A Symplectic Approach to Van Den Ban's Convexity Theorem

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Abstract. Let G be a complex semisimple Lie group and τ a complex antilinear involution that commutes with a Cartan involution. If H denotes the connected subgroup of τ-fixed points in G, and K is maximally compact, each H-orbit in G/K can be equipped with a Poisson structure as described by Evens and Lu. We consider symplectic leaves of certain such H-orbits with a natural Hamiltonian torus action. A symplectic convexity theorem then leads to van den Ban's convexity result for (complex) semisimple symmetric spaces.

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1. Introduction

In 1982, Atiyah [1] and independently Guillemin and Sternberg [4] discovered a surprising connection between results in Lie theory and symplectic geometry. They proved a general symplectic convexity theorem of which Kostant's linear convexity theorem (for complex semisimple Lie groups) is a corollary. In this context, the orbits relevant for Kostant's theorem carry the natural symplectic structure of coadjoint orbits. The symplectic convexity theorem states that the image under the moment map of a compact connected symplectic manifold with Hamiltonian torus action is a convex polytope. Subsequently, Duistermaat [2] extended the symplectic convexity theorem in a way that it could be used to prove Kostant's linear theorem for real semisimple Lie groups as well.
Lu and Ratiu [10] found a way to put Kostant's nonlinear theorem into a symplectic framework. For a complex semisimple Lie group G with Iwasawa decomposition $G = NAK$, they regard the relevant K-orbits as symplectic

leaves of the Poisson Lie group AN, carrying the Lu-Weinstein Poisson structure. Kostant's nonlinear theorem for both complex and certain real groups then follows from the AGS-theorem or Duistermaat's theorem.

In this paper, we want to give a symplectic interpretation of van den Ban's convexity theorem for a complex semisimple symmetric space $(\mathfrak{g}, \tau)$, which is a generalization of Kostant's nonlinear theorem for complex groups. The theorem describes the image of the projection of a coset of G^τ onto $\mathfrak{a}^-$, the (-1)-eigenspace of τ. The image is characterized as the sum of a convex polytope and a convex polyhedral cone. For the precise statement of van den Ban's result we refer to Section 2. The main difference in view of our symplectic approach is that van den Ban's theorem is concerned with orbits of a certain subgroup $H \subset G$ that are in general neither symplectic nor compact. Since G is complex we can use a method due to Evens and Lu [3] to equip H-orbits in G/K with a certain Poisson structure. An H-orbit foliates into symplectic leaves, and on each leaf some torus acts in a Hamiltonian way. The corresponding moment map Φ turns out to be proper, and therefore the symplectic convexity theorem of Hilgert-Neeb-Plank [6] can be applied, which describes the image under Φ in terms of local moment cones. An analysis of those local moment cones shows that the image of Φ is the sum of a compact convex polytope and a convex polyhedral cone, just as in van den Ban's theorem.

The case of van den Ban's theorem for a real semisimple symmetric space is dealt with in a separate paper [12]. It follows the symplectic approach of Lu and Ratiu towards Kostant's nonlinear convexity theorem. The main tool is a generalized version of Duistermaat's theorem for non-compact manifolds.

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2. Van den Ban's theorem

The purpose of this section is to fix notation and to recall the statement of van den Ban's theorem.

Let G be a real connected semisimple Lie group with finite center, equipped with an involution τ, i.e. τ is a smooth group homomorphism such that $\tau^2 = id$. Let $\mathfrak{g}$ be the Lie algebra of G. We write H for an open subgroup of G^τ, the τ-fixed points in G. Let K be a τ-stable maximal compact subgroup of G. The corresponding Cartan involution θ on $\mathfrak{g}$ commutes with τ and induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. If $\mathfrak{h}$ and $\mathfrak{q}$ denote the $(+1)$- and (-1)-eigenspace of $\mathfrak{g}$ with respect to τ one obtains

$$\mathfrak{g} = (\mathfrak{k} \cap \mathfrak{h}) + (\mathfrak{p} \cap \mathfrak{h}) + (\mathfrak{k} \cap \mathfrak{q}) + (\mathfrak{p} \cap \mathfrak{q}).$$

We fix a maximal abelian subalgebra $\mathfrak{a}^{-\tau}$ of $\mathfrak{p} \cap \mathfrak{q}$. (In [14] this subalgebra is denoted by $\mathfrak{a}_{pq}$.) In addition, we choose $\mathfrak{a}^\tau \subseteq \mathfrak{p} \cap \mathfrak{h}$ such that $\mathfrak{a} := \mathfrak{a}^\tau + \mathfrak{a}^{-\tau}$ is maximal abelian in $\mathfrak{p}$. Let $\Delta(\mathfrak{g}, \mathfrak{a}^{-\tau})$ and $\Delta(\mathfrak{g}, \mathfrak{a})$ denote the sets of roots for the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}^{-\tau}$ and $\mathfrak{a}$, respectively.

Next, we choose a system of positive roots $\Delta^+(\mathfrak{g},\mathfrak{a})$ and define

$$\Delta^+(\mathfrak{g},\mathfrak{a}^{-\tau}) = \{\alpha|_{\mathfrak{a}^{-\tau}} : \alpha \in \Delta^+(\mathfrak{g},\mathfrak{a}), \alpha|_{\mathfrak{a}^{-\tau}} \neq 0\}.$$

This leads to an Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k} = \mathfrak{n}^1 + \mathfrak{n}^2 + \mathfrak{a} + \mathfrak{k},$$

where

$$\begin{aligned} \mathfrak{n} &= \sum_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}^\alpha, \\ \mathfrak{n}^1 &= \sum_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{a}),\alpha|_{\mathfrak{a}^{-\tau}}\neq 0} \mathfrak{g}^\alpha = \sum_{\beta\in\Delta^+(\mathfrak{g},\mathfrak{a}^{-\tau})} \mathfrak{g}^\beta, \\ \mathfrak{n}^2 &= \sum_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{a}),\alpha|_{\mathfrak{a}^{-\tau}}=0} \mathfrak{g}^\alpha. \end{aligned}$$

Here $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : [H,X] = \alpha(H)X \ \forall H \in \mathfrak{a}\}$ for $\alpha \in \Delta(\mathfrak{g},\mathfrak{a})$, and similarly $\mathfrak{g}^\beta$ is defined for $\beta \in \Delta(\mathfrak{g},\mathfrak{a}^{-\tau})$.
Let N and A denote the analytic subgroups of G with Lie algebras $\mathfrak{n}$ and $\mathfrak{a}$, respectively. The Iwasawa decomposition $G = NAK$ on the group level has the middle projection $\mu : G \to A$. We write $pr_{\mathfrak{a}^{-\tau}} : \mathfrak{a} \to \mathfrak{a}^{-\tau}$ for the projection along $\mathfrak{a}^\tau$.
For $\beta \in \Delta^+(\mathfrak{g},\mathfrak{a}^{-\tau})$ define $H_\beta \in \mathfrak{a}^{-\tau}$ such that

$$H_\beta \perp \ker\beta, \quad \beta(H_\beta) = 1,$$

where $\perp$ means orthogonality with respect to the Killing form κ.
Note that the involution $\theta \circ \tau$ leaves each root space

$$\mathfrak{g}^\beta = \sum_{\alpha\in\Delta(\mathfrak{g},\mathfrak{a}),\alpha|_{\mathfrak{a}^{-\tau}}=\beta} \mathfrak{g}^\alpha$$

stable. Each $\mathfrak{g}^\beta = (\mathfrak{g}^\beta)_+ \oplus (\mathfrak{g}^\beta)_-$ decomposes into $(+1)$- and (-1)-eigenspace with respect to $\theta \circ \tau$.
For

$$\Delta_- := \{\beta \in \Delta(\mathfrak{g},\mathfrak{a}^{-\tau}) : (\mathfrak{g}^\beta)_- \neq 0\},$$

let $\Delta_-^+ = \Delta_- \cap \Delta^+(\mathfrak{g},\mathfrak{a}^{-\tau})$. Define the closed cone

$$\Gamma(\Delta_-^+) = \sum_{\beta\in\Delta_-^+} \mathbb{R}_+ H_\beta.$$

Write $\mathcal{W}_{K\cap H}$ for the Weyl group

$$\mathcal{W}_{K\cap H} = N_{K\cap H}(\mathfrak{a}^{-\tau})/Z_{K\cap H}(\mathfrak{a}^{-\tau}).$$

The convex hull of a Weyl group orbit through $X \in \mathfrak{a}^{-\tau}$ will be denoted by $\mathrm{conv}(\mathcal{W}_{K\cap H}.X)$.

Remark 2.1. *Consider the Lie algebra $\mathfrak{g}^{\theta\tau}$ of $\theta\tau$-fixed points in $\mathfrak{g}$. It is reductive and its semisimple part $\mathfrak{g}' = [\mathfrak{g}^{\theta\tau},\mathfrak{g}^{\theta\tau}]$ admits a Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$ with $\mathfrak{k}' \subset \mathfrak{k}$, $\mathfrak{p}' \subset \mathfrak{p}$. Due to our choice, $\mathfrak{a}^{-\tau}$ is a maximal abelian subalgebra of $\mathfrak{p}'$. The set of roots $\Delta(\mathfrak{g}',\mathfrak{a}^{-\tau})$ consists exactly of those reduced roots*

$\beta \in \Delta(\mathfrak{g}, \mathfrak{a}^{-\tau})$ for which $(\mathfrak{g}^\beta)_+ \neq 0$. Moreover, the Weyl group $\mathcal{W}'$ associated to $\mathfrak{g}'$ coincides with $\mathcal{W}_{K\cap H}$.

We can now state the central theorem.

THEOREM 2.2. (Van den Ban [14])
Let G be a real connected semisimple Lie group with finite center, equipped with an involution τ, and H a connected open subgroup of G^τ. For $X \in \mathfrak{a}^{-\tau}$, write $a = \exp X \in A^{-\tau}$. Then

$$(pr_{\mathfrak{a}^{-\tau}} \circ \log \circ \mu)(Ha) = \mathrm{conv}(\mathcal{W}_{K\cap H}.X) - \Gamma(\Delta_-^+).$$

REMARK 2.3.

- *The statement of the theorem above differs from the original in [14] by a minus sign in front of the conal part $\Gamma(\Delta_-^+)$. This is due to the fact that we consider the set Ha and an Iwasawa decomposition $G = NAK$, whereas in [14] the set $aH \subset G = KAN$ is considered. Indeed, if we denote the two middle projections by $\mu : NAK \to A$ and $\mu' : KAN \to A$, then $\Gamma(\Delta_-^+) = \log \circ \mu'(H) = -\log \circ \mu(H)$.*
- *Van den Ban proved his theorem under the weaker condition that H is an essentially connected open subgroup of G^τ (by reducing it to the connected case).*
- *If $\tau = \theta$ one obtains Kostant's (nonlinear) convexity theorem. Note that in this case the group H and the orbit Ha are compact.*

3. Poisson structure

Let G be a connected and simply connected semisimple complex Lie group with Lie algebra $\mathfrak{g}$. Cartan involutions on both group and Lie algebra level will be denoted by θ. In addition, let τ be a complex antilinear involution (on G and $\mathfrak{g}$) which commutes with θ.
The Lie algebra $\mathfrak{g}$ decomposes into $(+1)$- and (-1)-eigenspaces with respect to both involutions θ and τ.

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{h} + \mathfrak{q},$$

where $\mathfrak{k}$ and $\mathfrak{h}$ denote the $(+1)$-eigenspaces with respect to θ and τ, respectively, and $\mathfrak{p}$ and $\mathfrak{q}$ denote the (-1)-eigenspaces.
The maximal compact subgroup K of G with Lie algebra $\mathfrak{k}$ is τ-stable. Let H denote the connected subgroup of G consisting of τ-fixed points. We will be interested in certain H-orbits in the symmetric space G/K. Each such orbit can be equipped with a Poisson structure as introduced by Evens and Lu. We briefly describe their method which can be found in [3, Section 2.2]. For details on Poisson Lie groups see e.g. [11].
Let (U, π_U) be a connected Poisson Lie group with tangent Lie bialgebra $(\mathfrak{u}, \mathfrak{u}^*)$ and double Lie algebra $\mathfrak{d} = \mathfrak{u} \bowtie \mathfrak{u}^*$. The pairing

$$\langle v_1 + \lambda_1, v_2 + \lambda_2\rangle := \lambda_1(v_2) + \lambda_2(v_1) \quad \forall\ v_1, v_2 \in \mathfrak{u}, \lambda_1, \lambda_2 \in \mathfrak{u}^*,$$

defines a non-degenerate symmetric bilinear form and turns $(\mathfrak{d}, \mathfrak{u}, \mathfrak{u}^*)$ into a Manin triple. We will identify $\mathfrak{d}^*$ with $\mathfrak{d}$ via $\langle , \rangle$.

Consider the following bivector $R \in \wedge^2\mathfrak{d}$:

$$R(v_1+\lambda_1, v_2+\lambda_2) = \lambda_2(v_1) - \lambda_1(v_2) \quad \forall\ v_1, v_2 \in \mathfrak{u}, \lambda_1, \lambda_2 \in \mathfrak{u}^*.$$

In terms of a basis $\{v_1, \ldots, v_n\}$ for $\mathfrak{u}$ and a dual basis $\{\lambda_1, \ldots, \lambda_n\}$ for $\mathfrak{u}^*$ the bivector is represented by $R = \sum_{i=1}^n \lambda_i \wedge v_i$.
Assume that D is a connected Lie group with Lie algebra $\mathfrak{d}$, and assume that U is a connected subgroup of D with Lie algebra $\mathfrak{u}$. Then D acts on the Grassmannian $\mathrm{Gr}(n,\mathfrak{d})$ of n-dimensional subspaces of $\mathfrak{d}$ via the adjoint action of D on $\mathfrak{d}$ and therefore defines a Lie algebra antihomomorphism

$$\eta : \mathfrak{d} \to \mathcal{X}(\mathrm{Gr}(n,\mathfrak{d})),$$

into the vector fields on $\mathrm{Gr}(n,\mathfrak{d})$. Using the symbol η also for its multilinear extension we can define a bivector field Π on $\mathrm{Gr}(n,\mathfrak{d})$ by

$$\Pi = \frac{1}{2}\eta(R).$$

Note that Π in general does not define a Poisson structure on the entire $\mathrm{Gr}(n,\mathfrak{d})$. However, it does so on the subvariety $\mathfrak{L}(\mathfrak{d})$ of Lagrangian subspaces (with respect to $\langle,\rangle$) on $\mathfrak{d}$, and on each D-orbit $D.\mathfrak{l} \subset \mathfrak{L}(\mathfrak{d})$.
The bivector R also gives rise to a Poisson structure π_- on D that makes (D,π_-) a Poisson Lie group:

$$\pi_-(d) = \frac{1}{2}(r_dR - l_dR) \quad \forall d \in D. \tag{1}$$

Here r_d and l_d denote the differentials of right and left translations by d. Note that the restriction of π_- to the subgroup $U \subset D$ coincides with the original Poisson structure π_U on U, i.e. (U,π_U) is a Poisson subgroup of (D,π_-).
For $\mathfrak{l} \in \mathfrak{L}(\mathfrak{d})$ the D-orbit through l is not only a Poisson manifold with respect to Π but a homogeneous Poisson space under the action of (D,π_-). Moreover, the U-orbit $U.\mathfrak{l}$ is a homogeneous (U,π_U)-space, since the Poisson tensor Π at $\mathfrak{l}$ turns out to be tangent to $U.\mathfrak{l}$. In fact, the tangent space at $\mathfrak{l} \in D.\mathfrak{l}$ can be identified with $\mathfrak{d}/n(\mathfrak{l})$, where $n(\mathfrak{l})$ is the normalizer subalgebra of $\mathfrak{l}$. In the case when $n(\mathfrak{l}) = \mathfrak{l}$, we identify the cotangent space with $\mathfrak{l}$ itself, and for $X, Y \in \mathfrak{l}$ one obtains:

$$\Pi(\mathfrak{l})(X,Y) = \langle pr_{\mathfrak{u}}X, Y\rangle, \quad \text{i.e.} \quad \Pi(\mathfrak{l})^\sharp(X) = pr_{\mathfrak{u}}X, \tag{2}$$

where $pr_{\mathfrak{u}} : \mathfrak{d} \to \mathfrak{u}$ denotes the projection along $\mathfrak{u}^*$.
Let U^* be the connected subgroup of D with Lie algebra $\mathfrak{u}^*$. What has been said about the Poisson Lie group U is also true for its dual group U^*, i.e. (U^*,π_{U^*}) is a Poisson Lie subgroup of $(D,-\pi_-)$ and the orbit $U^*.\mathfrak{l}$ is a homogeneous (U^*,π_{U^*})-space. It follows in particular that $(U.\mathfrak{l}) \cap (U^*.\mathfrak{l})$ contains the symplectic leaf through $\mathfrak{l}$.

We now want to apply this construction to our complex semisimple Lie algebra $\mathfrak{g}$. In the above notation we will have $\mathfrak{d} = \mathfrak{g}$, and the pairing $\langle,\rangle$ will be given by the imaginary part, $\Im\kappa$, of the Killing form κ on $\mathfrak{g}$. Note that $\mathfrak{k} \in$

$\mathfrak{L}(\mathfrak{d})$. Throughout the paper, we will identify the G-orbit through $\mathfrak{k}$ with the symmetric space G/K. In particular, orbits in $G.\mathfrak{k}$ are identified with those in G/K. Then we set $\mathfrak{u} = \mathfrak{h}$, and it remains to define $\mathfrak{u}^*$.

First we choose an appropriate Iwasawa decomposition of $\mathfrak{g}$. Recall the τ-stable Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. We fix a maximal abelian subalgebra $\mathfrak{a}^{-\tau}$ in $\mathfrak{p} \cap \mathfrak{q}$. Then we can find an abelian subalgebra $\mathfrak{a}^{\tau}$ in $\mathfrak{p} \cap \mathfrak{h}$ such that $\mathfrak{a} = \mathfrak{a}^{-\tau} + \mathfrak{a}^{\tau}$ is maximal abelian in $\mathfrak{p}$. We choose a positive root system, $\Delta^+(\mathfrak{g}, \mathfrak{a})$ by the lexicographic ordering with respect to an ordering of a basis of $\mathfrak{a}$, which was constructed from a basis of $\mathfrak{a}^{-\tau}$ followed by a basis of $\mathfrak{a}^{\tau}$. This yields an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ which is compatible with the involution τ in the following sense.

Lemma 3.1. *For our choice of Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$, we have*

$$\mathfrak{h} \cap \mathfrak{n} = \{0\}.$$

Besides, the centralizer of $\mathfrak{a}^{-\tau}$ in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$.

Proof. Consider the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$,

$$\mathfrak{g} = (\mathfrak{a} + i\mathfrak{a}) + \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}^{\alpha}.$$

It is well-known [7, Proposition 6.70] that there are no real roots for a maximally compact Cartan subalgebra $(i\mathfrak{a}^{-\tau} + \mathfrak{a}^{\tau})$ of $\mathfrak{h}$, and therefore there are no $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$ such that $\alpha|_{\mathfrak{a}^{-\tau}} = 0$. By [5, Chapter VI, Lemma 3.3], this implies that $\tau(\mathfrak{g}^{\alpha}) \subset \bigoplus_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}^{-\alpha}$ for all $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$, and the claim $\mathfrak{h} \cap \mathfrak{n} = \{0\}$ follows immediately.

Since each $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$ does not vanish outside a hyperplane of $\mathfrak{a}^{-\tau}$, it follows that $\mathfrak{a}^{-\tau}$ contains regular elements and its centralizer in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$.

□

Consider the Cartan subalgebra $\mathfrak{c} = \mathfrak{z}(\mathfrak{a}^{-\tau})$ of $\mathfrak{g}$. Lemma 3.1 together with the properties of κ implies that $\mathfrak{g} = \mathfrak{h} \oplus (\mathfrak{c}^{-\tau} \oplus \mathfrak{n})$ is a Lagrangian splitting with respect to the bilinear form $\Im\kappa$. In other words, $(\mathfrak{g}, \mathfrak{h}, (\mathfrak{c}^{-\tau} + \mathfrak{n}))$ is a Manin triple.

We can now define the desired Poisson manifolds using the method of Evens and Lu outlined above. We set

$$\mathfrak{d} = \mathfrak{g},\ \mathfrak{u} = \mathfrak{h},\ \mathfrak{u}^* = \mathfrak{c}^{-\tau} + \mathfrak{n},\ \langle , \rangle = \Im\kappa.$$

Let C, $C^{-\tau}$, A and N denote the analytic subgroups of G with Lie algebras $\mathfrak{c}$, $\mathfrak{c}^{-\tau}$, $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The group H now has the structure of a Poisson Lie group. Its dual group is $H^* = C^{-\tau}N$. Fix $a \in A^{-\tau}$ and consider the base point $a.K \in G/K$. The H-orbit $P_a = Ha.K \in G/K$ is a Poisson homogeneous manifold with respect to the action by (H, π_H). Also, the dual group orbit $H^*a.K$ is Poisson homogeneous with respect to π_{H^*}. For the symplectic leaf in P_a through a, denoted by M_a, we have $M_a \subseteq Ha.K \cap H^*a.K$.

Lemma 3.2. *The Poisson manifold P_a is regular and equals the union of A^τ-translates of M_a, i.e. each $p \in P_a$ can be written $p = a'm$ with unique $a' \in A^\tau, m \in M_a$. Moreover, $M_a = Ha.K \cap H^*a.K$.*

Proof. Consider the map $M : A^\tau \times M_a \to P_a$.
First we will show that M is injective. The Poisson tensor $\pi_H = \pi_-$ as defined in (1) vanishes at each element $c \in C^\tau$, since $Ad(c)$ leaves both $\mathfrak{h}$ and $\mathfrak{h}^* = \mathfrak{c}^{-\tau} + \mathfrak{n}$ stable. Therefore $a' \in A^\tau$ acts on P_a by Poisson diffeomorphisms and maps the symplectic leaf M_a onto the symplectic leaf $M_{a'a}$. But $M_{a_1a} \neq M_{a_2a}$ for $a_1 \neq a_2 \in A^\tau$, following from the fact that M_{a_1a} lies in $H^*a_1a.K = C^{-\tau}Na_1a.K$ and the uniqueness of the Iwasawa decomposition.
At each point $p \in P_a$ one can explicitly calculate the codimension of the symplectic leaf through p in P_a, for instance by means of an infinitesimal version of Corollary 7.3 in [9] and Theorem 2.21 in [3]. It follows that the codimension of the leaf through the point $p = ha.K$ in the orbit P_a equals the dimension of the intersection of $Ad(a)\mathfrak{k}$ and $Ad(h^{-1})\mathfrak{h}^*$, which is easily seen to be independent from the point $p \in P_a$ and equal to the dimension of $\mathfrak{a}^\tau$. Here we used the fact that the dimension of $Ad(ha)\mathfrak{k} \cap \mathfrak{h}^*$ cannot exceed the dimension of $\mathfrak{a}^\tau$, since the Killing form is negative definite on $Ad(ha)\mathfrak{k}$ and a maximal negative definite subspace of $\mathfrak{h}^*$ is $i\mathfrak{a}^\tau$. This shows that P_a is a regular Poisson manifold, and that $A^\tau M_a$ is a full dimensional subset of P_a. Since A^τ acts freely on P_a and P_a is a regular Poisson manifold, it can be represented as the union of such open subsets. The connectedness of P_a then implies that $P_a = A^\tau M_a$.
Since A^τ is connected and the union of A^τ-translates of $Ha.K \cap H^*a.K$ equals $Ha.K$ and thus is also connected, it is easy to see that $Ha.K \cap H^*a.K$ is connected as well. Besides, from the transversality we see that

$$\dim(Ha.K \cap H^*a.K) = \dim(Ha.K) + \dim(H^*a.K) - \dim(G/K).$$

Note that the first part of the proof implies that $A^\tau a.K \cap H^*a.K = \{a.K\}$. Therefore, the codimension of $Ha.K \cap H^*a.K$ in $Ha.K$ is at least $\dim(\mathfrak{a}^\tau)$. But since M_a has codimension equal to $\dim(\mathfrak{a}^\tau)$, and $M_a \subseteq Ha.K \cap H^*a.K$, the last inclusion is actually an equality.

□

Consider the torus $T = \exp(i\mathfrak{a}^{-\tau}) \subset H$. It acts on M_a in a symplectic manner, since π_H vanishes at each $t \in T$. Moreover, the next lemma shows that this action is Hamiltonian with an associated moment map that is closely related to the middle projection $\mu : G = NAK \to A$ of the Iwasawa decomposition.

Lemma 3.3. *The action of $T = \exp(i\mathfrak{a}^{-\tau})$ on M_a is Hamiltonian with a moment map $\Phi = pr_{\mathfrak{a}^{-\tau}} \circ \log \circ \mu$. Here, $pr_{\mathfrak{a}^{-\tau}} : \mathfrak{a} \to \mathfrak{a}^{-\tau}$ denotes the projection along $\mathfrak{a}^\tau$, and $\mathfrak{t}^*$ is identified with $\mathfrak{a}^{-\tau}$ via $\Im\kappa$.*
Moreover, the moment map Φ is proper.

Proof. (1) $\Phi = pr_{\mathfrak{a}^{-\tau}} \circ \log \circ \mu$ is a moment map.
Let $b : G = NAK \longrightarrow B = NA$ be the B-projection in the Iwasawa decomposition. We write $pr_{\mathfrak{a}} : \mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k} \to \mathfrak{a}$ for the middle

projection on the Lie algebra level. Let $Z \in \mathfrak{t} = i\mathfrak{a}^{-\tau}$, $h \in H$ and $X \in \mathfrak{h}$. We denote by Φ_Z the function obtained by evaluating Φ at Z, by $\tilde{X}_{ha}$ the tangent vector of the vector field generated by X at the point $ha.K \in M_a$ (for brevity we will write $h.K$ simply as h henceforth, without fear of confusion) and by $D\Phi_{b(ha)}$ the derivative of Φ at the point $b(ha)$. We have:

$$\begin{aligned}
d\Phi_Z(ha).\tilde{X}_{ha} &= \frac{d}{ds}\Big|_{s=0}\Phi_Z(\exp(sX)ha) = \langle \frac{d}{ds}\Big|_{s=0}\Phi(\exp(sX)ha), Z\rangle \\
&= \langle \frac{d}{ds}\Big|_{s=0}\Phi(b(ha)\exp(sAd(b(ha)^{-1})X)), Z\rangle \\
&= \langle D\Phi_{b(ha)}Ad(b(ha)^{-1})X, Z\rangle \\
&= \langle pr_{\mathfrak{a}^{-\tau}} \circ pr_{\mathfrak{a}} Ad(b(ha)^{-1})X, Z\rangle = \langle Ad(b(ha)^{-1})X, Z\rangle \\
&= \langle X, Ad(b(ha))Z\rangle
\end{aligned}$$

The second last step follows from the fact that $\mathfrak{t}$ and $\mathfrak{k} + \mathfrak{a}^\tau + \mathfrak{n}$ are orthogonal with respect to $\langle , \rangle$.

Note that $Ad(b(ha))Z \in Z + \mathfrak{n}$. With (2) this implies

$$\Pi(ha)^\sharp(d\Phi_Z(ha)) = pr_{\mathfrak{h}} Ad(b(ha))Z = Z.$$

(2) Φ is proper.

This follows from Lemma 3.3 in [14], which states the properness of the map

$$F_a : (H \cap L_0)\backslash H \to \mathfrak{a}^{-\tau}, \quad F_a(x) = \Phi(xa).$$

In our case $L_0 = \exp(i\mathfrak{a})A^\tau$ (since $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^{-\tau}) = \mathfrak{c}$ by the argument in the proof of Lemma 3.1).

Properness of the map $F_a : TA^\tau\backslash H \to \mathfrak{a}^{-\tau}$ implies properness of the induced maps $F_a : A^\tau\backslash H \to \mathfrak{a}^{-\tau}$ and $F_a : A^\tau\backslash H/(H \cap aKa^{-1}) \to \mathfrak{a}^{-\tau}$. Since $A^\tau\backslash H/(H \cap aKa^{-1}) \cong M_a$ by Lemma 3.2, and since F_a becomes Φ under this identification, the claim follows.

□

Remark 3.4. *In case $\tau = \theta$ the Lu-Evens Poisson structure on $P_a = Ka.K$ coincides with the Lu-Weinstein symplectic structure, and Lemma 3.3 becomes Theorem 4.13 in* [10].

4. Symplectic convexity

Throughout this section we assume G to be complex and the involution τ to be complex antilinear. In this case we will interpret van den Ban's theorem in the symplectic framework developed in Section 3. More precisely, it can be viewed as a corollary of a symplectic convexity theorem for Hamiltonian torus actions.

Van den Ban's theorem describes the image of the group orbit Ha under the map $pr_{\mathfrak{a}^{-\tau}} \circ \log \circ \mu$. Recall from Section 3 the symplectic manifold $M_a \subseteq$

$Ha.K \subseteq G/K$ on which the torus $T = \exp(i\mathfrak{a}^{-\tau})$ acts in a Hamiltonian fashion (Lemma 3.3). The associated moment map is $\Phi = pr_{\mathfrak{a}^{-\tau}} \circ \log \circ \mu$. From Lemma 3.2 and from the A^τ-invariance of $pr_{\mathfrak{a}^{-\tau}} \circ \log \circ \mu$ it follows that

$$(pr_{\mathfrak{a}^{-\tau}} \circ \log \circ \mu)(Ha) = \Phi(M_a).$$

This means that van den Ban's theorem can be viewed as a description of the image of a symplectic manifold under an appropriate moment map.
The description of the image of the moment map is the content of a series of symplectic convexity theorems. Probably best known are the original theorems of Atiyah and Guillemin-Sternberg [1, 4]. The result needed here is a generalization of the AGS-theorems to a non-compact setting. Several versions can be found in the literature, e.g. [8, 13]. We will state the theorem as given in [6]. Recall that a subset C of a finite dimensional vector space V is called locally polyhedral iff for each $x \in C$ there is a neighborhood $U_x \subseteq V$ such that $C \cap U_x = (x + \Gamma_x) \cap U_x$ for some cone Γ_x. A cone Γ is called proper if it contains no lines, otherwise Γ is called improper.

Theorem 4.1. [6, Theorem 4.1(i)] *Consider a Hamiltonian torus action of T on the connected symplectic manifold M. Suppose the associated moment map $\Phi : M \to \mathfrak{t}^*$ is proper, i.e. Φ is a closed mapping and $\Phi^{-1}(Z)$ is compact for every $Z \in \mathfrak{t}^*$. Then $\Phi(M)$ is a closed, locally polyhedral, convex set.*

Remark 4.2. *Theorem 4.1 in* [6] *contains more detailed information, in particular a description of the cones that span $\Phi(M)$ locally (part (v)). More precisely, for each $m \in M$ there is a neighborhood $U_{\Phi(m)} \subseteq \mathfrak{t}^*$ of $\Phi(m)$ such that $\Phi(M) \cap U_{\Phi(m)} = (\Phi(m) + \Gamma_{\Phi(m)}) \cap U_\Phi(m)$, where $\Gamma_{\Phi(m)} = \mathfrak{t}_m^\perp + C_m$. Here, $\mathfrak{t}_m$ denotes the Lie algebra of the stabilizer T_m of m, and $C_m \subseteq \mathfrak{t}_m^*$ denotes the cone which is spanned by the weights of the linearized action of T_m. The (nontrivial) fact that the cone $\Gamma_{\Phi(m)} = \mathfrak{t}_m^\perp + C_m$ is actually independent of the choice of a preimage point of $\Phi(m)$ is also shown in* [6].

Coming back to the symplectic manifold M_a, Lemma 3.3 shows that the moment map $\Phi = pr_{\mathfrak{a}^{-\tau}} \circ \log \circ \mu$ on M_a is proper. Theorem 4.1 can therefore be applied and yields

$$\Phi(M_a) \quad \text{is a closed, locally polyhedral, convex set.}$$

We will now give a more detailed description of $\Phi(M_a)$. It turns out that the T-action on M_a has (finitely many) fixed points. At each fixed point we can calculate the cones that locally span $\Phi(M_a)$. From this description it will follow that the entire set $\Phi(M_a)$ lies in a proper cone and can therefore be described entirely by the local data at the fixed points.
We begin by determining the T-fixed points.

Proposition 4.3. *The T-fixed points in M_a are exactly those elements of the form $w(a).K \in G/K$ with $w \in \mathcal{W}_{K\cap H} = N_{K\cap H}(\mathfrak{a}^{-\tau})/Z_{K\cap H}(\mathfrak{a}^{-\tau})$.*

Proof. Recall that for $a \in A^{-\tau}$ we view the symplectic manifold M_a as a submanifold of the H-orbit in G/K through the base point $a.K \in G/K$. Clearly, each element $w(a).K \in G/K$ with $w \in \mathcal{W}_{K\cap H}$ is T-fixed. To see that $w(a).K$ lies in M_a, note that $w(a).K \in H^*a.K$ since $w(a) \in A^{-\tau}$. On the other hand, there exists $k \in K \cap H$ such that $w(a) = kak^{-1}$, which implies $w(a).K \in Ha.K$. Therefore, $w(a).K \in Ha.K \cap H^*a.K = M_a$ by Lemma 3.2.

Conversely, assume that $cpa.K \in M_a$ with $c \in K^\tau, p \in \exp(\mathfrak{p}^\tau)$ is T-fixed. Since M_a lies in the orbit of the dual group $H^* = NC^{-\tau}$ there are elements $n \in N, b \in A^{-\tau}, k \in K$ such that $cpa = nbk$. Since $nb.K \in G/K$ is a T-fixed point,

$$tnt^{-1}b \in nbK \qquad \forall\, t \in T.$$

The Lie subalgebra $\mathfrak{n}$ is T-invariant, so by the uniqueness of the Iwasawa decomposition $tnt^{-1} = n$ for all $t \in T$. But since $\alpha|_{\mathfrak{a}^{-\tau}} \neq 0$ for all $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$ this can happen only for $n = e$. This implies $cpa = bk$.

Symmetrizing the last equation yields

$$cpa\theta(cpa)^{-1} = cpa^2pc^{-1} = b^2. \tag{3}$$

Applying $\theta \circ \tau$ to (3) gives

$$cp^{-1}a^2p^{-1}c^{-1} = b^2. \tag{4}$$

We multiply (3) by (4) from the right and from the left and obtain

$$cpa^4p^{-1}c^{-1} = b^4 = cp^{-1}a^4pc^{-1}.$$

But then $pa^4p^{-1} = p^{-1}a^4p$, i.e. p^2 and a^4 commute (and are self-adjoint). Therefore, p and a^2 also commute, and we can combine equations (3) and (4) to

$$cp^2a^2c^{-1} = b^2 = cp^{-2}a^2c^{-1}.$$

This shows $p^2 = p^{-2}$ or $p = e$. But then (4) implies $cac^{-1} = b$. Since both a and b lie in $A^{-\tau}$ and since $c \in K^\tau = K \cap H$, there is some element $w \in \mathcal{W}_{K\cap H}$ such that $w(a) = b$ (Recall from Remark 2.1 that $\mathcal{W}_{K\cap H}$ is the Weyl group of the reductive Lie algebra $\mathfrak{g}^{\theta\tau} = (\mathfrak{k} \cap \mathfrak{h}) + (\mathfrak{p} \cap \mathfrak{q})$ of $\theta\tau$-fixed points of $\mathfrak{g}$).

The T-fixed point $cpa.K \in M_a$ can therefore be written as $cpa.K = b.K = w(a).K$ for some $w \in \mathcal{W}_{K\cap H}$. □

Recall our choice of base point $a = \exp(X)$ and the identification $\mathfrak{t}^* \cong \mathfrak{a}^{-\tau}$. We now describe the image of the moment map $\Phi(M_a) \in \mathfrak{a}^{-\tau}$ in the neighborhood of a fixed point image $\Phi(w(a).K) = w(X)$. From Theorem 4.1 (and Remark 4.2) we know that locally $\Phi(M_a)$ looks like $w(X)+\Gamma_{w(X)}$ for some cone $\Gamma_{w(X)} \in \mathfrak{a}^{-\tau}$. The next Lemma describes $\Gamma_{w(X)}$ in terms of the vectors H_β for (reduced) roots $\beta \in \Delta(\mathfrak{g}, \mathfrak{a}^{-\tau})$ defined in Section 2.

Lemma 4.4. *Let $a = \exp X$ with $X \in \mathfrak{a}^{-\tau}$ and $w \in \mathcal{W}_{K\cap H}$. The local cone $\Gamma_{\Phi(w(a).K)} = \Gamma_{w(X)} \subseteq \mathfrak{a}^{-\tau}$ is the cone spanned by the union of the following two sets.*

$$\{-\beta(w(X))H_\beta : \beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau}), (\mathfrak{g}^\beta)_+ \neq 0\}$$
$$\text{and} \quad \{-H_\beta : \beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau}), (\mathfrak{g}^\beta)_- \neq 0\}$$

Proof. We are adapting the argument from [6, page 155] to our setting. To determine the local cone $\Gamma_{w(X)}$ it is enough to consider the linearized action of T on the tangent space $V_{w(a).K} := T_{w(a).K}M_a$. Darboux's theorem guarantees the existence of a T-equivariant symplectomorphism of a neighborhood of $w(a).K \in M_a$ onto a neighborhood of $0 \in V_{w(a).K}$. This leads to a local normal form for the moment map.

$$\Phi_Z(Y) = \frac{1}{2}\Omega_{w(a).K}(Z.Y, Y) \quad \forall\, Y \in V_{w(a).K}, Z \in \mathfrak{t}.$$

Here, $\Omega_{w(a).K}$ denotes the symplectic form on the symplectic vector space $V_{w(a).K}$. Since T acts symplectically on $V_{w(a).K}$ the notation $Z.Y$ makes sense as the linear action of an element $Z \in \mathfrak{sp}(V_{w(a).K})$ on a vector $Y \in V_{w(a).K}$. In appropriate symplectic coordinates $q_1, p_1, \ldots, q_n, p_n$ we have $\Omega_{w(a).K} = \sum_i dq_i \wedge dp_i$ and the matrix representation for the linear map defined by $Z \in \mathfrak{t}$ is

$$Z.(q_1, p_1, \ldots, q_n, p_n) = \begin{pmatrix} 0 & \alpha_1(Z) & & & \\ -\alpha(Z) & 0 & & & \\ & & \ddots & & \\ & & & 0 & \alpha_n(Z) \\ & & & -\alpha_n(Z) & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ p_1 \\ \vdots \\ q_n \\ p_n \end{pmatrix}.$$

The moment map takes the form

$$\Phi(q_1, p_1, \ldots, q_n, p_n) = \Phi(w(a).K) + \sum_{i=1}^{n} \alpha_i \frac{1}{2}(q_i^2 + p_i^2).$$

In terms of the symplectic coordinates on $V_{w(a).K}$ chosen above, the matrix representations for the symplectic form $\Omega_{w(a).K}$ and the corresponding Poisson tensor $\Pi_{w(a).K}$ just differ by a factor of (-1). The moment map can then be expressed in terms of the Poisson tensor.

$$\Phi_Z(\varphi) = -\Pi_{w(a).K}(Z.\varphi, \varphi) \qquad \forall\, \varphi \in V^*_{w(a).K}, Z \in \mathfrak{t}.$$

(Recall the bijection $\Pi^\sharp : V^*_{w(a).K} \to V_{w(a).K}$. Then $Z.\varphi = (\Pi^\sharp)^{-1}(Z.(\Pi^\sharp(\varphi)))$, where the dot on the right hand side has been explained above.)
The local cone $\Gamma_{w(X)}$ is just $\Phi(V^*_{w(a).K})$, i.e. it consists exactly of the weights

$$\{\ Z \mapsto -\Pi_{w(a).K}(Z.\varphi, \varphi)\ :\ \varphi \in V^*_{w(a).K}\ \} \tag{5}$$

Recall that we identify the cotangent space $T^*_{w(a).K}(G/K)$ with $Ad(w(a)).\mathfrak{k}$. The formula for the Poisson tensor at $w(a).K$ says that for $Y_1, Y_2 \in \mathfrak{k}$,

$$\Pi_{w(a).K}(Ad(w(a))Y_1, Ad(w(a))Y_2) = \langle pr_{\mathfrak{h}} Ad(w(a))Y_1, Ad(w(a))Y_2\rangle.$$

Note that $T^*_{w(a).K}(G/K) = T^*_{w(a).K}M_a \oplus (T_{w(a).K}M_a)^\perp$. Both $T^*_{w(a).K}M_a$ and $(T_{w(a).K}M_a)^\perp$ are stable under the action of T. Moreover, $T_{w(a).K}M_a = \Pi^\sharp_{w(a).K}(T^*_{w(a).K}(G/K))$ by the definition of the symplectic leaf M_a. Hence, for $\varphi \in T^*_{w(a).K} M_a, \psi \in (T_{a_w} M_{a_w})^\perp$ and $Z \in \mathfrak{t}$, one obtains

$$\begin{aligned}
\Pi_{w(a).K}(Z.(\varphi+\psi),(\varphi+\psi)) &= (\varphi+\psi).\Pi^\sharp_{w(a).K}(Z.(\varphi+\psi)) \\
&= \varphi.\Pi^\sharp_{w(a).K}(Z(\varphi+\psi)) \\
&= \Pi_{w(a).K}(Z\varphi+Z\psi,\varphi) \\
&= -(Z\varphi+Z\psi).\Pi^\sharp_{w(a).K}(\varphi) \\
&= \Pi_{w(a).K}(Z\varphi,\varphi)
\end{aligned}$$

In view of (5) and (2) (from Section 3) it follows that the local cone is given by

$$\Gamma_{w(X)} = \{\ Z \mapsto -\langle pr_{\mathfrak{h}}[Z, Ad(w(a))Y], Ad(w(a))Y\rangle \ :\ Y \in \mathfrak{k}\ \}. \tag{6}$$

In order to determine the weights in (6) we will construct a basis $\{v_1,\dots,v_r\}$ for $\mathfrak{k}$ with two main features.

(1) For each v_i we determine explicitly an element $H_i \in \mathfrak{a}^{-\tau}$ such that

$$\langle pr_{\mathfrak{h}}[Z, Ad(w(a))v_i], Ad(w(a))v_i\rangle = \Im\kappa(H_i, Z) \quad \forall\ Z \in \mathfrak{t}.$$

(2) $\langle pr_{\mathfrak{h}}[Z, Ad(w(a))v_i], Ad(w(a))v_j\rangle = 0$ for all $Z \in \mathfrak{t}$ whenever $i \neq j$.

Once such a basis is found each $Y \in \mathfrak{k}$ can be written as a linear combination $Y = \sum_{i=1}^N c_i v_i$. Then, for $Z \in \mathfrak{t}$,

$$\begin{aligned}
&\langle pr_{\mathfrak{h}}[Z, Ad(w(a))Y], Ad(w(a))Y\rangle \\
&\qquad = \langle pr_{\mathfrak{h}}[Z, Ad(w(a))\sum_{i=1}^N c_i v_i], Ad(w(a))\sum_{i=1}^N c_i v_i\rangle \\
&\qquad = \sum_{i=1}^N c_i^2 \langle pr_{\mathfrak{h}}[Z, Ad(w(a))v_i], Ad(w(a))v_i\rangle \\
&\qquad = \sum_{i=1}^N c_i^2 \Im\kappa(H_i, Z)
\end{aligned}$$

In view of (6) it then follows that $\Gamma_{w(X)}$ is the cone spanned by the vectors H_i. Recall the weight space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}^{-\tau}$.

$$\mathfrak{g} = \mathfrak{a}^{-\tau} \oplus \mathfrak{a}^{\tau} \oplus i\mathfrak{a}^{-\tau} \oplus i\mathfrak{a}^{\tau} \oplus \sum_{\beta \in \Delta(\mathfrak{g},\mathfrak{a}^{-\tau})} \mathfrak{g}^\beta$$

Each $\mathfrak{g}^\beta$ is stable under the involution $\theta\tau$, hence decomposes into $(+1)$- and (-1)-eigenspaces $\mathfrak{g}^\beta = (\mathfrak{g}^\beta)_+ \oplus (\mathfrak{g}^\beta)_-$. We first consider certain bases for $\mathfrak{g}^\beta = (\mathfrak{g}^\beta)_+$ and $\mathfrak{g}^\beta = (\mathfrak{g}^\beta)_-$. Each $\mathfrak{g}^\beta$ is stable under the adjoint action of $\mathfrak{a}^\tau$. For the corresponding weight space decomposition we write

$$\mathfrak{g}^\beta = \sum_{\eta \in \Delta(\mathfrak{g}^\beta,\mathfrak{a}^\tau)} \mathfrak{g}^{\beta,\eta}$$

Note that $\mathfrak{g}^{\beta,\eta}$ is equal to the eigenspace $\mathfrak{g}^{\alpha} \subset \mathfrak{n}$ for $\alpha \in \Delta(\mathfrak{g},\mathfrak{a})$ if and only if $\alpha|_{\mathfrak{a}^{-\tau}} = \beta$ and $\alpha|_{\mathfrak{a}^{\tau}} = \eta$. Also, if $\mathfrak{g}^{\beta,\eta} = \mathfrak{g}^{\alpha}$, then $\beta \in \Delta^+(\mathfrak{g},\mathfrak{a}^{-\tau})$ if and only if $\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})$. The involutions τ and θ transform the eigenspaces as follows

$$\tau(\mathfrak{g}^{\beta,\eta}) = \mathfrak{g}^{-\beta,\eta}, \quad \theta(\mathfrak{g}^{\beta,\eta}) = \mathfrak{g}^{-\beta,-\eta}, \quad \theta\tau(\mathfrak{g}^{\beta,\eta}) = \mathfrak{g}^{\beta,-\eta}$$

For each eigenspace $\mathfrak{g}^{\beta,\eta}$ fix a vector $X_{\beta,\eta}$ that spans $\mathfrak{g}^{\beta,\eta}$ as a complex vector space. If $\eta \neq 0$ we define

$$A_{\beta,\eta} = X_{\beta,\eta} + \theta\tau X_{\beta,\eta}, \quad B_{\beta,\eta} = X_{\beta,\eta} - \theta\tau X_{\beta,\eta}.$$

We obtain the following (complex) basis for the reduced root space $\mathfrak{g}^{\beta}$

$$\{X_{\beta,0}\} \cup \{A_{\beta,\eta} : \eta \neq 0\} \cup \{B_{\beta,\eta} : \eta \neq 0\}$$

The important feature of this basis is that it consists of eigenvectors of the complex linear involution $\theta\tau$. Indeed, $\theta\tau A_{\beta,\eta} = A_{\beta,\eta}, \theta\tau B_{\beta,\eta} = -B_{\beta,\eta}$ and $X_{\beta,0}$ might be a $(+1)$- or a (-1)-eigenvector of $\theta\tau$. Therefore, a basis for $(\mathfrak{g}^{\beta})_+$ is given by the $A_{\beta,\eta}$'s and possibly $X_{\beta,0}$. A basis for $(\mathfrak{g}^{\beta})_-$ is given by the $B_{\beta,\eta}$'s and possibly $X_{\beta,0}$ (iff it is not contained in $\mathfrak{g}^{\beta} = (\mathfrak{g}^{\beta})_+$).
The desired (real) basis for $\mathfrak{k}$ now consists of a basis for $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}^{-\tau}) = i\mathfrak{a}^{-\tau} + i\mathfrak{a}^{\tau}$ and the following set.

$$\begin{aligned} \bigcup_{\beta\in\Delta^+(\mathfrak{g},\mathfrak{a}^{-\tau})} \Big(& \{X_{\beta,0} + \theta X_{\beta,0}\} \cup \{iX_{\beta,0} + \theta iX_{\beta,0}\} \\ & \cup \{A_{\beta,\eta} + \theta A_{\beta,\eta} : \eta \neq 0\} \cup \{iA_{\beta,\eta} + \theta iA_{\beta,\eta} : \eta \neq 0\} \\ & \cup \{B_{\beta,\eta} + \theta B_{\beta,\eta} : \eta \neq 0\} \cup \{iB_{\beta,\eta} + \theta iB_{\beta,\eta} : \eta \neq 0\} \Big) \end{aligned} \tag{7}$$

We can now calculate the weights appearing in (6) for each basis element. We fix $Z = iH \in \mathfrak{t} = i\mathfrak{a}_{-\tau}$. Recall that $a = \exp X$, therefore $w(a) = \exp(w(X))$. First we make two short auxiliary calculations. For a vector $C_\beta \in \mathfrak{g}^{\beta}$ which is also a $\theta\tau$-fixed point,

$$\begin{aligned} [Z, Ad(w(a)).(C_\beta + \theta C_\beta)] &= i\beta(H)w(a)^{\beta}C_\beta - i\beta(H)w(a)^{-\beta}\theta C_\beta \\ &= \beta(H)w(a)^{-\beta}(iC_\beta + \theta iC_\beta) + \beta(H)(w(a)^{\beta} - w(a)^{-\beta})iC_\beta \end{aligned}$$

In the second line, the first summand lies in $\mathfrak{h}$ the second in $\mathfrak{c}^{-\tau} + \mathfrak{n}$. For $D_\beta \in \mathfrak{g}^{\beta}$ such that $\theta\tau D_\beta = -D_\beta$, the $\mathfrak{h} \oplus (\mathfrak{c}^{-\tau} + \mathfrak{n})$ decomposition is different:

$$\begin{aligned} [Z, Ad(w(a)).(D_\beta + \theta D_\beta)] &= i\beta(H)w(a)^{\beta}D_\beta - i\beta(H)w(a)^{-\beta}\theta D_\beta \\ &= \beta(H)w(a)^{-\beta}(-iD_\beta + \theta iD_\beta) + \beta(H)(w(a)^{\beta} + w(a)^{-\beta})iD_\beta \end{aligned}$$

Now, for $A_{\beta,\eta}$, which lies in $\mathfrak{g}^\beta$ and satisfies $\theta\tau A_{\beta,\eta} = A_{\beta,\eta}$, we compute

$$\begin{aligned}
(8)\quad & \langle pr_{\mathfrak{h}}[Z, Ad(w(a)).(A_{\beta,\eta} + \theta A_{\beta,\eta})], Ad(w(a)).(A_{\beta,\eta} + \theta A_{\beta,\eta})\rangle \\
&= \langle \beta(H) w(a)^{-\beta}(iA_{\beta,\eta} + \theta i A_{\beta,\eta}), w(a)^{\beta} A_{\beta,\eta} + w(a)^{-\beta}\theta A_{\beta,\eta}\rangle \\
&= \beta(H) w(a)^{-2\beta}\langle iA_{\beta,\eta}, \theta A_{\beta,\eta}\rangle + \beta(H)\langle \theta i A_{\beta,\eta}, A_{\beta,\eta}\rangle \\
&= (w(a)^{-2\beta} - 1)\ \Re\kappa(A_{\beta,\eta}, \theta A_{\beta,\eta})\ \beta(H) \\
&= (w(a)^{-2\beta} - 1)\ \Re\kappa(A_{\beta,\eta}, \theta A_{\beta,\eta})\ \kappa(H_\beta, H) \\
&= (w(a)^{-2\beta} - 1)\ \Re\kappa(A_{\beta,\eta}, \theta A_{\beta,\eta})\ \Im\kappa(H_\beta, Z)
\end{aligned}$$

We can replace $A_{\beta,\eta}$ with $iA_{\beta,\eta}$ in the above calculation and obtain

$$\begin{aligned}
& \langle pr_{\mathfrak{h}}[Z, Ad(w(a)).(iA_{\beta,\eta} + \theta i A_{\beta,\eta})], Ad(w(a)).(iA_{\beta,\eta} + \theta i A_{\beta,\eta})\rangle \\
&\qquad = (w(a)^{-2\beta} - 1)\ \Re\kappa(iA_{\beta,\eta}, \theta i A_{\beta,\eta})\ \beta(H) \\
&\qquad = (w(a)^{-2\beta} - 1)\ \Re\kappa(A_{\beta,\eta}, \theta A_{\beta,\eta})\ \Im\kappa(H_\beta, Z)
\end{aligned}$$

Carrying out the calculation for $B_{\beta,\eta}$ (which are (-1)-eigenvectors of $\theta\tau$) we obtain a result of a different nature

$$\begin{aligned}
(9)\quad & \langle pr_{\mathfrak{h}}[Z, Ad(w(a)).(B_{\beta,\eta} + \theta B_{\beta,\eta})], Ad(w(a)).(B_{\beta,\eta} + \theta B_{\beta,\eta})\rangle \\
&\qquad = -(w(a)^{-2\beta} + 1)\ \Re\kappa(B_{\beta,\eta}, \theta B_{\beta,\eta})\ \Im\kappa(H_\beta, Z),
\end{aligned}$$

and

$$\begin{aligned}
& \langle pr_{\mathfrak{h}}[Z, Ad(w(a)).(iB_{\beta,\eta} + \theta i B_{\beta,\eta})], Ad(w(a)).(iB_{\beta,\eta} + \theta i B_{\beta,\eta})\rangle \\
&\qquad = -(w(a)^{-2\beta} + 1)\ \Re\kappa(B_{\beta,\eta}, \theta B_{\beta,\eta})\ \Im\kappa(H_\beta, Z).
\end{aligned}$$

If $X_{\beta,0}$ is fixed by $\theta\tau$, then

$$\begin{aligned}
(10)\quad & \langle pr_{\mathfrak{h}}[Z, Ad(w(a)).(X_{\beta,0} + \theta X_{\beta,0})], Ad(w(a)).(X_{\beta,0} + \theta X_{\beta,0})\rangle \\
&\qquad = (w(a)^{-2\beta} - 1)\ \Re\kappa(X_{\beta,0}, \theta X_{\beta,0})\ \Im\kappa(H_\beta, Z),
\end{aligned}$$

and

$$\begin{aligned}
& \langle pr_{\mathfrak{h}}[Z, Ad(w(a)).(iX_{\beta,0} + \theta i X_{\beta,0})], Ad(w(a)).(iX_{\beta,0} + \theta i X_{\beta,0})\rangle \\
&\qquad = (w(a)^{-2\beta} - 1)\ \Re\kappa(X_{\beta,0}, \theta X_{\beta,0})\ \Im\kappa(H_\beta, Z).
\end{aligned}$$

The case that $\theta\tau X_{\beta,0} = -X_{\beta,0}$ leads to

$$\begin{aligned}
(11)\quad & \langle pr_{\mathfrak{h}}[Z, Ad(w(a)).(X_{\beta,0} + \theta X_{\beta,0})], Ad(w(a)).(X_{\beta,0} + \theta X_{\beta,0})\rangle \\
&\qquad = -(w(a)^{-2\beta} + 1)\ \Re\kappa(X_{\beta,0}, \theta X_{\beta,0})\ \Im\kappa(H_\beta, Z),
\end{aligned}$$

and

$$\begin{aligned}
& \langle pr_{\mathfrak{h}}[Z, Ad(w(a)).(iX_{\beta,0} + \theta i X_{\beta,0})], Ad(w(a)).(iX_{\beta,0} + \theta i X_{\beta,0})\rangle \\
&\qquad = -(w(a)^{-2\beta}) + 1)\ \Re\kappa(X_{\beta,0}, \theta X_{\beta,0})\ \Im\kappa(H_\beta, Z).
\end{aligned}$$

Moreover, for $Y \in \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ one easily checks that

$$\langle pr_{\mathfrak{h}}[Z, Ad(w(a)).Y], Ad(w(a)).Y\rangle = 0.$$

Note that the coefficient of $\Im\kappa(H_\beta, Z)$ in (9) and (11) is always positive. Therefore, basis vectors of $\mathfrak{k}$ which are (-1)-eigenvectors of $\theta\tau$ contribute the set $\{-H_\beta : \beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau}), (\mathfrak{g}^\beta)_- \neq 0\}$ to $\Gamma_{w(X)}$.
On the other hand, the coefficient of $\Im\kappa(H_\beta, Z)$ in (8) and (10) depends on the value of $\beta(w(X))$. If $\beta(w(X)) = 0$ this coefficient is zero. If $\beta(w(X)) > 0$ the coefficient is positive, and if $\beta(w(X)) < 0$ it is negative. Therefore, basis vectors of $\mathfrak{k}$ which are $(+1)$-eigenvectors of $\theta\tau$ contribute the set $\{-\beta(w(X))H_\beta : \beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau}), (\mathfrak{g}^\beta)_+ \neq 0\}$ to $\Gamma_{w(X)}$.
The fact that $\langle pr_{\mathfrak{h}}[Z, Ad(w(a))v_i], Ad(w(a))v_j\rangle = 0$ holds for all $Z \in \mathfrak{t}$ whenever $i \neq j$ follows from general properties of the Killing form.
The conclusion is that the cone $\Gamma_{w(X)} = \Phi(V^*_{w(a).K})$ is generated by the weights

$$\{-\beta(w(X))H_\beta : \beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau}), (\mathfrak{g}^\beta)_+ \neq 0\}$$
$$\cup \{-H_\beta : \beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau}), (\mathfrak{g}^\beta)_- \neq 0\},$$

as asserted.

□

COROLLARY 4.5. *The image of the moment map $\Phi(M_a)$ is contained in the set $w'(X) + \Gamma_+$, where $w' \in \mathcal{W}_{K\cap H}$ is such that $\beta(w'(X)) \geq 0$ for all $\beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau})$ and Γ_+ is the proper cone $\Gamma_+ = \mathrm{cone}(-\mathrm{H}_\beta : \beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau}))$.*

Proof. ¿From Theorem 4.1 and Remark 4.2 we know that there is a neighborhood $U_{w'(X)} \subseteq \mathfrak{a}^{-\tau}$ of $w'(X)$ such that $\Phi(M_a) \cap U_{w'(X)} = (w'(X) + \Gamma_{w'(X)}) \cap U_{w'(X)}$. Lemma 4.4 implies that $\Gamma_{w'(X)} \subseteq \Gamma_+$. Suppose there exists some $Z \in \Phi(M_a)$ such that $Z \notin w'(X) + \Gamma_+$. Since $\Phi(M_a)$ is convex the line segment $\overline{w'(X)Z}$ lies entirely in $\Phi(M_a)$. Fix some $Y \in \overline{w'(X)Z} \cap U_{w'(X)}$ with $Y \neq w'(X)$. Then $Y \in \Phi(M_a) \cap U_{w'(X)} \subseteq w'(X) + \Gamma_{w'(X)} \subseteq w'(X) + \Gamma_+$. But this implies $Z \in w'(X) + \Gamma_+$ since Γ_+ is a cone and $Y \neq w'(X)$, a contradiction. Therefore, $\Phi(M_a) \subseteq w'(X) + \Gamma_+$. The cone Γ_+ is proper since it is spanned by vectors $-H_\beta$ associated to positive roots β. □

The special property of $\Phi(M_a)$ stated in the corollary allows us to describe $\Phi(M_a)$ entirely in terms of the local cones $\Gamma_{w(X)}$ associated to the fixed points, as the following proposition shows.

PROPOSITION 4.6. *Let C be a closed, convex, locally polyhedral set (in some finite dimensional vector space V). Denote by Γ_c the local cone at $c \in C$ (i.e. there is a neighborhood $U_c \subset V$ of c such that $C \cap U_c = (c + \Gamma_c) \cap U_c$). Suppose $C \subset x + \Gamma$ for some $x \in V$ and some proper cone $\Gamma \subset V$. Then*

$$C = \bigcap_{\Gamma_c \text{ proper}} (c + \Gamma_c),$$

i.e. C is completely determined by the local cones that are proper.

Proof. For any $c \in C$ we write d_c for the dimension of the maximal subspace contained in Γ_c. (In particular, $d_c = 0$ means that Γ_c is proper.) First we will show that if $d_c > 0$, then $c \in c' + \Gamma_{c'}$ for some c' with $d_{c'} < d_c$.
If $d_c > 0$, then Γ_c contains a line, say L. Since C lies in a proper cone, $(c+L)\cap C$ is semi-bounded. We pick an endpoint c' of $(c+L)\cap C$. Since C is closed $c' \in C$, and clearly $c \in c' + \Gamma_{c'}$. Convexity of C implies that if a line L' is contained in $\Gamma_{c'}$ then $L' \subset \Gamma_{\tilde{c}}$ for each inner point $\tilde{c}$ of $(c+L) \cap C$. In particular, $d_{c'} \leq d_c$. On the other hand, $\Gamma_{c'}$ does not contain the line $L \subset \Gamma_c$. Therefore, $d_{c'} < d_c$.
Now, the assumptions on C imply

$$C = \bigcap_{c \in C} (c + \Gamma_c)$$

If we set $n = \dim(\mathrm{V})$ the above arguments lead to

$$C = \bigcap_{d_c \leq n} (c + \Gamma_c) = \bigcap_{d_c \leq n-1} (c + \Gamma_c) = \cdots = \bigcap_{d_c = 0} (c + \Gamma_c)$$

□

We are now ready to give the desired description of $\Phi(M_a)$ which is the content of van den Ban's theorem.

THEOREM 4.7. *The set* $\Phi(M_a) = (pr_{\mathfrak{a}^{-\tau}} \circ \log \circ \mu)(Ha)$ *is the sum of a compact convex set and a closed (proper) cone* Γ. *More precisely, for* $a = \exp X$,

$$\Phi(M_a) = \mathrm{conv}(\mathcal{W}_{K\cap H}.X) + \Gamma,$$

with

$$\Gamma = \mathrm{cone}\{-H_\beta : \beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau}), (\mathfrak{g}^\beta)_- \neq 0\}$$

Proof. The image $\Phi(M_a)$ is closed, convex and locally polyhedral. Moreover, by Corollary 4.5, it is contained in $w'(X) + \Gamma_+$ for some proper cone Γ_+. Proposition 4.6 implies that $\Phi(M_a)$ is determined by the local cones that are proper. According to Remark 4.2, a local cone $\Gamma_{\Phi(m)}$ can be proper only if $\mathfrak{t}_m = \mathfrak{t}$, i.e. if m is a T-fixed point. The T-fixed points have been characterized in Proposition 4.3, so Proposition 4.6 yields

$$\Phi(M_a) = \bigcap_{w \in \mathcal{W}_{K\cap H}} (w(X) + \Gamma_{w(X)}),$$

with $\Gamma_{w(X)}$ as in Lemma 4.4.
The sum $\mathrm{conv}(\mathcal{W}_{K\cap H}.X) + \Gamma$ is closed, convex and locally polyhedral as well. As a sum of a compact set and the proper cone Γ it is contained in $x + \Gamma$ for some $x \in \mathfrak{a}^{-\tau}$, hence Proposition 4.6 is applicable. First we want to see at which points in $\mathrm{conv}(\mathcal{W}_{K\cap H}.X) + \Gamma$ the local cone is proper. Let $c \in \mathrm{conv}(\mathcal{W}_{K\cap H}.X)$ and $\gamma \in \Gamma$. Clearly, the local cone at $c + \gamma$ is improper unless $\gamma = 0$. But then $c + \gamma = c$ is contained in a convex set with extremal points $\{w(X) : w \in \mathcal{W}_{K\cap H}\}$. The local cone can be proper only if $c + \gamma$ is one of those extremal points. Proposition 4.6 now gives

$$\mathrm{conv}(\mathcal{W}_{K\cap H}.X) + \Gamma = \bigcap_{w \in \mathcal{W}_{K\cap H}} (w(X) + \Gamma'_{w(X)}).$$

Here, $\Gamma'_{w(X)}$ denotes the local cone of $\operatorname{conv}(\mathcal{W}_{K\cap H}.X) + \Gamma$ at $w(X)$. To finish the proof it is sufficient to show that $\Gamma'_{w(X)} = \Gamma_{w(X)}$.
Clearly, $\Gamma'_{w(X)} = \Gamma''_{w(X)} + \Gamma$, where $\Gamma = \operatorname{cone}\{-H_\beta : \beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau}), (\mathfrak{g}^\beta)_- \neq 0\}$ as before and $\Gamma''_{w(X)} = \operatorname{cone}\{w'(X) - w(X) : w' \in \mathcal{W}_{K\cap H}\}$. From Lemma 4.4 we know that $\Gamma_{w(X)}$ contains the cone Γ. Moreover, the set $\Phi(M_a)$ is convex and contains all points $w(X)$, and therefore contains $\operatorname{conv}(\mathcal{W}_{K\cap H}.X)$. This implies that its local cone at $w(X)$, i.e. $\Gamma_{w(X)}$, contains $\Gamma''_{w(X)}$ as well. Therefore, $\Gamma_{w(X)} \supseteq \Gamma''_{w(X)} + \Gamma = \Gamma'_{w(X)}$.
Each root $\beta \in \Delta(\mathfrak{g}, \mathfrak{a}^{-\tau})$ defines the isomorphism

$$s_\beta : \mathfrak{a}^{-\tau} \to \mathfrak{a}^{-\tau}, Z \mapsto Z - 2\frac{\beta Z}{\langle \beta, \beta \rangle} H_\beta.$$

In view of Remark 2.1 the Weyl group $\mathcal{W}' = \mathcal{W}_{K\cap H}$ consists exactly of those s_β for which $(\mathfrak{g}^\beta)_+ \neq 0$. In particular, $s_\beta(w(X)) \in \mathcal{W}_{K\cap H}$ for all $\beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau})$ for which $(\mathfrak{g}^\beta)_+ \neq 0$. The identity $s_\beta(w(X)) - w(X) = -2\frac{\beta(w(X))}{\langle \beta,\beta\rangle} H_\beta$ implies $\operatorname{cone}\{-\beta(w(X))H_\beta : \beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau}), (\mathfrak{g}^\beta)_+ \neq 0\} \subseteq \Gamma''_X$. With Lemma 4.4 we obtain $\Gamma_{w(X)} \subseteq \Gamma''_{w(X)} + \Gamma = \Gamma'_{w(X)}$. □

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