{Sm}/Y) \to \mathbf{Spt}(\mathbf{Sm}/X)$ by $f_*(E)_n := f_*(E_n)$, $f^*(F)_n := f^*(F_n)$. If f is smooth, we have $f_\sharp : \mathbf{Spt}(\mathbf{Sm}/X) \to \mathbf{Spt}(\mathbf{Sm}/Y)$ defined similarly by $f_\sharp(E)_n := f_\sharp(E_n)$.
We have the following result from [39]:

Proposition 7.1.2 (proposition 2.23 of [39]). *Let $f : X \to Y$ be a morphism in $\mathbf{Sch}_S$. Then (f_*, f^*) is a Quillen adjoint pair $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X) \leftrightarrow \mathbf{Spt}(\mathbf{Sm}/Y)$. If f is smooth, then $(f_\sharp, f^*)$ is a Quillen adjoint pair $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/Y) \leftrightarrow \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$. In particular:*

1. *f^* preserves cofibrations and trivial cofibration and f_* preserves fibrations and trivial fibrations.*
2. *if f is smooth, then f^* preserves fibrations and $f_\sharp$ preserves cofibrations*

It is clear that a cofibration in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$ is pointwise a cofibration in $\mathbf{Spt}$, hence a cofibration in $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}})$. As mentioned in [39] a fibrant object in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$ satisfies both Nisnevich excision and is $\mathbb{A}^1$-local, hence the weak equivalences between fibrant objects in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$ are weak equivalences in $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}})$ and are in fact pointwise weak equivalences in $\mathbf{Spt}(\mathbf{Sm}/X)$; similarly one shows that each fibration in $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}})$ is a fibration in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$ and each (trivial) cofibration in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$ is a (trivial) cofibration in $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}})$. Thus the identity on $\mathbf{Spt}(\mathbf{Sm}/X)$ defines a (left) Quillen equivalence $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X) \to \mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}})$. In particular, we have the equivalence of the homotopy categories

$$\mathcal{H}\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}}) \cong \mathcal{H}\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X).$$

We write $\mathcal{SH}_{\mathbb{A}^1}(X)$ for either $\mathcal{H}\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$ or $\mathcal{H}\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}})$, depending on the context.
One main result of [39] is

THEOREM 7.1.3 ([39, corollary 3.17]). *Sending $f : Y \to X$ in $\mathbf{Sch}_S$ to $Lf^* : \mathcal{SH}_{\mathbb{A}^1}(X) \to \mathcal{SH}_{\mathbb{A}^1}(X)$ satisfies the conditions of* [3, definition 1.4.1]. *In particular, the properties of a "2-foncteur homotopique stable" described in* [3] *are satisfied for $X \mapsto \mathcal{SH}_{\mathbb{A}^1}(X)$.*

Remark 7.1.4. Let $i : D \to X$ be a closed immersion in $\mathbf{Sch}_S$ with open complement $j : U \to X$. We have the functor

$$Li^*Rj_* : \mathcal{SH}_{\mathbb{A}^1}(X \setminus D) \to \mathcal{SH}_{\mathbb{A}^1}(D),$$

We would like to view our construction $E(\tau_\epsilon^{\hat{X}}(D)^0)$ as a weak version of Li^*Rj_*, in case D is a normal crossing divisor on a smooth k scheme X, the input E is the pull-back from $\mathbf{Spt}(\mathbf{Sm}/k)$, and the output $E(\tau_\epsilon^{\hat{X}}(D)^0)$ is in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$. In particular, $E(\tau_\epsilon^{\hat{X}}(D)^0)$ is only defined on Zariski open subsets of D, rather than on all of $\mathbf{Sm}/D$. In this section, we make this statement precise, defining an isomorphism of $E(\tau_\epsilon^{\hat{X}}(D)^0)$ with the restriction of $Li^*Rj_*(E)$ to $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$. □

7.2. THE SMOOTH CASE. Let $i : W \to X$ be a closed immersion in $\mathbf{Sch}_S$ with open complement $j : U \to X$. Let

$$\Theta : \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/U) \to \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/W)$$

be the functor representing Li^*Rj_*, i.e.

$$\Theta(E) := i^*(j_*(E^f)^c)^f.$$

Remark 7.2.1. Even for $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/U)$ bifibrant, one cannot simplify this expression for Li^*Rj_*E beyond replacing E^f with E. The inexplicit nature of the cofibrant and fibrant replacement functors make a concrete determination of Li^*Rj_*E difficult, which is one advantage of our approach using the punctured tubular neighborhood. □

LEMMA 7.2.2. *Let $i : W \to X$ be a closed immersion in $\mathbf{Sch}_S$ with open complement $j : U \to X$.*

(1) *For fibrant* $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/U)$ *all the maps in the square*

$$\begin{array}{ccc} j_*(E)^c & \longrightarrow & Rj_*(E)^c \\ \downarrow & & \downarrow \\ j_*(E) & \longrightarrow & Rj_*(E) \end{array}$$

are pointwise weak equivalences

(2) *Let* $X' \to X$ *be in* $\mathbf{Sm}/X$, *let* $W' := W \times_X X'$. *There is a canonical map*

$$\nu^0_{X'} : Rj_*(E)^c(X') \to \Theta E(W')$$

natural in X'.

Proof. (1) Since E is fibrant, the canonical map $E \to E^f$ is a trivial cofibration of fibrant objects in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/U)$, hence a homotopy equivalence. Thus $j_*E \to Rj_*E := j_*(E^f)$ is a homotopy equivalence of fibrant objects in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$, hence a pointwise weak equivalence. Applying the cofibrant replacement functor, we see that $(j_*E)^c \to (Rj_*E)^c$ is also a homotopy equivalance and a pointwise weak equivalence. Also the cofibrant replacement maps $(j_*E)^c \to j_*E$, $(Rj_*E)^c \to Rj_*E$ are trivial fibrations between fibrant objects of $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$, hence are both pointwise weak equivalences.

For (2), the unit $\mathrm{id} \to i_*i^*$ for the adjunction applied to $(Rj_*E)^c$ gives us the map

$$\nu^0_{X'} : (Rj_*E)^c(X') \to i_*i^*(Rj_*E)^c(X')$$

natural in X'. As $i_*i^*(Rj_*E)^c(X') = i^*(Rj_*E)^c(W')$, we have the natural transformation

$$\nu^0_{X'} : (Rj_*E)^c(X') \to i^*(Rj_*E)^c(W')$$

Composing with the canonical map $i^*(Rj_*E)^c \to (i^*(Rj_*E)^c)^f = \Theta(E)$ gives us the map we want. □

For $E \in \mathbf{Spt}(\mathbf{Sm}/B)$ or in $\mathbf{Spt}(B_{\mathrm{Nis}})$, we let E_{Zar} denote the restriction to $\mathbf{Spt}(B_{\mathrm{Zar}})$. Identifying $\mathcal{SH}_{\mathbb{A}^1}(B)$ with the homotopy category of bifibrant objects in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/B)$, we have the similarly defined restriction functor $\mathcal{SH}_{\mathbb{A}^1}(B) \to \mathcal{H}\mathbf{Spt}(B_{\mathrm{Zar}})$ sending E to E_{Zar}.

Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$ with open complement $j : U \to X$. We note that the "evaluation" maps

$$E \mapsto E(\tau^{\hat{X}}_\epsilon(W)),\ E \mapsto E(\tau^{\hat{X}}_\epsilon(W)^0)$$

are in fact defined for $E \in \mathbf{Spt}(\mathbf{Sm}/X)$. Similarly, the evaluation map $E \mapsto E(\tau^{\hat{X}}_\epsilon(W)^0)$ is defined for $E \in \mathbf{Spt}(\mathbf{Sm}/U)$. In addition, for $E \in \mathbf{Spt}(\mathbf{Sm}/U)$ we have a canonical isomorphism

$$E(\tau^{\hat{X}}_\epsilon(W)^0) \cong (j_*E)(\tau^{\hat{X}}_\epsilon(W)) \tag{7.2.1}$$

since $\hat{\Delta}^n_{X,W} \setminus \Delta^n_W \cong \hat{\Delta}^n_{X,W} \times_X U$ (as a pro-scheme).

Lemma 7.2.3. *Let* $i : W \to X$ *be a closed immersion in* $\mathbf{Sm}/k$ *with open complement* $j : U \to X$, *and let* $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/U)$ *be fibrant.*

(1) *There is a map*

$$\eta^c_E : (Rj_*E)^c(\tau^{\hat{X}}_\epsilon(W)) \to \Theta(E)_{\mathrm{Zar}}$$

in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$, *natural in* E.

(2) *Let*

$$\begin{tikzcd} & j_*(E)^c(\tau^{\hat{X}}_\epsilon(W)) \arrow[r] \arrow[d] & Rj_*(E)^c(\tau^{\hat{X}}_\epsilon(W)) \arrow[d] \\ E(\tau^{\hat{X}}_\epsilon(W)^0) \arrow[r, "\sim", "\phi"'] & j_*(E)(\tau^{\hat{X}}_\epsilon(W)) \arrow[r] & Rj_*(E)(\tau^{\hat{X}}_\epsilon(W)) \end{tikzcd}$$

be the diagram in $\mathbf{Spt}(W_{\mathrm{Zar}})$ *formed by evaluating the diagram of lemma 7.2.2(1) at* $\tau^{\hat{X}}_\epsilon(W)$, *and adding the isomorphism* (7.2.1). *Then all the maps in this diagram are pointwise weak equivalences.*

Proof. By lemma 7.2.2, we have maps

$$\eta^0_{X'} : (Rj_*E)^c(X') \to \Theta(E)(X' \times_X W)$$

natural in $X' \in \mathbf{Sm}/X$. For each open subscheme $U = D \setminus F \subset W$, the maps $\eta_{\hat{\Delta}^n_{X\setminus F,U}}$ define the map

$$\eta^0_{\Delta^*}(U) : (Rj_*E)^c(\tau^{\hat{X}}_\epsilon(W)(U)) \to \Theta(E)(\Delta^*_U).$$

Since $\Theta(E)$ is $\mathbb{A}^1$-local, the canonical map $\Theta(E)(U) \to \Theta(E)(\Delta^*_U)$ is a weak equivalence. This gives us the natural map in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$

$$\eta^0 : (Rj_*E)^c(\tau^{\hat{X}}_\epsilon(W)) \to \Theta(E)_{\mathrm{Zar}},$$

proving (1).

(2) follows immediately from lemma 7.2.2(1). □

Combining the morphism (1) with the diagram (2) gives us the canonical morphism in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$

$$\eta^0_E : E(\tau^{\hat{X}}_\epsilon(W)^0) \to \Theta(E)_{\mathrm{Zar}}.$$

Let $i : D \to X$ be a closed immersion in $\mathbf{Sch}_S$. We have the exact functor $i^! : \mathcal{SH}_{\mathbb{A}^1}(X) \to \mathcal{SH}_{\mathbb{A}^1}(D)$ which is characterized by the identity for fibrant $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$:

$$i_*i^!E(X' \to X) := \mathrm{hofib}(E(X') \to E(X' \times_X (X \setminus D)).$$

In fact, this operation gives the distinguished triangle, natural in fibrant $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$:

$$Ri_*i^!E \to E \to Rj_*j^*E \to i_*i^!E[1].$$

Applying Li^* (and noting that the counit $Li^*Ri_* \to \mathrm{id}$ is an isomorphism [3, definition 1.4.1]) gives the distinguished triangle in $\mathcal{SH}_{\mathbb{A}^1}(D)$

$$i^!E \to Li^*E \to \Theta(j^*E) \to i^!E[1] \tag{7.2.2}$$

We refer the reader to [3, proposition 1.4.9] for the construction of this triangle in the abstract setting.

Proposition 7.2.4. *Let $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/k)$ be fibrant, let $f : X \to \operatorname{Spec} k$ be in $\mathbf{Sm}/k$ and let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$ with open complement $j : U \to X$. Then*

$$\eta_E^0 : E(\tau_\epsilon^{\hat{X}}(W)^0) \to \Theta(j^* f^* E)_{\mathrm{Zar}}$$

is an isomorphism in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$.

Proof. Let $f_W : W \to \operatorname{Spec} k$ be the structure morphism. Since f and $fi = f_W$ are smooth, we have $Lf^* \cong f^*$, $f_W^* \cong L(fi)^* \cong Li^* f^*$, so $Li^* f^*$ is isomorphic to the restriction functor for $f_W \circ - : \mathbf{Sm}/W \to \mathbf{Sm}/k$. The definition of $i^!$ gives the commutative diagram for each $X' \to X$ in $\mathbf{Sm}/X$ (with $W' := X' \times_X W$)

$$\begin{tikzcd}
E^{W'}(X') \arrow[r] \arrow[d, "\phi_{X'}"'] & E(X') \arrow[d, "\eta_{X'}"] & \\
i^! f^* E(W') \arrow[r] & Li^* f^* E(W') \arrow[r, phantom, "\cong"] & E(W')
\end{tikzcd}$$

where $\eta_{X'}$ is just the restriction map $E(X') \to E(W')$ and $\phi_{X'}$ is the canonical isomorphism given by the definition of $i^!$. Using lemma 7.2.3, this gives us the map of distinguished triangles in $\mathcal{SH}$

$$\begin{tikzcd}
E^{W'}(X') \arrow[r] \arrow[d, "\phi_{X'}"'] & E(X') \arrow[r] \arrow[d, "\eta_{X'}"] & E(X' \setminus W') \arrow[r] \arrow[d, "\eta^0_{X'}"] & E^{W'}(X')[1] \arrow[d] \\
i^! f^* E(W') \arrow[r] & E(W') \arrow[r] & Li^* Rj_* j^* f^* E(W') \arrow[r] & i^! f^* E(W')[1]
\end{tikzcd}$$

Just as for η_E^0, these give rise to the natural map in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}/W_{\mathrm{Zar}})$

$$\eta_E : E(\tau_\epsilon^{\hat{X}}(W)) \to Li^* f^* E_{\mathrm{Zar}}$$

and the commutative diagram in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$

$$\begin{tikzcd}
E^{W_{\mathrm{Zar}}}(X) \arrow[r] \arrow[d, "\phi"'] & E(\tau_\epsilon^{\hat{X}}(W)) \arrow[r] \arrow[d, "\eta"] & E(\tau_\epsilon^{\hat{X}}(W)^0) \arrow[r] \arrow[d, "\eta^0"] & E^{W_{\mathrm{Zar}}}(X)[1] \arrow[d] \\
i^! f^* E_{\mathrm{Zar}} \arrow[r] & Li^* f^* E_{\mathrm{Zar}} \arrow[r] & Li^* Rj_* j^* f^* E_{\mathrm{Zar}} \arrow[r] & i^! f^* E_{\mathrm{Zar}}[1]
\end{tikzcd}$$

The bottom row is the distinguished triangle (7.2.2) for $f^* E$, restricted to W_{Zar}, and the top row is the distinguished triangle of corollary 4.1.3, after applying theorem 3.2.1. Similarly, theorem 3.2.1 shows that η is an isomorphism in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$. Since ϕ is an isomorphism in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$ η^0 is an isomorphism as well. □

7.3. The normal crossing case. We fix a reduced strict normal crossing divisor $i : D \to X$ on some $X \in \mathbf{Sm}/k$. Write $D = \sum_{i=1}^m D_i$ with the D_i smooth. For $X' \to X$ in $\mathbf{Sm}/X$, we write D' for $X' \times_X D$ and D'_I for $X' \times_X D_I$ and for $I \subset \{1, \ldots, m\}$. As in the previous section, we note that our definition of $E(\tau_\epsilon^{\hat{X}}(D))$ extends without change to $E \in \mathbf{Spt}(\mathbf{Sm}/X)$, and similarly, the construction of $E(\tau_\epsilon^{\hat{X}}(D)^0)$ extends without change to $E \in \mathbf{Spt}(\mathbf{Sm}/X \setminus D)$. The extension of proposition 7.2.4 to the normal crossing case follows essentially the same outline as before, with some additional patching results for the operation Li^*Rj_* that allow us give a description of Li^*Rj_* as a homotopy limit, matching our definition of $E(\tau_\epsilon^{\hat{X}}(W)^0)$.

Lemma 7.3.1. *Suppose that $F \in \mathbf{Spt}(\mathbf{Sm}/D)$ satisfies Nisnevich excision. For $I \subset \{1, \ldots, m\}$, $I \neq \emptyset$, let F_I be the presheaf on $\mathbf{Sm}/X$*

$$F_I(X') := F(\hat{X}'^h_{D'_I} \times_X D).$$

Then the canonical map

$$i_*F \to \operatorname*{holim}_{I \neq \emptyset} F_I$$

is a weak equivalence in $\mathbf{Spt}(\mathbf{Sm}/X)$.

Proof. Let $\{U_j \to D' \mid j \in M\}$ be a Nisnevich cover of D', with M a finite set. For $J \subset M$, set $U_J := \prod_{j \in J} U_j$, where the product is $\times_{D'}$. Since F satisfies Nisnevich excision, the canonical map

$$F(D') \to \operatorname*{holim}_{J \neq \emptyset} F(U_J)$$

is a weak equivalence. An argument similar to that of lemma 6.3.2 shows that one can replace the U_i with a pro-system of Nisnevich covers (with M fixed). Similarly, the Zariski stalk of $\operatorname{holim}_{I \neq \emptyset} F_I$ at $x \in X' \in \mathbf{Sm}/X$ is weakly equivalent to $\operatorname{holim}_{I \neq \emptyset} F(\hat{X}'^h_{x, D'_I} \times_X D)$, where $X'_x = \operatorname{Spec} \mathcal{O}_{X', x}$. Thus we need only show that for $X' \to X$ smooth, with X' local, the schemes $U_i := \hat{X}'^h_{D'_i} \times_X D$ form a pro-Nisnevich cover of D', and that

$$\prod_{i \in I} U_i \cong \hat{X}'^h_{D'_I} \times_X D$$

for each non-empty $I \subset \{1, \ldots, m\}$.
In fact the pro-schemes $\hat{X}'^h_{D'_i} \times_X D$, $i = 1, \ldots, m$, obvioiusly form a pro-Nisnevich cover of D'; it follows from lemma 6.3.4 that for each $I \subset \{1, \ldots, m\}$, $I \neq \emptyset$, we have natural isomorphisms (where $\prod$ is $\times_{X'}$)

$$\prod_{i \in I} \hat{X}'^h_{D'_i} \cong \hat{X}'^h_{D'_I}.$$

Thus (with the product over D')

$$\prod_{i \in I} \hat{X}'^h_{D'_i} \times_X D \cong \hat{X}'^h_{D'_I} \times_X D.$$

$\square$

Lemma 7.3.2. *Let $i : D \to X$ be a strict normal crossing divisor on some $f : X \to \operatorname{Spec} k$ in $\mathbf{Sm}/k$, and let $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/U)$ be fibrant. Then there is a canonical map in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$,*

$$\eta^0_E : E(\tau^{\hat{X}}_\epsilon(W)^0) \to \Theta(E)_{\mathrm{Zar}},$$

natural in E.

Proof. As in the smooth case, we construct η^0_E using lemmas 7.2.2 and 7.3.1. Indeed, let $j : X \setminus D \to X$ be the inclusion. Let $\Theta(E)_{I\mathrm{Zar}}$ denote the pull-back of $\Theta(E)$ to $\mathbf{Spt}(\hat{X}^h_{D_I} \times_X D_{\mathrm{Zar}})$. Let $\Theta(E)^*_{I\mathrm{Zar}}$ be the presheaf

$$\Theta(E)^*_{I\mathrm{Zar}}(U) := \Theta(E)_{I\mathrm{Zar}}(\Delta^*_U).$$

Similarly, let $\Theta(E)^*_{\mathrm{Zar}}$ denote the presheaf on D_{Zar}

$$\Theta(E)^*_{\mathrm{Zar}}(U) := \Theta(E)(\Delta^*_U)$$

and let $\Theta(E)_{\mathrm{Zar}}$ denote the restriction of $\Theta(E)$ to D_{Zar}.
The construction of lemma 7.2.3 gives us the diagram of maps

$$\tilde{\eta}^0_{E,I} : (Rj_*E)^c(\tau^{\hat{X}}_\epsilon(D_I)) \to \Theta(E)^*_{I\mathrm{Zar}}$$

and thus the map

$$\tilde{\eta}^0_E : (Rj_*E)^c(\tau^{\hat{X}}_\epsilon(D)) \to \operatorname*{holim}_{I \neq \emptyset}(I \mapsto \Theta(E)^*_{I\mathrm{Zar}})$$

By lemma 7.3.1 we have the canonical isomorphism in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$

$$\operatorname*{holim}_{I \neq \emptyset}(I \mapsto \Theta(E)^*_{I\mathrm{Zar}}) \cong \Theta(E)^*_{\mathrm{Zar}}.$$

Since $\Theta(E)$ is $\mathbb{A}^1$-homotopy invariant, the canonical map $\Theta(E)_{\mathrm{Zar}} \to \Theta(E)^*_{\mathrm{Zar}}$ is a pointwise weak equivalence, giving us the map in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$

$$\tilde{\eta}^0_E : (Rj_*E)^c(\tau^{\hat{X}}_\epsilon(D)) \to \Theta(E)_{\mathrm{Zar}}$$

Using the diagram of lemma 7.2.3, with $W = D_I$, and then taking the appropriate homotopy limit, we arrive at a canonical isomorphism in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$

$$(Rj_*E)^c(\tau^{\hat{X}}_\epsilon(D)) \cong E(\tau^{\hat{X}}_\epsilon(D)^0).$$

Combining $\tilde{\eta}^0_E$ with this isomorphism gives us the desired map η^0_E. $\square$

Lemma 7.3.3. *Let $i : W \to X$ be a closed immersion in $\mathbf{Sch}_S$. Suppose W is a union of closed subschemes, $W = W_1 \cup W_2$. Let $W_{12} := W_1 \cap W_2$ and let $i_j : W_j \to X$, $j = 1, 2$, $i_{12} : W_{12} \to X$ be the inclusions. Then for $E \in \mathcal{SH}_{\mathbb{A}^1}(\mathbf{Sm}/X)$ there is a canonical homotopy cartesian diagram in $\mathcal{SH}_{\mathbb{A}^1}(\mathbf{Sm}/X)$*

$$\begin{array}{ccc} Ri_*Li^*E & \longrightarrow & Ri_{1*}Li_1^*E \\ \downarrow & & \downarrow \\ Ri_{2*}Li_2^*E & \longrightarrow & Ri_{12*}L^*_{12}E \end{array}$$

Proof. Throughout the proof we use the canonical lifting of Li^*, Ri_*, etc., to functors on $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/-)$ by taking the appropriate cofibrant/fibrant replacement, but we use the same notation to denote these liftings.
Let $\iota : W_1 \to W$ be the inclusion. The unit $\mathrm{id} \to R\iota_* L\iota^*$ gives the map

$$Li^*E \to R\iota_* L\iota^* Li^*E \cong R\iota_* Li_1^*E$$

in $\mathcal{SH}_{A^1}(\mathbf{Sm}/W)$; applying Ri_* gives the map $Ri_*Li^*E \to Ri_{1*}Li_1^*E$. The other maps in the square are defined similarly; as the two compositions $Ri_*Li^*E \to Ri_{12*}L^*_{12}E$ are likewise defined by the adjoint property, these agree and the diagram commutes.
To show that the diagram is homotopy cartesian, let $j : U \to X$ be the complement of W, $j_1 : U_1 \to X$ the complement of W_1 and $j' : U \to U_1$, $i'_2 : D_2 \cap U_1 \to U_1$ the inclusions.
We have the distinguished triangles (see [3, Lemme 1.4.6])

$$\begin{aligned} &Lj_!j^*E \to E \to Ri_*Li^*E \to Lj_!j^*E[1]\\ &Lj_{1!}j_1^*E \to E \to Ri_{1*}Li_1^*E \to Lj_{1!}j_1^*E[1]\\ &Lj'_!j^*E \to j_1^*E \to Ri'_{2*}Li'^*_2j_1^*E \to Lj'_!j^*E[1]\end{aligned}$$

Applying $Lj_{1!}$ to the last line gives us the distinguished triangle

$$Lj_!j^*E \to Lj_{1!}j_1^*E \to Lj_{1!}Ri'_{2*}Li'^*_2j_1^*E \to Lj_!j^*E[1]$$

Thus we have the distinguished triangle

$$Lj_{1!}Ri'_{2*}Li'^*_2j_1^*E \to Ri_*Li^*E \to Ri_{1*}Li_1^*E \to Lj_{1!}Ri'_{2*}Li'^*_2j_1^*E[1]$$

The same argument applied to the complement $j_2 : U_2 \to X$ of W_2, the map $j'' : U_2 \to U'' := U \setminus W_{12}$, $j'_1 : U'' \to X$ and the inclusion $i''_2 : D_2 \cap U_1 \to U''$ gives the distinguished triangle

$$Lj'_{1!}Ri''_{2*}Li''^*_2j'^*_1E \to Ri_{2*}Li_2^*E \to Ri_{12*}Li_{12}^*E \to Lj'_{1!}Ri''_{2*}Li''^*_2j'^*_1E[1]$$

Since $D_2 \cap U_1$ is closed in U_1 and in U'', the natural map

$$Lj_{1!}Ri'_{2*}Li'^*_2j_1^*E \to Lj'_{1!}Ri''_{2*}Li''^*_2j'^*_1E$$

is an isomorphism. This shows that the diagram is homotopy cartesian. $\square$

Given a strict normal crossing divisor $i : D \to X$, $D = \sum_{i=1}^m D_i$, we have the inclusions $\iota_I : D_I \to D$, $\iota_{I,J} : D_J \to D_I$ for $I \subset J$ and $i_I : D_I \to X$. For $E \in \mathbf{Spt}(\mathbf{Sm}/X)$ we thus have the presheaves $i_I^*E \in \mathbf{Spt}(\mathbf{Sm}/D_I)$. The isomorphism $\iota^*_{I,J}i^*_IE \cong i^*_JE$ gives us the canonical maps $i^*_IE \to \iota_{I,J*}i^*_JE$; applying ι_{I*} to this map gives us the natural maps $\alpha_{J,I} : \iota_{I*}i^*_IE \to \iota_{J*}i^*_JE$. For $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$, using the cofibrant replacement of E, we see that the same procedure gives us the functor

$$I \mapsto \iota_{I*}(i_I^*E^c)^f \in \mathbf{Spt}(\mathbf{Sm}/D)$$

together with the natural map

$$\alpha : i^*(E^c)^f \to \operatorname*{holim}_{I\neq\emptyset} \iota_{I*}(i_I^*E^c)^f.$$

Lemma 7.3.4. *The map α is an isomorphism in $\mathcal{SH}_{\mathbb{A}^1}(D)$.*

Proof. As the co-unit $Li^*Ri_* \to \mathrm{id}$ is an isomorphism, Ri_* is faithful, so it suffices to show that $Ri_*(\alpha)$ is an isomorphism in $\mathcal{SH}_{\mathbb{A}^1}(X)$. This follows from lemma 7.3.3 and induction on m. □

Recall that for $E \in \mathbf{Spt}(\mathbf{Sm}/k)$ and $i : D \to X$ a strict normal crossing divisor, $D = \sum_{i=1}^m D_i$, we have the presheaf $E(D_{\mathrm{Zar}})$ on D_{Zar} defined by

$$E(D_{\mathrm{Zar}})(U) := \operatorname*{holim}_{I \neq \emptyset} E(D_I \cap U).$$

Proposition 7.3.5. *Let $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/k)$ be fibrant, $i : D \to X$ a strict normal crossing divisor on $X \in \mathbf{Sm}/k$, $f : X \to \operatorname{Spec} k$ the structure morphism. Then we have a natural isomorphism in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$*

$$E(D_{\mathrm{Zar}}) \cong Li^*(f^*E)_{\mathrm{Zar}}$$

Proof. Let $i_I : D_I \to X$ be the inclusion, $f_I : D_I \to \operatorname{Spec} k$ the structure morphism. By theorem 3.2.1, the canonical map

$$\eta_E : E(\tau_\epsilon^{\hat{X}}(D_I)) \to (f_I^*E)_{\mathrm{Zar}} \cong Li_I^*(f^*E)_{\mathrm{Zar}}$$

is an isomorphism in $\mathcal{H}\mathbf{Spt}(D_{I\mathrm{Zar}})$. By lemma 7.3.4 the induced map on the holim gives the desired isomorphism. □

Theorem 7.3.6. *Let $i : D \to X$ be a strict normal crossing divisor on $f : X \to \operatorname{Spec} k$ in $\mathbf{Sm}/k$, and let E be a fibrant object in $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/k_{\mathrm{Nis}})$. Then the map*

$$\eta_E^0 : E(\tau_\epsilon^{\hat{X}}(D)^0) \to \Theta(f^*E)_{\mathrm{Zar}} = [Li^*Rj_*(f^*E)]_{\mathrm{Zar}}$$

is an isomorphism in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$.

Proof. The proof is the same as the proof of proposition 7.2.4, using the distinguished triangle of theorem 6.3.7 together with the isomorphism of proposition 7.3.5 instead of the triangle of corollary 4.1.3. □

Remark 7.3.7. Fix a fibrant $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/k)$. Let $D' \to D$ be in $\mathbf{Sm}/D$ and suppose we have an $X' \to X$ in $\mathbf{Sm}/X$ and a D-isomorphism $D' \cong X' \times_X D$. Then we can replace $i : D \to X$ with $i' : D' \to X'$ and use theorem 7.3.6 to show that our tubular neighborhood construction gives the model $E(\tau_\epsilon^{\hat{X}'}(D')^0)$ for the restriction of $Li^*Rj_*(f^*E)$ to $\mathbf{Sm}/D'_{\mathrm{Zar}}$.
If D and D' are affine, then the theorem of [2] gives the existence of an X' as above, so our result gives at least a "local" description of the entire presheaf $Li^*Rj_*(f^*E)$. □

8. Limit objects

Let $p : X \to C$ be a morphism in $\mathbf{Sm}/k$, with C a smooth curve. Fix a k-point $0 \in C(k)$ and a parameter $t \in \mathcal{O}_{C,0}$. Ayoub combines the functor Li^*Rj_* with a cosimplicial version of the classical path space (i.e., the universal cover) construction to define the *unipotent specialization functor*

$$\mathrm{sp} : \mathcal{SH}(X \setminus p^{-1}(0)) \to \mathcal{SH}(p^{-1}(0))$$

Replacing Li^*Rj_* with the punctured tubular neighborhood, the same construction gives a model of this construction as a Zariski presheaf on X_0. In particular, we give a description of the "limiting values" $\lim_{t\to 0} E(X_t)$ for a semi-stable degeneration $\mathcal{X} \to (C, 0)$. As we mentioned in the introduction, we expect that this construction, applied to a suitable version of the de Rham complex (with weight and Hodge filtrations) as in [4] would yield the classical limit mixed Hodge structure of a semi-stable degeneration.

Remark 8.0.8. In [3, chapter 3] Ayoub describes a general theory of specialization structures; we concentrate on the unipotent structure, which Ayoub denotes Υ, and describes in [3, §3.4]. □

8.1. Path spaces. Before defining the cosimplicial models for various path spaces and homotopy fibers, we recall some basic operations of simplicial sets on schemes. We let $\mathbf{Spc}_f$ denote the full subcategory of $\mathbf{Spc}$ consisting of simplicial sets S with $S([n])$ finite for each n.
Let Y be a k-scheme. For a finite set S, let $Y^S := \prod_{s\in S} Y$, with the product being over $\operatorname{Spec} k$. This defines the contravariant functor $S \mapsto Y^S$ from finite sets to k-schemes. In particular, for $S \in \mathbf{Spc}_f$ we have the cosimplicial scheme Y^S with $Y^S([n]) := Y^{S([n])}$, giving the functor

$$Y^? : \mathbf{Spc}_f^{\mathrm{op}} \to \mathbf{Sch}_k^{\mathbf{Ord}}.$$

Similarly, if T is a simplicial set, we have the cosimplicial-simplicial set (cosimplicial space) T^S and the functor

$$T^? : \mathbf{Spc}^{\mathrm{op}} \to \mathbf{Spc}^{\mathbf{Ord}}.$$

Setting $Y \times S := \amalg_{s\in S} Y$, we have the functor $S \mapsto Y \times S$ from finite sets to k-schemes; if S is a simplicial set as above, we thus have the simplicial scheme $Y \times S$, giving the functor

$$Y \times ? : \mathbf{Spc}_f \to \mathbf{Sch}_k^{\mathbf{Ord}^{\mathrm{op}}}.$$

The adjunction

$$\mathrm{Hom}_{\mathbf{Sch}_k}(X \times S, Y) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(X, Y^S)$$

for S a finite set extends to S a simplicial set as above, giving the adjunction

$$\mathrm{Hom}_{\mathbf{Sch}_k^{\mathbf{Ord}^{\mathrm{op}}}}(X \times S, Y) \cong \mathrm{Hom}_{\mathbf{Sch}_k^{\mathbf{Ord}}}(X, Y^S)$$

where on the left, we consider Y as a constant simplicial scheme and on the right, X as a constant cosimplicial scheme. This is an analog of the adjunction for spaces

$$\mathrm{Hom}_{\mathbf{Spc}}(A \times S, T) \cong \mathrm{Hom}_{\mathbf{Spc}^{\mathbf{Ord}^{\mathrm{op}}}}(A \times S, T) \cong \mathrm{Hom}_{\mathbf{Spc}^{\mathbf{Ord}}}(A, T^S)$$

where the first isomorphism is the well-known identity relating maps of bisimplicial sets with maps of the corresponding diagonal simplicial sets.
For $E \in \mathbf{Spc}(k)$ and Y a simplicial object in $\mathbf{Sm}/k$, we have the cosimplicial space $E(Y)$ with n cosimplices $E(Y([n]))$. For s an element of a finite set S,

and a scheme $Y \in \mathbf{Sm}/k$, we have the inclusion $i_s : Y = Y \times s \to Y \times S$; the inclusions $i_s : Y \to Y \times S$, $s \in S$ induce the canonical natural map

$$E(Y \times S) \to E(Y)^S$$

which is an isomorphism if E is additive: $E(Y \amalg Y') \cong E(Y) \times E(Y')$. This isomorphsim extends immediately to finite simplicial sets $S \in \mathbf{Spc}_f$ and additive E.

Examples 8.1.1. (1) For a k-scheme Y, the *free path space* $\mathcal{P}_Y$ on Y is $Y^{[0,1]}$, where $[0,1]$ is just the 1-simplex $\Delta[1] := \mathrm{Hom}_{\mathbf{Ord}}(-,[1])$. Explicitly, $\mathcal{P}_Y$ has n-cosimplices Y^{n+2}, with structure maps as follows: Label the factors in Y^{n+2} from 0 to $n+1$. Send $\delta^n_i : [n] \to [n+1]$ to the diagonal

$$(y_0, \ldots, y_{n+1}) \mapsto (y_0, \ldots, y_{i-1}, y_i, y_i, y_{i+1}, \ldots, y_{n+1})$$

and send $s^n_i : [n] \to [n-1]$ to the projection

$$(y_0, \ldots, y_{n+1}) \mapsto (y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{n+1}).$$

The inclusion $\{0,1\} \to [0,1]$ gives rise to the projection $Y^{[0,1]} \to Y^{\{0,1\}}$, i.e. $\pi : \mathcal{P}_Y \to Y \times_k Y$; we thus have two structures of a cosimplicial Y-scheme on $\mathcal{P}_Y$: $\pi_1 : \mathcal{P}_Y \to Y$ and $\pi_2 : \mathcal{P}_Y \to Y$, with $\pi_i := p_i \circ \pi$.

(2) For a pointed k-scheme $(Y, y : \mathrm{Spec}\, k \to Y)$, we have the *pointed path space*

$$\mathcal{P}_Y(y) := \mathcal{P}_Y \times_{(\pi_2, y)} \mathrm{Spec}\, k.$$

(3) Now let $p : \mathcal{Y} \to Y$ be a Y-scheme,, $y : \mathrm{Spec}\, k \to Y$ a point. We have the *cosimplicial homotopy fiber of* p *over* y:

$$\mathcal{P}_{\mathcal{Y}/Y}(y) := \mathcal{Y} \times_{(p,\pi_1)} \mathcal{P}_Y(y)$$

We extend this definition to cosimplicial Y-schemes in the evident manner: if $\mathcal{Y}^\bullet \to Y$ is a cosimplicial Y-scheme, we have the bi-cosimplicial Y-scheme $\mathcal{P}_{\mathcal{Y}^\bullet/Y}(y)$; the extension to functors from some small category to cosimplicial Y-schemes is done in the same way. $\square$

Denoting the pointed k-scheme (Y, y) by Y_*, we sometimes write $\mathcal{P}_{Y_*}$ for $\mathcal{P}_Y(y)$ and $\mathcal{P}_{\mathcal{Y}^\bullet/Y_*}$ for $\mathcal{P}_{\mathcal{Y}^\bullet/Y}(y)$. For $E \in \mathbf{Spt}(k)$, we have the simplicial spectrum $E(\mathcal{P}_{\mathcal{Y}/Y_*})$.

The pointed path space $\mathcal{P}_Y(y)$ is contractible in the following sense:

Lemma 8.1.2. *Let (Y, y) be a pointed smooth k-scheme, U a smooth k-scheme. Then for $E \in \mathbf{Spt}(k)$, the projection $U \times \mathcal{P}_Y(y) \to U$ induces a weak equivalence*

$$E(U) \to E(U \times \mathcal{P}_Y(y)).$$

Proof. To prove the lemma, it suffices to show that, for $E \in \mathbf{Spc}(k)$, the projection $U \times \mathcal{P}_Y(y) \to U$ induces a homotopy equivalence

$$E(U) \to E(U \times \mathcal{P}_Y(y)).$$

We first show that $U \times \mathcal{P}_Y(y) \to U$ induces a homotopy equivalance of cosimplicial schemes.

The projection $[0,1] \to pt$ gives the map of cosimplicial schemes $s : Y = Y^{pt} \to Y^{[0,1]}$; composing with the k-point $y \to Y$ gives the point $s_y : \operatorname{Spec} k \to Y^{[0,1]}$ and thus the section $(y, s_y) : \operatorname{Spec} k \to \mathcal{P}_Y(y)$ to the projection $\mathcal{P}_Y(y) \to \operatorname{Spec} k$. This induces the section $s_U : U \to U \times \mathcal{P}_Y(y)$ to the projection $p_U : U \times \mathcal{P}_Y(y) \to U$.

We proceed to construct a homotopy between $p_U \circ s_U$ and the identity on $U \times \mathcal{P}_Y(y)$; it suffices to construct the homotopy for $U = \operatorname{Spec} k$.

For this, let $\sigma : Y^{[0,1]} \to Y^{[0,1]}$ be the map induced by the map of simplicial sets $[0,1] \to [0,1]$ sending $[0,1]$ to 1. Then $p_{\operatorname{Spec} k} \circ s_{\operatorname{Spec} k} : \mathcal{P}_Y(y) \to \mathcal{P}_Y(y)$ is the map $(\mathrm{id}_{\operatorname{Spec} k}, \sigma)$.

Let $p_0, p_1 : Y^{[0,1]\times[0,1]} \to Y^{[0,1]}$ be the maps induced by the inclusions $i_0, i_1 : [0,1] \to [0,1]\times[0,1]$, $i_0(x) = x \times 0$, $i_1(x) = x \times 1$, and let $\pi : Y \to Y^{[0,1]\times[0,1]}$ be the map induced by $[0,1]\times[0,1] \to pt$. Let $h : ([0,1]\times[0,1], 1\times[0,1]) \to ([0,1], 1)$ be any map of pairs of simplicial sets which is the identity on $[0,1] \times 0$ and the map to $1 \in [0,1]$ on $[0,1] \times 1$. Then h defines a map

$$H : Y^{[0,1]} \to Y^{[0,1]\times[0,1]}$$

with

$$\begin{aligned} p_0 \circ H &= \mathrm{id}_{Y^{[0,1]}} \\ p_1 \circ H &= \sigma \\ H \circ s &= \pi. \end{aligned}$$

From these identities, it follows that (H, id_y) induces a co-homotopy

$$H_y : \mathcal{P}_Y(y) \to \mathcal{P}_Y(y)^{[0,1]}$$

with $p_0 \circ H_y = \mathrm{id}$, $p_1 \circ H_y = p_{\operatorname{Spec} k} \circ s_{\operatorname{Spec} k}$. Taking the adjoint of H_y, we have the homotopy

$$h_y : \mathcal{P}_Y(y) \times [0,1] \to \mathcal{P}_Y(y);\ h_y \circ i_0 = \mathrm{id}, h_y \circ i_1 = p_{\operatorname{Spec} k} \circ s_{\operatorname{Spec} k},$$

where $\mathcal{P}_Y(y) \times [0,1]$ and $\mathcal{P}_Y(y)$ are to be considered as cosimplicial-simplicial schemes, with $\mathcal{P}_Y(y)$ constant in the simplicial direction.

Applying E to $\mathrm{id}_U \times h_y$ and composing with the canonical map

$$E(U \times \mathcal{P}_Y(y) \times [0,1]) \to E(U \times \mathcal{P}_Y(y))^{[0,1]}$$

gives us the co-homotopy

$$E(\mathrm{id}_U \times h_y) : E(U \times \mathcal{P}_Y(y)) \to E(U \times \mathcal{P}_Y(y))^{[0,1]}$$

between the identity and $E(p_U \circ s_U)$. Thus $E(U) \to E(U \times \mathcal{P}_Y(y))$ is a homotopy equivalence, as desired. □

8.2. Limit structures. For our purposes, a semi-stable degeneration need not be proper, so even if this is somewhat non-standard terminology, we use the following definition:

Definition 8.2.1. A *semi-stable degeneration* is a flat morphism $p : \mathcal{X} \to (C, 0)$, where $(C, 0)$ is a smooth pointed local curve over k, $C = \operatorname{Spec} \mathcal{O}_{C,0}$, $\mathcal{X}$ is a smooth irreducible k-scheme, p is smooth over $C \setminus 0$ and $X_0 := p^{-1}(0)$ is a reduced strict normal crossing divisor on $\mathcal{X}$. □

For the rest of this section, we fix a semi-stable degeneration $\mathcal{X} \to (C, 0)$. We denote the open complement of X_0 in $\mathcal{X}$ by $\mathcal{X}^0$. We write $\mathbb{G}_m$ for the pointed k-scheme $(\mathbb{A}^1_k \setminus \{0\}, 1)$.
Fix a uniformizing parameter $t \in \mathcal{O}_{C,0}$, giving the morphism $t : (C, 0) \to (\mathbb{A}^1_k, 0)$, which restricts to $t : C\setminus 0 \to \mathbb{G}_m$. Let $p[t] : \mathcal{X} \to \mathbb{A}^1$ be the composition $t \circ p$, and let $p[t]^0 : \mathcal{X}^0 \to \mathbb{G}_m$ be the restriction of $p[t]$. Composing $p[t]$ with the canonical morphism $\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0 \to \mathcal{X}^0$ yields the map

$$\hat{p}[t]^0 : \tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0 \to \mathbb{G}_m.$$

Let $X_0^1, \ldots, X_0^m$ be the irreducible components of X_0. Recalling the construction of $\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0$ as a diagram (see (6.3.1)), let us denote, for $I \subset \{1, \ldots, m\}$, the co-presheaf $\iota_{I*}(\tau_\epsilon^{\hat{\mathcal{X}}}(X_{0I})^0)$ by $\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I$. The map $\hat{p}[t]^0$ makes $\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0$ into a diagram of co-presheaves (on $X_{0\mathrm{Zar}}$) of cosimplicial pro-schemes over $\mathbb{G}_m$. We thus have the diagram of cosimplicial co-presheaves on $X_{0\mathrm{Zar}}$:

$$I \mapsto \mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m}.$$

We denote this diagram by

$$\lim_{t\to 0} X_t. \tag{8.2.1}$$

Now let E be in $\mathbf{Spt}(k)$. For each $I \subset \{1, \ldots, m\}$, we have the presheaf of bisimplicial spectra on $X_{0\mathrm{Zar}}$, $E(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m})$, giving us the functor

$$I \mapsto \tilde{E}(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m}).$$

where ˜ means fibrant model. Taking the homotopy limit over I of the associated diagram of presheaves of total spectra gives us the fibrant presheaf of spectra

$$E(\lim_{t\to 0} X_t) := \operatorname*{holim}_{I\neq\emptyset} \mathrm{Tot}\tilde{E}(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m}).$$

Taking the global sections gives us the spectrum $E(\lim_{t\to 0} X_t)(X_0)$, which we denote by $\lim_{t\to 0} E(X_t)$.

Remark 8.2.2. Suppose $E \in \mathbf{Spt}(k)$ is homotopy invariant and satisfies Nisnevich excision. We can form the homotopy limit $\bar{E}(\lim_{t\to 0} X_t)$ of the diagram of presheaves

$$I \mapsto E(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m}).$$

Since E is quasi-fibrant (see remark 2.3.2) the map

$$E(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m}) \to \tilde{E}(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m})$$

is a pointwise weak equivalence, hence the map $\bar{E}(\lim_{t\to 0} X_t) \to E(\lim_{t\to 0} X_t)$ is a pointwise weak equivalence. In particular, $\bar{E}(\lim_{t\to 0} X_t)(X)_0) \to$

$\lim_{t\to 0} E(X_t)$ is a weak equivalence. In short, if E is homotopy invariant and satisfies Nisnevich excision, then it is not necessary to take the fibrant model $\tilde{E}$ in the construction of $E(\lim_{t\to 0} X_t)$ or $\lim_{t\to 0} E(X_t)$. $\square$

We remind the reader of the presheaves $E(X_{0\bullet})$ and $E(\Delta_{X_{0\bullet}})$ on $X_{0\mathrm{Zar}}$ described in definition 6.2.1.

Proposition 8.2.3. *Suppose E is homotopy invariant and satisfies Nisnevich excision. Then*
(1) There is a canonical map in $\mathcal{H}\mathbf{Spt}(X_{0\mathrm{Zar}})$:

$$E(X_{0\bullet}) \xrightarrow{\gamma_X} E(\lim_{t\to 0} X_t).$$

(2) If X_0 is smooth, then $E(X_{0\bullet}) = E(X_{0\mathrm{Zar}})$ and γ_X is an isomorphism.

Proof. We have the maps

$$E(X_{0\bullet}) \xrightarrow{p_\bullet^*} E(\Delta^*_{X_{0\bullet}}) \xleftarrow{\hat{\iota}^*} E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)).$$

which by proposition 6.2.2 are Zariski-local weak equivalences. Similarly, we have the diagram of open immersions

$$\hat{j} : \tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0 \to \tau_\epsilon^{\hat{\mathcal{X}}}(X_0)$$

inducing

$$\hat{j}^* : E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)) \to E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0).$$

Thus we have the map

$$p_0^* : E(X_{0\bullet}) \to E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0);$$

$p_0^* := \hat{j}^*(\hat{\iota}^*)^{-1}p_\bullet^*$.
Similarly, we have the projection

$$\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0/\mathbb{G}_m} \to \tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0,$$

giving the map

$$q^* : E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0) \to E(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0/\mathbb{G}_m});$$

we set $\gamma_X := q^* \circ p^*$.
For (2), the diagram $X_\bullet$ is just the identity copresheaf $X_{0\mathrm{Zar}}$, hence $E(X_{0\bullet}) = E(X_{0\mathrm{Zar}})$. To show γ_X is an isomorphism, fix a point $x \in X_0$. There is a Zariski neighborhood U of x in X_0 and a Nisnevich neighborhood $\mathcal{X}' \to \mathcal{X}$ of U in $\mathcal{X}$ which is isomorphic to a Nisnevich neighborhood of U in $U \times \mathbb{A}^1$. Thus it suffices to prove the result in the case $\mathcal{X} = X_0 \times \mathbb{A}^1$, $(C, 0) = (\mathbb{A}^1, 0)$ and $p = p_2 : \mathcal{X} \to \mathbb{A}^1$.
For each smooth k-scheme T, it follows from homotopy invariance and theorem 3.2.1 that the canonical map $p : \tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0 \times 0) \to X_0 \times \mathbb{A}^1$ induces a weak equivalence

$$p^* : E(T \times_k X_{0\mathrm{Zar}} \times \mathbb{A}^1) \to E(T \times_k \tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0 \times 0))$$

The Morel-Voevodsky purity theorem [34, theorem 2.23] plus Nisnevich excision and the homotopy property for E implies that p induces a weak equivalence

$$p^*: E^{T\times X_{0\mathrm{Zar}}\times 0}(T\times_k X_{0\mathrm{Zar}}\times\mathbb{A}^1)\to E^{T\times\Delta^*_{X_{0\mathrm{Zar}}\times 0}}(T\times_k\tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0\times 0)).$$

This gives us the map of homotopy fiber sequences

$$\begin{tikzcd}
E^{T\times X_{0\mathrm{Zar}}\times 0}(T\times_k X_{0\mathrm{Zar}}\times\mathbb{A}^1) \arrow[r, "p^*"] \arrow[d] & E^{T\times\Delta^*_{X_{0\mathrm{Zar}}\times 0}}(T\times_k\tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0\times 0)) \arrow[d] \\
E(T\times_k X_{0\mathrm{Zar}}\times\mathbb{A}^1) \arrow[r, "p^*"] \arrow[d, "j^*"'] & E(T\times_k\tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0\times 0)) \arrow[d, "\hat{j}^*"] \\
E(T\times_k X_{0\mathrm{Zar}}\times\mathbb{G}_m) \arrow[r, "p^{0*}"] & E(T\times_k\tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0\times 0)^0)
\end{tikzcd}$$

with p^{0*} induced by the restriction of p,

$$p^0: \tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0\times 0)^0\to X_0\times\mathbb{G}_m.$$

Thus p^{0*} is a weak equivalence.
Applying these term-by-term with respect to the cosimplicial schemes defining the respective path spaces, we have the weak equivalence (assuming $\mathcal{X}=X_0\times\mathbb{A}^1$)

$$E(U\times\mathcal{P}_{\mathbb{G}_m})\to E(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0/\mathbb{G}_m})(U).$$

Thus we need only show that the projection $U\times\mathcal{P}_{\mathbb{G}_m}\to U$ induces a weak equivalence

$$E(U)\to E(U\times\mathcal{P}_{\mathbb{G}_m})$$

for all smooth k-schemes U. This is lemma 8.1.2 □

8.3. Comparison. We conclude this section by connecting our construction with the specialization functor sp for the specialization structure Υ defined by Ayoub [3, chapter 3].
Let $E\in\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/k)$ be fibrant, let $p:\mathcal{X}\to(C,0)$ be a semi-stable degeneration and choose a parameter $t\in\mathcal{O}_{C,0}$. In this setting, Ayoub's functor sp applied to some $E\in\mathbf{Spt}(\mathbf{Sm}/\mathcal{X}^0)$ is defined as follows: First form the presheaf $E(\mathcal{P}_{-/\mathbb{G}_m})$ on $\mathbf{Sm}/\mathcal{X}^0$ by taking the total spectrum

$$E(\mathcal{P}_{-/\mathbb{G}_m})(X'\to\mathcal{X}^0):=\mathrm{Tot}(E(\mathcal{P}_{X'/\mathbb{G}_m})).$$

where we use the composition $X'\to\mathcal{X}^0\xrightarrow{t}\mathbb{G}_m$ as structure morphism. Then $\mathrm{sp}(E)\in\mathcal{SH}_{\mathbb{A}^1}(\mathbf{Sm}/X_0)$ is represented by the presheaf

$$\mathrm{sp}(E):=i^*\left(j_*\left(E(\mathcal{P}_{-/\mathbb{G}_m})^f\right)^c\right).$$

Similarly, we have the simplicial presheaf on $\mathbf{Sm}/X_0$ with n-simplices

$$\mathrm{sp}(E)_n:=i^*\left(j_*\left(E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f\right)^c\right).$$

Let $\mathrm{Tot}(\mathrm{sp}(E)_*)$ denote the presheaf formed by taking the total spectrum of $n \mapsto \mathrm{sp}(E)_n$.

Lemma 8.3.1. *Suppose E is fibrant. Then there is a natural isomorphism in $\mathcal{SH}_{\mathbb{A}^1}(X_0)$*

$$\mathrm{Tot}(\mathrm{sp}(E)_*) \cong \mathrm{sp}(E)$$

Proof. Since $E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f$ is fibrant, the presheaf $E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f$ on $\mathcal{X}^0$ satisfies Nisnevich excision and is $\mathbb{A}^1$ homotopy invariant. Thus the same holds for the total spectrum of the simplicial spectrum $n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f$, hence

$$\mathrm{Tot}(n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f) \to \left(\mathrm{Tot}(n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f)\right)^f$$

is a pointwise weak equivalence in $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/\mathcal{X}^0)$, and thus we still have a pointwise weak equivalence after applying j_*. Similarly, the evident map

$$\left(\mathrm{Tot}(n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n])\right)^f \to \left(\mathrm{Tot}(n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f)\right)^f$$

is a pointwise weak equivalence. Taking cofibrant models and applying i^* gives the isomorphism in $\mathcal{SH}_{\mathbb{A}^1}(X_0)$

$$\mathrm{sp}(E) \cong i^*\left(\left(\mathrm{Tot}(n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f)\right)^c\right).$$

On the other hand, taking the total complex commutes with taking the cofibrant model, and with the functor i^*, so we have the isomorphism in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X_0)$

$$\begin{aligned}\mathrm{sp}(E) = i^*&\left(\left(\mathrm{Tot}(n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n]))^f\right)^c\right)\\ &\cong \mathrm{Tot}\left(n \mapsto i^*\left(\left(E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f\right)^c\right)\right) = \mathrm{Tot}(\mathrm{sp}_*(E)).\end{aligned}$$

□

Using the diagram of lemma 7.2.3 for the n-cosimplices $\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0 \times \mathbb{G}_m^n$ of $\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0/\mathbb{G}_m}$, and taking the total spectrum, we arrive at a natural map

$$E(\lim_{t\to 0} X_t) \to \mathrm{Tot}(\mathrm{sp}(E)_*)_{\mathrm{Zar}}$$

in $\mathcal{H}\mathbf{Spt}(X_{0\mathrm{Zar}})$; combining this with lemma 8.3.1 gives us the comparison map

$$\theta_E : E(\lim_{t\to 0} X_t) \to \mathrm{sp}(E)_{\mathrm{Zar}}$$

in $\mathcal{H}\mathbf{Spt}(X_{0\mathrm{Zar}})$.

Proposition 8.3.2. *The map $\theta_E : E(\lim_{t\to 0} X_t) \to \mathrm{sp}(E)_{\mathrm{Zar}}$ is an isomorphism in $\mathcal{H}\mathbf{Spt}(X_{0\mathrm{Zar}})$.*

Proof. By theorem 7.3.6, the map

$$\theta_E(n) : E(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0/\mathbb{G}_m}([n])) \to \mathrm{sp}_n(E)$$

is an isomorphism in $\mathcal{H}\mathbf{Spt}(X_{0\mathrm{Zar}})$ for each n, thus the map θ_E on the total spectra is also an isomorphism in $\mathcal{H}\mathbf{Spt}(X_{0\mathrm{Zar}})$. □

9. The monodromy sequence

In this section, we construct the monodromy sequence for the limit object $E(\lim_{t\to 0} X_t)$ (see corollary 9.3.5). As pointed out to us by Ayoub, one needs to restrict E quite a bit. We give here a theory valid for presheaves of complexes of $\mathbb{Q}$-vector space on $\mathbf{Sm}/k$ which are homotopy invariant and satisfy Nisnevich excision, and satisfy an additional "alternating" property (definition 9.2.2).
Ayoub [3, Chap. 3] constructs the monodromy sequence in a more general setting; our construction is based on his ideas applied to our tubular neighborhood construction. In particular, our mondromy sequence agrees with the monodromy sequence of *loc. cite.*

9.1. Presheaves of complexes. For a noetherian ring R, we let C_R denote the category of (unbounded) homological complexes of R-modules, $C_{R\geq 0}$ the full subcategory of C_R consisting of complexes which are zero in strictly negative degrees.
By the Dold-Kan equivalence, we may identify $C_{R\geq 0}$ with the category of simplicial R-modules $\mathbf{Spc}_R$. The forgetful functor $\mathbf{Spc}_R \to \mathbf{Spc}_*$ allows us to use the standard model structure on $\mathbf{Spc}_*$ to induce a model structure on $\mathbf{Spc}_R$, i.e., cofibrations are degreewise monomorphisms, weak equivalences are homotopy equivalences on the geometric realization and fibrations are maps with the RLP for trivial cofibrations. This induces a model structure on $C_{R\geq 0}$ with weak equivalence the quasi-isomorphisms; the suspension functor is the usual (homological) shift operator: $\Sigma C := C[1]$, $C[1]_n := C_{n-1}$, $d_{C[1],n} = -d_{C,n-1}$. This model structure is extended to C_R by identifying C_R with the category of "spectra in $C_{R\geq 0}$", i.e., sequences $(C^0, C^1, \ldots)$ with bonding maps $\epsilon_n : C^n[1] \to C^{n+1}$. Following Hovey [17], the model structure on $\mathbf{Spt}$ induces a model structure on spectra of simplicial R-modules, and thus a model structure on C_R, with weak equivalences the quasi-isomorphisms. In particular, the homotopy category $\mathcal{H}C_R$ is just the unbounded derived category D_R.
Similarly, for a category $\mathcal{C}$, the model structure for the presheaf category $\mathbf{Spt}(\mathcal{C})$ gives a model structure for presheaves of complexes on $\mathcal{C}$, $C_R(\mathcal{C})$ with weak equivalences the pointwise quasi-isomorphisms, and homotopy category the derived category $D_R(\mathcal{C})$. We may introduce a topology (e.g., the Zariski or Nisnevich topology), giving the model categories $C_R(X_{\mathrm{Zar}})$, $C_R(\mathbf{Sm}/B_{\mathrm{Zar}})$, $C_R(X_{\mathrm{Nis}})$, $C_R(\mathbf{Sm}/B_{\mathrm{Nis}})$. These have homotopy categories equivalent to the derived categories (on the small or big sites) $D_R(X_{\mathrm{Zar}})$, $D_R(\mathbf{Sm}/S_{\mathrm{Zar}})$, $D_R(X_{\mathrm{Nis}})$, $D_R(\mathbf{Sm}/S_{\mathrm{Nis}})$, respectively. Finally, we may consider the $\mathbb{A}^1$-localization, giving the Nisnevich-local $\mathbb{A}^1$-model structure $C_{R,\mathbb{A}^1}(\mathbf{Sm}/B_{\mathrm{Nis}})$ with homotopy category $D_{R\mathbb{A}^1}(B)$.
Let I be a small category, $F : I \to C_R$ a functor. Since we can consider F as a spectrum-valued functor by the various equivalences described above, we may form the complex $\mathrm{holim}_I F$. Explicitly, this is the following complex: One first

forms the cosimplicial complex $\underline{\mathrm{holim}}_I F$ with n-cosimplices

$$\underline{\mathrm{holim}}_I F^n := \prod_{\sigma=(\sigma_0\to\ldots\to\sigma_n)\in\mathcal{N}(I)_n} F(\sigma_n).$$

For $g : [m] \to [n]$, with $g(m) = m' \leq n$, the σ-component of the map $\underline{\mathrm{holim}}_I F^n(g)$ sends $\prod x_\tau$ to $F(\sigma_{m'} \to \sigma_n)(x_{g^*(\sigma)})$. The complex $\mathrm{holim}_I F$ is then the total complex of the double complex $n \mapsto \underline{\mathrm{holim}}_I F^n$, with second differential the alternating sum of the coface maps. This construction being functorial and preserving quasi-isomorphisms, it passes to the derived category $D_R(\mathcal{C})$. If I is a finite category, the construction commutes with filtered colimits, hence passes to the Zariski- and Nisnevich-local derived categories, as well as the $\mathbb{A}^1$-local versions.

Remarks 9.1.1. (1) For a set S, let RS denote the free R-module on S. Sending a pointed space $(S,*)$ to the simplicial R-module RS, with $RS(n) := RS_n/R\{*\}$ defines the *R-localization functor* $\mathbf{Spc}_* \to \mathbf{Spc}_R$. This extends to the spectrum categories, and gives us the exact R-localization functor on homotopy category $\otimes R : \mathcal{SH} \to D_R$. The R-localization functor $\otimes R$ extends to all the model categories we have been considering, in particular, we have the R-localization

$$\otimes R : \mathcal{SH}_{\mathbb{A}^1}(B) \to D_{R,\mathbb{A}^1}(B),$$

For $R = \mathbb{Q}$, we can also take the $\mathbb{Q}$-localization of $\mathcal{SH}$ by performing a Bousfield localization, i.e., define $Z \in \mathbf{Spt}$ to be $\mathbb{Q}$-local if $\pi_n(Z)$ is a $\mathbb{Q}$-vector space for all n, and $E \to F$ a $\mathbb{Q}$ weak equivalence if $\mathrm{Hom}_{\mathbf{Spt}}(F,Z) \to \mathrm{Hom}_{\mathbf{Spt}}(E,Z)$ is an isomorphism for all $\mathbb{Q}$-local Z. Inverting the $\mathbb{Q}$-weak equivalences defines the $\mathbb{Q}$-local homotopy category $\mathcal{SH}_{\mathbb{Q}}$, and $\otimes\mathbb{Q} : \mathcal{SH} \to D_{\mathbb{Q}}$ identifies $\mathcal{SH}_{\mathbb{Q}}$ with $D_{\mathbb{Q}}$. This passes to the other homotopy categories we have defined, in particular, $\otimes\mathbb{Q} : \mathcal{SH}_{\mathbb{A}^1}(B) \to D_{\mathbb{Q},\mathbb{A}^1}(B)$ identifies $SH_{\mathbb{A}^1}(B)_{\mathbb{Q}}$ with $D_{\mathbb{Q},\mathbb{A}^1}(B)$.

(2) $D_{\mathbb{Q},\mathbb{A}^1}(k)$ is *not* the same as the ($\mathbb{Q}$-localized) big category of motives over k, $DM(k)_{\mathbb{Q}}$; the $\mathbb{Q}$-localization does not give rise to transfers. □

9.2. The log complex. Let $\mathrm{sgn} : S_n \to \{\pm 1\}$ be the sign representation of the symmetric group S_n. Consider a presheaf of $\mathbb{Q}$-vector spaces E on $\mathbf{Sm}/k$. For $X, Y \in \mathbf{Sm}/k$, let $\mathrm{alt}_n : E(Y \times X^n) \to E(Y \times X^n)$ be the alternating projector

$$\mathrm{alt}_n = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)(\mathrm{id}_Y \times \sigma)^*,$$

with σ operating on X^n by permuting the factors. Let $E(Y \times X^n)^{\mathrm{alt}} \subset E(Y \times X^n)$ be the image of alt_n and $E(Y, X^n)^{\mathrm{alt}}_{\perp}$ the kernel. We extend these constructions to presheaves of complexes E by operating degreewise.

If $(X,*)$ is a pointed k-scheme, we have the inclusions $i_j : Y \times X^{n-1} \to Y \times X^n$ inserting the point $*$ in the jth factor. For E a presheaf of $\mathbb{Q}$-vector spaces, we let $E(Y \wedge X^{\wedge n})$ be the intersection of the kernels of the restriction maps

$$(\mathrm{id}_Y \times i_j)^* : E(Y \times X^n) \to E(Y \times X^{n-1}).$$

Letting $p_j : X^n \to X^{n-1}$ be the projection omitting the jth factor, the composition $(\mathrm{id} - p_n^* i_n^*) \circ \ldots \circ (\mathrm{id} - p_1^* i_1^*)$ gives a splitting

$$\pi_n : E(Y \times X^n) \to E(Y \wedge X^{\wedge n})$$

to the inclusion $E(Y \wedge X^{\wedge n}) \to E(Y \times X^n)$.
Clearly S_n acts on $E(Y \wedge X^{\wedge n})$ through its action on X^n; we let $E(Y, X^{\wedge n})^{\mathrm{alt}}$ and $E(Y \wedge X^{\wedge n})^{\mathrm{alt}}_{\perp}$ be the image and kernel of alt_n on $E(Y \wedge X^{\wedge n})$.
Let $f : X \to \mathbb{G}_m$ be a morphism, E a presheaf of $\mathbb{Q}$-vector spaces on $\mathbf{Sm}/k$. Let $f_n : X \times \mathbb{G}_m^n \to X \times \mathbb{G}_m^{n+1}$ be the morphism

$$f_n(x, t_1, \ldots, t_n) := (x, f(x), t_1, \ldots, t_n).$$

Denote the map $\mathrm{alt}_n \circ \pi_n \circ f_n^* : E(X \wedge \mathbb{G}_m^{\wedge n+1})^{\mathrm{alt}} \to E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$ by

$$\cup f : E(X \wedge \mathbb{G}_m^{\wedge n+1})^{\mathrm{alt}} \to E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$$

One checks that

Lemma 9.2.1. $(\cup f)^2 = 0$.

Proof. We work in the $\mathbb{Q}$-linear category $\mathbb{Q}\mathbf{Sm}/k$, with the same objects as $\mathbf{Sm}/k$, disjoint union being direct sum, and, for X, Y connected, $\mathrm{Hom}_{\mathbb{Q}\mathbf{Sm}/k}(X, Y)$ is the $\mathbb{Q}$-vector space freely generated by the set $\mathrm{Hom}_{\mathbf{Sm}/k}(X, Y)$. Product over k makes $\mathbb{Q}\mathbf{Sm}/k$ a tensor category. The map $\cup f$ is gotten by applying E to the map $\cup f^\vee : X \times \mathbb{G}_m^n \to X \times \mathbb{G}_m^{n+1}$ in $\mathbb{Q}\mathbf{Sm}/k$:

$$(x, t_1, \ldots, t_{n-1}) \mapsto \mathrm{alt}[((x, f(x)) - (x, 1)) \otimes t_1 - 1 \otimes \ldots \otimes t_n - 1)]$$

and restricting to $E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$. But $(\cup f^\vee)^2$ is

$$\begin{aligned} &(x, t_1, \ldots, t_n) \mapsto \\ &\quad \mathrm{alt}[((x, f(x), f(x)) - (x, 1, f(x)) - (x, f(x), 1) + (x, 1, 1)) \otimes \ldots \otimes t_n - 1) \end{aligned}$$

which is evidently the zero map. $\square$

Form the complex $E(\log_f)$ by

$$E(\log_f)_n := E(X \wedge \mathbb{G}_m^{\wedge n}))^{\mathrm{alt}}$$

with differential $\cup f$. Since $E(\log_f)_0 = E(X)$, we have the canonical map $\iota_X : E(X) \to E(\log_f)$.
We extend this definition to an I-diagram of schemes over $\mathbb{G}_m$, $f^\bullet : X^\bullet \to \mathbb{G}_m$ (with the $X^n \in \mathbf{Sm}/k$) by

$$E(\log_{f^\bullet}) := \operatorname*{holim}_{i \in I} E(\log_{f^i});$$

similarly, we extend to E a presheaf of complexes on $\mathbf{Sm}/k$ by taking the total complex of the double complex $n \mapsto E_n(\log_{f^\bullet})$. The map ι_X extends to

$$\iota_{X^\bullet} : E(X^\bullet) \to E(\log_{f^\bullet}),$$

where

$$E(X^\bullet) := \operatorname*{holim}_{i \in I} E(X^i).$$

We consider as well a truncation of $E(\log_f)$. Recall that the stupid truncation $\sigma_{\geq n}C$ of a homological complex C is the quotient complex of C with

$$\sigma_{\geq n}C_m := \begin{cases} C_m & \text{for } m \geq n \\ 0 & \text{for } m < n. \end{cases}$$

For E a presheaf of abelian groups and $f : X \to \mathbb{G}_m$ a morphism in $\mathbf{Sm}/k$, set

$$E(\sigma_{\geq 1}\log_f) := \sigma_{\geq 1}E(\log_f).$$

We have the quotient map $N : E(\log_f) \to E(\sigma_{\geq 1}\log_f)$, natural in f and E. We extend to I-diagrams $f^\bullet : X^\bullet \to \mathbb{G}_m$ and to presheaves of complexes as for $E(\log_f)$. The quotient map N defined above extends to the natural map

$$N : E(\log_{f^\bullet}) \to E(\sigma_{\geq 1}\log_{f^\bullet}).$$

for $f^\bullet : X^\bullet \to \mathbb{G}_m$ an I-diagram of morphisms in $\mathbf{Sm}/k$, and $E \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$. Finally, for $E \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$, define $E(-1)$ to be the presheaf of complexes

$$E(-1)(X) := E(X \wedge \mathbb{G}_m)[1] := \ker\left(E(X \times \mathbb{G}_m) \xrightarrow{i_1^*} E(X)\right)[1].$$

Definition 9.2.2. Let E be be in $C_{\mathbb{Q}}(\mathbf{Sm}/k)$. Call E *alternating* if for every $X \in \mathbf{Sm}/k$ and every $n \geq 0$, the alternating projection

$$\mathrm{alt}_n : E(X \wedge \mathbb{G}_m^{\wedge n}) \to E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$$

is a quasi-isomorphism. □

Remarks 9.2.3. (1) Clearly, E is alternating if and only if S_n acts via the sign representation on $H_pE(X \wedge \mathbb{G}_m^{\wedge n})$ for all X, n and p.

(2) Fix integers $1 \leq i \leq n$. We have the split injection $\iota_{i,i+1} : E(X \wedge \mathbb{G}_m^{\wedge n}) \to E((X \times \mathbb{G}_m^{n-2}) \wedge \mathbb{G}_m^{\wedge 2})$ by shuffling the $i, i+1$ coordinates to position $n-1, n$. In particular, we have the injection

$$H_p(\iota_{i,i+1}) : H_pE(X \wedge \mathbb{G}_m^{\wedge n}) \to H_pE((X \times \mathbb{G}_m^{n-2}) \wedge \mathbb{G}_m^{\wedge 2}).$$

Since S_n is generated by simple transpositions, this shows that E is alternating if and only if the exchange of factors in $\mathbb{G}_m \wedge \mathbb{G}_m$ acts by -1 on $H_pE(X \wedge \mathbb{G}_m \wedge \mathbb{G}_m)$ for all X and p.

(3) Suppose that $E \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$ is homotopy invariant and satisfies Nisnevich excision. Consider $\mathbb{P}^1$ as pointed by ∞. Then $E(X \wedge \mathbb{P}^1)$ is quasi-isomorphic to the suspension $E(X \wedge \mathbb{G}_m)[-1]$, hence E is alternating if and only if the exchange of factors in $\mathbb{P}^1 \wedge \mathbb{P}^1$ induces the identity on $H_pE(X \wedge \mathbb{P}^1 \wedge \mathbb{P}^1)$ for all X and p.
The homotopy invariance and Nisnevich excision properties of E give a natural quasi-isomorphism of $E(X \wedge \mathbb{P}^1 \wedge \mathbb{P}^1)$ with $E(X \wedge (\mathbb{A}^2/\mathbb{A}^2 \setminus \{0\}))$, with the exchange of factors in $\mathbb{P}^1 \wedge \mathbb{P}^1$ going over to the linear transformation $(x, y) \mapsto (y, x)$. If the characterstic of k is different from 2, this transformation is conjugate to $(x, y) \mapsto (-x, y)$. Thus E is alternating if and only if the map

$[-1] : \mathbb{P}^1 \to \mathbb{P}^1$, $[-1](x_0, x_1) = (x_0, -x_1)$, acts by the identity on $H_pE(X \wedge \mathbb{P}^1)$ for all X and p.

(4) Call E *oriented* if E is an associative graded-commutative ring:

$$\mu : E \otimes_{\mathbb{Q}} E \to E$$

and (roughly speaking) E admits a natural Chern class transformation

$$c_1 : \mathrm{Pic} \to H^2E$$

satisfying the projective bundle formula: For $\mathcal{E} \to X$ a rank r vector bundle with associated projective space bundle $\mathbb{P}(\mathcal{E}) \to X$ and tautological line bundle $\mathcal{O}(1)$, $H^*E(\mathbb{P}(\mathcal{E}))$ is a free $H^*E(X)$-module with basis $1, \xi, \ldots, \xi^{r-1}$, where $\xi = c_1(\mathcal{O}(1)) \in H^2E(\mathbb{P}(\mathcal{E}))$. We do not assume that c_1 is a group homomorphism. The projective bundle formula and the fact that $[-1]^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(1)$ implies that an oriented E is alternating. In particular, rational motivic cohomology, $\mathbb{Q}_\ell(*)$ étale cohomlogy, $\mathbb{Q}$-singular cohomology (with respect to a chosen embedding $k \to \mathbb{C}$) and rational algebraic cobordism $\mathrm{MGL}^{**}_{\mathbb{Q}}$ are all alternating.
On the other hand, rational motivic co-homotopy is alternating if -1 is a square in k, but is not alternating for $k = \mathbb{R}$. This is pointed out in [31]: if $-1 = i^2$, $[-1]$ is represented by the 2×2 matrix with diagonal entries i and $-i$. As this is a product of elementary matrices, one has an $\mathbb{A}^1$-homotopy connecting $[-1]$ and id. To see the non-triviality of $[-1]$ for $k = \mathbb{R}$, let $[X, Y]$ denote the set of morphisms $X \to Y$ in $\mathcal{H}_{\mathbb{A}^1}\mathbf{Spc}_*(k)$. Morel defines a map (of sets) $[\mathbb{P}^1, \mathbb{P}^1] \to \phi(k)$, where $\phi(k)$ is the set of isomorphism classes of quadratic forms over k, and notes that the map $[u]$,

$$[u](x_0, x_1) := (x_0, ux_1),$$

goes to the class of the form ux^2. This map extends to a ring homomorphism

$$\mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(\mathbb{P}^1, \mathbb{P}^1) \to \mathrm{GW}(k),$$

where $\mathrm{GW}(k)$ is the Grothendieck-Witt ring (see also [32, Lemma 3.2.4] for details). Identifying $\mathrm{GW}(\mathbb{R})$ with $\mathbb{Z} \times \mathbb{Z}$ by rank and signature, we see that $[-1]$ goes to the non-torsion element $(1, -1)$.
The example of motivic (co)homotopy is in fact universal for this phenomenon, so if $[-1]$ vanishes in $[\mathbb{P}^1, \mathbb{P}^1]$, then every $E \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$ satisfying homotopy invariance and Nisnevich excision is alternating.
We are grateful to F. Morel for explaining the computation of the transposition action on $\mathbb{P}^1 \wedge \mathbb{P}^1$ in terms of quadratic forms and the Grothendieck-Witt group.
(4) Looking at the $\mathbb{A}^1$-stable homotopy category of T-spectra over k, $\mathcal{SH}(k)$, one can decompose the $\mathbb{Q}$-linearization $\mathcal{SH}(k)_{\mathbb{Q}}$ into the symmetric part $\mathcal{SH}(k)_+$ and alternating part $\mathcal{SH}(k)_-$ with respect to the exchange of factors on $\mathbb{G}_m \wedge \mathbb{G}_m$. Morel [33] has announced a result stating that $\mathcal{SH}(k)_-$ is in general equivalent to Voevodsky's big motivic category $DM(k)_{\mathbb{Q}}$, and that $\mathcal{SH}(k)_+$ is zero if -1 is a sum of squares. This suggests that the alternating part

$\mathcal{SH}_{S^1}(k)$ of the category of rational S^1-spectra $\mathcal{SH}_{S^1}(k)_{\mathbb{Q}}$ is closely related to the big category of effective motives (with $\mathbb{Q}$-coefficients) $DM^{\mathrm{eff}}(k)_{\mathbb{Q}}$. □

Proposition 9.2.4. *Let E be in $C_{\mathbb{Q}}(\mathbf{Sm}/k)$, $f : X \to \mathbb{G}_m$ an I-diagram of morphisms in $\mathbf{Sm}/k$.*

(1) *The sequence*

$$E(X) \xrightarrow{\iota_X} E(\log_f) \xrightarrow{N} E(\sigma_{\geq 1}\log_f)$$

identifies $E(\sigma_{\geq 1}\log_f)$ with the quotient complex $E(\log_f)/E(X)$.

(2) *Suppose E is alternating. Then there is a natural quasi-isomorphism* $\mathrm{alt} : E(-1)(\log_f) \to E(\sigma_{\geq 1}\log_f)$.

Proof. It suffices to prove (1) for E a presheaf of $\mathbb{Q}$-vector spaces, and $f : X \to \mathbb{G}_m$ a morphism in $\mathbf{Sm}/k$, where the assertion is obvious. Similarly, it suffices to construct a natural map $\theta_{E,X} : E(-1)(\log_f) \to E(\sigma_{\geq 1}\log_f)$ for E a presheaf of $\mathbb{Q}$-vector spaces, extend as above to a map in general, and show that $\theta_{E,X}$ is a quasi-isomorphism for $E \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$ alternating and $f : X \to \mathbb{G}_m$ a morphism in $\mathbf{Sm}/k$.

In fact, for E a presheaf of $\mathbb{Q}$-vector spaces and $n \geq 1$,

$$E(-1)(\log_f)_n = \ker[(\mathrm{id}_X \times i)^* : E(X \times \mathbb{G}_m, \mathbb{G}_m^{\wedge n-1})^{\mathrm{alt}} \to E(X, \mathbb{G}_m^{\wedge n-1})^{\mathrm{alt}}]$$

so $E(-1)(\log_f)_n$ is a subspace of $E(X, \mathbb{G}_m^{\wedge n})$; thus alt_n defines a map $E(-1)(\log_f)_n \to E(\log_f)_n$. One easily checks that this defines a map of complexes

$$\mathrm{alt}_* : E(-1)(\log_f) \to E(\sigma_{\geq 1}\log_f),$$

as desired.

Now suppose that E is alternating, i.e., that

$$(a) \qquad E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}} \to E(X \wedge \mathbb{G}_m^{\wedge n})$$

is a quasi-isomorphism for all n and X. This implies that the maps

$$E(X \times \mathbb{G}_m, \mathbb{G}_m^{\wedge n-1})^{\mathrm{alt}} \to E(X \times \mathbb{G}_m, \mathbb{G}_m^{\wedge n-1})$$
$$E(X \wedge \mathbb{G}_m^{\wedge n-1})^{\mathrm{alt}} \to E(X \wedge \mathbb{G}_m^{\wedge n-1})$$

are quasi-isomorphisms, hence

$$(b) \qquad \mathrm{id}_{X\wedge\mathbb{G}_m} \times \mathrm{alt}_{n-1} : E((X \wedge \mathbb{G}_m) \wedge \mathbb{G}_m^{\wedge n-1}) \to E((X \wedge \mathbb{G}_m) \wedge \mathbb{G}_m^{\wedge n-1})^{\mathrm{alt}}$$

is a quasi-isomorphism. Since $E((X \wedge \mathbb{G}_m) \wedge \mathbb{G}_m^{\wedge n-1}) = E(X \wedge \mathbb{G}_m^{\wedge n})$, (a) and (b) imply that

$$\mathrm{id}_X \times \mathrm{alt}_n : E((X \wedge \mathbb{G}_m) \wedge \mathbb{G}_m^{\wedge n-1})^{\mathrm{alt}} \to E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$$

is a quasi-isomorphism. As $\mathrm{alt}_* : E(-1)(\log_f) \to E(\sigma_{\geq 1}\log_f)$ is the map on the total complex of the double complexes

$$n \mapsto \mathrm{id}_X \times \mathrm{alt}_n : E((X \wedge \mathbb{G}_m) \wedge \mathbb{G}_m^{\wedge n-1})^{\mathrm{alt}} \to E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$$

we see that alt_* is a quasi-isomorphism. □

9.3. The log complex and path spaces. Let $f : X \to \mathbb{G}_m$ be a morphism in $\mathbf{Sm}/k$ arising from a semi-stable degeneration $\mathcal{X} \to (C, 0)$ and choice of parameter in $\mathcal{O}_{C,0}$. The monodromy sequence for $E(\lim_{t\to 0} X_t)$ arises from the sequence of Proposition 9.2.4 by comparing the path space $E(\mathcal{P}_{X/\mathbb{G}_m})$ with $E(\log_f)$.

We use the Dold-Kan correspondence to rewrite $E(\mathcal{P}_{X/\mathbb{G}_m})$ as a complex, namely: take for each p the associated complex $E_p(\mathcal{P}^*_{X/\mathbb{G}_m})$ of the simplicial abelian group $n \mapsto E_p(\mathcal{P}^n_{X/\mathbb{G}_m})$, with differential the alternating sum of the face maps, and then take the total complex of the double complex

$$p \mapsto E_p(\mathcal{P}^*_{X/\mathbb{G}_m}).$$

We write this complex as $E(\mathcal{P}_{X/\mathbb{G}_m})$.

We also have the normalized subcomplex $NE(\mathcal{P}_{X/\mathbb{G}_m})$ of $E(\mathcal{P}_{X/\mathbb{G}_m})$, quasi-isomorphic to $E(\mathcal{P}_{X/\mathbb{G}_m})$ via the inclusion. Recall that, for a simplicial abelian group $n \mapsto A_n$, the normalized complex NA_* has

$$NA_n := \cap_{i=1}^n \ker d_i : A_n \to A_{n-1}$$

with differential $d_0 : NA_n \to NA_{n-1}$. We define $NE(\mathcal{P}_{X/\mathbb{G}_m})$ by first taking the normalized subcomplex $NE_p(\mathcal{P}_{X/\mathbb{G}_m})$ of $E_p(\mathcal{P}^*_{X/\mathbb{G}_m})$ for each p, and then forming the total complex of the double complex $p \mapsto NE_p(\mathcal{P}_{X/\mathbb{G}_m})$.

In particular, we have the inclusion of double complexes

$$NE_*(\mathcal{P}^*_{X/\mathbb{G}_m}) \subset E_*(\mathcal{P}^*_{X/\mathbb{G}_m});$$

which gives for each n the inclusion of single complexes

$$NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \subset E_*(\mathcal{P}^n_{X/\mathbb{G}_m});$$

Recalling that $\mathcal{P}^n_{X/\mathbb{G}_m} = X \times \mathbb{G}_m^n$, we thus have for each n the inclusion of complexes

$$NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \subset E_*(X \times \mathbb{G}_m^n),$$

We may therefore apply the projections $\pi_n : E_*(X \times \mathbb{G}_m^n) \to E_*(X \wedge \mathbb{G}_m^{\wedge n})$ and alt_n, giving the map

$$\mathrm{alt}_n \circ \pi_n : NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \to E_*(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}.$$

Lemma 9.3.1. *Suppose that E is alternating. Then*

$$\mathrm{alt}_n \circ \pi_n : NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \to E_*(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$$

is a quasi-isomorphism.

Proof. The map $p_1^* : E(X) \to E(X \times \mathbb{G}_m)$ splits $i_1^* : E(X \times \mathbb{G}_m) \to E(X)$, so we have the natural splitting

$$E(X \times \mathbb{G}_m) = E(X) \oplus E(X \wedge \mathbb{G}_m).$$

Extending this to $E(X \times \mathbb{G}_m^n)$ by using the maps i_j^* and p_j^*, we have the natural splitting

$$E(X \times \mathbb{G}_m^n) = \oplus_{m=0}^n \oplus_{\substack{I\subset\{1,\ldots,n\}\\ |I|=m}} E(X \wedge \mathbb{G}_m^{\wedge I}). \tag{9.3.1}$$

To explain the notation: For $I \subset \{1,\dots,n\}$, $E(X \wedge \mathbb{G}_m^{\wedge I}) = E(X \wedge \mathbb{G}_m^{\wedge |I|})$, included in $E(X \times \mathbb{G}_m^n)$ by the composition

$$E(X \wedge \mathbb{G}_m^{\wedge |I|}) \subset E(X \times \mathbb{G}_m^{|I|}) \xrightarrow{(\mathrm{id}_X \times p_I)^*} E(X \times \mathbb{G}_m^n)$$

where $p_I : \mathbb{G}_m^n \to \mathbb{G}_m^{|I|}$ is the projection on the factors $i_1, \dots, i_m$ if $I = \{i_1, \dots, i_m\}$ with $i_1 < \ldots < i_m$.
The action of S_n on $E(X \times \mathbb{G}_m^n)$ preserves this decomposition, with $\sigma \in S_n$ mapping $E(X \wedge \mathbb{G}_m^{\wedge I})$ to $E(X \wedge \mathbb{G}_m^{\wedge \sigma^{-1}(I)})$ in the evident manner.
Now, for a simplicial abelian group A, the inclusion $NA_n \to A_n$ is split by universal expressions in the face and degeneracy maps. If $n \mapsto C_{*n}$ is a simplicial complex, we can form the complex of normalized subgroups (with respect to the simplicial variable) $N_{*_2}(C_{*_1,*_2})$ and take the homology $H_p(N_{*_2}(C_{*_1,*_2}), d_1)$, or we can form the simplicial abelian group $n \mapsto H_p(C_{*n})$ and take the normalized subgroup $N_{*_2} H_p(C_{*_1,*_2}, d_1) \subset H_p(C_{*_1,n}, d_1)$. Using the universal spitting mentioned above, we see that the two are the same:

$$H_p(N_{*_2}(C_{*_1,*_2}), d_1) = N_{*_2} H_p(C_{*_1,*_2}, d_1)$$

Since S_n acts by the sign representation on $H_p E(X \wedge \mathbb{G}_m^{\wedge n})$, it follows that, for $1 \le j < n$, the diagonal map

$$\begin{aligned} \delta_j &: \mathbb{G}_m^{n-1} \to \mathbb{G}_m^n \\ (t_1, \dots, t_{n-1}) &\mapsto (t_1, \dots t_j, t_j, t_{j+1}, \dots t_{n-1}) \end{aligned}$$

induces the zero map on $H_p E(X \wedge \mathbb{G}_m^{\wedge n})$. Similarly, the inclusion

$$\begin{aligned} i_n &: \mathbb{G}_m^{n-1} \to \mathbb{G}_m^n \\ (t_1, \dots, t_{n-1}) &\mapsto (t_1, \dots t_{n-1}, 1) \end{aligned}$$

is the zero map on $H_p E(X \wedge \mathbb{G}_m^I)$ if $n \in I$.
From this, it is not hard to see that

$$NH_p E_*(NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) = H_p E_*(X \wedge \mathbb{G}_m^{\wedge n}),$$

with respect to the decomposition of $E_*(\mathcal{P}^n_{X/\mathbb{G}_m}) = E_*(X \times \mathbb{G}_m^n)$ given by (9.3.1). Indeed,

$$\begin{aligned} \ker(H_p(d_n)) &= \ker(i_n^* : H_p E_*(X \times \mathbb{G}_m^n) \to H_p E_*(X \times \mathbb{G}_m^{n-1}) \\ &= \oplus_{\substack{I \subset \{1,\dots,n\} \\ n \in I}} H_p E_*(X \wedge \mathbb{G}_m^{\wedge I}) \end{aligned}$$

It is then easy to show by descending induction on i that

$$\cap_{j=i}^n \ker H_p(d_j) = \oplus_{\substack{I \subset \{1,\dots,n\} \\ \{i,\dots,n\} \subset I}} H_p E_*(X \wedge \mathbb{G}_m^{\wedge I})$$

from which our claim follows taking $i = 1$. Thus the projection

$$p_n : NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \to E_*(X \wedge \mathbb{G}_m^{\wedge n})$$

is a quasi-isomorphism for each n. As E is alternating, the alternating projection

$$\mathrm{alt}_n : E_*(X \wedge \mathbb{G}_m^{\wedge n}) \to E_*(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$$

is a quasi-isomorphism as well, completing the proof. □

Lemma 9.3.2. *Let E be in $C_{\mathbb{Q}}(\mathbf{Sm}/k)$. Let $\delta_0 : E(X \times \mathbb{G}_m^n) \to E(X \times \mathbb{G}_m^{n-1})$ be the map $[(\mathrm{id}_X, f), \mathrm{id}_{\mathbb{G}_m^{n-1}}]^*$. Then the diagram*

$$\begin{tikzcd} E(X \times \mathbb{G}_m^n) \arrow[r, "\delta_0"] \arrow[d, "\mathrm{alt}_n \circ \pi_n"'] & E(X \times \mathbb{G}_m^{n-1}) \arrow[d, "\mathrm{alt}_{n-1} \circ \pi_{n-1}"] \\ E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}} \arrow[r, "\cup f"'] & E(X \wedge \mathbb{G}_m^{\wedge n-1})^{\mathrm{alt}} \end{tikzcd}$$

commutes.

Proof. This follows directly from the definition of $\cup f$ and the fact that $\mathrm{alt}_n \circ \pi_n$ is the identity on $E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$. □

Proposition 9.3.3. *Let $E \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$ be alternating. Then the maps $\mathrm{alt}_n \circ \pi_n : NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \to E_*(X \times \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$ define a quasi-isomorphism of total complexes*

$$\mathrm{alt} \circ \pi : NE(\mathcal{P}_{X/\mathbb{G}_m}) \to E(\log_f)$$

Proof. That the maps $\mathrm{alt}_n \circ \pi_n : NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \to E_*(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$ define a map of total complex $NE(\mathcal{P}_{X/\mathbb{G}_m}) \to E(\log_f)$ follows from Lemma 9.3.2 and the fact that, for each n, the differential d_0 on $NE(\mathcal{P}^n_{X/\mathbb{G}_m})$ is the restriction of $\delta_0 : E(X \times \mathbb{G}_m^n) \to E(X \times \mathbb{G}_m^{n-1})$. Lemma 9.3.1 implies that $\mathrm{alt} \circ \pi$ is a quasi-isomorphism. □

We collect our results in

Theorem 9.3.4. *Let $E \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$ be alternating, $f^\bullet : X^\bullet \to \mathbb{G}_m$ an I-diagram of morphisms in $\mathbf{Sm}/k$. Consider the diagram*

$$\begin{tikzcd} & & E(\mathcal{P}_{X/\mathbb{G}_m}) & E(-1)(\mathcal{P}_{X/\mathbb{G}_m}) & \\ & E(X^\bullet) \arrow[r, "\iota_{X^\bullet}"] \arrow[dd, equal] & NE(\mathcal{P}_{X/\mathbb{G}_m}) \arrow[u, "i"'] \arrow[dd, "\mathrm{alt}\circ\pi"'] & NE(-1)(\mathcal{P}_{X/\mathbb{G}_m}) \arrow[u, "i"'] \arrow[d, "\mathrm{alt}\circ\pi"'] & \\ & & & E(-1)(\log_{f^\bullet}) & \\ 0 \arrow[r] & E(X^\bullet) \arrow[r, "\iota_{X^\bullet}"'] & E(\log_{f^\bullet}) \arrow[r, "N"'] & E(\sigma_{\geq 1} \log_{f^\bullet}) \arrow[r] \arrow[u, "\mathrm{alt}"] & 0 \end{tikzcd}$$

Here the maps i are the canonical inclusions and the maps $\iota_{X^\bullet}$ are the canonical maps given by the identities $E_n(\mathcal{P}_{X^i/\mathbb{G}_m})_0 = E_n(\log_{f^i})_0 = E_n(X^i)$. Then

(1) *The diagram commutes and is natural in E and $f^\bullet$.*
(2) *All the maps in the diagram are maps of complexes.*

(3) *All the vertical maps are quasi-isomorphisms*
(4) *The bottom sequence is termwise exact.*

Proof. The first point follows by construction, the remaining assertions follow from the Dold-Kan correspondence, Proposition 9.3.3 and Proposition 9.2.4. □

Corollary 9.3.5 (Monodromy sequence). *Let $W \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$ be alternating, $p : \mathcal{X} \to (C, 0)$ a semi-stable degeneration, $t \in \mathcal{O}_{C,0}$ a uniformizing parameter. Then there is a distinguished triangle in $D(X_{0\mathrm{Zar}})$*

$$E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0) \to E(\lim_{t\to 0} X_t) \xrightarrow{N} E(-1)(\lim_{t\to 0} X_t) \to E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0)[1],$$

natural in (p, t) and in E.

Proof. The commutative diagram of Theorem 9.3.4 being natural in the choice of I-diagram and in E, one can extend the diagram directly to the case of a co-presheaf of cosimplicial I-diagrams

$$U \mapsto f^\bullet(U) : X^\bullet(U) \to \mathbb{G}_m.$$

If we take I to be finite, we can extend further to a co-presheaf of cosimplicial I-diagrams $f^\bullet : X^\bullet \to \mathbb{G}_m$, with $X^\bullet(U)$ a pro-scheme smooth over k, and still preserve the quasi-isomorphisms and exactness. Feeding the I-diagram

$$t \circ p : \tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0 \to \mathbb{G}_m$$

to this machine and taking the distinguished triangle induced by the exact sequence of log complexes at the bottom of the diagram completes the proof. □

Remark 9.3.6. If we splice together the long exact homotopy sequence for the monodromy distinguished triangle

$$E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0) \to E(\lim_{t\to 0} X_t) \xrightarrow{N} E(-1)(\lim_{t\to 0} X_t)$$

with the localization distinguished triangle of Theorem 6.3.7

$$E^{D_{\mathrm{Zar}}}(X) \xrightarrow{\alpha_D} E(D_{\mathrm{Zar}}) \xrightarrow{\beta_D} E(\tau_\epsilon^{\hat{X}}(D)^0)$$

(both evaluated on $D = X_0$), we have the complex

$$\begin{aligned} \ldots \to E_n^{X_{0\mathrm{Zar}}}(X) \to E_n(X_{0\mathrm{Zar}}) \to E_n(\lim_{t\to 0} X_t) \xrightarrow{N} \\ E(-1)_n(\lim_{t\to 0} X_t) \to E_{n-2}^{X_{0\mathrm{Zar}}}(X) \to E_{n-2}(X_{0\mathrm{Zar}}) \to \ldots \end{aligned} \tag{9.3.2}$$

If $k = \mathbb{C}$ and E represents singular cohomology (for the classical topology)

$$E_n(Y) = H^{-n}(Y(\mathbb{C}), \mathbb{Q})$$

then Steenbrink's theorem [44] states that the above sequence is exact. The argument uses the mixed Hodge structure on all the terms together with a weight argument.

One should be able to define a natural geometric "weight filtration" on $E(\lim_{t\to 0} X_t)$ by using the stratification of X_0 by faces. However, for general E, this additional structure might not suffice to force the exactness of the above sequence. It would be interesting to give a general additional structure on E that would imply this exactness. □

9.4. Ayoub's monodromy sequence. The monodromy sequence of corollary 9.3.5 agrees with the monodromy sequence constructed by Ayoub in [3, section 3.6] after making the identification described in proposition 8.3.2 and working throughout in the category of rational motives $DM(k)_{\mathbb{Q}}$. Indeed, it is easy to check that our complex $E(\log_{f^\bullet})$ agrees with the contruction $E\otimes f_\eta^*\mathcal{L}og^\vee$ of [3, section 3.6.3], and that our isomorphism $E(\mathcal{P}_{X/\mathbb{G}_m}) \cong E(\log_{f^\bullet})$ agrees with the map $E\otimes f_\eta^*\mathcal{L}og^\vee \to E\otimes f_\eta^*\mathcal{U}$ induced by the map $\ell : \mathcal{L}og^\vee \to \mathcal{U}$ of [3, definition 3.6.42]. From there, one can easily compare with Ayoub's monodromy sequence [3, definition 3.6.37]. We give a sketch of these comparisons.

Ayoub's construction begins with the *Kummer motive* $\mathcal{K}$. We denote the object in $DM(S)_{\mathbb{Q}}$ represented by a smooth S-scheme X as $m_S(X)$ and write $_S$ for $m_S(S)$, the unit for the tensor structure in $DM(S)_{\mathbb{Q}}$; we delete the subscript S from the notation for $S = \operatorname{Spec} k$. The 1-section $i_1 : S \to \mathbb{G}_{mS}$ induces the splitting

$$m_S(\mathbb{G}_{mS}) = \ _S \oplus \ _S(1)[1]$$

and thus the projection $\pi : m_S(\mathbb{G}_{mS}) \to \ _S(1)[1]$
We take $S = \mathbb{G}_m$. The diagonal $\Delta : \mathbb{G}_m \to \mathbb{G}_m \times_k \mathbb{G}_m = \mathbb{G}_{mS}$ induces $m_S(\Delta) : \ _S \to m_S(\mathbb{G}_{mS})$; composing with π and twisting and shifting gives the map

$$\cup t_* : \ _{\mathbb{G}_m}(-1)[-1] \to \ _{\mathbb{G}_m}$$

The Kummer motive $\mathcal{K} \in DM(\mathbb{G}_m)_{\mathbb{Q}}$ is defined as the "cone" of $\cup t_*$: Ayoub shows there is a canonial distinguished triangle in $DM(\mathbb{G}_m)_{\mathbb{Q}}$

$$_{\mathbb{G}_m}(-1)[-1] \xrightarrow{\cup t_*} \ _{\mathbb{G}_m} \to \mathcal{K} \to \ _{\mathbb{G}_m}(-1)$$

Next, Ayoub defines the object $\mathcal{L}og^\vee$ of $DM(\mathbb{G}_m)_{\mathbb{Q}}$. Viewing $\mathcal{K}$ as the two-term complex $[m_S(\mathbb{G}_{mS})(-1)[-1] \xrightarrow{\cup t_*} m_S(\mathbb{G}_{mS})]$ with $m_S(\mathbb{G}_{mS})$ in degree zero, one sees that the nth symmetric product $\operatorname{Sym}^n\mathcal{K}$ is the complex

$$_{\mathbb{G}_m}(-n)[-n] \xrightarrow{\cup t_*} \ _{\mathbb{G}_m}(-n+1)[-n+1] \xrightarrow{\cup t_*} \ldots \xrightarrow{\cup t_*} \ _{\mathbb{G}_m}(-1)[-1] \xrightarrow{\cup t_*} \ _{\mathbb{G}_m},$$

where we write the map $\cup t_*(-i)[-i]$ as $\cup t_*$ for short. The map $_{\mathbb{G}_m} \to \mathcal{K}$ gives rise to the map $\operatorname{Sym}^n\mathcal{K} \to \operatorname{Sym}^{n+1}\mathcal{K}$. We can take the limit $\mathcal{L}og^\vee$ in $DM(\mathbb{G}_m)_{\mathbb{Q}}$

$$\mathcal{L}og^\vee := \lim_n \operatorname{Sym}^n\mathcal{K}$$

As a complex, $\mathcal{L}og^\vee$ is just

$$\ldots \xrightarrow{\cup t_*} {}_{\mathbb{G}_m}(-n)[-n] \xrightarrow{\cup t_*} {}_{\mathbb{G}_m}(-n+1)[-n+1] \xrightarrow{\cup t_*} \ldots$$
$$\xrightarrow{\cup t_*} {}_{\mathbb{G}_m}(-1)[-1] \xrightarrow{\cup t_*} {}_{\mathbb{G}_m}.$$

Now suppose we have a semi-stable degeneration $f : \mathcal{X} \to \mathbb{A}^1$ and an object $E \in DM(\mathcal{X}^0)_\mathbb{Q}$. Let $f^0 : \mathcal{X}^0 \to \mathbb{G}_m$ be the restriction of f; since f^0 is smooth, we have $Lf^{0*} = f^*$. Let $i : X_0 \to \mathcal{X}$, $j : \mathcal{X}^0 \to \mathcal{X}$ be the inclusions. The *logarithmic* specialization functor $\log_f$ is defined by

$$\log_f(E) := Li^* Rj_*(E \otimes f^{0*}\mathcal{L}og^\vee)$$

Remark 9.4.1. If we replace $\mathcal{X}$ with $\mathcal{X} \times_{\mathbb{A}^1} \operatorname{Spec} \mathcal{O}_{\mathbb{A}^1,0}$, we have the canonical identification of $\mathcal{X}^0$ with the generic fiber $\mathcal{X}_\eta$ and f^0 with f_η. We avoid doing this to keep with the notation of our earlier sections. □

The first step in our comparison is

Lemma 9.4.2. *Take $E \in DM(\mathcal{X}^0)_\mathbb{Q}$, represented by a fibrant object $\tilde{E} \in C_\mathbb{Q}(\mathbf{Sm}/\mathcal{X}^0)$. Then $E \otimes f^{0*}\mathcal{L}og^\vee$ is represented by $\tilde{E}(\log_{f\bullet})$.*

Proof. Note that we may assume that $\tilde{E}$ is alternating, since E is a motive. Letting $\mathcal{H}om$ denote the internal Hom in $DM(\mathcal{X}^0)_\mathbb{Q}$. We have the distinguished triangle

$$E(-1)[-1] \to \mathcal{H}om(\mathbb{G}_{m\mathcal{X}_0}, E) \xrightarrow{i_1^*} \mathcal{H}om(\quad, E) = E \to E(-1)$$

Thus $E(-1)[-1]$ is represented by the presheaf

$$X' \mapsto \mathrm{fib}[\tilde{E}(X' \times_k \mathbb{G}_m) \xrightarrow{\mathrm{id}\times i_1^*} \tilde{E}(X')]$$

Similarly, for $n \geq 1$, $E(-n)[-n]$ is represented by the presheaf

$$X' \mapsto \tilde{E}(X' \wedge \mathbb{G}_m^{\wedge n}).$$

SInce $\tilde{E}$ is alternating, this latter presheaf is equivalent to

$$X' \mapsto \tilde{E}(X', \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}.$$

Finally, the map $\cup f : \tilde{E}(X'_+ \wedge \mathbb{G}_m) \to \tilde{E}(X')$ is just the map induced by the pull-back by f and $f \times \mathrm{id}$ of the diagonal map $\mathbb{G}_m \to \mathbb{G}_m \times_k \mathbb{G}_m$, hence $\cup f$ represents the map $f^{0*}(\cup t_*)$. The comparison follows easily from this. □

Next, Ayoub considers the object $\mathcal{U}$ of $DM(\mathbb{G}_m)_\mathbb{Q}$. Interpreting his general construction in the case of $DM(\mathbb{G}_m)_\mathbb{Q}$, $\mathcal{U}$ is the motive associated to the simplicial object

$$n \mapsto \mathcal{H}om_{DM(\mathbb{G}_m)_\mathbb{Q}}(\mathcal{P}^n_{\mathbb{G}_m}, {}_{\mathbb{G}_m}),$$

i.e., the homological complex which is $\mathcal{H}om_{DM(\mathbb{G}_m)_\mathbb{Q}}(\mathcal{P}^n_{\mathbb{G}_m}, {}_{\mathbb{G}_m})$ in degree n, and with differential the alternating sum of the maps induced by the coface maps in $\mathcal{P}_{\mathbb{G}_m}$. Naturally, to make sense of this, we need to lift this construction to the appropriate category of complexes. In any case, the same proof as for lemma 9.4.2 gives us a canonical isomorphism of $E \otimes f^{0*}\mathcal{U}$ with $E(\mathcal{P}_{\mathcal{X}^0/\mathbb{G}_m})$. Similarly, the uni-potent specialization functor Υ is given by

$$\Upsilon(E) = Li^* Rj_*(E \otimes f^{0*}\mathcal{U}).$$

Finally, we have

$$\mathcal{H}om_{DM(\mathbb{G}_m)_\mathbb{Q}}(\mathcal{P}^n_{\mathbb{G}_m}, \ {}_{\mathbb{G}_m}) \cong (\ {}_{\mathbb{G}_m}(-1)[-1] \oplus \ {}_{\mathbb{G}_m})^{\otimes n}$$

and the first differential $\mathcal{U}_1 \to \mathcal{U}_0$ is

$${}_{\mathbb{G}_m}(-1)[-1] \oplus \ {}_{\mathbb{G}_m} \xrightarrow{\cup t_* + \mathrm{id}} \ {}_{\mathbb{G}_m}$$

Thus we have the evident map $\mathcal{K} \to \mathcal{U}$. The diagonal map on $\mathcal{P}_{\mathbb{G}_m}$ dualizes to make $\mathcal{U}$ a commutive ring object in $DM(\mathbb{G}_m)_\mathbb{Q}$. Ayoub notes that $\mathcal{K} \to \mathcal{L}og^\vee$ is universal for maps of $\mathcal{K}$ to a commutative ring in $DM(\mathbb{G}_m)_\mathbb{Q}$, hence there is a unique ring map $\ell : \mathcal{L}og^\vee \to \mathcal{U}$ making

$$\begin{tikzcd}
\mathcal{K} \arrow[r] \arrow[dr] & \mathcal{L}og^\vee \arrow[d, "\ell"] \\
 & \mathcal{U}
\end{tikzcd}$$

commute. It is not hard to see that our map $\mathrm{alt} \circ \pi$ is induced by a map of complexes in $\mathbb{Q}\mathbf{Sm}/\mathbb{G}_m$ which commutes with the co-multiplications dual to the ring multiplications for $\mathcal{U}$ and $\mathcal{L}og^\vee$. Since both

$$E \otimes f^{0*}\mathcal{L}og^\vee \xrightarrow{\mathrm{id}\otimes\ell} E \otimes f^{0*}\mathcal{U}$$

and

$$E(\mathcal{P}_{\mathcal{X}^0/\mathbb{G}_m}) \xrightarrow{\mathrm{alt}\circ\pi} E(\log_{f^\bullet})$$

are isomorphisms in $DM(\mathcal{X}^0)_\mathbb{Q}$, it follows that these maps are inverse to each other.

Once we have the pair of compatible isomorphisms $E \otimes f^{0*}\mathcal{L}og^\vee \cong E(\log_{f^\bullet})$ and $E \otimes \mathcal{U} \cong E(\mathcal{P}_{\mathcal{X}^0/\mathbb{G}_m})$, it is easy to see that Ayoub's construction of the monodromy sequence and ours are compatible: Ayoub's construction follows from the obvious identification of $\mathcal{L}og^\vee(-1)$ with $\sigma_{\le -1}\mathcal{L}og^\vee$ (cohomological notation) giving the distinguished triangle

$${}_{\mathbb{G}_m} \to \mathcal{L}og^\vee \xrightarrow{N} \mathcal{L}og^\vee(-1) \to \ {}_{\mathbb{G}_m}[1]$$

which clearly passes over to our identification $E(\sigma_{\ge 1}\log_{f^\bullet}) \cong E(-1)$ and the monodromy distinguished triangle of corollary 9.3.5.

10. Limit motives

We use our construction of limit cohomology, slightly modified, to give a construction of the limit motive of a semi-stable degeneration, as an object in the big category of motives $DM(k)$.

10.1. The big category of motives. Voevodsky has defined the category of effective motives as the full subcategory $DM_-^{\mathrm{eff}}(k)$ of the derived category of Nisnevich sheaves with transfer $D_-(\mathbf{NST}(k))$ consisting of those complexes with strictly homotopy invariant cohomology sheaves.

In his thesis, Spitzweck [43] defines a "big" category of motives over a field k. Other constructions of a big category of motives over a noetherian base scheme S have been given by Østvar-Röndigs [40] and also by Cisinski-Deglise [9]. To give the reader the main idea of all these constructions, we quote from a recent letter from Röndigs [41]:

"One may construct a model category of simplicial presheaves with transfers on $\mathbf{Sm}/k$, in which the weak equivalences and fibrations are defined via the functor forgetting transfers. Via the Dold-Kan correspondence, there is an induced model structure on nonnegative chain complexes of presheaves with transfers. Both may be stabilized with respect to T or $\mathbb{P}^1$, in the sense of [18]. The Dold-Kan correspondence extends accordingly. Since T is a suspension already, one can then pass to a model category of $\mathbb{G}_m$-spectra of integer-indexed chain complexes as well. For k a perfect field, results from [46] show that the homotopy category of the latter model category contains Voevodsky's DM_{gm} as a full subcategory. "

We will use the $\mathbb{P}^1$-spectrum model. For details, we refer the reader to [40] and [35].

10.2. The cohomological motive. We start with the category of presheaves with transfer $\mathbf{PST}(k)$ on $\mathbf{Sm}/k$, which is defined as in [46] as the category of presheaves on the correspondence category $\mathbf{Cor}(k)$. We let $C_{\geq 0}(\mathbf{PST}(k))$ denote the model category of non-negative chain complexes in $\mathbf{PST}(k)$, with model structure induced from simplicial presheaves on $\mathbf{Sm}/k$, as described above. For $P \in C_{\geq 0}(\mathbf{PST}(k))$, let $P(-1)$ denote the presheaf

$$Y \mapsto \ker[P(Y \times \mathbb{P}^1) \xrightarrow{i_\infty^*} P(Y \times \infty)][2].$$

where "ker" means the termwise kernel of the termwise split surjection i_∞^*. Let $\mathbb{Z}_X^{tr}$ denote the presheaf on $\mathbf{Cor}(k)$ represented by $X \in \mathbf{Sm}/k$, and let

$$\tilde{\mathbb{Z}}_{\mathbb{P}^1}^{tr} := \mathrm{coker}(\mathbb{Z}_{\mathrm{Spec}\, k}^{tr} \xrightarrow{i_{\infty *}} \mathbb{Z}_{\mathbb{P}^1}^{tr}).$$

One has the adjoint isomorphism

$$\mathrm{Hom}_{C_{\geq 0}(\mathbf{PST}(k))}(C \otimes \tilde{\mathbb{Z}}_{\mathbb{P}^1}^{tr}, C') \cong \mathrm{Hom}_{C_{\geq 0}(\mathbf{PST}(k))}(C, C'(-1)[-2])$$

so the bonding maps for $\mathbb{P}^1$-spectra in $C_{\geq 0}(\mathbf{PST}(k))$ can be just as well defined via maps

$$C_n \to C_{n+1}(-1)[-2].$$

We will use this normalization of the bonding morphisms from now on.

For an integer $q \geq 0$, we have the (homological) Friedlander-Suslin presheaf $\mathbb{Z}_{FS}(q)$. To define this, one starts with the presheaf with transfers of quasi-finite cycles $z_{q.fin}(\mathbb{A}^q)$, with value on $Y \in \mathbf{Sm}/k$ the cycles on $Y \times \mathbb{A}^q$ which are quasi-finite over Y. One then forms the Suslin complex $C_*(z_{q.fin}(\mathbb{A}^q))$ and reindexes:

$$\mathbb{Z}_{FS}(q)(Y)_n := C_{n-2q}(z_{q.fin}(\mathbb{A}^q))(Y) := z_{q.fin}(\mathbb{A}^q)(Y \times \Delta^{n-2q}).$$

(see [26, §2.4] for a precise definition). This represents motivic cohomology Zariski-locally:

$$H^p(X, \mathbb{Z}(q)) = \mathbb{H}^p(X_{\mathrm{Zar}}, \mathbb{Z}_{FS}(q)).$$

More generally, for $X \in \mathbf{Sm}/k$, define $\mathbb{Z}^X_{FS}(q)$ by

$$\mathbb{Z}^X_{FS}(q)(Y) := \mathbb{Z}_{FS}(q)(X \times Y).$$

We define

$$\delta_n : \mathbb{Z}^X_{FS}(n) \to \mathbb{Z}^X_{FS}(n+1)(-1)$$

by sending a cycle W on $X \times Y \times \Delta^m \times \mathbb{A}^n$ to $W \times \Delta$, where $\Delta \subset \mathbb{A}^1 \times \mathbb{P}^1$ is the graph of the inclusion $\mathbb{A}^1 \subset \mathbb{P}^1$, and then reordering the factors to yield a cycle on $X \times Y \times \mathbb{P}^1 \times \Delta^m \times \mathbb{A}^{n+1}$.

Definition 10.2.1. Let X be in $\mathbf{Sm}/k$. The *cohomological motive* of X is the sequence

$$\tilde{h}(X) := (\mathbb{Z}^X_{FS}(0), \mathbb{Z}^X_{FS}(1)[2], \ldots, \mathbb{Z}^X_{FS}(n)[2n], \ldots)$$

with the bonding morphisms $\delta_n[2n]$. □

Remark 10.2.2. One can also define the cohomological motive $h(X) \in DM_{gm}(k)$ as the dual of the usual (homological) motive $m(X) := C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}_X)$. For X of dimension d, $h(X)(n)$ is actually in $DM^{\mathrm{eff}}_-(k)$ for all $n \geq d$, and is represented by $\mathbb{Z}^X_{FS}(n)$. From this, one sees that the image of $\tilde{h}(X)$ in $DM(k)$ is canonically isomorphic to $h(X)$.
Also, one can work in $DM^{\mathrm{eff}}_-(k)$ if one wants to define the cohomological motive of a diagram in $\mathbf{Sm}/k$ if the varieties involved have uniformly bounded dimension. Since our construction of limit cohomology uses varieties of arbitrarily large dimension, we need to work in $DM(k)$. □

10.3. The limit motive. It is now an easy matter to define the limit motive for a semi-stable degeneration. Let $\mathcal{X} \to (C, 0)$ be a semi-stable degeneration with parameter t at 0; suppose the special fiber X_0 has irreducible components $X_0^1, \ldots, X_0^m$. We have the diagram (8.2.1) of cosheaves on $X_{0\mathrm{Zar}}$, $\lim_{t\to 0} X_t$, indexed by the non-empty subsets $I \subset \{1, \ldots, m\}$, which we write as

$$I \mapsto [\lim_{t\to 0} X_t]_I.$$

Taking global sections on X_0 yields the diagram of cosimplicial schemes

$$I \mapsto [\lim_{t\to 0} X_t]_I(X_0).$$

Applying $\tilde{h}$ gives us the diagram of $\mathbb{P}^1$-spectra in $C_{\geq 0}(\mathbf{PST}(k))$

$$I \mapsto \tilde{h}([\lim_{t\to 0} X_t]_I(X_0)).$$

We then take the homotopy limit over this diagram forming the complex

$$\lim_{t\to 0} \tilde{h}(X_t) := \operatorname*{holim}_I \{I \mapsto \tilde{h}([\lim_{t\to 0} X_t]_I(X_0))\}.$$

Definition 10.3.1. Let $\mathcal{X} \to (C, 0)$ be a semi-stable degeneration with parameter t at 0. The limit cohomological motive $\lim_{t\to 0} h(X_t)$ is the image of $\lim_{t\to 0} \tilde{h}(X_t)$ in $DM(k)$. □

Using the same procedure, we have, for $D \subset X$ a normal crossing scheme, the motive of the tubular neighborhood $h(\tau_\epsilon^{\hat{X}}(D))$ and the motive of the punctured tubular neighborhood $h(\tau_\epsilon^{\hat{X}}(D)^0)$. All the general results now apply for these cohomological motives. In particular, from corollary 9.3.5 we have the monodromy distinguished triangle (for the $\mathbb{Q}$-motive)

$$h(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0)_{\mathbb{Q}} \to \lim_{t\to 0} h(X_t)_{\mathbb{Q}} \to \lim_{t\to 0} h(X_t)_{\mathbb{Q}}(-1)$$

and theorem 6.3.7 gives the localization distinguished triangle

$$h^{X_0}(\mathcal{X}) \to h(X_0) \to h(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0),$$

where $h^{X_0}(\mathcal{X})$ is represented by

$$\mathrm{Cone}(\tilde{h}(\mathcal{X}) \xrightarrow{j^*} \tilde{h}(\mathcal{X} \setminus X_0))[-1].$$

From this latter triangle, we see that $h(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0)$ is in $DM_{gm}(k)$.

11. Gluing smooth curves

We use the exponential map defined in §5 to define an algebraic version of gluing smooth curves along boundary components. We begin by recalling the construction of the moduli space of smooth curves with boundary components; for details we refer the reader to the article by Hain [15].

11.1. Curves with boundary components. For a k-scheme Y, a *smooth curve over* Y is a smooth proper morphism of finite type $p : \mathcal{C} \to Y$ with geometrically irreducible fibers of dimension one. We say that $\mathcal{C}$ has genus g if all the geometric fibers of p are curves of genus g. A *boundary component* of $\mathcal{C} \to Y$ consists of a section $x : Y \to \mathcal{C}$ together with an isomorphism $v : \mathcal{O}_Y \to x^* T_{\mathcal{C}/Y}$, where $T_{\mathcal{C}/Y}$ is the relative tangent bundle on $\mathcal{C}$. Equivalently, v is a nowhere vanishing section of $T_{\mathcal{C}/Y}$ along x. A smooth curve with n boundary components is $(\mathcal{C} \to Y, (x_1, v_1), \ldots, (x_n, v_n))$ with all the x_i disjoint. One has the evident notion of isomorphism of such tuples, so we can consider the functor $\mathcal{M}_g^n$ on $\mathbf{Sch}_k$:

$$\mathcal{M}_g^n(Y) := \{\text{smooth genus } g \text{ curves over } Y \text{ with } n \text{ boundary components}\}/\cong$$

For $n = 0$, this is just the well-know functor of moduli of smooth curves, which admits the coarse moduli space M_g. For $n \geq 1$ and $g \geq 1$, it is easy to show that a smooth curve over Y with n boundary components admits no non-identity automorphisms (over Y), from which it follows that $\mathcal{M}_g^n$ is representable; we denote the representing scheme by $\mathcal{M}_g^n$ as well. The same holds for genus 0 if $n \geq 2$; in fact the data of a genus zero curve C with two points $0, \infty$ together with a tangent vector $v \neq 0$ in $T_0(C)$ has no non-identity automorphisms.

One can form a partial compactification of $\mathcal{M}_g^n$ by allowing *stable* curves with boundary components. As we will not require the full extent of this theory, we restrict ourselves to connected curves C with a single singularity, this being an ordinary double point p. We require that the boundary components are in the smooth locus of C. If C is reducible, then C has two irreducible components C_1, C_2; we also require that both C_1 and C_2 have at least one boundary component. As above, such data has no non-trivial automorphisms, which leads to the existence of a fine moduli space $\bar{\mathcal{M}}_g^n$. We let $\mathcal{C}_g^n \to \mathcal{M}_g^n$ be the universal curve with universal boundary components $(x_1, v_1), \ldots, (x_n, v_n)$, and $\bar{\mathcal{C}}_g^n \to \bar{M}_g^n$ the extended universal curve.
The boundary $\partial\bar{\mathcal{M}}_g^n := \bar{\mathcal{M}}_g^n \setminus \mathcal{M}_g^n$ is a disjoint union of divisors

$$\partial\bar{\mathcal{M}}_g^n := D_{(g,n)} \amalg \coprod_{(g_1,g_2),(n_1,n_2)} D_{(g_1,g_2),(n_1,n_2)},$$

where $D_{(g_1,g_2),(n_1,n_2)}$ consists of the curves $C_1 \cup C_2$ with $g(C_i) = g_i$, and with C_i having n_i boundary components (we specify which component is C_1 by requiring C_1 to contain the boundary component (x_1, v_1)) and $D_{(g,n)}$ is the locus of irreducible singular curves.
Let $(C, (x_1, v_1), \ldots)$ be a curve in $\partial\bar{\mathcal{M}}_g^n$ with singular point p. By standard deformation theory, it follows that $\partial\bar{\mathcal{M}}_g^n$ is a smooth divisor in $\bar{\mathcal{M}}_g^n$; let $N_{(g_1,g_2),(n_1,n_2)}$ denote the normal bundle of $D_{(g_1,g_2),(n_1,n_2)}$. Deformation theory gives a canonical identification of the fiber of the punctured normal bundle $N^0_{g_1,g_2,n_1,n_2} := N_{(g_1,g_2),(n_1,n_2)} \setminus 0$ at $(C, (x_1, v_1), \ldots)$ with $\mathbb{G}_m$-torsor of isomorphisms

$$\Lambda^2 T_{C,p} \cong k(p).$$

11.2. Algebraic gluing. We can now describe our algebraic construction of gluing curves. Fix integers $g_1, g_2,\ n_1, n_2 \geq 1$. We define the morphism

$$\bar{\mu} : \mathcal{M}_{g_1,n_1} \times \mathcal{M}_{g_2,n_2} \to D_{g_1,g_2,n_1-1,n_2-1}.$$

by gluing $(C_1, (x_1, v_1), \ldots, (x_{n_1}, v_{n_1}))$ and $(C_2, (y_1, w_1), \ldots, (y_{n_2}, w_{n_2}))$ along x_{n_1} and y_1, forming the curve $C := C_1 \cup C_2$ with boundary components $(x_1, v_1), \ldots, (x_{n_1-1}, v_{n_1-1})$, $(y_2, w_2), \ldots, (y_{n_2}, w_{n_2})$ and singular point p. We lift $\bar{\mu}$ to

$$\mu : \mathcal{M}_{g_1,n_1} \times \mathcal{M}_{g_2,n_2} \to N^0_{g_1,g_2,n_1,n_2}$$

using the isomorphism $\Lambda^2 T_{C,p} \to k(p)$ which sends $v_{n-1} \wedge w_1$ to 1 and the identification of $(N^0_{g_1,g_2,n_1,n_2})_{C_1 \cup C_2, \ldots}$ described above.
We now pass to the category $\mathcal{SH}_{\mathbb{A}^1}(k)$. Taking the infinite suspension, the map μ defines the map

$$\Sigma^\infty \mu : \Sigma^\infty \mathcal{M}_{g_1,n_1+} \wedge \Sigma^\infty \mathcal{M}_{g_2,n_2+} \to \Sigma^\infty N^0_{g_1,g_2,n_1,n_2+}.$$

Composing with our exponential map defined in §5 gives us our gluing map

$$\oplus : \Sigma^\infty \mathcal{M}_{g_1,n_1+} \wedge \Sigma^\infty \mathcal{M}_{g_2,n_2+} \to \Sigma^\infty \mathcal{M}_{g_1+g_2,n_1+n_2-2+}.$$

Remarks 11.2.1. (1) If one fixes a curve $\mathcal{E} := (E, (x_1, v_1), (x_2, v_2)) \in \mathcal{M}_{1,2}$, one can form the tower under $\mathcal{E}\oplus$

$$\ldots \to \Sigma^\infty \mathcal{M}_{g,n} \to \Sigma^\infty \mathcal{M}_{g+1,n} \to \ldots,$$

and form the homotopy colimit $\Sigma^\infty \mathcal{M}_{\infty,n}$. If E is an object of $\mathcal{SH}_{\mathbb{A}^1}(k)$, one thus has the E-cohomology $E^*(\mathcal{M}_{\infty,n})$. For intance, this gives a possible definition of stable motivic cohomology or algebraic K-theory of smooth curves. However, it is not at all clear if this limit is independent of the choice of $\mathcal{E}$. In the topological setting, one notes that the space $\mathcal{M}_{1,2}(\mathbb{C})$ is connected, so the limit cohomology, for example, is independent of the choice of $\mathcal{E}$. On the contrary, $\mathcal{M}_{1,2}(\mathbb{R})$ is not connected (the number of connected components in the real points of the curve corresponding to a real point of $\mathcal{M}_{1,2}$ splits $\mathcal{M}_{1,2}(\mathbb{R})$ into disconnected pieces), so even there, the choice of $\mathcal{E}$ plays a role. It is also not clear if $\mathcal{M}_{\infty,n}$ is independent of n (up to isomorphism in $\mathcal{SH}_{\mathbb{A}^1}(k)$).
(2) In the topological setting, the map $\oplus$ is the infinite suspension of a map

$$\phi : \mathcal{M}_{g_1,n_1}(\mathbb{C}) \times \mathcal{M}_{g_2,n_2}(\mathbb{C}) \to \mathcal{M}_{g_1+g_2,n_1+n_2-2}(\mathbb{C}),$$

making $\amalg_{g,n}\mathcal{M}_{g,n+2}(\mathbb{C})$ into a topological monoid; the group completion is homotopy equivalent to the plus construction on the stable moduli space $\lim_{g\to\infty} \mathcal{M}_{g,1}(\mathbb{C})$ formed as in (1). Letting $\mathcal{M}_\infty(\mathbb{C})^+$ denote this group completion, the group structure induces on $\Sigma^\infty\mathcal{M}_\infty(\mathbb{C})^+$ the structure of a Hopf algebra (this was pointed out to me by Fabian Morel), the co-algebra structure being the canonical one on a suspension spectrum. The functoriality of the exponential map $\exp^0$ as described in Remark 5.2.2 shows that the maps $\oplus$ make $\bigvee_{g,n} \Sigma^\infty \mathcal{M}_{g,n+2}$ into a biaglebra object in $\mathcal{SH}_{\mathbb{A}^1}(k)$. It is not clear if there is an analogous "Hopf algebra completion" of $\bigvee_{g,n} \Sigma^\infty \mathcal{M}_{g,n+2}$ in $\mathcal{SH}_{\mathbb{A}^1}(k)$. □

12. Tangential base-points

Since, by work of Østvar-Röndigs [35], motivic cohomology is represented in $\mathcal{SH}_{\mathbb{A}^1}(k)$, our methods are applicable to this theory. However, one can simplify the construction somewhat, since we are dealing with complexes of abelian groups rather than spectra. One can also achieve a refinement incorporating the multiplicative structure; this allows for a motivic definition of tangential base-points for the category of mixed Tate motives from the point of view of cycle algebras. Of course, the unipotent specialization functor of Ayoub [3], when restricted to the triangulated category of Tate motives in $DM(-)$ also gives tangential base-points for mixed Tate motives, but we hope our construction will be useful for applications of this operation.

12.1. Cubical complexes. If we work with presheaves of complexes rather than presheaves of spectra, we can replace all our simplicial constructions with cubical versions. This enables an easy extension to the setting of differential graded algebras (d.g.a.'s), or even graded-commutative d.g.a.'s (c.d.g.a.'s) if we work with complexes of $\mathbb{Q}$-vector spaces. We list the main results without proof here; the methods discussed in [26, §2.5] carry over without difficulty.

For a commutative ring R, we denote the model category of complexes of R-modules on the big Nisnevich site, $\mathbf{C}_R(\mathbf{Sm}/S_{\mathrm{Nis}})$ by $\mathbf{C}_{R,\mathrm{Nis}}(S)$ and the derived category by $\mathbf{D}_{R,\mathrm{Nis}}(S)$.
The *cubical category* **Cube** has objects $\underline{n}$, $n = 0, 1, \ldots$. **Cube** is a subcategory of the category of finite sets, with $\underline{n}$ standing for the set $\{0,1\}^n$, with morphisms making **Cube** the smallest subcategory of finite sets containing the following maps:

(1) all inclusions $s_{i,n,\epsilon} : \{0,1\}^n \to \{0,1\}^{n+1}$, $\epsilon \in \{0,1\}$, $i = 1, \ldots, n+1$, where $s_{i,n,\epsilon}$ is the inclusion inserting ϵ in the ith factor.
(2) all projections $p_{i,n} : \{0,1\}^n \to \{0,1\}^{n-1}$, $i = 1, \ldots, n$, where $p_{i,n}$ is the projection deleting the ith factor.
(3) all maps $q_{i,n} : \{0,1\}^n \to \{0,1\}^{n-1}$, $i = 1, \ldots, n-1$, $n \geq 2$, defined by

$$q_{i,n}(\epsilon_1, \ldots, \epsilon_n) := (\epsilon_1, \ldots, \epsilon_{i-1}, \delta, \epsilon_{i+2}, \ldots, \epsilon_n)$$

with

$$\delta := \begin{cases} 0 & \text{if } (\epsilon_i, \epsilon_{i+1}) = (0,0) \\ 1 & \text{else.} \end{cases}$$

A cubical object in a category $\mathcal{C}$ is a functor $\mathbf{Cube} \to \mathcal{C}$.
The basic cubical object in **Sch** is the sequence of n-cubes $\square^* : \mathbf{Cube} \to \mathbf{Sm}/k$. The operations of the projections $p_{i,n}$ and inclusions $s_{i,n}$ are the evident ones; $q_{i,n}$ acts by

$$q_{i,n}(x_1, \ldots, x_n) := (x_1, \ldots, x_{i-1}, 1 - (x_i - 1)(x_{i+1} - 1), x_{i+2}, \ldots, x_n).$$

Now let $P : \mathbf{Cube} \to \mathbf{Mod}_R$ be a cubical R-module. We have the *cubical realization* $|P|^c \in \mathbf{C}_R$ with

$$|P|^c_n := P(\underline{n}) / \sum_{i=1}^n p^*_{i,n}(P(\underline{n-1})).$$

The differential $d^c_n : |P|^c_n \to |P|^c_{n-1}$ is

$$d^c_n := \sum_{i=1}^n (-1)^i s^*_{i,1} - \sum_{i=1}^n (-1)^i s^*_{i,0}.$$

$| - |^c$ is clearly a functor from the R-linear category of cubical R-modules to $\mathbf{C}(R)$; in particular, if we apply $| - |^c$ to a complex of R-modules, we end up with a double complex. For a complex C, also write $|C|^c$ for the total complex of this double complex, letting the context make the meaning clear.

Example 12.1.1. For a presheaf of abelian groups P on $\mathbf{Sm}/k$, we have cubical presheaf $\mathcal{C}^c(P)$ with

$$\mathcal{C}^c(P)(Y) := P(Y \times \square^*).$$

Taking the cubical realization yields the *cubical Suslin complex* $C_*(P)^c$ with

$$C_*(P)^c(Y) := |\mathcal{C}^c(P)(Y)|^c.$$

$\square$

The symmetric group S_n acts on $C_n(P)^c$, we let $C_n(P)^c_{\mathrm{alt}}$ denote the subpresheaf of alternating sections. If P is a presheaf of $\mathbb{Q}$-vector spaces, $C_n(P)^c_{\mathrm{alt}}$ is a canonical summand of $C_n(P)^c$, with projection given by the idempotent $\mathrm{Alt}_n := \frac{1}{n!}\sum_g \mathrm{sgn}(g)g$; one checks that the $C_n(P)^c_{\mathrm{alt}}$ form a subcomplex of $C_*(P)^c$.
The main result on these constructions is

Proposition 12.1.2. *(1) There is a canonical homotopy equivalence of functors*

$$C_* \to C_*^c : \mathbf{C}(k) \to \mathbf{C}(k)$$

(2) If P is a complex of presheaves of $\mathbb{Q}$-vector spaces, the inclusion

$$C_*(P)^c_{\mathrm{alt}} \to C_*(P)^c$$

is a quasi-isomorphism

Sketch of proof; see [27, §5] *for details.* For (1), one uses the algebraic maps

$$\square^n \to \Delta^n$$

which collapse the faces $x_i = 1$ to the vertex $(0, \ldots, 0, 1)$ to get a map $C_* \to C_*^c$. The homotopy inverse is given by triangulating the $\square^n$. For (2) one checks that S_n acts by the sign representation on the homology sheaves of $C_*(P)^c$. The projections Alt_n define a map of complexes $\mathrm{Alt}_* : C_*(P)^c \to C_*(P)^c_{\mathrm{alt}}$ which thus gives the inverse in homology. $\square$

12.2. Cubical tubular neighborhoods. For a closed immersion $i : W \to X$ in $\mathbf{Sm}/k$, set $\hat{\square}^n_{X,W} := (\widehat{\square^n_X})^h_{\square^n_W}$, giving us the cubical pro-scheme

$$\hat{\square}^*_{X,W} : \mathbf{Cube} \to \mathbf{Pro\text{-}Sm}/k$$

We use the same notation for morphisms in the cubical setting as in the simplicial version, e.g., $\hat{i}_W : \square^*_W \to \hat{\square}^*_{X,W}$. We have as well the co-presheaf on W_{Zar}

$$\hat{\square}^n_{X,W_{\mathrm{Zar}}}(W \setminus F) := \hat{\square}^n_{X\setminus,W\setminus F}$$

and the cubical co-presheaf

$$\tau^{\hat{X}}_\epsilon(W)^c := \hat{\square}^*_{X,W_{\mathrm{Zar}}}.$$

Now let P be in $\mathbf{C}(k)$. We define $P(\tau^{\hat{X}}_\epsilon(W)^c)_*$ to be the complex of presheaves

$$P(\tau^{\hat{X}}_\epsilon(W)^c)_* := |P(\tau^{\hat{X}}_\epsilon(W)^c)|^c.$$

We also have the alternating subcomplex $P(\tau^{\hat{X}}_\epsilon(W)^c)^{\mathrm{alt}} \subset P(\tau^{\hat{X}}_\epsilon(W)^c)$.
We have as well the punctured tubular neighborhood in cubical form

$$\tau^{\hat{X}}_\epsilon(W)^{0c} := \tau^{\hat{X}}_\epsilon(W)^c \setminus \square^*_{W_{\mathrm{Zar}}}$$

on which we can evaluate P:

$$P(\tau^{\hat{X}}_\epsilon(W)^{0c})_* := |P(\tau^{\hat{X}}_\epsilon(W)^{0c})|^c.$$

Let $P(\tau^{\hat{X}}_\epsilon(W)^{0c})^{\mathrm{alt}} \subset P(\tau^{\hat{X}}_\epsilon(W)^{0c})$ be the alternating subcomplex.

We let $EM : \mathbf{C} \to \mathbf{Spt}$ be a choice of the Eilenberg-Maclane spectrum functor. Our main comparison result is

Theorem 12.2.1. *(1) Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$. For $P \in \mathbf{C}(k)$, there are natural isomorphisms in $\mathcal{SH}(W_{\mathrm{Zar}})$*

$$EM(P(\tau_\epsilon^{\hat{X}}(W)^c)) \cong EM(P)(\tau_\epsilon^{\hat{X}}(W))$$
$$EM(P(\tau_\epsilon^{\hat{X}}(W)^{0c})) \cong EM(P)(\tau_\epsilon^{\hat{X}}(W)^0)$$

(2) If P is a presheaf of complexes of $\mathbb{Q}$-vector spaces, then the inclusion

$$P(\tau_\epsilon^{\hat{X}}(W)^c)^{\mathrm{alt}} \to P(\tau_\epsilon^{\hat{X}}(W)^c)$$

is a quasi-isomorphism.

Proof. Define $P(\tau_\epsilon^{\hat{X}}(W))$ to be the total complex of the double complex associated to the simplicial complex $n \mapsto P(\tau_\epsilon^{\hat{X}}(W)^n)$. The homotopy equivalence used in Proposition 12.1.2(1) extends, by the functoriality of the Nisnevich neighborhood, to a homotopy equivalence

$$P(\tau_\epsilon^{\hat{X}}(W))^c \sim P(\tau_\epsilon^{\hat{X}}(W))$$

This yields a weak equivalence on the associated Eilenberg-Maclane spectra. Since the functor EM passes to the homotopy category, we have a canonical isomorphism

$$EM(P)(\tau_\epsilon^{\hat{X}}(W))) \cong EM(P(\tau_\epsilon^{\hat{X}}(W))).$$

Putting these isomorphisms together completes the proof of the first assertion for the tubular neighborhood. The proof for the punctured tubular neighborhood is essentially the same. The second assertion follows from Proposition 12.1.2(2). □

12.3. The motivic c.d.g.a. There are a number of different complexes which represent motivic cohomology; we will use the strictly functorial one of Friedlander-Suslin, $\mathbb{Z}_{FS}(q)$ (see the description in §10.2) reindexed as a cohomological complex:

$$\mathbb{Z}_{FS}(q)^n := \mathbb{Z}_{FS}(q)_{-n}.$$

We will use the cubical version $\mathbb{Z}_{FS}(q)^c$:

$$\mathbb{Z}_{FS}(q)^{c,n}(Y) := C_{2q-n}(z_{q.fin}(\mathbb{A}^q))^c(Y).$$

By Proposition 12.1.2, $\mathbb{Z}_{FS}(q)^c$ is quasi-isomorphic to $\mathbb{Z}_{FS}(q)$.
Passing to $\mathbb{Q}$-coefficients, we have the quasi-isomorphic alternating subcomplex $\mathbb{Q}_{FS}(q)^c_{\mathrm{alt}} \subset \mathbb{Q}_{FS}(q)^c$. We may also symmetrize with respect to the coordinates in the $\mathbb{A}^q$ in $z_{q.fin}(\mathbb{A}^q)$; it is shown in [26] that the inclusion

$$\mathbb{Q}_{FS}(q)^c_{\mathrm{alt,sym}} \subset \mathbb{Q}_{FS}(q)^c_{\mathrm{alt}}$$

is also a quasi-isomorphism.
The product map

$$z_{q.fin}(\mathbb{A}^q)(\square^n \times Y) \otimes z_{q.fin}(\mathbb{A}^{q'})(\square^{n'} \times Y) \to z_{q.fin}(\mathbb{A}^{q+q'})(\square^{n+n'} \times Y)$$

makes the graded complex

$$\tilde{\mathcal{N}}_{\mathbb{Z}} := \oplus_{q\geq 0}\mathbb{Z}_{FS}(q)^c$$

into a presheaf of Adams-graded d.g.a.'s on $\mathbf{Sm}/k$ (with Adams grading q). Passing to $\mathbb{Q}$-coefficients, and following the product with the alternating and symmetric projections makes

$$\mathcal{N} := \oplus_{q\geq 0}\mathbb{Q}_{FS}(q)^c_{\text{alt,sym}}$$

a presheaf of Adams-graded c.d.g.a.'s, the *motivic* c.d.g.a. on $\mathbf{Sm}/k$.
We let $\mathcal{N} \to \mathcal{N}^{\text{fib}}$ denote a fibrant model of $\mathcal{N}$ in the model category of (Adams-graded) c.d.g.a.'s on $\mathbf{Sm}/k$, where the weak equivalences are Adams-graded quasi-isomorphisms of c.d.g.a.'s for the Zariski topology.

Remarks 12.3.1. (1) Since $\mathcal{N}$ is strictly homotopy invariant [46, Theorem 4.2], $\mathcal{N}^{\text{fib}}$ is homotopy invariant.
(2) In case k admits resolution of singularities (i.e., char$k = 0$) the canonical map $\mathbb{Z}_{FS}(q) \to \mathbb{Z}_{FS}(q)^{\text{fib}}$ is a pointwise weak equivalence [46, Theorem 7.4]. Thus, in this case, we can use $\mathcal{N}$ instead of $\mathcal{N}^{\text{fib}}$. □

12.4. The specialization map. We consider the situation of a smooth curve C over our base-field k with a k-point x. We let $\mathcal{O}$ denote the local ring of x in C, K the quotient field of $\mathcal{O}$ and choose a uniformizing parameter t, which we view as giving a map

$$t : \operatorname{Spec}\mathcal{O} \to \mathbb{A}^1.$$

sending x to 0.
Letting $i_x : x \to \operatorname{Spec}\mathcal{O}$ be the inclusion, we have the restriction map

$$i_x^* : \mathcal{N}(\mathcal{O}) \to \mathcal{N}(k(x)),$$

which is a morphism of Adams-graded c.d.g.a.'s. In this section, we extend i_x^* to a map

$$sp_t : \mathcal{N}(K) \to \mathcal{N}(k(x))$$

in the homotopy category of Adams-graded c.d.g.a.'s over $\mathbb{Q}$ (denoted $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$). This is essentially our construction of the tubular neighborhood, where we use cubical constructions throughout to keep track of the multiplication.
First, if we apply $\mathcal{N}$ to $\square^* \times Y$ and take the alternating projection again, we have the presheaf of c.d.g.a.'s $\mathcal{N}(\square^*_{\text{alt}})$ and the quasi-isomorphism of presheaves of c.d.g.a.'s

$$\iota : \mathcal{N} \to \mathcal{N}(\square^*_{\text{alt}}).$$

Next, write $\hat{\square}^{m0}_{C,x}$ for $\hat{\square}^m_{C,x} \setminus \square^m_x$, and consider the cubical punctured tubular neighborhood $\mathbb{Z}_{FS}(q)^c(\tau^{\hat{C}}_\epsilon(x)^{0c})$. The product map

$$z_{q.fin}(\mathbb{A}^q)(\square^n \times \hat{\square}^{m0}_{C,x}) \otimes z_{q.fin}(\mathbb{A}^{q'})(\square^{n'} \times \hat{\square}^{m'0}_{C,x}) \to z_{q.fin}(\mathbb{A}^{q+q'})(\square^{n+n'} \times \hat{\square}^{m+m'0}_{C,x})$$

makes $\oplus_{q\geq 0}\mathbb{Z}_{FS}(q)^c(\tau_\epsilon^{\widehat{\mathrm{Spec}\,\mathcal{O}}}(x)^{0c})$ into an Adams-graded d.g.a.; taking the alternating projection in both the $\square^n$ and $\hat{\square}^{m0}_{C,x}$ variables, and the symmetric projection in $\mathbb{A}^q$ and applying the fibrant model gives a presheaf of Adams-graded c.d.g.a.'s, denoted $\mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\widehat{\mathrm{Spec}\,\mathcal{O}}}(x)^{0c}_{\mathrm{alt}})$.

Similarly, we perform this construction using the full tubular neighborhood, giving the presheaf $\mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{C}}(x)^c_{\mathrm{alt}})$, and the commutative diagram of Adams-graded c.d.g.a.'s:

$$\begin{tikzcd}
\mathcal{N}(k(x)) \arrow[d, "\iota"'] & \mathcal{N}(\mathcal{O}) \arrow[l, "i_x^*"'] \arrow[r, "\mathrm{res}"] \arrow[d, "\pi_{\mathcal{O}}^*"'] & \mathcal{N}(K) \arrow[d, "\pi_K^*"] \\
\mathcal{N}^{\mathrm{fib}}(\square^*_{\mathrm{alt}})(k(x)) & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{C}}(x)^c_{\mathrm{alt}}) \arrow[l, "i_x^*"] \arrow[r, "\mathrm{res}"'] & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{C}}(x)^{0c}_{\mathrm{alt}})
\end{tikzcd}$$

Replacing (C,x) with $(\mathbb{A}^1,0)$ and using $\mathbb{A}^1$ and $\mathbb{G}_m$ instead of $\mathrm{Spec}\,\mathcal{O}$ and $\mathrm{Spec}\,K$ yields the commutative diagram of Adams-graded c.d.g.a.'s

$$\begin{tikzcd}
\mathcal{N}(k(0)) \arrow[d, "\iota"'] & \mathcal{N}^{\mathrm{fib}}(\mathbb{A}^1) \arrow[l, "i_0^*"'] \arrow[r, "\mathrm{res}"] \arrow[d, "\pi_{\mathbb{A}^1}^*"'] & \mathcal{N}^{\mathrm{fib}}(\mathbb{G}_m) \arrow[d, "\pi_{\mathbb{G}_m}^*"] \\
\mathcal{N}^{\mathrm{fib}}(\square^*_{\mathrm{alt}})(k(x)) & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{\mathbb{A}^1}}(0)^c_{\mathrm{alt}}) \arrow[l, "i_0^*"] \arrow[r, "\mathrm{res}"'] & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{\mathbb{A}^1}}(0)^{0c}_{\mathrm{alt}}).
\end{tikzcd}$$

By Corollary 3.3.3 and Corollary 4.1.4, the maps $\pi_{\mathbb{A}^1}^*$ and $\pi_{\mathbb{G}_m}^*$ are quasi-isomorphisms of complexes, hence quasi-isomorphisms of Adams-graded c.d.g.a.'s. Since $\mathcal{N}^{\mathrm{fib}}$ is homotopy invariant, the maps ι are quasi-isomorphisms of Adams-graded c.d.g.a.'s.

Finally, the map t induces the commutative diagram of Adams-graded c.d.g.a.'s

$$\begin{tikzcd}
\mathcal{N}(k(x)) & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{C}}(x)^c_{\mathrm{alt}}) \arrow[l, "i_x^*"'] \arrow[r, "\mathrm{res}"] & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{C}}(x)^{0c}_{\mathrm{alt}}) \\
\mathcal{N}(k(0)) \arrow[u, "t^*"] & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{\mathbb{A}^1}}(x)^c_{\mathrm{alt}}) \arrow[l, "i_0^*"] \arrow[r, "\mathrm{res}"'] \arrow[u, "t^*"] & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{\mathbb{A}^1}}(x)^{0c}_{\mathrm{alt}}). \arrow[u, "t^*"']
\end{tikzcd}$$

Since $t:(C,x)\to(\mathbb{A}^1,0)$ is a Nisnevich neighborhood of 0 in $\mathbb{A}^1$, all three maps t^* are isomorphisms. Putting these diagrams together and inverting the quasi-isomorphisms ι, t^*, $\pi_{\mathbb{A}^1}^*$ and $\pi_{\mathbb{G}_m}^*$ yields the commutative diagram in $\mathcal{H}(\mathrm{c.d.g.a}_{\mathbb{Q}})$:

(12.4.1)
$$\begin{tikzcd}
\mathcal{N}(k(x)) & \mathcal{N}(\mathcal{O}) \arrow[l, "i_x^*"'] \arrow[r, "\mathrm{res}"] \arrow[d, "\phi_{\mathcal{O}}^*"'] & \mathcal{N}(K) \arrow[d, "\phi_K^*"] \\
\mathcal{N}(k(0)) \arrow[u, "t^*"] & \mathcal{N}^{\mathrm{fib}}(\mathbb{A}^1) \arrow[l, "i_0^*"] \arrow[r, "\mathrm{res}"'] & \mathcal{N}^{\mathrm{fib}}(\mathbb{G}_m)
\end{tikzcd}$$

DEFINITION 12.4.1. Let $i_1 : \operatorname{Spec} k \to \mathbb{G}_m$ be the inclusion. The map $sp_t : \mathcal{N}(K) \to \mathcal{N}(k(x))$ in $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$ is defined to be the composition

$$\mathcal{N}(K) \xrightarrow{\phi_K^*} \mathcal{N}^{\text{fib}}(\mathbb{G}_m) \xrightarrow{i_1^*} \mathcal{N}^{\text{fib}}(k) \cong \mathcal{N}(k) = \mathcal{N}(k(0)) \xrightarrow{t^*} \mathcal{N}(k(x)).$$

□

PROPOSITION 12.4.2. *The diagram in* $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$

$$\begin{tikzcd} \mathcal{N}(\mathcal{O}) \arrow[r, "\text{res}"] \arrow[dr, "i_x^*"'] & \mathcal{N}(K) \arrow[d, "sp_t"] \\ & \mathcal{N}(k(x)) \end{tikzcd}$$

commutes.

Proof. Since $\mathcal{N}^{\text{fib}}$ is homotopy invariant, the maps

$$i_0^*, i_1^* : \mathcal{N}^{\text{fib}}(\mathbb{A}^1) \to \mathcal{N}(k)$$

are equal in $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$. The proposition follows directly from this and a chase of the commutative diagrams defined above. □

Remark 12.4.3. In the situation we are considering, we already have the following diagram:

$$\mathcal{N}(K) \to \mathcal{N}(\lim_{t\to 0} \operatorname{Spec} K) \cong \mathcal{N}(k(0)).$$

However, the above diagram is only a diagram in the homotopy category of complexes of $\mathbb{Q}$-vector spaces, which is thus equivalent to the same diagram for cohomology of the complexes involved. We have gone to the trouble of redoing our theory using cubes throughout because we need to keep track of the multiplication, i.e. our construction lifts the above diagram in $D^b(\mathbb{Q})$ to one in $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$. □

12.5. THE SPECIALIZATION FUNCTOR. For a field k, we have the triangulated category $\text{DMT}(k)$ of *mixed Tate motives* over k, this being the full triangulated subcategory of Voevodsky's triangulated category of motives (with $\mathbb{Q}$-coefficients), $DM_{gm}(k)_{\mathbb{Q}}$, generated by the Tate objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$.
We will also use in this section the derived category of *finite cell modules* over an Adams-graded c.d.g.a. $\mathcal{A}$, $\text{DCM}(\mathcal{A})$. This construction was introduced in [23]; we refer the reader to the discussion in [26, §5] for the properties of DCM we will be using below.
Let $\mathcal{O}$ be as in the previous section the local ring of a k-point x on a smooth curve C over k, with quotient field K. The map $sp_t : \mathcal{N}(K) \to \mathcal{N}(k(x))$ yields an exact tensor functor

$$sp_t : \text{DMT}(K) \to \text{DMT}(k(x))$$

Indeed, as discussed in [26, §5.5], Spitzweck's representation theorem gives a natural equivalence of $\text{DMT}(k)$ with the derived category $\text{DCM}(\mathcal{N}(k))$ of finite cell modules over the Adams-graded c.d.g.a. $\mathcal{N}(k)$, as triangulated tensor $\mathbb{Q}$-tensor categories.

The functor DCM associating to an Adams-graded $\mathbb{Q}$-c.d.g.a. $\mathcal{A}$ the triangulated tensor category $\mathrm{DCM}(\mathcal{A})$ takes quasi-isomorphisms to triangulated tensor equivalences, hence DCM descends to a well-defined pseudo-functor on $\mathcal{H}(\mathrm{c.d.g.a}_{\mathbb{Q}})$. Thus, we may make the following

Definition 12.5.1. Let $\mathcal{O}$ be the local ring of a k-point x on a smooth curve C over k, with quotient field K and uniformizing parameter t. Let $sp_t : \mathrm{DMT}(K) \to \mathrm{DMT}(k(x))$ be the exact tensor functor induced by $\mathrm{DCM}(sp_t) : \mathrm{DCM}(\mathcal{N}(K)) \to \mathrm{DCM}(\mathcal{N}(k(x))$, using Spitzweck's representation theorem to identify the derived categories of cell modules with the appropriate category of mixed Tate motives. □

Remark 12.5.2. (1) The discussion in [26, §5.5], in particular, the statement and proof of Spitzweck's representation theorem, is in the setting of motives over a field. However, we now have available a reasonable triangulated category $DM(S)$ of motives over an arbitrary base-scheme S (see [48]), and we can thus define the triangulated category of mixed Tate motives over S, $\mathrm{DMT}(S)$, as in the case of a field.
Furthermore, if S is in $\mathbf{Sm}/k$ for k a field of characteristic zero, then $\mathcal{N}(S)$ has the correct cohomology, i.e.

$$H^n(\mathcal{N}(S)) = \oplus_{q\geq 0} H^n(S, \mathbb{Q}(q)),$$

and one has the isomorphism

$$H^n(S, \mathbb{Z}(q)) \cong \mathrm{Hom}_{DM(S)}(\mathbb{Z}, \mathbb{Z}(q)).$$

This is all that is required for the argument in [26, §5.5] to go through, yielding the equivalence of the triangulated tensor category of cell modules $\mathrm{DCM}(\mathcal{N}(S))$ with $\mathrm{DMT}(S)$.

(2) Joshua [20] has *defined* the triangulated category of $\mathbb{Q}$ mixed Tate motives over S as $\mathrm{DCM}(\mathcal{N}(S))$; the discussion in (1) shows that this agrees with the definition as a subcategory of $DM(S)_{\mathbb{Q}}$. □

With these remarks, we can now state the main compatibility property of the functor $sp_t : \mathrm{DMT}(K) \to \mathrm{DMT}(k(x))$.

Proposition 12.5.3. *Let $\mathcal{O}$ be the local ring of a k-point x on a smooth curve C over k, with quotient field K and uniformizing parameter t. Let $i_x^* : \mathrm{DMT}(\mathcal{O}) \to \mathrm{DMT}(k)$ and $j^* : \mathrm{DMT}(\mathcal{O}) \to \mathrm{DMT}(K)$ be the functors induced by the inclusions $i_x : \mathrm{Spec}\, k \to \mathrm{Spec}\, \mathcal{O}$ and $j : \mathrm{Spec}\, K \to \mathrm{Spec}\, \mathcal{O}$, respectively. Then the diagram*

$$\begin{tikzcd} \mathrm{DMT}(\mathcal{O}) \arrow[r, "j^*"] \arrow[dr, "i_x^*"'] & \mathrm{DMT}(K) \arrow[d, "sp_t"] \\ & \mathrm{DMT}(k(x)) \end{tikzcd}$$

commutes up to natural isomorphism.

Proof. This follows from Proposition 12.4.2 and the functoriality (up to natural isomorphism) of the equivalence $\mathrm{DCM}(\mathcal{N}(S)) \sim \mathrm{DMT}(S)$. $\square$

12.6. Compatibility with specialization on motivic cohomology. As above, let $\mathcal{O}$ be the local ring of a closed point x on a smooth curve C over k, with quotient field K and uniformizing parameter t. We have the localization sequence for motivic cohomology

$$\ldots \to H^n(\mathcal{O}, \mathbb{Z}(q)) \xrightarrow{j^*} H^n(K, \mathbb{Z}(q)) \xrightarrow{\partial} H^{n-1}(k(x), \mathbb{Z}(q-1)) \xrightarrow{i_{x*}} \ldots$$

In addition, the parameter t determines the element $[t] \in H^1(K, \mathbb{Z}(1))$. One defines the *specialization homomorphism*

$$\widetilde{\mathrm{sp}}_t : H^n(K, \mathbb{Z}(q)) \to H^n(k(x), \mathbb{Z}(q))$$

by the formula

$$\widetilde{\mathrm{sp}}_t(\alpha) := \partial([t] \cup \alpha).$$

On the other hand, if $k(x) = k$, we have the specialization functor

$$sp_t : \mathrm{DMT}(K) \to \mathrm{DMT}(k(x))$$

and the natural identifications

$$H^n(K, \mathbb{Q}(q)) \cong \mathrm{Hom}_{\mathrm{DMT}(K)}(\mathbb{Q}, \mathbb{Q}(q)[n])$$
$$H^n(k, \mathbb{Q}(q)) \cong \mathrm{Hom}_{\mathrm{DMT}(k)}(\mathbb{Q}, \mathbb{Q}(q)[n]).$$

Thus the functor sp_t induces the homomorphism

$$sp_t : \mathrm{Hom}_{\mathrm{DMT}(K)}(\mathbb{Q}, \mathbb{Q}(q)[n]) \to \mathrm{Hom}_{\mathrm{DMT}(k)}(\mathbb{Q}, \mathbb{Q}(q)[n])$$

and hence a new homomorphism

$$sp'_t : H^n(K, \mathbb{Q}(q)) \to H^n(k, \mathbb{Q}(q)).$$

Proposition 12.6.1. *sp'_t agrees with the $\mathbb{Q}$-extension of $\widetilde{\mathrm{sp}}_t$.*

Proof. Using the equivalence $\mathrm{DMT}(K) \sim \mathrm{DCM}(\mathcal{N}(K))$ and the canonical identifications

$$\mathrm{Hom}_{\mathrm{DCM}(K)}(\mathbb{Q}, \mathbb{Q}(q)[n]) \cong H^n(\mathcal{N}(K)) \cong \oplus_{q\geq 0} H^n(K, \mathbb{Q}(q))$$

(and similarly for k) we need to show that the $\mathbb{Q}$-linear extension of $\widetilde{\mathrm{sp}}_t$ agrees with the map

$$H^n(sp_t) : H^n(\mathcal{N}(K)) \to H^n(\mathcal{N}(k))$$

induced by $sp_t : \mathcal{N}(K) \to \mathcal{N}(k)$.

For this, take an element $\alpha \in H^n(K, \mathbb{Z}(q))$ and set

$$\bar{\beta} := \partial\alpha \in H^{n-1}(k, \mathbb{Z}(q-1)).$$

Since $i_x : x \to \mathrm{Spec}\,\mathcal{O}$ is split by the structure morphism $\pi : \mathrm{Spec}\,\mathcal{O} \to \mathrm{Spec}\,k$, we can lift $\bar{\beta}$ to $\beta : \pi^*(\bar{\beta}) \in H^{n-1}(\mathcal{O}, \mathbb{Z}(q-1))$. Then

$$\partial([t] \cup \beta) = \partial([t]) \cup i_x^*\beta = \bar{\beta},$$

the first identity following from the Leibniz rule and the second from the fact that $\partial([t]) = 1 \in H^0(k, \mathbb{Z}(0))$. Thus

$$\partial(\alpha - [t] \cup \beta) = 0,$$

hence there is a class $\gamma \in H^n(\mathcal{O}, \mathbb{Z}(q))$ with

$$j^*\gamma = \alpha - [t] \cup \beta.$$

We consider γ as an element of $H^n(\mathcal{N}(\mathcal{O}))$.
By Proposition 12.4.2, we have

$$H^n(i_x^*)(\gamma) = H^n(sp_t)(\alpha - [t] \cup \beta).$$

By the functoriality of the identification

$$H^n(\mathcal{N}(-)) \cong \oplus_{q\geq 0}\mathrm{Hom}_{\mathrm{DCM}}(\mathcal{N}(-))(\mathbb{Q}, \mathbb{Q}(q))$$

and Proposition 12.4.2 it follows that

$$\widetilde{\mathrm{sp}}_t(j^*\gamma) = H^n(i_x^*)(\gamma) = H^n(sp_t)(j^*\gamma)$$

so we reduce to showing

$$\widetilde{\mathrm{sp}}_t([t] \cup \beta) = 0 = H^n(sp_t)([t] \cup \beta).$$

The first identity follows from $[t] \cup [t] = 0$ in $H^2(K, \mathbb{Q}(2))$. For the second, because sp_t is a morphism in $\mathcal{H}(\text{c.d.g.a}_q)$, the map $H^*(sp_t)$ is multiplicative, hence it suffices to show that $H^1(sp_t)([t]) = 0$.
For this, it follows from the constuction of the map $sp_t : \mathcal{N}(K) \to \mathcal{N}(k(x))$ in $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$ that sp_t is natural with respect to Nisnevich neighborhoods $f : (C', x') \to (C, x)$ of x, i.e.,

$$sp_{f^*(t)} \circ f^* = sp_t.$$

Now, the map $t : (C, x) \to (\mathbb{A}^1, 0)$ is clearly a Nisnevich neighborhood of 0 (after shrinking C if necessary) and

$$[t] = t^*([T])$$

where $\mathbb{A}^1 = \operatorname{Spec} k[T]$. Thus, we may assume that $C = \mathbb{A}^1$ and $t = T$. But then $[T]$ is a well-defined element of $H^1(\mathcal{N}(\mathbb{G}_m))$ hence

$$H^1(sp_t)([T]) = i_1^*([T]) = [1] = 0$$

by definition of $sp_t : \mathcal{N}(\mathcal{O}_{\mathbb{A}^1,0}) \to \mathcal{N}(k)$. This completes the proof. □

Remark 12.6.2. Since sp'_t is multiplicative, as we have already remarked, Proposition 12.6.1 gives a rather long-winded re-proof of the multiplicativity of the specialization homomorphism $\widetilde{\mathrm{sp}}_t$ □

12.7. Tangential base-points. As shown in [29], the category $\mathrm{DMT}(k)$ carries a canonical exact *weight filtration*. For an Adams-graded c.d.g.a. $\mathcal{A}$, the derived category of cell modules $\mathrm{DCM}(\mathcal{A})$ carries a natural weight filtration as well; the equivalence $\mathrm{DCM}(\mathcal{N}(k)) \sim \mathrm{DMT}(k)$ given by Spitzweck's representation theorem is compatible with the weight filtrations [26, Theorem 5.24].
If $\mathcal{A}$ is cohomologically connected ($H^n(\mathcal{A}) = 0$ for $n < 0$ and $H^0(\mathcal{A}) = \mathbb{Q} \cdot \mathrm{id}$), then $\mathrm{DCM}(\mathcal{A})$ carries a t-structure, natural among cohomologically connected $\mathcal{A}$. Finally, if $\mathcal{A}$ is *1-minimal* then $\mathrm{DCM}(\mathcal{A})$ is equivalent to the derived category of the heart of this t-structure (see [26, §5]).
Thus, if $\mathcal{N}(F)$ is cohomologically connected, then $\mathrm{DMT}(F)$ has a t-structure; the heart is called the category of mixed Tate motives over F, denoted $\mathrm{MT}(F)$. In fact, $\mathrm{MT}(F)$ is a Tannakian category , with natural fiber functor given by the weight filtration; let $\mathrm{Gal}_\mu(F)$ denote the pro-algebraic group scheme over $\mathbb{Q}$ associated to $\mathrm{MT}(F)$ by the Tannakian formalism. If $\mathcal{N}(F)$ is 1-minimal, then $\mathrm{DMT}(F)$ is equivalent to $D^b(\mathrm{MT}(F))$, but we won't be using this.
Now let x be a k-point on a smooth curve C over k, and t a parameter in $\mathcal{O}_{C,x}$. The specialization functor

$$sp_t : \mathrm{DMT}(k(C)) \to \mathrm{DMT}(k(x))$$

arises from the map $sp_t : \mathcal{N}(k(C)) \to \mathcal{N}(k(x))$ in $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$, hence sp_t is compatible with the weight filtrations. When $\mathcal{N}(k(C))$ and $\mathcal{N}(k(x))$ are cohomologically connected, sp_t is compatible with the t-structures, hence induces an exact functor of Tannakian categories

$$sp_t : \mathrm{MT}(k(C)) \to \mathrm{MT}(k(x))$$

compatible with the fiber functors gr^W. By Tannakian duality, sp_t is equivalent to a homomorphism

$$\frac{\partial}{\partial t}_* : \mathrm{Gal}_\mu(k(x)) \to \mathrm{Gal}_\mu(k(C)),$$

which is the *tangential base-point* associated to the parameter t. This gives a purely "motivic" construction of the tangential base-point construction of Deligne-Goncharov [10]; the construction in [10] relies on realization functors.

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