On the Equivariant Tamagawa Number Conjecture for Abelian Extensions of a Quadratic Imaginary Field

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Abstract. Let k be a quadratic imaginary field, p a prime which splits in $k/\mathbb{Q}$ and does not divide the class number h_k of k. Let L denote a finite abelian extension of k and let K be a subextension of L/k. In this article we prove the p-part of the Equivariant Tamagawa Number Conjecture for the pair $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[\mathrm{Gal}(L/K)])$.

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1 Introduction

The aim of this paper is to provide new evidence for the validity of the Equivariant Tamagawa Number Conjectures (for short ETNC) as formulated by Burns and Flach in [4]. We recall that these conjectures generalize and refine the Tamagawa Number Conjectures of Bloch, Kato, Fontaine, Perrin-Riou et al. In the special case of the untwisted Tate motive the conjecture also refines and generalizes the central conjectures of classical Galois module theory as developed by Fröhlich, Chinburg, Taylor et al (see [2]). Moreover, in many cases it implies refinements of Stark-type conjectures formulated by Rubin and Popescu and the 'refined class number formulas' of Gross. For more details in this direction see [3].

Let k denote a quadratic imaginary field. Let L be a finite abelian extension of k and let K be any subfield of L/k. Let p be a prime number which does not divide the class number h_k of k and which splits in $k/\mathbb{Q}$. Then we prove the 'p-part' of the ETNC for the pair $(h^0(\mathrm{Spec}(L), \mathbb{Z}[\mathrm{Gal}(L/K)]))$ (see Theorem 4.2).

To help put the main result of this article into context we recall that so far the ETNC for Tate motives has only been verified for abelian extensions of the rational numbers $\mathbb{Q}$ and certain quaternion extensions of $\mathbb{Q}$. The most important result in this context is due to Burns and Greither [5] and establishes the validity of the ETNC for the pair $(h^0(\mathrm{Spec}(L)(r), \mathbb{Z}[\frac{1}{2}][\mathrm{Gal}(L/K)]))$, where $L/\mathbb{Q}$ is abelian, $\mathbb{Q} \subseteq K \subseteq L$ and $r \leq 0$. The 2-part was subsequently dealt with by Flach [8], who also gives a nice survey on the general theory of the ETNC, including a detailed outline of the proof of Burns and Greither. Shortly after Burns and Greither, the special case $r = 0$ was independently shown (up to the 2-part) by Ritter and Weiss [22] using different methods.

In order to prove our main result we follow very closely the strategy of Burns and Greither, which was inspired by previous work of Bloch, Kato, Fontaine and Perrin-Riou. In particular, in [13] Kato formulates a conjecture whose proof is one of the main achievements in the work of Burns and Greither.

Roughly speaking, we will replace cyclotomic units by elliptic units. More concretely, the ETNC for the pair $(h^0(\mathrm{Spec}(L), \mathbb{Z}[\mathrm{Gal}(L/K)]))$ conjecturally describes the leading coefficient in the Laurant series of the equivariant Dirichlet L-function at $s = 0$ as the determinant of a canonical complex. By Kronecker's limit formula we replace L-values by sums of logarithms of elliptic units. In this formulation we may pass to the limit along a $\mathbb{Z}_p$-extension and recover (an analogue) of a conjecture which was formulated by Kato in [13]. As in [5] we will deduce this limit conjecture from the Main Conjecture of Iwasawa Theory and the triviality of certain Iwasawa μ-invariants (see Theorem 5.1). Combining the validity of the limit theorem with Iwasawa-theoretic descent considerations we then achieve the proof of our main result.

The Main Conjecture in the elliptic setting was proved by Rubin in [24], but only in semi-simple case (i.e. $p \nmid [L : k]$). Following Greither's exposition [10] we adapt Rubin's proof and obtain the full Main Conjecture (see Theorem 3.1) for ray class fields L and primes p which split in $k/\mathbb{Q}$ and do not divide the class number h_k of k.

The triviality of μ-invariants in the elliptic setting is known from work of Gillard [9], but again only in the ordinary case when p is split in $k/\mathbb{Q}$.

The descent considerations are particularly involved in the presence of 'trivial zeros' of the associated p-adic L-functions. In this case we make crucial use of a recently published result of the author [1] concerning valuative properties of certain elliptic p-units.

As in the cyclotomic case it is possible to use the Iwasawa-theoretic result of Theorem 5.1 and Iwasawa descent to obtain the p-part of the ETNC for $(h^0(\mathrm{Spec}(L)(r), \mathbb{Z}[\mathrm{Gal}(L/K)]))$, $r < 0$. We refer to thesis of Johnson [12] who deals with this case.

We conclude this introduction with some remarks on the non-split situation. Generically this case is more complicated because the corresponding Iwasawa extension is of type $\mathbb{Z}_p^2$. The main issue, if one tries to apply the above described strategy in the non-split case, is to prove $\mu = 0$. Note that we already use the triviality of μ in our proof of the Iwasawa Main Conjecture (see Remark 3.9).

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2 Elliptic units

The aim of this section is to define the elliptic units that we will use in this paper. Our main references are [20], [21] and [1].
We let $L \subseteq \mathbb{C}$ denote a $\mathbb{Z}$-lattice of rank 2 with complex multiplication by the ring of integers of a quadratic imaginary field k. We write $N = N_{k/\mathbb{Q}}$ for the norm map from k to $\mathbb{Q}$. For any $\mathcal{O}_k$-ideal $\mathfrak{a}$ satisfying $(N(\mathfrak{a}), 6) = 1$ we define a meromorphic function

$$\psi(z; L, \mathfrak{a}) := \tilde{F}(z; L, \mathfrak{a}^{-1}L), \quad z \in \mathbb{C},$$

where $\tilde{F}$ is defined in [20, Théorème principale, (15)]. This function ψ coincides with the function $\theta(z;\mathfrak{a})$ used by Rubin in [23, Appendix] and it is a canonical 12th root of the function $\theta(z; L, \mathfrak{a})$ defined in [6, II.2].
The basic arithmetical properties of special values of ψ are summarized in [1, §2].
We choose a $\mathbb{Z}$-basis w_1, w_2 of the complex lattice L such that $\mathrm{Im}(w_1/w_2) > 0$ and write $\eta(\tau)$, $\mathrm{Im}(\tau) > 0$, for the Dedekind η-function. Let η_1, η_2 denote the quasi-periods of the Weierstrass ζ-function and for any $z = a_1 w_1 + a_2 w_2 \in \mathbb{C}, a_1, a_2 \in \mathbb{R}$, put $z^* = a_1\eta_1 + a_2\eta_2$. Writing $\sigma(z; L)$ for the Weierstrass σ-function attached to L we define

$$\varphi(z; w_1, w_2) := 2\pi i e^{-zz^*/2} \sigma(z; L) \eta^2\left(\frac{w_1}{w_2}\right) w_2^{-1}. \tag{1}$$

Note that φ is exactly the function defined in [20, (4)]. The function φ is not a function of lattices but depends on the choice of a basis w_1, w_2. Its 12th power does not depend on this choice and we will also write $\varphi^{12}(z; L)$. We easily deduce from [20, Sec. 3, Lemme] and its proof that the relation between φ and ψ is given by

$$\psi^{12}(z; L, \mathfrak{a}) = \frac{\varphi^{12N(\mathfrak{a})}(z; L)}{\varphi^{12}(z; \mathfrak{a}^{-1}L)}. \tag{2}$$

3 The Iwasawa main conjecture

For any $\mathcal{O}_k$-ideal $\mathfrak{b}$ we write $k(\mathfrak{b})$ for the ray class field of conductor $\mathfrak{b}$. In this notation $k(1)$ denotes the Hilbert class field. We let $w(\mathfrak{b})$ denote the number of roots of unity in k which are congruent to 1 modulo $\mathfrak{b}$. Hence $w(1)$ is the number of roots of unity in k. This number will also be denoted by w_k.

Let p denote an odd rational prime which splits in $k/\mathbb{Q}$, and let $\mathfrak{p}$ be a prime ideal of k lying over p. We assume $p \nmid h_k$. For each $n \geq 0$ we write

$$\mathrm{Gal}(k(\mathfrak{p}^{n+1})/k) = \mathrm{Gal}(k(\mathfrak{p}^{n+1})/k(\mathfrak{p})) \times H,$$

where H is isomorphic to $\mathrm{Gal}(k(\mathfrak{p})/k)$ by restriction. We set

$$k_n := k(\mathfrak{p}^{n+1})^H, \quad k_\infty := \bigcup_{n \geq 0} k_n,$$

and note that k_∞/k is a $\mathbb{Z}_p$-extension. More precisely, k_∞/k is the unique $\mathbb{Z}_p$-extension of k which is unramified outside $\mathfrak{p}$. The prime $\mathfrak{p}$ is totally ramified in k_∞/k.

Let now $\mathfrak{f}$ be any integral ideal of k such that $(\mathfrak{f}, \mathfrak{p}) = 1$. Let $F = k(\mathfrak{f}\mathfrak{p})$ denote the ray class field of conductor $\mathfrak{f}\mathfrak{p}$. We set $K_n := Fk_n = k(\mathfrak{f}\mathfrak{p}^{n+1})$ and $K_\infty := \cup_{n \geq 0} K_n$. Then K_∞/K_0 is a $\mathbb{Z}_p$-extension in which each prime divisor of $\mathfrak{p}$ is totally ramified.

For any number field L we denote the p-part of the ideal class group of L by $A(L)$. Set $A_\infty := \varprojlim_n A(K_n)$, the inverse limit formed with respect to the norm maps. We write $\mathcal{E}_n$ for the group of global units of K_n. For a divisor $\mathfrak{g}$ of $\mathfrak{f}$ we let $\mathcal{C}_{n,\mathfrak{g}}$ denote the subgroup of primitive Robert units of conductor $\mathfrak{f}\mathfrak{p}^{n+1}$, $n \geq 0$. If $\mathfrak{g} \neq (1)$, then $\mathcal{C}_{n,\mathfrak{g}}$ is generated by all $\psi(1; \mathfrak{g}\mathfrak{p}^{n+1}, \mathfrak{a})$ with $(\mathfrak{a}, \mathfrak{g}\mathfrak{p}) = 1$ and the roots of unity in K_n. If $\mathfrak{g} = (1)$, then the elements $\psi(1; \mathfrak{p}^{n+1}, \mathfrak{a})$ are no longer units. By [1, Th. 2.4] a product of the form $\prod \psi(1; \mathfrak{p}^{n+1}, \mathfrak{a})^{m(\mathfrak{a})}$ is a unit, if and only if $\sum m(\mathfrak{a})(N(\mathfrak{a}) - 1) = 0$. We let $\mathcal{C}_{n,\mathfrak{g}}$ denote the group generated by all such products and the roots of unity in K_n. We let $\mathcal{C}_n$ be the group of units generated by the subgroups $\mathcal{C}_{n,\mathfrak{g}}$ with $\mathfrak{g}$ running over the divisors of $\mathfrak{f}$.

We let U_n denote the semi-local units of $K_n \otimes_k k_\mathfrak{p}$ which are congruent to 1 modulo all primes above $\mathfrak{p}$, and let $\bar{\mathcal{E}}_n$ and $\bar{\mathcal{C}}_n$ denote the closures of $\mathcal{E}_n \cap U_n$ and $\mathcal{C}_n \cap U_n$, respectively, in U_n. Finally we define

$$\bar{\mathcal{E}}_\infty := \varprojlim_n \bar{\mathcal{E}}_n, \quad \bar{\mathcal{C}}_\infty := \varprojlim_n \bar{\mathcal{C}}_n,$$

both inverse limits formed with respect to the norm maps.

We let

$$\Lambda = \varprojlim_n \mathbb{Z}_p[\mathrm{Gal}(K_n/k)]$$

denote the completed group ring and for a finitely generated Λ-module and any abelian character χ of $\Delta := \mathrm{Gal}(K_0/k)$ we define the χ-quotient of M by

$$M_\chi := M \otimes_{\mathbb{Z}_p[\Delta]} \mathbb{Z}_p(\chi),$$

where $\mathbb{Z}_p(\chi)$ denotes the ring extension of $\mathbb{Z}_p$ generated by the values of χ. For the basic properties of the functor $M \mapsto M_\chi$ the reader is referred to [30, §2]. The ring Λ_χ is (non-canonically) isomorphic to the power series ring $\mathbb{Z}_p(\chi)[[T]]$. If M_χ is a finitely generated torsion Λ_χ-module, then we write $\mathrm{char}(M_\chi)$ for the characteristic ideal.

Theorem 3.1 *Let p be an odd rational prime which splits into two distinct primes in $k/\mathbb{Q}$. Then*

$$\mathrm{char}(A_{\infty,\chi}) = \mathrm{char}((\bar{\mathcal{E}}_\infty/\bar{\mathcal{C}}_\infty)_\chi).$$

Remarks 3.2 a) If $p \nmid [F : k]$ and p does not divide the number of roots of unity in $k(1)$, then the result of Theorem 3.1 is already proved by Rubin, see [24, Th. 4.1(i)].
b) The Main Conjecture of Iwasawa theory for abelian extensions of $\mathbb{Q}$ was first proved by Mazur and Wiles [16] using deep methods from algebraic geometry. They proved the version which identifies the characteristic power series of the projective limit over the p-class groups with a p-adic L-function. These methods were further developed by Wiles [32] who in 1990 established the Main Conjecture for $p \neq 2$ and Galois extensions L/K of a totally real base field K. Under the condition that $p \nmid |\mathrm{Gal}(L/\mathbb{Q})|$ the result of Mazur and Wiles implies a second version of the Main Conjecture where the p-adic L-function is replaced by the characteristic power series of "units modulo cyclotomic units". It is this version which is needed in the context of this manuscript.
Due to work of Kolyvagin and Rubin there is a much more elementary proof of the Main Conjecture for abelian extension $L/\mathbb{Q}$ with $p \nmid |\mathrm{Gal}(L/\mathbb{Q})|$. This approach uses the Euler system of cyclotomic units. Replacing cyclotomic units by elliptic units (amongst many other things) Rubin achieves the result mentioned in part a) of this remark.
In 1992 Greither [10] refined the method of Rubin and used the Euler system of cyclotomic units to give an elementary (but technical) proof of the second version of the Main Conjecture for $L/\mathbb{Q}$ abelian and all primes p. Our proof of Theorem 3.1 will closely follow Greither's exposition.
Finally we mention recent work of Huber and Kings [11]. They apply the machinery of Euler systems and simultaneously prove the Main Conjecture and the Bloch-Kato conjecture for all primes $p \neq 2$ and all abelian extensions $L/\mathbb{Q}$.

The rest of this section is devoted to the proof of Theorem 3.1. Let $\mathcal{C}(\mathfrak{f})$ denote the Iwasawa module of elliptic units as defined in [6, III.1.6]. Then $\mathcal{C}(\mathfrak{f}) \subseteq \bar{\mathcal{C}}_\infty$, so that $\mathrm{char}((\bar{\mathcal{E}}_\infty/\bar{\mathcal{C}}_\infty)_\chi)$ divides $\mathrm{char}((\bar{\mathcal{E}}_\infty/\mathcal{C}(\mathfrak{f}))_\chi)$. By [6, III.2.1, Theorem] it suffices to show that $\mathrm{char}(A_{\infty,\chi})$ divides $\mathrm{char}((\bar{\mathcal{E}}_\infty/\mathcal{C}(\mathfrak{f}))_\chi)$ for all characters χ of $\Delta = \mathrm{Gal}(K_0/k)$ in order to prove the equality $\mathrm{char}(A_{\infty,\chi}) = \mathrm{char}((\bar{\mathcal{E}}_\infty/\mathcal{C}(\mathfrak{f}))_\chi)$. Hence it is enough for us to prove

$$\mathrm{char}(A_{\infty,\chi}) \text{ divides } \mathrm{char}((\bar{\mathcal{E}}_\infty/\bar{\mathcal{C}}_\infty)_\chi) \tag{3}$$

for all characters χ of Δ.
For an abelian character χ of Δ we write

$$e_\chi := \frac{1}{|\Delta|}\sum_{\delta\in\Delta} \mathrm{Tr}(\chi(\delta))\delta^{-1}$$

for the idempotent of $\mathbb{Q}_p[\Delta]$ corresponding to χ with Tr denoting the trace map from $\mathbb{Z}_p(\chi)$ to $\mathbb{Z}_p$. We also set $\mathrm{Tr}_\Delta = \sum_{\delta\in\Delta}\delta$.
For any $\mathbb{Z}_p[\Delta]$-module M we have an epimorphism

$$M_\chi = M\otimes_{\mathbb{Z}_p[\Delta]}\mathbb{Z}_p(\chi) \longrightarrow |\Delta|e_\chi M, \quad m\otimes\alpha \mapsto |\Delta|\lambda_\alpha e_\chi m,$$

where $\lambda_\alpha \in \mathbb{Z}_p[\Delta]$ is an element which maps to α under $\mathbb{Z}_p[\Delta] \to \mathbb{Z}_p(\chi)$. If Z denotes the kernel, then it is easily seen that $|\Delta|Z = 0$.
Let now $M = A_\infty$ and $\chi = 1$. Then

$$Z \longrightarrow A_{\infty,\chi} \longrightarrow \mathrm{Tr}_\Delta A_\infty \longrightarrow 0$$

is exact. Since $\mathrm{Tr}_\Delta A_n$ is contained in the p-Sylow subgroup of the ideal class group of k_n, which is trivial by our assumption $p \nmid h_k$ and [31, Th. 10.4], we see that $A_{\infty,\chi}$ is annihilated by $|\Delta|$. By the main result of [9] the Iwasawa μ-invariant of $A_{\infty,\chi}$ is trivial. From this we deduce $\mathrm{char}(A_{\infty,\chi}) = (1)$, thus establishing (3) for the trivial character.

The rest of this section is devoted to the proof of the divisibility relation (3) for non-trivial characters χ. As already mentioned we will closely follow Greither's exposition [10]. Whenever there are only minor changes we shall be very brief, but emphasize those arguments which differ from the cyclotomic situation.
To see the Euler system method applied in an easy setting the reader is advised to have a look at [26]. The strategy of the proof of our Theorem 3.1 is essentially the same, but there are additional difficulties because we allow p to divide $|\Delta|$. If $p \nmid |\Delta|$, the functor $M \mapsto M_\chi$ is exact and the Euler system machinery directly produces a divisibility result of the form $\mathrm{char}(A_{\infty,\chi}) \mid \mathrm{char}((\bar{\mathcal{E}}_\infty/\bar{\mathcal{C}}_\infty)_\chi)$. If $p \mid |\Delta|$, the functor $M \mapsto M_\chi$ is no longer exact, but Greither's paper [10] shows how to adapt the Euler system method to produce a weaker divisibility relation of the form $\mathrm{char}(A_{\infty,\chi}) \mid \eta\mathrm{char}((\bar{\mathcal{E}}_\infty/\bar{\mathcal{C}}_\infty)_\chi)$ with an additional factor $\eta \in \Lambda_\chi$which is essentially a product of powers of p and $\gamma-1$. Because of Lemma 3.7 and the triviality of the μ-invariant of $A_{\infty,\chi}$, the factor η is coprime with $\mathrm{char}(A_{\infty,\chi})$, so that we again derive a clean divisibility result as in the case $p \nmid |\Delta|$.
We will need some notation from Kolyvagin's theory. Let M be a large power of p and define $\mathcal{L} = \mathcal{L}_{F,M}$ to be the set of all primes $\mathfrak{l}$ of k satisfying

$$\begin{array}{ll} (i) & \mathfrak{l} \text{ splits completely in } F/k, \\ (ii) & N_{k/\mathbb{Q}}(\mathfrak{l}) \equiv 1 (\mathrm{mod}\ M). \end{array}$$

By [24, Lem. 1.1] there exists a unique extension $F(\mathfrak{l})$ of F of degree M in $Fk(\mathfrak{l})$. Further $F(\mathfrak{l})/F$ is cyclic, totally ramified at all primes above $\mathfrak{l}$ and unramified at all other primes.
We write $J = \oplus_\lambda \mathbb{Z}\lambda$ for the group of fractional ideals of F and for every prime $\mathfrak{l}$ of k we let $J_\mathfrak{l} = \oplus_{\lambda|\mathfrak{l}}\mathbb{Z}\lambda$ denote the subgroup of J generated by the prime divisors of $\mathfrak{l}$. If $y \in F^\times$ we let $(y)_\mathfrak{l} \in J_\mathfrak{l}$ denote the support of the principal ideal $(y) = y\mathcal{O}_F$ above $\mathfrak{l}$. Analogously we write $[y] \in J/MJ$ and $[y]_\mathfrak{l} \in J_\mathfrak{l}/MJ_\mathfrak{l}$.

For $\mathfrak{l} \in \mathcal{L}$ we let

$$\varphi_{\mathfrak{l}} : \frac{(\mathcal{O}_F/\mathfrak{l}\mathcal{O}_F)^\times}{\left((\mathcal{O}_F/\mathfrak{l}\mathcal{O}_F)^\times\right)^M} \longrightarrow J_{\mathfrak{l}}/MJ_{\mathfrak{l}}$$

denote the $\mathrm{Gal}(F/k)$-equivariant isomorphism defined by [24, Prop. 2.3]. For every $\mathfrak{l} \in \mathcal{L}$ we also write $\varphi_{\mathfrak{l}}$ for the induced map

$$\varphi_{\mathfrak{l}} : \{y \in F^\times/\left(F^\times\right)^M : [y]_{\mathfrak{l}} = 0\} \longrightarrow J_{\mathfrak{l}}/MJ_{\mathfrak{l}}, \quad y \mapsto \varphi_{\mathfrak{l}}(u),$$

where $y = z^M u, z \in F^\times$, u a unit at all places above $\mathfrak{l}$.
We write $\mathcal{S} = \mathcal{S}_{F,M}$ for the set of squarefree integral ideals of k which are only divisible by primes $\mathfrak{l} \in \mathcal{L}$. If $\mathfrak{a} \in \mathcal{S}$, $\mathfrak{a} = \prod_{i=1}^k \mathfrak{l}_i$, we write $F(\mathfrak{a})$ for the compositum $F(\mathfrak{l}_1)\cdots F(\mathfrak{l}_k)$ and $F(\mathcal{O}_k) = F$. For every ideal $\mathfrak{g}$ of $\mathcal{O}_k$ let $\mathcal{S}(\mathfrak{g}) \subseteq \mathcal{S}$ be the subset $\{\mathfrak{a} \in \mathcal{S} : (\mathfrak{a}, \mathfrak{g}) = 1\}$. We write $\bar{F}$ for the algebraic closure of F and let $\mathcal{U}(\mathfrak{g})$ denote the set of all functions

$$\alpha : \mathcal{S}(\mathfrak{g}) \longrightarrow \bar{F}^\times$$

satisfying the properties (1a)-(1d) of [24]. Any such function will be called an Euler system. Define $\mathcal{U}_F = \mathcal{U}_{F,M} = \coprod \mathcal{U}(\mathfrak{g})$. For $\alpha \in \mathcal{U}_F$ we write $\mathcal{S}(\alpha)$ for the domain of α, i.e. $\mathcal{S}(\alpha) = \mathcal{S}(\mathfrak{g})$ if $\alpha \in \mathcal{U}(\mathfrak{g})$.
Given any Euler system $\alpha \in \mathcal{U}_F$, we let $\kappa = \kappa_\alpha : \mathcal{S}(\alpha) \longrightarrow F^\times/\left(F^\times\right)^M$ be the map defined in [24, Prop. 2.2].
Then we have:

Proposition 3.3 *Let $\alpha \in \mathcal{U}_F$, $\kappa = \kappa_\alpha$, $\mathfrak{a} \in \mathcal{S}(\alpha), \mathfrak{a} \neq 1$, and $\mathfrak{l}$ a prime of k. If $\mathfrak{a} = \mathfrak{l}$ we also assume that $\alpha(1)$ satisfies $v_\lambda(\alpha(1)) \equiv 0 (\mathrm{mod}\ M)$ for all $\lambda \mid \mathfrak{l}$ in F/k. Then:*

$$\text{If } \mathfrak{l} \nmid \mathfrak{a}, \text{ then } [\kappa(\mathfrak{a})]_{\mathfrak{l}} = 0.$$
$$\text{If } \mathfrak{l} \mid \mathfrak{a}, \text{ then } [\kappa(\mathfrak{a})]_{\mathfrak{l}} = \varphi_{\mathfrak{l}}(\kappa(\mathfrak{a}/\mathfrak{l})).$$

Proof See [24, Prop. 2.4]. Note that the additional assumption in the case $\mathfrak{a} = \mathfrak{l}$ is needed in (ii), both for its statement ($\varphi_{\mathfrak{l}}(\kappa(1))$ may not be defined in general) and for its proof. □

We now come to the technical heart of Kolyvagin's induction procedure, the application of Chebotarev's theorem.

Theorem 3.4 *Let K/k be an abelian extension, $G = \mathrm{Gal}(K/k)$. Let M denote a (large enough) power of p. Assume that we are given an ideal class $\mathfrak{c} \in A(K)$, a finite $\mathbb{Z}[G]$-module $W \subseteq K^\times/\left(K^\times\right)^M$, and a G-homomorphism*

$$\psi : W \longrightarrow (\mathbb{Z}/M\mathbb{Z})[G].$$

Let $\bar{\mathfrak{p}}^c$ be the precise power of $\bar{\mathfrak{p}}$ which divides the conductor $\mathfrak{f}$ of K. Then there are infinitely many primes λ of K such that

(1) $[\lambda] = p^{3c+3}\mathfrak{c}$ *in* $A(K)$.

(2) If $\mathfrak{l} = k \cap \lambda$, *then* $N\mathfrak{l} \equiv 1(\mathrm{mod}\ M)$, *and* $\mathfrak{l}$ *splits completely in* K.

(3) For all $w \in W$ *one has* $[w]_\mathfrak{l} = 0$ *in* $J_\mathfrak{l}/MJ_\mathfrak{l}$ *and there exists a unit* $u \in (\mathbb{Z}/M\mathbb{Z})^\times$ *such that*

$$\varphi_\mathfrak{l}(w) = p^{3c+3}u\psi(w)\lambda.$$

Proof We follow the strategy of Greither's proof of [10, Th. 3.7], but have to change some technical details. Let H denote the Hilbert p-class field of K. For a natural number n we write μ_n for the nth roots of unity in an algebraic closure of K. We consider the following diagram of fields

$$\begin{tikzcd}[arrows=dash]
K'' = K(\mu_M, W^{1/M}) \arrow[d] & \\
K' = K(\mu_M) \arrow[d] & H \arrow[dl] \\
K &
\end{tikzcd}$$

Claim (a) $[H \cap K' : K] \leq p^c$

Proof: The situation is clarified by the following diagram

$$\begin{tikzcd}[arrows=dash]
 & & & K' \arrow[dl] \arrow[d] \\
 & & K' \cap H \arrow[dl] & k(\mu_M) \arrow[d] \arrow[ddll] \\
 & K \arrow[d] & & \mathbb{Q}(\mu_M) \arrow[dddlll] \\
 & \bullet \arrow[dl] & & \\
k \arrow[d] & & & \\
\mathbb{Q} & & &
\end{tikzcd}$$

We write $\varphi_\mathbb{Z}$ (resp. $\varphi_{\mathcal{O}_k}$) for the Euler function in $\mathbb{Z}$ (resp. $\mathcal{O}_k$). Obviously $\bar{\mathfrak{p}}$ is totally ramified in $k(\mu_M)/k$. Hence $\bar{\mathfrak{p}}$ ramifies in K'/k of exponent at least $\varphi_\mathbb{Z}(M)$. On the other hand, $\bar{\mathfrak{p}}$ is ramified in K/k of exponent at most

$\varphi_{\mathcal{O}_k}(\bar{\mathfrak{p}}^c)$. Therefore any prime divisor of $\bar{\mathfrak{p}}$ ramifies in K'/K of degree at least $\varphi_{\mathbb{Z}}(M)/\varphi_{\mathcal{O}_k}(\bar{\mathfrak{p}}^c)$. Since $K' \cap H/K$ is unramified and $[K' : K] \leq \varphi_{\mathbb{Z}}(M)$, we derive $[K' \cap H : K] \leq \varphi_{\mathcal{O}_k}(\bar{\mathfrak{p}}^c)$. Since p is split in $k/\mathbb{Q}$ we obtain $\varphi_{\mathcal{O}_k}(\bar{\mathfrak{p}}^c) = (p-1)p^{c-1} < p^c$, so that the claim is shown.

In order to follow Greither's core argument for the proof of Theorem 3.4 we establish the following two claims.

Claim (b) $\mathrm{Gal}(H \cap K''/K)$ is annihilated by p^{2c+1}.

Claim (c) The cokernel of the canonical map from Kummer theory

$$\mathrm{Gal}(K''/K') \hookrightarrow \mathrm{Hom}(W, \mu_M)$$

is annihilated by p^{c+2}.

We write $M = p^m$. Since divisors of $\bar{\mathfrak{p}}$ are totally ramified in $k(\mu_M)/k$ of degree $\varphi_{\mathbb{Z}}(M)$ and at most ramified in K/k of degree $\varphi_{\mathcal{O}_k}(\bar{\mathfrak{p}}^c)$, one has

$$[k(\mu_M) : K \cap k(\mu_M)] \geq \frac{\varphi_{\mathbb{Z}}(M)}{\varphi_{\mathcal{O}_k}(\bar{\mathfrak{p}}^c)} = \begin{cases} p^{m-c}, & \text{if } c \geq 1, \\ (p-1)p^{m-1}, & \text{if } c = 0. \end{cases}$$

Since $k(\mu_M)/k$ is cyclic, there exists an element $j \in \mathrm{Gal}(k(\mu_M)/K \cap k(\mu_M))$ of exact order $a = p^{m-c-1}$. Let $r \in \mathbb{Z}$ such that $j(\zeta_M) = \zeta_M^r$. Then $r^a \equiv 1 (\mathrm{mod}\ M)$ and $r^b \not\equiv 1 (\mathrm{mod}\ M)$ for all $0 < b < a$. We also write $j \in \mathrm{Gal}(K'/K)$ for the unique extension of j to K' with $j|_K = id$. Let $\sigma \in \mathrm{Gal}(K''/K')$ and $\alpha \in K''$ such that $\alpha^M = w \in W$. Then there exists an integer t_w such that $\sigma(\alpha) = \zeta_M^{t_w}\alpha$. Since $W \subseteq K^\times/\left(K^\times\right)^M$, there is an extension of j to K''/K such that $j(\alpha) = \alpha$ for all $\alpha \in K''$ such that $\alpha^M \in W$. Therefore, for any such α,

$$j\sigma j^{-1}(\alpha) = j\sigma(\alpha) = j(\zeta_M^{t_w}\alpha) = \zeta_M^{rt_w}\alpha.$$

Hence j acts as $\sigma \mapsto \sigma^r$ on $\mathrm{Gal}(K''/K')$. Since $\mathrm{Gal}(K'/K)$ acts trivially on $\mathrm{Gal}(K'' \cap K'H/K')$ this implies that $r-1$ annihilates $\mathrm{Gal}(K'' \cap K'H/K')$. On the other hand $\mathrm{Gal}(K'' \cap K'H/K')$ is an abelian group of exponent M, so that also $\gcd(M, r-1)$ annihilates. Suppose that p^d divides $r-1$ with $d \geq 1$. By induction one easily shows that $r^{p^{m-d}} \equiv 1 (\mathrm{mod}\ p^m)$. Hence $a = p^{m-c-1}$ divides p^{m-d}, which implies $d \leq c+1$. As a consequence, p^{c+1} annihilates $\mathrm{Gal}(K'' \cap K'H/K') \simeq \mathrm{Gal}(K'' \cap H/K' \cap H)$. Together with claim (a) this proves (b).

We now proceed to demonstrate claim (c). Let $W' \subseteq K'^\times/\left(K'^\times\right)^M$ denote the image of W under the homomorphism

$$K^\times/\left(K^\times\right)^M \longrightarrow K'^\times/\left(K'^\times\right)^M. \tag{4}$$

Since $\mathrm{Gal}(K''/K') \simeq \mathrm{Hom}(W', \mu_M)$, it suffices to show that the kernel U of the map in (4) is annihilated by p^{c+2}. By Kummer theory U is isomorphic to $H^1(K'/K, \mu_M)$.

The extension K'/K is cyclic and a Herbrand quotient argument shows

$$\#H^1(K'/K,\mu_M) = \#H^0(K'/K,\mu_M) = \#\frac{\mu_M(K)}{N_{K'/K}(\mu_M)}.$$

From [20, Lem. 7] we deduce that $\#\mu_M(K)$ divides p^{c+2}. Hence U is annihilated by p^{c+2}.

Now that claim (b) and (c) are proved, the core argument runs precisely as in [10, pg.473/474] (using Greiter's notation the proof has to be adapted in the following way: $p^{c+2}\iota\psi$ has preimage $\gamma \in \mathrm{Gal}(K''/K')$; $\gamma_1 = p^{c+2}\left(\frac{\mathfrak{c}}{H/K}\right) \in \mathrm{Gal}(H/K)$; $\delta \in \mathrm{Gal}(K''H/K)$ with $\delta|_H = p^{2c+1}\gamma_1, \delta|_{K''} = p^{2c+1}\gamma$.)

$\square$

Recall the notation introduced at the beginning of this section. In addition, we let $\Delta = \mathrm{Gal}(K_0/k)$, $G_n = \mathrm{Gal}(K_n/k)$, $G_\infty = \mathrm{Gal}(K_\infty/k)$ and $\Gamma_n = \mathrm{Gal}(K_n/K_0)$. We fix a topological generator γ of $\Gamma = \mathrm{Gal}(K_\infty/K_0)$, and abbreviate the p^nth power of γ by γ_n.

For any abelian character χ of Δ we write $\Lambda_\chi = \mathbb{Z}_p(\chi)[[T]]$ for the usual Iwasawa algebra. Note that $\Lambda \otimes_{\mathbb{Z}_p[\Delta]} \mathbb{Z}_p(\chi) \simeq \mathbb{Z}_p(\chi)[[T]]$, so that our notation is consistent. We choose a generator $h_\chi \in \Lambda_\chi$ of $\mathrm{char}\left((\bar{\mathcal{E}}_\infty/\bar{\mathcal{C}}_\infty)_\chi\right)$. By the general theory of finitely generated Λ_χ-modules there is a quasi-isomorphism

$$\tau : A_{\infty,\chi} \longrightarrow \bigoplus_{i=1}^{k} \Lambda_\chi/(g_i)$$

with $g_i \in \Lambda_\chi$, and by definition, $\mathrm{char}(A_{\infty,\chi}) = (g)$ with $g := g_1 \cdots g_k$.

As in [10] we need the following lemmas providing the link to finite levels.

Lemma 3.5 *Let $\chi \neq 1$ be an abelian character of Δ. Then there exist constants $n_0 = n_0(F), c_i = c_i(F), i = 1,2$, a divisor h'_χ of h_χ (all independent of n) and G_n-homomorphisms*

$$\vartheta_n : \bar{\mathcal{E}}_{n,\chi} \longrightarrow \Lambda_{n,\chi} := \Lambda_\chi/(1-\gamma_n)\Lambda_\chi$$

such that

(i) h'_χ is relatively prime to $\gamma_n - 1$ for all n

(ii) $(\gamma_{n_0} - 1)^{c_1} p^{c_2} h'_\chi \Lambda_{n,\chi} \subseteq \vartheta_n(\mathrm{im}(\bar{\mathcal{C}}_{n,\chi}))$

where here $\mathrm{im}(\bar{\mathcal{C}}_{n,\chi})$ denotes the image of $\bar{\mathcal{C}}_{n,\chi}$ in $\bar{\mathcal{E}}_{n,\chi}$.

Proof We mainly follow Greither's proof of [10, Lem. 3.9].

We let

$$\pi_n : \bar{\mathcal{E}}_\infty/(1-\gamma_n)\bar{\mathcal{E}}_\infty \longrightarrow \bar{\mathcal{E}}_n$$

denote the canonical map and first prove

Claim 1: There exists an integer κ (independent of n) such that

$$(\gamma-1)p^{\kappa}\ker(\pi_n)=0 \text{ and } (\gamma-1)p^{\kappa}\mathrm{cok}(\pi_n)=0$$

This is shown as in Greither's proof of [10, Lem. 3.9]. He uses [25, Lem. 1.2], which is stated under the additional assumption $p\nmid|\Delta|$. As already remarked by Greither, this hypothesis is not necessary.

Next we define $U_\infty := \varprojlim_n U_n$ and proceed to prove

Claim 2 $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}U_\infty\simeq\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\Lambda=\Lambda[\frac{1}{p}]$.
This can be proved similarly as [18, Th. 11.2.5]. The assumption $p\nmid|\Delta|$ of loc.cit. is not needed, since we invert p. Alternatively, Claim 2 follows from [6, Prop. III.1.3], together with Exercise (iii) of [6, III.1.1].

It follows that $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}U_{\infty,\chi}$ is free cyclic over $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\Lambda_\chi=\Lambda_\chi[\frac{1}{p}]$. Since $\Lambda_\chi[\frac{1}{p}]$ is a principal ideal domain, the submodule $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\bar{\mathcal{E}}_{\infty,\chi}$ is also free cyclic over $\Lambda_\chi[\frac{1}{p}]$. It follows that there exists a pseudo-isomorphism

$$f:\bar{\mathcal{E}}_{\infty,\chi}\longrightarrow C:=\bigoplus_i\Lambda_\chi/p^{n_i}\Lambda_\chi\oplus\Lambda_\chi.$$

If we apply the snake lemma to the diagram

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \bar{\mathcal{E}}_{\infty,\chi} \arrow[r, "="] \arrow[d, "f"'] & \bar{\mathcal{E}}_{\infty,\chi} \arrow[r] \arrow[d, "pr\circ f=:\alpha"] & 0 \\
0 \arrow[r] & \oplus\Lambda_\chi/p^{n_i}\Lambda_\chi \arrow[r] & C \arrow[r, "pr"] & \Lambda_\chi \arrow[r] & 0
\end{tikzcd}$$

we see that $\ker(\alpha)$ is annihilated by some power of p and $\mathrm{cok}(\alpha)$ is finite.
We note that for any G_∞-module X one has

$$\left(X/(1-\gamma_n)X\right)_\chi\simeq X_\chi/(1-\gamma_n)X_\chi.$$

Let W_n denote the image of π_n and set $T:=\mathrm{Tor}_{\mathbb{Z}_p[\Delta]}(\mathrm{cok}(\pi_n),\mathbb{Z}_p(\chi))$. Then we have a commutative diagram (with exact lines)

$$\begin{tikzcd}
 & T \arrow[r, "\varphi"] & W_{n,\chi} \arrow[r] \arrow[d, "="] & \bar{\mathcal{E}}_{n,\chi} \arrow[r] & \mathrm{cok}(\pi_n)_\chi \arrow[r] & 0 \\
\ker(\pi_n)_\chi \arrow[r, "\tau"] & \frac{\bar{\mathcal{E}}_{\infty,\chi}}{(1-\gamma_n)\bar{\mathcal{E}}_{\infty,\chi}} \arrow[r, "\pi_n"] & W_{n,\chi} \arrow[r] & 0
\end{tikzcd}$$

We write $\pi_{n,\chi}$ for the composite map and obtain the exact sequence

$$0 \longrightarrow \ker(\pi_{n,\chi}) \longrightarrow \frac{\bar{\mathcal{E}}_{\infty,\chi}}{(1-\gamma_n)\bar{\mathcal{E}}_{\infty,\chi}} \xrightarrow{\pi_{n,\chi}} \bar{\mathcal{E}}_{n,\chi} \longrightarrow \operatorname{cok}(\pi_n)_\chi \longrightarrow 0$$

We claim that $\ker(\pi_{n,\chi})$ is annihilated by $(\gamma-1)^2p^{2\kappa}$: Let $e \in \ker(\pi_{n,\chi})$. Then

$$\begin{aligned} & \pi_n(e) = \varphi(t) \text{ for some } t \in \operatorname{Tor}_{\mathbb{Z}_p[\Delta]}(\operatorname{cok}(\pi_n), \mathbb{Z}_p(\chi)) \\ \Longrightarrow \quad & \pi_n((\gamma-1)p^\kappa e) = \varphi((\gamma-1)p^\kappa t) = 0 \\ \Longrightarrow \quad & \tau(c) = (\gamma-1)p^\kappa e \text{ for some } c \in \ker(\pi_n)_\chi \\ \Longrightarrow \quad & 0 = \tau((\gamma-1)p^\kappa c) = (\gamma-1)^2 p^{2\kappa} e \end{aligned}$$

So both $\ker(\pi_{n,\chi})$ and $\operatorname{cok}(\pi_{n,\chi})$ are annihilated by $(\gamma-1)^2p^{2\kappa}$.
Consider now the following commutative diagram

$$\begin{tikzcd} \bar{\mathcal{E}}_{\infty,\chi} \arrow[rr, "(\gamma-1)^4p^{4\kappa}\alpha"] \arrow[d, "\pi_{n,\chi}"] & & \Lambda_\chi \arrow[d] \\ \bar{\mathcal{E}}_{n,\chi} \arrow[rr, "\vartheta_n"] & & \Lambda_{n,\chi} = \Lambda_\chi/(1-\gamma_n)\Lambda_\chi \end{tikzcd}$$

where we define ϑ_n in the following manner: for $e \in \bar{\mathcal{E}}_{n,\chi}$ there exists $z \in \bar{\mathcal{E}}_{\infty,\chi}$ such that $\pi_{n,\chi}(z) = (\gamma-1)^2p^{2\kappa}e$. We then set

$$\vartheta_n(e) := (\gamma-1)^2p^{2\kappa}\alpha(z) (\text{mod } (1-\gamma_n)\Lambda_\chi).$$

On the other hand, we have the exact sequence

$$\bar{\mathcal{C}}_{\infty,\chi} \longrightarrow \bar{\mathcal{E}}_{\infty,\chi} \longrightarrow \left(\bar{\mathcal{E}}_\infty/\bar{\mathcal{C}}_\infty\right)_\chi \longrightarrow 0$$

so that

$$\bar{\mathcal{E}}_{\infty,\chi}/\operatorname{im}(\bar{\mathcal{C}}_{\infty,\chi}) \hookrightarrow \left(\bar{\mathcal{E}}_\infty/\bar{\mathcal{C}}_\infty\right)_\chi.$$

The structure theorem of Λ_χ-torsion modules implies that $h_\chi\left(\bar{\mathcal{E}}_\infty/\bar{\mathcal{C}}_\infty\right)_\chi$ is finite. Since $\alpha(\bar{\mathcal{E}}_{\infty,\chi})/\alpha(\operatorname{im}\bar{\mathcal{C}}_{\infty,\chi})$ is a quotient of $\bar{\mathcal{E}}_{\infty,\chi}/\operatorname{im}(\bar{\mathcal{C}}_{\infty,\chi})$, the module $h_\chi\left(\alpha(\bar{\mathcal{E}}_{\infty,\chi})/\alpha(\operatorname{im}\bar{\mathcal{C}}_{\infty,\chi})\right)$ is also finite. Since $\operatorname{cok}(\alpha)$ is finite, there exists a power p^s such that $p^s \in \alpha(\bar{\mathcal{E}}_{\infty,\chi})$ and $p^s h_\chi \alpha(\bar{\mathcal{E}}_{\infty,\chi}) \subseteq \alpha(\operatorname{im}(\bar{\mathcal{C}}_{\infty,\chi})))$. Therefore $p^{2s}h_\chi \in \alpha(\operatorname{im}(\bar{\mathcal{C}}_{\infty,\chi}))$ and we conclude further:

$$\begin{aligned} & p^{2s+4\kappa}(\gamma-1)^4h_\chi = p^{4\kappa}(\gamma-1)^4\alpha(z) \text{ for some } z \in \operatorname{im}(\bar{\mathcal{C}}_{\infty,\chi}) \\ \Longrightarrow \quad & \vartheta_n(z_n) = p^{2s+4\kappa}(\gamma-1)^4h_\chi \text{ with } z_n = \pi_{n,\chi}(z) \in \operatorname{im}(\bar{\mathcal{C}}_{n,\chi}) \\ \Longrightarrow \quad & p^{2s+4\kappa}(\gamma-1)^4h_\chi\Lambda_{n,\chi} \subseteq \vartheta_n(\operatorname{im}(\bar{\mathcal{C}}_{n,\chi})) \end{aligned} \tag{5}$$

Since $\gamma_n - 1$ divides $\gamma_{n+1}-1$ for all n there exists a positive integer n_0 and a divisor h'_χ of h_χ such that h_χ divides $(\gamma_{n_0}-1)h'_\chi$ and such that h'_χ is relatively prime with $\gamma_n - 1$ for all n. The assertions of the lemma are now immediate from (5).

$\square$

Lemma 3.6 *Let $\chi \neq 1$ be a character of Δ. Then there exists a constant $c_3 = c_3(F)$ (independent of n) and G_n-homomorphisms*

$$\tau_n : A_{n,\chi} \longrightarrow \bigoplus_{i=1}^{k} \Lambda_{n,\chi}/(\bar{g}_i)$$

such that $p^{c_3}\mathrm{cok}\tau_n = 0$ for all $n \geq 0$. Here $\bar{g}_i$ denotes the image of $g_i \in \Lambda_\chi$ in $\Lambda_{n,\chi}$.

Proof The proof is identical to Greither's proof of [10, Lem. 3.10]. It is based on the following sublemma which will be used again at the end of the section.

$\square$

Lemma 3.7 *For $n \geq 0$ the kernel and cokernel of multiplication with $\gamma_n - 1$ on A_∞ are finite.*

Proof See [25, pg. 705]. It is remarkable that one uses the known validity of Leopoldt's conjecture in this proof. $\square$

The following technical lemma is the analogue of [10, Lem. 3.12].

Lemma 3.8 *Let K/k be an abelian extension, $G = \mathrm{Gal}(K/k)$ and Δ a subgroup of G. Let χ denote a character of Δ, M a power of p, $\mathfrak{a} = \mathfrak{l}_1 \cdots \mathfrak{l}_i \in \mathcal{S}_{M,K}$. Let $\mathfrak{l} = \mathfrak{l}_i$ and let λ be a fixed prime divisor of $\mathfrak{l}$ in K. We write $\mathfrak{c}$ for the class of λ and assume that $\mathfrak{c} \in A = A(K)$, where as usual $A(K)$ denotes the p-Sylow subgroup of the ideal class group of K.*
Let $B \subseteq A$ denote the subgroup generated by classes of prime divisors of $\mathfrak{l}_1, \dots, \mathfrak{l}_{i-1}$. Let $x \in K^\times/\left(K^\times\right)^M$ such that $[x]_\mathfrak{q} = 0$ for all primes $\mathfrak{q}$ not dividing $\mathfrak{a}$, and let $W \subseteq K^\times/\left(K^\times\right)^M$ denote the $\mathbb{Z}_p[G]$-span of x. Assume that there exist elements

$$E, g, \eta \in \mathbb{Z}_p[G]$$

satisfying

(i) $E \cdot \mathrm{ann}_{(\mathbb{Z}_p[G])_\chi}(\bar{\mathfrak{c}}_\chi) \subseteq g \cdot \left(\mathbb{Z}_p[G]\right)_\chi$, where $\bar{\mathfrak{c}}_\chi$ is the image of $\mathfrak{c}$ under $A \to A/B \to (A/B)_\chi$.

(ii) $\#\left(\left(\mathbb{Z}_p[G]\right)_\chi / g \left(\mathbb{Z}_p[G]\right)_\chi\right) < \infty$

(iii) $M \geq |A_\chi| \left| \eta \left(\frac{J_\mathfrak{l}/MJ_\mathfrak{l}}{[W]_\mathfrak{l}}\right)_\chi \right|$, where $[W]_\mathfrak{l}$ denotes the subgroup of $J_\mathfrak{l}/MJ_\mathfrak{l}$ generated by elements $[w]_\mathfrak{l}, w \in W$.

Then there exists a G-homomorphism

$$\psi : W_\chi \longrightarrow ((\mathbb{Z}/M\mathbb{Z})[G])_\chi$$

such that

$$g\psi(x)\lambda_\chi = (E \cdot \eta[x]_{\mathfrak{l}})_\chi$$

in $(J_{\mathfrak{l}}/MJ_{\mathfrak{l}})_\chi$.

Proof Completely analoguous to the proof of [10, Lem. 3.12]. ∎

We will now sketch the main argument of the proof of Theorem 3.1. We fix a natural number $n \geq 1$ and let $K = K_n = Fk_n$. We view Δ as a subgroup of $G = \mathrm{Gal}(K/k)$.
We let M denote a large power of p which we will specify in course of the proof. By Lemma 3.6 there exists for each $i = 1, \ldots, k$ an ideal class $\mathfrak{c}_i \in A_\chi$ such that

$$\tau_n(\mathfrak{c}_i) = (0, \ldots, 0, p^{c_3}, 0, \ldots, 0)$$

in $\bigoplus_{i=1}^k \Lambda_{n,\chi}/(\bar{g}_i)$ with p^{c_3} at the ith position. Choose $\mathfrak{c}_{k+1}$ arbitrary. By Lemma 3.5 there exists an element $\xi' \in \mathrm{im}(\bar{\mathcal{C}}_{n,\chi})$ such that $\vartheta_n(\xi') = (\gamma_{n_0} - 1)^{c_1} p^{c_2} h'_\chi$ in $\Lambda_{n,\chi}$. It is now easy to show that there exists an actual elliptic unit $\xi \in \mathcal{C}_n$ such that

$$\vartheta_n(\xi) = (\gamma_{n_0} - 1)^{c_1} p^{c_2} h'_\chi (\mathrm{mod}\ M\Lambda_{n,\chi}). \tag{6}$$

By [24, Prop. 1.2] there exists an Euler system $\alpha \in \mathcal{U}_{K,M}$ such that $\alpha(1) = \xi$. Set $d := 3c+3$, where c was defined in Theorem 3.4. Following Greither we will use Theorem 3.4 to construct inductively prime ideals λ_i of K, $1 \leq i \leq k+1$, such that

(a) $[\lambda_i]_\chi = p^d \mathfrak{c}_i$

(b) $\mathfrak{l}_i = \lambda_i \cap k \subseteq \mathcal{S}_{M,K}$

(c) one has the equalities

$$\begin{aligned} (v_{\lambda_1}(\kappa(\mathfrak{l}_1)))_\chi &= u_1|\Delta|(\gamma_{n_0} - 1)^{c_1} p^{d+c_2} h'_\chi, \\ (g_{i-1} v_{\lambda_i}(\kappa(\mathfrak{l}_1 \cdots \mathfrak{l}_i)))_\chi &= u_i|\Delta|(\gamma_{n_0} - 1)^{c_1^{i-1}} p^{d+c_3} \left(v_{\lambda_{i-1}}(\kappa(\mathfrak{l}_1 \cdots \mathfrak{l}_{i-1}))\right)_\chi \end{aligned}$$

for $2 \leq i \leq k+1$. These are equalities in $\Lambda_{n,\chi}/M\Lambda_{n,\chi}$. The elements u_i are units in $\mathbb{Z}/M\mathbb{Z}$ and $v_\lambda(x) \in (\mathbb{Z}/M\mathbb{Z})[G] \simeq \Lambda_n/M\Lambda_n$ is defined by $v_\lambda(x)\lambda = [x]_{\mathfrak{l}}$ in $J_{\mathfrak{l}}/MJ_{\mathfrak{l}}$, if $\mathfrak{l} = \lambda \cap k \in \mathcal{L}_{M,K}$.

We briefly descibe this induction process. For $i = 1$ we let $\mathfrak{c} \in A$ be a preimage of $\mathfrak{c}_1$ under the canonical epimorphism $A \to A_\chi$. We apply Theorem 3.4 with the data $\mathfrak{c}$, $W = \mathcal{E}/\mathcal{E}^M$ (with $\mathcal{E} := \mathcal{O}_K^\times$) and

$$\psi : W \xrightarrow{v} \bar{\mathcal{E}}_{n,\chi}/\bar{\mathcal{E}}_{n,\chi}^M \xrightarrow{\vartheta_n} \Lambda_{n,\chi}/M\Lambda_{n,\chi} \xrightarrow{\varepsilon_\chi} (\mathbb{Z}/M\mathbb{Z})[G]$$

where $v \in (\mathbb{Z}/M\mathbb{Z})^\times$ is such that each unit $x \in K \otimes k_\mathfrak{p}$ satisfies $x^v \equiv 1$ modulo all primes above $\mathfrak{p}$. The map ε_χ is defined in [10, Lemma 3.13]. Theorem 3.4 provides a prime ideal $\lambda = \lambda_1$ which obviously satisfies (a) and (b) and, in addition,

$$\varphi_\mathfrak{l}(w) = p^d u\psi(w)\lambda \text{ for all } w \in \mathcal{E}/\mathcal{E}^M.$$

From this equality we conclude further

$$\begin{aligned} v_\lambda(\kappa(\mathfrak{l}))\lambda &= [\kappa(\mathfrak{l})]_\mathfrak{l} = \varphi_\mathfrak{l}(\kappa(1)) = \varphi_\mathfrak{l}(\xi) \\ &= p^d u\psi(\xi)\lambda = \left(p^d uv(\varepsilon_\chi \circ \vartheta_n)(\xi)\right)\lambda \end{aligned}$$

in $J_\mathfrak{l}/MJ_\mathfrak{l} = (\mathbb{Z}/M\mathbb{Z})[G]\lambda$. Projecting the equality $v_\lambda(\kappa(\mathfrak{l})) = p^d uv(\varepsilon_\chi \circ \vartheta_n)(\xi)$ to $((\mathbb{Z}/M\mathbb{Z})[G])_\chi = \Lambda_{n,\chi}/M\Lambda_{n,\chi}$ and using [10, Lemma 3.13] together with (6) we obtain equality (c) for $i = 1$.
For the induction step $i-1 \mapsto i$ we set $\mathfrak{a}_{i-1} := \mathfrak{l}_1 \cdots \mathfrak{l}_{i-1}$. Using (c) inductively we see that $\left(v_{\lambda_{i-1}}(\kappa(\mathfrak{a}_{i-1}))\right)_\chi$ divides

$$\left(\underbrace{|\Delta|^{i-1}p^{(i-2)(d+c_3)+(d+c_2)}}_{=:D_i}(\gamma_{n_0}-1)^{c_1+\sum_{s=1}^{i-2}c_1^s}h'_\chi\right)_\chi.$$

Without loss of generality we may assume that $c_1 \geq 2$. Then one has $c_1 + \sum_{i=1}^{i-2} c_1^s \leq c_1^{i-1}$, so that $\left(v_{\lambda_{i-1}}(\kappa(\mathfrak{a}_{i-1}))\right)_\chi$ also divides $D_i(\gamma_{n_0}-1)^{t_i}h'_\chi$ with $t_i := c_1^{i-1}$. The module

$$N = (\gamma_{n_0}-1)^{t_i}\left(J_{\mathfrak{l}_{i-1}}/\left(M, [\kappa(\mathfrak{a}_{i-1})]_{\mathfrak{l}_{i-1}}\right)\right)_\chi$$

is a cyclic as a $\Lambda_{n,\chi}$-module and annihilated by $D_i h'_\chi$. Consequently

$$|N| \leq |\Lambda_{n,\chi}/(D_i)| \cdot |\Lambda_{n,\chi}/(h'_\chi)|.$$

Note that by the definition of h'_χ the quotient $\Lambda_{n,\chi}/(h'_\chi)$ is finite. If we choose M such that

$$M \geq \max\left(|A_\chi| \cdot |\Lambda_{n,\chi}/(D_{k+1})| \cdot |\Lambda_{n,\chi}/(h'_\chi)|, p^n\right)$$

then one has $|N| \leq M|A_\chi|^{-1}$.
We now apply Lemma 3.8 with $\mathfrak{a} = \mathfrak{a}_{i-1}, g = g_{i-1}, x = \kappa(\mathfrak{a}_{i-1}), E = p^{c_3}$ and $\eta = (\gamma_{n_0}-1)^{t_i}$. Following Greither it is straight forward to check the hypothesis (a), (b) and (c) of Lemma 3.8. Note that for (b) one has to use the fact that $\mathrm{char}(A_{\infty,\chi})$ is relatively prime to $\gamma_n - 1$ for all n, which is an immediate consequence of Lemma 3.5. We let W denote the $\mathbb{Z}_p[G]$-span of $\kappa(\mathfrak{a}_{i-1})$ in $K^\times/(K^\times)^M$ and obtain a homomorphism

$$\psi_i : W_\chi \longrightarrow ((\mathbb{Z}/M\mathbb{Z})[G])_\chi$$

such that $g_{i-1}\psi_i(\kappa(\mathfrak{a}_{i-1})) = \left(p^{c_3}(\gamma_{n_0}-1)^{t_i}v_{\lambda_{i-1}}(\kappa(\mathfrak{a}_{i-1}))\right)_\chi$. We let $\mathfrak{c}$ denote a preimage of $\mathfrak{c}_i$ and consider the homomorphism

$$\psi : W \longrightarrow W_\chi \xrightarrow{\psi_i} \Lambda_{n,\chi}/M\Lambda_{n,\chi} \xrightarrow{\varepsilon_\chi} (\mathbb{Z}/M\mathbb{Z})\,[G]$$

We again apply Theorem 3.4 and obtain λ_i satifying (a), (b) and also

$$\varphi_{\mathfrak{l}_i}(\kappa(\mathfrak{a}_{i-1})) = p^d u\psi(\kappa(\mathfrak{a}_{i-1}))\lambda_i.$$

As in the case $i = 1$ one now establishes equality (c). This concludes the inductive construction of $\lambda_1, \ldots, \lambda_{k+1}$.
Using (c) successively we obtain (suppressing units in $\mathbb{Z}/M\mathbb{Z}$)

$$\left(g_1 \cdots g_k v_{\lambda_{k+1}}(\kappa(\mathfrak{l}_1 \cdots \mathfrak{l}_{k+1}))\right) = \eta h'_\chi$$

(as an equality in $\Lambda_{n,\chi}/M\Lambda_{n,\chi}$) with

$$\eta = \left(|\Delta|^{k+1} p^{k(d+c_3)+d+c_2}(\gamma_{n_0}-1)^{c_1+\sum_{s=1}^k c_1^s}\right)_\chi.$$

Therefore $g = g_1 \cdots g_k$ divides $\eta h'_\chi$ in $\Lambda_{n,\chi}/M\Lambda_{n,\chi}$, and since $p^n \mid M$ we also see that g divides $\eta h'_\chi$ in $\Lambda_{n,\chi}/p^n\Lambda_{n,\chi}$. As in [31, page 371, last but one paragraph] we deduce that g divides $\eta h'_\chi$ in Λ_χ.
By [6, III.2.1, Theorem] (together with [6, III.1.7, (13)]) we know that the μ-invariant of $A_{\infty,\chi}$ is trivial. Hence $g = \mathrm{char}(A_{\infty,\chi})$ is coprime with p. By Lemma 3.7 it is also coprime with $\gamma_{n_0} - 1$, and consequently $|\Lambda_\chi/(g,\eta)| < \infty$. Therefore there exist $\alpha, \beta \in \Lambda_\chi$ and $N \in \mathbb{N}$ such that $p^N = \alpha g + \beta\eta$ and we see that g divides $p^N h'_\chi$. Since g is prime to p we obtain $g \mid h'_\chi$.

Remark 3.9 There are several steps in the proof where we use the assumption that p splits in $k/\mathbb{Q}$. Among these the vanishing of $\mu(A_{\infty,\chi})$ is most important. The proof of this uses an important result of Gillard [9]. If p is not split in $k/\mathbb{Q}$ our knowledge about $\mu(A_{\infty,\chi})$ seems to be quite poor.

4 The conjecture

In this section we fix an integral $\mathcal{O}_k$-ideal $\mathfrak{f}$ such that $w(\mathfrak{f}) = 1$ and write

$$M = h^0(\mathrm{Spec}(k(\mathfrak{f}))), \quad A = \mathbb{Q}[G_\mathfrak{f}], \quad \mathfrak{A} = \mathbb{Z}[G_\mathfrak{f}],$$

where for any $\mathcal{O}_k$-ideal $\mathfrak{m}$ we let $G_\mathfrak{m}$ denote the Galois group $\mathrm{Gal}(k(\mathfrak{m})/k)$.
For any commutative ring R we write $\mathcal{D}(R)$ for the derived category of the homotopy category of bounded complexes of R-modules and $\mathcal{D}^p(R)$ for the full triangulated subcategory of perfect complexes of R-modules. We write $\mathcal{D}^{pis}(R)$ for the subcategory of $\mathcal{D}^p(R)$ in which the objects are the same, but the morphisms are restricted to quasi-isomorphisms.

We let $\mathcal{P}(R)$ denote the category of graded invertible R-modules. If R is reduced, we write Det_R for the functor from $\mathcal{D}^{pis}(R)$ to $\mathcal{P}(R)$ introduced by Knudsen and Mumford [14]. To be more precise, we define

$$\mathrm{Det}_R(P) := \left(\bigwedge_R^{\mathrm{rk}_R(P)} P, \mathrm{rk}_R(P) \right) \quad \in \quad \mathrm{Ob}(\mathcal{P}(R))$$

for any finitely generated projective R-module P and for a bounded complex $P^\bullet$ of such modules we set

$$\mathrm{Det}_R(P^\bullet) := \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_R^{(-1)^i}(P^i).$$

If R is reduced, then this functor extends to a functor from $\mathcal{D}^{pis}(R)$ to $\mathcal{P}(R)$. For more information and relevant properties the reader is refered to [5, §2], or the original papers [14] and [15].

For any finite set S of places of k we define $Y_S = Y_S(k(\mathfrak{f})) = \oplus_{w\in S(k(\mathfrak{f}))}\mathbb{Z}w$. Here $S(k(\mathfrak{f}))$ denotes the set of places of $k(\mathfrak{f})$ lying above places in S. We let $X_S = X_S(k(\mathfrak{f}))$ denote the kernel of the augmentation map $Y_S \to \mathbb{Z}, w \mapsto 1$. The fundamental line $\Xi({}_AM)$ is given by

$$\Xi({}_AM)^\# = \mathrm{Det}_A^{-1}\left(\mathcal{O}_{k(\mathfrak{f})}^\times \otimes_\mathbb{Z}\mathbb{Q}\right) \otimes_A \mathrm{Det}_A\left(X_{\{v|\infty\}}\otimes_\mathbb{Z}\mathbb{Q}\right),$$

where the superscript $\#$ means twisting the action of $G_\mathfrak{f}$ by $g \mapsto g^{-1}$. We let

$$\begin{array}{rcl} R = R_{k(\mathfrak{f})} : \mathcal{O}_{k(\mathfrak{f})}^\times \otimes_\mathbb{Z}\mathbb{R} & \longrightarrow & X_{\{v|\infty\}}\otimes_\mathbb{Z}\mathbb{R}, \\ u & \mapsto & -\sum_{v|\infty} \log|u|_v \cdot v \end{array}$$

denote the Dirichlet regulator map. Let

$${}_A\vartheta_\infty : \mathbb{R}[G_\mathfrak{f}] \longrightarrow \Xi({}_AM)^\# \otimes_\mathbb{Q}\mathbb{R}$$

be the inverse of the canonical isomorphism

$$\begin{array}{rl} & \mathrm{Det}_{\mathbb{R}[G_\mathfrak{f}]}^{-1}\left(\mathcal{O}_{k(\mathfrak{f})}^\times\otimes_\mathbb{Z}\mathbb{R}\right) \otimes_{\mathbb{R}[G_\mathfrak{f}]} \mathrm{Det}_{\mathbb{R}[G_\mathfrak{f}]}\left(X_{\{v|\infty\}}\otimes_\mathbb{Z}\mathbb{R}\right) \\ \xrightarrow{\det(R)\otimes 1} & \mathrm{Det}_{\mathbb{R}[G_\mathfrak{f}]}^{-1}\left(X_{\{v|\infty\}}\otimes_\mathbb{Z}\mathbb{R}\right) \otimes_{\mathbb{R}[G_\mathfrak{f}]} \mathrm{Det}_{\mathbb{R}[G_\mathfrak{f}]}\left(X_{\{v|\infty\}}\otimes_\mathbb{Z}\mathbb{R}\right) \\ \xrightarrow{\mathrm{eval}} & (\mathbb{R}[G_\mathfrak{f}], 0). \end{array}$$

Following [19] we define for integral $\mathcal{O}_k$-ideals $\mathfrak{g}, \mathfrak{g}_1$ with $\mathfrak{g} \mid \mathfrak{g}_1$ and each abelian character η of $G_\mathfrak{g} \simeq \mathrm{cl}(\mathfrak{g})$ ($\mathrm{cl}(\mathfrak{g})$ denoting the ray class group modulo $\mathfrak{g}$)

$$S_\mathfrak{g}(\eta, \mathfrak{g}_1) = \sum_{c\in\mathrm{cl}(\mathfrak{g}_1)} \eta(c^{-1}) \log|\varphi_\mathfrak{g}(c)|,$$

where η is regarded as a character of $\mathrm{cl}(\mathfrak{g}_1)$ via inflation. For the definition of the ray class invariants $\varphi_{\mathfrak{g}}(c)$ we choose an integral ideal $\mathfrak{c}$ in the class c and set

$$\varphi_{\mathfrak{g}}(c) = \varphi_{\mathfrak{g}}(\mathfrak{c}) = \begin{cases} \varphi^{12N(\mathfrak{g})}(1;\mathfrak{g}\mathfrak{c}^{-1}), & \text{if } \mathfrak{g} \neq 1, \\ \left| \frac{N(\mathfrak{c}^{-1})^6 \Delta(\mathfrak{c}^{-1})}{(2\pi)^{12}} \right|, & \text{if } \mathfrak{g} = 1, \end{cases}$$

where φ was defined in (1). Note that this definition does not depend on the choice of the ideal $\mathfrak{c}$ (see [20, pp. 15/16]).

For an abelian character η of $\mathrm{cl}(\mathfrak{g})$ we write $\mathfrak{f}_\eta$ for its conductor. We write $L^*(\eta)$ for the leading term of the Taylor expansion of the Dirichlet L-function $L(s,\eta)$ at $s=0$.

From [20, Th. 3] and the functional equation satisfied by Dirichlet L-functions we deduce

$$L^*(\eta^{-1}) = -\frac{S_{\mathfrak{f}_\eta}(\eta, \mathfrak{f}_\eta)}{6N(\mathfrak{f}_\eta)w(\mathfrak{f}_\eta)}. \tag{7}$$

We denote by $\hat{G}_{\mathfrak{f}}^{\mathbb{Q}}$ the set of $\mathbb{Q}$-rational characters associated with the $\mathbb{Q}$-irreducible representations of $G_{\mathfrak{f}}$. For $\chi \in \hat{G}_{\mathfrak{f}}^{\mathbb{Q}}$ we set $e_\chi = \sum_{\eta\in\chi} e_\eta \in A$, where we view χ as an $\mathrm{Gal}(\mathbb{Q}^c/\mathbb{Q})$-orbit of absolutely irreducible characters of $G_{\mathfrak{f}}$. Then the Wedderburn decompostion of A is given by

$$A \simeq \prod_{\chi\in\hat{G}_{\mathfrak{f}}^{\mathbb{Q}}} \mathbb{Q}(\chi). \tag{8}$$

Here, by a slight abuse of notation, $\mathbb{Q}(\chi)$ denotes the extension generated by the values of η for any $\eta \in \chi$. For any character $\chi \in \hat{G}_{\mathfrak{f}}^{\mathbb{Q}}$ the conductor $\mathfrak{f}_\chi$, defined by $\mathfrak{f}_\chi := \mathfrak{f}_\eta$ for any $\eta \in \chi$, is well defined.

We put $L^*(\chi) := \sum_{\eta\in\chi} L^*(\eta)e_\eta$ and note that $L^*(\chi)^{\#} := \sum_{\eta\in\chi} L^*(\eta^{-1})e_\eta$. The statement $L^*(\chi)^{\#} \in Ae_\chi$ (compare to [8, page 8]) is not obvious, but needs to be proved. This is essentially Stark's conjecture.

We fix a prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$ and also choose an auxiliary ideal $\mathfrak{a}$ of $\mathcal{O}_k$ such that $(\mathfrak{a}, 6\mathfrak{f}\mathfrak{p}) = 1$. For each $\eta \neq 1$ we define elements

$$\xi_\eta := \begin{cases} \psi(1;\mathfrak{f}_\eta,\mathfrak{a}), & \text{if } \mathfrak{f}_\eta \neq 1, \\ \frac{\delta(\mathcal{O}_k,\mathfrak{a}^{-1})}{\delta(\mathfrak{p},\mathfrak{p}\mathfrak{a}^{-1})}, & \text{if } \mathfrak{f}_\eta = 1, \eta \neq 1, \end{cases} \tag{9}$$

where δ denotes the function of lattices defined in [21, Th. 1]. We set $\xi_\chi := \xi_\eta$ for any $\eta \in \chi$.

We fix an embedding $\sigma : \mathbb{Q}^c \hookrightarrow \mathbb{C}$ and write $w_\infty = \sigma|_{k(\mathfrak{f})}$. A standard computation leads to

$$\begin{aligned} & R(e_\eta\xi_\eta) \\ = & \begin{cases} (N\mathfrak{a} - \eta(\mathfrak{a}))w(\mathfrak{f}_\eta)[k(\mathfrak{f}):k(\mathfrak{f}_\eta)]L^*(\eta^{-1})e_\eta w_\infty, & \mathfrak{f}_\eta \neq 1, \\ (1-\eta(\mathfrak{p})^{-1})(N\mathfrak{a} - \eta(\mathfrak{a}))w(1)[k(\mathfrak{f}):k(1)]L^*(\eta^{-1})e_\eta w_\infty, & \mathfrak{f}_\eta = 1, \eta \neq 1. \end{cases} \end{aligned} \tag{10}$$

For the reader's convenience we briefly sketch the computation for characters $\eta \neq 1$ with $\mathfrak{f}_\eta = 1$. By definition of the Dirichlet regulator map and [21, Cor. 2] we obtain

$$R(e_\eta \xi_\eta) = -\frac{1}{6}[k(\mathfrak{f}) : k(1)] \sum_{\mathfrak{c} \in \mathrm{cl}(1)} \log \left| \frac{\Delta(\mathfrak{c})^{N\mathfrak{a}} \Delta(\mathfrak{a}^{-1}\mathfrak{c}\mathfrak{p})}{\Delta(\mathfrak{a}^{-1}\mathfrak{c}) \Delta(\mathfrak{c}\mathfrak{p})^{N\mathfrak{a}}} \right| \eta(\mathfrak{c}) e_\eta w_\infty. \qquad (11)$$

Since $\sum_{\mathfrak{c} \in \mathrm{cl}(1)} C\eta(\mathfrak{c}) = 0$ for any constant C we compute further

$$\begin{aligned}
&\sum_{\mathfrak{c} \in \mathrm{cl}(1)} \log \left| \frac{\Delta(\mathfrak{c})^{N\mathfrak{a}} \Delta(\mathfrak{a}^{-1}\mathfrak{c}\mathfrak{p})}{\Delta(\mathfrak{a}^{-1}\mathfrak{c}) \Delta(\mathfrak{c}\mathfrak{p})^{N\mathfrak{a}}} \right| \eta(\mathfrak{c}) \\
= \;& \sum_{\mathfrak{c} \in \mathrm{cl}(1)} \log \left| \left(\frac{(N\mathfrak{c})^6 \Delta(\mathfrak{c})}{(2\pi)^{12}} \right)^{N\mathfrak{a}} \right| \eta(\mathfrak{c}) + \sum_{\mathfrak{c} \in \mathrm{cl}(1)} \log \left| \frac{(N\mathfrak{a}^{-1}\mathfrak{c}\mathfrak{p})^6 \Delta(\mathfrak{a}^{-1}\mathfrak{c}\mathfrak{p})}{(2\pi)^{12}} \right| \eta(\mathfrak{c}) - \\
& - \sum_{\mathfrak{c} \in \mathrm{cl}(1)} \log \left| \left(\frac{(N\mathfrak{c}\mathfrak{p})^6 \Delta(\mathfrak{c}\mathfrak{p})}{(2\pi)^{12}} \right)^{N\mathfrak{a}} \right| \eta(\mathfrak{c}) - \sum_{\mathfrak{c} \in \mathrm{cl}(1)} \log \left| \frac{(N\mathfrak{a}^{-1}\mathfrak{c})^6 \Delta(\mathfrak{a}^{-1}\mathfrak{c})}{(2\pi)^{12}} \right| \eta(\mathfrak{c}) \\
= \;& N\mathfrak{a} \sum_{\mathfrak{c} \in \mathrm{cl}(1)} \log \left| \varphi_1(\mathfrak{c}^{-1}) \right| \eta(\mathfrak{c}) + \sum_{\mathfrak{c} \in \mathrm{cl}(1)} \log \left| \varphi_1(\mathfrak{a}\mathfrak{c}^{-1}\mathfrak{p}^{-1}) \right| \eta(\mathfrak{c}) - \\
& - N\mathfrak{a} \sum_{\mathfrak{c} \in \mathrm{cl}(1)} \log \left| \varphi_1(\mathfrak{p}^{-1}\mathfrak{c}^{-1}) \right| \eta(\mathfrak{c}) - \sum_{\mathfrak{c} \in \mathrm{cl}(1)} \log \left| \varphi_1(\mathfrak{a}\mathfrak{c}^{-1}) \right| \eta(\mathfrak{c}).
\end{aligned}$$

Recalling that $\varphi_\mathfrak{g}(c)$ is a class invariant we obtain

$$\sum_{\mathfrak{c} \in \mathrm{cl}(1)} \log \left| \frac{\Delta(\mathfrak{c})^{N\mathfrak{a}} \Delta(\mathfrak{a}^{-1}\mathfrak{c}\mathfrak{p})}{\Delta(\mathfrak{a}^{-1}\mathfrak{c}) \Delta(\mathfrak{c}\mathfrak{p})^{N\mathfrak{a}}} \right| \eta(\mathfrak{c}) = (N\mathfrak{a} - \eta(\mathfrak{a}))(1 - \eta(\mathfrak{p})^{-1}) S_1(\eta, 1) e_\eta w_\infty,$$

so that (10) is an immediate consequence of (7) and (11).
According to the decomposition (8) we decompose $\Xi({}_A M)^\#$ character by character and obtain a canonical isomorphism

$$\Xi({}_A M)^\# \longrightarrow \left(\prod_{\chi \in \hat{G}_\mathfrak{f}^\mathbb{Q}} \left(\mathrm{Det}^{-1}_{\mathbb{Q}(\chi)} (\mathcal{O}^\times_{k(\mathfrak{f})} \otimes_\mathfrak{A} \mathbb{Q}(\chi)) \otimes_{\mathbb{Q}(\chi)} \mathrm{Det}_{\mathbb{Q}(\chi)} (X_{\{v|\infty\}} \otimes_\mathfrak{A} \mathbb{Q}(\chi)) \right) \right).$$

As in the cyclotomic case one has

$$\dim_{\mathbb{Q}(\chi)} e_\chi \left(\mathcal{O}^\times_{k(\mathfrak{f})} \otimes_\mathfrak{A} \mathbb{Q}(\chi) \right) = \dim_{\mathbb{Q}(\chi)} e_\chi \left(X_{\{v|\infty\}} \otimes_\mathfrak{A} \mathbb{Q}(\chi) \right) = \begin{cases} 1, & \chi \neq 1, \\ 0, & \chi = 1. \end{cases} \qquad (12)$$

Upon recalling that $\mathrm{Det}_\mathbb{Q}(0) = (\mathbb{Q}, 0)$ in $\mathcal{P}(\mathbb{Q})$ we get a canonical isomorphism

$$\Xi({}_A M)^\# \quad \longrightarrow \quad \mathbb{Q} \times \left(\prod_{\chi \neq 1} \left((\mathcal{O}^\times_{k(\mathfrak{f})} \otimes_\mathfrak{A} \mathbb{Q}(\chi))^{(-1)} \otimes_{\mathbb{Q}(\chi)} (X_{\{v|\infty\}} \otimes_\mathfrak{A} \mathbb{Q}(\chi)) \right) \right).$$

From (10) we deduce

$$\begin{aligned}&\left({}_A\vartheta_\infty(L^*({}_AM,0)^{-1})\right)_\chi\\ =&\begin{cases} w(\mathfrak{f}_\chi)[k(\mathfrak{f}):k(\mathfrak{f}_\chi)](N\mathfrak{a}-\sigma(\mathfrak{a}))e_\chi\xi_\chi^{-1}\otimes w_\infty, & \mathfrak{f}_\chi\neq 1,\\ w(1)[k(\mathfrak{f}):k(1)](1-\sigma(\mathfrak{p})^{-1})(N\mathfrak{a}-\sigma(\mathfrak{a}))e_\chi\xi_\chi^{-1}\otimes w_\infty, & \mathfrak{f}_\chi=1,\chi\neq 1\\ L(\chi,0)^{-1}, & \chi=1.\end{cases}\end{aligned}$$

In particular, this proves the equivariant version of [8, Conjecture 2].
We fix a prime p and put $A_p := A\otimes_{\mathbb{Q}}\mathbb{Q}_p = \mathbb{Q}_p[G_\mathfrak{f}]$, $\mathfrak{A}_p := \mathfrak{A}\otimes_{\mathbb{Z}}\mathbb{Z}_p = \mathbb{Z}_p[G_\mathfrak{f}]$. Let $S = S_{\mathrm{ram}}\cup S_\infty$ be the union of the set of ramified places and the set of archimedean places of k. Let $S_p = S\cup\{\mathfrak{p}\mid p\}$ and put

$$\Delta(k(\mathfrak{f})) := R\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{k(\mathfrak{f}),S_p},\mathbb{Z}_p),\mathbb{Z}_p)[-3]$$

Then $\Delta(k(\mathfrak{f}))$ can be represented by a perfect complex of $\mathfrak{A}_p$-modules whose cohomology groups $H^i(\Delta(k(\mathfrak{f}))$ are trivial for $i\neq 1,2$. For $i=1$ one finds

$$H^1(\Delta(k(\mathfrak{f}))\simeq \mathcal{O}_{k(\mathfrak{f}),S_p}^\times\otimes_{\mathbb{Z}}\mathbb{Z}_p,$$

and H^2 fits into an short exact sequence

$$0\longrightarrow \mathrm{Pic}(\mathcal{O}_{k(\mathfrak{f}),S_p})\otimes_{\mathbb{Z}}\mathbb{Z}_p\longrightarrow H^2(\Delta(k(\mathfrak{f})))\longrightarrow X_{\{w\mid\mathfrak{f}p\infty\}}\otimes_{\mathbb{Z}}\mathbb{Z}_p\longrightarrow 0$$

We have an isomorphism

$${}_A\vartheta_p : \Xi({}_AM)^{\#}\otimes_{\mathbb{Q}}\mathbb{Q}_p\longrightarrow \mathrm{Det}_{A_p}\left(\Delta(k(\mathfrak{f}))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\right)$$

given by the composite

$$\begin{aligned}&\mathrm{Det}_{A_p}^{-1}(\mathcal{O}_{k(\mathfrak{f})}^\times\otimes_{\mathbb{Z}}\mathbb{Q}_p)\otimes_{A_p}\mathrm{Det}_{A_p}(X_{\{v\mid\infty\}}\otimes_{\mathbb{Z}}\mathbb{Q}_p)\\ \xrightarrow{\varphi_1}\;&\mathrm{Det}_{A_p}^{-1}(\mathcal{O}_{k(\mathfrak{f}),S_p}^\times\otimes_{\mathbb{Z}}\mathbb{Q}_p)\otimes_{A_p}\mathrm{Det}_{A_p}(X_{\{v\mid\mathfrak{f}p\infty\}}\otimes_{\mathbb{Z}}\mathbb{Q}_p)\\ \xrightarrow{\varphi_2}\;&\mathrm{Det}_{A_p}^{-1}(\mathcal{O}_{k(\mathfrak{f}),S_p}^\times\otimes_{\mathbb{Z}}\mathbb{Q}_p)\otimes_{A_p}\mathrm{Det}_{A_p}(X_{\{v\mid\mathfrak{f}p\infty\}}\otimes_{\mathbb{Z}}\mathbb{Q}_p)\\ \xrightarrow{\varphi_3}\;&\mathrm{Det}_{A_p}\left(\Delta(k(\mathfrak{f}))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\right).\end{aligned}$$

Here φ_1 is induced by the split short exact sequences

$$0\longrightarrow\mathcal{O}_{k(\mathfrak{f})}^\times\otimes_{\mathbb{Z}}\mathbb{Q}_p\longrightarrow\mathcal{O}_{k(\mathfrak{f}),S_p}^\times\otimes_{\mathbb{Z}}\mathbb{Q}_p\longrightarrow Y_{\{w\mid\mathfrak{f}p\}}\otimes_{\mathbb{Z}}\mathbb{Q}_p\longrightarrow 0 \tag{13}$$

$$0\longrightarrow X_{\{w\mid\infty\}}\otimes_{\mathbb{Z}}\mathbb{Q}_p\longrightarrow X_{\{w\mid\mathfrak{f}p\infty\}}\otimes_{\mathbb{Z}}\mathbb{Q}_p\longrightarrow Y_{\{w\mid\mathfrak{f}p\}}\otimes_{\mathbb{Z}}\mathbb{Q}_p\longrightarrow 0 \tag{14}$$

The isomorphism φ_2 is multiplication with the Euler factor $\prod_{v\in S_p}\mathcal{E}_v^{\#}\in A^\times$ where $\mathcal{E}_v$ is defined by

$$\mathcal{E}_v = \sum_{\eta|_{D_v}=1}|D_v/I_v|\,e_\eta+\sum_{\eta|_{D_v}\neq 1}(1-\eta(f_v))^{-1}\,e_\eta, \tag{15}$$

where $f_v \in D_v$ denotes a lift of the Frobenius element in D_v/I_v and $I_v \subseteq D_v \subseteq G_{\mathfrak{f}}$ are the inertia and decomposition subgroups for a place $w \mid v$ in $k(\mathfrak{f})/k$. Finally φ_3 arises from the explicit description of the cohomology groups $H^i(\Delta(k(\mathfrak{f})))$, $i = 1, 2$, and the canonical isomorphism

$$\mathrm{Det}_{A_p}(\Delta(k(\mathfrak{f}))\otimes_{\mathfrak{A}_p}\mathbb{Q}_p) \simeq \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_{A_p}^{(-1)^i}\left(H^i(\Delta(k(\mathfrak{f}))\otimes_{\mathfrak{A}_p}\mathbb{Q}_p)\right) \tag{16}$$

([14, Rem. b) following Th. 2]).
We are now in position to give a very explicit description of the equivariant version of [8, Conjecture 3].

Conjecture 4.1 ${}_A\vartheta_p\left({}_A\vartheta_\infty(L^*({}_AM,0)^{-1})\right)\mathfrak{A}_p = \mathrm{Det}_{\mathfrak{A}_p}(\Delta(k(\mathfrak{f})))$.

The main result of this article reads:

Theorem 4.2 *Let k denote a quadratic imaginary field and let p be an odd prime which splits in $k/\mathbb{Q}$ and which does not divide the class number h_k of k. Then Conjecture 4.1 holds.*

Corollary 4.3 *Let k denote a quadratic imaginary field and let p be an odd prime which splits in $k/\mathbb{Q}$ and which does not divide the class number h_k of k. Let L be a finite abelian extension of k and $k \subseteq K \subseteq L$. Then the p-part of the ETNC holds for the pair $(h^0(\mathrm{Spec}(L), \mathbb{Z}[\mathrm{Gal}(L/K)]))$.*

Proof This is implied by well known functorial properties of the ETNC. □

5 The limit theorem

Following [8] or [5] we will deduce Theorem 4.2 from an Iwasawa theoretic result which we will describe next. Let now $p = \mathfrak{p}\bar{\mathfrak{p}}$ denote a split rational prime and $\mathfrak{f}$ an integral $\mathcal{O}_k$-ideal such that $w(\mathfrak{f}) = 1$. In addition, we assume that $\bar{\mathfrak{p}}$ divides $\mathfrak{f}$ whenever $\mathfrak{p}$ divides $\mathfrak{f}$. We write $\mathfrak{f} = \mathfrak{f}_0\mathfrak{p}^\nu, \mathfrak{p} \nmid \mathfrak{f}_0$. We put $\Delta := \mathrm{Gal}(k(\mathfrak{f}_0\mathfrak{p})/k) = G_{\mathfrak{f}_0\mathfrak{p}}$ and let

$$\Lambda = \varprojlim_n \mathbb{Z}_p[G_{\mathfrak{f}\mathfrak{p}^n}] \simeq \mathbb{Z}_p[\Delta][[T]]$$

denote the completed group ring. The element $T = \gamma - 1$ depends on the choice of a topological generator γ of $\Gamma := \mathrm{Gal}(k(\mathfrak{f}_0\mathfrak{p}^\infty)/k(\mathfrak{f}_0\mathfrak{p})) \simeq \mathbb{Z}_p$.
We will work in the derived category $\mathcal{D}^p(\Lambda)$ and define

$$\Delta^\infty := \varprojlim_n \Delta(k(\mathfrak{f}_0\mathfrak{p}^n)).$$

Then Δ_∞ can be represented by a perfect complex of Λ-modules. For its cohomology groups one obtains $H^i(\Delta^\infty) = 0$ for $i \neq 1, 2$,

$$H^1(\Delta^\infty) \simeq U^\infty_{S_p} := \varprojlim_n \left(\mathcal{O}^\times_{k(\mathfrak{f}_0\mathfrak{p}^n),S_p}\otimes_{\mathbb{Z}}\mathbb{Z}_p\right)$$

and $H^2(\Delta^\infty)$ fits into the short exact sequence

$$0 \longrightarrow P^\infty_{S_p} \longrightarrow H^2(\Delta^\infty) \longrightarrow X^\infty_{\{w|\mathfrak{f}_0\mathfrak{p}\infty\}} \longrightarrow 0,$$

where

$$\begin{aligned} P^\infty_{S_p} &:= \lim_{\substack{\leftarrow \\ n}} \left(\mathrm{Pic}(\mathcal{O}_{k(\mathfrak{f}_0\mathfrak{p}^n),S_p}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \right), \\ X^\infty_{\{w|\mathfrak{f}_0\mathfrak{p}\infty\}} &:= \lim_{\substack{\leftarrow \\ n}} \left(X_{\{w|\mathfrak{f}_0\mathfrak{p}\infty\}}(k(\mathfrak{f}_0\mathfrak{p}^n)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \right). \end{aligned}$$

The limits over the unit and Picard groups are taken with respect to the norm maps; the transition maps for the definition of $X^\infty_{\{w|\mathfrak{f}_0\mathfrak{p}\infty\}}$ are defined by sending each place to its restriction.
For $\mathfrak{g} \mid \mathfrak{f}_0$ we put

$$\begin{aligned} \eta_\mathfrak{g} &:= \left\{ \psi(1; \mathfrak{g}\mathfrak{p}^{n+1}, \mathfrak{a}) \right\}_{n \ge 0} \in U^\infty_{S_p}, \\ \sigma_\infty &:= \left\{ \sigma|_{k(\mathfrak{f}_0\mathfrak{p}^{n+1})} \right\}_{n\ge 0} \in Y^\infty_{\{w|\mathfrak{f}p\infty\}}, \end{aligned}$$

where σ is our fixed embedding $\mathbb{Q}^c \hookrightarrow \mathbb{C}$.
For any commutative ring R we write $Q(R)$ for its total ring of fractions. Then $Q(\Lambda)$ is a finite product of fields,

$$Q(\Lambda) \simeq \prod_{\psi \in \hat{\Delta}^{\mathbb{Q}_p}} Q(\psi), \tag{17}$$

where $\hat{\Delta}^{\mathbb{Q}_p}$ denotes the set of $\mathbb{Q}_p$-rational characters of Δ which are associated with the set of $\mathbb{Q}_p$-irreducible representations of Δ. For each $\psi \in \hat{\Delta}^{\mathbb{Q}_p}$ one has

$$Q(\psi) = Q\left(\mathbb{Z}_l(\psi)[[T]][\frac{1}{p}] \right).$$

As in [8] one shows that for each $\psi \in \hat{\Delta}^{\mathbb{Q}_p}$ one has

$$\dim_{Q(\psi)} \left(U^\infty_{S_p} \otimes_\Lambda Q(\psi) \right) = \dim_{Q(\psi)} \left(Y^\infty_{\{w|\mathfrak{f}p\infty\}} \otimes_\Lambda Q(\psi) \right) = 1$$

It follows that the element $e_\psi(\eta_{\mathfrak{f}_0}^{-1} \otimes \sigma_\infty)$ is a $Q(\psi)$-basis of

$$\mathrm{Det}_{Q(\psi)}(\Delta^\infty \otimes_\Lambda Q(\psi)) \simeq \mathrm{Det}^{-1}_{Q(\psi)}(U^\infty_{S_p} \otimes_\Lambda Q(\psi)) \otimes \mathrm{Det}_{Q(\psi)}(X^\infty_{\{w|\mathfrak{f}p\infty\}} \otimes_\Lambda Q(\psi)).$$

Theorem 5.1 $\Lambda \cdot \mathcal{L} = \mathrm{Det}_\Lambda(\Delta^\infty)$ *with* $\mathcal{L} = (N\mathfrak{a} - \sigma(\mathfrak{a}))\left(\eta_{\mathfrak{f}_0}^{-1} \otimes \sigma_\infty \right)$.

Proof By [8, Lem. 5.3] it suffices to show that the equality

$$\Lambda_\mathfrak{q} \cdot \mathcal{L} = \mathrm{Det}_{\Lambda_\mathfrak{q}}(\Delta^\infty \otimes_\Lambda \Lambda_\mathfrak{q}) \tag{18}$$

holds for all height 1 prime ideals of Λ. Such a height 1 prime is called regular (resp. singular) if $p \notin \mathfrak{q}$ (resp. $p \in \mathfrak{q}$).
We first assume that $\mathfrak{q}$ is a regular prime. Then $\Lambda_{\mathfrak{q}}$ is a discrete valuation ring, in particular, a regular ring. Hence we can work with the cohomology groups of Δ^∞, and in this way, the equality $\Lambda_{\mathfrak{q}} \cdot \mathcal{L} = \mathrm{Det}_{\Lambda_{\mathfrak{q}}}(\Delta^\infty \otimes_\Lambda \Lambda_{\mathfrak{q}})$ is equivalent to

$$\begin{aligned} & (N\mathfrak{a} - \sigma(\mathfrak{a}))\mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}(\mathbb{Z}_{p,\mathfrak{q}})\,\mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}\left(U^\infty_{S_p,\mathfrak{q}}/\eta_{\mathfrak{f}_0}\Lambda_{\mathfrak{q}}\right) \\ = \quad & \mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}\left(P^\infty_{S_p,\mathfrak{q}}\right)\mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}\left(Y^\infty_{\{w|\mathfrak{f}_0\mathfrak{p}\infty\},\mathfrak{q}}/\Lambda_{\mathfrak{q}}\sigma_\infty\right). \end{aligned} \tag{19}$$

Attached to each regular prime $\mathfrak{q}$ there is a unique character $\psi = \psi_{\mathfrak{q}} \in \hat{\Delta}^{\mathbb{Q}_p}$. To understand this notion we recall that

$$\Lambda[\frac{1}{p}] \simeq \prod_{\psi \in \hat{\Delta}^{\mathbb{Q}_p}} (\mathbb{Z}_p(\psi)[[T]])\,[\frac{1}{p}].$$

If $p \notin \mathfrak{q}$, then $\Lambda_{\mathfrak{q}}$ is just a further localisation of $\Lambda[\frac{1}{p}]$, so that exactly one of the above components survives the localization process.
We set

$$\begin{aligned} U^\infty &:= \lim_{\substack{\leftarrow \\ n}} \left(\mathcal{O}^\times_{k(\mathfrak{f}_0\mathfrak{p}^n)} \otimes_{\mathbb{Z}} \mathbb{Z}_p\right), \\ P^\infty &:= \lim_{\substack{\leftarrow \\ n}} \left(\mathrm{Pic}(\mathcal{O}_{k(\mathfrak{f}_0\mathfrak{p}^n)}) \otimes_{\mathbb{Z}} \mathbb{Z}_p\right). \end{aligned}$$

Remark 5.2 Note that, using the notation of Section 3, one has $P^\infty = A_\infty$. We put $K_n := k(\mathfrak{f}_0\mathfrak{p}^{n+1})$. Mimicking the proof of Leopoldt's conjecture, one can show that for each $n \geq 0$ the natural map $\mathcal{O}^\times_{K_n} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to U_n$ (semi-local units in $K_n \otimes_k k_{\mathfrak{p}}$ which are congruent to 1 mod $\mathfrak{p}$) is an injection. It follows that $U^\infty = \bar{\mathcal{E}}_\infty$, where $\bar{\mathcal{E}}_\infty$ is, as in Section 3, the projective limit over the closures of the global units.

There is an exact sequence of Λ-modules

$$0 \longrightarrow U^\infty \longrightarrow U^\infty_{S_p} \longrightarrow Y^\infty_{\{w|\mathfrak{f}_0\mathfrak{p}\},\beta} \longrightarrow P^\infty \longrightarrow P^\infty_{S_p} \longrightarrow 0, \tag{20}$$

where

$$Y^\infty_{\{w|\mathfrak{f}_0\mathfrak{p}\},\beta} = \lim_{\substack{\leftarrow \\ n}} \left(Y_{\{w|\mathfrak{f}_0\mathfrak{p}\}}(k(\mathfrak{f}_0\mathfrak{p}^n)) \otimes \mathbb{Z}_p\right)$$

with respect to the transition maps

$$Y_{\{w|\mathfrak{f}_0\mathfrak{p}\}}(k(\mathfrak{f}_0\mathfrak{p}^{n+1}) \stackrel{\beta_{n+1/n}}{\longrightarrow} Y_{\{w|\mathfrak{f}_0\mathfrak{p}\}}(k(\mathfrak{f}_0\mathfrak{p}^n))$$

induced by $w \mapsto f_{w|v}v$, if v denotes the restriction of w and $f_{w|v}$ the residue degree.
If now $\mathfrak{b}$ is a prime divisor of $\mathfrak{f}_0$ and $n_0 \in \mathbb{N}$ such that there is no further splitting of primes above $\mathfrak{b}$ in $k(\mathfrak{f}_0\mathfrak{p}^\infty)/k(\mathfrak{f}_0\mathfrak{p}^{n_0})$, then $\beta_{m|n}(w) = p^{m-n}w|_{k(\mathfrak{f}_0\mathfrak{p}^{n+1})}$ for all

$m \geq n \geq n_0$. Letting m tend to infinity this shows that $Y^\infty_{\{w|\mathfrak{b}\},\beta} = 0$. Hence we have an exact sequence of Λ-modules

$$0 \longrightarrow U^\infty \longrightarrow U^\infty_{S_p} \longrightarrow Y^\infty_{\{w|\mathfrak{p}\},\beta} \longrightarrow P^\infty \longrightarrow P^\infty_{S_p} \longrightarrow 0. \tag{21}$$

In addition, one has the exact sequence

$$0 \longrightarrow X^\infty_{\{w|\mathfrak{f}_0\}} \longrightarrow X^\infty_{\{w|\mathfrak{f}_0\mathfrak{p}\infty\}} \longrightarrow Y^\infty_{\{w|\mathfrak{p}\}} \oplus Y^\infty_{\{w|\infty\}} \longrightarrow 0. \tag{22}$$

Remark 5.3 Note that the transition maps in the first two limits are induced by restriction, which coincides with $\beta_{n+1|n}$ for the places above $\mathfrak{p}$ and ∞. Hence $Y_{\{w|\infty\}} = Y_{\{w|\infty\},\beta}$ and $Y_{\{w|\mathfrak{p}\}} = Y_{\{w|\mathfrak{p}\},\beta}$.

We observe that $Y^\infty_{\{w|\infty\},\mathfrak{q}} = \Lambda_\mathfrak{q} \cdot \sigma_\infty$. Putting together (21) and (22) we therefore deduce that (19) is equivalent to

$$(N\mathfrak{a} - \sigma(\mathfrak{a}))\mathrm{Fitt}_{\Lambda_\mathfrak{q}}\left(U^\infty_\mathfrak{q}/\eta_{\mathfrak{f}_0}\Lambda_\mathfrak{q}\right) = \mathrm{Fitt}_{\Lambda_\mathfrak{q}}\left(P^\infty_\mathfrak{q}\right)\mathrm{Fitt}_{\Lambda_\mathfrak{q}}\left(X_{\{w|\mathfrak{f}_0\},\mathfrak{q}}\right). \tag{23}$$

Let $\mathfrak{d}$ be a divisor of $\mathfrak{f}_0$ such that $\psi_\mathfrak{q}$ has conductor $\mathfrak{d}$ or $\mathfrak{d}\mathfrak{p}$. For any prime divisor $\mathfrak{l} \mid \mathfrak{f}_0$ we write $I_\mathfrak{l} \subseteq D_\mathfrak{l} \subseteq G_{\mathfrak{f}_0\mathfrak{p}^\infty}$ for the inertia and decomposition subgroups at $\mathfrak{l}$. Let $\mathrm{Fr}_\mathfrak{l}$ denote a lift of the Frobenius element in $D_\mathfrak{l}/I_\mathfrak{l}$. We view ψ as a character of $G_{\mathfrak{f}_0\mathfrak{p}^\infty}$ via inflation and note that if $\mathfrak{l} \nmid \mathfrak{d}$ (i.e. $\psi|_{I_\mathfrak{l}} = 1$), then $\mathrm{Fr}_\mathfrak{l}$ is a well defined element in $\Lambda_\mathfrak{q}$.

Lemma 5.4 *Let*

$$\varepsilon = \begin{cases} 0, & \psi \neq 1, \\ 1, & \psi = 1. \end{cases}$$

Then:

$$\mathrm{Fitt}_{\Lambda_\mathfrak{q}}(\Lambda_\mathfrak{q} T^\varepsilon \eta_\mathfrak{d}/\Lambda_\mathfrak{q}\eta_{\mathfrak{f}_0}) = T^{-\varepsilon} \prod_{\mathfrak{l}|\mathfrak{f}_0, \mathfrak{l}\nmid\mathfrak{d}} (1 - \mathrm{Fr}_\mathfrak{l}^{-1})\Lambda_\mathfrak{q} = \mathrm{Fitt}_{\Lambda_\mathfrak{q}}(X^\infty_{\{w|\mathfrak{f}_0\},\mathfrak{q}}).$$

Lemma 5.5

$$\mathrm{Fitt}_{\Lambda_\mathfrak{q}}(U^\infty_\mathfrak{q}/\Lambda_\mathfrak{q} T^\varepsilon \eta_\mathfrak{d}) = (N\mathfrak{a} - \sigma(\mathfrak{a}))\mathrm{Fitt}_{\Lambda_\mathfrak{q}}(P^\infty_\mathfrak{q})$$

Proof of Lemma 5.5: Let $\psi = \psi_\mathfrak{q}$. By the Iwasawa main conjecture (Theorem 3.1) and Remark 5.2 we have

$$\mathrm{char}(P^\infty_\psi) = \mathrm{char}\left((U^\infty/\bar{\mathcal{C}}_\infty)_\psi\right),$$

where (again by a slight abuse of notation) for a Λ-module M we set $M_\psi := M_\eta$ for any $\eta \in \psi$.
The corollary to [16, App. Prop. 2] implies that

$$\mathrm{Fitt}_{\Lambda_\mathfrak{q}}(P^\infty_\mathfrak{q}) = \mathrm{Fitt}_{\Lambda_\mathfrak{q}}\left((U^\infty/\bar{\mathcal{C}}_\infty)_\mathfrak{q}\right).$$

Hence it suffices to show that

$$\begin{aligned} \bar{\mathcal{C}}_\infty(\mathfrak{a})_\mathfrak{q} &= \Lambda_\mathfrak{q} \cdot T^\varepsilon \eta_\mathfrak{d}, & (24)\\ \mathrm{Fitt}_{\Lambda_\mathfrak{q}}(\bar{\mathcal{C}}_{\infty,\mathfrak{q}}/\bar{\mathcal{C}}_\infty(\mathfrak{a})_\mathfrak{q}) &= (N\mathfrak{a} - \sigma(\mathfrak{a}))\Lambda_\mathfrak{q}. & (25)\end{aligned}$$

Here $\bar{\mathcal{C}}_\infty(\mathfrak{a})$ is the projective limit over

$$\bar{\mathcal{C}}_n(\mathfrak{a}) = \text{ closure of } \langle \psi(1;\mathfrak{g}\mathfrak{p}^{n+1},\mathfrak{a})\colon \mathfrak{g} \mid \mathfrak{f}_0\rangle_{\mathbb{Z}[\mathrm{Gal}(k(\mathfrak{f}_0\mathfrak{p}^{n+1})/k)]} \cap \mathcal{E}_n.$$

(Note that $\Lambda_\mathfrak{q}\eta_\mathfrak{d}$ is for $\psi \neq 1$ a group of units. This is true even for $\mathfrak{d} = 1$, because $\Lambda_\mathfrak{q}\eta_1 = \Lambda_\mathfrak{q} e_\psi \eta_1$ and e_ψ has augmentation 0.)
In order to prove (24) we set

$$\psi_n := \psi(1;\mathfrak{d}\mathfrak{p}^{n+1},\mathfrak{a}), \quad G_n := \mathrm{Gal}(k(\mathfrak{f}_0\mathfrak{p}^{n+1})/k), \quad \Lambda_n := \mathbb{Z}_p[G_n].$$

If $\mathfrak{b}_n$ denotes the annihilator of ψ_n in Λ_n, then we have the following exact sequence of inverse systems of finitely generated $\mathbb{Z}_p$-modules

$$0 \longrightarrow (\Lambda_n/\mathfrak{b}_n)_n \longrightarrow \big(\bar{\mathcal{C}}_n(\mathfrak{a})\big)_n \longrightarrow \big(\bar{\mathcal{C}}_n(\mathfrak{a})/\Lambda_n\psi_n\big)_n \longrightarrow 0.$$

The topology of $\mathbb{Z}_p$ induces on each of these modules the structure of a compact topological group, so that [27, Prop. B.1.1] implies that $\varprojlim_n$ is exact. Hence we obtain the short exact sequence of Λ-modules

$$0 \longrightarrow \varprojlim_n (\Lambda_n/\mathfrak{b}_n) \longrightarrow \bar{\mathcal{C}}_\infty(\mathfrak{a}) \longrightarrow \varprojlim_n \big(\bar{\mathcal{C}}_n(\mathfrak{a})/\Lambda_n\psi_n\big) \longrightarrow 0.$$

Again by [27, Prop. B.1.1] we obtain

$$\varprojlim_n (\Lambda_n/\mathfrak{b}_n) \simeq \Lambda/\varprojlim_n \mathfrak{b}_n \simeq \Lambda\eta_\mathfrak{d},$$

so that

$$\bar{\mathcal{C}}_\infty(\mathfrak{a})/\Lambda\eta_\mathfrak{d} \simeq \varprojlim_n(\bar{\mathcal{C}}_n(\mathfrak{a})/\Lambda\psi_n). \tag{26}$$

For $\mathfrak{d} \mid \mathfrak{f}_0$ we identify $\mathrm{Gal}(k(\mathfrak{f}_0\mathfrak{p}^{n+1})/k(\mathfrak{d}\mathfrak{p}^{n+1}))$ and $\mathrm{Gal}(k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{d}\mathfrak{p}))$. Then one has (in additive notation) for any $\mathfrak{g}$ with $\mathfrak{d} \mid \mathfrak{g} \mid \mathfrak{f}_0$ the distribution relation

$$\begin{aligned} & N_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{d}\mathfrak{p})}\big(\psi(1;\mathfrak{g}\mathfrak{p}^{n+1},\mathfrak{a})\big) \\ = \;& [k(\mathfrak{f}_0\mathfrak{p}) : k(\mathfrak{g}\mathfrak{p})]\left(\prod_{\mathfrak{l}\mid\mathfrak{g},\mathfrak{l}\nmid\mathfrak{d}}(1-\mathrm{Fr}_\mathfrak{l}^{-1})\right)\psi(1;\mathfrak{d}\mathfrak{p}^{n+1},\mathfrak{a}). \end{aligned} \tag{27}$$

In addition, one obviously has

$$[k(\mathfrak{f}_0\mathfrak{p}) : k(\mathfrak{g}\mathfrak{p})]\psi(1;\mathfrak{g}\mathfrak{p}^{n+1},\mathfrak{a}) = N_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{g}\mathfrak{p})}\big(\psi(1;\mathfrak{g}\mathfrak{p}^{n+1},\mathfrak{a})\big). \tag{28}$$

Note that for $\psi \neq 1$ and $\mathfrak{d} \nmid \mathfrak{g}$ one has $\psi(N_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{g}\mathfrak{p})}) = 0$. Hence, if $\psi \neq 1$, then (27), (28) and (26) show that

$$A := \left(\prod_{\mathfrak{g} | \mathfrak{f}_0, \mathfrak{d} \nmid \mathfrak{g}} [k(\mathfrak{f}_0\mathfrak{p}) : k(\mathfrak{g}\mathfrak{p})] \right) \cdot N_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{d}\mathfrak{p})}$$

annihilates $\bar{\mathcal{C}}_\infty(\mathfrak{a})/\Lambda\eta_{\mathfrak{d}}$. Since $\psi(A) \in \mathbb{Z}_p$ is non-trivial and p is invertible in $\Lambda_{\mathfrak{q}}$, the element A is a actually a unit in $\Lambda_{\mathfrak{q}}$, which implies $\bar{\mathcal{C}}_\infty(\mathfrak{a})_{\mathfrak{q}} = \Lambda_{\mathfrak{q}}\eta_{\mathfrak{d}}$.
If $\psi = 1$ we proceed in almost the same way, but now set $\psi_n := \psi(1; \mathfrak{p}^{n+1}, \mathfrak{a})^{\gamma - 1}$. In this case we have $\mathfrak{d} = 1$.

Sublemma: Let $\{C_n, f_n\}_{n \geq 0}$ be a projective system of finitely generated $\mathbb{Z}_p[G_n]$-modules and set $C_\infty = \varprojlim_n C_n$. Let $\mathfrak{q}$ denote a regular prime and let $\psi = \psi_{\mathfrak{q}}$. Then:

$$C_{\infty,\mathfrak{q}} \simeq (\varprojlim_n C_{n,\psi})_{\mathfrak{q}}.$$

Proof of Sublemma: The natural map $C_n \longrightarrow \oplus_{\chi \in \hat{\Delta}^{\mathbb{Q}_p}} C_{n,\chi}$ has kernel and cokernel annihilated by $|\Delta|$. Passing to the limit we obtain (again by [27, B.1.1]) an exact sequence of Λ-modules

$$0 \longrightarrow W_\infty \longrightarrow C_\infty \longrightarrow \bigoplus_{\chi \in \hat{\Delta}^{\mathbb{Q}_p}} \varprojlim_n C_{n,\chi} \longrightarrow X_\infty \longrightarrow 0,$$

where W_∞ and X_∞ are annihilated by $|\Delta|$. Since $|\Delta| \in \Lambda_{\mathfrak{q}}^\times$ we obtain

$$C_{\infty,\mathfrak{q}} \simeq \left(\bigoplus_{\chi \in \hat{\Delta}^{\mathbb{Q}_p}} \varprojlim_n C_{n,\chi} \right)_{\mathfrak{q}} = \left(\varprojlim_n C_{n,\psi} \right)_{\mathfrak{q}}.$$

$\square$

Arguing as in the case $\psi \neq 1$ and applying the Sublemma we obtain

$$\left(\bar{\mathcal{C}}_\infty(\mathfrak{a})/\Lambda T \eta_{\mathfrak{d}}\right)_{\mathfrak{q}} \simeq \left(\varprojlim_n (\bar{\mathcal{C}}_n(\mathfrak{a})/\Lambda_n \psi_n) \right)_{\mathfrak{q}} \simeq \left(\varprojlim_n (\bar{\mathcal{C}}_n(\mathfrak{a})/\Lambda_n \psi_n)_\psi \right)_{\mathfrak{q}}.$$

Hence it suffices to show that each of the modules $(\bar{\mathcal{C}}_n(\mathfrak{a})/\Lambda_n\psi_n)_\psi$ is annihilated by the unit $N_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{p})}$. If

$$\prod_{\mathfrak{g} \neq 1} \psi(1; \mathfrak{g}\mathfrak{p}^{n+1}, \mathfrak{a})^{\alpha_{\mathfrak{g}}} \cdot \psi(1; \mathfrak{p}^{n+1}, \mathfrak{a})^{\alpha_1} \text{ with } \alpha_1, \alpha_{\mathfrak{g}} \in \mathbb{Z}_p[G_n]$$

is a unit in $K_n = k(\mathfrak{f}_0\mathfrak{p}^{n+1})$, then the prime ideal factorization of the singular values $\psi(1; \mathfrak{g}\mathfrak{p}^{n+1}, \mathfrak{a})$ (see [1, Th. 2.4]) implies that α_1 has augmentation 0. It follows that $\psi(\alpha_1) \in \mathbb{Z}_p[\mathrm{Gal}(K_n/K_0)]$ is divisible by $\gamma - 1$. For any element $\sigma \in G_n$ we write $\sigma = \gamma(\sigma)\delta(\sigma)$ according to the decomposition $G_n = \mathrm{Gal}(K_n/K_0) \times \Delta$. If $\mathfrak{g} \neq 1$ each of the factors $\psi(1 - \mathrm{Fr}_{\mathfrak{l}}^{-1}) = 1 - \gamma(\mathrm{Fr}_{\mathfrak{l}})^{-1}$ in (27) is divisible by $\gamma - 1$.

Altogether this implies that $N_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{p})}$ annihilates $\left(\bar{\mathcal{C}}_\infty(\mathfrak{a})/\Lambda T \eta_{\mathfrak{d}}\right)_{\mathfrak{q}}$, hence $\bar{\mathcal{C}}_\infty(\mathfrak{a})_{\mathfrak{q}} = \Lambda_{\mathfrak{q}} T \eta_{\mathfrak{d}}$.

It finally remains to prove (25). For any integral ideal $\mathfrak{m}$ and any two integral ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $(\mathfrak{ab}, 6\mathfrak{m}) = 1$ one has the relation

$$\psi(1; \mathfrak{m}, \mathfrak{a})^{N\mathfrak{b} - \sigma(\mathfrak{b})} = \psi(1; \mathfrak{m}, \mathfrak{b})^{N\mathfrak{a} - \sigma(\mathfrak{a})}. \tag{29}$$

This is a straightforward consequence of [1, Prop. 2.2] and the definition of ψ, see in particular [20, Théorème principal (b) and Remarque 1 (g)]. Equality (29) shows that $N\mathfrak{a} - \sigma(\mathfrak{a})$ annihilates $\bar{\mathcal{C}}_{\infty,\mathfrak{q}}/\bar{\mathcal{C}}_\infty(\mathfrak{a})_{\mathfrak{q}}$. Using the same arguments as in the proof of Lemma 3.5 (see that paragraph following Claim 2), one shows that this module is generated by one element. By [16, App. 3 and 8] it therefore suffices to show that $(N\mathfrak{a} - \sigma(\mathfrak{a}))\Lambda_{\mathfrak{q}}$ is the exact annihilator. From Lemma 5.6 below we obtain finitely many ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ and $n_1, \ldots, n_s \in \Lambda_{\mathfrak{q}}$ such that

$$1 = \sum_{i=1}^{s} n_i (N\mathfrak{a}_i - \sigma(\mathfrak{a}_i)).$$

Consider the element $\eta := T^\varepsilon \prod_{i=1}^s \eta_{\mathfrak{d}}(\mathfrak{a}_i)^{n_i}$, where $\eta_{\mathfrak{d}}(\mathfrak{a}_i) := \{\psi(1; \mathfrak{d}\mathfrak{p}^{n+1}, \mathfrak{a}_i)\}_{n=0}^\infty$. One has

$$\eta^{N\mathfrak{a} - \sigma(\mathfrak{a})} = T^\varepsilon \eta_{\mathfrak{d}}.$$

As a consequence of Lemma 3.5, Claim 2, the module $\bar{\mathcal{C}}_\infty(\mathfrak{a})_{\mathfrak{q}} = \Lambda_{\mathfrak{q}} T^\varepsilon \eta_{\mathfrak{d}}$ is $\Lambda_{\mathfrak{q}}$-free. It follows that no divisor of $N\mathfrak{a} - \sigma(\mathfrak{a})$ annihilates the quotient $\bar{\mathcal{C}}_{\infty,\mathfrak{q}}/\bar{\mathcal{C}}_\infty(\mathfrak{a})_{\mathfrak{q}}$.

To complete the proof for the localization at regular primes $\mathfrak{q}$ we add the following

Lemma 5.6 *Let $\psi \in \hat{\Delta}^{\mathbb{Q}_p}$, $\eta \in \psi$ and write $R = \mathbb{Z}_l(\psi) = \mathbb{Z}_l(\eta)$. Let I denote the ideal of $\Lambda_\psi = R[[\Gamma]]$ generated by the elements $N\mathfrak{a} - \sigma(\mathfrak{a}) = N\mathfrak{a} - \eta(\mathfrak{a})\gamma(\mathfrak{a})$, where $\mathfrak{a}$ runs through the integral ideals of $\mathcal{O}_k$ such that $(\mathfrak{a}, 6\mathfrak{f}\mathfrak{p}) = 1$. Then $I\Lambda_\psi[\frac{1}{p}] = \Lambda_\psi[\frac{1}{p}]$.*

Proof As usual we identify $R[[\Gamma]]$ with $R[[T]]$ by identifying γ with $1 + T$. We note that $\Lambda_\psi[\frac{1}{p}]$ is a principal ideal domain whose irreducible elements are given by the irreducible distinguished polynomials $f \in R[T]$. We fix such f and write

$$f(T) = \gamma^s + a_{s-1}\gamma^{s-1} + \ldots + a_1\gamma + a_0, \quad a_i \in R.$$

For any n there exist ideals $\mathfrak{a}_0, \ldots, \mathfrak{a}_s$ (depending on n) such that $(\mathfrak{a}_i, 6\mathfrak{f}\mathfrak{p}) = 1$ and $\sigma(\mathfrak{a}_i)|_{K_n} = \gamma^i|_{K_n}$. In particular, this implies $\eta(\mathfrak{a}_i) = \gamma^i$ and

$$\sum_{i=0}^{s} a_i(N\mathfrak{a}_i - \sigma(\mathfrak{a}_i)) \equiv \sum_{i=0}^{s} a_i N\mathfrak{a}_i - f(T) (\mathrm{mod}\ (\gamma^{p^n} - 1)\Lambda_\psi).$$

Inverting p we derive

$$\sum_{i=0}^{s} a_i'(N\mathfrak{a}_i - \sigma(\mathfrak{a}_i)) \equiv 1 - cf(T)(\mathrm{mod}\ (\gamma^{p^n} - 1)\Lambda_\psi[\frac{1}{p}])$$

with $a_1', \dots, a_s', c \in \mathbb{Q}_p(\psi) = \mathbb{Q}_p(\eta)$. Therefore

$$1 \in I\Lambda_\psi[\frac{1}{p}] + f\Lambda_\psi[\frac{1}{p}] + \bigcap_n (\gamma^{p^n} - 1)\Lambda_\psi[\frac{1}{p}].$$

Since $(\gamma^{p^n} - 1)\Lambda_\psi[\frac{1}{p}]$ is a strictly decreasing sequence of ideals in a principal ideal domain we obtain $\bigcap_n (\gamma^{p^n} - 1)\Lambda_\psi[\frac{1}{p}] = (0)$. Consequently, $I\Lambda_\psi[\frac{1}{p}] + f\Lambda_\psi[\frac{1}{p}] = \Lambda_\psi[\frac{1}{p}]$ for every irreducible distinguished polynomial f and the lemma is proved. $\square$

We now assume that $\mathfrak{q}$ is a singular prime. We write $\Delta = \Delta' \times P$ with $p \nmid |\Delta'|$ and note that the singular primes $\mathfrak{q}$ are in one-to-one correspondence with the $\mathbb{Q}_p$-rational irreducible characters of Δ' ([5, Lem. 6.2(i)]). Assume that in this way $\mathfrak{q}$ is associated with $\psi \in \hat{\Delta}'^{\mathbb{Q}_p}$ and set $\chi = \psi \times \eta$, where $\eta \in \hat{P}$ is arbitrarily chosen. From [6, III,2.1 Theorem] and [6, III,1.7 (13)] we know that the μ-invariant of $P_\chi^\infty := P^\infty \otimes_{\mathbb{Z}_p[\Delta]} \mathbb{Z}_p(\chi)$ vanishes. By [8, Lem. 5.6] it follows that $P_\mathfrak{q}^\infty = 0$. The module $X_{\{w|\mathfrak{f}_0\mathfrak{p}\}}^\infty$ is $\mathbb{Z}_p[[T]]$-torsion and free over $\mathbb{Z}_p$, hence has vanishing μ-invariant (as $\mathbb{Z}_p[[T]]$-module). Again by [8, Lem. 5.6] we derive $X_{\{w|\mathfrak{f}_0\mathfrak{p}\},\mathfrak{q}}^\infty = 0$. Since $P_{S_p}^\infty$ is an epimorphic image of P^∞ and because of the exactness of

$$0 \longrightarrow X_{\{w|\mathfrak{f}_0\mathfrak{p}\}}^\infty \longrightarrow X_{\{w|\mathfrak{f}_0\mathfrak{p}\infty\}}^\infty \longrightarrow Y_{\{w|\infty\}}^\infty \longrightarrow 0$$

we derive

$$H^2(\Delta^\infty)_\mathfrak{q} = Y_{\{w|\infty\},\mathfrak{q}}^\infty \simeq \Lambda_\mathfrak{q}\sigma_\infty.$$

We now compute $H^1(\Delta^\infty)_\mathfrak{q}$. Consider the filtration

$$\Lambda \cdot \eta_{\mathfrak{f}_0} \subseteq \bar{\mathcal{C}}_\infty(\mathfrak{a}) \subseteq \bar{\mathcal{C}}_\infty \subseteq U^\infty \subseteq U_{S_p}^\infty = H^1(\Delta^\infty).$$

By (21) the quotient $U_{S_p}^\infty/U^\infty$ injects into $Y_{\{w|\mathfrak{p}\}}^\infty$. This module is a finite free $\mathbb{Z}_p$-module and hence has vanishing μ-invariant. The module $U^\infty/\bar{\mathcal{C}}_\infty$ (or rather any of its χ-components) also has vanishing μ-invariant by [6, III, 2.1 Theorem and 1.7 (13)]. As shown above, the graded piece $\bar{\mathcal{C}}_\infty/\bar{\mathcal{C}}_\infty(\mathfrak{a})$ is annihilated by $N\mathfrak{a} - \sigma(\mathfrak{a})$. We claim that $N\mathfrak{a} - \sigma(\mathfrak{a}) \in \Lambda_\mathfrak{q}^\times$. In order to prove the claim we note that $N\mathfrak{a} - \sigma(\mathfrak{a}) = N\mathfrak{a} - \delta(\mathfrak{a})(1+T)^w$ with $w \in \mathbb{Z}_p$ and $w \neq 0$ (since $\sigma(\mathfrak{a})$ has infinite order in $G_{\mathfrak{f}_0\mathfrak{p}^\infty}$). Let π denote a prime element in $\mathbb{Z}_p(\psi)$. Then the explicit description of $\mathfrak{q}$ given in [5, Lem. 6.2] easily implies $\mathfrak{q} = (\pi, \Delta P)[[T]]$, where ΔP is the kernel of the augmentation map $\mathbb{Z}_p(\psi)[P] \to \mathbb{Z}_p(\psi)$. Therefore $\Lambda/\mathfrak{q} \simeq (\mathbb{Z}_p(\psi)/\pi))[[T]]$. Hence it suffices to show that the image of $N\mathfrak{a} - \sigma(\mathfrak{a})$ under

$$\Lambda \longrightarrow \mathbb{Z}_p(\psi)[[T]] \longrightarrow (\mathbb{Z}_p(\psi)/\pi)\,[[T]] = \Lambda/\mathfrak{q}$$

given by $N\mathfrak{a} - \psi(\mathfrak{a})(1+T)^w$ is non-trivial. This, in turn, is an easy exercise. Finally we will use the distribution relation

$$N_{k(\mathfrak{f}_0\mathfrak{p}^{n+1})/k(frg\mathfrak{p}^{n+1})}\psi(1;\mathfrak{f}_0\mathfrak{p}^{n+1},\mathfrak{a}) = \left(\prod_{\mathfrak{l}|\mathfrak{f}_0,\mathfrak{l}\nmid\mathfrak{g}}(1-\mathrm{Fr}_{\mathfrak{l}}^{-1})\right)\psi(1;\mathfrak{g}\mathfrak{p}^{n+1},\mathfrak{a}) \quad (30)$$

to show that $\bar{\mathcal{C}}_\infty(\mathfrak{a})_\mathfrak{q}/\Lambda_\mathfrak{q}\eta_{\mathfrak{f}_0}$ is trivial. Indeed, a statement similar to (26) shows that this quotient is annihilated by $\prod_{\mathfrak{l}|\mathfrak{f}_0}(1-\mathrm{Fr}_{\mathfrak{l}}^{-1})$, which is a unit in $\Lambda_\mathfrak{q}$ (same argument as with $N\mathfrak{a} - \sigma(\mathfrak{a})$ as above).
In conclusion, we have now shown that $\Delta_\mathfrak{q}^\infty$ has perfect cohomology, so that again (18) is equivalent to (19), which is trivially valid because all modules involved have trivial μ-invariants. ∎

In the following we want to deduce Conjecture 4.1 from Theorem 5.1. Again we can almost word by word rely on Flach's exposition [8].
We have a ring homomorphism

$$\Lambda \longrightarrow \mathbb{Z}_p[G_\mathfrak{f}] = \mathfrak{A}_p \subseteq A_p = \prod_{\chi\in\hat{G}_\mathfrak{f}^{\mathbb{Q}_p}} \mathbb{Q}_p(\chi),$$

a canonical isomorphism of complexes

$$\Delta^\infty \otimes^{\mathbf{L}}_\Lambda \mathfrak{A}_p \simeq \Delta(k(\mathfrak{f})), \quad (31)$$

and a canonical isomorphism of determinants

$$(\mathrm{Det}_\Lambda \Delta^\infty)\otimes_\Lambda \mathfrak{A}_p \simeq \mathrm{Det}_{\mathfrak{A}_p}(\Delta(k(\mathfrak{f})))$$

It remains to verify that the image of the element $\mathcal{L}\otimes 1$ in $\mathrm{Det}_{\mathfrak{A}_p}(\Delta(k(\mathfrak{f}))) \subseteq \mathrm{Det}_{A_p}\left(\Delta(k(\mathfrak{f}))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\right)$ agrees with ${}_A\vartheta_p\left({}_A\vartheta_\infty(L^*({}_AM,0)^{-1})\right)$. Let δ denote the morphism such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Det}_{Q(\Lambda)}(\Delta^\infty\otimes_\Lambda Q(\Lambda))\otimes_{Q(\Lambda)}A_p \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathrm{Det}_{Q(\Lambda)}(H^\bullet(\Delta^\infty\otimes_\Lambda Q(\Lambda)))\otimes_{Q(\Lambda)}A_p \arrow[d, "\delta"] \\
\mathrm{Det}_{A_p}(\Delta(k(\mathfrak{f}))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p) \arrow[r, "\simeq"] & \mathrm{Det}_{A_p}(H^\bullet(\Delta(k(\mathfrak{f}))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p))
\end{tikzcd}$$

We let

$$\begin{aligned}
\phi \quad : \quad & \mathrm{Det}_{\mathbb{Q}_p(\chi)}\left(\Delta(k(\mathfrak{f}))\otimes_{\mathfrak{A}_p}\mathbb{Q}_p(\chi)\right) \\
\simeq \quad & \begin{cases} \mathrm{Det}^{-1}_{\mathbb{Q}_p(\chi)}(\mathcal{O}^\times_{k(\mathfrak{f})}\otimes_{\mathfrak{A}_p}\mathbb{Q}_p(\chi))\otimes_{\mathbb{Q}_p(\chi)}\mathrm{Det}_{\mathbb{Q}_p(\chi)}(X_{\{v|\infty\}}\otimes_{\mathfrak{A}_p}\mathbb{Q}_p(\chi)), & \chi\neq 1, \\ \mathbb{Q}, & \chi = 1 \end{cases}
\end{aligned}$$

denote the isomorphism induced by φ_1^{-1} and φ_3^{-1} (see (13), (14) and (16)). Note that ϕ is defined in terms of cohomology. Then we have to show that

$$\left(\prod_{v\in S_p}\left(\mathcal{E}_v^{\#}\right)^{-1}\right)\phi(\delta(\mathcal{L}\otimes 1)) \tag{32}$$
$$= \begin{cases} w(\mathfrak{f}_\chi)[k(\mathfrak{f}):k(\mathfrak{f}_\chi)](N\mathfrak{a}-\chi(\mathfrak{a}))e_\chi\xi_\chi^{-1}\otimes w_\infty, & \mathfrak{f}_\chi\neq 1,\\ w(1)[k(\mathfrak{f}):k(1)](1-\chi(\mathfrak{p})^{-1})(N\mathfrak{a}-\chi(\mathfrak{a}))e_\chi\xi_\chi^{-1}\otimes w_\infty, & \mathfrak{f}_\chi = 1, \chi\neq 1\\ L(\chi,0)^{-1}, & \chi = 1.\end{cases}$$

By abuse of notation we also write χ for the composite ring homomorphism $\Lambda\to\mathbb{Q}_p(\chi)$ and denote its kernel by $\mathfrak{q}_\chi$. Then $\mathfrak{q}_\chi$ is a regular prime of Λ and $\Lambda_{\mathfrak{q}_\chi}$ is a discrete valuation ring with residue field $\mathbb{Q}_p(\chi)$. We consider χ as a character of $\mathrm{Gal}(k(\mathfrak{f}_0\mathfrak{p}^\infty)/k)$. If $\chi=\psi\times\eta$ with $\psi\in\hat{\Delta}$ and η a character of $\mathrm{Gal}(k(\mathfrak{f}_0\mathfrak{p}^\infty)/k(\mathfrak{f}_0\mathfrak{p}))$, then the quotient field of $\Lambda_{\mathfrak{q}_\chi}$ is given by $Q(\psi)$ (notation as in (17)). We set

$$\mathfrak{f}_1 = \begin{cases}\mathfrak{f}, & \text{if } \mathfrak{p}\mid\mathfrak{f},\\ \mathfrak{f}\mathfrak{p}, & \text{if } \mathfrak{p}\nmid\mathfrak{f}.\end{cases}$$

Let p^n be the degree of $k(\mathfrak{f}_1)/k(\mathfrak{f}_0\mathfrak{p})$.

Lemma 5.7 *The element $\bar\omega := 1-\gamma^{p^n}$ is a uniformizing element for $\Lambda_{\mathfrak{q}_\chi}$.*

Proof We have to show that after localisation at $\mathfrak{q}_\chi$ the kernel of χ is generated by $\bar\omega$. Since the idempotents e_ψ and e_η associated with ψ and η, respectively, are units in $\Lambda_{\mathfrak{q}_\chi}$, one has $\left(\Lambda[\frac{1}{p}]\right)_{\mathfrak{q}_\chi} = \left((\mathbb{Z}_p(\psi)[[T]][\frac{1}{p}]\right)_{\mathfrak{q}_\chi}$ and $(\mathbb{Q}_p(\psi)][\Gamma_n])_{\mathfrak{q}_\chi} = \mathbb{Q}_p(\chi)$. This immediately implies the result. ∎

We apply [8, Lem. 5.7] to

$$R=\Lambda_{\mathfrak{q}_\chi},\quad \Delta=\Delta^\infty_{\mathfrak{q}_\chi},\quad \bar\omega = 1-\gamma^{p^n}.$$

For a R-module M we put $M_{\bar\omega} := \{m\in M \mid \bar\omega m = 0\}$ and $M/\bar\omega := M/\bar\omega M$. As we already know, the cohomology of Δ is concentrated in degrees 1 and 2. We will see that the R-torsion subgroup of $H^i(\Delta), i=1,2$, is annihilated by $\bar\omega$, hence $H^i(\Delta)_{\mathrm{tors}} = H^i(\Delta)_{\bar\omega}$. We define free R-modules M^i, $i=1,2$, by the short exact sequences

$$0\longrightarrow H^i(\Delta)_{\bar\omega}\longrightarrow H^i(\Delta)\longrightarrow M^i\longrightarrow 0,$$

and consider the exact sequences of $\mathbb{Q}_p(\chi)$-vector spaces

$$0\longrightarrow H^i(\Delta)/\bar\omega\longrightarrow H^i(\Delta\otimes^{\mathbf{L}}_R\mathbb{Q}_p(\chi))\longrightarrow H^{i+1}(\Delta)_{\bar\omega}\longrightarrow 0$$

induced by the distinguished triangle

$$\Delta\xrightarrow{\bar\omega}\Delta\longrightarrow\Delta\otimes^{\mathbf{L}}_R\mathbb{Q}_p(\chi)\longrightarrow\Delta[1].$$

Then the map $\phi_{\bar\omega}$ of [8, Lem. 5.7] is induced by the exact sequence of $\mathbb{Q}_p(\chi)$-vector spaces

$$0 \longrightarrow M^1/\bar\omega \longrightarrow H^1(\Delta\otimes^{\mathbf{L}}_R\mathbb{Q}_p(\chi)) \xrightarrow{\beta_{\bar\omega}} H^2(\Delta\otimes^{\mathbf{L}}_R\mathbb{Q}_p(\chi)) \longrightarrow M^2/\bar\omega \longrightarrow 0, \tag{33}$$

where the Bockstein map $\beta_{\bar\omega}$ is given by the composite

$$H^1(\Delta\otimes^{\mathbf{L}}_R\mathbb{Q}_p(\chi)) \longrightarrow H^2(\Delta)_{\bar\omega} \longrightarrow H^2(\Delta)/\bar\omega \longrightarrow H^2(\Delta\otimes^{\mathbf{L}}_R\mathbb{Q}_p(\chi)).$$

Note that for the exactness of (33) on the left we need to show that $H^1(\Delta)$ is torsion-free.
We recall that $\mathrm{Gal}(k(\mathfrak{f}_0\mathfrak{p}^{n+1})/k(\mathfrak{f}_0\mathfrak{p})) = \mathrm{Gal}(K_n/K_0)$ is isomorphic to $(1+\mathfrak{f}_0\mathfrak{p})/(1+\mathfrak{f}_0\mathfrak{p}^{n+1}) \simeq (1+p\mathbb{Z}_p)/(1+p^{n+1}\mathbb{Z}_p)$ via the Artin map. As before we denote this isomorphism by $\sigma : (1+p\mathbb{Z}_p)/(1+p^{n+1}\mathbb{Z}_p) \to \mathrm{Gal}(K_n/K_0)$ and also write $\sigma : 1+p\mathbb{Z}_p \to \Gamma$. Passing to the limit we obtain a character

$$\chi_{\mathrm{ell}} : \Gamma \longrightarrow 1+p\mathbb{Z}_p$$

uniquely defined by $\sigma(\chi_{\mathrm{ell}}(\tau) \bmod (1+p^{n+1}\mathbb{Z}_p)) = \tau|_{K_n}$ for all $\tau \in \Gamma$. Note that one has

$$\psi(1;\mathfrak{f}_0\mathfrak{p}^{n+1},\mathfrak{a})^\tau = \psi(\chi_{\mathrm{ell}}(\tau);\mathfrak{f}_0\mathfrak{p}^{n+1},\mathfrak{a})$$

for all $n \geq 0$ and $\tau \in \Gamma$.
For a place $w \mid \mathfrak{p}$ in $k(\mathfrak{f})/k$ and $u \in k(\mathfrak{f})$ we write $u_w = \sigma_w(u)$, where $\sigma_w : \mathbb{Q}^c \to \mathbb{Q}_p^c$ defines w.

Lemma 5.8 *Define for $\mathfrak{l} \mid \mathfrak{f}_0$ the element $c_\mathfrak{l} \in \mathbb{Z}_p$ by $\gamma^{c_\mathfrak{l} p^n} = \mathrm{Fr}_\mathfrak{l}^{-f_\mathfrak{l}}$, where $f_\mathfrak{l} \in \mathbb{Z}$ is the residue degree at $\mathfrak{l}$ of $k(\mathfrak{f})/k$. Put $c_\mathfrak{p} = \log_p(\chi_{\mathrm{ell}}(\gamma^{p^n}))^{-1} \in \mathbb{Q}_p$. Then $\beta_{\bar\omega}$ is induced by the map*

$$H^1(\Delta(k(\mathfrak{f})))\otimes\mathbb{Q}_p = \mathcal{O}^\times_{k(\mathfrak{f}),S_p}\otimes\mathbb{Q}_p \longrightarrow X_{\{w\mid\mathfrak{f}\mathfrak{p}\infty\}}\otimes\mathbb{Q}_p = H^2(\Delta(k(\mathfrak{f})))\otimes\mathbb{Q}_p$$

given by

$$u \mapsto \sum_{\mathfrak{l}\mid\mathfrak{f}_0} c_\mathfrak{l} \sum_{w\mid\mathfrak{l}} \mathrm{ord}_w(u)\cdot w + c_\mathfrak{p} \sum_{w\mid\mathfrak{p}} \mathrm{Tr}_{k(\mathfrak{f})_w/\mathbb{Q}_p}\left(\log_p(u_w)\right)\cdot w.$$

Proof As in [8, Lem. 5.8]. □

Let $\mathfrak{a}_1, \mathfrak{a}_2$ denote integral $\mathcal{O}_k$-ideals and set $\mathfrak{b} = \mathrm{lcm}(\mathfrak{a}_1,\mathfrak{a}_2), \mathfrak{c} = \gcd(\mathfrak{a}_1,\mathfrak{a}_2)$. In the following we will frequently apply the formulas

$$[k(\mathfrak{b}) : k(\mathfrak{a}_1)k(\mathfrak{a}_2)] = \frac{w(\mathfrak{b})w(\mathfrak{c})}{w(\mathfrak{a}_1)w(\mathfrak{a}_2)}, \quad k(\mathfrak{a}_1)\cap k(\mathfrak{a}_2) = k(\mathfrak{c}),$$

which follow easily from [28, (15)]. Without loss of generality we may assume that $w(\mathfrak{f}_0) = 1$. We also note that $w(\mathfrak{p}) = 1$, because $\mathfrak{p} \nmid 2$ and $\mathfrak{p} \neq \bar{\mathfrak{p}}$. This implies $w(\mathfrak{g}) = 1$ for any multiple $\mathfrak{g}$ of $\mathfrak{f}_0$ or $\mathfrak{p}$.

After these preparations we will now prove equality (32). Recall that this equality is equivalent to the statement that the image of $\mathcal{L}\otimes 1$ in $\mathrm{Det}_{\mathfrak{A}_p}(\Delta(k(\mathfrak{f}))) \subseteq \mathrm{Det}_{A_p}\left(\Delta(k(\mathfrak{f}))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\right)$ is equal to ${}_A\vartheta_p\left({}_A\vartheta_\infty(L^*({}_AM,0)^{-1})\right)$. This suggests to think of $\mathcal{L}$ as a p-adic L-function. The connection with the usual p-adic L-function is hidden in the fact that for each character χ the characteristic power series of the module of "semi-local units modulo elliptic units" is generated by the p-adic L-function (see [24, Remark after Theorem 4.1]).
It is therefore intuitively clear that our descent considerations will be most difficult (and interesting) in the case of "trivial zeros" of the associated p-adic L-function.

The case of $\chi|_{D_\mathfrak{p}} \neq 1$. We let $\chi \in \hat{G}_\mathfrak{f}^{\mathbb{Q}_p}$ be a non-trivial character such that $\chi|_{D_\mathfrak{p}} \neq 1$. This should be considered as the case of no trivial zeros.
We first show that $P_{\mathfrak{q}_\chi}^\infty = 0$. From Lemma 3.7 we know that multiplication by $\gamma^{p^n} - 1$ on P^∞ has finite kernel and cokernel. It follows that the characteristic power series $h \in \mathbb{Z}_p[[\Gamma]]$ of P^∞ (considered as a module over $\mathbb{Z}_p[[\Gamma]]$) is coprime with $\gamma^{p^n} - 1$. Hence h is a unit in $\Lambda_{\mathfrak{q}_\chi}$ which annihilates $P_{\mathfrak{q}_\chi}^\infty$.
From (21) and Remark 5.3 we obtain the short exact sequence

$$0 \longrightarrow U_{\mathfrak{q}_\chi}^\infty \longrightarrow U_{S_p,\mathfrak{q}_\chi}^\infty \longrightarrow Y_{\{w|\mathfrak{p}\},\mathfrak{q}_\chi}^\infty \longrightarrow 0$$

Moreover, $Y_{\{w|\mathfrak{p}\}}^\infty = \mathbb{Z}_p[G_\infty/D_\mathfrak{p}]$, so that $\chi|_{D_\mathfrak{p}} \neq 1$ implies $Y_{\{w|\mathfrak{p}\},\mathfrak{q}_\chi}^\infty = 0$. It follows that $H^1(\Delta) = U_{S_p,\mathfrak{q}_\chi}^\infty \simeq U_{\mathfrak{q}_\chi}^\infty$ and Lemma 5.5 implies

$$U_{\mathfrak{q}_\chi}^\infty = (N\mathfrak{a} - \sigma(\mathfrak{a}))(1-\gamma)^\varepsilon \eta_{\mathfrak{f}_{\chi,0}} \cdot \Lambda_{\mathfrak{q}_\chi},$$

where $\mathfrak{f}_{\chi,0}$ is the divisor of $\mathfrak{f}_0$ such that ψ has conductor $\mathfrak{f}_{\chi,0}$ or $\mathfrak{f}_{\chi,0}\mathfrak{p}$. Recall also that

$$\varepsilon = \begin{cases} 0, & \psi \neq 1, \\ 1, & \psi = 1. \end{cases}$$

If $\psi = 1$, then $\eta \neq 1$ and $1 - \chi(\gamma) = 1 - \eta(\gamma) \neq 0$, so that $1-\gamma$ is a unit in $\Lambda_{\mathfrak{q}_\chi}$. Since also $N\mathfrak{a} - \sigma(\mathfrak{a}) \in \Lambda_{\mathfrak{q}_\chi}^\times$, we may choose $\beta_1 = \eta_{\mathfrak{f}_{\chi,0}}$ as $\Lambda_{\mathfrak{q}_\chi}$-basis of $M^1 = U_{\mathfrak{q}_\chi}^\infty$.
Since $P_{\{w|\mathfrak{f}\mathfrak{p}\}}^\infty$ is a quotient of P^∞ we obtain $P_{\{w|\mathfrak{f}\mathfrak{p}\},\mathfrak{q}_\chi}^\infty = 0$. Therefore $H^2(\Delta) = X_{\{w|\mathfrak{f}_0\mathfrak{p}\infty\},\mathfrak{q}_\chi}^\infty$. From the short exact sequence

$$0 \longrightarrow X_{\{w|\mathfrak{f}_0\mathfrak{p}\}}^\infty \longrightarrow X_{\{w|\mathfrak{f}_0\mathfrak{p}\infty\}}^\infty \longrightarrow Y_{\{w|\infty\}}^\infty \longrightarrow 0$$

together with the fact that $X_{\{w|\mathfrak{f}_0\mathfrak{p}\}}^\infty$ is Λ-torsion, we derive

$$M^2 = Y_{\{w|\infty\}}^\infty = \Lambda_{\mathfrak{q}_\chi} \cdot \beta_2 \text{ with } \beta_2 = \sigma_\infty.$$

We now apply [8, Lem. 5.7] with $\bar{\omega} = 1 - \gamma^{p^n}$. Recall that $H^2(\Delta)_{\mathrm{tors}} = X_{\{w|\mathfrak{f}_0\mathfrak{p}\},\mathfrak{q}_\chi}^\infty$ and this module is annihilated by $\bar{\omega}$. Indeed, $\bar{\omega} \sim 1 - \gamma^{p^m}$ for

$m \geq n$. For large m one has $\gamma^{p^m} \in D_{\mathfrak{l}}$ for each $\mathfrak{l} \mid \mathfrak{f}_0\mathfrak{p}$. It follows that $1-\gamma^{p^m}$ annihilates $X^{\infty}_{\{w|\mathfrak{f}_0\mathfrak{p}\}}$, so that the assumptions of [8, Lem. 5.7] are satisfied. The element $\bar{\beta}_1 \in M^1/\bar{\omega}$ is the image of the norm-compatible system

$$\eta_{\mathfrak{f}_{\chi,0}} = \left\{\psi(1;\mathfrak{f}_{\chi,0}\mathfrak{p}^{n+1},\mathfrak{a})\right\}_{n\geq 0}$$

in $M^1/\bar{\omega} \subseteq \mathcal{O}^{\times}_{k(\mathfrak{f}),S_p}\otimes_{\mathbb{Z}[G_{\mathfrak{f}}]}\mathbb{Q}_p(\chi)$. We write

$$\mathfrak{f} = \mathfrak{f}_0\mathfrak{p}^{\nu}, \quad \mathfrak{f}_\chi = \mathfrak{f}_{\chi,0}\mathfrak{p}^{\nu'}.$$

and recall the definition of ξ_χ in (9). We will show that

$$\bar{\beta}_1 = T_\chi\xi_\chi\otimes[k(\mathfrak{f}) : k(\mathfrak{f}_0\mathfrak{p}^{\nu'})]^{-1}$$

with

$$T_\chi = \begin{cases}(1-\chi^{-1}(\mathfrak{p})), & \text{if } \mathfrak{f}_\chi \neq 1,\\ 1, & \text{if } \mathfrak{f}_\chi = 1.\end{cases}$$

If $\nu = 0$, then $\mathfrak{f}_1 = \mathfrak{f}\mathfrak{p}, \mathfrak{f}_{\chi,0} = \mathfrak{f}_\chi$ and we have the following diagram of fields

$$\begin{tikzcd}
 & & k(\mathfrak{f}_1) \\
 & k(\mathfrak{f})k(\mathfrak{f}_\chi\mathfrak{p}) \arrow[ur, dash, "w(\mathfrak{f}_\chi)"] & \\
k(\mathfrak{f}) \arrow[ur, dash] & k(\mathfrak{f}_\chi\mathfrak{p}) \arrow[u, dash] & \\
k(\mathfrak{f}_\chi) \arrow[u, dash] \arrow[ur, dash] & &
\end{tikzcd}$$

Hence we obtain from[1, Th. 2.3]

$$\bar{\beta}_1 \quad = \quad N_{k(\mathfrak{f}_1)/k(\mathfrak{f})}\psi(1;\mathfrak{f}_\chi\mathfrak{p},\mathfrak{a})\otimes 1 = T_\chi\xi_\chi\otimes 1.$$

Note that in this case $[k(\mathfrak{f}) : k(\mathfrak{f}_0\mathfrak{p}^{\nu'})] = 1$.
If $\nu > 0$ and $\nu' = 0$ we obtain the diagram

$$\begin{tikzcd}
 & & k(\mathfrak{f}_0\mathfrak{p}^{\nu}) = k(\mathfrak{f}) \\
 & k(\mathfrak{f}_0)k(\mathfrak{f}_\chi\mathfrak{p}^{\nu}) \arrow[ur, dash, "\frac{w(\mathfrak{f}_\chi)}{w(\mathfrak{f}_\chi\mathfrak{p}^{\nu})}"] & \\
k(\mathfrak{f}_0) \arrow[ur, dash] & k(\mathfrak{f}_\chi\mathfrak{p}^{\nu}) \arrow[u, dash] & \\
k(\mathfrak{f}_{\chi,0}) = k(\mathfrak{f}_\chi) \arrow[u, dash] \arrow[ur, dash] & &
\end{tikzcd}$$

Writing $|G_\mathfrak{f}|e_\chi = t_\chi$ and $\bar{t}_\chi$ for the image of t_χ in $\mathbb{Z}[\mathrm{Gal}(k(\mathfrak{f}_\chi)/k)]$ we therefore have

$$\begin{aligned}
\bar{\beta}_1 &= \psi(1;\mathfrak{f}_\chi\mathfrak{p}^\nu,\mathfrak{a})\otimes 1\\
&= t_\chi\psi(1;\mathfrak{f}_\chi\mathfrak{p}^\nu,\mathfrak{a})\otimes 1/|G_\mathfrak{f}|\\
&= \bar{t}_\chi T_\chi\xi_\chi\otimes\frac{w(\mathfrak{f}_\chi\mathfrak{p}^\nu)}{w(\mathfrak{f}_\chi)}\frac{1}{[k(\mathfrak{f}_\chi\mathfrak{p}^\nu):k]}\\
&= T_\chi\xi_\chi\otimes\frac{w(\mathfrak{f}_\chi\mathfrak{p}^\nu)}{w(\mathfrak{f}_\chi)}\frac{1}{[k(\mathfrak{f}_\chi\mathfrak{p}^\nu):k(\mathfrak{f}_\chi)]}\\
&= T_\chi\xi_\chi\otimes[k(\mathfrak{f}):k(\mathfrak{f}_0)]^{-1}.
\end{aligned}$$

The case $\nu,\nu'>0$ is similar. Note that in this case $\chi(\mathfrak{p})=0$.
For each $\mathfrak{l}\mid\mathfrak{f}_0$ we choose a place $w_\mathfrak{l}$ above $\mathfrak{l}$ in $k(\mathfrak{f})/k$. It is easy to see that

$$Y_{\{w|\mathfrak{l}\}}\otimes_\mathfrak{A}\mathbb{Q}_p(\chi)=\begin{cases}0, & \chi|_{D_\mathfrak{l}}\neq 1,\\ \mathbb{Q}_p(\chi)\cdot w_\mathfrak{l}, & \chi|_{D_\mathfrak{l}}=1.\end{cases}$$

We choose for each $\mathfrak{l}\mid\mathfrak{f}_0$ with $\chi|_{D_\mathfrak{l}}=1$ an element $x_\mathfrak{l}\in k(\mathfrak{f})^\times$ such that

$$\begin{aligned}
\mathrm{ord}_{w_\mathfrak{l}}(x_\mathfrak{l}) &\neq 0\\
\mathrm{ord}_w(x_\mathfrak{l}) &= 0 \text{ for all } w\neq w_\mathfrak{l}.
\end{aligned}$$

Then $\mathbb{Q}_p(\chi)x_\mathfrak{l}\xrightarrow{\mathrm{val}}Y_{\{w|\mathfrak{l}\}}\otimes_{\mathbb{Z}_p[G_\mathfrak{f}]}\mathbb{Q}_p(\chi)=\mathbb{Q}_p(\chi)w_\mathfrak{l}$ is an isomorphism. We set

$$J=\{\mathfrak{l}\mid\mathfrak{f}_0\colon\chi|_{D_\mathfrak{l}}=1\},\quad x_J:=\bigwedge_{\mathfrak{l}\in J}x_\mathfrak{l},\quad w_J:=\bigwedge_{\mathfrak{l}\in J}w_\mathfrak{l}\text{ and } c_\chi:=\prod_{\mathfrak{l}\in J}c_\mathfrak{l}.$$

Since $\mathcal{O}_{k(\mathfrak{f})}^\times\otimes_\mathfrak{A}\mathbb{Q}_p(\chi)$ is a $\mathbb{Q}_p(\chi)$-vector space of of dimension 1, the element $\bar{\beta}_1$ is necessarily a generator. Therefore $\{\bar{\beta}_1\}\cup\{x_\mathfrak{l}\colon\mathfrak{l}\in J\}$ is a $\mathbb{Q}_p(\chi)$-basis of $H^1(\Delta\otimes^{\mathbf{L}}_R\mathbb{Q}_p(\chi))=\mathcal{O}^\times_{k(\mathfrak{f}),S_p}\otimes_\mathfrak{A}\mathbb{Q}_p(\chi)$. Moreover, $\{\bar{\beta}_2\}\cup\{w_\mathfrak{l}\colon\mathfrak{l}\in J\}$ is a $\mathbb{Q}_p(\chi)$-basis of $Y_{\{w|\mathfrak{f}_0\mathfrak{p}\infty\}}\otimes_\mathfrak{A}\mathbb{Q}_p(\chi)$. Finally note that $\bar{\beta}_2=\sigma|_{k(\mathfrak{f})}$. From (33) we deduce

$$(\phi\circ\phi_{\bar{\omega}}^{-1})(\bar{\beta}_1^{-1}\otimes\bar{\beta}_2)=\phi(\bar{\beta}_1^{-1}\wedge x_J^{-1}\otimes\beta_{\bar{\omega}}(x_J)\wedge\bar{\beta}_2)$$

Applying Lemma 5.8 we obtain further

$$\begin{aligned}
(\phi\circ\phi_{\bar{\omega}}^{-1})(\bar{\beta}_1^{-1}\otimes\bar{\beta}_2) &= c_\chi\phi(\bar{\beta}_1^{-1}\wedge x_J^{-1}\otimes\mathrm{val}(x_J)\wedge\bar{\beta}_2)\\
&= c_\chi(\bar{\beta}_1^{-1}\otimes\bar{\beta}_2)\\
&= \underbrace{c_\chi[k(\mathfrak{f}):k(\mathfrak{f}_0\mathfrak{p}^{\nu'})]T_\chi^{-1}\xi_\chi^{-1}\otimes\sigma|_{k(\mathfrak{f})}}_{=:A}.
\end{aligned}\tag{34}$$

In order to apply [8, Lem. 5.7] we compute the exponent e such that $\bar{\omega}^e\beta_1^{-1}\otimes\beta_2$ is a $\Lambda_{\mathfrak{q}_\chi}$-basis of $\mathrm{Det}_{\Lambda_{\mathfrak{q}_\chi}}(\Delta^\infty_{\mathfrak{q}_\chi})$. By the proof of [8, Lem. 5.7] one has

$$\begin{aligned} e &= \sum_{i\in\mathbb{Z}}(-1)^{i+1}\dim_{\mathbb{Q}_p(\chi)}\left(H^i(\Delta)_{\bar{\omega}}\right) \\ &= -\dim_{\mathbb{Q}_p(\chi)}\left(X^\infty_{\{w|\mathfrak{f}_0\mathfrak{p}\}}\otimes_{\mathfrak{A}}\mathbb{Q}_p(\chi)\right) \\ &\overset{\chi\neq 1}{=} -\dim_{\mathbb{Q}_p(\chi)}\left(\bigoplus_{\mathfrak{l}|\mathfrak{f}_0\mathfrak{p}}\mathbb{Z}_p[G_\infty/D_\mathfrak{l}]\otimes_{\mathfrak{A}}\mathbb{Q}_p(\chi)\right) \\ &= -|J|. \end{aligned}$$

As elements of $\left(\mathrm{Det}_\Lambda(\Delta^\infty)\right)_{\mathfrak{q}_\chi}$ we have

$$\begin{aligned} \mathcal{L} &= (N\mathfrak{a}-\sigma(\mathfrak{a}))\eta_{\mathfrak{f}_0}^{-1}\otimes\sigma_\infty \\ &= (N\mathfrak{a}-\sigma(\mathfrak{a}))[k(\mathfrak{f}_0\mathfrak{p}):k(\mathfrak{f}_{\chi,0}\mathfrak{p})][\mathrm{Tr}_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{f}_{\chi,0}\mathfrak{p})}\eta_{\mathfrak{f}_0}]^{-1}\otimes\sigma_\infty, \end{aligned}$$

because $\mathrm{Tr}_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{f}_{\chi,0}\mathfrak{p})} = [k(\mathfrak{f}_0\mathfrak{p}):k(\mathfrak{f}_{\chi,0}\mathfrak{p})]$ as elements of $\Lambda_{\mathfrak{q}_\chi}$ (multiply both sides with e_χ). From the distribution relation we derive further

$$\begin{aligned} \mathcal{L} &= (N\mathfrak{a}-\sigma(\mathfrak{a}))[k(\mathfrak{f}_0\mathfrak{p}):k(\mathfrak{f}_{\chi,0}\mathfrak{p})]\prod_{\mathfrak{l}|\mathfrak{f}_0,\mathfrak{l}\nmid\mathfrak{f}_{\chi,0}}\frac{1}{1-\mathrm{Fr}_\mathfrak{l}^{-1}}\eta_{\mathfrak{f}_{\chi,0}}^{-1}\otimes\sigma_\infty \\ &= \underbrace{(N\mathfrak{a}-\sigma(\mathfrak{a}))[k(\mathfrak{f}_0\mathfrak{p}):k(\mathfrak{f}_{\chi,0}\mathfrak{p})]\prod_{\substack{\mathfrak{l}|\mathfrak{f}_0,\mathfrak{l}\nmid\mathfrak{f}_{\chi,0}\\ \chi(\mathfrak{l})\neq 1}}\frac{1}{1-\mathrm{Fr}_\mathfrak{l}^{-1}}\prod_{\mathfrak{l}\in J}\frac{\bar{\omega}}{1-\mathrm{Fr}_\mathfrak{l}^{-1}}}_{=:B}\left(\bar{\omega}^e\beta_1^{-1}\otimes\beta_2\right). \end{aligned}$$

Now [8, Lem. 5.7] implies

$$\phi_{\bar{\omega}}(B^{-1}(\mathcal{L}\otimes 1)) = \bar{\beta}_1^{-1}\otimes\bar{\beta}_2,$$

which in conjunction with (34) shows that $\phi(B^{-1}(\mathcal{L}\otimes 1)) = A$ or $\phi(\mathcal{L}\otimes 1) = AB$. For $\mathfrak{l}\in J$ we have by definition of $c_\mathfrak{l}$ the equality $\mathrm{Fr}_\mathfrak{l}^{-f_\mathfrak{l}} = \gamma^{c_\mathfrak{l}p^n}$ and therefore

$$\chi\left(\frac{\bar{\omega}}{1-\mathrm{Fr}_\mathfrak{l}^{-1}}\right) = \chi\left(\frac{(1-\gamma^{p^n})(1+\mathrm{Fr}_\mathfrak{l}^{-1}+\ldots+\mathrm{Fr}_\mathfrak{l}^{-f_\mathfrak{l}+1})}{1-\gamma^{c_\mathfrak{l}p^n}}\right) = \frac{f_\mathfrak{l}}{c_\mathfrak{l}}. \tag{35}$$

Using $[k(\mathfrak{f}):k(\mathfrak{f}_0\mathfrak{p}^{\nu'})][k(\mathfrak{f}_0\mathfrak{p}):k(\mathfrak{f}_{\chi,0}\mathfrak{p})] = w(\mathfrak{f}_\chi)[k(\mathfrak{f}):k(\mathfrak{f}_\chi)]$ it follows that

$$AB =$$

$$(N\mathfrak{a}-\sigma(\mathfrak{a}))w(\mathfrak{f}_\chi)[k(\mathfrak{f}):k(\mathfrak{f}_\chi)]\left(\prod_{\substack{\mathfrak{l}|\mathfrak{f}_0,\mathfrak{l}\nmid\mathfrak{f}_{\chi,0}\\ \chi(\mathfrak{l})\neq 1}}\frac{1}{1-\mathrm{Fr}_\mathfrak{l}^{-1}}\right)\left(\prod_{\mathfrak{l}\in J}f_\mathfrak{l}\right)T_\chi^{-1}\xi_\chi^{-1}\otimes\sigma|_{k(\mathfrak{f})}.$$

Recalling the definition of the elements $\mathcal{E}_v$ from (15) we observe that this is exactly the equality (32).

The case of $\chi \neq 1$ and $\chi|_{D_\mathfrak{p}} = 1$. We let $\chi \in \hat{G}_\mathfrak{f}^{\mathbb{Q}_p}$ be a non-trivial character such that $\chi|_{D_\mathfrak{p}} = 1$. This should be considered as the case of trivial zeros. Note that in this case $\mathfrak{p} \nmid \mathfrak{f}_\chi$, i.e. $\mathfrak{f}_{\chi,0} = \mathfrak{f}_\chi$.

Before going into detail we brievely explain the strategy of the proof. We first point out that in one respect the elliptic case is easier than the cyclotomic case: there is no distinction between odd and even characters. Indeed, in the elliptic setting every non-trivial character behaves like an even character. Nevertheless, the strategy of the proof becomes most clear, if one recalls what happens in the cyclotomic case for odd characters. In order to avoid the trivial zero one divides the p-adic L-function by $\gamma - 1$. As a consequence of a theorem of Ferrero and Greenberg [7] (which gives a formula for the first derivative of the p-adic L-function) one obtains that this quotient interpolates essentially the global L-function $L(\chi^{-1}, s)$ at $s = 0$ (for more details see [8, Lemma 5.11]).

For even characters the strategy can be motivated by the fact that the p-adic L-function is closely related to norm-coherent sequences of cyclotomic (or in our case, of elliptic) units. In order to "avoid the trivial zero" we again divide by $\gamma - 1$, which means that we have to take the $(\gamma - 1)$-st root of a norm-coherent sequence of cyclotomic or elliptic units. In the cyclotomic case this is achieved by using a result of Solomon [29] which also provides enough information to work out the relation to the value of $L(\chi^{-1}, s)$ at $s = 0$. In the elliptic case we will use an analoguous result proved by the author in [1].

For any subgroup H of G_∞ we define J_H to be the kernel of the canonical map $\mathbb{Z}_p[[G_\infty]] \to \mathbb{Z}_p[[G_\infty/H]]$.

As in the case of no trivial zeros we can show that $P_{\mathfrak{q}_\chi}^\infty = 0$. From (21) we obtain the short exact sequence

$$0 \longrightarrow U_{\mathfrak{q}_\chi}^\infty \longrightarrow U_{S_p,\mathfrak{q}_\chi}^\infty \longrightarrow Y_{\{w|\mathfrak{p}\},\mathfrak{q}_\chi}^\infty \longrightarrow 0 \tag{36}$$

where now $Y_{\{w|\mathfrak{p}\},\mathfrak{q}_\chi}^\infty \simeq \mathbb{Z}_p[G_\infty/D_\mathfrak{p}] \otimes_\Lambda \Lambda_{\mathfrak{q}_\chi} \simeq \Lambda/J_{D_\mathfrak{p}} \otimes_\Lambda \Lambda_{\mathfrak{q}_\chi} \simeq \Lambda_{\mathfrak{q}_\chi}/J_{D_\mathfrak{p}}\Lambda_{\mathfrak{q}_\chi}$. Since $\Gamma \subseteq D_\mathfrak{p}$ one has $\gamma^{p^n} - 1 \sim \gamma - 1$. It follows that $Y_{\{w|\mathfrak{p}\},\mathfrak{q}_\chi}^\infty \simeq \mathbb{Q}_p(\chi)$, and in addition, the structure theorem for modules over principal ideal rings implies $(\gamma - 1)U_{S_p,\mathfrak{q}_\chi}^\infty = U_{\mathfrak{q}_\chi}^\infty$.

For a finite set S of places of k we set $U_{k(\mathfrak{f}_\chi),S}^\infty = \varprojlim_n \left(\mathcal{O}_{k(\mathfrak{f}_\chi\mathfrak{p}^{n+1}),S}^\times \otimes_\mathbb{Z} \mathbb{Z}_p\right)$.

Lemma 5.9 *a) The sequence*

$$0 \longrightarrow U_{k(\mathfrak{f}_\chi),S_p}^\infty \xrightarrow{\gamma-1} U_{k(\mathfrak{f}_\chi),S_p}^\infty \longrightarrow U_{k(\mathfrak{f}_\chi),S_p,\Gamma}^\infty \longrightarrow 0$$

is exact.

b) The canonical map $U_{k(\mathfrak{f}_\chi),S_p,\Gamma}^\infty \longrightarrow \mathcal{O}_{k(\mathfrak{f}_\chi\mathfrak{p}),S_p}^\times \otimes_\mathbb{Z} \mathbb{Z}_p$ *is injective.*

PROOF One has $\left(U^{\infty}_{k(\mathfrak{f}_\chi),S_p}\right)^{\Gamma} = \varprojlim_n \left(\mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{p}),S_p}\otimes_{\mathbb{Z}}\mathbb{Z}_p\right) = 0$. Hence a) is immediate. For b) one has to prove

$$(\gamma-1)U^{\infty}_{k(\mathfrak{f}_\chi),S_p} = \{u \in U^{\infty}_{k(\mathfrak{f}_\chi),S_p} \mid u_0 = 1\}.$$

The inclusion "$\subseteq$" is obvious. Suppose that $u_0 = 1$. Then for each n Hilbert's Theorem 90 provides an element $\beta_n \in k(\mathfrak{f}_\chi\mathfrak{p}^{n+1})^{\times}/k(\mathfrak{f}_\chi\mathfrak{p})^{\times}$ such that

$$\beta_n^{\gamma-1} = u_n \quad \text{and} \quad N_{k(\mathfrak{f}_\chi\mathfrak{p}^{n+2})/k(\mathfrak{f}_\chi\mathfrak{p}^{n+1})}(\beta_{n+1}) \equiv \beta_n (\mathrm{mod}\ k(\mathfrak{f}_\chi\mathfrak{p})^{\times}).$$

Let S be a finite set of places of k containing S_p and such that $\mathrm{Pic}(\mathcal{O}_{k(\mathfrak{f}_\chi\mathfrak{p}),S}) = 0$. Then we may assume that

$$\beta_n \in \mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{p}^{n+1}),S}/\mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{p}),S}.$$

In the following diagram all vertical maps are induced by the norm,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{p}),S}\otimes\mathbb{Z}_p \arrow[r] \arrow[d] & \mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{p}^{n+2}),S}\otimes\mathbb{Z}_p \arrow[r] \arrow[d] & \dfrac{\mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{p}^{n+2}),S}\otimes\mathbb{Z}_p}{\mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{p}),S}\otimes\mathbb{Z}_p} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{p}),S}\otimes\mathbb{Z}_p \arrow[r] & \mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{p}^{n+1}),S}\otimes\mathbb{Z}_p \arrow[r] & \dfrac{\mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{p}^{n+1}),S}\otimes\mathbb{Z}_p}{\mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{p}),S}\otimes\mathbb{Z}_p} \arrow[r] & 0
\end{tikzcd}$$

The natural topology of $\mathbb{Z}_p$ induces on each finitely generated $\mathbb{Z}_p$-module the structure of a compact topological group. By [17, Satz IV.2.7] the functor $\varprojlim_n$ is therefore exact on the above exact sequence of projective systems. In addition, the projective limit over the modules on the left hand side is obviously trivial and therefore

$$U^{\infty}_{k(\mathfrak{f}_\chi),S} \simeq \varprojlim_n \frac{\mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{p}^{n+1}),S}\otimes\mathbb{Z}_p}{\mathcal{O}_{k(\mathfrak{f}_\chi\mathfrak{p}),S}\otimes\mathbb{Z}_p}.$$

Moreover, the argument used to prove (21) also shows that $U^{\infty}_{k(\mathfrak{f}_\chi),S} \simeq U^{\infty}_{k(\mathfrak{f}_\chi),S_p} \simeq U^{\infty}_{k(\mathfrak{f}_\chi),\{w|\mathfrak{p}\infty\}}$ for any set $S \supseteq S_p$, so that the inclusion "$\supseteq$" follows. □

We now choose an auxiliary prime ideal $\mathfrak{b}$ of $\mathcal{O}_k$ such that

$$(\mathfrak{b},\mathfrak{f}\mathfrak{p}) = 1, \quad w(\mathfrak{b}) = 1, \quad \chi(\mathfrak{b}) \neq 1.$$

In order to be able to deal also with the case $\mathfrak{f}_\chi = 1$ we introduce the element

$$\eta = \{\psi(1;\mathfrak{f}_\chi\mathfrak{b}\mathfrak{p}^{n+1},\mathfrak{a})\}_{n=0}^{\infty} \in \varprojlim_n \mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{b}\mathfrak{p}^{n+1})}.$$

With respect to the injection $U^{\infty}_{k(\mathfrak{f}_\chi),S_p,\Gamma} \longrightarrow \mathcal{O}^{\times}_{k(\mathfrak{f}_\chi\mathfrak{p}),S_p}\otimes\mathbb{Z}_p$ the element $N_{k(\mathfrak{f}_\chi\mathfrak{b}\mathfrak{p})/F}(\eta)$ maps to $N_{k(\mathfrak{f}_\chi\mathfrak{b}\mathfrak{p})/F}(\eta^0)$, where here F denotes the decomposition subfield at $\mathfrak{p}$ in $k(\mathfrak{f}_\chi)/k$. One has the following diagram of fields

$$\begin{tikzcd}
& & k(\mathfrak{f}) \\
& \bullet \arrow[ur, dash] & \\
k(\mathfrak{f})^{D_\mathfrak{p}} \arrow[ur, dash] & k(\mathfrak{f}_\chi) \arrow[u, dash] & \\
F \arrow[u, dash] \arrow[ur, dash] & & \\
k(\mathfrak{f})^{\ker(\chi)} \arrow[u, dash] & &
\end{tikzcd}$$

Since by definition of F one has $\sigma(\mathfrak{p})|_F = id$, we derive from the distribution relation

$$N_{k(\mathfrak{f}_\chi \mathfrak{b}\mathfrak{p})/F}(\eta^0) = (1 - \sigma(\mathfrak{p})^{-1}) N_{k(\mathfrak{f}_\chi \mathfrak{b})/F} \psi(1; \mathfrak{f}_\chi \mathfrak{b}, \mathfrak{a}) = 1,$$

so that Lemma 5.9 yields a unique element $z^\infty \in U^\infty_{k(\mathfrak{f}_\chi), S_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ such that

$$(\gamma - 1) z^\infty = \frac{1}{[k(\mathfrak{f}_\chi \mathfrak{b}\mathfrak{p}) : F]} N_{k(\mathfrak{f}_\chi \mathfrak{b}\mathfrak{p})/F}(\eta). \tag{37}$$

From Lemma 5.5 and $N\mathfrak{a} - \sigma(\mathfrak{a}) \sim 1$ we deduce $U^\infty_{\mathfrak{q}_\chi} = \Lambda_{\mathfrak{q}_\chi} \eta_{\mathfrak{f}_\chi}$. Again from the distribution relations [1, Th. 2.3] we deduce

$$N_{k(\mathfrak{f}_\chi \mathfrak{b}\mathfrak{p})/F} \eta = (1 - Fr_\mathfrak{b}^{-1}) N_{k(\mathfrak{f}_\chi \mathfrak{p})/F} \eta_{\mathfrak{f}_\chi}.$$

Combining (36) and (37) we see that

$$H^1(\Delta) = U^\infty_{S_p, \mathfrak{q}_\chi} = \Lambda_{\mathfrak{q}_\chi} \cdot \beta_1 \text{ with } \beta_1 = z^\infty.$$

Note that

$$\bar{\beta}_1 = \begin{cases} z_\mu, & \text{if } \mathfrak{p} \mid \mathfrak{f}, \mathfrak{f} = \mathfrak{f}_0 \mathfrak{p}^{\mu+1}, \mu \geq 0, \\ N_{k(\mathfrak{f}\mathfrak{p})/k(\mathfrak{f})}(z_0), & \text{if } \mathfrak{p} \nmid \mathfrak{f}, \end{cases}$$

when we regard $\bar{\beta}_1$ as an element in $\mathcal{O}^\times_{k(\mathfrak{f}), S_p} \otimes \mathbb{Z}_p$.

Let v denote a place of $k(\mathfrak{f})$ above w, where $w \mid \mathfrak{p}$ in F/k. Using the above diagram we compute

$$\begin{aligned}
& \mathrm{Tr}_{k(\mathfrak{f})_v/\mathbb{Q}_p} \left(\log_p \left(N_{k(\mathfrak{f}_\chi \mathfrak{b})/k(\mathfrak{f}_\chi)} \left(\psi(1; \mathfrak{f}_\chi \mathfrak{b}, \mathfrak{a}) \right) \right) \right) \\
= \; & \frac{|D_\mathfrak{p}|}{[k(\mathfrak{f}_\chi) : F]} \log_p \left(N_{k(\mathfrak{f}_\chi \mathfrak{b})/F} (\psi(1; \mathfrak{f}_\chi \mathfrak{b}, \mathfrak{a})) \right) \\
= \; & \frac{|D_\mathfrak{p}|}{[k(\mathfrak{f}_\chi) : F]} \log_p(\chi_{\mathrm{ell}}(\gamma)) \underbrace{\frac{1}{\log_p(\chi_{\mathrm{ell}}(\gamma))} \log_p \left(N_{k(\mathfrak{f}_\chi \mathfrak{b})/F} (\psi(1; \mathfrak{f}_\chi \mathfrak{b}, \mathfrak{a})) \right)}_{=:B}
\end{aligned}$$

By the main result of [1] the quantity B is well known. We briefly recall the construction of [1]. Let k_∞ denote the unique $\mathbb{Z}_p$-extension of k which is unramified outside $\mathfrak{p}$. Let $k_n \subseteq k_\infty$ denote the extension of degree p^n above k. We put $F_n := Fk_n$ and consider the diagram of fields

$$\begin{tikzcd}
 & k(\mathfrak{f}_\chi \mathfrak{b}\mathfrak{p}^{n+1}) \arrow[dl, dash] \arrow[d, dash] \\
k(\mathfrak{f}_\chi \mathfrak{b}\mathfrak{p}) \arrow[d, dash] & \bullet \arrow[dl, dash] \arrow[d, dash] \\
k(\mathfrak{f}_\chi \mathfrak{b}) \arrow[d, dash] & F_n \arrow[dl, dash] \\
F = F_0 &
\end{tikzcd}$$

For each n Hilbert's Theorem 90 provides an element $\beta_n \in F_n^\times / F^\times$ such that

$$\beta_n^{\gamma-1} = N_{k(\mathfrak{f}_\chi \mathfrak{b}\mathfrak{p}^{n+1})/F_n} \left(\psi(1; \mathfrak{f}_\chi \mathfrak{b}\mathfrak{p}^{n+1}, \mathfrak{a}) \right).$$

If we put $\kappa_n := N_{F_n/F}(\beta_n) \in F^\times / (F^\times)^{p^n}$ and $\kappa^\infty := \{\kappa_n\}_{n=0}^\infty \in \lim F^\times / (F^\times)^{p^n}$, then the main result of [1] says

$$B = \mathrm{ord}_w(\kappa^\infty).$$

From the construction of z^∞ it is clear that one has

$$\beta_n = N_{k(\mathfrak{f}_\chi \mathfrak{b}\mathfrak{p}^{n+1})/F_n}(z_n) \text{ in } F_n^\times / F^\times,$$

and consequently,

$$\kappa^\infty = \{ N_{k(\mathfrak{f}_\chi \mathfrak{b}\mathfrak{p})/F}(z_0) \}_{n=0}^\infty.$$

We let $w' \mid w$ denote the place in $k(\mathfrak{f}_\chi)/F$ defined by v and set $c_\mathfrak{p}(\gamma) :=$

$\log_p(\chi_{\mathrm{ell}}(\gamma))^{-1}$. Then

$$
\begin{aligned}
& \mathrm{Tr}_{k(\mathfrak{f})_v/\mathbb{Q}_p}\left(\log_p\left(N_{k(\mathfrak{f}_\chi\mathfrak{b})/k(\mathfrak{f}_\chi)}\left(\psi(1;\mathfrak{f}_\chi\mathfrak{b},\mathfrak{a})\right)\right)\right) \\
= \; & \frac{|D_\mathfrak{p}|}{[k(\mathfrak{f}_\chi):F]} c_\mathfrak{p}(\gamma)^{-1}\mathrm{ord}_w(N_{k(\mathfrak{f}_\chi\mathfrak{b}\mathfrak{p})/F}z_0) \\
= \; & \frac{|D_\mathfrak{p}|c_\mathfrak{p}(\gamma)^{-1}}{[k(\mathfrak{f}_\chi):F]}\mathrm{ord}_w(N_{k(\mathfrak{f}_\chi\mathfrak{p})/F}N_{k(\mathfrak{f}_\chi\mathfrak{b}\mathfrak{p})/k(\mathfrak{f}_\chi\mathfrak{p})}z_0) \\
= \; & \frac{|D_\mathfrak{p}|c_\mathfrak{p}(\gamma)^{-1}[k(\mathfrak{f}_\chi\mathfrak{b}\mathfrak{p}):k(\mathfrak{f}_\chi\mathfrak{p})]}{[k(\mathfrak{f}_\chi):F]}\mathrm{ord}_w(N_{k(\mathfrak{f}_\chi\mathfrak{p})/F}z_0) \\
= \; & \frac{|D_\mathfrak{p}|c_\mathfrak{p}(\gamma)^{-1}w(1)[k(\mathfrak{b}):k(1)]}{[k(\mathfrak{f}_\chi):F]}\mathrm{ord}_w(N_{k(\mathfrak{f}_\chi)/F}(N_{k(\mathfrak{f}_\chi\mathfrak{p})/k(\mathfrak{f}_\chi)}z_0)) \\
= \; & \frac{|D_\mathfrak{p}|c_\mathfrak{p}(\gamma)^{-1}w(1)[k(\mathfrak{b}):k(1)]}{[k(\mathfrak{f}_\chi):F]}f_{w'/w}\mathrm{ord}_{w'}(N_{k(\mathfrak{f}_\chi\mathfrak{p})/k(\mathfrak{f}_\chi)}z_0) \\
= \; & |D_\mathfrak{p}|c_\mathfrak{p}(\gamma)^{-1}w(1)[k(\mathfrak{b}):k(1)]e_{v/w'}^{-1}\mathrm{ord}_v(N_{k(\mathfrak{f}_\chi\mathfrak{p})/k(\mathfrak{f}_\chi)}z_0) \\
= \; & f_\mathfrak{p}c_\mathfrak{p}(\gamma)^{-1}w(1)[k(\mathfrak{b}):k(1)]\mathrm{ord}_v(N_{k(\mathfrak{f}_\chi\mathfrak{p})/k(\mathfrak{f}_\chi)}z_0)
\end{aligned}
$$

We now apply Lemma 5.8. The congruence in the following computation is modulo $Y_{\{w|\mathfrak{f}_0\}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.

$$
\begin{aligned}
& \beta_{\bar\omega}\left(N_{k(\mathfrak{f}_\chi\mathfrak{b})/k(\mathfrak{f}_\chi)}(\psi(1;\mathfrak{f}_\chi\mathfrak{b},\mathfrak{a}))\right) \\
\equiv \; & c_\mathfrak{p}\sum_{v|\mathfrak{p}}\mathrm{Tr}_{k(\mathfrak{f})_v/\mathbb{Q}_p}\left(\log_p(N_{k(\mathfrak{f}_\chi\mathfrak{b})/k(\mathfrak{f}_\chi)}(\psi(1;\mathfrak{f}_\chi\mathfrak{b},\mathfrak{a})))\right)\cdot v \\
= \; & \frac{c_\mathfrak{p}}{c_\mathfrak{p}(\gamma)}f_\mathfrak{p}w(1)[k(\mathfrak{b}):k(1)]\sum_{v|\mathfrak{p}}\mathrm{ord}_v\left(N_{k(\mathfrak{f}_\chi\mathfrak{p})/k(\mathfrak{f}_\chi)}(z_0)\right)\cdot v \\
= \; & \begin{cases} \frac{f_\mathfrak{p}}{p^n}[k(\mathfrak{b}):k(1)]\frac{w(1)}{w(\mathfrak{f}_\chi)}\sum_{v|\mathfrak{p}}\mathrm{ord}_v(N_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{f}_0)}z_0)\cdot v, & \text{if } \mathfrak{p}\nmid\mathfrak{f}, \\ \frac{f_\mathfrak{p}}{p^n}w(1)[k(\mathfrak{b}):k(1)][k(\mathfrak{f}_\chi\mathfrak{p}):k(\mathfrak{f}_\chi)]\sum_{v|\mathfrak{p}}\mathrm{ord}_v(z_0)\cdot v, & \text{if } \mathfrak{p}\mid\mathfrak{f} \end{cases} \\
= \; & \begin{cases} \frac{f_\mathfrak{p}}{p^n}[k(\mathfrak{b}):k(1)]\frac{w(1)}{w(\mathfrak{f}_\chi)}\sum_{v|\mathfrak{p}}\mathrm{ord}_v(N_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{f}_0)}z_0)\cdot v, & \text{if } \mathfrak{p}\nmid\mathfrak{f}, \\ \frac{f_\mathfrak{p}}{p^n}w(1)[k(\mathfrak{b}):k(1)][k(\mathfrak{f}_\chi\mathfrak{p}):k(\mathfrak{f}_\chi)][k(\mathfrak{f}_\chi\mathfrak{p}^{\mu+1}):k(\mathfrak{f}_\chi\mathfrak{p})]\sum_{v|\mathfrak{p}}\mathrm{ord}_v(z_\mu)\cdot v, & \\ & \text{if } \mathfrak{p}\mid\mathfrak{f} \end{cases} \\
= \; & \begin{cases} \frac{f_\mathfrak{p}}{p^n}[k(\mathfrak{b}):k(1)]\frac{w(1)}{w(\mathfrak{f}_\chi)}\sum_{v|\mathfrak{p}}\mathrm{ord}_v(N_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{f}_0)}z_0)\cdot v, & \text{if } \mathfrak{p}\nmid\mathfrak{f}, \\ \frac{f_\mathfrak{p}}{p^n}w(1)[k(\mathfrak{b}):k(1)][k(\mathfrak{f}_\chi\mathfrak{p}^{\mu+1}):k(\mathfrak{f}_\chi)]\sum_{v|\mathfrak{p}}\mathrm{ord}_v\left(\bar\beta_1\right)\cdot v, & \text{if } \mathfrak{p}\mid\mathfrak{f} \end{cases} \\
= \; & \begin{cases} \frac{f_\mathfrak{p}}{p^n}\frac{w(1)}{w(\mathfrak{f}_\chi)}[k(\mathfrak{b}):k(1)]\sum_{v|\mathfrak{p}}\mathrm{ord}_v(\bar\beta_1)\cdot v, & \text{if } \mathfrak{p}\nmid\mathfrak{f}, \\ \frac{f_\mathfrak{p}}{p^n}w(1)[k(\mathfrak{b}):k(1)][k(\mathfrak{f}_\chi\mathfrak{p}^{\nu}):k(\mathfrak{f}_\chi)]\sum_{v|\mathfrak{p}}\mathrm{ord}_v\left(\bar\beta_1\right)\cdot v, & \text{if } \mathfrak{p}\mid\mathfrak{f} \end{cases}
\end{aligned}
$$

We first assume that $\mathfrak{p} \mid \mathfrak{f}$ and use this data to compute

$$\begin{aligned}
& (\phi \circ \phi_{\bar{\omega}}^{-1})(\bar{\beta}_1^{-1} \otimes \bar{\beta}_2) \\
= \; & \phi\left(\bar{\beta}_1^{-1} \wedge [N_{k(\mathfrak{f}_\chi \mathfrak{b})/k(\mathfrak{f}_\chi)} \psi(1; \mathfrak{f}_\chi \mathfrak{b}, \mathfrak{a})]^{-1} \right. \\
& \qquad \left. \wedge x_J^{-1} \otimes \beta_{\bar{\omega}}(x_J) \wedge \beta_{\bar{\omega}}(N_{k(\mathfrak{f}_\chi \mathfrak{b})/k(\mathfrak{f}_\chi)}(\psi(1; \mathfrak{f}_\chi \mathfrak{b}, \mathfrak{a}))) \wedge \bar{\beta}_2\right) \\
= \; & c_\chi \frac{f_\mathfrak{p}}{p^n} w(1)[k(\mathfrak{b}) : k(1)][k(\mathfrak{f}_\chi \mathfrak{p}^\nu) : k(\mathfrak{f}_\chi)] \times \\
& \phi\left(\bar{\beta}_1^{-1} \wedge [N_{k(\mathfrak{f}_\chi \mathfrak{b})/k(\mathfrak{f}_\chi)}(\psi(1; \mathfrak{f}_\chi \mathfrak{b}, \mathfrak{a}))]^{-1} \wedge x_J^{-1} \otimes \mathrm{val}(x_J) \wedge \mathrm{val}(\bar{\beta}_1) \wedge \bar{\beta}_2\right) \\
= \; & \underbrace{-c_\chi \frac{f_\mathfrak{p}}{p^n} w(1)[k(\mathfrak{b}) : k(1)][k(\mathfrak{f}_\chi \mathfrak{p}^\nu) : k(\mathfrak{f}_\chi)][N_{k(\mathfrak{f}_\chi \mathfrak{b})/k(\mathfrak{f}_\chi)}(\psi(1; \mathfrak{f}_\chi \mathfrak{b}, \mathfrak{a}))]^{-1} \otimes \sigma_\infty|_{k(\mathfrak{f})}}_{=:A}.
\end{aligned}$$

On the other hand we note that $\mathfrak{f}_\chi = \mathfrak{f}_{\chi,0}$ and compute

$$\mathcal{L} \quad = \quad (N\mathfrak{a} - \sigma(\mathfrak{a}))[k(\mathfrak{f}_0 \mathfrak{p}) : k(\mathfrak{f}_\chi \mathfrak{p})] \prod_{\mathfrak{l} \mid \mathfrak{f}_0, \mathfrak{l} \nmid \mathfrak{f}_\chi} \frac{1}{1 - \mathrm{Fr}_\mathfrak{l}^{-1}} \eta_{\mathfrak{f}_\chi}^{-1} \otimes \sigma_\infty.$$

In addition, one has

$$(\gamma - 1)\beta_1 = (\gamma - 1)z^\infty = \frac{1}{[k(\mathfrak{f}_\chi \mathfrak{b} \mathfrak{p}) : F]} N_{k(\mathfrak{f}_\chi \mathfrak{b} \mathfrak{p})/F}(\eta)$$

and

$$\frac{\bar{\omega}}{1 - \gamma} = T := 1 + \gamma + \ldots + \gamma^{p^n - 1}.$$

This implies the equality

$$\begin{aligned}
\bar{\omega}\beta_1 \quad &= \quad (1 - \gamma)T\beta_1 = -T \frac{1}{[k(\mathfrak{f}_\chi \mathfrak{b} \mathfrak{p}) : k(\mathfrak{f}_\chi \mathfrak{p})]} N_{k(\mathfrak{f}_\chi \mathfrak{b} \mathfrak{p})/k(\mathfrak{f}_\chi \mathfrak{p})}(\eta) \\
&= \quad -T \frac{1}{[k(\mathfrak{f}_\chi \mathfrak{b} \mathfrak{p}) : k(\mathfrak{f}_\chi \mathfrak{p})]} (1 - \sigma(\mathfrak{b})^{-1}) \eta_{\mathfrak{f}_\chi} \\
&= \quad -T \frac{1}{w(1)[k(\mathfrak{b}) : k(1)]} (1 - \sigma(\mathfrak{b})^{-1}) \eta_{\mathfrak{f}_\chi}
\end{aligned}$$

in $U^\infty_{S_p, \mathfrak{q}_\chi}$. Since $e = -(|J| + 1)$ we obtain $\mathcal{L} = B\bar{\omega}^e(\beta_1^{-1} \otimes \beta_2)$ with

$$\begin{aligned}
B \quad = \quad & -T(N\mathfrak{a} - \sigma(\mathfrak{a}))[k(\mathfrak{f}_0 \mathfrak{p}) : k(\mathfrak{f}_\chi \mathfrak{p})](w(1)[k(\mathfrak{b}) : k(1)])^{-1} \times \\
& (1 - \sigma(\mathfrak{b})^{-1}) \prod_{\substack{\mathfrak{l} \mid \mathfrak{f}_0, \mathfrak{l} \nmid \mathfrak{f}_\chi \\ \chi(\mathfrak{l}) \neq 1}} \frac{1}{1 - \mathrm{Fr}_\mathfrak{l}^{-1}} \prod_{\mathfrak{l} \in J} \frac{\bar{\omega}}{1 - \mathrm{Fr}_\mathfrak{l}^{-1}}.
\end{aligned}$$

Again we deduce from [8, Lem. 5.7] that $\phi(\mathcal{L} \otimes 1) = AB$. From

$$[k(\mathfrak{f}_\chi \mathfrak{p}^\nu) : k(\mathfrak{f}_\chi)][k(\mathfrak{f}_0 \mathfrak{p} : k(\mathfrak{f}_\chi \mathfrak{p})] = [k(\mathfrak{f}) : k(\mathfrak{f}_\chi)]$$

(recall again that $w(\mathfrak{p}) = w(\mathfrak{f}) = 1$) and

$$N_{k(\mathfrak{f}_\chi\mathfrak{b})/k(\mathfrak{f}_\chi)}(\psi(1;\mathfrak{f}_\chi\mathfrak{b},\mathfrak{a})^{w(\mathfrak{f}_\chi)}) = \begin{cases}(1-\sigma(\mathfrak{b})^{-1})\psi(1;\mathfrak{f}_\chi,\mathfrak{a}), & \text{if } \mathfrak{f}_\chi \neq 1,\\ \frac{\delta(\mathcal{O}_k,\mathfrak{a}^{-1})}{\delta(\mathfrak{b},\mathfrak{a}^{-1}\mathfrak{b})}, & \text{if } \mathfrak{f}_\chi = 1,\end{cases}$$

we compute

$$\begin{aligned} & AB = \\ = \quad & w(\mathfrak{f}_\chi)[k(\mathfrak{f}):k(\mathfrak{f}_\chi)](N\mathfrak{a}-\sigma(\mathfrak{a}))\prod_{\substack{\mathfrak{l}\mid\mathfrak{f}_0\\ \chi(\mathfrak{l})\neq 1}}\frac{1}{1-\mathrm{Fr}_\mathfrak{l}^{-1}}\prod_{\mathfrak{l}\in J\cup\{\mathfrak{p}\}} f_\mathfrak{l}\times \\ & \begin{cases}\psi(1;\mathfrak{f}_\chi,\mathfrak{a})^{-1}\otimes\sigma_\infty|_{k(\mathfrak{f})}, & \text{if } \mathfrak{f}_\chi\neq 1,\\ (1-\sigma(\mathfrak{b})^{-1})\frac{\delta(\mathcal{O}_k,\mathfrak{a}^{-1})}{\delta(\mathfrak{b},\mathfrak{a}^{-1}\mathfrak{b})}\otimes\sigma_\infty|_{k(\mathfrak{f})}, & \text{if } \mathfrak{f}_\chi = 1.\end{cases}\end{aligned}$$

Finally we use in the case $\mathfrak{f}_\chi = 1$ the relation

$$\left(\frac{\delta(\mathcal{O}_k,\mathfrak{a}^{-1})}{\delta(\mathfrak{b},\mathfrak{a}^{-1}\mathfrak{b})}\right)^{1-\sigma(\mathfrak{p})^{-1}} = \left(\frac{\delta(\mathcal{O}_k,\mathfrak{a}^{-1})}{\delta(\mathfrak{p},\mathfrak{a}^{-1}\mathfrak{p})}\right)^{1-\sigma(\mathfrak{b})^{-1}}$$

and recover the equation (32). The case $\mathfrak{p}\nmid\mathfrak{f}$ is completely analogous.

The case of the trivial character In this case $\beta_1 = \eta_1$ and we first have to compute $\bar\beta_1$. If $\mathfrak{p}\nmid\mathfrak{f}$, the $\bar\beta_1 = N_{k(\mathfrak{f}\mathfrak{p})/k(\mathfrak{f})}(\psi(1;\mathfrak{q},\mathfrak{a}))$ and the distribution relation [1, Th. 2.3 b)] implies

$$\bar\beta_1 = N_{k(\mathfrak{q})/k(1)}(\psi(1;\mathfrak{q},\mathfrak{a})^{w(1)}) = \frac{\delta(\mathcal{O}_k,\mathfrak{a}^{-1})}{\delta(\mathfrak{p},\mathfrak{p}\mathfrak{a}^{-1})},$$

where δ denotes the function of lattices defined in [21, Th. 1]. We recall that

$$\delta(L,\underline{L})^{12} = \frac{\Delta(L)^{[\underline{L}:L]}}{\Delta(\underline{L})}.$$

If $\mathfrak{p}\mid\mathfrak{f}$, then $\bar\beta_1 = \psi(1;\mathfrak{p}^\nu,\mathfrak{a})$, where again $\mathfrak{f} = \mathfrak{f}_0\mathfrak{p}^\nu$.
We now want to compute $(\phi\circ\phi_{\bar\omega}^{-1})(\bar\beta_1^{-1}\otimes\bar\beta_2)$. Since χ is now trivial we no longer have $X_{\{w|\infty\}}\otimes_\mathfrak{A}\mathbb{Q}_p(\chi) = Y_{\{w|\infty\}}\otimes_\mathfrak{A}\mathbb{Q}_p(\chi)$, and therefore have to take into account the short exact sequence

$$0\longrightarrow X^\infty_{\{w|\mathfrak{f}_0\mathfrak{p}\}}\longrightarrow X^\infty_{\{w|\mathfrak{f}_0\mathfrak{p}\infty\}}\longrightarrow Y^\infty_{\{w|\infty\}}\longrightarrow 0 \tag{38}$$

in the definition of $\phi_{\bar\omega}$. Recall here that $H^2(\Delta) = X^\infty_{\{w|\mathfrak{f}_0\mathfrak{p}\infty\}}$ and $Y^\infty_{\{w|\infty\}} = M^2$. A lift of $\sigma|_{k(\mathfrak{f})}\in M^2/\bar\omega = Y_{\{w|\infty\}}$ is given by $\sigma|_{k(\mathfrak{f})} - w_\mathfrak{p}$, where $w_\mathfrak{p}$ denotes a fixed place of $k(\mathfrak{f})$ above $\mathfrak{p}$. We obtain

$$\begin{aligned}(\phi\circ\phi_{\bar\omega}^{-1})(\bar\beta_1^{-1}\otimes\bar\beta_2) &= \phi(\bar\beta_1^{-1}\wedge x_J^{-1}\otimes\beta_{\bar\omega}(x_J)\wedge(\sigma|_{k(\mathfrak{f})}-w_\mathfrak{p}))\\ &= c_\chi\phi(\bar\beta_1^{-1}\wedge x_J^{-1}\otimes\mathrm{val}(x_J)\wedge(\sigma|_{k(\mathfrak{f})}-w_\mathfrak{p}))\end{aligned}$$

Next, we compute $\mathrm{val}(\bar{\beta}_1)$ and express the result in terms of $\sigma|_{k(\mathfrak{f})} - w_\mathfrak{p}$. If $\mathfrak{p} \nmid \mathfrak{f}$, then

$$\mathrm{val}(\bar{\beta}_1) = \frac{1}{12}(N\mathfrak{a} - 1)\mathrm{val}\left(\frac{\Delta(\mathcal{O}_k)}{\Delta(\mathfrak{p})}\right).$$

We use $\Delta(\mathcal{O}_k)/\Delta(\mathfrak{p}) \sim \mathfrak{p}^{12}$ and obtain in $Y_{\{w|\mathfrak{f}_0\mathfrak{p}\}} \otimes_\mathfrak{A} \mathbb{Q}_p(\chi)$

$$\begin{aligned} \mathrm{val}(\bar{\beta}_1) &= (N\mathfrak{a} - 1)\sum_{w|\mathfrak{p}} \mathrm{ord}_w(\mathfrak{p}) \cdot w \\ &= (N\mathfrak{a} - 1)|I_\mathfrak{p}|\sum_{w|\mathfrak{p}} w \\ &= (N\mathfrak{a} - 1)|I_\mathfrak{p}|\frac{|G_\mathfrak{f}|}{|D_\mathfrak{p}|} w_\mathfrak{p} \\ &= (N\mathfrak{a} - 1)\frac{[k(\mathfrak{f}) : k]}{f_\mathfrak{p}} w_\mathfrak{p} \end{aligned}$$

An explicit splitting of the short exact sequence (14) is given by

$$w \mapsto w - \frac{1}{[k(\mathfrak{f}) : k]}\mathrm{Tr}_{k(\mathfrak{f})/k}\sigma|_{k(\mathfrak{f})}.$$

Under this map $\mathrm{val}(\bar{\beta}_1)$ maps to $-(N\mathfrak{a} - 1)\frac{[k(\mathfrak{f}):k]}{f_\mathfrak{p}}(\sigma|_{k(\mathfrak{f})} - w_\mathfrak{p})$ in $X_{\{w|\mathfrak{f}\mathfrak{p}\infty\}} \otimes_\mathfrak{A} \mathbb{Q}_p(\chi)$.

Recall that $\varphi_{\mathcal{O}_k}$ denotes the Euler function attached to the ring $\mathcal{O}_k$. In the case $\mathfrak{p} \mid \mathfrak{f}$ we compute from [1, Th. 2.4]

$$\begin{aligned} \mathrm{val}(\bar{\beta}_1) &= \frac{N\mathfrak{a} - 1}{\varphi_{\mathcal{O}_k}(\mathfrak{p}^\nu)}\sum_{w|\mathfrak{p}} \mathrm{ord}_w(\mathfrak{p}) \cdot w \\ &= \frac{N\mathfrak{a} - 1}{\varphi_{\mathcal{O}_k}(\mathfrak{p}^\nu)}\frac{[k(\mathfrak{f}) : k]}{f_\mathfrak{p}} w_\mathfrak{p}. \end{aligned}$$

So we derive the closed formula

$$\mathrm{val}(\bar{\beta}_1) = -\frac{N\mathfrak{a} - 1}{\varphi_{\mathcal{O}_k}(\mathfrak{p}^\nu)}\frac{[k(\mathfrak{f}) : k]}{f_\mathfrak{p}}(\sigma|_{k(\mathfrak{f})} - w_\mathfrak{p})$$

as elements of $X_{\{w|\mathfrak{f}\mathfrak{p}\infty\}} \otimes_\mathfrak{A} \mathbb{Q}_p(\chi)$.

This implies

$$(\phi \circ \phi_{\bar{\omega}}^{-1})(\bar{\beta}_1^{-1}) = \underbrace{-c_\chi \frac{\varphi_{\mathcal{O}_k}(\mathfrak{p}^\nu)}{N\mathfrak{a} - 1}\frac{f_\mathfrak{p}}{[k(\mathfrak{f}) : k]}}_{=:A}$$

On the other hand we compute for $\mathcal{L}\otimes 1$

$$\begin{aligned}
\mathcal{L}\otimes 1 &= (N\mathfrak{a}-\sigma(\mathfrak{a}))\eta_{\mathfrak{f}_0}^{-1}\otimes\sigma \\
&= (N\mathfrak{a}-\sigma(\mathfrak{a}))\,[k(\mathfrak{f}_0):k(1)]\,\frac{w(1)}{w(\mathfrak{f}_0)}\left[\mathrm{Tr}_{k(\mathfrak{f}_0\mathfrak{p}^{n+1})/k(\mathfrak{p}^{n+1})}\eta_{\mathfrak{f}_0}^{-1}\right]^{-1}\otimes\sigma \\
&= (N\mathfrak{a}-\sigma(\mathfrak{a}))\,[k(\mathfrak{f}_0):k(1)]\,\frac{w(1)}{w(\mathfrak{f}_0)}\prod_{\mathfrak{l}|\mathfrak{f}_0}\frac{1}{1-\mathrm{Fr}_{\mathfrak{l}}^{-1}}\eta_1^{-1}\otimes\sigma \\
&= \underbrace{(N\mathfrak{a}-\sigma(\mathfrak{a}))\,[k(\mathfrak{f}_0):k(1)]\,\frac{w(1)}{w(\mathfrak{f}_0)}\prod_{\mathfrak{l}|\mathfrak{f}_0}\frac{\bar{\omega}}{1-\mathrm{Fr}_{\mathfrak{l}}^{-1}}}_{=:B}\,\bar{\omega}^e(\beta_1^{-1}\otimes\bar{\beta}_2).
\end{aligned}$$

It follows from (35) together with

$$[k(\mathfrak{f}):k] = h_k\frac{w(\mathfrak{f})}{w(1)}\varphi_{\mathcal{O}_k}(\mathfrak{f}),\quad [k(\mathfrak{f}_0):k(1)] = \frac{w(\mathfrak{f}_0)}{w(1)}\varphi_{\mathcal{O}_k}(\mathfrak{f}_0)$$

that $\phi(\mathcal{L}\otimes 1) = AB = -f_{\mathfrak{p}}\left(\prod_{\mathfrak{l}|\mathfrak{f}_0} f_{\mathfrak{l}}\right)\frac{w(1)}{h_k}$. Since $\zeta_k^*(0) = -\frac{h_k}{w(1)}$ this concludes the proof.

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