The Supersingular Loci and Mass Formulas on Siegel Modular Varieties

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Abstract. We describe the supersingular locus of the Siegel 3-fold with a parahoric level structure. We also study its higher dimensional generalization. Using this correspondence and a deep result of Li and Oort, we evaluate the number of irreducible components of the supersingular locus of the Siegel moduli space $\mathcal{A}_{g,1,N}$ for arbitrary g.

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1. Introduction

In this paper we discuss some extensions of works of Katsura and Oort [5], and of Li and Oort [8] on the supersingular locus of a mod p Siegel modular variety. Let p be a rational prime number, $N \geq 3$ a prime-to-p positive integer. We choose a primitive N-th root of unity ζ_N in $\overline{\mathbb{Q}} \subset \mathbb{C}$ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\mathcal{A}_{g,1,N}$ be the moduli space over $\mathbb{Z}_{(p)}[\zeta_N]$ of g-dimensional principally polarized abelian varieties (A, λ, η) with a symplectic level-N structure (See Subsection 2.1).

Let $\mathcal{A}_{2,1,N,(p)}$ be the cover of $\mathcal{A}_{2,1,N}$ which parametrizes isomorphism classes of objects (A, λ, η, H), where (A, λ, η) is an object in $\mathcal{A}_{2,1,N}$ and $H \subset A[p]$ is a finite flat subgroup scheme of rank p. It is known that the moduli scheme $\mathcal{A}_{2,1,N,(p)}$ has semi-stable reduction and the reduction $\mathcal{A}_{2,1,N,(p)} \otimes \overline{\mathbb{F}}_p$ modulo p has two irreducible components. Let $\mathcal{S}_{2,1,N,(p)}$ (resp. $\mathcal{S}_{2,1,N}$) denote the supersingular locus of the moduli space $\mathcal{A}_{2,1,N,(p)} \otimes \overline{\mathbb{F}}_p$ (resp. $\mathcal{A}_{2,1,N} \otimes \overline{\mathbb{F}}_p$). Recall

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that an abelian variety A in characteristic p is called *supersingular* if it is isogenous to a product of supersingular elliptic curves over an algebraically closed field k; it is called *superspecial* if it is isomorphic to a product of supersingular elliptic curves over k.
The supersingular locus $\mathcal{S}_{2,1,N}$ of the Siegel 3-fold is studied in Katsura and Oort [5]. We summarize the main results for $\mathcal{S}_{2,1,N}$ (the local results obtained earlier in Koblitz [7]):

THEOREM 1.1.
(1) *The scheme $\mathcal{S}_{2,1,N}$ is equi-dimensional and each irreducible component is isomorphic to $\mathbf{P}^1$.*
(2) *The scheme $\mathcal{S}_{2,1,N}$ has*

$$|\operatorname{Sp}_4(\mathbb{Z}/N\mathbb{Z})|\frac{(p^2-1)}{5760} \tag{1.1}$$

irreducible components.
(3) *The singular points of $\mathcal{S}_{2,1,N}$ are exactly the superspecial points and there are*

$$|\operatorname{Sp}_4(\mathbb{Z}/N\mathbb{Z})|\frac{(p-1)(p^2+1)}{5760} \tag{1.2}$$

of them. Moreover, at each singular point there are $p+1$ irreducible components passing through and intersecting transversely.

PROOF. See Koblitz [7, p.193] and Katsura-Oort [5, Section 2, Theorem 5.1, Theorem 5.3].

In this paper we extend their results to $\mathcal{S}_{2,1,N,(p)}$. We show

THEOREM 1.2.
(1) *The scheme $\mathcal{S}_{2,1,N,(p)}$ is equi-dimensional and each irreducible component is isomorphic to $\mathbf{P}^1$.*
(2) *The scheme $\mathcal{S}_{2,1,N,(p)}$ has*

$$|\operatorname{Sp}_4(\mathbb{Z}/N\mathbb{Z})|\cdot\frac{(-1)\zeta(-1)\zeta(-3)}{4}\left[(p^2-1)+(p-1)(p^2+1)\right] \tag{1.3}$$

irreducible components, where $\zeta(s)$ is the Riemann zeta function.
(3) *The scheme $\mathcal{S}_{2,1,N,(p)}$ has only ordinary double singular points and there are*

$$|\operatorname{Sp}_4(\mathbb{Z}/N\mathbb{Z})|\cdot\frac{(-1)\zeta(-1)\zeta(-3)}{4}(p-1)(p^2+1)(p+1) \tag{1.4}$$

of them.
(4) *The natural morphism $\mathcal{S}_{2,1,N,(p)}\to\mathcal{S}_{2,1,N}$ contracts*

$$|\operatorname{Sp}_4(\mathbb{Z}/N\mathbb{Z})|\cdot\frac{(-1)\zeta(-1)\zeta(-3)}{4}(p-1)(p^2+1) \tag{1.5}$$

projective lines onto the superspecial points of $\mathcal{S}_{2,1,N}$.

Remark 1.3. (1) By the basic fact that

$$\zeta(-1) = \frac{-1}{12} \quad \text{and} \quad \zeta(-3) = \frac{1}{120},$$

the number (1.5) (of the vertical components) equals the number (1.2) (of superspecial points), and the number (1.3) (sum of vertical and horizontal components) equals the sum of the numbers (1.1) (of irreducible components) and (1.2) (of superspecial points). Thus, the set of horizontal irreducible components of $\mathcal{S}_{2,1,N,(p)}$ is in bijection with the set of irreducible components of $\mathcal{S}_{2,1,N}$

(2) Theorem 1.2 (4) says that $\mathcal{S}_{2,1,N,(p)}$ is a "desingularization" or a "blow-up" of $\mathcal{S}_{2,1,N}$ at the singular points. Strictly speaking, the desingularization of $\mathcal{S}_{2,1,N}$ is its normalization, which is the (disjoint) union of horizontal components of $\mathcal{S}_{2,1,N,(p)}$. The vertical components of $\mathcal{S}_{2,1,N,(p)}$ should be the exceptional divisors of the blowing up of a suitable ambient surface of $\mathcal{S}_{2,1,N}$ at superspecial points.

In the proof of Theorem 1.2 (Section 4) we see that

- the set of certain superspecial points (the set Λ in Subsection 4.1) in $\mathcal{S}_{2,p,N}$ (classifying degree-p^2 polarized supersingular abelian surfaces) is in bijection with the set of irreducible components of $\mathcal{S}_{2,1,N}$, and
- the set of superspecial points in $\mathcal{S}_{2,1,N}$ is in bijection with the set of irreducible components of $\mathcal{S}_{2,p,N}$, furthermore
- the supersingular locus $\mathcal{S}_{2,1,N,(p)}$ provides the explicit link of this duality as a correspondence that performs simply through the "blowing-ups" and "blowing-downs".

In the second part of this paper we attempt to generalize a similar picture to higher (even) dimensions.

Let $g = 2D$ be an even positive integer. Let $\mathcal{H}$ be the moduli space over $\mathbb{Z}_{(p)}[\zeta_N]$ which parametrizes equivalence classes of objects $(\varphi : \underline{A}_1 \to \underline{A}_2)$, where

- $\underline{A}_1 = (A_1, \lambda_1, \eta_1)$ is an object in $\mathcal{A}_{g,1,N}$,
- $\underline{A}_2 = (A_2, \lambda_2, \eta_2)$ is an object in $\mathcal{A}_{g,p^D,N}$, and
- $\varphi : A_1 \to A_2$ is an isogeny of degree p^D satisfying $\varphi^*\lambda_2 = p\lambda_1$ and $\varphi_*\eta_1 = \eta_2$.

The moduli space $\mathcal{H}$ with natural projections gives the following correspondence:

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathcal{A}_{g,1,N} & & \mathcal{A}_{g,p^D,N}. \end{tikzcd}$$

Let $\mathcal{S}$ be the supersingular locus of $\mathcal{H} \otimes \overline{\mathbb{F}}_p$, which is the reduced closed subscheme consisting of supersingular points (either A_1 or A_2 is supersingular, or equivalently both are so).

In the special case where $g=2$, $\mathcal{H}$ is isomorphic to $\mathcal{A}_{2,1,N,(p)}$, and $\mathcal{S} \simeq \mathcal{S}_{2,1,N,(p)}$ under this isomorphism (See Subsection 4.5).

As the second main result of this paper, we obtain

Theorem 1.4. *Let C be the number of irreducible components of $\mathcal{S}_{g,1,N}$. Then*

$$C = |\operatorname{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})| \cdot \frac{(-1)^{g(g+1)/2}}{2^g} \left\{ \prod_{i=1}^{g} \zeta(1-2i) \right\} \cdot L_p\,,$$

where

$$L_p = \begin{cases} \prod_{i=1}^{g} \left\{(p^i + (-1)^i\right\} & \text{if } g \text{ is odd;} \\ \prod_{i=1}^{D} (p^{4i-2}-1) & \text{if } g = 2D \text{ is even.} \end{cases}$$

In the special case where $g=2$, Theorem 1.4 recovers Theorem 1.1 (2).

We give the idea of the proof. Let $\Lambda_{g,1,N}$ denote the set of superspecial (geometric) points in $\mathcal{A}_{g,1,N} \otimes \overline{\mathbb{F}}_p$. For $g=2D$ is even, let $\Lambda^*_{g,p^D,N}$ denote the set of superspecial (geometric) points (A,λ,η) in $\mathcal{A}_{g,p^D,N} \otimes \overline{\mathbb{F}}_p$ satisfying $\ker \lambda = A[F]$, where $F : A \to A^{(p)}$ is the relative Frobenius morphism on A. These are finite sets and each member is defined over $\overline{\mathbb{F}}_p$. By a result of Li and Oort [8] (also see Section 5), we know

$$C = \begin{cases} |\Lambda_{g,1,N}| & \text{if } g \text{ is odd;} \\ |\Lambda^*_{g,p^D,N}| & \text{if } g \text{ is even.} \end{cases}$$

One can use the geometric mass formula due to Ekedahl [2] and some others (see Section 3) to compute $|\Lambda_{g,1,N}|$. Therefore, it remains to compute $|\Lambda^*_{g,p^D,N}|$ when g is even. We restrict the correspondence $\mathcal{S}$ to the product $\Lambda_{g,1,N} \times \Lambda^*_{g,p^D,N}$ of superspecial points, and compute certain special points in $\mathcal{S}$. This gives us relation between $\Lambda^*_{g,p^D,N}$ and $\Lambda_{g,1,N}$. See Section 6 for details.

Theorem 1.4 tells us how the number $C = C(g,N,p)$ varies when p varies. For another application, one can use this result to compute the dimension of the space of Siegel cusp forms of certain level at p by the expected Jacquet-Langlands correspondence for symplectic groups. As far as the author knows, the latter for general g is not available yet in the literature.

The paper is organized as follows. In Section 2, we recall the basic definitions and properties of the Siegel moduli spaces and supersingular abelian varieties. In Section 3, we state the mass formula for superspecial principally polarized abelian varieties due to Ekedahl (and some others). The proof of Theorem 1.2 is given in Section 4. In Section 5, we describe the results of Li and Oort on irreducible components of the supersingular locus. In Section 6, we introduce a correspondence and use this to evaluate the number of irreducible components of the supersingular locus.

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2. Notation and preliminaries

2.1. Throughout this paper we fix a rational prime p and a prime-to-p positive integer $N \geq 3$. Let d be a positive integer with $(d, N) = 1$. We choose a primitive N-th root of unity ζ_N in $\overline{\mathbb{Q}} \subset \mathbb{C}$ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. The element ζ_N gives rise to a trivialization $\mathbb{Z}/N\mathbb{Z} \simeq \mu_N$ over any $\mathbb{Z}_{(p)}[\zeta_N]$-scheme. For a polarized abelian variety (A, λ) of degree d^2, a full symplectic level-N structure with respect to the choice ζ_N is an isomorphism

$$\eta : (\mathbb{Z}/N\mathbb{Z})^{2g} \simeq A[N]$$

such that the following diagram commutes

$$\begin{array}{ccc} (\mathbb{Z}/N\mathbb{Z})^{2g} \times (\mathbb{Z}/N\mathbb{Z})^{2g} & \xrightarrow{(\eta,\eta)} & A[N] \times A[N] \\ {\scriptstyle \langle\, ,\, \rangle}\downarrow & & {\scriptstyle e_\lambda}\downarrow \\ \mathbb{Z}/N\mathbb{Z} & \xrightarrow{\zeta_N} & \mu_N, \end{array}$$

where $\langle\, ,\, \rangle$ is the standard non-degenerate alternating form on $(\mathbb{Z}/N\mathbb{Z})^{2g}$ and e_λ is the Weil pairing induced by the polarization λ.
Let $\mathcal{A}_{g,d,N}$ denote the moduli space over $\mathbb{Z}_{(p)}[\zeta_N]$ of g-dimensional polarized abelian varieties (A, λ, η) of degree d^2 with a full symplectic level N structure with respect to ζ_N. Let $\mathcal{S}_{g,d,N}$ denote the supersingular locus of the reduction $\mathcal{A}_{g,d,N} \otimes \overline{\mathbb{F}}_p$ modulo p, which is the closed reduced subscheme of $\mathcal{A}_{g,d,N} \otimes \overline{\mathbb{F}}_p$ consisting of supersingular points in $\mathcal{A}_{g,d,N} \otimes \overline{\mathbb{F}}_p$. Let $\Lambda_{g,d,N}$ denote the set of superspecial (geometric) points in $\mathcal{S}_{g,d,N}$; this is a finite set and every member is defined over $\overline{\mathbb{F}}_p$.

For a scheme X of finite type over a field K, we denote by $\Pi_0(X)$ the set of geometrically irreducible components of X.
Let k be an algebraically closed field of characteristic p.

2.2. Over a ground field K of characteristic p, denote by α_p the finite group scheme of rank p that is the kernel of the Frobenius endomorphism from the additive group $\mathbb{G}_{\mathrm{a}}$ to itself. One has

$$\alpha_p = \mathrm{Spec} K[X]/X^p, \quad m(X) = X \otimes 1 + 1 \otimes X,$$

where m is the group law.
By definition, an elliptic curve E over K is called *supersingular* if $E[p](\overline{K}) = 0$. An abelian variety A over K is called *supersingular* if it is isogenous to a product of supersingular elliptic curves over $\overline{K}$; A is called *superspecial* if it is isomorphic to a product of supersingular elliptic curves over $\overline{K}$.
For any abelian variety A over K where K is perfect, the *a-number* of A is defined by

$$a(A) := \dim_K \mathrm{Hom}(\alpha_p, A).$$

The following interesting results are well-known; see Subsection 1.6 of [8] for a detail discussion.

THEOREM 2.1 (Oort). *If $a(A) = g$, then A is superspecial.*

THEOREM 2.2 (Deligne, Ogus, Shioda). *For $g \geq 2$, there is only one g-dimensional superspecial abelian variety up to isomorphism over k.*

3. THE MASS FORMULA

Let Λ_g denote the set of isomorphism classes of g-dimensional principally polarized superspecial abelian varieties over $\overline{\mathbb{F}}_p$. Write

$$M_g := \sum_{(A,\lambda)\in\Lambda_g} \frac{1}{|\operatorname{Aut}(A,\lambda)|}$$

for the mass attached to Λ_g. The following mass formula is due to Ekedahl [2, p.159] and Hashimoto-Ibukiyama [3, Proposition 9].

THEOREM 3.1. *Notation as above. One has*

$$M_g = \frac{(-1)^{g(g+1)/2}}{2^g} \left\{ \prod_{k=1}^{g} \zeta(1-2k) \right\} \cdot \prod_{k=1}^{g} \left\{ (p^k + (-1)^k \right\}. \tag{3.1}$$

Similarly, we set

$$M_{g,1.N} := \sum_{(A,\lambda,\eta)\in\Lambda_{g,1,N}} \frac{1}{|\operatorname{Aut}(A,\lambda,\eta)|}.$$

LEMMA 3.2. *We have $M_{g,1,N} = |\Lambda_{g,1,N}| = |\operatorname{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})| \cdot M_g$.*

PROOF. The first equality follows from a basic fact that (A,λ,η) has no non-trivial automorphism. The proof of the second equality is elementary; see Subsection 4.6 of [11].

COROLLARY 3.3. *One has*

$$|\Lambda_{2,1,N}| = |\operatorname{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \cdot \frac{(-1)\zeta(-1)\zeta(-3)}{4}(p-1)(p^2+1).$$

4. PROOF OF THEOREM 1.2

4.1. In this section we consider the case where $g = 2$. Let

$$\Lambda := \{(A,\lambda,\eta) \in \mathcal{S}_{2,p,N};\ \ker\lambda \simeq \alpha_p \times \alpha_p\}.$$

Note that every member $\underline{A}$ of Λ is superspecial (because $A \supset \alpha_p \times \alpha_p$), that is, $\Lambda \subset \Lambda_{2,p,N}$. For a point ξ in Λ, consider the space S_ξ which parametrizes the isogenies of degree p

$$\varphi : (A_\xi, \lambda_\xi, \eta_\xi) \to \underline{A} = (A,\lambda,\eta)$$

which preserve the polarizations and level structures. Let

$$\psi_\xi : S_\xi \to \mathcal{S}_{2,1,N}$$

be the morphism which sends $(\varphi : \xi \to \underline{A})$ to $\underline{A}$. Let $V_\xi \subset \mathcal{S}_{2,1,N}$ be the image of S_ξ under ψ_ξ.
The following theorem is due to Katsura and Oort [5, Theorem 2.1 and Theorem 5.1]:

Theorem 4.1 (Katsura-Oort). *Notation as above. The map $\xi \mapsto V_\xi$ gives rise to a bijection $\Lambda \simeq \Pi_0(\mathcal{S}_{2,1,N})$ and one has*

$$|\Lambda| = |\operatorname{Sp}_4(\mathbb{Z}/N\mathbb{Z})|(p^2-1)/5760.$$

We will give a different way of evaluating $|\Lambda|$ that is based on the geometric mass formula (see Corollary 4.6).

4.2. Dually we can consider the space S'_ξ for each $\xi \in \Lambda$ that parametrizes the isogenies of degree p

$$\varphi' : \underline{A} = (A, \lambda, \eta) \to \xi = (A_\xi, \lambda_\xi, \eta_\xi),$$

with $\underline{A} \in \mathcal{A}_{2,1,N} \otimes \overline{\mathbb{F}}_p$, such that $\varphi'_*\eta = \eta_\xi$ and $\varphi'^*\lambda_\xi = p\,\lambda$. Let

$$\psi'_\xi : S'_\xi \to \mathcal{S}_{2,1,N}$$

be the morphism which sends $(\varphi' : \underline{A} \to \xi)$ to $\underline{A}$. Let $V'_\xi \subset \mathcal{S}_{2,1,N}$ be the image of S'_ξ under ψ'_ξ.
For a degree p isogeny $(\varphi : \underline{A}_1 \to \underline{A}_2)$ with $\underline{A}_2$ in $\mathcal{A}_{2,1,N}$, $\varphi^*\lambda_2 = \lambda_1$ and $\varphi_*\eta_1 = \eta_2$, we define

$$(\varphi : \underline{A}_1 \to \underline{A}_2)^* = (\varphi' : \underline{A}'_2 \to \underline{A}'_1),$$

where $\varphi' = \varphi^t$ and

$$\begin{aligned} \underline{A}'_2 &= (A_2^t, \lambda_2^{-1}, \lambda_2 \circ \eta_2), \\ \underline{A}'_1 &= (A_1^t, p\,\lambda_1^{-1}, \lambda_1 \circ \eta_1). \end{aligned}$$

Note that $\varphi'_*\eta'_2 = \eta'_1$ as we have the commutative diagram:

$$\begin{array}{ccccc} (\mathbb{Z}/N\mathbb{Z})^4 & \xrightarrow{\eta_2} & A_2[N] & \xrightarrow{\lambda_2} & A_2^t[N] \\ \big\downarrow{=} & & \big\uparrow{\varphi} & & \big\downarrow{\varphi^t} \\ (\mathbb{Z}/N\mathbb{Z})^4 & \xrightarrow{\eta_1} & A_1[N] & \xrightarrow{\lambda_1} & A_1^t[N]. \end{array}$$

If $\underline{A}_1 \in \Lambda$, then $\underline{A}'_1$ is also in Λ. Therefore, the map $\xi \mapsto V'_\xi$ also gives rise to a bijection $\Lambda \simeq \Pi_0(\mathcal{S}_{2,1,N})$.

4.3. We use the classical contravariant Dieudonné theory. We refer the reader to Demazure [1] for a basic account of this theory. Let K be a perfect field of characteristic p, $W := W(K)$ the ring of Witt vectors over K, $B(K)$ the fraction field of $W(K)$. Let σ be the Frobenius map on $B(K)$. A quasi-polarization on a Dieudonné module M here is a non-degenerate (meaning of non-zero discriminant) alternating pairing

$$\langle\,,\rangle : M \times M \to B(K),$$

such that $\langle Fx, y\rangle = \langle x, Vy\rangle^\sigma$ for $x, y \in M$ and $\langle M^t, M^t\rangle \subset W$. Here we regard the dual M^t of M as a Dieudonné submodule in $M \otimes B(K)$ using the pairing. A

quasi-polarization is called *separable* if $M^t = M$. Any polarized abelian variety (A,λ) over K naturally gives rise to a quasi-polarized Dieudonné module. The induced quasi-polarization is separable if and only if $(p, \deg\lambda) = 1$.
Recall (Subsection 2.1) that k denotes an algebraically closed field of characteristic p.

Lemma 4.2.
(1) *Let M be a separably quasi-polarized superspecial Dieudonné module over k of rank 4. Then there exists a basis f_1, f_2, f_3, f_4 for M over $W := W(k)$ such that*

$$Ff_1 = f_3, Ff_3 = pf_1, \quad Ff_2 = f_4, Ff_4 = pf_2$$

and the non-zero pairings are

$$\langle f_1, f_3\rangle = -\langle f_3, f_1\rangle = \beta_1, \quad \langle f_2, f_4\rangle = -\langle f_4, f_2\rangle = \beta_1,$$

where $\beta_1 \in W(\mathbb{F}_{p^2})^\times$ with $\beta_1^\sigma = -\beta_1$.
(2) *Let ξ be a point in Λ, and let M_ξ be the Dieudonné module of ξ. Then there is a W-basis e_1, e_2, e_3, e_4 for M_ξ such that*

$$Fe_1 = e_3, \quad Fe_2 = e_4, \quad Fe_3 = pe_1, \quad Fe_4 = pe_2,$$

and the non-zero pairings are

$$\langle e_1, e_2\rangle = -\langle e_2, e_1\rangle = \frac{1}{p}, \quad \langle e_3, e_4\rangle = -\langle e_4, e_3\rangle = 1.$$

Proof. (1) This is a special case of Proposition 6.1 of [8].
(2) By Proposition 6.1 of [8], $(M_\xi, \langle\,,\rangle)$ either is indecomposable or decomposes into a product of two quasi-polarized supersingular Dieudonné modules of rank 2. In the indecomposable case, one can choose such a basis e_i for M_ξ. Hence it remains to show that $(M_\xi, \langle\,,\rangle)$ is indecomposable. Let $(A_\xi[p^\infty], \lambda_\xi)$ be the associated polarized p-divisible group. Suppose it decomposes into $(H_1, \lambda_1) \times (H_2, \lambda_2)$. Then the kernel of λ is isomorphic to $E[p]$ for a supersingular elliptic curve E. Since $E[p]$ is a nontrivial extension of α_p by α_p, one gets contradiction. This completes the proof. ∎

4.4. Let (A_0, λ_0) be a superspecial principally polarized abelian surface and $(M_0, \langle\,,\rangle_0)$ be the associated Dieudonné module. Let $\varphi' : (A_0, \lambda_0) \to (A, \lambda)$ be an isogeny of degree p with $\varphi'^*\lambda = p\,\lambda_0$. Write $(M, \langle\,,\rangle)$ for the Dieudonné module of (A, λ). Choose a basis f_1, f_2, f_3, f_4 for M_0 as in Lemma 4.2. We have the inclusions

$$(F,V)M_0 \subset M \subset M_0.$$

Modulo $(F,V)M_0$, a module M corresponds a one-dimensional subspace $\overline{M}$ in $\overline{M}_0 := M_0/(F,V)M_0$. As $\overline{M}_0 = k < f_1, f_2 >$, $\overline{M}$ is of the form

$$\overline{M} = k < af_1 + bf_2 >, \quad [a:b] \in \mathbf{P}^1(k).$$

The following result is due to Moret-Bailly [9, p.138-9]. We include a proof for the reader's convenience.

Lemma 4.3. *Notation as above,* $\ker\lambda \simeq \alpha_p\times\alpha_p$ *if and only if the corresponding point* $[a:b]$ *satisfies* $a^{p+1}+b^{p+1}=0$. *Consequently, there are* $p+1$ *isogenies* φ' *so that* $\ker\lambda\simeq\alpha_p\times\alpha_p$.

Proof. As $\varphi'^*\lambda = p\lambda_0$, we have $\langle\,,\rangle = \frac{1}{p}\langle\,,\rangle_0$. The Dieudonné module $M(\ker\lambda)$ of the subgroup $\ker\lambda$ is equal to M/M^t. Hence the condition $\ker\lambda\simeq\alpha_p\times\alpha_p$ is equivalent to that F and V vanish on $M(\ker\lambda)=M/M^t$.
Since $\langle\,,\rangle$ is a perfect pairing on FM_0, that is, $(FM_0)^t = FM_0$, we have

$$pM_0\subset M^t\subset FM_0\subset M\subset M_0.$$

Changing the notation, put $\overline{M}_0 := M_0/pM_0$ and let

$$\langle\,,\rangle_0 : \overline{M}_0\times\overline{M}_0\to k.$$

be the induced perfect pairing. In $\overline{M}_0$, the subspace $\overline{M^t}$ is equal to $\overline{M}^{\perp}$. Indeed,

$$\begin{aligned} M^t &= \{m\in M_0; \langle m,x\rangle_0\in pW\ \forall\, x\in M\},\\ \overline{M^t} &= \{m\in\overline{M}_0; \langle m,x\rangle_0 = 0\ \forall\, x\in\overline{M}\} = \overline{M}^{\perp}. \end{aligned} \tag{4.1}$$

From this we see that the condition $\ker\lambda\simeq\alpha_p\times\alpha_p$ is equivalent to $\langle\overline{M},F\overline{M}\rangle = \langle\overline{M},V\overline{M}\rangle = 0$. Since $\overline{FM_0} = k < f_3, f_4 >$, one has $\overline{M} = k < f_1', f_3, f_4 >$ where $f_1' = af_1 + bf_2$. The condition $\langle\overline{M},F\overline{M}\rangle = \langle\overline{M},V\overline{M}\rangle = 0$, same as $\langle f_1', Ff_1'\rangle = \langle f_1', Vf_1'\rangle = 0$, gives the equation $a^{p+1}+b^{p+1}=0$. This completes the proof. ∎

Conversely, fix a polarized superspecial abelian surface (A,λ) such that $\ker\lambda\simeq\alpha_p\times\alpha_p$. Then there are p^2+1 degree-p isogenies $\varphi' : (A_0,\lambda_0)\to(A,\lambda)$ such that A_0 is superspecial and $\varphi'^*\lambda = p\,\lambda_0$. Indeed, each isogeny φ' always has the property $\varphi'^*\lambda = p\,\lambda_0$ for a principal polarization λ_0, and there are $|\mathbf{P}^1(\mathbb{F}_{p^2})|$ isogenies with A_0 superspecial.

4.5. We denote by $\mathcal{A}'_{2,1,N,(p)}$ the moduli space which parametrizes equivalence classes of isogenies $(\varphi' : \underline{A}_0\to\underline{A}_1)$ of degree p, where $\underline{A}_1$ is an object in $\mathcal{A}_{2,p,N}$ and $\underline{A}_0$ is an object in $\mathcal{A}_{2,1,N}$, such that $\varphi'^*\lambda_1 = p\,\lambda_0$ and $\varphi'_*\eta_0 = \eta_1$.
There is a natural isomorphism from $\mathcal{A}_{2,1,N,(p)}$ to $\mathcal{A}'_{2,1,N,(p)}$. Given an object (A,λ,η,H) in $\mathcal{A}_{2,1,N,(p)}$, let $\underline{A}_0 := \underline{A}$, $A_1 := A/H$ and $\varphi' : A_0\to A_1$ be the natural projection. The polarization $p\,\lambda_0$ descends to one, denoted by λ_1, on A_1. Put $\eta_1 := \varphi'_*\eta_0$ and $\underline{A}_1 = (A_1,\lambda_1,\eta_1)$. Then $(\varphi' : \underline{A}_0\to\underline{A}_1)$ lies in $\mathcal{A}'_{2,1,N,(p)}$ and the morphism

$$q : \mathcal{A}_{2,1,N,(p)}\to\mathcal{A}'_{2,1,N,(p)},\quad (A,\lambda,\eta,H)\mapsto(\varphi' : \underline{A}_0\to\underline{A}_1)$$

is an isomorphism.
We denote by $\mathcal{S}'_{2,1,N,(p)}$ the supersingular locus of $\mathcal{A}'_{2,1,N,(p)}\otimes\overline{\mathbb{F}}_p$. Thus we have $\mathcal{S}'_{2,1,N,(p)}\simeq\mathcal{S}_{2,1,N,(p)}$. It is clear that $S'_\xi\subset\mathcal{S}'_{2,1,N,(p)}$ for each $\xi\in\Lambda$, and $S'_\xi\cap S'_{\xi'}=\emptyset$ if $\xi\neq\xi'$. For each $\gamma\in\Lambda_{2,1,N}$, let S''_γ be the subspace of $\mathcal{S}'_{2,1,N,(p)}$ that consists of objects $(\varphi' : \underline{A}_0\to\underline{A}_1)$ with $\underline{A}_0 = \underline{A}_\gamma$. One also has $S''_\gamma\cap S''_{\gamma'}=\emptyset$ if $\gamma\neq\gamma'$.

Lemma 4.4. (1) *Let $(M_0, \langle\, ,\rangle_0)$ be a separably quasi-polarized supersingular Dieudonné module of rank 4 and suppose $a(M_0)=1$. Let $M_1 := (F,V)M_0$ and N be the unique Dieudonné module containing M_0 with $N/M_0 = k$. Let $\langle\, ,\rangle_1 := \frac{1}{p}\langle\, ,\rangle_0$ be the quasi-polarization for M_1. Then one has $a(N)=a(M_1)=2$, $VN = M_1$, and $M_1/M_1^t \simeq k\oplus k$ as Dieudonné modules.*
(2) *Let $(M_1, \langle\, ,\rangle_1)$ be a quasi-polarized supersingular Dieudonné module of rank 4. Suppose that M_1/M_1^t is of length 2, that is, the quasi-polarization has degree p^2.*

(i) *If $a(M_1)=1$, then letting $M_2 := (F,V)M_1$, one has that $a(M_2)=2$ and $\langle\, ,\rangle_1$ is a separable quasi-polarization on M_2.*
(ii) *Suppose $(M_1, \langle\, ,\rangle_1)$ decomposes as the product of two quasi-polarized Dieudonné submodules of rank 2. Then there are a unique Dieudonné submodule M_2 of M_1 with $M_1/M_2=k$ and a unique Dieudonné module M_0 containing M_1 with $M_0/M_1=k$ so that $\langle\, ,\rangle_1$ (resp. $p\langle\, ,\rangle_1$) is a separable quasi-polarization on M_2 (resp. M_0).*
(iii) *Suppose $M_1/M_1^t \simeq k\oplus k$ as Dieudonné modules. Let $M_2 \subset M_1$ be any Dieudonné submodule with $M_1/M_2 = k$, and $M_0 \supset M_1$ be any Dieudonné overmodule with $M_0/M_1 = k$. Then $\langle\, ,\rangle_1$ (resp. $p\langle\, ,\rangle_1$) is a separable quasi-polarization on M_2 (resp. M_0).*

This is well-known; the proof is elementary and omitted.

Proposition 4.5. *Notation as above.*
(1) *One has*

$$\mathcal{S}'_{2,1,N,(p)} = \left(\coprod_{\xi\in\Lambda} S'_\xi\right) \cup \left(\coprod_{\gamma\in\Lambda_{2,1,N}} S''_\gamma\right).$$

(2) *The scheme $\mathcal{S}'_{2,1,N,(p)}$ has ordinary double singular points and*

$$(\mathcal{S}'_{2,1,N,(p)})^{\rm sing} = \left(\coprod_{\xi\in\Lambda} S'_\xi\right) \cap \left(\coprod_{\gamma\in\Lambda_{2,1,N}} S''_\gamma\right).$$

Moreover, one has

$$|(\mathcal{S}'_{2,1,N,(p)})^{\rm sing}| = |\Lambda_{2,1,N}|(p+1) = |\Lambda|(p^2+1).$$

Proof. (1) Let $(\varphi' : \underline{A}_0 \to \underline{A}_1)$ be a point of $\mathcal{S}'_{2,1,N,(p)}$. If $a(A_0)=1$, then $\ker\varphi'$ is the unique α-subgroup of $A_0[p]$ and thus $\underline{A}_1\in\Lambda$. Hence this point lies in S'_ξ for some ξ. Suppose that $\underline{A}_1$ is not in Λ, then there is a unique lifting $(\varphi'_1 : \underline{A}'_0 \to \underline{A}_1)$ in $\mathcal{S}'_{2,1,N,(p)}$ and the source $\underline{A}'_0$ is superspecial. Hence $\underline{A}_0 = \underline{A}'_0$ is superspecial and the point $(\varphi' : \underline{A}_0 \to \underline{A}_1)$ lies in S''_γ for some γ.
(2) It is clear that the singularities only occur at the intersection of S'_ξ's and S''_γ's, as S'_ξ and S''_γ are smooth. Let $x = (\varphi' : \underline{A}_\gamma \to \underline{A}_\xi) \in S'_\xi \cap S''_\gamma$. We know that the projection ${\rm pr}_0 : \mathcal{S}'_{2,1,N,(p)} \to \mathcal{S}_{2,1,N}$ induces an isomorphism from S'_ξ to V'_ξ. Therefore, ${\rm pr}_0$ maps the one-dimensional subspace $T_x(S'_\xi)$ of $T_x(\mathcal{A}'_{2,1,N,(p)}\otimes\overline{\mathbb{F}}_p)$ onto the one-dimensional subspace $T_{{\rm pr}_0(x)}(V'_\xi)$ of $T_{{\rm pr}_0(x)}(\mathcal{A}_{2,1,N}\otimes\overline{\mathbb{F}}_p)$, where

$T_x(X)$ denotes the tangent space of a variety X at a point x. On the other hand, pr_0 maps the subspace $T_x(S''_\gamma)$ to zero. This shows $T_x(S'_\xi) \neq T_x(S''_\gamma)$ in $T_x(\mathcal{A}'_{2,1,N,(p)} \otimes \overline{\mathbb{F}}_p)$; particularly $\mathcal{S}'_{2,1,N,(p)}$ has ordinary double singularity at x. Since every singular point lies in both S'_ξ and S''_γ for some ξ, γ, by Subsection 4.4 each S'_ξ has p^2+1 singular points and each S''_γ has $p+1$ singular points. We get

$$|(\mathcal{S}'_{2,1,N,(p)})^{\mathrm{sing}}| = |\Lambda_{2,1,N}|(p+1), \quad \text{and} \quad |(\mathcal{S}'_{2,1,N,(p)})^{\mathrm{sing}}| = |\Lambda|(p^2+1).$$

This completes the proof. ∎

Corollary 4.6. *We have*

$$|(\mathcal{S}'_{2,1,N,(p)})^{\mathrm{sing}}| = |\operatorname{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \cdot \frac{(-1)\zeta(-1)\zeta(-3)}{4}(p-1)(p^2+1)(p+1)$$

and

$$|\Lambda| = |\operatorname{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \cdot \frac{(-1)\zeta(-1)\zeta(-3)}{4}(p^2-1).$$

Proof. This follows from Corollary 3.3 and (2) of Proposition 4.5. ∎

Note that the evaluation of $|\Lambda|$ here is different from that given in Katsura-Oort [5]. Their method does not rely on the mass formula but the computation is more complicated.

Since $\mathcal{S}_{2,1,N,(p)} \simeq \mathcal{S}'_{2,1,N,(p)}$ (Subsection 4.5), Theorem 1.2 follows from Proposition 4.5 and Corollary 4.6.

As a byproduct, we obtain the description of the supersingular locus $\mathcal{S}_{2,p,N}$.

Theorem 4.7.

(1) *The scheme $\mathcal{S}_{2,p,N}$ is equi-dimensional and each irreducible component is isomorphic to $\mathbf{P}^1$.*

(2) *The scheme $\mathcal{S}_{2,p,N}$ has $|\Lambda_{2,1,N}|$ irreducible components.*

(3) *The singular locus of $\mathcal{S}_{2,p,N}$ consists of superspecial points (A, λ, η) with $\ker \lambda \simeq \alpha_p \times \alpha_p$, and thus $|\mathcal{S}^{\mathrm{sing}}_{2,p,N}| = |\Lambda|$. Moreover, at each singular point there are p^2+1 irreducible components passing through and intersecting transversely.*

(4) *The natural morphism $\mathrm{pr}_1 : \mathcal{S}_{2,1,N,(p)} \to \mathcal{S}_{2,p,N}$ contracts $|\Lambda|$ projective lines onto the singular locus of $\mathcal{S}_{2,p,N}$.*

5. The class numbers $H_n(p,1)$ and $H_n(1,p)$

In this section we describe the arithmetic part of the results in Li and Oort [8]. Our references are Ibukiyama-Katsura-Oort [4, Section 2] and Li-Oort [8, Section 4].

Let B be the definite quaternion algebra over $\mathbb{Q}$ with discriminant p, and $\mathcal{O}$ be a maximal order of B. Let $V = B^{\oplus n}$, regarded as a left B-module of row vectors, and let $\psi(x,y) = \sum_{i=1}^n x_i \bar{y}_i$ be the standard hermitian form on V, where $y_i \mapsto \bar{y}_i$ is the canonical involution on B. Let G be the group of ψ-similitudes over $\mathbb{Q}$; its group of $\mathbb{Q}$-points is

$$G(\mathbb{Q}) := \{ h \in M_n(B) \mid h\bar{h}^t = rI_n \text{ for some } r \in \mathbb{Q}^\times \}.$$

Two $\mathcal{O}$-lattices L and L' in $B^{\oplus n}$ are called *globally equivalent* (denoted by $L \sim L'$) if $L' = Lh$ for some $h \in G(\mathbb{Q})$. For a finite place v of $\mathbb{Q}$, we write $B_v := B \otimes \mathbb{Q}_v$, $\mathcal{O}_v := \mathcal{O} \otimes \mathbb{Z}_v$ and $L_v := L \otimes \mathbb{Z}_v$. Two $\mathcal{O}$-lattices L and L' in $B^{\oplus n}$ are called *locally equivalent at* v (denoted by $L_v \sim L'_v$) if $L'_v = L_v h_v$ for some $h_v \in G(\mathbb{Q}_v)$. A *genus* of $\mathcal{O}$-lattices is a set of (global) $\mathcal{O}$-lattices in $B^{\oplus n}$ which are equivalent to each other locally at every finite place v.
Let

$$N_p = \mathcal{O}_p^{\oplus n} \cdot \begin{pmatrix} I_r & 0 \\ 0 & \pi I_{n-r} \end{pmatrix} \cdot \xi \subset B_p^{\oplus n},$$

where r is the integer $[n/2]$, π is a uniformizer in $\mathcal{O}_p$, and ξ is an element in $\mathrm{GL}_n(B_p)$ such that

$$\xi \bar{\xi}^t = \text{anti-diag}(1, 1, \dots, 1).$$

DEFINITION 5.1. (1) Let $\mathcal{L}_n(p,1)$ denote the set of global equivalence classes of $\mathcal{O}$-lattices L in $B^{\oplus n}$ such that $L_v \sim \mathcal{O}_v^{\oplus n}$ at every finite place v. The genus $\mathcal{L}_n(p,1)$ is called the *principal genus*, and let $H_n(p,1) := |\mathcal{L}_n(p,1)|$.
(2) Let $\mathcal{L}_n(1,p)$ denote the set of global equivalence classes of $\mathcal{O}$-lattices L in $B^{\oplus n}$ such that $L_p \sim N_p$ and $L_v \sim \mathcal{O}_v^{\oplus n}$ at every finite place $v \neq p$. The genus $\mathcal{L}_n(p,1)$ is called the *non-principal genus*, and let $H_n(1,p) := |\mathcal{L}_n(1,p)|$.

Recall (Section 3) that Λ_g is the set of isomorphism classes of g-dimensional principally polarized superspecial abelian varieties over $\overline{\mathbb{F}}_p$. When $g = 2D > 0$ is even, we denote by Λ^*_{g,p^D} the set of isomorphism classes of g-dimensional polarized superspecial abelian varieties (A, λ) of degree p^{2D} over $\overline{\mathbb{F}}_p$ satisfying $\ker \lambda = A[F]$.
Let $\mathcal{A}_{g,1}$ be the *coarse moduli scheme* of g-dimensional principally polarized abelian varieties, and let $\mathcal{S}_{g,1}$ be the supersingular locus of $\mathcal{A}_{g,1} \otimes \overline{\mathbb{F}}_p$. Recall (Subsection 2.1) that $\Pi_0(\mathcal{S}_{g,1})$ denotes the set of irreducible components of $\mathcal{S}_{g,1}$.

THEOREM 5.2 (Li-Oort). *We have*

$$|\Pi_0(\mathcal{S}_{g,1})| = \begin{cases} |\Lambda_g| & \text{if } g \text{ is odd;} \\ |\Lambda^*_{g,p^D}| & \text{if } g = 2D \text{ is even.} \end{cases}$$

The arithmetic part for $\Pi_0(\mathcal{S}_{g,1})$ is given by the following

PROPOSITION 5.3.
(1) *For any positive integer g, one has $|\Lambda_g| = H_g(p,1)$.*
(2) *For any even positive integer $g = 2D$, one has $|\Lambda^*_{g,p^D}| = H_g(1,p)$.*

PROOF. (1) See [4, Theorem 2.10]. (2) See [8, Proposition 4.7].

6. CORRESPONDENCE COMPUTATION

6.1. Let M_0 be a superspecial Dieudonné module over k of rank $2g$, and call

$$\tilde{M}_0 := \{x \in M_0; F^2 x = px\},$$

the skeleton of M_0 (cf. [8, 5.7]). We know that $\tilde{M}_0$ is a Dieudonné module over $\mathbb{F}_{p^2}$ and $\tilde{M}_0 \otimes_{W(\mathbb{F}_{p^2})} W(k) = M_0$. The vector space $\tilde{M}_0/V\tilde{M}_0$ defines an $\mathbb{F}_{p^2}$-structure of the k-vector space M_0/VM_0.
Let $\mathrm{Gr}(n,m)$ be the Grassmannian variety of n-dimensional subspaces in an m-dimensional vector space. Suppose M_1 is a Dieudonné submodule of M_0 such that

$$VM_0 \subset M_1 \subset M_0, \quad \dim_k M_1/VM_0 = r,$$

for some integer $0 \le r \le g$. As $\dim M_0/VM_0 = g$, the subspace $\overline{M}_1 := M_1/VM_0$ corresponds to a point in $\mathrm{Gr}(r,g)(k)$.

Lemma 6.1. *Notation as above. Then M_1 is superspecial (i.e. $F^2M_1 = pM_1$) if and only if $\overline{M}_1 \in \mathrm{Gr}(r,g)(\mathbb{F}_{p^2})$.*

Proof. If M_1 is generated by $V\tilde{M}_0$ and $x_1, x_2, \dots, x_r$, $x_i \in \tilde{M}_0$ over W. Then $\tilde{M}_1$ generates M_1 and thus $F^2M_1 = pM_1$. Therefore, M_1 is superspecial. Conversely if M_1 is superspecial, then we have

$$V\tilde{M}_0 \ \subset \ \tilde{M}_1 \ \subset \tilde{M}_0.$$

Therefore, $\tilde{M}_1$ gives rise to an element in $\mathrm{Gr}(r,g)(\mathbb{F}_{p^2})$. ∎

6.2. Let $L(n,2n) \subset \mathrm{Gr}(n,2n)$ be the Lagrangian variety of maximal isotropic subspaces in a $2n$-dimensional vector space with a non-degenerate alternating form.
From now on $g = 2D$ is an even positive integer. Recall (in Introduction and Section 5) that $\Lambda^*_{g,p^D,N}$ denotes the set of superspecial (geometric) points (A,λ,η) in $\mathcal{A}_{g,p^D,N} \otimes \overline{\mathbb{F}}_p$ satisfying $\ker\lambda = A[F]$.

Lemma 6.2. *Let $(A_2,\lambda_2,\eta_2) \in \Lambda^*_{g,p^D,N}$ and $(M_2, \langle\, ,\rangle_2)$ be the associated Dieudonné module. There is a W-basis $e_1, \dots, e_{2g}$ for M_2 such that for $1 \le i \le g$*

$$Fe_i = e_{g+i}, \quad Fe_{g+i} = pe_i,$$

and the non-zero pairings are

$$\langle e_i, e_{D+i}\rangle_2 = -\langle e_{D+i}, e_i\rangle_2 = \frac{1}{p},$$

$$\langle e_{g+i}, e_{g+D+i}\rangle_2 = -\langle e_{g+D+i}, e_{g+i}\rangle_2 = 1,$$

for $1 \le i \le D$.

Proof. Use the same argument of Lemma 4.2 (2).

6.3. Let $\mathcal{H}$ be the moduli space over $\mathbb{Z}_{(p)}[\zeta_N]$ which parametrizes equivalence classes of objects $(\varphi : \underline{A}_1 \to \underline{A}_2)$, where

- $\underline{A}_1 = (A_1,\lambda_1,\eta_1)$ is an object in $\mathcal{A}_{g,1,N}$,
- $\underline{A}_2 = (A_2,\lambda_2,\eta_2)$ is an object in $\mathcal{A}_{g,p^D,N}$, and
- $\varphi : A_1 \to A_2$ is an isogeny of degree p^D satisfying $\varphi^*\lambda_2 = p\lambda_1$ and $\varphi_*\eta_1 = \eta_2$.

The moduli space $\mathcal{H}$ with two natural projections gives the following correspondence:

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathcal{A}_{g,1,N} & & \mathcal{A}_{g,p^D,N}. \end{tikzcd}$$

Let $\mathcal{S}$ be the supersingular locus of $\mathcal{H}\otimes\overline{\mathbb{F}}_p$, which is the reduced closed subscheme consisting of supersingular points (either A_1 or A_2 is supersingular, or equivalently both are supersingular). Restricting the natural projections on $\mathcal{S}$, we have the following correspondence

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathcal{S}_{g,1,N} & & \mathcal{S}_{g,p^D,N}. \end{tikzcd}$$

Suppose that $\underline{A}_2 \in \Lambda^*_{g,p^D,N}$. Let $(\varphi : \underline{A}_1 \to \underline{A}_2) \in \mathcal{S}(k)$ be a point in the pre-image $\mathrm{pr}_2^{-1}(\underline{A}_2)$, and let $(M_1, \langle\,,\rangle_1)$ be the Dieudonné module associated to $\underline{A}_1$. We have

$$M_1 \subset M_2, \quad p\langle\,,\rangle_2 = \langle\,,\rangle_1, \quad FM_2 = M_2^t.$$

Since $\underline{A}_2$ is superspecial and $\langle\,,\rangle_1$ is a perfect pairing on M_1, we get

$$FM_2 = VM_2 = M_2^t, \quad M_2^t \subset M_1^t = M_1.$$

Therefore, we have

$$FM_2 = VM_2 \subset M_1 \subset M_2, \quad \dim_k M_1/VM_2 = D.$$

Put $\langle\,,\rangle := p\langle\,,\rangle_2$. The pairing

$$\langle\,,\rangle : M_2 \times M_2 \to W$$

induces a pairing

$$\langle\,,\rangle : M_2/VM_2 \times M_2/VM_2 \to k$$

which is perfect (by Lemma 6.2). Furthermore, M_1/VM_2 is a maximal isotropic subspace for the pairing $\langle\,,\rangle$. This is because $\langle\,,\rangle_1$ is a perfect pairing on M_1 and $\dim M_1/VM_2 = D$ is the maximal dimension of isotropic subspaces. We conclude that the point $(\varphi : \underline{A}_1 \to \underline{A}_2)$ lies in $\mathrm{pr}_2^{-1}(\underline{A}_2)$ if and only if $VM_2 \subset M_1 \subset M_2$ and M_1/VM_2 is a maximal isotropic subspace of the symplectic space $(M_2/VM_2, \langle\,,\rangle)$. By Lemma 6.1, we have proved

Proposition 6.3. *Let $\underline{A}_2$ be a point in $\Lambda^*_{g,p^D,N}$.*
(1) *The pre-image $\mathrm{pr}_2^{-1}(\underline{A}_2)$ is naturally isomorphic to the projective variety $L(D,2D)$ over k.*
(2) *The set $\mathrm{pr}_1^{-1}(\Lambda_{g,1,N}) \cap \mathrm{pr}_2^{-1}(\underline{A}_2)$ is in bijection with $L(D,2D)(\mathbb{F}_{p^2})$, where the $W(\mathbb{F}_{p^2})$-structure of M_2 is given by the skeleton $\tilde{M}_2$.*

6.4. We compute $\mathrm{pr}_1^{-1}(\underline{A}_1) \cap \mathrm{pr}_2^{-1}(\Lambda^*_{g,p^D,N})$ for a point $\underline{A}_1$ in $\Lambda_{g,1,N}$. Let $\mathcal{T}$ be the closed subscheme of $\mathcal{S}$ consisting of the points $(\varphi : \underline{A}_1 \to \underline{A}_2)$ such that $\ker \lambda_2 = A_2[F]$. We compute the closed subvariety $\mathrm{pr}_1^{-1}(\underline{A}_1) \cap \mathcal{T}$ first.
Let $\underline{A}_1 \in \Lambda_{g,1,N}$, and let $(\varphi : \underline{A}_1 \to \underline{A}_2) \in \mathcal{S}(k)$ be a point in the pre-image $\mathrm{pr}_1^{-1}(\underline{A}_1)$. Let $(M_1, \langle\, ,\rangle_1)$ and $(M_2, \langle\, ,\rangle_2)$ be the Dieudonné modules associated to $\underline{A}_1$ and $\underline{A}_2$, respectively.
One has

$$M_1^t = M_1 \supset M_2^t = FM_2,$$

and thus has

$$FM_2 \subset M_1 \subset M_2 \subset p^{-1}VM_1.$$

Since M_1 is superspecial, $p^{-1}VM_1 = p^{-1}FM_1$. Put $M_0 := p^{-1}VM_1$ and $\langle\, ,\rangle := p\langle\, ,\rangle_2$. We have

$$pM_0 \subset M_2^t = FM_2 \subset M_1 = VM_0 \subset M_2 \subset M_0$$

and that $\langle\, ,\rangle$ is a perfect pairing on M_0. By Proposition 6.1 of [8] (cf. Lemma 4.2 (1)), there is a W-basis $f_1 \dots, f_{2g}$ for M_0 such that for $1 \le i \le g$

$$Ff_i = f_{g+i}, \quad Ff_{g+i} = pf_i,$$

and the non-zero pairings are

$$\langle f_i, f_{g+i}\rangle = -\langle f_{g+i}, f_i\rangle = \beta_1, \quad \forall\, 1 \le i \le g$$

where $\beta_1 \in W(\mathbb{F}_{p^2})^\times$ with $\beta_1^\sigma = -\beta_1$. In the vector space $\overline{M}_0 := M_0/pM_0$, $\overline{M}_2$ is a vector subspace over k of dimension $g + D$ with

$$\overline{M}_2 \supset \overline{VM}_0 = k < f_{g+1}, \dots, f_{2g} > \quad \text{and} \quad \langle \overline{M}_2, F\overline{M}_2\rangle = 0.$$

We can write

$$\overline{M}_2 = k < v_1, \dots, v_D > + \overline{VM}_0, \quad v_i = \sum_{r=1}^{g} a_{ir} f_r.$$

One computes

$$\langle v_i, Fv_j\rangle = \langle \sum_{r=1}^{g} a_{ir} f_r, F(\sum_{q=1}^{g} a_{jq} f_q)\rangle = \langle \sum_{r=1}^{g} a_{ir} f_r, \sum_{q=1}^{g} a_{jq}^p f_{g+q}\rangle = \beta_1 \sum_{r=1}^{g} a_{ir} a_{jr}^p.$$

This computation leads us to the following definition.

6.5. Let $V := \mathbb{F}_{p^2}^{2n}$. For any field $K \supset \mathbb{F}_{p^2}$, we put $V_K := V \otimes_{\mathbb{F}_{p^2}} K$ and define a pairing on V_K

$$\langle\, ,\rangle' : V_K \times V_K \to K, \quad \langle (a_i), (b_i)\rangle' := \sum_{i=1}^{2n} a_i\, b_i^p.$$

Let $\mathbf{X}(n, 2n) \subset \mathrm{Gr}(n, 2n)$ be the subvariety over $\mathbb{F}_{p^2}$ which parametrizes n-dimensional (maximal) isotropic subspaces in V with respect to the pairing $\langle\, ,\rangle'$.
With the computation in Subsection 6.4 and Lemma 6.1, we have proved

Proposition 6.4. *Let $\underline{A}_1$ be a point in $\Lambda_{g,1,N}$ and $g=2D$.*
(1) *The intersection $\mathrm{pr}_1^{-1}(\underline{A}_1)\cap\mathcal{T}$ is naturally isomorphic to the projective variety $\mathbf{X}(n,2n)$ over k.*
(2) *The set $\mathrm{pr}_1^{-1}(\underline{A}_1)\cap\mathrm{pr}_2^{-1}(\Lambda^*_{g,p^D,N})$ is in bijection with $\mathbf{X}(D,2D)(\mathbb{F}_{p^2})$.*

When $g=2$, Proposition 6.4 (2) is a result of Moret-Bailly (Lemma 4.3).

6.6. To compute $\mathbf{X}(n,2n)(\mathbb{F}_{p^2})$, we show that it is the set of rational points of a homogeneous space under the quasi-split group $U(n,n)$.
Let $V=\mathbb{F}_{p^2}^{2n}$ and let $x\mapsto\bar{x}$ be the involution of $\mathbb{F}_{p^2}$ over $\mathbb{F}_p$. Let $\psi((x_i),(y_i))=\sum_i x_i\bar{y}_i$ be the standard hermitian form on V. For any field $K\supset\mathbb{F}_p$, we put $V_K:=V\otimes_{\mathbb{F}_p}K$ and extend ψ to a from

$$\psi: V_K\otimes V_K\to\mathbb{F}_{p^2}\otimes K$$

by K-linearity.
Let $U(n,n)$ be the group of automorphisms of V that preserve the hermitian form ψ. Let $LU(n,2n)(K)$ be the space of n-dimensional (maximal) isotropic K-subspaces in V_K with respect to ψ. We know that $LU(n,2n)$ is a projective scheme over $\mathbb{F}_p$ of finite type, and this is a homogeneous space under $U(n,n)$. It follows from the definition that

$$LU(n,2n)(\mathbb{F}_p)=\mathbf{X}(n,2n)(\mathbb{F}_{p^2}).$$

However, the space $LU(n,2n)$ is not isomorphic to the space $\mathbf{X}(n,2n)$ over k.
Let $\widetilde{\Lambda}$ be the subset of $\mathcal{S}$ consisting of elements $(\varphi:\underline{A}_1\to\underline{A}_2)$ such that $\underline{A}_2\in\Lambda^*_{g,p^D,N}$ and $\underline{A}_1\in\Lambda_{g,1,N}$. We have natural projections

$$\begin{tikzcd} & \widetilde{\Lambda} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \Lambda_{g,1,N} & & \Lambda^*_{g,p^D,N}. \end{tikzcd}$$

By Propositions 6.3 and 6.4, and Subsection 6.6, we have proved

Proposition 6.5. *Notation as above, one has*

$$|\widetilde{\Lambda}|=|L(D,2D)(\mathbb{F}_{p^2})|\cdot|\Lambda^*_{g,p^D,N}|=|LU(D,2D)(\mathbb{F}_p)|\cdot|\Lambda_{g,1,N}|.$$

Theorem 6.6. *We have*

$$|\Lambda^*_{g,p^D,N}|=|\,\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})|\cdot\frac{(-1)^{g(g+1)/2}}{2^g}\left\{\prod_{i=1}^{g}\zeta(1-2i)\right\}\cdot\prod_{i=1}^{D}(p^{4i-2}-1).$$

Proof. We compute in Section 7 that

$$|L(D,2D)(\mathbb{F}_{p^2})|=\prod_{i=1}^{D}(p^{2i}+1),$$

$$|LU(D,2D)(\mathbb{F}_p)|=\prod_{i=1}^{D}(p^{2i-1}+1).$$

Using Proposition 6.5, Theorem 3.1 and Lemma 3.2, we get the value of $|\Lambda^*_{g,p^D,N}|$. ∎

6.7. Proof of Theorem 1.4. By a theorem of Li and Oort (Theorem 5.2), we know

$$|\Pi_0(\mathcal{S}_{g,1,N})| = \begin{cases} |\Lambda_{g,1,N}| & \text{if } g \text{ is odd;} \\ |\Lambda^*_{g,p^D,N}| & \text{if } g = 2D \text{ is even.} \end{cases}$$

Note that the result of Li and Oort is formulated for the coarse moduli space $\mathcal{S}_{g,1}$. However, it is clear that adding the level-N structure yields a modification as above. Theorem 1.4 then follows from Theorem 3.1, Lemma 3.2 and Theorem 6.6. ∎

7. $L(n, 2n)(\mathbb{F}_q)$ and $LU(n, 2n)(\mathbb{F}_q)$

Let $L(n, 2n)$ be the Lagrangian variety of maximal isotropic subspaces in a $2n$-dimensional vector space V_0 with a non-degenerate alternating form ψ_0.

Lemma 7.1. $|L(n, 2n)(\mathbb{F}_q)| = \prod_{i=1}^n (q^i + 1)$.

Proof. Let $e_1, \dots, e_{2n}$ be the standard symplectic basis for V_0. The group $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ acts transitively on the space $L(n, 2n)(\mathbb{F}_q)$. For $h \in \mathrm{Sp}_{2n}(\mathbb{F}_q)$, the map $h \mapsto \{he_1, \dots he_{2n}\}$ induces a bijection between $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ and the set $\mathcal{B}(n)$ of ordered symplectic bases $\{v_1, \dots, v_{2n}\}$ for V_0. The first vector v_1 has $q^{2n} - 1$ choices. The first companion vector v_{n+1} has $(q^{2n} - q^{2n-1})/(q - 1)$ choices as it does not lie in the hyperplane $v_1^\perp$ and we require $\psi_0(v_1, v_{n+1}) = 1$. The remaining ordered symplectic basis can be chosen from the complement $\mathbb{F}_q < v_1, v_{n+1} >^\perp$. Therefore, we have proved the recursive formula

$$|\mathrm{Sp}_{2n}(\mathbb{F}_q)| = (q^{2n} - 1)q^{2n-1} |\mathrm{Sp}_{2n-2}(\mathbb{F}_q)|.$$

From this, we get

$$|\mathrm{Sp}_{2n}(\mathbb{F}_q)| = q^{n^2} \prod_{i=1}^n (q^{2i} - 1).$$

Let P be the stabilizer of the standard maximal isotropic subspace $\mathbb{F}_q < e_1, \dots, e_n >$. It is easy to see that

$$P = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} ; AD^t = I_n,\ BA^t = AB^t \right\}.$$

The matrix BA^t is symmetric and the space of $n \times n$ symmetric matrices has dimension $(n^2 + n)/2$. This yields

$$|P| = q^{\frac{n^2+n}{2}} |\mathrm{GL}_n(\mathbb{F}_q)| = q^{n^2} \prod_{i=1}^n (q^i - 1)$$

as one has

$$|\mathrm{GL}_n(\mathbb{F}_q)| = q^{\frac{n^2-n}{2}} \prod_{i=1}^n (q^i - 1).$$

Since $L(n, 2n)(\mathbb{F}_q) \simeq \mathrm{Sp}_{2n}(\mathbb{F}_q)/P$, we get $|L(n, 2n)(\mathbb{F}_q)| = \prod_{i=1}^n (q^i + 1)$. ∎

7.1. Let $V=\mathbb{F}_{q^2}^{2n}$ and let $x\mapsto \bar{x}=x^q$ be the involution of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. Let $\psi((x_i),(y_i))=\sum_i x_i\bar{y}_i$ be the standard hermitian form on V. For any field $K\supset \mathbb{F}_q$, we put $V_K:=V\otimes_{\mathbb{F}_p} K$ and extend ψ to a from

$$\psi: V_K\otimes V_K\to \mathbb{F}_{q^2}\otimes_{\mathbb{F}_q} K$$

by K-linearity.
Let $U(n,n)$ be the group of automorphisms of V that preserve the hermitian form ψ. Let $LU(n,2n)(K)$ be the set of n-dimensional (maximal) isotropic K-subspaces in V_K with respect to ψ. We know that $LU(n,2n)$ is a homogeneous space under $U(n,n)$. Let

$$I_m:=\{\underline{a}=(a_1,\dots,a_m)\in \mathbb{F}_{q^2}^m\,;\ Q(\underline{a})=0\},$$

where $Q(\underline{a})=a_1^{q+1}+\dots+a_m^{q+1}$.

Lemma 7.2. *We have* $|I_m|=q^{2m-1}+(-1)^m q^m+(-1)^{m-1}q^{m-1}$.

Proof. For $m>1$, consider the projection $p: I_m\to \mathbb{F}_{q^2}^{m-1}$ which sends $(a_1,\dots,a_m)$ to $(a_1,\dots,a_{m-1})$. Let I_{m-1}^c be the complement of I_{m-1} in $\mathbb{F}_{q^2}^{m-1}$. If $x\in I_{m-1}$, then the pre-image $p^{-1}(x)$ consists of one element. If $x\in I_{m-1}^c$, then the pre-image $p^{-1}(x)$ consists of solutions of the equation $a_m^{q+1}=-Q(x)\in \mathbb{F}_q^\times$ and thus $p^{-1}(x)$ has $q+1$ elements. Therefore, $|I_m|=|I_{m-1}|+(q+1)|I_{m-1}^c|$. From this we get the recursive formula

$$|I_m|=(q+1)q^{2(m-1)}-q|I_{m-1}|.$$

We show the lemma by induction. When $m=1$, $|I_m|=1$ and the statement holds. Suppose the statement holds for $m=k$, i.e. $|I_k|=q^{2k-1}+(-1)^k q^k+(-1)^{k-1}q^{k-1}$. When $m=k+1$,

$$\begin{aligned}|I_{k+1}|&=(q+1)q^{2k}-q[q^{2k-1}+(-1)^k q^k+(-1)^{k-1}q^{k-1}]\\&=q^{2k+1}+(-1)^{k+1}q^{k+1}+(-1)^k q^k.\end{aligned}$$

This completes the proof. ∎

Proposition 7.3. $|LU(n,2n)(\mathbb{F}_q)|=\prod_{i=1}^n (q^{2i-1}+1)$.

Proof. We can choose a new basis $e_1,\dots,e_{2n}$ for V such that the non-zero pairings are

$$\psi(e_i,e_{n+i})=\psi(e_{n+i},e_i)=1,\quad \forall\, 1\le i\le n.$$

The representing matrix for ψ with respect to $\{e_1,\dots,e_{2n}\}$ is

$$J=\begin{pmatrix}0 & I_n\\ I_n & 0\end{pmatrix}.$$

Let P be the stabilizer of the standard maximal isotropic subspace $\mathbb{F}_{q^2}<e_1,\dots,e_n>$. It is easy to see that

$$P=\left\{\begin{pmatrix}A & B\\ 0 & D\end{pmatrix}; AD^*=I_n,\ BA^*+AB^*=0\right\}.$$

The matrix BA^* is skew-symmetric hermitian. The space of $n \times n$ skew-symmetric hermitian matrices has dimension n^2 over $\mathbb{F}_q$. Indeed, the diagonal consists of entries in the kernel of the trace; this gives dimension n. The upper triangular has $(n^2-n)/2$ entries in $\mathbb{F}_{q^2}$; this gives dimension n^2-n. Hence,

$$|P| = q^{n^2}|\operatorname{GL}_n(\mathbb{F}_{q^2})| = q^{2n^2-n}\prod_{i=1}^{n}(q^{2i}-1).$$

We compute $|U(n,n)(\mathbb{F}_q)|$. For $h \in U(n,n)(\mathbb{F}_q)$, the map

$$h \mapsto \{he_1, \dots, he_{2n}\}$$

gives a bijection between $U(n,n)(\mathbb{F}_q)$ and the set $\mathcal{B}(n)$ of ordered bases $\{v_1, \dots, v_{2n}\}$ for which the representing matrix of ψ is J. The first vector v_1 has

$$|I_{2n}| - 1 = q^{4n-1} + q^{2n} - q^{2n-1} - 1 = (q^{2n}-1)(q^{2n-1}+1).$$

choices (Lemma 7.2). For the choices of the companion vector v_{n+1} with $\psi(v_{n+1}, v_{n+1}) = 0$ and $\psi(v_1, v_{n+1}) = 1$, consider the set

$$Y := \{v \in V; \psi(v_1, v) = 1\}.$$

Clearly, $|Y| = q^{4n-2}$. The additive group $\mathbb{F}_{q^2}$ acts on Y by $a \cdot v = v + av_1$ for $a \in \mathbb{F}_{q^2}$, $v \in Y$. It follows from

$$\psi(v + av_1, v + av_1) = \psi(v,v) + \bar{a} + a$$

that every orbit $O(v)$ contains an isotropic vector v_0 and any isotropic vector in $O(v)$ has the form $v_0 + av_1$ with $\bar{a} + a = 0$. Hence, the vector v_{n+1} has

$$\frac{|\,Y|\,q}{q^2} = q^{4n-3}$$

choices. In conclusion, we have proved the recursive formula

$$|U(n,n)(\mathbb{F}_q)| = (q^{2n}-1)(q^{2n-1}+1)q^{4n-3}|U(n-1,n-1)(\mathbb{F}_q)|.$$

It follows that

$$|U(n,n)(\mathbb{F}_q)| = q^{2n^2-n}\prod_{i=1}^{n}(q^{2i}-1)(q^{2i-1}+1).$$

Since $LU(n,2n)(\mathbb{F}_q) \simeq U(n,n)(\mathbb{F}_q)/P$, we get $|LU(n,2n)(\mathbb{F}_q)| = \prod_{i=1}^{n}(q^{2i-1}+1)$. ∎

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