# Volumes of Symmetric Spaces via Lattice Points

Henri Gillet and Daniel R. Grayson

Received: February 15, 2004
Revised: August 2, 2006

Communicated by Ulf Rehmann

Abstract. We show how to use elementary methods to compute the volume of $\mathrm{Sl}_k \mathbb{R}/\mathrm{Sl}_k \mathbb{Z}$. We compute the volumes of certain unbounded regions in Euclidean space by counting lattice points and then appeal to the machinery of Dirichlet series to get estimates of the growth rate of the number of lattice points appearing in the region as the lattice spacing decreases. We also present a proof of the closely related result that the Tamagawa number is 1.

2000 Mathematics Subject Classification: 11F06, 11H06, 11M45.
Keywords and Phrases: special linear group, volume, lattice, Tauberian theorem, arithmetic group, Tamagawa number.

## Introduction

In this paper we show how to use elementary methods to prove that the volume of $\mathrm{Sl}_k \mathbb{R}/\mathrm{Sl}_k \mathbb{Z}$ is $\zeta(2)\zeta(3)\cdots\zeta(k)/k$; see Corollary 3.16. Using a version of reduction theory presented in this paper, we can compute the volumes of certain unbounded regions in Euclidean space by counting lattice points and then appeal to the machinery of Dirichlet series to get estimates of the growth rate of the number of lattice points appearing in the region as the lattice spacing decreases.

In section 4 we present a proof of the closely related result that the Tamagawa number of $\mathrm{Sl}_{k,\mathbb{Q}}$ is 1 that is somewhat simpler and more arithmetic than Weil's in [37]. His proof proceeds by induction on $k$ and appeals to the Poisson summation formula, whereas the proof here brings to the forefront local versions (5) of the formula, one for each prime $p$, which help to illuminate the appearance of values of zeta functions in formulas for volumes.

The volume computation above is known; see, for example, [26] (with important corrections in [30]), formula (24) in [29], and Theorem 10.4 in [22]. The methods used in the computation of the volume of $\mathrm{Sl}_k \mathbb{R}/\mathrm{Sl}_k \mathbb{Z}$ in the book [31, Lecture XV] have a different flavor from ours and do not involve counting lattice points. One positive point about the proof there is that it proceeds by induction on $k$, making clear how the factor $\zeta(k)$ enters in at $k$-th stage. See also [36, §14.12, formula (2)]. The proof offered there seems to have a gap which consists of assuming that a certain region (denoted by $T$ there) is bounded, thereby allowing the application of [36, §14.4, Theorem 3][1]. The region in Example 2.7 below shows that filling the gap is not easy, hence if we want to compute the volume by counting lattice points, something like our use of reduction theory in Section 3 is needed.

An almost equivalent result was proved by Minkowski in formula (85.) of [16], where he computed the volume of $SO(k)\backslash \mathrm{Sl}_k \mathbb{R}/\mathrm{Sl}_k \mathbb{Z}$. The relationship between the two volume computations is made clear in the proof of [36, §14.12, Theorem 2].

Some of the techniques we use were known to Siegel, who used similar methods in his investigation of representability of integers by quadratic forms in [24, 25, 27]. See especially [25, Hilfssatz 6, p. 242], which is analogous to our Lemma 2.5 and the reduction theory of Section 3, where we show how to compute the volume of certain unbounded domains in Euclidean space by counting lattice points; see also the computations in [24, §9], which have the same general flavor as ours. See also [28, p. 581] where Siegel omits the laborious study, using reduction theory, of points at infinity; it is those details that concern us here.

We thank Harold Diamond for useful information about Dirichlet series and Ulf Rehmann for useful suggestions, advice related to Tamagawa numbers, and clarifications of Siegel's work. We also thank the National Science Foundation for support provided by NSF grants DMS 01-00587 and 05-00762 (Gillet), and 99-70085 and 03-11378 (Grayson).

## 1 Counting with zeta functions

As in [8] we define the zeta function of a group $G$ by summing over the subgroups $H$ in $G$ of finite index.

$$\zeta(G,s) = \sum_{H \subseteq G} [G:H]^{-s} \tag{1}$$

Evidently, $\zeta(\mathbb{Z},s) = \zeta(s)$ and the series converges for $s > 1$. For good groups $G$ the number of subgroups of index at most $T$ grows slowly enough as a function of $T$ that $\zeta(G,s)$ will converge for $s$ sufficiently large.

Let's pick $k \geq 0$ and compute $\zeta(\mathbb{Z}^k, s)$. Any subgroup $H$ of $\mathbb{Z}^k$ of finite index is isomorphic to $\mathbb{Z}^k$; choosing such an isomorphism amounts to finding a matrix $A : \mathbb{Z}^k \to \mathbb{Z}^k$ whose determinant is nonzero and whose image is $H$.

[1] called Dirichlet's Principle in [3, §5.1, Theorem 3]

Any two matrices $A$, $A'$ with the same image $H$ are related by an equation $A' = AS$ where $S \in \mathrm{Gl}_k \mathbb{Z}$.

Thus the terms in the sum defining $\zeta(\mathbb{Z}^k, s)$ correspond to the orbits for the action of $\mathrm{Gl}_k \mathbb{Z}$ via column operations on the set of $k \times k$-matrices with integer entries and nonzero determinant. A unique representative from each orbit is provided by the matrices $A$ that are in *Hermite normal form* (see [4, p. 66] or [19, II.6]), i.e., those matrices $A$ with $A_{ij} = 0$ for $i > j$, $A_{ii} > 0$ for all $i$, and $0 \le A_{ij} < A_{ii}$ for $i < j$.

Let HNF be the set of integer $k \times k$ matrices in Hermite normal form. Given positive integers $n_1, \dots, n_k$, consider the set of matrices $A$ in HNF with $A_{ii} = n_i$ for all $i$. The number of matrices in it is $n_1^{k-1} n_2^{k-2} \cdots n_{k-1}^1 n_k^0$. Using that, we compute formally as follows.

$$\begin{aligned}
\zeta(\mathbb{Z}^k, s) &= \sum_{H \subseteq \mathbb{Z}^k} [\mathbb{Z}^k : H]^{-s} \\
&= \sum_{A \in \mathrm{HNF}} (\det A)^{-s} \\
&= \sum_{n_1 > 0, \dots, n_k > 0} (n_1^{k-1} n_2^{k-2} \cdots n_{k-1}^1 n_k^0)(n_1 \cdots n_k)^{-s} \\
&= \sum_{n_1 > 0, \dots, n_k > 0} n_1^{k-1-s} n_2^{k-2-s} \cdots n_{k-1}^{1-s} n_k^{-s} \\
&= \sum_{n_1 > 0} n_1^{k-1-s} \sum_{n_2 > 0} n_2^{k-2-s} \cdots \sum_{n_{k-1} > 0} n_{k-1}^{1-s} \sum_{n_k > 0} n_k^{-s} \\
&= \zeta(s-k+1)\zeta(s-k+2) \cdots \zeta(s-1)\zeta(s)
\end{aligned} \tag{2}$$

The result $\zeta(s-k+1)\zeta(s-k+2)\cdots\zeta(s-1)\zeta(s)$ is a product of Dirichlet series with positive coefficients that converge for $s > k$, and thus $\zeta(\mathbb{Z}^k, s)$ also converges for $s > k$. This computation is old, and appears in various guises. See, for example: proof 2 of Proposition 1.1 in [8]; Lemma 10 in [15]; formula (1.1) in [32]; page 64 in [23]; formula (5) and the lines following it in [26], where the counting argument is attributed to Eisenstein, and its generalization to number rings is attributed to Hurwitz; and pages 37–38 in [37].

Lemma 1.1. *$\#\{H \subseteq \mathbb{Z}^k \mid [\mathbb{Z}^k : H] \le T\} \sim \zeta(2)\zeta(3)\cdots\zeta(k)T^k/k$ for $k \ge 1$.*

The right hand side is interpreted as $T$ when $k = 1$. The notation $f(T) \sim g(T)$ means that $\lim_{T\to\infty} f(T)/g(T) = 1$.

*Proof.* We give two proofs.

The first one is more elementary, and was told to us by Harold Diamond. Writing $\zeta(s-k+1) = \sum n^{k-1} n^{-s}$ and letting $B(T) = \sum_{n \le T} n^{k-1}$ be the corresponding coefficient summatory function we see that $B(T) = T^k/k + O(T^{k-1})$. If $k \ge 3$ we may apply Theorem A.2 to show that the coefficient summatory function for the Dirichlet series $\zeta(s)\zeta(s-k+1)$ behaves as $\zeta(k)T^k/k + O(T^{k-1})$.

Applying it several more times shows that the coefficient summatory function for the Dirichlet series $\zeta(s)\zeta(s-1)\cdots\zeta(s-k+3)\zeta(s-k+1)$ behaves as $\zeta(k)\zeta(k-1)\cdots\zeta(3)T^k/k+O(T^{k-1})$. Applying it one more time we see that the coefficient summatory function for $\zeta(\mathbb{Z}^k,s)=\zeta(s)\cdots\zeta(s-k+1)$ behaves as $\zeta(k)\zeta(k-1)\cdots\zeta(2)T^k/k+O(T^{k-1}\log T)$, which in turn implies the result.

The second proof is less elementary, since it uses a Tauberian theorem. From (2) we know that the rightmost (simple) pole of $\zeta(\mathbb{Z}^k,s)$ occurs at $s=k$, that the residue there is the product $\zeta(2)\zeta(3)\cdots\zeta(k)$, and that Theorem A.4 can be applied to get the result. □

Now we point out a weaker version of lemma 1.1 whose proof is even more elementary.

Lemma 1.2. *If* $T>0$ *then* $\#\{H\subseteq\mathbb{Z}^k \mid [\mathbb{Z}^k:H]\le T\}\le T^k$.

*Proof.* As above, we obtain the following formula.

$$\begin{aligned}\#\{H\subseteq\mathbb{Z}^k \mid [\mathbb{Z}^k:H]\le T\} &= \#\{A\in\mathrm{HNF} \mid \det A\le T\}\\ &= \sum_{\substack{n_1>0,\dots,n_k>0\\ n_1\cdot\dots\cdot n_k\le T}} n_1^{k-1}n_2^{k-2}\cdots n_{k-1}^1 n_k^0\end{aligned}$$

We use it to prove the desired inequality by induction on $k$, the case $k=0$ being clear.

$$\begin{aligned}\#\{H\subseteq\mathbb{Z}^k \mid [\mathbb{Z}^k:H]\le T\} &= \sum_{n_1=1}^{\lfloor T\rfloor} n_1^{k-1}\sum_{\substack{n_2>0,\dots,n_k>0\\ n_2\cdots n_k\le T/n_1}} n_2^{k-2}\cdots n_{k-1}^1 n_k^0\\ &= \sum_{n_1=1}^{\lfloor T\rfloor} n_1^{k-1}\cdot\#\{H\subseteq\mathbb{Z}^{k-1} \mid [\mathbb{Z}^{k-1}:H]\le T/n_1\}\\ &\le \sum_{n_1=1}^{\lfloor T\rfloor} n_1^{k-1}(T/n_1)^{k-1} \qquad \text{[by induction on } k\text{]}\\ &= \sum_{n_1=1}^{\lfloor T\rfloor} T^{k-1} = \lfloor T\rfloor\cdot T^{k-1}\le T^k\end{aligned}$$

□

## 2 Volumes

Recall that a bounded subset $U$ of Euclidean space $\mathbb{R}^k$ is said to have *Jordan content* if its volume can be approximated arbitrarily well by unions of boxes contained in it or by unions of boxes containing it, or in other words, that the the characteristic function $\chi_U$ is Riemann integrable. Equivalently, the boundary $\partial U$ of $U$ has (Lebesgue) measure zero (see [21, Theorem 105.2, Lemma

105.2, and the discussion above it]). If $U$ is a possibly unbounded subset of $\mathbb{R}^k$ whose boundary has measure zero, its intersection with any ball will have Jordan content.

Now let's consider the Lie group $G = \mathrm{Sl}_k \mathbb{R}$ as a subspace of the Euclidean space $M_k\mathbb{R}$ of $k \times k$ matrices. Siegel defines a Haar measure on $G$ as follows (see page 341 of [29]). Let $E$ be a subset of $G$. Letting $I = [0, 1]$ be the unit interval and considering a number $T > 0$, we may consider the following cones.

$$\begin{aligned} I \cdot E &= \{t \cdot B \mid B \in E, 0 \le t \le 1\} \\ T \cdot I \cdot E &= \{t \cdot B \mid B \in E, 0 \le t \le T\} \\ \mathbb{R}^+ \cdot E &= \{t \cdot B \mid B \in E, 0 \le t\} \end{aligned}$$

Observe that if $B \in T \cdot I \cdot E$, then $0 \le \det B \le T^k$.

Definition 2.1. *We say that $E$ is measurable if $I \cdot E$ is, and in that case we define $\mu_\infty(E) = \mathrm{vol}(I \cdot E) \in [0, \infty]$.*

The Jacobian of left or right multiplication by a matrix $\gamma$ on $M_k\mathbb{R}$ is $(\det B)^k$, so for $\gamma \in \mathrm{Sl}_k \mathbb{R}$ volume is preserved. Thus the measure is invariant under $G$, by multiplication on either side. According to Siegel, the introduction of such invariant measures on Lie groups goes back to Hurwitz (see [10, p. 546] or [9]).

Let $F \subseteq G$ be the fundamental domain for the action of $\Gamma = \mathrm{Sl}_k \mathbb{Z}$ on the right of $G$ presented in [15, section 7]; it's an elementary construction of a fundamental domain which is a Borel set without resorting to Minkowski's reduction theory. In each orbit they choose the element which is closest to the identity matrix in the standard Euclidean norm on $M_k\mathbb{R} \cong \mathbb{R}^{k^2}$, and ties are broken by ordering $M_k\mathbb{R}$ lexicographically. This set $F$ is the union of an open subset of $G$ (consisting of those matrices with no ties) and a countable number of sets of measure zero.

The intersection of $T \cdot I \cdot F$ with a ball has Jordan content. To establish that, it is enough to show that the measure of the boundary $\partial F$ in $G$ is zero. Suppose $g \in \partial F$. Then it is a limit of points $g_i \notin F$, each of which has another point $g_i h_i$ in its orbit which is at least as close to 1. Here $h_i$ is in $\mathrm{Sl}_k(\mathbb{Z})$ and is not 1. The sequence $i \mapsto g_i h_i$ is bounded, and thus so is the sequence $h_i$; since $\mathrm{Sl}_k(\mathbb{Z})$ is discrete, that implies that $h_i$ takes only a finite number of values. So we may assume $h_i = h$ is independent of $i$, and is not 1. By continuity, $gh$ is at least as close to 1 as $g$ is. Now $g$ is also a limit of points $f_i$ in $F$, each of which has $f_i h$ not closer to 1 than $f_i$ is. Hence $gh$ is not closer to 1 than $g$ is, by continuity. Combining, we see that $gh$ and $g$ are equidistant from 1. The locus of points $g$ in $\mathrm{Sl}_k(\mathbb{R})$ such that $gh$ and $g$ are equidistant from 1 is given by the vanishing of a nonzero quadratic polynomial, hence has measure zero. The boundary $\partial F$ is contained in a countable number of such sets, because $\mathrm{Sl}_k(\mathbb{Z})$ is countable, hence has measure zero, too.

We remark that HNF contains a unique representative for each orbit of the action of $\mathrm{Sl}_k \mathbb{Z}$ on $\{A \in M_k\mathbb{Z} \mid \det A > 0\}$. The same is true for $\mathbb{R}^+ \cdot F$.

Restricting our attention to matrices $B$ with $\det B \leq T^k$ we see that $\#(T \cdot I \cdot F \cap M_k\mathbb{Z}) = \#\{A \in \mathrm{HNF} \mid \det A \leq T^k\}$.

Warning: HNF is not contained in $\mathbb{R}^+ \cdot F$. To convince yourself of this, consider the matrix $A = \begin{pmatrix} 5 & -8 \\ 3 & 5 \end{pmatrix}$ of determinant 49. Column operations with integer coefficients reduce it to $B = \begin{pmatrix} 49 & 18 \\ 0 & 1 \end{pmatrix}$, but $(1/7)A$ is closer to the identity matrix than $(1/7)B$ is, so $B \in HNF$, but $B \notin \mathbb{R}^+ \cdot F$.

We want to approximate the volume of $T \cdot I \cdot F$ by counting the lattice points it contains, i.e., by using the number $\#(T \cdot I \cdot F \cap M_k\mathbb{Z})$, at least when $T$ is large. Alternatively, we may use $\#(I \cdot F \cap r \cdot M_k\mathbb{Z})$, when $r$ is small.

Definition 2.2. *Suppose $U$ is a subset of $\mathbb{R}^n$. Let*

$$N_r(U) = r^n \cdot \#\{U \cap r \cdot \mathbb{Z}^n\}$$

*and let*

$$\mu_{\mathbb{Z}}(U) = \lim_{r \to 0} N_r(U),$$

*if the limit exists, possibly equal to $+\infty$. An equation involving $\mu_{\mathbb{Z}}(U)$ is to be regarded as true only if the limit exists.*

Lemma 2.3. $\mu_{\mathbb{Z}}(I \cdot F) = \zeta(2)\zeta(3) \cdots \zeta(k)/k$

*Proof.* We replace $r$ above with $1/T$:

$$\begin{aligned} \mu_{\mathbb{Z}}(I \cdot F) &= \lim_{T \to \infty} T^{-k^2} \cdot \#(T \cdot I \cdot F \cap M_k\mathbb{Z}) \\ &= \lim_{T \to \infty} T^{-k^2} \cdot \#\{A \in \mathrm{HNF} \mid \det A \leq T^k\} \\ &= \lim_{T \to \infty} T^{-k^2} \cdot \#\{H \subseteq \mathbb{Z}^k \mid [\mathbb{Z}^k : H] \leq T^k\} \\ &= \zeta(2)\zeta(3) \cdots \zeta(k)/k \qquad \text{[using lemma 1.1]} \end{aligned}$$

□

Lemma 2.4. *If $U$ is a bounded subset of $\mathbb{R}^n$ with Jordan content, then $\mu_{\mathbb{Z}}(U) = \operatorname{vol} U$.*

*Proof.* Subdivide $\mathbb{R}^n$ into cubes of width $r$ (and of volume $r^n$) centered at the points of $r\mathbb{Z}^n$. The number $\#\{U \cap r \cdot \mathbb{Z}^n\}$ lies between the number of cubes contained in $U$ and the number of cubes meeting $U$, so $r^n \cdot \#\{U \cap r \cdot \mathbb{Z}^n\}$ is captured between the total volume of the cubes contained in $U$ and the total volume of the cubes meeting $U$, hence approaches the same limit those two quantities do, namely $\operatorname{vol} U$. □

Lemma 2.5. *Let $B_R$ be the ball of radius $R > 0$ centered at the origin, and let $U$ be a subset of $\mathbb{R}^n$ whose boundary has measure zero.*

1. *For all $R$, the quantity $\mu_{\mathbb{Z}}(U)$ exists if and only if $\mu_{\mathbb{Z}}(U - B_R)$ exists, and in that case, $\mu_{\mathbb{Z}}(U) = \operatorname{vol}(U \cap B_R) + \mu_{\mathbb{Z}}(U - B_R)$.*
2. *If $\mu_{\mathbb{Z}}(U)$ exists then $\mu_{\mathbb{Z}}(U) = \operatorname{vol}(U) + \lim_{R\to\infty} \mu_{\mathbb{Z}}(U - B_R)$.*
3. *If $\operatorname{vol}(U) = +\infty$, then $\mu_{\mathbb{Z}}(U) = +\infty$.*
4. *If $\lim_{R\to\infty} \limsup_{r\to 0} N_r(U - B_R) = 0$, then $\mu_{\mathbb{Z}}(U) = \operatorname{vol}(U)$.*

*Proof.* Writing $U = (U \cap B_R) \cup (U - B_R)$ we have

$$N_r(U) = N_r(U \cap B_R) + N_r(U - B_R).$$

For each $R > 0$, the set $U \cap B_R$ is a bounded set with Jordan content, and thus lemma 2.4 applies to it. We deduce that

$$\liminf_{r\to 0} N_r(U) = \operatorname{vol}(U \cap B_R) + \liminf_{r\to 0} N_r(U - B_R)$$

and

$$\limsup_{r\to 0} N_r(U) = \operatorname{vol}(U \cap B_R) + \limsup_{r\to 0} N_r(U - B_R),$$

from which we can deduce (1), because $\operatorname{vol}(U \cap B_R) < \infty$. We deduce (2) from (1) by taking limits. Letting $R \to \infty$ in the equalities above we see that

$$\liminf_{r\to 0} N_r(U) = \operatorname{vol}(U) + \lim_{R\to\infty} \liminf_{r\to 0} N_r(U - B_R)$$

and

$$\limsup_{r\to 0} N_r(U) = \operatorname{vol}(U) + \lim_{R\to\infty} \limsup_{r\to 0} N_r(U - B_R),$$

in which some of the terms might be $+\infty$. Now (3) follows from $\liminf_{r\to 0} N_r(U) \geq \operatorname{vol}(U)$, and (4) follows because if

$$\lim_{R\to\infty} \limsup_{r\to 0} N_r(U - B_R) = 0,$$

then

$$\lim_{R\to\infty} \liminf_{r\to 0} N_r(U - B_R) = 0$$

also. □

Lemma 2.6. *If $U$ is a subset of $\mathbb{R}^n$ whose boundary has measure zero, and $\mu_{\mathbb{Z}}(U) = \operatorname{vol}(U)$, then $\operatorname{vol}(T \cdot U) \sim \#(T \cdot U \cap \mathbb{Z}^n)$ as $T \to \infty$.*

*Proof.* The statement follows immediately from the definitions. □

Care is required in trying to compute the volume of $I \cdot F$ by counting lattice points in it, for it is not a bounded set (even for $k = 2$, because $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \in F$).

Example 2.7. It's easy to construct an unbounded region where counting lattice points does not determine the volume, by concentrating infinitely many very thin spikes along rays of rational slope with small numerator and denominator. Consider, for example, a bounded region $B$ in $\mathbb{R}^2$ with Jordan content and nonzero area $v = \operatorname{vol} B$, for which (by Lemma 2.4) $\mu_{\mathbb{Z}} B = \operatorname{vol} B$. Start by replacing $B$ by its intersection $B'$ with the lines through the origin of rational (or infinite) slope – this doesn't change the value of $\mu_{\mathbb{Z}}$, because every lattice point is contained in a line of rational slope, but now the boundary $\partial B'$ does not have measure zero. To repair that, we enumerate the lines $M_1, M_2, \dots$ through the origin of rational slope, and for each $i = 1, 2, 3, \dots$ we replace $R_i = B \cap M_i$ by a suitably scaled and rotated version $L_i$ of it contained in the line $N_i$ of slope $i$ through the origin, with scaling factor chosen precisely so $L_i$ intersects each $r \cdot \mathbb{Z}^2$ in the same number of points as does $R_i$, for every $r > 0$. The scaling factor is the ratio of the lengths of the shortest lattice points in the lines $M_i$ and $N_i$. The union $L = \bigcup L_i$ has $\mu_{\mathbb{Z}} L = \mu_{\mathbb{Z}} B = v \neq 0$, but it and its boundary have measure zero.

## 3 Reduction Theory

In this section we apply reduction theory to show that the volume of $I \cdot F$ can be computed by counting lattice points.

We introduce a few basic notions about lattices. For a more leisurely introduction see [7].

Definition 3.1. *A lattice is a free abelian group $L$ of finite rank equipped with an inner product on the vector space $L \otimes \mathbb{R}$.*

We will regard $\mathbb{Z}^k$ or one of its subgroups as a lattice by endowing it with the standard inner product on $\mathbb{R}^k$.

Definition 3.2. *If $L$ is a lattice, then a sublattice $L' \subseteq L$ is a subgroup with the induced inner product. The quotient $L/L'$, if it's torsion free, is made into a lattice by equipping it with the inner product on the orthogonal complement of $L'$.*

There's a way to handle lattices with torsion, but we won't need them.

Definition 3.3. *If $L$ is a lattice, then* $\operatorname{covol} L$ *denotes the volume of a fundamental domain for $L$ acting on $L \otimes \mathbb{R}$.*

The covolume can be computed as $|\det(\theta v_1, \cdots, \theta v_k)|$, where $\theta : L \otimes \mathbb{R} \to \mathbb{R}^k$ is an isometry, $\{v_1, \dots, v_k\}$ is a basis of $L$, and $(\theta v_1, \dots, \theta v_k)$ denotes the matrix whose $i$-th column is $\theta v_i$. We have the identity $\operatorname{covol}(L) = \operatorname{covol}(L') \cdot \operatorname{covol}(L/L')$ when $L/L'$ is torsion free.

If $L$ is a subgroup of $\mathbb{Z}^k$ of finite index, then $\operatorname{covol} L = [\mathbb{Z}^k : L]$.

Definition 3.4. *If $L$ is a nonzero lattice, then* $\min L$ *denotes the smallest length of a nonzero vector in $L$.*

If $L$ is a lattice of rank 1, then $\min L = \operatorname{covol} L$.

Proposition 3.5. *For any natural number $k > 0$, there is a constant $c$ such that for any $S \geq 1$ and for any $T > 0$ the following inequality holds.*

$$cS^{-k}T^{k^2} \geq \#\{L \subseteq \mathbb{Z}^k \mid [\mathbb{Z}^k : L] \leq T^k \text{ and } \min L \leq T/S\}.$$

*Proof.* For $k = 1$ we may take $c = 2$, so assume $k \geq 2$. Letting $N$ be the number of these lattices $L$, we bound $N$ by picking within each $L$ a nonzero vector $v$ of minimal length, and counting the pairs $(v, L)$ instead. For each $v$ occurring in such pair we write $v$ in the form $v = n_1 v_1$ where $n_1 \in \mathbb{N}$ and $v_1$ is a primitive vector of $\mathbb{Z}^k$, and then we extend $\{v_1\}$ to a basis $B = \{v_1, \ldots, v_k\}$ of $\mathbb{Z}^k$. We count the lattices $L$ occurring in such pairs with $v$ by putting a basis $C$ for $L$ into Hermite normal form with respect to $B$, i.e., it will have the form $C = \{n_1 v_1, A_{12}v_1 + n_2 v_2, \ldots, A_{1k}v_1 + \cdots + A_{k-1,k}v_{k-1} + n_k v_k\}$, with $n_i > 0$ and $0 \leq A_{ij} < n_i$. Notice that $n_1$ has been determined in the previous step by the choice of $v$. The number of vectors $v \in \mathbb{Z}^k$ satisfying $\|v\| \leq T/S$ is bounded by a number of the form $c(T/S)^k$; for $c$ we may take a large enough multiple of the volume of the unit ball. With notation as above, and counting the bases for $C$ in Hermite normal form as before, we see that

$$\begin{aligned}
N &\leq \sum_{\|v\| \leq T/S} \sum_{\substack{n_2 > 0, \ldots, n_k > 0 \\ n_1 \cdots n_k \leq T^k}} n_1^{k-1} n_2^{k-2} \cdots n_{k-1}^1 n_k^0 \\
&= \sum_{\|v\| \leq T/S} n_1^{k-1} \sum_{\substack{n_2 > 0, \ldots, n_k > 0 \\ n_2 \cdots n_k \leq T^k/n_1}} n_2^{k-2} \cdots n_{k-1}^1 n_k^0 \\
&= \sum_{\|v\| \leq T/S} n_1^{k-1} \cdot \#\{H \subseteq \mathbb{Z}^{k-1} \mid [\mathbb{Z}^{k-1} : H] \leq T^k/n_1\} \\
&\leq \sum_{\|v\| \leq T/S} n_1^{k-1} (T^k/n_1)^{k-1} \qquad \text{[by Lemma 1.2]} \\
&= \sum_{\|v\| \leq T/S} T^{k(k-1)} \\
&\leq c(T/S)^k T^{k(k-1)} \\
&= cS^{-k}T^{k^2}.
\end{aligned}$$

□

Corollary 3.6. *The following equality holds.*

$$0 = \lim_{S \to \infty} \limsup_{T \to \infty} T^{-k^2} \cdot \#\{L \subseteq \mathbb{Z}^k \mid [\mathbb{Z}^k : L] \leq T^k \text{ and } \min L \leq T/S\}$$

The following two lemmas are standard facts. Compare them, for example, with [2, 1.4 and 1.5].

Lemma 3.7. *Let $L$ be a lattice and let $v \in L$ be a primitive vector. Let $\bar{L} = L/\mathbb{Z}v$, let $\bar{w} \in \bar{L}$ be any vector, and let $w \in L$ be a vector of minimal length among all those that project to $\bar{w}$. Then $\|w\|^2 \leq \|\bar{w}\|^2 + (1/4)\|v\|^2$.*

*Proof.* The vectors $w$ and $w \pm v$ project to $\bar{w}$, so $\|w\|^2 \leq \|w \pm v\|^2 = \|w\|^2 + \|v\|^2 \pm 2\langle w, v\rangle$, and thus $|\langle w, v\rangle| \leq (1/2)\|v\|^2$. We see then that

$$\begin{aligned}\|\bar{w}\|^2 &= \|w - \frac{\langle w, v\rangle}{\|v\|^2} v\|^2 \\ &= \|w\|^2 - \frac{\langle w, v\rangle^2}{\|v\|^2} \\ &\geq \|w\|^2 - \frac{1}{4}\|v\|^2.\end{aligned}$$

□

Lemma 3.8. *Let $L$ be a lattice of rank 2 with a nonzero vector $v \in L$ of minimal length. Let $L' = \mathbb{Z}v$ and $L'' = L/L'$. Then* $\operatorname{covol} L'' \geq (\sqrt{3}/2) \operatorname{covol} L'$.

*Proof.* Let $\bar{w} \in L''$ be a nonzero vector of minimal length, and lift it to a vector $w \in L$ of minimal length among possible liftings. By lemma 3.7 $\|w\|^2 \leq \|\bar{w}\|^2 + (1/4)\|v\|^2$. Combining that with $\|v\|^2 \leq \|w\|^2$ we deduce that $\operatorname{covol} L'' = \|\bar{w}\| \geq (\sqrt{3}/2)\|v\| = (\sqrt{3}/2) \operatorname{covol} L'$. □

Definition 3.9. *If $L$ is a lattice, then* $\operatorname{minbasis} L$ *denotes the smallest value possible for $(\|v_1\|^2 + \cdots + \|v_k\|^2)^{1/2}$, where $\{v_1, \ldots, v_k\}$ is a basis of $L$.*

Proposition 3.10. *Given $k \in \mathbb{N}$ and $S \geq 1$, for all $R \gg 0$, for all $T > 0$, and for all lattices $L$ of rank $k$ with* $\operatorname{covol} L \leq T^k$, *if* $\operatorname{minbasis} L \geq RT$ *then* $\min L \leq T/S$.

*Proof.* We show instead the contrapositive: provided $\operatorname{covol} L \leq T^k$, if $\min L > T/S$ then $\operatorname{minbasis} L < RT$. There is an obvious procedure for producing an economical basis of a lattice $L$, namely: we let $v_1$ be a nonzero vector in $L$ of minimal length; we let $v_2$ be a vector in $L$ of minimal length among those projecting onto a nonzero vector in $L/(\mathbb{Z}v_1)$ of minimal length; we let $v_3$ be a vector in $L$ of minimal length among those projecting onto a vector in $L/(\mathbb{Z}v_1)$ of minimal length among those projecting onto a nonzero vector in $L/(\mathbb{Z}v_1 + \mathbb{Z}v_2)$ of minimal length; and so on. A vector of minimal length is primitive, so one can show by induction that the quotient group $L/(\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_i)$ is torsion free; the case where $i = k$ tells us that $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k$. Let $L_i = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_i$, and let $\alpha_i = \operatorname{covol}(L_i/L_{i-1})$, so that $\alpha_1 = \|v_1\| = \min L > T/S$.

Applying Lemma 3.8 to the rank 2 lattice $L_i/L_{i-2}$ shows that $\alpha_i \geq A\alpha_{i-1}$, where $A = \sqrt{3}/2$, and repeated application of Lemma 3.7 shows that $\|v_i\|^2 \leq$

$\alpha_i^2 + (1/4)(\alpha_{i-1}^2 + \cdots + \alpha_1^2)$, so of course $\|v_i\|^2 \le (1/4)(\alpha_k^2 + \cdots + \alpha_{i+1}^2) + \alpha_i^2 + (1/4)(\alpha_{i-1}^2 + \cdots + \alpha_1^2)$. We deduce that

$$\operatorname{minbasis} L \le (\sum_{i=1}^{k} \|v_i\|^2)^{1/2} \le \left(\frac{k+3}{4} \sum \alpha_i^2\right)^{1/2}. \tag{3}$$

Going a bit further, we see that

$$\begin{aligned} T^k &\ge \operatorname{covol} L \\ &= \alpha_1 \cdots \alpha_k \\ &\ge A^{0+1+2+\cdots+(i-2)} \alpha_1^{i-1} \cdot A^{0+1+2+\cdots+(k-i)} \alpha_i^{k-i+1} \\ &> c_1 (T/S)^{i-1} \alpha_i^{k-i+1} \end{aligned}$$

where $c_1$ is some constant depending on $S$ which we may take to be independent of $i$. Dividing through by $T^{i-1}$ we get $T^{k-i+1} > c_2 \alpha_i^{k-i+1}$, from which we deduce that $T > c_3 \alpha_i$, where $c_2$ and $c_3$ are new constants (depending only on $S$). Combining these latter inequalities for each $i$, we find that $(((k+3)/4) \sum \alpha_i^2)^{1/2} < RT$, where $R$ is a new constant (depending only on $S$); combining that with (3) yields the result. $\square$

Corollary 3.11. *The following equality holds.*

$$0 = \lim_{R\to\infty} \limsup_{T\to\infty} T^{-k^2} \cdot \#\{L \subseteq \mathbb{Z}^k \mid [\mathbb{Z}^k : L] \le T^k \text{ and } \operatorname{minbasis} L \ge RT\}$$

*Proof.* Combine (3.6) and (3.10). $\square$

If in the definition of our fundamental domain $F$ we had taken the smallest element of each orbit, rather than the one nearest to 1, we would have been almost done now. The next lemma takes care of that discrepancy.

Definition 3.12. *If $L$ is a (discrete) lattice of rank $k$ in $\mathbb{R}^k$, then* $\operatorname{size} L$ *denotes the value of* $(\|w_1\|^2 + \cdots + \|w_k\|^2)^{1/2}$, *where* $\{w_1, \ldots, w_k\}$ *is the (unique) basis of $L$ satisfying* $(w_1, \ldots, w_k) \in \mathbb{R}^+ \cdot F$.

Lemma 3.13. *For any (discrete) lattice $L \subseteq \mathbb{R}^k$ of rank $k$ the inequalities*

$$\operatorname{minbasis} L \le \operatorname{size} L \le \operatorname{minbasis} L + 2\sqrt{k}(\operatorname{covol} L)^{1/k}$$

*hold.*

*Proof.* Let $\{v_1, \ldots, v_k\}$ be the basis envisaged in the definition of $\operatorname{minbasis} L$, let $\{w_1, \ldots, w_k\}$ be the basis of $L$ envisaged the definition of $\operatorname{size} L$, and let $U = (\operatorname{covol} L)^{1/k} = (\det(v_1, \ldots, v_k))^{1/k} = (\det(w_1, \ldots, w_k))^{1/k}$. The following chain of inequalities gives the result.

$$\begin{aligned} \operatorname{minbasis} L &= \|(v_1, \ldots, v_k)\| \le \operatorname{size} L \\ &= \|(w_1, \ldots, w_k)\| \le \|(w_1, \ldots, w_k) - U \cdot 1_k\| + U\sqrt{k} \\ &\le \|(v_1, \ldots, v_k) - U \cdot 1_k\| + U\sqrt{k} \\ &\le \|(v_1, \ldots, v_k)\| + 2U\sqrt{k} = \operatorname{minbasis} L + 2U\sqrt{k} \end{aligned}$$

□

COROLLARY 3.14. *The following equality holds.*

$$0 = \lim_{Q\to\infty} \limsup_{T\to\infty} T^{-k^2} \cdot \#\{L \subseteq \mathbb{Z}^k \mid [\mathbb{Z}^k : L] \leq T^k \text{ and } \operatorname{size} L \geq QT\}$$

*Proof.* It follows from (3.13) that given $R > 0$, for all $Q \gg 0$ (namely $Q \geq R + 2\sqrt{k}$) if $\operatorname{covol} L \leq T^k$ and $\operatorname{size} L \geq QT$ then $\operatorname{minbasis} L \geq RT$. Now apply (3.11). □

THEOREM 3.15. $\operatorname{vol}(I \cdot F) = \mu_{\mathbb{Z}}(I \cdot F)$.

*Proof.* Observe that $\#\{L \subseteq \mathbb{Z}^k \mid \operatorname{covol} L \leq T^k \text{ and } \operatorname{size} L \geq QT\} = \#((T \cdot I \cdot F - B_{QT}) \cap M_k\mathbb{Z}) = \#((I \cdot F - B_Q) \cap T^{-1}M_k\mathbb{Z})$, so replacing $1/T$ by $r$, Corollary 3.14 implies that $\lim_{Q\to\infty} \limsup_{r\to 0} N_r(I \cdot F - B_Q) = 0$, which allows us to apply Lemma 2.5 (4). □

The theorem allows us to compute the volume of $F$ arithmetically, simultaneously showing it's finite.

COROLLARY 3.16. $\mu_\infty(G/\Gamma) = \zeta(2)\zeta(3)\cdots\zeta(k)/k$

*Proof.* Combine the theorem with lemma 2.3 as follows.

$$\mu_\infty(G/\Gamma) = \mu_\infty(F) = \operatorname{vol}(I \cdot F) = \mu_{\mathbb{Z}}(I \cdot F) = \zeta(2)\zeta(3)\cdots\zeta(k)/k$$

□

REMARK 3.17. *Theorem 10.4 in [22] states that the volume of $G/\Gamma$ is $\zeta(2)\zeta(3)\cdots\zeta(k)\sqrt{k}$. The difference arises from a different choice of Haar measure on $G$. Theirs assigns volume $\sqrt{k}$ to $\mathfrak{sl}_k(\mathbb{R})/\mathfrak{sl}_k(\mathbb{Z})$, whereas ours assigns volume $1/k$ to it, as we see in formula (14) below. The ambiguity is unavoidable, because there is no canonical choice of Haar measure. (The Tamagawa number resolves that ambiguity.)*

## 4 $p$-ADIC VOLUMES

In this section we reformulate the computation of the volume of $G/\Gamma$ to yield a natural and informative computation of the Tamagawa number of $\mathrm{Sl}_k$. We are interested in the form of the proof, not its length, so we incorporate the proofs of (3.16) and (2) rather than their statements. The standard source for information about $p$-adic measures and Tamagawa measures is Chapter II of [37], and the proof we simplify occurs there in sections 3.1 through 3.4. See also [11] and [20].

We let $\mu_p$ denote the standard translation invariant measure on $\mathbb{Q}_p$ normalized so that $\mu_p(\mathbb{Z}_p) = 1$. Let $\mu_p$ also denote the product measure on the ring of $k$ by $k$ matrices, $M_k(\mathbb{Q}_p)$. Observe that $\mu_p(M_k(\mathbb{Z}_p)) = 1$.

For $x \in \mathbb{Q}_p$, let $|x|_p$ denote the standard valuation normalized so that $|p|_p = 1/p$

If $A \in M_k(\mathbb{Q}_p)$ and $U \subseteq \mathbb{Q}_p^k$, then $\mu_p(A \cdot U) = |\det A|_p \cdot \mu_p(U)$. (To prove this, first diagonalize $A$ using row and column operations, and then assume that $U$ is a cube.) It follows that if $V \subseteq M_k(\mathbb{Q}_p)$, then $\mu_p(A \cdot V) = |\det A|_p^k \cdot \mu_p(V)$.

Consider $\mathrm{Gl}_k(\mathbb{Z}_p)$ as an open subset of $M_k(\mathbb{Z}_p)$. The following computation occurs on page 31 of [37].

$$\begin{aligned} \mu_p(\mathrm{Gl}_k(\mathbb{Z}_p)) &= \#(\mathrm{Gl}_k(\mathbb{F}_p))/p^{k^2} \\ &= (p^k - 1)(p^k - p)\cdots(p^k - p^{k-1})/p^{k^2} \\ &= (1 - p^{-k})(1 - p^{-k+1})\cdots(1 - p^{-1}) \end{aligned} \tag{4}$$

Weil considers the open set $M_k(\mathbb{Z}_p)^* = \{A \in M_k(\mathbb{Z}_p) \mid \det A \neq 0\}$.

Lemma 4.1. $\mu_p(M_k(\mathbb{Z}_p)^*) = 1$

*Proof.* Let $Z = M_k(\mathbb{Z}_p) \setminus M_k(\mathbb{Z}_p)^*$ be the set of singular matrices. If $A \in Z$, then one of the columns of $A$ is a linear combination of the others. (This depends on $\mathbb{Z}_p$ being a discrete valuation ring – take any linear dependency with coefficients in $\mathbb{Q}_p$ and multiply the coefficients by a suitable power of $p$ to put all of them in $\mathbb{Z}_p$, with at least one of them being invertible.) For each $n \geq 0$ we can get an upper bound for the number of equivalence classes of elements of $Z$ modulo $p^n$ by enumerating the possibly dependent columns, the possible vectors in the other columns, and the possible coefficients in the linear combination: $\mu_p(Z) \leq \lim_{n\to\infty} k \cdot (p^{nk})^{k-1} \cdot (p^n)^{k-1}/(p^n)^{k^2} = \lim_{n\to\infty} k \cdot p^{-n} = 0$. ☐

We call rank $k$ submodules $J$ of $\mathbb{Z}_p^k$ *lattices*. To each $A \in M_k(\mathbb{Z}_p)^*$ we associate the lattice $J = A\mathbb{Z}_p^k \subseteq \mathbb{Z}_p^k$. This sets up a bijection between the lattices $J$ and the orbits of $\mathrm{Gl}_k(\mathbb{Z}_p)$ acting on $M_k(\mathbb{Z}_p)^*$. The measure of the orbit corresponding to $J$ is $\mu_p(A \cdot \mathrm{Gl}_k(\mathbb{Z}_p)) = |\det A|_p^k \cdot \mu_p(\mathrm{Gl}_k(\mathbb{Z}_p)) = [\mathbb{Z}_p^k : J]^{-k} \cdot \mu_p(\mathrm{Gl}_k(\mathbb{Z}_p))$. Now we sum over the orbits.

$$\begin{aligned} 1 &= \mu_p(M_k(\mathbb{Z}_p)^*) \\ &= \sum_J \Big( [\mathbb{Z}_p^k : J]^{-k} \cdot \mu_p(\mathrm{Gl}_k(\mathbb{Z}_p)) \Big) \\ &= \Big( \sum_J [\mathbb{Z}_p^k : J]^{-k} \Big) \cdot \mu_p(\mathrm{Gl}_k(\mathbb{Z}_p)) \end{aligned} \tag{5}$$

An alternative way to prove (5) would be to use the local analogue of (2), which holds and asserts that $\sum_J [\mathbb{Z}_p^k : J]^{-s} = (1 - p^{k-1-s})^{-1}(1 - p^{k-2-s})^{-1}\cdots(1 - p^{-s})^{-1}$; we could substitute $k$ for $s$ and compare with the number in (4). The approach via lemma 4.1 and (5) is preferable because $M_k(\mathbb{Z}_p)^*$ provides natural glue that makes the computation seem more natural.

The product $\prod_p \mu_p(\mathrm{Gl}_k(\mathbb{Z}_p))$ doesn't converge because $\prod_p (1-p^{-1})$ doesn't converge, so consider the following formula instead.

$$1 = \Big( (1-p^{-1}) \sum_J [\mathbb{Z}_p^k : J]^{-k} \Big) \cdot \Big( (1-p^{-1})^{-1} \mu_p(\mathrm{Gl}_k(\mathbb{Z}_p)) \Big)$$

Now we can multiply these formulas together.

$$1 = \Big( \prod_p (1-p^{-1}) \sum_J [\mathbb{Z}_p^k : J]^{-k} \Big) \cdot \prod_p \Big( (1-p^{-1})^{-1} \mu_p(\mathrm{Gl}_k(\mathbb{Z}_p)) \Big) \tag{6}$$

We've parenthesized the formula above so it has one factor for each place of $\mathbb{Q}$, and now we connect each of them with a volume involving $\mathrm{Sl}_k$ at that place.

We use the Haar measure on $\mathrm{Sl}_k(\mathbb{Z}_p)$ normalized to have total volume

$$\#\, \mathrm{Sl}_k(\mathbb{F}_p)/p^{\dim \mathrm{Sl}_k}.$$

The normalization anticipates (13), which shows how a gauge form could be used to construct the measure, or alternatively, it ensures that the exact sequence $1 \to \mathrm{Sl}_k(\mathbb{Z}_p) \to \mathrm{Gl}_k(\mathbb{Z}_p) \to \mathbb{Z}_p^\times \to 1$ of groups leads to the desired assertion $\mu_p(\mathrm{Gl}_k(\mathbb{Z}_p)) = \mu_p(\mathbb{Z}_p^\times) \cdot \mu_p(\mathrm{Sl}_k(\mathbb{Z}_p))$ about multiplicativity of measures. We rewrite the factor of the right hand side of (6) corresponding to the prime $p$ as follows.

$$\begin{aligned} (1-p^{-1})^{-1} \mu_p(\mathrm{Gl}_k(\mathbb{Z}_p)) &= \mu_p(\mathbb{Z}_p^\times)^{-1} \cdot \mu_p(\mathrm{Gl}_k(\mathbb{Z}_p)) \\ &= \mu_p(\mathrm{Sl}_k(\mathbb{Z}_p)). \end{aligned} \tag{7}$$

To evaluate the left hand factor of the right hand side of (6), we insert the complex variable $s$. Because the ring $\mathbb{Z}$ is a principal ideal domain, any finitely generated sub-$\mathbb{Z}$-module $H \subseteq \mathbb{Z}^k$ is free. Hence a lattice $H \subseteq \mathbb{Z}^k$ is determined freely by its localizations $H_p = H \otimes_{\mathbb{Z}} \mathbb{Z}_p \subseteq \mathbb{Z}_p^k$ (where $H_p = \mathbb{Z}_p^k$ for all but finitely many $p$), and its index is given by the formula

$$[\mathbb{Z}^k : H] = \prod_p [\mathbb{Z}_p^k : H_p], \tag{8}$$

in which only a finite number of terms are not equal to 1.

$$
\begin{aligned}
&\operatorname{res}_{s=k}\zeta(\mathbb{Z}^k, s)\\
&\qquad = \operatorname{res}_{s=k}\sum_H [\mathbb{Z}^k : H]^{-s} \qquad \text{[by (1)]}\\
&\qquad = \lim_{s\to k+} \zeta(s-k+1)^{-1}\cdot \sum_H [\mathbb{Z}^k : H]^{-s}\\
&\qquad = \lim_{s\to k+}\Big(\zeta(s-k+1)^{-1}\big(\sum_{H\subseteq \mathbb{Z}^k}\prod_p [\mathbb{Z}_p^k : H_p]^{-s}\big)\Big) \qquad \text{[by (8)]}\\
&\qquad = \lim_{s\to k+}\Big(\zeta(s-k+1)^{-1}\big(\prod_p \sum_{J\subseteq \mathbb{Z}_p^k} [\mathbb{Z}_p^k : J]^{-s}\big)\Big) \qquad \text{[positive terms]}\\
&\qquad = \lim_{s\to k+}\prod_p \big((1-p^{-s+k-1})\sum_J [\mathbb{Z}_p^k : J]^{-s}\big)\\
&\qquad = \prod_p (1-p^{-1})\sum_J [\mathbb{Z}_p^k : J]^{-k}
\end{aligned}
\tag{9}
$$

Starting again we get the following chain of equalities.

$$
\begin{aligned}
\operatorname{res}_{s=k}\zeta(\mathbb{Z}^k, s) &= \operatorname{res}_{s=k}\zeta(s-k+1)\zeta(s-k+2)\cdots\zeta(s-1)\zeta(s)\\
&= \zeta(2)\cdots\zeta(k-1)\zeta(k)\\
&= k\cdot \lim_{T\to\infty} T^{-k}\#\{H\subseteq \mathbb{Z}^k \mid [\mathbb{Z}^k : H]\le T\} \qquad \text{[by 1.1]}\\
&= k\cdot \lim_{T\to\infty} T^{-k^2}\#\{H\subseteq \mathbb{Z}^k \mid [\mathbb{Z}^k : H]\le T^k\}\\
&= k\cdot \lim_{T\to\infty} T^{-k^2}\#\{A\in \mathrm{HNF} \mid \det A\le T^k\}\\
&= k\cdot \mu_{\mathbb{Z}}(I\cdot F) \qquad \text{[by definition 2.2]}\\
&= k\cdot \mu_\infty(\mathrm{Sl}_k(\mathbb{R})/\mathrm{Sl}_k(\mathbb{Z})) \qquad \text{[by 3.15 and 2.1]}
\end{aligned}
\tag{10}
$$

Combining (9) and (10) we get the following equation.

$$
\prod_p (1-p^{-1})\sum_J [\mathbb{Z}_p^k : J]^{-k} = k\cdot \mu_\infty(\mathrm{Sl}_k(\mathbb{R})/\mathrm{Sl}_k(\mathbb{Z})) \tag{11}
$$

We combine (6), (7) and (11) to obtain the following equation.

$$
1 = k\cdot \mu_\infty(\mathrm{Sl}_k(\mathbb{R})/\mathrm{Sl}_k(\mathbb{Z}))\cdot \prod_p \mu_p(\mathrm{Sl}_k(\mathbb{Z}_p)) \tag{12}
$$

To relate this to the Tamagawa number we have to introduce a gauge form $\omega$ on the algebraic group $\mathrm{Sl}_k$ over $\mathbb{Q}$, invariant by left translations, as in sections 2.2.2 and 2.4 of [37]. We can even get gauge forms over $\mathbb{Z}$. Let $X$ be a generic element of $\mathrm{Gl}_k$. The entries of the matrix $X^{-1}dX$ provide a basis for the 1-forms invariant by left translation on $\mathrm{Gl}_k$. On $\mathrm{Sl}_k$ we see that $\operatorname{tr}(X^{-1}dX) = d(\det X) = 0$, so omitting the element in the $(n,n)$ spot will provide a basis of the invariant forms on $\mathrm{Sl}_k$. We let $\omega$ be the exterior product of these forms. Just as in the proof of Theorem 2.2.5 in [37] we obtain the following equality.

$$\int_{\mathrm{Sl}_k(\mathbb{Z}_p)} \omega_p = \mu_p(\mathrm{Sl}_k(\mathbb{Z}_p)) \tag{13}$$

The measure $\omega_p$ is defined in [37, 2.2.1] in a neighborhood of a point $P$ by writing $\omega = f\,dx_1 \wedge \cdots \wedge dx_n$ and setting $\omega_p = |f(P)|_p (dx_1)_p \ldots (dx_n)_p$, where $(dx_i)_p$ is the Haar measure on $\mathbb{Q}_p$ normalized so that $\int_{\mathbb{Z}_p} (dx_i)_p = 1$, and $|c|_p$ is the $p$-adic valuation normalized so that $d(cx)_p = |c|_p (dx)_p$.

Now we want to determine the constant that relates our original Haar measure $\mu_\infty$ on $\mathrm{Sl}_k(\mathbb{R})$ to the one determined by $\omega_\infty$. For this purpose, it will suffice to evaluate both measures on the infinitesimal parallelepiped $B$ in $\mathrm{Sl}_k(\mathbb{R})$ centered at the identity matrix and spanned by the tangent vectors $\varepsilon e_{ij}$ for $i \neq j$ and $\varepsilon(e_{ii} - e_{kk})$ for $i < k$. Here $\varepsilon$ is an infinitesimal number, and $e_{ij}$ is the matrix with a 1 in position $(i,j)$ and zeroes elsewhere. For the purpose of this computation, we may even take $\varepsilon = 1$. We remark that $B$ is a fundamental domain for $\mathfrak{sl}_k(\mathbb{Z})$ acting on the Lie algebra $\mathfrak{sl}_k(\mathbb{R})$. We compute easily that $\int_B \omega_\infty = 1$ and

$$\begin{aligned} \mu_\infty(B) &= \operatorname{vol}(I \cdot B) \\ &= (1/k^2) \cdot |\det(e_{11} - e_{kk}, \cdots, e_{k-1,k-1} - e_{kk}, \textstyle\sum e_{ii})| \\ &= (1/k^2) \cdot |\det(e_{11} - e_{kk}, \cdots, e_{k-1,k-1} - e_{kk}, k e_{kk})| \\ &= (1/k^2) \cdot |\det(e_{11}, \cdots, e_{k-1,k-1}, k e_{kk})| \\ &= 1/k \end{aligned} \tag{14}$$

We obtain the following equation.

$$\mu_\infty(\mathrm{Sl}_k(\mathbb{R})/\mathrm{Sl}_k(\mathbb{Z})) = \frac{1}{k} \int_{\mathrm{Sl}_k(\mathbb{R})/\mathrm{Sl}_k(\mathbb{Z})} \omega_\infty \tag{15}$$

See [36, §14.12, (3)] for an essentially equivalent proof of this equation. We may now rewrite (12) as follows.

$$1 = \int_{\mathrm{Sl}_k(\mathbb{R})/\mathrm{Sl}_k(\mathbb{Z})} \omega_\infty \cdot \prod_p \int_{\mathrm{Sl}_k(\mathbb{Z}_p)} \omega_p \tag{16}$$

(If done earlier, this computation would have justified normalizing $\mu_\infty$ differently.)

The Tamagawa number $\tau(\mathrm{Sl}_{k,\mathbb{Q}}) = \int_{\mathrm{Sl}_k(\mathbb{A}_\mathbb{Q})/\mathrm{Sl}_k(\mathbb{Q})} \omega$ is the same as the right hand side of (16) because $F \times \prod_p \mathrm{Sl}_k(\mathbb{Z}_p)$ is a fundamental domain for the action of $\mathrm{Sl}_k(\mathbb{Q})$ on $\mathrm{Sl}_k(\mathbb{A}_\mathbb{Q})$. Thus $\tau(\mathrm{Sl}_{k,\mathbb{Q}}) = 1$. This was originally proved by Weil in Theorem 3.3.1 of [37]. See also [14], [12], and [36, §14.11, Corollary to Langlands' Theorem].

See also [33, §8] for an explanation that Siegel's measure formula amounts to the first determination that $\tau(SO) = 2$.

## A Dirichlet series

Theorem A.1. *Suppose we are given a Dirichlet series* $f(s) := \sum_{n=1}^\infty a_n n^{-s}$ *with nonnegative coefficients. Let* $A(T) := \sum_{n\le T} a_n$. *If* $A(T) = O(T^k)$ *as* $T \to \infty$, *then* $\sum_{n=T}^\infty a_n n^{-s} = O(T^{k-s})$ *as* $T \to \infty$, *and thus* $f(s)$ *converges for all complex numbers* $s$ *with* $\mathrm{Re}\, s > k$.

*Proof.* Write $\sigma = \mathrm{Re}\, s$ and assume $\sigma > k$. We estimate the tail of the series as follows.

$$\begin{aligned}\sum_{n=T}^\infty a_n n^{-s} &= \int_T^\infty x^{-s}\, dA(x)\\ &= x^{-s}A(x)\big]_T^\infty - \int_T^\infty A(x)\, d(x^{-s})\\ &= x^{-s}A(x)\big]_T^\infty + s\int_T^\infty x^{-s-1}A(x)\, dx\\ &= O(x^{k-\sigma})\big]_T^\infty + s\int_T^\infty x^{-s-1}O(x^k)\, dx\\ &= O(T^{k-\sigma}) + s\int_T^\infty O(x^{k-\sigma-1})\, dx\\ &= O(T^{k-\sigma})\end{aligned}$$

□

Theorem A.2. *Suppose we are given two Dirichlet series*

$$f(s) := \sum_{n=1}^\infty a_n n^{-s} \qquad g(s) := \sum_{n=1}^\infty b_n n^{-s}$$

*with nonnegative coefficients and corresponding coefficient summatory functions*

$$A(T) := \sum_{n\le T} a_n \qquad B(T) := \sum_{n\le T} b_n$$

*Assume that* $A(T) = O(T^i)$ *and* $B(T) = cT^k + O(T^j)$, *where* $i \le j < k$. *Let* $h(s) := f(s)g(s) = \sum_{n=1}^\infty c_n n^{-s}$, *and let* $C(T) := \sum_{n\le T} c_n$. *Then* $C(T) = cf(k)T^k + O(T^j \log T)$ *if* $i = j$, *and* $C(T) = cf(k)T^k + O(T^j)$ *if* $i < j$.

*Proof.* The basic idea for this proof was told to us by Harold Diamond.

Observe that Theorem A.1 ensures that $f(k)$ converges. Let's fix the notation $\beta(T) = O(\gamma(T))$ to mean that there is a constant $C$ so that $|\beta(T)| \leq C\gamma(T)$ for all $T \in [1, \infty)$, and simultaneously replace $O(T^j \log T)$ in the statement by $O(T^j(1 + \log T))$ in order to avoid the zero of $\log T$ at $T = 1$. We will use the notation in an infinite sum only with a uniform value of the implicit constant $C$.

We examine $C(T)$ as follows.

$$\begin{aligned}
C(T) &= \sum_{n\leq T} c_n = \sum_{n\leq T}\sum_{pq=n} a_p b_q = \sum_{pq\leq n} a_p b_q \\
&= \sum_{p\leq T} a_p \sum_{q\leq T/p} b_q = \sum_{p\leq T} a_p B(T/p) \\
&= \sum_{p\leq T} a_p\{c(T/p)^k + O((T/p)^j)\} \\
&= cT^k \sum_{p\leq T} a_p p^{-k} + O(T^j)\sum_{p\leq T} a_p p^{-j} \\
&= cT^k\{f(k) + O(T^{i-k})\} + O(T^j)\sum_{p\leq T} a_p p^{-j} \\
&= cf(k)T^k + O(T^i) + O(T^j)\sum_{p\leq T} a_p p^{-j}
\end{aligned}$$

If $i < j$ then $\sum_{p\leq T} a_p p^{-j} \leq f(j) = O(1)$. Alternatively, if $i = j$, then

$$\begin{aligned}
\sum_{p\leq T} a_p p^{-j} = \sum_{p\leq T} a_p p^{-i} &= \int_{1-}^{T} p^{-i}\, d(A(p)) \\
&= p^{-i}A(p)\Big]_{1-}^{T} - \int_{1-}^{T} A(p)\, d(p^{-i}) \\
&= T^{-i}A(T) + i\int_{1-}^{T} A(p)p^{-i-1}\, dp \\
&= O(1) + O(\int_{1-}^{T} p^{-1}\, dp) = O(1 + \log T)
\end{aligned}$$

In both cases the result follows. □

The proof of the following "Abelian" theorem for generalized Dirichlet series is elementary.

THEOREM A.3. *Suppose we are given numbers $R$, $k \geq 1$, and $1 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$. Suppose that*

$$N(T) := \sum_{\lambda_n\leq T} 1 = (R + o(1))\frac{T^k}{k} \qquad (T \to \infty)$$

*for some number $R$. Then the generalized Dirichlet series $\psi(s) := \sum \lambda_n^{-s}$ converges for all real numbers $s > k$, and $\lim_{s\to k+}(s-k)\psi(s) = R$.*

*Proof.* In the case $R \neq 0$, the proof can be obtained by adapting the argument in the last part of the proof of [3, Chapter 5, Section 1, Theorem 3]: roughly, one reduces to the case where $k = 1$ by a simple change of variables, shows $\lambda_n \sim n/R$, uses that to compare a tail of $\sum \lambda_n^{-s}$ to a tail of $\zeta(s) = \sum n^{-s}$, and then uses $\lim_{s\to 1+}(s-1)\zeta(s) = 1$.

Alternatively, one can refer to [34, Theorem 10, p. 114] for the statement about convergence, and then to [34, Theorem 2, p. 219] for the statement about the limit. Actually, those two theorems are concerned with Dirichlet series of the form $F(s) = \sum a_n n^{-s}$, but the first step there is to consider the growth rate of $\sum_{n\le x} a_n$ as $x \to \infty$. Essentially the same proof works for $F(s) = \psi(s)$ by considering the growth rate of $N(x)$ instead.

The result also follows from the following estimate, provided to us by Harold Diamond. Assume $s > k$.

$$\begin{aligned}
\psi(s) &:= \sum \lambda_n^{-s} \\
&= \int_{1-}^{\infty} x^{-s}\, dN(x) \\
&= x^{-s}N(x)\Big]_{1-}^{\infty} + s\int_1^{\infty} x^{-s-1}N(x)\, dx \\
&= O(x^{k-s})\Big]^{\infty} + s\int_1^{\infty} x^{-s-1}(R+o(1))\frac{x^k}{k}\, dx \qquad (x\to\infty) \\
&= \frac{s(R+o(1))}{k}\int_1^{\infty} x^{-s-1+k}\, dx \qquad (s\to k+) \\
&= \frac{s(R+o(1))}{k(s-k)} \qquad (s\to k+)
\end{aligned}$$

Notice the shift in the meaning of $o(1)$ from one line to the next, verified by writing $\int_1^\infty = \int_1^b + \int_b^\infty$ and letting $b$ go to $\infty$; it turns out that for sufficiently small $\epsilon$ the major contribution to $\int_1^\infty x^{-1-\epsilon}\, dx$ comes from $\int_b^\infty x^{-1-\epsilon}\, dx$. $\square$

The following Wiener-Ikehara "Tauberian" theorem is a converse to the previous theorem, but the proof is much harder.

Theorem A.4. *Suppose we are given numbers $R > 0$, $k > 0$, $1 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty$, and nonnegative numbers $a_1, a_2, \ldots$. Suppose that the Dirichlet series $\psi(s) = \sum a_n\lambda_n^{-s}$ converges for all complex numbers with $\operatorname{Re} s > k$, and that the function $\psi(s) - R/(s-k)$ can be extended to a function defined and continuous for $\operatorname{Re} s \ge k$. Then*

$$\sum_{\lambda_n\le T} a_n \sim RT^k/k.$$

*Proof.* Replacing $s$ by $ks$ allows us to reduce to the case where $k = 1$, which can be deduced directly from the Landau-Ikehara Theorem in [1], from Theorem 2.2 on p. 93 of [35], from Theorem 1 on p. 464 of [17], or from Theorem 1 on p. 534 of [18]. See also Theorem 17 on p. 130 of [40] for the case where $\lambda_n = n$, which suffices for our purposes. A weaker prototype of this theorem was first proved by Landau in 1909 [13, §241]. Other relevant papers include [39], [6], and [5]. See also Bateman's discussion in [13, Appendix, page 931] and the good exposition of Abelian and Tauberian theorems in chapter 5 of [38]. $\square$

## References


[1] S. Bochner. Ein Satz von Landau und Ikehara. *Math. Z.*, 37:1–9, 1933.

[2] Armand Borel. *Introduction aux groupes arithmétiques.* Publications de l'Institut de Mathématique de l'Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, No. 1341. Hermann, Paris, 1969.

[3] A. I. Borevich and I. R. Shafarevich. *Number theory.* Academic Press, New York, 1966. Translated from the Russian by Newcomb Greenleaf. Pure and Applied Mathematics, Vol. 20.

[4] Henri Cohen. *A course in computational algebraic number theory*, volume 138 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 1993.

[5] Hubert Delange. Sur le théorème taubérien de Ikéhara. *C. R. Acad. Sci. Paris*, 232:465–467, 1951.

[6] Hubert Delange. Généralisation du théorème de Ikehara. *Ann. Sci. Ecole Norm. Sup. (3)*, 71:213–242, 1954.

[7] Daniel R. Grayson. Reduction theory using semistability. *Comment. Math. Helv.*, 59:600–634, 1984.

[8] F. J. Grunewald, D. Segal, and G. C. Smith. Subgroups of finite index in nilpotent groups. *Invent. Math.*, 93:185–223, 1988.

[9] Adolf Hurwitz. Ueber die Erzeugung der Invarianten durch Integration. *Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse*, pages 71–90, 1897.

[10] Adolf Hurwitz. *Mathematische Werke. Bd. II: Zahlentheorie, Algebra und Geometrie.* Birkhäuser Verlag, Basel, 1963. Herausgegeben von der Abteilung für Mathematik und Physik der Eidgenössischen Technischen Hochschule in Zürich.

[11] M. Kneser. Semisimple algebraic groups. In *Algebraic Number. Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 250–265. Thompson, Washington, D.C., 1967.

[12] Robert E. Kottwitz. Tamagawa numbers. *Ann. of Math. (2)*, 127:629–646, 1988.

[13] Edmund Landau. *Handbuch der Lehre von der Verteilung der Primzahlen. 2 Bände*. Chelsea Publishing Co., New York, 1953. 2d ed, With an appendix by Paul T. Bateman.

[14] R. P. Langlands. The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 143–148. Amer. Math. Soc., Providence, R.I., 1966.

[15] A. M. Macbeath and C. A. Rogers. Siegel's mean value theorem in the geometry of numbers. *Proc. Cambridge Philos. Soc.*, 54:139–151, 1958.

[16] Hermann Minkowski. Diskontinuitätsbereich für arithmetische Äquivalenz. *J. Reine Angew. Math.*, 192:220–274, 1905.

[17] Władysław Narkiewicz. *Elementary and analytic theory of algebraic numbers*. PWN—Polish Scientific Publishers, Warsaw, 1974. Monografie Matematyczne, Tom 57.

[18] Władysław Narkiewicz. *Elementary and analytic theory of algebraic numbers*. Springer-Verlag, Berlin, second edition, 1990.

[19] Morris Newman. *Integral matrices*. Academic Press, New York, 1972. Pure and Applied Mathematics, Vol. 45.

[20] Takashi Ono. On the relative theory of Tamagawa numbers. *Ann. of Math. (2)*, 82:88–111, 1965.

[21] Hans Sagan. *Advanced calculus of real-valued functions of a real variable and vector-valued functions of a vector variable*. Houghton Mifflin Co., Boston, Mass., 1974.

[22] Winfried Scharlau and Hans Opolka. *From Fermat to Minkowski*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1985. Lectures on the theory of numbers and its historical development, Translated from the German by Walter K. Bühler and Gary Cornell.

[23] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo, 1971. Kanô Memorial Lectures, No. 1.

[24] Carl Ludwig Siegel. Über die analytische Theorie der quadratischen Formen, I. *Ann. Math.*, 36:527–606, 1935.

[25] Carl Ludwig Siegel. Über die analytische Theorie der quadratischen Formen, II. *Ann. Math.*, 37:230–263, 1936.

[26] Carl Ludwig Siegel. The volume of the fundamental domain for some infinite groups. *Trans. Amer. Math. Soc.*, 39:209–218, 1936.

[27] Carl Ludwig Siegel. Über die analytische Theorie der quadratischen Formen, III. *Ann. Math.*, 38:212–291, 1937.

[28] Carl Ludwig Siegel. On the theory of indefinite quadratic forms. *Ann. of Math. (2)*, 45:577–622, 1944.

[29] Carl Ludwig Siegel. A mean value theorem in the geometry of numbers. *Ann. Math.*, 46:340–347, 1945.

[30] Carl Ludwig Siegel. Zur Bestimmung des Volumens des Fundamentalbereichs der unimodularen Gruppe. *Math. Ann.*, 137:427–432, 1959.

[31] Carl Ludwig Siegel. *Lectures on the geometry of numbers.* Springer-Verlag, Berlin, 1989. Notes by B. Friedman, Rewritten by Komaravolu Chandrasekharan with the assistance of Rudolf Suter, With a preface by Chandrasekharan.

[32] Louis Solomon. Zeta functions and integral representation theory. *Advances in Math.*, 26:306–326, 1977.

[33] Tsuneo Tamagawa. Adèles. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 113–121. Amer. Math. Soc., Providence, R.I., 1966.

[34] Gérald Tenenbaum. *Introduction to analytic and probabilistic number theory.* Cambridge University Press, Cambridge, 1995. Translated from the second French edition (1995) by C. B. Thomas.

[35] J. van de Lune. *An introduction to Tauberian theory: from Tauber to Wiener.* Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1986.

[36] V. E. Voskresenskiĭ. *Algebraic groups and their birational invariants.* American Mathematical Society, Providence, RI, 1998. Translated from the Russian manuscript by Boris Kunyavski [Boris È. Kunyavskiĭ].

[37] André Weil. *Adeles and algebraic groups.* Birkhäuser, Boston, Mass., 1982. With appendices by M. Demazure and Takashi Ono.

[38] David Vernon Widder. *The Laplace Transform.* Princeton University Press, Princeton, N. J., 1941.

[39] N. Wiener. Tauberian theorems. *Ann. of Math.*, 33:1–100, 1932.

[40] Norbert Wiener. *The Fourier integral and certain of its applications.* Cambridge University Press, London, 1933.

Henri Gillet
University of Illinois at Chicago
http://www.math.uic.edu/~henri/
henri@math.uic.edu

Daniel R. Grayson
University of Illinois
at Urbana-Champaign
http://www.math.uiuc.edu/~dan/
dan@math.uiuc.edu