Acyclicity Versus Total Acyclicity for Complexes over Noetherian Rings

Srikanth Iyengar[1], Henning Krause

Received: October 4, 2005
Revised: June 14, 2006

Communicated by Max Karoubi

Abstract. It is proved that for a commutative noetherian ring with dualizing complex the homotopy category of projective modules is equivalent, as a triangulated category, to the homotopy category of injective modules. Restricted to compact objects, this statement is a reinterpretation of Grothendieck's duality theorem. Using this equivalence it is proved that the (Verdier) quotient of the category of acyclic complexes of projectives by its subcategory of totally acyclic complexes and the corresponding category consisting of injective modules are equivalent. A new characterization is provided for complexes in Auslander categories and in Bass categories of such rings.

2000 Mathematics Subject Classification: Primary: 16E05; Secondary: 13D05, 16E10, 18E30
Keywords and Phrases: Totally acyclic complex, dualizing complex, Gorenstein dimension, Auslander category, Bass category

Introduction

Let R be a commutative noetherian ring with a dualizing complex D; in this article, this means, in particular, that D is a bounded complex of injective R-modules; see Section 3 for a detailed definition. The starting point of the work described below was a realization that $\mathbf{K}(\operatorname{Prj} R)$ and $\mathbf{K}(\operatorname{Inj} R)$, the homotopy categories of complexes of projective R-modules and of injective R-modules, respectively, are equivalent. This equivalence comes about as follows: D consists of injective modules and, R being noetherian, direct sums of injectives are injective, so $D \otimes_R -$ defines a functor from $\mathbf{K}(\operatorname{Prj} R)$ to $\mathbf{K}(\operatorname{Inj} R)$. This

[1]S. I. was partly supported by NSF grant DMS 0442242

functor factors through $\mathbf{K}(\operatorname{Flat} R)$, the homotopy category of flat R-modules, and provides the lower row in the following diagram:

$$\begin{tikzcd} \mathbf{K}(\operatorname{Prj} R) \arrow[r, shift right, "\mathsf{inc}"'] & \mathbf{K}(\operatorname{Flat} R) \arrow[l, shift right, "\mathsf{q}"'] \arrow[r, shift right, "D\otimes_R -"'] & \mathbf{K}(\operatorname{Inj} R) \arrow[l, shift right, "\operatorname{Hom}_R(D,-)"'] \end{tikzcd}$$

The triangulated structures on the homotopy categories are preserved by inc and $D \otimes_R -$. The functors in the upper row of the diagram are the corresponding right adjoints; the existence of q is proved in Proposition (2.4). Theorem (4.2) then asserts:

Theorem I. *The functor* $D \otimes_R - \colon \mathbf{K}(\operatorname{Prj} R) \to \mathbf{K}(\operatorname{Inj} R)$ *is an equivalence of triangulated categories, with quasi-inverse* $\mathsf{q} \circ \operatorname{Hom}_R(D, -)$.

This equivalence is closely related to, and may be viewed as an extension of, Grothendieck's duality theorem for $\mathbf{D}^f(R)$, the derived category of complexes whose homology is bounded and finitely generated. To see this connection, one has to consider the classes of compact objects – the definition is recalled in (1.2) – in $\mathbf{K}(\operatorname{Prj} R)$ and in $\mathbf{K}(\operatorname{Inj} R)$. These classes fit into a commutative diagram of functors:

$$\begin{tikzcd} \mathbf{K}^c(\operatorname{Prj} R) \arrow[r, "D\otimes_R -"] \arrow[d, "\mathsf{P}"', "\wr"] & \mathbf{K}^c(\operatorname{Inj} R) \arrow[d, "\mathsf{I}", "\wr"'] \\ \mathbf{D}^f(R) \arrow[r, "\mathbf{R}\operatorname{Hom}_R(-,D)"] & \mathbf{D}^f(R) \end{tikzcd}$$

The functor P is induced by the composite

$$\mathbf{K}(\operatorname{Prj} R) \xrightarrow{\operatorname{Hom}_R(-,R)} \mathbf{K}(R) \xrightarrow{\mathsf{can}} \mathbf{D}(R)\,,$$

and it is a theorem of Jørgensen [11] that P is an equivalence of categories. The equivalence I is induced by the canonical functor $\mathbf{K}(R) \to \mathbf{D}(R)$; see [14]. Given these descriptions it is not hard to verify that $D \otimes_R -$ preserves compactness; this explains the top row of the diagram. Now, Theorem I implies that $D \otimes_R -$ restricts to an equivalence between compact objects, so the diagram above implies $\mathbf{R}\operatorname{Hom}_R(-, D)$ is an equivalence; this is one version of the duality theorem; see Hartshorne [9]. Conversely, given that $\mathbf{R}\operatorname{Hom}_R(-, D)$ is an equivalence, so is the top row of the diagram; this is the crux of the proof of Theorem I.

Theorem I appears in Section 4. The relevant definitions and the machinery used in the proof of this result, and in the rest of the paper, are recalled in Sections 1 and 2. In the remainder of the paper we develop Theorem (4.2) in two directions. The first one deals with the difference between the category of acyclic complexes in $\mathbf{K}(\operatorname{Prj} R)$, denoted $\mathbf{K}_{\mathrm{ac}}(\operatorname{Prj} R)$, and its subcategory consisting of totally acyclic complexes, denoted $\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R)$. We consider also the injective counterparts. Theorems (5.3) and (5.4) are the main new results in this context; here is an extract:

Theorem II. *The quotients*

$$\mathbf{K}_{\mathrm{ac}}(\operatorname{Prj} R)/\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R) \text{ and } \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R)/\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} R)$$

are compactly generated, and there are, up to direct factors, equivalences

$$\begin{aligned} \operatorname{Thick}(R,D)/\operatorname{Thick}(R) &\xrightarrow{\sim} \big[\big(\mathbf{K}_{\mathrm{ac}}(\operatorname{Prj} R)/\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R)\big)^{c}\big]^{\mathrm{op}} \\ \operatorname{Thick}(R,D)/\operatorname{Thick}(R) &\xrightarrow{\sim} \big(\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R)/\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} R)\big)^{c}. \end{aligned}$$

In this result, $\operatorname{Thick}(R,D)$ is the thick subcategory of $\mathbf{D}^f(R)$ generated by R and D, while $\operatorname{Thick}(R)$ is the thick subcategory generated by R; that is to say, the subcategory of complexes of finite projective dimension. The quotient $\operatorname{Thick}(R,D)/\operatorname{Thick}(R)$ is a subcategory of the category $\mathbf{D}^f(R)/\operatorname{Thick}(R)$, which is sometimes referred to as the stable category of R. Since a dualizing complex has finite projective dimension if and only if R is Gorenstein, one corollary of the preceding theorem is that R is Gorenstein if and only if every acyclic complex of projectives is totally acyclic, if and only if every acyclic complex of injectives is totally acyclic.

Theorem II draws attention to the category $\operatorname{Thick}(R,D)/\operatorname{Thick}(R)$ as a measure of the failure of a ring R from being Gorenstein. Its role is thus analogous to that of the full stable category with regards to regularity: $\mathbf{D}^f(R)/\operatorname{Thick}(R)$ is trivial if and only if R is regular. See (5.6) for another piece of evidence that suggests that $\operatorname{Thick}(R,D)/\operatorname{Thick}(R)$ is an object worth investigating further.

In Section 6 we illustrate the results from Section 5 on local rings whose maximal ideal is square-zero. Their properties are of interest also from the point of view of Tate cohomology; see (6.5).

Sections 7 and 8 are a detailed study of the functors induced on $\mathbf{D}(R)$ by those in Theorem I. This involves two different realizations of the derived category as a subcategory of $\mathbf{K}(R)$, both obtained from the localization functor $\mathbf{K}(R) \to \mathbf{D}(R)$: one by restricting it to $\mathbf{K}_{\mathrm{prj}}(R)$, the subcategory of K-projective complexes, and the other by restricting it to $\mathbf{K}_{\mathrm{inj}}(R)$, the subcategory of K-injective complexes. The inclusion $\mathbf{K}_{\mathrm{prj}}(R) \to \mathbf{K}(\operatorname{Prj} R)$ admits a right adjoint p; for a complex X of projective modules the morphism $\mathsf{p}(X) \to X$ is a K-projective resolution. In the same way, the inclusion $\mathbf{K}_{\mathrm{inj}}(R) \to \mathbf{K}(\operatorname{Inj} R)$ admits a left adjoint i, and for a complex Y of injectives the morphism $Y \to \mathsf{i}(Y)$ is a K-injective resolution. Consider the functors $\mathsf{G} = \mathsf{i} \circ (D \otimes_R -)$ restricted to $\mathbf{K}_{\mathrm{prj}}(R)$, and $\mathsf{F} = \mathsf{p} \circ \mathsf{q} \circ \operatorname{Hom}_R(D,-)$ restricted to $\mathbf{K}_{\mathrm{inj}}(R)$. These functors better visualized as part of the diagram below:

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathbf{K}(\operatorname{Prj} R) \arrow[r, shift right, "D\otimes_R -"', "\sim"] \arrow[d, shift left, "\mathsf{p}"] & \mathbf{K}(\operatorname{Inj} R) \arrow[l, shift right, "\mathsf{q}\circ \operatorname{Hom}_R(D,-)"'] \arrow[d, shift right, "\mathsf{i}"'] \\
\mathbf{K}_{\mathrm{prj}}(R) \arrow[u, shift left, "\mathsf{inc}"] \arrow[r, shift right, "\mathsf{G}"'] & \mathbf{K}_{\mathrm{inj}}(R) \arrow[l, shift right, "\mathsf{F}"'] \arrow[u, shift right, "\mathsf{inc}"']
\end{tikzcd}$$

It is clear that (G, F) is an adjoint pair of functors. However, the equivalence in the upper row of the diagram does not imply an equivalence in the lower one. Indeed, given Theorem I and the results in Section 5 it is not hard to prove:

The natural morphism $X \to \mathsf{FG}(X)$ is an isomorphism if and only if the mapping cone of the morphism $(D \otimes_R X) \to \mathsf{i}(D \otimes_R X)$ is totally acyclic.

The point of this statement is that the mapping cones of resolutions are, in general, only acyclic. Complexes in $\mathbf{K}_{\mathrm{inj}}(R)$ for which the morphism $\mathsf{GF}(Y) \to Y$ is an isomorphism can be characterized in a similar fashion; see Propositions (7.3) and (7.4). This is the key observation that allows us to describe, in Theorems (7.10) and (7.11), the subcategories of $\mathbf{K}_{\mathrm{prj}}(R)$ and $\mathbf{K}_{\mathrm{inj}}(R)$ where the functors G and F restrict to equivalences.

Building on these results, and translating to the derived category, we arrive at:

Theorem III. *A complex X of R-modules has finite G-projective dimension if and only if the morphism $X \to \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R X)$ in $\mathbf{D}(R)$ is an isomorphism and $H(D \otimes^{\mathbf{L}}_R X)$ is bounded on the left.*

The notion of finite G-projective dimension, and finite G-injective dimension, is recalled in Section 8. The result above is part of Theorem (8.1); its counterpart for G-injective dimensions is Theorem (8.2). Given these, it is clear that Theorem I restricts to an equivalence between the category of complexes of finite G-projective dimension and the category of complexes of finite G-injective dimension.

Theorems (8.1) and (8.2) recover recent results of Christensen, Frankild, and Holm [6], who arrived at them from a different perspective. The approach presented here clarifies the connection between finiteness of G-dimension and (total) acyclicity, and uncovers a connection between Grothendieck duality and the equivalence between the categories of complexes of finite G-projective dimension and of finite G-injective dimension by realizing them as different shadows of the same equivalence: that given by Theorem I.

So far we have focused on the case where the ring R is commutative. However, the results carry over, with suitable modifications in the statements and with nearly identical proofs, to non-commutative rings that possess dualizing complexes; the appropriate comments are collected towards the end of each section. We have chosen to present the main body of the work, Sections 4–8, in the commutative context in order to keep the underlying ideas transparent, and unobscured by notational complexity.

Notation. The following symbols are used to label arrows representing functors or morphisms: $\sim$ indicates an equivalence (between categories), $\cong$ an isomorphism (between objects), and $\simeq$ a quasi-isomorphism (between complexes).

1. Triangulated categories

This section is primarily a summary of basic notions and results about triangulated categories used frequently in this article. For us, the relevant examples of triangulated categories are homotopy categories of complexes over noetherian rings; they are the focus of the next section. Our basic references are Weibel [23], Neeman [19], and Verdier [22].

1.1. Triangulated categories. Let $\mathcal{T}$ be a triangulated category. We refer the reader to [19] and [22] for the axioms that define a triangulated category. When we speak of subcategories, it is implicit that they are full.

A non-empty subcategory $\mathcal{S}$ of $\mathcal{T}$ is said to be *thick* if it is a triangulated subcategory of $\mathcal{T}$ that is closed under retracts. If, in addition, $\mathcal{S}$ is closed under all coproducts allowed in $\mathcal{T}$, then it is *localizing*; if it is closed under all products in $\mathcal{T}$ it is *colocalizing.*

Let $\mathcal{C}$ be a class of objects in $\mathcal{T}$. The intersection of the thick subcategories of $\mathcal{T}$ containing $\mathcal{C}$ is a thick subcategory, denoted $\operatorname{Thick}(\mathcal{C})$. We write $\operatorname{Loc}(\mathcal{C})$, respectively, $\operatorname{Coloc}(\mathcal{C})$, for the intersection of the localizing, respectively, colocalizing, subcategories containing $\mathcal{C}$. Note that $\operatorname{Loc}(\mathcal{C})$ is itself localizing, while $\operatorname{Coloc}(\mathcal{C})$ is colocalizing.

1.2. Compact objects and generators. Let $\mathcal{T}$ be a triangulated category admitting arbitrary coproducts. An object X of $\mathcal{T}$ is *compact* if $\operatorname{Hom}_{\mathcal{T}}(X,-)$ commutes with coproducts; that is to say, for each coproduct $\coprod_i Y_i$ of objects in $\mathcal{T}$, the natural morphism of abelian groups

$$\coprod_i \operatorname{Hom}_{\mathcal{T}}(X,Y_i) \longrightarrow \operatorname{Hom}_{\mathcal{T}}\big(X,\coprod_i Y_i\big)$$

is bijective. The compact objects form a thick subcategory that we denote $\mathcal{T}^c$. We say that a class of objects $\mathcal{S}$ *generates* $\mathcal{T}$ if $\operatorname{Loc}(\mathcal{S}) = \mathcal{T}$, and that $\mathcal{T}$ is *compactly generated* if there exists a generating set consisting of compact objects.

Let $\mathcal{S}$ be a class of compact objects in $\mathcal{T}$. Then $\mathcal{S}$ generates $\mathcal{T}$ if and only if for any object Y of $\mathcal{T}$, we have $Y = 0$ provided that $\operatorname{Hom}_{\mathcal{T}}(\Sigma^n S, Y) = 0$ for all S in $\mathcal{S}$ and $n \in \mathbb{Z}$; see [18, (2.1)].

Adjoint functors play a useful, if technical, role in this work, and pertinent results on these are collected in the following paragraphs. MacLane's book [15, Chapter IV] is the basic reference for this topic; see also [23, (A.6)].

1.3. Adjoint functors. Given categories $\mathcal{A}$ and $\mathcal{B}$, a diagram

$$\mathcal{A} \underset{\mathsf{F}}{\overset{\mathsf{G}}{\leftrightarrows}} \mathcal{B}$$

indicates that F and G are adjoint functors, with F left adjoint to G; that is to say, there is a natural isomorphism $\operatorname{Hom}_{\mathcal{B}}(\mathsf{F}(A),B) \cong \operatorname{Hom}_{\mathcal{A}}(A,\mathsf{G}(B))$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

1.4. Let $\mathcal{T}$ be a category, $\mathcal{S}$ a full subcategory of $\mathcal{T}$, and $\mathsf{q}\colon \mathcal{T} \to \mathcal{S}$ a right adjoint of the inclusion $\mathsf{inc}\colon \mathcal{S} \to \mathcal{T}$. Then $\mathsf{q} \circ \mathsf{inc} \cong \mathsf{id}_{\mathcal{S}}$. Moreover, for each T in $\mathcal{T}$, an object P in $\mathcal{S}$ is isomorphic to $\mathsf{q}(T)$ if and only if there is a morphism $P \to T$ with the property that the induced map $\operatorname{Hom}_{\mathcal{T}}(S,P) \to \operatorname{Hom}_{\mathcal{T}}(S,T)$ is bijective for each $S \in \mathcal{S}$.

1.5. Let $\mathsf{F}\colon \mathcal{S} \to \mathcal{T}$ be an exact functor between triangulated categories such that $\mathcal{S}$ is compactly generated.

(1) The functor F admits a right adjoint if and only if it preserves coproducts.
(2) The functor F admits a left adjoint if and only if it preserves products.
(3) If F admits a right adjoint G, then F preserves compactness if and only if G preserves coproducts.

For (1), we refer to [18, (4.1)]; for (2), see [19, (8.6.1)]; for (3), see [18, (5.1)].

1.6. ORTHOGONAL CLASSES. Given a class $\mathcal{C}$ of objects in a triangulated category $\mathcal{T}$, the full subcategories

$$\begin{aligned} \mathcal{C}^{\perp} &= \{Y \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(\Sigma^n X, Y) = 0 \quad \text{for all } X \in \mathcal{C} \text{ and } n \in \mathbb{Z}\}\,, \\ {}^{\perp}\mathcal{C} &= \{X \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n Y) = 0 \quad \text{for all } Y \in \mathcal{C} \text{ and } n \in \mathbb{Z}\}\,. \end{aligned}$$

are called the classes *right orthogonal* and *left orthogonal* to $\mathcal{C}$, respectively. It is elementary to verify that $\mathcal{C}^{\perp}$ is a colocalizing subcategory of $\mathcal{T}$, and equals $\operatorname{Thick}(\mathcal{C})^{\perp}$. In the same vein, ${}^{\perp}\mathcal{C}$ is a localizing subcategory of $\mathcal{T}$, and equals ${}^{\perp}\operatorname{Thick}(\mathcal{C})$.
Caveat: Our notation for orthogonal classes conflicts with the one in [19].

An additive functor $\mathsf{F}\colon \mathcal{A} \to \mathcal{B}$ between additive categories is an *equivalence up to direct factors* if F is full and faithful, and every object in $\mathcal{B}$ is a direct factor of some object in the image of F.

PROPOSITION 1.7. *Let $\mathcal{T}$ be a compactly generated triangulated category and let $\mathcal{C} \subseteq \mathcal{T}$ be a class of compact objects.*

(1) *The triangulated category $\mathcal{C}^{\perp}$ is compactly generated. The inclusion $\mathcal{C}^{\perp} \to \mathcal{T}$ admits a left adjoint which induces, up to direct factors, an equivalence*
$$\mathcal{T}^c/\operatorname{Thick}(\mathcal{C}) \xrightarrow{\sim} (\mathcal{C}^{\perp})^c\,.$$
(2) *For each class $\mathcal{B} \subseteq \mathcal{C}$, the triangulated category $\mathcal{B}^{\perp}/\mathcal{C}^{\perp}$ is compactly generated. The canonical functor $\mathcal{B}^{\perp} \to \mathcal{B}^{\perp}/\mathcal{C}^{\perp}$ induces, up to direct factors, an equivalence*
$$\operatorname{Thick}(\mathcal{C})/\operatorname{Thick}(\mathcal{B}) \xrightarrow{\sim} (\mathcal{B}^{\perp}/\mathcal{C}^{\perp})^c\,.$$

Proof. First observe that $\mathcal{C}$ can be replaced by a set of objects because the isomorphism classes of compact objects in $\mathcal{T}$ form a set. Neeman gives in [17, (2.1)] a proof of (1); see also [17, p. 553 ff]. For (2), consider the following diagram

$$\begin{tikzcd} \mathcal{T}^c \arrow[r, "\mathsf{can}"] \arrow[d, "\mathsf{inc}"] & \mathcal{T}^c/\operatorname{Thick}(\mathcal{B}) \arrow[r, "\mathsf{can}"] \arrow[d] & \mathcal{T}^c/\operatorname{Thick}(\mathcal{C}) \arrow[d] \\ \mathcal{T} \arrow[r, shift right, "\mathsf{a}"'] & \mathcal{B}^{\perp} \arrow[l, shift right, "\mathsf{inc}"'] \arrow[r, shift right, "\mathsf{b}"'] & \mathcal{C}^{\perp} \arrow[l, shift right, "\mathsf{inc}"'] \end{tikzcd}$$

where a and b denote adjoints of the corresponding inclusion functors and unlabeled functors are induced by a and b respectively. The localizing subcategory

$\mathrm{Loc}(\mathcal{C})$ of $\mathcal{T}$ is generated by $\mathcal{C}$ and hence it is compactly generated and its full subcategory of compact objects is precisely $\mathrm{Thick}(\mathcal{C})$; see [17, (2.2)]. Moreover, the composite

$$\mathrm{Loc}(\mathcal{C}) \xrightarrow{\mathsf{inc}} \mathcal{T} \xrightarrow{\mathsf{can}} \mathcal{T}/\mathcal{C}^{\perp}$$

is an equivalence. From the right hand square one obtains an analogous description of $\mathcal{B}^{\perp}/\mathcal{C}^{\perp}$, namely: the objects of $\mathcal{C}$ in $\mathcal{T}^c/\mathrm{Thick}(\mathcal{B})$ generate a localizing subcategory of $\mathcal{B}^{\perp}$, and this subcategory is compactly generated and equivalent to $\mathcal{B}^{\perp}/\mathcal{C}^{\perp}$. Moreover, the full subcategory of compact objects in $\mathcal{B}^{\perp}/\mathcal{C}^{\perp}$ is equivalent to the thick subcategory generated by $\mathcal{C}$ which is, up to direct factors, equivalent to $\mathrm{Thick}(\mathcal{C})/\mathrm{Thick}(\mathcal{B})$. □

2. Homotopy categories

We begin this section with a recapitulation on the homotopy category of an additive category. Then we introduce the main objects of our study: the homotopy categories of projective modules, and of injective modules, over a noetherian ring, and establish results which prepare us for the development in the ensuing sections.

Let $\mathcal{A}$ be an additive category; see [23, (A.4)]. We grade complexes cohomologically, thus a complex X over $\mathcal{A}$ is a diagram

$$\cdots \longrightarrow X^n \xrightarrow{\partial^n} X^{n+1} \xrightarrow{\partial^{n+1}} X^{n+2} \longrightarrow \cdots$$

with X^n in $\mathcal{A}$ and $\partial^{n+1} \circ \partial^n = 0$ for each integer n. For such a complex X, we write ΣX for its suspension: $(\Sigma X)^n = X^{n+1}$ and $\partial_{\Sigma X} = -\partial_X$.

Let $\mathbf{K}(\mathcal{A})$ be the homotopy category of complexes over $\mathcal{A}$; its objects are complexes over $\mathcal{A}$, and its morphisms are morphisms of complexes modulo homotopy equivalence. The category $\mathbf{K}(\mathcal{A})$ has a natural structure of a triangulated category; see [22] or [23].

Let R be a ring. Unless stated otherwise, modules are left modules; right modules are sometimes referred to as modules over R^{op}, the opposite ring of R. This proclivity for the left carries over to properties of the ring as well: when we say noetherian without any further specification, we mean left noetherian, etc. We write $\mathbf{K}(R)$ for the homotopy category of complexes over R; it is $\mathbf{K}(\mathcal{A})$ with $\mathcal{A}$ the category of R-modules. The paragraphs below contain basic facts on homotopy categories required in the sequel.

2.1. Let $\mathcal{A}$ be an additive category, and let X and Y complexes over $\mathcal{A}$. Set $\mathbf{K} = \mathbf{K}(\mathcal{A})$. Let d be an integer. We write $X^{\geqslant d}$ for the subcomplex

$$\cdots \to 0 \to X^d \to X^{d+1} \to \cdots$$

of X, and $X^{\leqslant d-1}$ for the quotient complex $X/X^{\geqslant d}$. In $\mathbf{K}$ these fit into an exact triangle

$$X^{\geqslant d} \longrightarrow X \longrightarrow X^{\leqslant d-1} \longrightarrow \Sigma X^{\geqslant d} \tag{$*$}$$

This induces homomorphisms $\mathrm{Hom}_{\mathbf{K}}(X,Y) \to \mathrm{Hom}_{\mathbf{K}}(X^{\geqslant d},Y)$ and $\mathrm{Hom}_{\mathbf{K}}(X^{\leqslant d-1},Y) \to \mathrm{Hom}_{\mathbf{K}}(X,Y)$ of abelian groups. These have the following properties.

(1) One has isomorphisms of abelian groups:

$$H^d(\mathrm{Hom}_{\mathcal{A}}(X,Y)) \cong \mathrm{Hom}_{\mathbf{K}}(X,\Sigma^d Y) \cong \mathrm{Hom}_{\mathbf{K}}(\Sigma^{-d}X,Y)\,.$$

(2) If $Y^n = 0$ for $n \geq d$, then the map $\mathrm{Hom}_{\mathbf{K}}(X^{\leqslant d},Y) \to \mathrm{Hom}_{\mathbf{K}}(X,Y)$ is bijective.

(3) If $Y^n = 0$ for $n \leq d$, then the map $\mathrm{Hom}_{\mathbf{K}}(X,Y) \to \mathrm{Hom}_{\mathbf{K}}(X^{\geqslant d},Y)$ is bijective.

There are also versions of (2) and (3), where the hypothesis is on X.
Indeed, these remarks are all well-known, but perhaps (2) and (3) less so than (1). To verify (2), note that (1) implies

$$H^0(\mathrm{Hom}_{\mathcal{A}}(X^{\geqslant d+1},Y)) = 0 = H^1(\mathrm{Hom}_{\mathcal{A}}(X^{\geqslant d+1},Y))\,,$$

so applying $\mathrm{Hom}_{\mathcal{A}}(-,Y)$ to the exact triangle $(*)$ yields that the induced homomorphism of abelian groups

$$H^0(\mathrm{Hom}_{\mathcal{A}}(X^{\leqslant d},Y)) \longrightarrow H^0(\mathrm{Hom}_{\mathcal{A}}(X,Y))$$

is bijective, which is as desired. The argument for (3) is similar.

Now we recall, with proof, a crucial observation from [14, (2.1)]:

2.2. Let R be a ring, M an R-module, and let $\mathsf{i}M$ be an injective resolution of M. Set $\mathbf{K} = \mathbf{K}(R)$. If Y is a complex of injective R-modules, the induced map

$$\mathrm{Hom}_{\mathbf{K}}(\mathsf{i}M,Y) \longrightarrow \mathrm{Hom}_{\mathbf{K}}(M,Y)$$

is bijective. In particular, $\mathrm{Hom}_{\mathbf{K}}(\mathsf{i}R,Y) \cong H^0(Y)$.
Indeed, one may assume $(\mathsf{i}M)^n = 0$ for $n \leq -1$, since all injective resolutions of M are isomorphic in $\mathbf{K}$. The inclusion $M \to \mathsf{i}M$ leads to an exact sequence of complexes

$$0 \longrightarrow M \longrightarrow \mathsf{i}M \longrightarrow X \longrightarrow 0$$

with $X^n = 0$ for $n \leq -1$ and $H(X) = 0$. Therefore for $d = -1, 0$ one has isomorphisms

$$\mathrm{Hom}_{\mathbf{K}}(\Sigma^d X,Y) \cong \mathrm{Hom}_{\mathbf{K}}(\Sigma^d X,Y^{\geqslant -1}) = 0\,,$$

where the first one holds by an analogue of (2.1.2), and the second holds because $Y^{\geqslant -1}$ is a complex of injectives bounded on the left. It now follows from the exact sequence above that the induced map $\mathrm{Hom}_{\mathbf{K}}(\mathsf{i}M,Y) \to \mathrm{Hom}_{\mathbf{K}}(M,Y)$ is bijective.

The results below are critical ingredients in many of our arguments. We write $\mathbf{K}^{-,b}(\operatorname{prj} R)$ for the subcategory of $\mathbf{K}(R)$ consisting of complexes X of finitely generated projective modules with $H(X)$ bounded and $X^n = 0$ for $n \gg 0$, and $\mathbf{D}^f(R)$ for its image in $\mathbf{D}(R)$, the derived category of R-modules.

2.3. Let R be a (not necessarily commutative) ring.

(1) When R is coherent on both sides and flat R-modules have finite projective dimension, the triangulated category $\mathbf{K}(\operatorname{Prj} R)$ is compactly generated and the functors $\operatorname{Hom}_R(-,R)\colon \mathbf{K}(\operatorname{Prj} R) \to \mathbf{K}(R^{\mathrm{op}})$ and $\mathbf{K}(R^{\mathrm{op}}) \to \mathbf{D}(R^{\mathrm{op}})$ induce equivalences

$$\mathbf{K}^c(\operatorname{Prj} R) \xrightarrow{\sim} \mathbf{K}^{-,b}(\operatorname{prj} R^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{D}^f(R^{\mathrm{op}})^{\mathrm{op}}.$$

(2) When R is noetherian, the triangulated category $\mathbf{K}(\operatorname{Inj} R)$ is compactly generated, and the canonical functor $\mathbf{K}(\operatorname{Inj} R) \to \mathbf{D}(R)$ induces an equivalence

$$\mathbf{K}^c(\operatorname{Inj} R) \xrightarrow{\sim} \mathbf{D}^f(R)$$

Indeed, (1) is a result of Jørgensen [11, (2.4)] and (2) is a result of Krause [14, (2.3)].

In the propositions below $d(R)$ denotes the supremum of the projective dimensions of all flat R-modules.

Proposition 2.4. *Let R be a two-sided coherent ring such that $d(R)$ is finite. The inclusion $\mathbf{K}(\operatorname{Prj} R) \to \mathbf{K}(\operatorname{Flat} R)$ admits a right adjoint:*

$$\begin{tikzcd} \mathbf{K}(\operatorname{Prj} R) \arrow[r, shift right, "\mathsf{inc}"'] & \mathbf{K}(\operatorname{Flat} R) \arrow[l, shift right, "\mathsf{q}"'] \end{tikzcd}$$

Moreover, the category $\mathbf{K}(\operatorname{Prj} R)$ admits arbitrary products.

Proof. By Proposition (2.3.1), the category $\mathbf{K}(\operatorname{Prj} R)$ is compactly generated. The inclusion inc evidently preserves coproducts, so (1.5.1) yields the desired right adjoint q. The ring R is right coherent, so the (set-theoretic) product of flat modules is flat, and furnishes $\mathbf{K}(\operatorname{Flat} R)$ with a product. Since inc is an inclusion, the right adjoint q induces a product on $\mathbf{K}(\operatorname{Prj} R)$: the product of a set of complexes $\{P_\lambda\}_{\lambda\in\Lambda}$ in $\mathbf{K}(\operatorname{Prj} R)$ is the complex $\mathsf{q}\big(\prod_\lambda P_\lambda\big)$. □

The proof of Theorem 2.7 below uses homotopy limits in the homotopy category of complexes; its definition is recalled below.

2.5. Homotopy limits. Let R be a ring and let $\cdots \to X(r+1) \to X(r)$ be a sequence of morphisms in $\mathbf{K}(R)$. The *homotopy limit* of the sequence $\{X(i)\}$, denoted $\operatorname{holim} X(i)$, is defined by an exact triangle

$$\operatorname{holim} X(i) \longrightarrow \prod_{i\geqslant r} X(i) \xrightarrow{\mathrm{id}-\mathrm{shift}} \prod_{i\geqslant r} X(i) \longrightarrow \Sigma \operatorname{holim} X(i)\,.$$

The homotopy limit is uniquely defined, up to an isomorphism in $\mathbf{K}(R)$; see [4] for details.

The result below identifies, in some cases, a homotopy limit in the homotopy category with a limit in the category of complexes.

Lemma 2.6. *Let R be a ring. Consider a sequence of complexes of R-modules:*

$$\cdots \longrightarrow X(i) \xrightarrow{\varepsilon(i)} X(i-1) \longrightarrow \cdots \longrightarrow X(r+1) \xrightarrow{\varepsilon(r+1)} X(r)\,.$$

If for each degree n, there exists an integer s_n such that $\varepsilon(i)^n$ is an isomorphism for $i \geq s_n + 1$, then there exists a degree-wise split-exact sequence of complexes

$$0 \longrightarrow \varprojlim X(i) \longrightarrow \prod_i X(i) \xrightarrow{\mathrm{id}-\mathrm{shift}} \prod_i X(i) \longrightarrow 0\,.$$

In particular, it induces in $\mathbf{K}(R)$ an isomorphism holim $X(i) \cong \varprojlim X(i)$.

Proof. To prove the desired degree-wise split exactness of the sequence, it suffices to note that if $\cdots \longrightarrow M(r+1) \overset{\delta(r+1)}{\longrightarrow} M(r)$ is a sequence of R-modules such that $\delta(i)$ is an isomorphism for $i \geq s+1$, for some integer s, then one has a split exact sequence of R-modules:

$$0 \longrightarrow M(s) \xrightarrow{\eta} \prod_i M(i) \xrightarrow{\mathrm{id}-\mathrm{shift}} \prod_i M(i) \longrightarrow 0\,,$$

where the morphism η is induced by $\eta_i \colon M(s) \to M(i)$ with

$$\eta_i = \begin{cases} \delta(i+1)\cdots\delta(s) & \text{if } i \leq s-1 \\ \mathrm{id} & \text{if } i = s \\ \delta(i)^{-1}\cdots\delta(s+1)^{-1} & \text{if } i \geq s+1\,. \end{cases}$$

Indeed, in the sequence above, the map $(\mathrm{id}-\mathrm{shift})$ is surjective since the system $\{M_i\}$ evidently satisfies the Mittag-Leffler condition, see [23, (3.5.7)]. Moreover, a direct calculation shows that $\mathrm{Im}(\eta) = \mathrm{Ker}(\mathrm{id}-\mathrm{shift})$. It remains to note that the morphism $\pi\colon \prod M(i) \to M(s)$ defined by $\pi(a_i) = a_s$ is such that $\pi\eta = \mathrm{id}$.
Finally, it is easy to verify that degree-wise split exact sequences of complexes induce exact triangles in the homotopy category. Thus, by the definition of homotopy limits, see (2.5), and the already established part of the lemma, we deduce: holim $X(i) \cong \varprojlim X(i)$ in $\mathbf{K}(R)$, as desired. □

The result below collects some properties of the functor $\mathsf{q}\colon \mathbf{K}(\mathrm{Flat}\,R) \to \mathbf{K}(\mathrm{Prj}\,R)$. It is noteworthy that the proof of part (3) describes an explicit method for computing the value of q on complexes bounded on the left. As usual, a morphism of complexes is called a *quasi-isomorphism* if the induced map in homology is bijective.

Theorem 2.7. *Let R be a two-sided coherent ring with $d(R)$ finite, and let F be a complex of flat R-modules.*

(1) *The morphism $\mathsf{q}(F) \to F$ is a quasi-isomorphism.*
(2) *If $F^n = 0$ for $n \gg 0$, then $\mathsf{q}(F)$ is a projective resolution of F.*
(3) *If $F^n = 0$ for $n \leq r$, then $\mathsf{q}(F)$ is isomorphic to a complex P with $P^n = 0$ for $n \leq r - d(R)$.*

Proof. (1) For each integer n, the map $\mathrm{Hom}_{\mathbf{K}}(\Sigma^n R, \mathsf{q}(F)) \to \mathrm{Hom}_{\mathbf{K}}(\Sigma^n R, F)$, induced by the morphism $\mathsf{q}(F) \to F$, is bijective; this is because R is in $\mathbf{K}(\mathrm{Prj}\,R)$. Therefore (2.1.1) yields $H^{-n}(\mathsf{q}(F)) \cong H^{-n}(F)$, which proves (1).

(2) When $F^n = 0$ for $n \geq r$, one can construct a projective resolution $P \to F$ with $P^n = 0$ for $n \geq r$. Thus, for each $X \in \mathbf{K}(\operatorname{Prj} R)$ one has the diagram below

$$\operatorname{Hom}_{\mathbf{K}}(X^{\leqslant r}, P) = \operatorname{Hom}_{\mathbf{K}}(X, P) \to \operatorname{Hom}_{\mathbf{K}}(X, F) = \operatorname{Hom}_{\mathbf{K}}(X^{\leqslant r}, F)\,.$$

where equalities hold by (2.1.2). The complex $X^{\leqslant r}$ is K-projective, so the composed map is an isomorphism; hence the same is true of the one in the middle. This proves that $\mathsf{q}(F) \cong P$; see (1.4).

(3) We may assume $d(R)$ is finite. The construction of the complex P takes place in the category of complexes of R-modules. Note that $F^{>i}$ is a subcomplex of F for each integer $i \geq r$; denote $F(i)$ the quotient complex $F/F^{>i}$. One has surjective morphisms of complexes of R-modules

$$\cdots \longrightarrow F(i) \xrightarrow{\varepsilon(i)} F(i-1) \longrightarrow \cdots \longrightarrow F(r+1) \xrightarrow{\varepsilon(r+1)} F(r) = 0$$

with $\operatorname{Ker}(\varepsilon(i)) = \Sigma^i F^i$. The surjections $F \to F(i)$ are compatible with the $\varepsilon(i)$, and the induced map $F \to \varprojlim F(i)$ is an isomorphism. The plan is to construct a commutative diagram in the category of complexes of R-modules

$$(\dagger)\qquad \begin{tikzcd} \cdots \arrow[r] & P(i) \arrow[r, "\delta(i)"] \arrow[d, "\kappa(i)"'] & P(i-1) \arrow[r] \arrow[d, "\kappa(i-1)"'] & \cdots \arrow[r] & P(r+1) \arrow[r, "\delta(r+1)"] \arrow[d, "\kappa(r+1)"'] & P(r) = 0 \\ \cdots \arrow[r] & F(i) \arrow[r, "\varepsilon(i)"] & F(i-1) \arrow[r] & \cdots \arrow[r] & F(r+1) \arrow[r, "\varepsilon(r+1)"] & F(r) = 0 \end{tikzcd}$$

with the following properties: for each integer $i \geq r+1$ one has that

(a) $P(i)$ consists of projectives R-modules and $P(i)^n = 0$ for $n \notin (r-d(R), i]$;
(b) $\delta(i)$ is surjective, and $\operatorname{Ker}\delta(i)^n = 0$ for $n < i - d(R)$;
(c) $\kappa(i)$ is a surjective quasi-isomorphism.

The complexes $P(i)$ and the attendant morphisms are constructed iteratively, starting with $\kappa(r+1)\colon P(r+1) \to F(r+1) = \Sigma^{r+1}F^{r+1}$ a surjective projective resolution, and $\delta(r+1) = 0$. One may ensure $P(r+1)^n = 0$ for $n \geq r+2$, and also for $n \leq r - d(R)$, because the projective dimension of the flat R-module F^{r+1} is at most $d(R)$. Note that $P(r+1)$, $\delta(r+1)$, and $\kappa(r+1)$ satisfy conditions (a)–(c).

Let $i \geq r+2$ be an integer, and let $\kappa(i-1)\colon P(i-1) \to F(i-1)$ be a homomorphism with the desired properties. Build a diagram of solid arrows

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r, dashed] \arrow[d, "\theta"] & P(i) \arrow[r, dashed, "\delta(i)"] \arrow[d, dashed, "\kappa(i)"] & P(i-1) \arrow[r] \arrow[d, "\kappa(i-1)"] & 0 \\ 0 \arrow[r] & \Sigma^i F^i \arrow[r, "\iota"] & F(i) \arrow[r, "\varepsilon(i)"] & F(i-1) \arrow[r] & 0 \end{tikzcd}$$

where ι is the canonical injection, and $\theta\colon Q \to \Sigma^i F^i$ is a surjective projective resolution, chosen such that $Q^n = 0$ for $n < i - d(R)$. The Horseshoe Lemma now yields a complex $P(i)$, with underlying graded R-module $Q \oplus P(i-1)$, and dotted morphisms that form the commutative diagram above; see [23, (2.2.8)].

It is clear that $P(i)$ and $\delta(i)$ satisfy conditions (a) and (b). As to (c): since both θ and $\kappa(i-1)$ are surjective quasi-isomorphisms, so is $\kappa(i)$. This completes the construction of the diagram (†).

Set $P = \varprojlim P(i)$; the limit is taken in the category of complexes. We claim that P is a complex of projectives and that $\mathsf{q}(F) \cong P$ in $\mathbf{K}(\operatorname{Prj} R)$.

Indeed, by property (b), for each integer n the map $P(i+1)^n \to P(i)^n$ is bijective for $i > n + d(R)$, so $P^n = P(n+d(R))^n$, and hence the R-module P^n is projective. Moreover $P^n = 0$ for $n \leq r - d(R)$, by (a).

The sequences of complexes $\{P(i)\}$ and $\{F(i)\}$ satisfy the hypotheses of Lemma (2.6); the former by construction, see property (b), and the latter by definition. Thus, Lemma (2.6) yields the following isomorphisms in $\mathbf{K}(R)$:

$$\operatorname{holim} P(i) \cong P \qquad \text{and} \qquad \operatorname{holim} F(i) \cong F\,.$$

Moreover, the $\kappa(i)$ induce a morphism $\kappa\colon \operatorname{holim} P(i) \to \operatorname{holim} F(i)$ in $\mathbf{K}(R)$. Let X be a complex of projective R-modules. To complete the proof of (3), it suffices to prove that for each integer i the induced map

$$\operatorname{Hom}_{\mathbf{K}}(X, \kappa(i))\colon \operatorname{Hom}_{\mathbf{K}}(X, P(i)) \longrightarrow \operatorname{Hom}_{\mathbf{K}}(X, F(i))$$

is bijective. Then, a standard argument yields that $\operatorname{Hom}_{\mathbf{K}}(X, \kappa)$ is bijective, and in turn this implies $P \cong \operatorname{holim} P(i) \cong \mathsf{q}(\operatorname{holim} F(i)) \cong \mathsf{q}(F)$, see (1.4).

Note that, since $\kappa(i)$ is a quasi-isomorphism and $P(i)^n = 0 = F(i)^n$ for $n \geq i+1$, the morphism $\kappa(i)\colon P(i) \to F(i)$ is a projective resolution. Since projective resolutions are isomorphic in the homotopy category, it follows from (2) that $P(i) \cong \mathsf{q}(F(i))$, and hence that the map $\operatorname{Hom}_{\mathbf{K}}(X, \kappa(i))$ is bijective, as desired. Thus, (3) is proved. □

3. Dualizing complexes

Let R be a commutative noetherian ring. In this article, a *dualizing complex* for R is a complex D of R-modules with the following properties:

(a) the complex D is bounded and consists of injective R-modules;
(b) the R-module $H^n(D)$ is finitely generated for each n;
(c) the canonical map $R \to \operatorname{Hom}_R(D, D)$ is a quasi-isomorphism.

See Hartshorne [9, Chapter V] for basic properties of dualizing complexes. The presence of a dualizing complex for R implies that its Krull dimension is finite. As to the existence of dualizing complexes: when R is a quotient of a Gorenstein ring Q of finite Krull dimension, it has a dualizing complex: a suitable representative of the complex $\mathbf{R}\operatorname{Hom}_Q(R, Q)$ does the job. On the other hand, Kawasaki [13] has proved that if R has a dualizing complex, then it is a quotient of a Gorenstein ring.

3.1. A dualizing complex induces a contravariant equivalence of categories:

$$\mathbf{D}^f(R) \underset{\operatorname{Hom}_R(-,D)}{\overset{\operatorname{Hom}_R(-,D)}{\rightleftarrows}} \mathbf{D}^f(R)$$

This property characterizes dualizing complexes: if C is a complex of R-modules such that $\mathbf{R}\mathrm{Hom}_R(-,C)$ induces a contravariant self-equivalence of $\mathbf{D}^f(R)$, then C is isomorphic in $\mathbf{D}(R)$ to a dualizing complex for R; see [9, (V.2)]. Moreover, if D and E are dualizing complexes for R, then E is quasi-isomorphic to $P \otimes_R D$ for some complex P which is locally free of rank one; that is to say, for each prime ideal $\mathfrak{p}$ in R, the complex $P_\mathfrak{p}$ is quasi-isomorphic $\Sigma^n R_\mathfrak{p}$ for some integer n; see [9, (V.3)].

Remark 3.2. Let R be a ring with a dualizing complex. Then, as noted above, the Krull dimension of R is finite, so a result of Gruson and Raynaud [20, (II.3.2.7)] yields that the projective dimension of each flat R-module is at most the Krull dimension of R. The upshot is that Proposition (2.4) yields an adjoint functor

$$\mathbf{K}(\mathrm{Prj}\, R) \underset{\mathsf{inc}}{\overset{\mathsf{q}}{\rightleftarrows}} \mathbf{K}(\mathrm{Flat}\, R)$$

and this has properties described in Theorem (2.7). In the remainder of the article, this remark will be used often, and usually without comment.

In [6], Christensen, Frankild, and Holm have introduced a notion of a dualizing complex for a pair of, possibly non-commutative, rings:

3.3. Non-commutative rings. In what follows $\langle S, R\rangle$ denotes a pair of rings, where S is left noetherian and R is left coherent and right noetherian. This context is more restrictive than that considered in [6, Section 1], where it is not assumed that R is left coherent. We make this additional hypothesis on R in order to invoke (2.3.1).

3.3.1. A *dualizing complex* for the pair $\langle S, R\rangle$ is complex D of S-R bimodules with the following properties:

(a) D is bounded and each D^n is an S-R bimodule that is injective both as an S-module and as an R^{op}-module;
(b) $H^n(D)$ is finitely generated as an S-module and as an R^{op}-module for each n;
(c) the following canonical maps are quasi-isomorphisms:

$$R \longrightarrow \mathrm{Hom}_S(D,D) \quad \text{and} \quad S \longrightarrow \mathrm{Hom}_{R^{\mathrm{op}}}(D,D)$$

When R is commutative and $R = S$ this notion of a dualizing complex coincides with the one recalled in the beginning of this section. The appendix in [6] contains a detailed comparison with other notions of dualizing complexes in the non-commutative context.
The result below implies that the conclusion of Remark (3.2): existence of a functor q with suitable properties, applies also in the situation considered in (3.3).

Proposition 3.4. *Let D be a dualizing complex for the pair of rings $\langle S, R\rangle$, where S is left noetherian and R is left coherent and right noetherian.*

(1) *The projective dimension of each flat R-module is finite.*

(2) *The complex D induces a contravariant equivalence:*

$$\begin{tikzcd} \mathbf{D}^f(R^{\mathrm{op}}) \arrow[r, shift right, "\mathrm{Hom}_{R^{\mathrm{op}}}(-,D)"'] & \mathbf{D}^f(S) \arrow[l, shift right, "\mathrm{Hom}_S(-,D)"'] \end{tikzcd}$$

Indeed, (1) is contained in [6, (1.5)]. Moreover, (2) may be proved as in the commutative case, see [9, (V.2.1)], so we provide only a

Sketch of a proof of (2). By symmetry, it suffices to prove that for each complex X of right R-modules if $H(X)$ is bounded and finitely generated in each degree, then so is $H(\mathrm{Hom}_{R^{\mathrm{op}}}(X,D))$, as an S-module, and that the biduality morphism

$$\theta(X)\colon X \longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_{R^{\mathrm{op}}}(X,D),D))$$

is a quasi-isomorphism. To begin with, since $H(X)$ is bounded, we may pass to a quasi-isomorphic complex and assume X is itself bounded, in which case the complex $\mathrm{Hom}_{R^{\mathrm{op}}}(X,D)$, and hence its homology, is bounded.
For the remainder of the proof, by replacing X by a suitable projective resolution, we assume that each X^i is a finitely generated projective module, with $X^i = 0$ for $i \gg 0$. In this case, for any bounded complex Y of S-R bimodules, if the S-module $H(Y)$ is finitely generated in each degree, then so is the S-module $H(\mathrm{Hom}_{R^{\mathrm{op}}}(X,Y))$; this can be proved by an elementary induction argument, based on the number

$$\sup\{i \mid H^i(Y) \neq 0\} - \inf\{i \mid H^i(Y) \neq 0\}\,,$$

keeping in mind that S is noetherian. Applied with $Y = D$, one obtains that each $H^i(\mathrm{Hom}_{R^{\mathrm{op}}}(X,D))$ is finitely generated, as desired.
As to the biduality morphism: fix an integer n, and pick an integer $d \leq n$ such that the morphism of complexes

$$\mathrm{Hom}_S(\mathrm{Hom}_{R^{\mathrm{op}}}(X^{\geqslant d},D),D)) \longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_{R^{\mathrm{op}}}(X,D),D))$$

is bijective in degrees $\geq n-1$; such a d exists because D is bounded. Therefore, $H^n(\theta(X))$ is bijective if and only if $H^n(\theta(X^{\geqslant d}))$ is bijective. Thus, passing to $X^{\geqslant d}$, we may assume that $X^i = 0$ when $|i| \gg 0$. One has then a commutative diagram of morphisms of complexes

$$\begin{tikzcd} X \otimes_R R \arrow[r, "X\otimes_R\theta(R)"] \arrow[d, "\cong"'] & X \otimes_R \mathrm{Hom}_S(D,D) \arrow[d, "\cong"'] \\ X \arrow[r, "\theta(X)", "\simeq"'] & \mathrm{Hom}_S(\mathrm{Hom}_{R^{\mathrm{op}}}(X,D),D) \end{tikzcd}$$

The isomorphism on the right holds because X is a finite complex of finitely generated projectives; for the same reason, since $\theta(R)$ is a quasi-isomorphism, see (3.3.1.c), so is $X \otimes_R \theta(R)$. Thus, $\theta(X)$ is a quasi-isomorphism. This completes the proof. $\square$

4. An equivalence of homotopy categories

The standing assumption in the rest of this article is that R is a *commutative* noetherian ring. Towards the end of each section we collect remarks on the extensions of our results to the non-commutative context described in (3.3).

The main theorem in this section is an equivalence between the homotopy categories of complexes of projectives and complexes of injectives. As explained in the discussion following Theorem I in the introduction, it may be viewed as an extension of the Grothendieck duality theorem, recalled in (3.1). Theorem (4.2) is the basis for most results in this work.

Remark 4.1. Let D be a dualizing complex for R; see Section 3.

For any flat module F and injective module I, the R-module $I \otimes_R F$ is injective; this is readily verified using Baer's criterion. Thus, $D \otimes_R -$ is a functor between $\mathbf{K}(\operatorname{Prj} R)$ and $\mathbf{K}(\operatorname{Inj} R)$, and it factors through $\mathbf{K}(\operatorname{Flat} R)$. If I and J are injective modules, the R-module $\operatorname{Hom}_R(I, J)$ is flat, so $\operatorname{Hom}_R(D, -)$ defines a functor from $\mathbf{K}(\operatorname{Inj} R)$ to $\mathbf{K}(\operatorname{Flat} R)$; evidently it is right adjoint to $D \otimes_R -\colon \mathbf{K}(\operatorname{Flat} R) \to \mathbf{K}(\operatorname{Inj} R)$.

Here is the announced equivalence of categories. The existence of q in the statement below is explained in Remark (3.2), and the claims implicit in the right hand side of the diagram are justified by the preceding remark.

Theorem 4.2. *Let R be a noetherian ring with a dualizing complex D. The functor $D \otimes_R -\colon \mathbf{K}(\operatorname{Prj} R) \to \mathbf{K}(\operatorname{Inj} R)$ is an equivalence. A quasi-inverse is $\mathsf{q} \circ \operatorname{Hom}_R(D, -)$:*

$$\mathbf{K}(\operatorname{Prj} R) \underset{\mathsf{inc}}{\overset{\mathsf{q}}{\rightleftarrows}} \mathbf{K}(\operatorname{Flat} R) \underset{D\otimes_R -}{\overset{\operatorname{Hom}_R(D,-)}{\rightleftarrows}} \mathbf{K}(\operatorname{Inj} R)$$

where q denotes the right adjoint of the inclusion $\mathbf{K}(\operatorname{Prj} R) \to \mathbf{K}(\operatorname{Flat} R)$.

4.3. The functors that appear in the theorem are everywhere dense in the remainder of this article, so it is expedient to abbreviate them: set

$$\begin{aligned} \mathsf{T} &= D \otimes_R - : \mathbf{K}(\operatorname{Prj} R) \longrightarrow \mathbf{K}(\operatorname{Inj} R) \qquad \text{and} \\ \mathsf{S} &= \mathsf{q} \circ \operatorname{Hom}_R(D, -)\colon \mathbf{K}(\operatorname{Inj} R) \longrightarrow \mathbf{K}(\operatorname{Prj} R)\,. \end{aligned}$$

The notation 'T' should remind one that this functor is given by a tensor product. The same rule would call for an 'H' to denote the other functor; unfortunately, this letter is bound to be confounded with an 'H', so we settle for an 'S'.

Proof. By construction, $(\mathsf{inc}, \mathsf{q})$ and $(D \otimes_R -, \operatorname{Hom}_R(D, -))$ are adjoint pairs of functors. It follows that their composition (T, S) is an adjoint pair of functors as well. Thus, it suffices to prove that T is an equivalence: this would imply that S is its quasi-inverse, and hence also an equivalence.

Both $\mathbf{K}(\operatorname{Prj} R)$ and $\mathbf{K}(\operatorname{Inj} R)$ are compactly generated, by Proposition (2.3), and T preserves coproducts. It follows, using a standard argument, that it suffices to verify that T induces an equivalence $\mathbf{K}^c(\operatorname{Prj} R) \to \mathbf{K}^c(\operatorname{Inj} R)$. Observe that each complex P of finitely generated projective R-modules satisfies

$$\operatorname{Hom}_R(P, D) \cong D \otimes_R \operatorname{Hom}_R(P, R)\,.$$

Thus one has the following commutative diagram

$$\begin{tikzcd}
\mathbf{K}^{-,b}(\operatorname{prj} R) \arrow[r, "\operatorname{Hom}_R(-,R)", "\sim"'] \arrow[d, "\wr"] & \mathbf{K}^{c}(\operatorname{Prj} R) \arrow[r, "\mathsf{T}"] & \mathbf{K}^{+}(\operatorname{Inj} R) \arrow[d, "\wr"] \\
\mathbf{D}^{f}(R) \arrow[rr, "\operatorname{Hom}_R(-,D)"'] & & \mathbf{D}^{+}(R)
\end{tikzcd}$$

By (2.3.2), the equivalence $\mathbf{K}^{+}(\operatorname{Inj} R) \to \mathbf{D}^{+}(R)$ identifies $\mathbf{K}^{c}(\operatorname{Inj} R)$ with $\mathbf{D}^{f}(R)$, while by (3.1), the functor $\operatorname{Hom}_R(-, D)$ induces an auto-equivalence of $\mathbf{D}^{f}(R)$. Hence, by the commutative diagram above, T induces an equivalence $\mathbf{K}^{c}(\operatorname{Prj} R) \to \mathbf{K}^{c}(\operatorname{Inj} R)$. This completes the proof. □

In the proof above we utilized the fact that $\mathbf{K}(\operatorname{Prj} R)$ and $\mathbf{K}(\operatorname{Inj} R)$ admit coproducts compatible with T. The categories in question also have products; this is obvious for $\mathbf{K}(\operatorname{Inj} R)$, and contained in Proposition (2.4) for $\mathbf{K}(\operatorname{Prj} R)$. The equivalence of categories established above implies:

COROLLARY 4.4. *The functors T and S preserve coproducts and products.*

Remark 4.5. Let $\mathsf{i}R$ be an injective resolution of R, and set $D^* = \mathsf{S}(\mathsf{i}R)$. Injective resolutions of R are uniquely isomorphic in $\mathbf{K}(\operatorname{Inj} R)$, so the complex $\mathsf{S}(\mathsf{i}R)$ is independent up to isomorphism of the choice of $\mathsf{i}R$, so one may speak of D^* without referring to $\mathsf{i}R$.

LEMMA 4.6. *The complex D^* is isomorphic to the image of D under the composition*

$$\mathbf{D}^{f}(R) \xrightarrow{\sim} \mathbf{K}^{-,b}(\operatorname{prj} R) \xrightarrow{\operatorname{Hom}_R(-,R)} \mathbf{K}(\operatorname{Prj} R)\,.$$

Proof. The complex D is bounded and has finitely generated homology modules, so we may choose a projective resolution P of D with each R-module P^n finitely generated, and zero for $n \gg 0$. In view of Theorem (4.2), it suffices to verify that $\mathsf{T}(\operatorname{Hom}_R(P, R))$ is isomorphic to $\mathsf{i}R$. The complex $\mathsf{T}(\operatorname{Hom}_R(P, R))$, that is to say, $D \otimes_R \operatorname{Hom}_R(P, R)$ is isomorphic to the complex $\operatorname{Hom}_R(P, D)$, which consists of injective R-modules and is bounded on the left. Therefore $\operatorname{Hom}_R(P, D)$ is K-injective. Moreover, the composite

$$R \longrightarrow \operatorname{Hom}_R(D, D) \longrightarrow \operatorname{Hom}_R(P, D)$$

is a quasi-isomorphism, and one obtains that in $\mathbf{K}(\operatorname{Inj} R)$ the complex $\operatorname{Hom}_R(P, D)$ is an injective resolution of R. □

The objects in the subcategory $\operatorname{Thick}(\operatorname{Prj} R)$ of $\mathbf{K}(\operatorname{Prj} R)$ are exactly the complexes of finite projective dimension; those in the subcategory $\operatorname{Thick}(\operatorname{Inj} R)$ of $\mathbf{K}(\operatorname{Inj} R)$ are the complexes of finite injective dimension. It is known that the functor $D \otimes_R -$ induces an equivalence between these categories; see, for instance, [1, (1.5)]. The result below may be read as the statement that this equivalence extends to the full homotopy categories.

Proposition 4.7. *Let R be a noetherian ring with a dualizing complex D. The equivalence $\mathsf{T}\colon \mathbf{K}(\operatorname{Prj} R) \to \mathbf{K}(\operatorname{Inj} R)$ restricts to an equivalence between* $\operatorname{Thick}(\operatorname{Prj} R)$ *and* $\operatorname{Thick}(\operatorname{Inj} R)$. *In particular,* $\operatorname{Thick}(\operatorname{Inj} R)$ *equals* $\operatorname{Thick}(\operatorname{Add} D)$.

Proof. It suffices to prove that the adjoint pair of functors (T, S) in Theorem (4.2) restrict to functors between $\operatorname{Thick}(\operatorname{Prj} R)$ and $\operatorname{Thick}(\operatorname{Inj} R)$.
The functor T maps R to D, which is a bounded complex of injectives and hence in $\operatorname{Thick}(\operatorname{Inj} R)$. Therefore T maps $\operatorname{Thick}(\operatorname{Prj} R)$ into $\operatorname{Thick}(\operatorname{Inj} R)$.
Conversely, given injective R-modules I and J, the R-module $\operatorname{Hom}_R(I, J)$ is flat. Therefore $\operatorname{Hom}_R(D, -)$ maps $\operatorname{Thick}(\operatorname{Inj} R)$ into $\operatorname{Thick}(\operatorname{Flat} R)$, since D is a bounded complex of injectives. By Theorem (2.7.2), for each flat R-module F, the complex $\mathsf{q}(F)$ is a projective resolution of F. The projective dimension of F is finite since R has a dualizing complex; see (3.2). Hence q maps $\operatorname{Thick}(\operatorname{Flat} R)$ to $\operatorname{Thick}(\operatorname{Prj} R)$. ☐

4.8. Non-commutative rings. Consider a pair of rings $\langle S, R\rangle$ as in (3.3), with a dualizing complex D. Given Proposition (3.4), the proof of Theorem (4.2) carries over verbatim to yield:

Theorem. *The functor $D \otimes_R -\colon \mathbf{K}(\operatorname{Prj} R) \to \mathbf{K}(\operatorname{Inj} S)$ is an equivalence, and the functor $\mathsf{q} \circ \operatorname{Hom}_S(D, -)$ is a quasi-inverse.* ☐

This basic step accomplished, one can readily transcribe the remaining results in this section, and their proofs, to apply to the pair $\langle S, R\rangle$; it is clear what the corresponding statements should be.

5. Acyclicity versus total acyclicity

This section contains various results concerning the classes of (totally) acyclic complexes of projectives, and of injectives. We start by recalling appropriate definitions.

5.1. Acyclic complexes. A complex X of R-modules is *acyclic* if $H^n X = 0$ for each integer n. We denote $\mathbf{K}_{\mathrm{ac}}(R)$ the full subcategory of $\mathbf{K}(R)$ formed by acyclic complexes of R-modules. Set

$$\mathbf{K}_{\mathrm{ac}}(\operatorname{Prj} R) = \mathbf{K}(\operatorname{Prj} R) \cap \mathbf{K}_{\mathrm{ac}}(R) \quad \text{and} \quad \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R) = \mathbf{K}(\operatorname{Inj} R) \cap \mathbf{K}_{\mathrm{ac}}(R)\,.$$

Evidently acyclicity is a property intrinsic to the complex under consideration. Next we introduce a related notion which depends on a suitable subcategory of $\operatorname{Mod} R$.

5.2. Total acyclicity. Let $\mathcal{A}$ be an additive category. A complex X over $\mathcal{A}$ is *totally acyclic* if for each object $A \in \mathcal{A}$ the following complexes of abelian groups are acyclic.

$$\operatorname{Hom}_{\mathcal{A}}(A, X) \quad \text{and} \quad \operatorname{Hom}_{\mathcal{A}}(X, A)$$

We denote by $\mathbf{K}_{\mathrm{tac}}(\mathcal{A})$ the full subcategory of $\mathbf{K}(\mathcal{A})$ consisting of totally acyclic complexes. Specializing to $\mathcal{A} = \operatorname{Prj} R$ and $A = \operatorname{Inj} R$ one gets the notion of a *totally acyclic complex of projectives* and a *totally acyclic complex of injectives*, respectively.

Theorems (5.3) and (5.4) below describe various properties of (totally) acyclic complexes. In what follows, we write $\mathbf{K}^c_{\mathrm{ac}}(\operatorname{Prj} R)$ and $\mathbf{K}^c_{\mathrm{ac}}(\operatorname{Inj} R)$ for the class of compact objects in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Prj} R)$ and $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R)$, respectively; in the same way, $\mathbf{K}^c_{\mathrm{tac}}(\operatorname{Prj} R)$ and $\mathbf{K}^c_{\mathrm{tac}}(\operatorname{Inj} R)$ denote compacts among the corresponding totally acyclic objects.

Theorem 5.3. *Let R be a noetherian ring with a dualizing complex D.*

(1) *The categories $\mathbf{K}_{\mathrm{ac}}(\operatorname{Prj} R)$ and $\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R)$ are compactly generated.*
(2) *The equivalence $\mathbf{D}^f(R) \to \mathbf{K}^c(\operatorname{Prj} R)^{\mathrm{op}}$ induces, up to direct factors, equivalences*

$$\mathbf{D}^f(R)/\operatorname{Thick}(R) \xrightarrow{\sim} \mathbf{K}^c_{\mathrm{ac}}(\operatorname{Prj} R)^{\mathrm{op}}$$
$$\mathbf{D}^f(R)/\operatorname{Thick}(R,D) \xrightarrow{\sim} \mathbf{K}^c_{\mathrm{tac}}(\operatorname{Prj} R)^{\mathrm{op}}.$$

(3) *The quotient $\mathbf{K}_{\mathrm{ac}}(\operatorname{Prj} R)/\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R)$ is compactly generated, and one has, up to direct factors, an equivalence*

$$\operatorname{Thick}(R,D)/\operatorname{Thick}(R) \xrightarrow{\sim} \left[\big(\mathbf{K}_{\mathrm{ac}}(\operatorname{Prj} R)/\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R)\big)^c\right]^{\mathrm{op}}.$$

The proof of this result, and also of the one below, which is an analogue for complexes of injectives, is given in (5.10). It should be noted that, in both cases, part (1) is not new: for the one above, see the proof of [12, (1.9)], and for the one below, see [14, (7.3)].

Theorem 5.4. *Let R be a noetherian ring with a dualizing complex D.*

(1) *The categories $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R)$ and $\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} R)$ are compactly generated.*
(2) *The equivalence $\mathbf{D}^f(R) \to \mathbf{K}^c(\operatorname{Inj} R)$ induces, up to direct factors, equivalences*

$$\mathbf{D}^f(R)/\operatorname{Thick}(R) \xrightarrow{\sim} \mathbf{K}^c_{\mathrm{ac}}(\operatorname{Inj} R)$$
$$\mathbf{D}^f(R)/\operatorname{Thick}(R,D) \xrightarrow{\sim} \mathbf{K}^c_{\mathrm{tac}}(\operatorname{Inj} R).$$

(3) *The quotient $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R)/\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} R)$ is compactly generated, and we have, up to direct factors, an equivalence*

$$\operatorname{Thick}(R,D)/\operatorname{Thick}(R) \xrightarrow{\sim} \big(\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R)/\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} R)\big)^c.$$

Here is one consequence of the preceding results. In it, one cannot restrict to complexes (of projectives or of injectives) of finite modules; see the example in Section 6.

Corollary 5.5. *Let R be a noetherian ring with a dualizing complex. The following conditions are equivalent.*

(a) *The ring R is Gorenstein.*
(b) *Every acyclic complex of projective R-modules is totally acyclic.*

(c) *Every acyclic complex of injective R-modules is totally acyclic.*

Proof. Theorems (5.3.3) and (5.4.3) imply that (b) and (c) are equivalent, and that they hold if and only if D lies in $\operatorname{Thick}(R)$, that is to say, if and only if D has finite projective dimension. This last condition is equivalent to R being Gorenstein; see [5, (3.3.4)]. □

Remark 5.6. One way to interpret Theorems (5.3.3) and (5.4.3) is that the category $\operatorname{Thick}(R, D)/\operatorname{Thick}(R)$ measures the failure of the Gorenstein property for R. This invariant of R appears to possess good functorial properties. For instance, let R and S be local rings with dualizing complexes D_R and D_S, respectively. If a local homomorphism $R \to S$ is quasi-Gorenstein, in the sense of Avramov and Foxby [1, Section 7], then tensoring with S induces an equivalence of categories, up to direct factors:

$$- \otimes^{\mathbf{L}}_R S\colon \operatorname{Thick}(R, D_R)/\operatorname{Thick}(R) \xrightarrow{\sim} \operatorname{Thick}(S, D_S)/\operatorname{Thick}(S)$$

This is a quantitative enhancement of the ascent and descent of the Gorenstein property along such homomorphisms.

The notion of total acyclicity has a useful expression in the notation of (1.6).

LEMMA 5.7. *Let $\mathcal{A}$ be an additive category. One has $\mathbf{K}_{\mathrm{tac}}(\mathcal{A}) = \mathcal{A}^{\perp} \cap {}^{\perp}\mathcal{A}$, where $\mathcal{A}$ is identified with complexes concentrated in degree zero.*

Proof. By (2.1.1), for each A in $\mathcal{A}$ the complex $\operatorname{Hom}_{\mathcal{A}}(X, A)$ is acyclic if and only if $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X, \Sigma^n A) = 0$ for every integer n; in other words, if and only if X is in ${}^{\perp}\mathcal{A}$. By the same token, $\operatorname{Hom}_{\mathcal{A}}(A, X)$ is acyclic if and only if X is in $\mathcal{A}^{\perp}$. □

5.8. Let R be a ring. The following identifications hold:

$$\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R) = \mathbf{K}_{\mathrm{ac}}(\operatorname{Prj} R) \cap {}^{\perp}(\operatorname{Prj} R)$$
$$\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} R) = (\operatorname{Inj} R)^{\perp} \cap \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R)\,.$$

Indeed, both equalities are due to (5.7), once it is observed that for any complex X of R-modules, the following conditions are equivalent: X is acyclic; $\operatorname{Hom}_R(P, X)$ is acyclic for each projective R-module P; $\operatorname{Hom}_R(X, I)$ is acyclic for each injective R-module I.

In the presence of a dualizing complex total acyclicity can be tested against a pair of objects, rather than against the entire class of projectives, or of injectives, as called for by the definition. This is one of the imports of the result below. Recall that $\mathsf{i}R$ denotes an injective resolution of R, and that $D^* = \mathsf{S}(\mathsf{i}R)$; see (4.5).

PROPOSITION 5.9. *Let R be a noetherian ring with a dualizing complex D.*

(1) *The functor T restricts to an equivalence of $\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R)$ with $\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} R)$.*
(2) $\mathbf{K}_{\mathrm{ac}}(\operatorname{Prj} R) = \{R\}^{\perp}$ *and* $\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R) = \{R, D^*\}^{\perp}$.
(3) $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R) = \{\mathsf{i}R\}^{\perp}$ *and* $\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} R) = \{\mathsf{i}R, D\}^{\perp}$.

Proof. (1) By Proposition (4.7), the equivalence induced by T identifies Thick(Prj R) with Thick(Inj R). This yields the equivalence below:

$$\begin{aligned}\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R) &= \operatorname{Thick}(\operatorname{Prj} R)^{\perp} \cap {}^{\perp}\operatorname{Thick}(\operatorname{Prj} R)\\ &\xrightarrow{\sim} \operatorname{Thick}(\operatorname{Inj} R)^{\perp} \cap {}^{\perp}\operatorname{Thick}(\operatorname{Inj} R) = \mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} R)\end{aligned}$$

The equalities are by Lemma (5.7).
(3) That $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R)$ equals $\{\mathsf{i}R\}^{\perp}$ follows from (2.2). Given this, the claim on $\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} R)$ is a consequence of (5.8) and the identifications

$$\{D\}^{\perp} = \operatorname{Thick}(\operatorname{Add} D)^{\perp} = \operatorname{Thick}(\operatorname{Inj} R)^{\perp} = (\operatorname{Inj} R)^{\perp},$$

where the second one is due to Proposition (4.7).
(2) The equality involving $\mathbf{K}_{\mathrm{ac}}(\operatorname{Prj} R)$ is immediate from (2.1.1). Since $R \otimes_R D \cong D$ and $D^* \otimes_R D \cong \mathsf{i}R$, the second claim follows from (1) and (3). □

5.10. PROOF OF THEOREMS (5.4) AND (5.3). The category $\mathcal{T} = \mathbf{K}(\operatorname{Inj} R)$ is compactly generated, the complexes $\mathsf{i}R$ and D are compact, and one has a canonical equivalence $\mathcal{T}^c \xrightarrow{\sim} \mathbf{D}^f(R)$; see (2.3.2). Therefore, Theorem (5.4) is immediate from Proposition (5.9.3), and Proposition (1.7) applied with $\mathcal{B} = \{\mathsf{i}R\}$ and $\mathcal{C} = \{\mathsf{i}R, D\}$.
To prove Theorem (5.3), set $\mathcal{T} = \mathbf{K}(\operatorname{Prj} R)$. By (2.3.1), this category is compactly generated, and in it R and D^* are compact; for D^* one requires also the identification in (4.5). Thus, in view of Proposition (5.9.2), Proposition (1.7) applied with $\mathcal{B} = \{R\}$ and $\mathcal{C} = \{R, D^*\}$ yields that the categories $\mathbf{K}_{\mathrm{ac}}(\operatorname{Prj} R)$ and $\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R)$, and their quotient, are compactly generated. Furthermore, it provides equivalences up to direct factors

$$\mathbf{K}^c(\operatorname{Prj} R)/\operatorname{Thick}(R) \xrightarrow{\sim} \mathbf{K}^c_{\mathrm{ac}}(\operatorname{Prj} R)$$

$$\mathbf{K}^c(\operatorname{Prj} R)/\operatorname{Thick}(R, D^*) \xrightarrow{\sim} \mathbf{K}^c_{\mathrm{tac}}(\operatorname{Prj} R)$$

$$\operatorname{Thick}(R, D^*)/\operatorname{Thick}(R) \xrightarrow{\sim} \big(\mathbf{K}_{\mathrm{ac}}(\operatorname{Prj} R)/\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R)\big)^c .$$

Combining these with the equivalence $\mathbf{D}^f(R) \to \mathbf{K}^c(\operatorname{Prj} R)^{\mathrm{op}}$ in (2.3.1) yields the desired equivalences. □

Remark 5.11. Proposition (5.9.3) contains the following result: a complex of injectives X is totally acyclic if and only if both X and $\operatorname{Hom}_R(D, X)$ are acyclic. We should like to raise the question: if both $\operatorname{Hom}_R(X, D)$ and $\operatorname{Hom}_R(D, X)$ are acyclic, is then X acyclic, and hence totally acylic? An equivalent formulation is: if X is a complex of projectives and X and $\operatorname{Hom}_R(X, R)$ are acyclic, is then X totally acyclic?
In an earlier version of this article, we had claimed an affirmative answer to this question, based on a assertion that if X is a complex of R-modules such that $\operatorname{Hom}_R(X, D)$ is acyclic, then X is acyclic. This assertion is false. Indeed, let R be a complete local domain, with field of fractions Q. A result of Jensen [10, Theorem 1] yields $\operatorname{Ext}^i_R(Q, R) = 0$ for $i \geq 1$, and it is easy to check that $\operatorname{Hom}_R(Q, R) = 0$ as well. Thus, $\operatorname{Hom}_R(Q, \mathsf{i}R)$ is acyclic. It remains to recall that when R is Gorenstein, $\mathsf{i}R$ is a dualizing complex for R.

5.12. Non-commutative rings. Theorems (5.3) and (5.4), and Proposition (5.9), all carry over, again with suitable modifications in the statements, to the pair of rings $\langle S, R\rangle$ from (3.3). The analogue of Corollary (5.5) is especially interesting:

Corollary. *The following conditions are equivalent.*

(a) *The projective dimension of D is finite over R^{op}.*
(b) *The projective dimension of D is finite over S.*
(c) *Every acyclic complex of projective R-modules is totally acyclic.*
(d) *Every acyclic complex of injective S-modules is totally acyclic.* □

6. An example

Let A be a commutative noetherian local ring, with maximal ideal $\mathfrak{m}$, and residue field $k = A/\mathfrak{m}$. Assume that $\mathfrak{m}^2 = 0$, and that $\operatorname{rank}_k(\mathfrak{m}) \geq 2$. Observe that A is *not* Gorenstein; for instance, its socle is $\mathfrak{m}$, and hence of rank at least 2. Let E denote the injective hull of the R-module k; this is a dualizing complex for A.

Proposition 6.1. *Set $\mathbf{K} = \mathbf{K}(\operatorname{Prj} A)$ and let X be a complex of projective A-modules.*

(1) *If X is acyclic and the A-module X^d is finite for some d, then $X \cong 0$ in $\mathbf{K}$.*
(2) *If X is totally acyclic, then $X \cong 0$ in $\mathbf{K}$.*
(3) *The cone of the homothety $A \to \operatorname{Hom}_A(P, P)$, where P is a projective resolution of D, is an acyclic complex of projectives, but it is not totally acyclic.*
(4) *In the derived category of A, one has $\operatorname{Thick}(A, D) = \mathbf{D}^f(A)$, and hence*

$$\operatorname{Thick}(A, D)/\operatorname{Thick}(A) = \mathbf{D}^f(A)/\operatorname{Thick}(A)\,.$$

The proof is given in (6.4). It hinges on some properties of minimal resolutions over A, which we now recall. Since A is local, each projective A-module is free. The Jacobson radical $\mathfrak{m}$ of A is square-zero, and in particular, nilpotent. Thus, Nakayama's lemma applies to each A-module M, hence it has a projective cover $P \to M$, and hence a minimal projective resolution; see [7, Propositions 3 and 15]. Moreover, $\Omega = \operatorname{Ker}(P \to M)$, the first syzygy of M, satisfies $\Omega \subseteq \mathfrak{m}P$, so that $\mathfrak{m}\Omega \subseteq \mathfrak{m}^2 P = 0$, so $\mathfrak{m}\Omega = 0$. In what follows, $\ell_A(-)$ denotes length.

Lemma 6.2. *Let M be an A-module; set $b = \ell_A(M)$, $c = \ell_A(\Omega)$.*

(1) *If M is finite, then its Poincaré series is*

$$P_M^A(t) = b + \frac{ct}{1 - et}$$

In particular, $\beta_n^A(M)$, the nth Betti number of M, equals ce^{n-1}, for $n \geq 1$.

(2) *If $\operatorname{Ext}_A^n(M, A) = 0$ for some $n \geq 2$, then M is free.*

Proof. (1) This is a standard calculation, derived from the exact sequences

$$0 \longrightarrow \mathfrak{m} \longrightarrow A \longrightarrow k \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \Omega \longrightarrow P \longrightarrow M \longrightarrow 0$$

The one on the left implies $P_k^A(t) = 1 + etP_k^A(t)$, so $P_k^A(t) = (1-et)^{-1}$, while the one on the right yields $P_M^A(t) = b + ctP_k^A(t)$, since $\mathfrak{m}\Omega = 0$.
(2) If M is not free, then $\Omega \neq 0$ and hence has k as a direct summand. In this case, since $\operatorname{Ext}_A^{n-1}(\Omega, A) \cong \operatorname{Ext}_A^n(M, A) = 0$, one has $\operatorname{Ext}_A^{n-1}(k, A) = 0$, which in turn implies that A is Gorenstein; a contradiction. □

The following test to determine when an acyclic complex is homotopically trivial is surely known. Note that it applies to any (commutative) noetherian ring of finite Krull dimension, and, in particular, to the ring A that is the focus of this section.

LEMMA 6.3. *Let R be a ring whose finitistic global dimension is finite. An acyclic complex X of projective R-modules is homotopically trivial if and only if for some integer s the R-module* $\operatorname{Coker}(X^{s-1} \to X^s)$ *is projective.*

Proof. For each integer n set $M(n) = \operatorname{Coker}(X^{n-1} \to X^n)$. It suffices to prove that the R-module $M(n)$ is projective for each n. This is immediate for $n \leq s$ because $M(s)$ is projective so that the sequence $\cdots \to X^{s-1} \to X^s \to M(s) \to 0$ is split exact.
We may now assume that $n \geq s+1$. By hypothesis, there exists an integer d with the following property: for any R-module M, if its projective dimension, $\operatorname{pd}_R M$ is finite, then $\operatorname{pd}_R M \leq d$. It follows from the exact complex

$$0 \longrightarrow M(s) \longrightarrow X^{s+1} \longrightarrow \cdots \longrightarrow X^{n+d} \longrightarrow M(n+d) \longrightarrow 0$$

that $\operatorname{pd}_R M(n+d)$ is finite. Thus, $\operatorname{pd}_R M(n+d) \leq d$, and another glance at the exact complex above reveals that $M(n)$ must be projective, as desired. □

Now we are ready for the

6.4. PROOF OF PROPOSITION (6.1). In what follows, set $M(s) = \operatorname{Coker}(X^{s-1} \to X^s)$.
(1) Pick an integer $n \geq 1$ with $e^{n-1} \geq \operatorname{rank}_A(X^d) + 1$. Since X is acyclic, $\Sigma^{-d-n} X^{\leqslant d+n}$ is a free resolution of the A-module $M(n+d)$. Let Ω be the first syzygy of $M(n+d)$. One then obtains the first one of the following equalities:

$$\operatorname{rank}_A(X^d) \geq \beta_n^A(M(n+d)) \geq \ell_A(\Omega)e^{n-1} \geq \ell_A(\Omega)(\operatorname{rank}_A(X^d)+1)$$

The second equality is Lemma (6.2.1) applied to $M(n+d)$ while the last one is by the choice of n. Thus $\ell_A(\Omega) = 0$, so $\Omega = 0$ and $M(n+d)$ is free. Now Lemma (6.3) yields that X is homotopically trivial.
(2) Fix an integer d. Since $\Sigma^{-d} X^{\leqslant d}$ is a projective resolution of $M(d)$, total acyclicity of X implies that the homology of $\operatorname{Hom}_A(\Sigma^{-d}X^{\leqslant d}, A)$ is zero in degrees ≥ 1, so $\operatorname{Ext}_A^n(M(d), A) = 0$ for $n \geq 1$. Lemma (6.2.2) established above implies $M(d)$ is free. Once again, Lemma (6.3) completes the proof.
(3) Suppose that the cone of $A \to \operatorname{Hom}_A(P,P)$ is totally acyclic. This leads to a contradiction: (2) implies that the cone is homotopic to zero, so $A \cong$

$\operatorname{Hom}_A(P,P)$ in $\mathbf{K}$. This entails the first of the following isomorphisms in $\mathbf{K}(A)$; the others are standard.

$$\begin{aligned}\operatorname{Hom}_A(k,A) &\cong \operatorname{Hom}_A(k,\operatorname{Hom}_A(P,P))\\ &\cong \operatorname{Hom}_A(P\otimes_A k,P)\\ &\cong \operatorname{Hom}_k(P\otimes_A k,\operatorname{Hom}_A(k,P))\\ &\cong \operatorname{Hom}_k(P\otimes_A k,\operatorname{Hom}_A(k,A)\otimes_A P)\\ &\cong \operatorname{Hom}_k(P\otimes_A k,\operatorname{Hom}_A(k,A)\otimes_k (k\otimes_A P))\end{aligned}$$

Passing to homology and computing ranks yields $H(k\otimes_A P)\cong k$, and this implies $D\cong A$. This cannot be for $\operatorname{rank}_k\operatorname{soc}(D)=1$, while $\operatorname{rank}_k\operatorname{soc}(A)=e$ and $e\geq 2$.
(4) Combining Theorem (5.3.2) and (3) gives the first part. The second part then follows from the first. A direct and elementary argument is also available: As noted above the A-module D is not free; thus, the first syzygy module Ω of D is non-zero, so has k as a direct summand. Since Ω is in $\operatorname{Thick}(A,D)$, we deduce that k, and hence every homologically finite complex of A-modules, is in $\operatorname{Thick}(A,D)$.

Remark 6.5. Let A be the ring introduced at the beginning of this section, and let X and Y be complexes of A-modules.
The Tate cohomology of X and Y, in the sense of Jørgensen [12], is the homology of the complex $\operatorname{Hom}_A(T,Y)$, where T is a complete projective resolution of X; see (7.6). By Proposition (6.1.2) any such T, being totally acyclic, is homotopically trivial, so the Tate cohomology modules of X and Y are all zero. The same is true also of the version of Tate cohomology introduced by Krause [14, (7.5)] via complete injective resolutions. This is because A has no non-trivial totally acyclic complexes of injectives either, as can be verified either directly, or by appeal to Proposition (5.9.1).
These contrast drastically with another generalization of Tate cohomology over the ring A, introduced by Vogel and described by Goichot [8]. Indeed, Avramov and Veliche [3, (3.3.3)] prove that for an arbitrary commutative local ring R with residue field k, if the Vogel cohomology with $X=k=Y$ has finite rank even in a *single* degree, then R is Gorenstein.

7. Auslander categories and Bass categories

Let R be a commutative noetherian ring with a dualizing complex D. We write $\mathbf{K}_{\mathrm{prj}}(R)$ for the subcategory of $\mathbf{K}(\operatorname{Prj}R)$ consisting of K-projective complexes, and $\mathbf{K}_{\mathrm{inj}}(R)$ for the subcategory of $\mathbf{K}(\operatorname{Inj}R)$ consisting of K-injective complexes. This section is motivated by the following considerations: One has adjoint pairs of functors

$$\mathbf{K}_{\mathrm{prj}}(R) \underset{\mathsf{inc}}{\overset{\mathsf{p}}{\rightleftarrows}} \mathbf{K}(\operatorname{Prj}R) \quad\text{and}\quad \mathbf{K}(\operatorname{Inj}R) \underset{\mathsf{i}}{\overset{\mathsf{inc}}{\rightleftarrows}} \mathbf{K}_{\mathrm{inj}}(R)$$

and composing these functors with those in Theorem (4.2) gives functors

$$\mathsf{G} = (\mathsf{i} \circ \mathsf{T})\colon \mathbf{K}_{\mathrm{prj}}(R) \longrightarrow \mathbf{K}_{\mathrm{inj}}(R) \quad \text{and} \quad \mathsf{F} = (\mathsf{p} \circ \mathsf{S})\colon \mathbf{K}_{\mathrm{inj}}(R) \longrightarrow \mathbf{K}_{\mathrm{prj}}(R)\,.$$

These functors fit into the upper half of the picture below:

$$\begin{tikzcd}
\mathbf{K}(\operatorname{Prj} R) \arrow[d, shift left, "\mathsf{p}"] \arrow[r, shift right, "\sim", "\mathsf{T}"'] & \mathbf{K}(\operatorname{Inj} R) \arrow[l, shift right, "\mathsf{S}"'] \arrow[d, shift right, "\mathsf{i}"'] \\
\mathbf{K}_{\mathrm{prj}}(R) \arrow[u, shift left, "\mathsf{inc}"] \arrow[r, shift right, "\mathsf{G}"'] \arrow[d, "\wr"] & \mathbf{K}_{\mathrm{inj}}(R) \arrow[u, shift right, "\mathsf{inc}"'] \arrow[l, shift right, "\mathsf{F}"'] \arrow[d, "\wr"] \\
\mathbf{D}(R) \arrow[r, shift right, "D\otimes^{\mathbf{L}}_R -"'] & \mathbf{D}(R) \arrow[l, shift right, "\mathbf{R}\mathrm{Hom}_R(D,-)"']
\end{tikzcd}$$

The vertical arrows in the lower half are obtained by factoring the canonical functor $\mathbf{K}(\operatorname{Prj} R) \to \mathbf{D}(R)$ through p, and similarly $\mathbf{K}(\operatorname{Inj} R) \to \mathbf{D}(R)$ through i. A straightforward calculation shows that the functors in the last row of the diagram are induced by those in the middle. Now, while T and S are equivalences – by Theorem (4.2) – the functors G and F need not be; indeed, they are equivalences if and only if R is Gorenstein; see Corollary (7.5) ahead. The results in this section address the natural:

Question. Identify subcategories of $\mathbf{K}_{\mathrm{prj}}(R)$ and $\mathbf{K}_{\mathrm{inj}}(R)$ on which G and F restrict to equivalences.

Given the equivalences in the lower square of the diagram an equivalent problem is to characterize subcategories of $\mathbf{D}(R)$ on which the functors $D \otimes^{\mathbf{L}}_R -$ and $\mathbf{R}\mathrm{Hom}_R(D,-)$ induce equivalences. This leads us to the following definitions:

7.1. Auslander category and Bass category. Consider the categories

$$\begin{aligned}
\widehat{\mathcal{A}}(R) =&\{X \in \mathbf{D}(R) \mid \text{the natural map} \\
&\qquad X \to \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R X) \text{ is an isomorphism.}\} \\
\widehat{\mathcal{B}}(R) =&\{Y \in \mathbf{D}(R) \mid \text{the natural map} \\
&\qquad D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, Y) \to Y \text{ is an isomorphism.}\}
\end{aligned}$$

The notation is intended to be reminiscent of the ones for the *Auslander category* $\mathcal{A}(R)$ and the *Bass category* $\mathcal{B}(R)$, introduced by Avramov and Foxby [1], which are the following subcategories of the derived category:

$$\mathcal{A}(R) = \{X \in \widehat{\mathcal{A}}(R) \mid X \text{ and } D \otimes^{\mathbf{L}}_R X \text{ are homologically bounded.}\}$$

$$\mathcal{B}(R) = \{Y \in \widehat{\mathcal{B}}(R) \mid Y \text{ and } \mathbf{R}\mathrm{Hom}_R(D, Y) \text{ are homologically bounded.}\}$$

The definitions are engineered to lead immediately to the following

Proposition 7.2. *The adjoint pair of functors* (G,F) *restrict to equivalences of categories between* $\widehat{\mathcal{A}}(R)$ *and* $\widehat{\mathcal{B}}(R)$*, and between* $\mathcal{A}(R)$ *and* $\mathcal{B}(R)$. □

In what follows, we identify $\widehat{\mathcal{A}}(R)$ and $\widehat{\mathcal{B}}(R)$ with the subcategories of $\mathbf{K}_{\mathrm{prj}}(R)$ and $\mathbf{K}_{\mathrm{inj}}(R)$ on which $\mathsf{S}\circ\mathsf{T}$ and $\mathsf{T}\circ\mathsf{S}$, respectively, restrict to equivalences. The Auslander category and the Bass category are identified with appropriate subcategories.

The main task then is describe the complexes in the categories being considered. In this section we provide an answer in terms of the categories of K-projectives and K-injectives; in the next one, it is translated to the derived category. Propositions (7.3) and (7.4) below are the first step towards this end. In them, the *cone* of a morphism $U\to V$ in a triangulated category refers to an object W obtained by completing the morphism to an exact triangle: $U\to V\to W\to \Sigma U$. We may speak of *the* cone because they exist and are all isomorphic.

Proposition 7.3. *Let X be a complex of projective R-modules. If X is K-projective, then it is in $\widehat{\mathcal{A}}(R)$ if and only if the cone of the morphism* $\mathsf{T}(X)\to \mathsf{iT}(X)$ *in* $\mathbf{K}(\operatorname{Inj} R)$ *is totally acyclic.*

Remark. The cone in question is always acyclic, because $\mathsf{T}(X)\to\mathsf{iT}(X)$ is an injective resolution; the issue thus is the difference between acyclicity and total acyclicity.

Proof. Let $\eta\colon \mathsf{T}(X)\to\mathsf{iT}(X)$ be a K-injective resolution. In $\mathbf{K}(\operatorname{Prj} R)$ one has then a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\kappa"] \arrow[d, "\cong"'] & \mathsf{FG}(X) \arrow[d, "\simeq"] \\ \mathsf{ST}(X) \arrow[r, "\mathsf{S}(\eta)"] & \mathsf{SiT}(X) \end{tikzcd}$$

of adjunction morphisms, where the isomorphism is by Theorem (4.2). It is clear from the diagram above that

$$\begin{aligned} X \text{ is in } \widehat{\mathcal{A}}(R) &\iff \kappa \text{ is a quasi-isomorphism} \\ &\iff \mathsf{S}(\eta) \text{ is a quasi-isomorphism} \end{aligned}$$

It thus remains to prove that the last condition is equivalent to total acyclicity of the cone of η. In $\mathbf{K}(\operatorname{Inj} R)$ complete η to an exact triangle:

$$\mathsf{T}(X) \xrightarrow[\simeq]{\eta} \mathsf{iT}(X) \longrightarrow C \longrightarrow \Sigma\mathsf{T}(X)$$

From this triangle one obtains that $\mathsf{S}(\eta)$ is a quasi-isomorphism if and only if $\mathsf{S}(C)$ is acyclic. Now $\mathsf{S}(C)$ is quasi-isomorphic to $\operatorname{Hom}_R(D,C)$, see Theorem (2.7.1), and the acyclicity of $\operatorname{Hom}_R(D,C)$ is equivalent to C being in $\{D\}^{\perp}$, in $\mathbf{K}(\operatorname{Inj} R)$. However, C is already acyclic, and hence in $\{\mathsf{i}R\}^{\perp}$. Therefore Proposition (5.9.3) implies that $\mathsf{S}(C)$ is acyclic if and only if C is totally acyclic, as desired. □

An analogous proof yields:

Proposition 7.4. *Let Y be a complex of injective R-modules. If Y is K-injective, then it is in $\widehat{\mathcal{B}}(R)$ if and only if the cone of the morphism $\mathsf{pS}(Y) \to \mathsf{S}(Y)$ in $\mathbf{K}(\operatorname{Prj} R)$ is totally acyclic.* □

Corollary 7.5. *Let R be a noetherian ring with a dualizing complex. The ring R is Gorenstein if and only if $\widehat{\mathcal{A}}(R) = \mathbf{K}_{\mathrm{prj}}(R)$, if and only if $\widehat{\mathcal{B}}(R) = \mathbf{K}_{\mathrm{inj}}(R)$.*

Proof. Combine Propositions (7.3) and (7.4) with Corollary (5.5). □

One shortcoming in Propositions (7.3) and (7.4) is they do not provide a structural description of objects in the Auslander and Bass categories. Addressing this issue requires a notion of complete resolutions.

7.6. Complete resolutions. The subcategory $\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R)$ of $\mathbf{K}(\operatorname{Prj} R)$ is closed under coproducts; moreover, it is compactly generated, by Theorem (5.3.1). Thus, the inclusion $\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R) \to \mathbf{K}(\operatorname{Prj} R)$ admits a right adjoint:

$$\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R) \underset{\mathsf{inc}}{\overset{\mathsf{t}}{\rightleftarrows}} \mathbf{K}(\operatorname{Prj} R)$$

For each complex X in $\mathbf{K}(\operatorname{Prj} R)$ we call $\mathsf{t}(X)$ the *complete projective resolution* of X. In $\mathbf{K}(\operatorname{Prj} R)$, complete the natural morphism $\mathsf{t}(X) \to X$ to an exact triangle:

$$\mathsf{t}(X) \longrightarrow X \longrightarrow \mathsf{u}(X) \longrightarrow \Sigma\mathsf{t}(X)$$

Up to an isomorphism, this triangle depends only on X.
Similar considerations show that the inclusion $\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} R) \to \mathbf{K}(\operatorname{Inj} R)$ admits a left adjoint. We denote it s, and for each complex Y of injectives call $\mathsf{s}(Y)$ the *complete injective resolution* of Y. This leads to an exact triangle in $\mathbf{K}(\operatorname{Inj} R)$:

$$\mathsf{v}(Y) \longrightarrow Y \longrightarrow \mathsf{s}(Y) \longrightarrow \Sigma\mathsf{v}(Y)$$

Relevant properties of complete resolutions and the corresponding exact triangles are summed up in the next two result; the arguments are standard, and details are given for completeness.

Lemma 7.7. *Let X be a complex of projectives R-modules.*

(1) *The morphism $X \to \mathsf{u}(X)$ is a quasi-isomorphism and $\mathsf{u}(X)$ is in $\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R)^{\perp}$.*
(2) *Any exact triangle $T \to X \to U \to \Sigma T$ in $\mathbf{K}(\operatorname{Prj} R)$ where T is totally acyclic and U is in $\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R)^{\perp}$ is isomorphic to $\mathsf{t}(X) \to X \to \mathsf{u}(X) \to \Sigma\mathsf{t}(X)$.*

Proof. (1) By definition, one has an exact triangle

$$\mathsf{t}(X) \longrightarrow X \longrightarrow \mathsf{u}(X) \longrightarrow \Sigma\mathsf{t}(X)\,.$$

Since the complex $\mathsf{t}(X)$ is acyclic, the homology long exact sequence arising from this triangle proves that $X \to \mathsf{u}(X)$ is an quasi-isomorphism, as claimed.

Moreover, for each totally acyclic complex T the induced map below is bijective:

$$(\dagger) \qquad \mathrm{Hom}_{\mathbf{K}}(T, \mathsf{t}(X)) \longrightarrow \mathrm{Hom}_{\mathbf{K}}(T, X)$$

This holds because t is a right adjoint to the inclusion $\mathbf{K}_{\mathrm{tac}}(\mathrm{Prj}\, R) \to \mathbf{K}(\mathrm{Prj}\, R)$. Since $\mathsf{t}(-)$ commutes with translations, the morphism $\Sigma^n \mathsf{t}(X) \to \Sigma^n X$ coincides with the morphism $\mathsf{t}(\Sigma^n X) \to \Sigma^n X$. Thus, from (†) one deduces that the induced map

$$\mathrm{Hom}_{\mathbf{K}}(T, \mathsf{t}(\Sigma^n X)) \longrightarrow \mathrm{Hom}_{\mathbf{K}}(T, \Sigma^n X)$$

is bijective for each integer n. It is now immediate from the exact triangle above that $\mathrm{Hom}_{\mathbf{K}}(T, \mathsf{u}(X)) = 0$; this settles (1), since $\mathbf{K}_{\mathrm{tac}}(\mathrm{Prj}\, R)$ is stable under translations.

(2) Given such an exact triangle, the induced map $\mathrm{Hom}_{\mathbf{K}}(-, T) \to \mathrm{Hom}_{\mathbf{K}}(-, X)$ is bijective on $\mathbf{K}_{\mathrm{tac}}(\mathrm{Prj}\, R)$, since $\mathrm{Hom}_{\mathbf{K}}(-, U)$ vanishes on $\mathbf{K}_{\mathrm{tac}}(\mathrm{Prj}\, R)$. Thus, there is an isomorphism $\alpha\colon T \to \mathsf{t}(X)$, by (1.4), and one obtains a commutative diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[d, "\alpha"] & X \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d, dashed, "\beta"] & \Sigma T \arrow[r] \arrow[d, "\Sigma\alpha"] & \cdots \\
\mathsf{t}(X) \arrow[r] & X \arrow[r] & \mathsf{u}(X) \arrow[r] & \Sigma \mathsf{t}(X) \arrow[r] & \cdots
\end{tikzcd}$$

of morphisms in $\mathbf{K}(\mathrm{Prj}\, R)$. Since the rows are exact triangles, and we are in a triangulated category, there exists a β as above that makes the diagram commute. Moreover, since α is an isomorphism, so is β; this is the desired result. □

One has also a version of Lemma (7.7) for complexes of injectives; proving it calls for a new ingredient, provided by the next result. Recall that $\mathsf{i}R$ denotes an injective resolution of R and $D^* = \mathsf{S}(\mathsf{i}R)$; see (4.5).

LEMMA 7.8. ${}^{\perp}\mathbf{K}_{\mathrm{tac}}(\mathrm{Inj}\, R) = \mathrm{Loc}(\mathsf{i}R, D)$

Proof. Proposition (5.9.3) implies that $\mathsf{i}R$ and D are contained in ${}^{\perp}\mathbf{K}_{\mathrm{tac}}(\mathrm{Inj}\, R)$, and hence so is $\mathrm{Loc}(\mathsf{i}R, D)$. To see that the reverse inclusion also holds note that $\mathrm{Loc}(\mathsf{i}R, D)$ is compactly generated (by $\mathsf{i}R$ and D) and closed under coproducts. Thus, by (1.5.1), the inclusion $\mathrm{Loc}(\mathsf{i}R, D) \to \mathbf{K}(\mathrm{Inj}\, R)$ admits a right adjoint, say r. Let X be a complex of injectives. Complete the canonical morphism $\mathsf{r}(X) \to X$ to an exact triangle

$$\mathsf{r}(X) \longrightarrow X \longrightarrow C \longrightarrow \Sigma \mathsf{r}(X)$$

For each integer n the induced map $\mathrm{Hom}_{\mathbf{K}}(-, \Sigma^n \mathsf{r}(X)) \to \mathrm{Hom}_{\mathbf{K}}(-, \Sigma^n X)$ is bijective on $\{\mathsf{i}R, D\}$, so the exact triangle above yields that $\mathrm{Hom}_{\mathbf{K}}(\mathsf{i}R, \Sigma^n C) = 0 = \mathrm{Hom}_{\mathbf{K}}(D, \Sigma^n C)$. Therefore, C is totally acyclic, by Proposition (5.9.3). In particular, when X is in ${}^{\perp}\mathbf{K}_{\mathrm{tac}}(\mathrm{Inj}\, R)$, one has $\mathrm{Hom}_{\mathbf{K}}(X, C) = 0$, so the exact triangle above is split, that is to say, X is a direct summand of $\mathsf{r}(X)$, and hence in $\mathrm{Loc}(\mathsf{i}R, D)$, as claimed. □

Here is the analogue of Lemma (7.7) for complexes of injectives; it is a better result for it provides a structural description of $\mathsf{v}(Y)$.

Lemma 7.9. *Let Y be a complex of injective R-modules.*

(1) *The morphism $\mathsf{v}(Y) \to Y$ is a quasi-isomorphism and $\mathsf{v}(Y)$ is in* $\mathrm{Loc}(\mathsf{i}R, D)$.

(2) *Any exact triangle $V \to X \to T \to \Sigma V$ in $\mathbf{K}(\mathrm{Inj}\, R)$ where T is totally acyclic and V is in $\mathrm{Loc}(\mathsf{i}R, D)$ is isomorphic to $\mathsf{v}(Y) \to Y \to \mathsf{s}(Y) \to \Sigma\mathsf{v}(Y)$.*

Proof. An argument akin to the proof of Lemma (7.7.1) yields that $\mathsf{v}(Y) \to Y$ is a quasi-isomorphism and that $\mathsf{v}(Y)$ is in ${}^{\perp}\mathbf{K}_{\mathrm{tac}}(\mathrm{Inj}\, R)$, which equals $\mathrm{Loc}(\mathsf{i}R, D)$, by Lemma (7.8). Given this, the proof of part (2) is similar to that of Lemma (7.7.2). □

Our interest in complete resolutions is due to Theorems (7.11) and (7.10), which provide one answer to the question raised at the beginning of this section.

Theorem 7.10. *Let R be a noetherian ring with a dualizing complex D, and let X be a complex of projective R-modules. If X is K-projective, then the following conditions are equivalent.*

(a) *The complex X is in $\widehat{\mathcal{A}}(R)$.*

(b) *The complex $\mathsf{u}(X)$ is in* $\mathrm{Coloc}(\mathrm{Prj}\, R)$.

(c) *In $\mathbf{K}(\mathrm{Prj}\, R)$, there exists an exact triangle $T \to X \to U \to \Sigma U$ where T is totally acyclic and U is in* $\mathrm{Coloc}(\mathrm{Prj}\, R)$.

Proof. Let $\mathsf{t}(X) \to X \to \mathsf{u}(X) \to \Sigma\mathsf{t}(X)$ be the exact triangle associated to the complete projective resolution of X; see (7.6). Let $\eta\colon \mathsf{T}(X) \to \mathsf{iT}(X)$ be a K-injective resolution, and consider the commutative diagram

$$\begin{tikzcd} \mathsf{T}(X) \arrow[r, "\eta", "\simeq"'] \arrow[d, "\simeq"'] & \mathsf{iT}(X) \arrow[d, equal] \\ \mathsf{Tu}(X) \arrow[r, "\kappa", "\simeq"'] & \mathsf{iT}(X) \end{tikzcd}$$

arising as follows: the vertical map on the left is a quasi-isomorphism because it sits in the exact triangle with third vertex $\mathsf{Tt}(X)$, which is acyclic since $\mathsf{t}(X)$ is totally acyclic; see Proposition (5.9.1). Since $\mathsf{iT}(X)$ is K-injective, η extends to yield κ, which is a quasi-isomorphism because the other maps in the square are.

Note that the cone of the morphism $\mathsf{T}(X) \to \mathsf{Tu}(X)$ is $\Sigma\,\mathsf{Tt}(X)$, so applying the octahedral axiom to the commutative square above gives us an exact triangle

$$\Sigma\mathsf{Tt}(X) \longrightarrow \mathrm{Cone}(\eta) \longrightarrow \mathrm{Cone}(\kappa) \longrightarrow \Sigma^2\,\mathsf{Tt}(X)$$

where $\mathrm{Cone}(-)$ refers to the cone of the morphism in parenthesis. Since $\mathsf{t}(X)$ is totally acyclic, so is $\mathsf{Tt}(X)$, by Proposition (5.9.1). Hence the exact triangle above yields:

(†) Cone(η) is totally acyclic if and only if Cone(κ) is totally acyclic.

This observation is at the heart of the equivalence one has set out to establish.
(a) $\Rightarrow$ (b): Proposition (7.3) yields that Cone(η) is totally acyclic, and hence so is Cone(κ), by (†). Consider the exact triangle

$$\mathsf{Tu}(X) \xrightarrow[\simeq]{\kappa} \mathsf{iT}(X) \longrightarrow \operatorname{Cone}(\kappa) \longrightarrow \Sigma\mathsf{Tu}(X)$$

According to Lemma (7.7.1) the complex $\mathsf{u}(X)$ is in $\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R)^{\perp}$, so Proposition (5.9) yields that $\mathsf{Tu}(X)$ is in $\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} R)^{\perp}$, and hence the total acyclicity of Cone(κ) implies

$$\operatorname{Hom}_{\mathbf{K}}(\operatorname{Cone}(\kappa), \mathsf{Tu}(X)) = 0$$

Thus the triangle above is split exact, and $\mathsf{Tu}(X)$ is a direct summand of $\mathsf{iT}(X)$. Consequently $\mathsf{Tu}(X)$ is in $\operatorname{Coloc}(\operatorname{Inj} R)$, so, by Theorem (4.2) and Corollary (4.4), one obtains that $\mathsf{u}(X)$ is in $\operatorname{Coloc}(\operatorname{Prj} R)$, as desired.
(b) $\Rightarrow$ (a): By Corollary (4.4), as $\mathsf{u}(X)$ is in $\operatorname{Coloc}(\operatorname{Prj} R)$ the complex $\mathsf{Tu}(X)$ is in $\operatorname{Coloc}(\operatorname{Inj} R)$, that is to say, it is K-injective. The map $\kappa\colon \mathsf{Tu}(X) \to \mathsf{iT}(X)$, being a quasi-isomorphism between K-injectives, is an isomorphism. Therefore $\operatorname{Cone}(\kappa) \cong 0$ so (†) implies that Cone(η) is totally acyclic. It remains to recall Proposition (7.3).
That (b) implies (c) is patent, and (c) $\Rightarrow$ (b) follows from Lemma (7.7), because $\mathbf{K}_{\mathrm{tac}}(\operatorname{Prj} R)^{\perp} \supseteq \operatorname{Coloc}(\operatorname{Prj} R)$. The completes the proof of the theorem. □

An analogous argument yields a companion result for complexes of injectives:

Theorem 7.11. *Let R be a noetherian ring with a dualizing complex D, and let Y be a complex of injective R-modules. If Y is K-injective, then the following conditions are equivalent.*

(a) *The complex Y is in $\widehat{\mathcal{B}}(R)$.*
(b) *The complex $\mathsf{v}(Y)$ is in* $\operatorname{Loc}(D)$.
(c) *In $\mathbf{K}(\operatorname{Inj} R)$, there exists an exact triangle $V \to Y \to T \to \Sigma V$ where V is in $\operatorname{Loc}(D)$ and T is totally acyclic.* □

Section 8 translates Theorems (7.11) and (7.10) to the derived category of R.

7.12. Non-commutative rings. Consider a pair of rings $\langle S, R\rangle$ with a dualizing complex D, defined in (3.3). As in (7.1), one can define the Auslander category of R and the Bass category of S; these are equivalent via the adjoint pair of functors $(D \otimes_R -, \mathsf{q} \circ \operatorname{Hom}_S(D, -))$. The analogues of Theorems (7.10) and (7.11) extend to the pair $\langle S, R\rangle$, and they describe the complexes in $\widehat{\mathcal{A}}(R)$ and $\widehat{\mathcal{B}}(S)$.

8. Gorenstein dimensions

Let R be a commutative noetherian ring, and let X be a complex of R-modules. We say that X has *finite Gorenstein projective dimension*, or, in short: *finite G-projective dimension*, if there exists an exact sequence of complexes of projective

R-modules

$$0 \longrightarrow U \longrightarrow T \longrightarrow \mathsf{p}X \longrightarrow 0$$

where T is totally acyclic, $\mathsf{p}X$ is a K-projective resolution of X, and $U^n = 0$ for $n \ll 0$.

Similarly, a complex Y of R-modules has *finite G-injective dimension* if there exists an exact sequence of complexes of injective R-modules

$$0 \longrightarrow \mathsf{i}Y \longrightarrow T \longrightarrow V \longrightarrow 0$$

where T is totally acyclic, $\mathsf{i}Y$ is a K-injective resolution of Y, and $V^n = 0$ for $n \gg 0$.

The preceding definitions are equivalent to the usual ones, in terms of G-projective and G-injective resolutions; see Veliche [21], and Avramov and Martsinkovsky [2].

The theorem below contains a recent result of Christensen, Frankild, and Holm; more precisely, the equivalence of (i) and (ii) in [6, (4.1)], albeit in the case when R is commutative; however, see (8.3).

Theorem 8.1. *Let R be a noetherian ring with a dualizing complex D, and X a complex of R-modules. The following conditions are equivalent:*

(a) *X has finite G-projective dimension.*
(b) *$\mathsf{p}X$ is in $\widehat{\mathcal{A}}(R)$ and $D \otimes^{\mathbf{L}}_R X$ is homologically bounded on the left.*
(c) *$\mathsf{u}(\mathsf{p}X)$ is isomorphic, in $\mathbf{K}(\operatorname{Prj} R)$, to a complex U with $U^n = 0$ for $n \ll 0$.*

When $H(X)$ is bounded, these conditions are equivalent to: X is in $\mathcal{A}(R)$.

Proof. Substituting X with $\mathsf{p}X$, one may assume that X is K-projective and that $D \otimes^{\mathbf{L}}_R X$ is quasi-isomorphic to $D \otimes_R X$, that is to say, to $\mathsf{T}(X)$.

(a) $\Rightarrow$ (b): By definition, there is an exact sequence of complexes of projectives $0 \to U \to T \to X \to 0$ where T is totally acyclic and $U^n = 0$ for $n \ll 0$. Passing to $\mathbf{K}(\operatorname{Prj} R)$ gives rise to an exact triangle

$$U \longrightarrow T \longrightarrow X \longrightarrow \Sigma U$$

Since T is totally acyclic, $\mathsf{T}(X)$ is quasi-isomorphic to $\mathsf{T}(\Sigma U)$; the latter is bounded on the left as a complex, hence the former is homologically bounded on the left, as claimed. This last conclusion yields also that $\mathsf{T}(\Sigma U)$ is in $\operatorname{Coloc}(\operatorname{Inj} R)$. Thus, by Theorem (4.2) and Corollary (4.4), the complex ΣU is in $\operatorname{Coloc}(\operatorname{Prj} R)$, so the exact triangle above and Theorem (7.10) imply that X is in $\widehat{\mathcal{A}}(R)$.

(b) $\Rightarrow$ (c): By Theorem (7.10), there is an exact triangle

$$T \longrightarrow X \longrightarrow U \longrightarrow \Sigma T$$

with T totally acyclic and U in $\operatorname{Coloc}(\operatorname{Prj} R)$. The first condition implies that $\mathsf{T}(U)$ is quasi-isomorphic to $\mathsf{T}(X)$, and hence homologically bounded on the left, while the second implies, thanks to Corollary (4.4), that it is in $\operatorname{Coloc}(\operatorname{Inj} R)$, that is to say, it is K-injective. Consequently $\mathsf{T}(U)$ is isomorphic to a complex of injectives I with $I^n = 0$ for $n \ll 0$. This implies that the complex of flat R-modules $\mathrm{Hom}_R(D, \mathsf{T}(U))$ is bounded on the left. Theorem (2.7.3) now yields that the complex $\mathsf{q}(\mathrm{Hom}_R(D, \mathsf{T}(U)))$, that is to say, $\mathsf{ST}(U)$, is bounded on the left; thus, the same is true of U as it is isomorphic to $\mathsf{ST}(U)$, by Theorem (4.2). It remains to note that $\mathrm{Coloc}(\mathrm{Prj}\, R) \subseteq \mathbf{K}_{\mathrm{tac}}(\mathrm{Prj}\, R)^{\perp}$, so $\mathsf{u}(X) \cong U$ by Lemma (7.7).

(c) $\Rightarrow$ (a): Lift the morphism $X \to \mathsf{u}(X) \cong U$ in $\mathbf{K}(\mathrm{Prj}\, R)$ to a morphism $\alpha\colon X \to U$ of complexes of R-modules. In the mapping cone exact sequence

$$0 \longrightarrow U \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow \Sigma X \longrightarrow 0$$

$\mathrm{Cone}(\alpha)$ is homotopic to $\mathsf{t}(X)$, and hence totally acyclic, while $U^n = 0$ for $n \ll 0$, by hypothesis. Thus, the G-projective dimension of ΣX, and hence of X, is finite.

Finally, when $H(X)$ is bounded, $D \otimes^{\mathbf{L}}_R X$ is always bounded on the right. It is now clear from definitions that the condition that X is in $\mathcal{A}(R)$ is equivalent to (b). □

Here is a characterization of complexes in $\mathbf{D}(R)$ that are in the Bass category. For commutative rings, it recovers [6, (4.4)]; see (8.3). The basic idea of the proof is akin the one for the theorem above, but the details are dissimilar enough to warrant exposition.

Theorem 8.2. *Let R be a noetherian ring with a dualizing complex D, and Y a complex of R-modules. The following conditions are equivalent:*

(a) *Y has finite G-injective dimension.*
(b) *$\mathsf{i}Y$ is in $\widehat{\mathcal{B}}(R)$ and $\mathbf{R}\mathrm{Hom}_R(D, Y)$ is homologically bounded on the right.*
(c) *$\mathsf{v}(\mathsf{i}Y)$ is isomorphic, in $\mathbf{K}(\mathrm{Inj}\, R)$, to a complex V with $V^n = 0$ for $n \gg 0$.*

When $H(Y)$ is bounded, these conditions are equivalent to: Y is in $\mathcal{B}(R)$.

Proof. Replacing Y with $\mathsf{i}Y$ we assume that Y is K-injective, so $\mathbf{R}\mathrm{Hom}_R(D, Y)$ is quasi-isomorphic to $\mathrm{Hom}_R(D, Y)$. In the argument below the following remark is used without comment: in $\mathbf{K}(\mathrm{Inj}\, R)$, given an exact triangle

$$Y_1 \longrightarrow Y_2 \longrightarrow T \longrightarrow \Sigma Y_1$$

if T is totally acyclic, then one has a sequence

$$\mathrm{Hom}_R(D, Y_1) \xleftarrow{\simeq} \mathsf{S}(Y_1) \xrightarrow{\simeq} \mathsf{S}(Y_2) \xrightarrow{\simeq} \mathrm{Hom}_R(D, Y_2)\,.$$

of quasi-isomorphisms. Indeed, the first and the last quasi-isomorphism hold by Theorem (2.7.1), while the middle one holds because $\mathsf{S}(T)$ is totally acyclic, by Theorem (4.2).

(a) $\Rightarrow$ (b): The defining property of complexes of finite G-injective dimension provides an exact sequence of complexes of injectives $0 \to Y \to T \to V \to 0$ where T is totally acyclic and $V^n = 0$ for $n \gg 0$. Passing to $\mathbf{K}(\mathrm{Inj}\, R)$ gives rise to an exact triangle

$$\Sigma^{-1} V \longrightarrow Y \longrightarrow T \longrightarrow V$$

Since T is totally acyclic, $\mathrm{Hom}_R(D, \Sigma^{-1}V)$ is quasi-isomorphic to $\mathrm{Hom}_R(D, Y)$; the former is bounded on the right as a complex, so the latter is homologically bounded on the right, as claimed. Furthermore, since V is bounded on the right, so is $\mathrm{Hom}_R(D, \Sigma^{-1}V)$. Theorem (2.7.2) then yields that $\mathsf{S}(\Sigma^{-1}V)$ is its projective resolution, and hence it is in $\mathrm{Loc}(R)$. Thus, by Theorem (4.2), the complex $\Sigma^{-1}V$ is in $\mathrm{Loc}(D)$, so the exact triangle above and Theorem (7.11) imply that Y is in $\widehat{\mathcal{B}}(R)$.

(b) $\Rightarrow$ (c): By hypothesis and Theorem (7.11) there exists and exact triangle

$$V \longrightarrow Y \longrightarrow T \longrightarrow \Sigma V$$

in $\mathbf{K}(\mathrm{Inj}\, R)$, where V lies in $\mathrm{Loc}(D)$ and T is totally acyclic. Thus $\mathsf{S}(V)$ is in $\mathrm{Loc}(R)$, that is to say, it is K-projective, and it is quasi-isomorphic to $\mathrm{Hom}_R(D, Y)$, and hence it is homologically bounded on the right. Therefore, $\mathsf{S}(V)$ is isomorphic to a complex of projectives P with $P^n = 0$ for $n \gg 0$. By Theorem (4.2), this implies that V is isomorphic to $\mathsf{T}(P)$, which is bounded on the right.

(c) $\Rightarrow$ (a): Lift the morphism $V \cong \mathsf{v}(Y) \to Y$ in $\mathbf{K}(\mathrm{Inj}\, R)$ to a morphism $\alpha\colon V \to Y$ of complexes of R-modules. In the mapping cone exact sequence

$$0 \longrightarrow Y \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow \Sigma V \longrightarrow 0$$

the complex $\mathrm{Cone}(\alpha)$ is homotopic to $\mathsf{s}(Y)$, and hence totally acyclic, while $V^n = 0$ for $n \gg 0$, by hypothesis. Thus, the G-injective dimension of Y is finite.

Finally, when Y is homologically bounded, $\mathbf{R}\mathrm{Hom}_R(D, Y)$ is bounded on the left, so Y is in $\mathcal{B}(R)$ if and only if it satisfies condition (b). $\square$

8.3. Non-commutative rings. Following the thread in (3.3), (4.8), (5.12), and (7.12), the development of this section also carries over to the context of a pair of rings $\langle S, R\rangle$ with a dualizing complex D. In this case, the analogues of Theorems (8.1) and (8.2) identify complexes of finite G-projective dimension over R and of finite G-injective dimension over S as those in the Auslander category of R and the Bass category of S, respectively. These results contain [6, (4.1),(4.4)], but only when one assumes that the ring R is left coherent as well; the reason for this has already been given in (3.3).

Acknowledgments

This project was initiated in June 2004, when Srikanth Iyengar was visiting the University of Paderborn. He thanks the Department of Mathematics at Paderborn and its algebra group for hospitality. The authors thank Lucho Avramov for his comments and suggestions, Lars Winther Christensen for numerous discussions on [6] which motivates the results in Section 8, and Amnon Neeman for a careful reading of the manuscript.

References

[1] L. Avramov, H.-B. Foxby: Ring homomorphisms and finite Gorenstein dimension. Proc. London Math. Soc. (3) 75 (1997), no. 2, 241–270.

[2] L. Avramov, A. Martsinkovsky: Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension. Proc. London Math. Soc. (3) 85 (2002), 393-440.

[3] L. Avramov, O. Veliche: Stable cohomology over local rings. ArXiv: math.AC/0508021.

[4] M. Bökstedt, A. Neeman: Homotopy limits in triangulated categories. Compositio Math. 86 (1993), 209–234.

[5] L. W. Christensen: Gorenstein dimensions. Lecture Notes Math. 1747, Springer, Berlin, 2000.

[6] L. W. Christensen, A. Frankild, H. Holm: On Gorenstein projective, injective and flat dimensions – A functorial description with applications. J. Algebra 302 (2006), 231–279.

[7] S. Eilenberg: Homological dimensions of syzygies. Ann. of Math. 64 (1956), 328–336.

[8] F. Goichot:, Homologie de Tate-Vogel équivariante. J. Pure Appl. Algebra 82 (1992), 39–64.

[9] R. Hartshorne: Residues and Duality. Springer Lecture Notes Math. 20 (1966).

[10] C. U. Jensen: On the vanishing of $\varprojlim^{(i)}$. J. Algebra 15 (1970), 151–166.

[11] P. Jørgensen: The homotopy category of complexes of projective modules. Adv. Math. 193 (2005), 223–232.

[12] P. Jørgensen: Existence of Gorenstein projective resolutions and Tate cohomology. J. Eur. Math. Soc. (to appear).

[13] T. Kawasaki: On arithmetic Macaulayfication of Noetherian rings. Trans. Amer. Math. Soc. 354 (2002), 123–149.

[14] H. Krause: The stable derived category of a noetherian scheme. Compositio Math. 141 (2005), 1128–1162.

[15] S. MacLane: Categories for the working mathematician. second ed. Graduate Texts Math., 5, Springer-Verlag, New York, 1998

[16] A. Neeman: The chromatic tower for $D(R)$, With an appendix by Marcel Bökstedt. Topology 31 (1992), 519–532.

[17] A. Neeman: The Connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel. Ann. Sci. École Norm. Sup. 25 (1992), 547–566.

[18] A. Neeman: The Grothendieck duality theorem via Bousfield's techniques and Brown representability. J. Amer. Math. Soc. 9 (1996), 205–236.

[19] A. Neeman: Triangulated categories. Annals of Mathematics Studies 148, Princeton University Press (2001).

[20] M. Raynaud, L. Gruson: Critères de platitude et de projectivité. Techniques de "platification" d'un module. Invent. Math. 13 (1971), 1–89.

[21] O. Veliche: Gorenstein projective dimension for complexes. Trans. Amer. Math. Soc. 358 (2006), 1257–1283.

[22] J. L. Verdier: Des catégories dérivées des catégories abéliennes. Astérisque 239 (1996).

[23] C. Weibel: An introduction to homological algebra, Cambridge Stud. Adv. Math. 38, Cambridge Univ. Press, Cambridge, 1994.

Srikanth Iyengar
Department of Mathematics
University of Nebraska
Lincoln, NE - 68588
U.S.A.
iyengar@math.unl.edu

Henning Krause
Institut für Mathematik
Universität Paderborn
33095 Paderborn
Germany.
hkrause@math.uni-paderborn.de