Statistics of Lattice Points in Thin Annuli for Generic Lattices

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Abstract. We study the statistical properties of the counting function of lattice points inside thin annuli. By a conjecture of Bleher and Lebowitz, if the width shrinks to zero, but the area converges to infinity, the distribution converges to the Gaussian distribution. If the width shrinks slowly to zero, the conjecture was proven by Hughes and Rudnick for the standard lattice, and in our previous paper for generic rectangular lattices. We prove this conjecture for arbitrary lattices satisfying some generic Diophantine properties, again assuming the width of the annuli shrinks slowly to zero. One of the obstacles of applying the technique of Hughes-Rudnick on this problem is the existence of so-called close pairs of lattice points. In order to overcome this difficulty, we bound the rate of occurence of this phenomenon by extending some of the work of Eskin-Margulis-Mozes on the quantitative Openheim conjecture.

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1 Introduction

We consider a variant of the circle problem. Let $\Lambda \subset \mathbb{R}^2$ be a planar lattice, with $\det \Lambda$ the area of its fundamental cell. Let

$$N_\Lambda(t) = \{x \in \Lambda : \, |x| \leq t\},$$

denote its counting function, that is, we are counting Λ-points inside a disc of radius t.
As well known, as $t \to \infty$, $N_\Lambda(t) \sim \frac{\pi}{\det \Lambda} t^2$. Denoting the remainder or the error term

$$\Delta_\Lambda(t) = N_\Lambda(t) - \frac{\pi}{\det \Lambda} t^2,$$

it is a conjecture of Hardy that

$$|\Delta_\Lambda(t)| \ll_\epsilon t^{1/2+\epsilon}.$$

Another problem one could study is the *statistical* behavior of the value distribution of Δ_Λ normalized by $\sqrt{t}$, namely of

$$F_\Lambda(t) := \frac{\Delta_\Lambda(t)}{\sqrt{t}}.$$

Heath-Brown [HB] shows that for the standard lattice $\Lambda = \mathbb{Z}^2$, the value distribution of F_Λ, weakly converges to a non-Gaussian distribution with density $p(x)$. Bleher [BL3] established an analogue of this theorem for a more general setting, where in particular it implies a non-Gaussian limiting distribution of F_Λ, for any lattice $\Lambda \subset \mathbb{Z}^2$.
However, the object of our interest is slightly different. Rather than counting lattice points in the circle of varying radius t, we will do the same for *annuli*. More precisely, we define

$$N_\Lambda(t,\, \rho) := N_\Lambda(t+\rho) - N_\Lambda(t),$$

that is, the number of Λ-points inside the annulus of inner radius t and width ρ. The "expected" value is the area $\frac{\pi}{\det \Lambda}(2t\rho + \rho^2)$, and the corresponding normalized remainder term is

$$S_\Lambda(t,\, \rho) := \frac{N_\Lambda(t+\rho) - N_\Lambda(t) - \frac{\pi}{\det \Lambda}(2t\rho + \rho^2)}{\sqrt{t}}.$$

The statistics of $S_\Lambda(t,\, \rho)$ vary depending to the size of $\rho(t)$. Of our particular interest is the *intermediate* or *macroscopic regime*. Here $\rho \to 0$, but $\rho t \to \infty$. A particular case of the conjecture of Bleher and Lebowitz [BL4] states that $S_\Lambda(t,\, \rho)$ has a Gaussian distribution. In 2004 Hughes and Rudnick [HR] established the Gaussian distribution for the unit circle, under an additional assumption that $\rho(t) \gg t^{-\epsilon}$ for every $\epsilon > 0$.
By a rotation and dilation (which does not effect the counting function), we may assume, with no loss of generality, that Λ admits a basis one of whose elements is the vector $(1, 0)$, that is $\Lambda = \langle 1,\, \alpha + i\beta \rangle$ (we make the natural identification of i with $(0,\, 1)$). In a previous paper [W] we already dealt with the problem of investigating the statistical properties of the error term for rectangular lattice $\Lambda = \langle 1,\, i\beta \rangle$. We established the limiting Gaussian distribution for the "generic" case in this 1-parameter family.

Some of the work done in [W] extends quite naturally for the 2-parameter family of planar lattices $\langle 1, \alpha + i\beta \rangle$. That is, in the current work we will require the algebraic independence of α and β, as well as a strong Diophantine property of the pair (α, β) (to be defined), rather than the transcendence and a strong Diophantine property of the aspect ratio of the ellipse, as in [W].
We say that a real number ξ is *strongly Diophantine*, if for every *fixed* natural n, there exists $K_1 > 0$, such that for integers a_j with $\sum\limits_{j=0}^{n} a_j \xi^j \neq 0$,

$$\left| \sum_{j=0}^{n} a_j \xi^j \right| \gg_n \frac{1}{\left(\max\limits_{0 \le j \le n} |a_j| \right)^{K_1}}.$$

It was shown by Mahler [MAH], that this property holds for a "generic" real number. We say that a pair of numbers (α, β) is *strongly Diophantine*, if for every *fixed* natural n, there exists a number $K_1 > 0$, such that for every integral polynomial $p(x, y) = \sum\limits_{i+j \le n} a_{i,j} x^i y^j$ of degree $\le n$, we have

$$|p(\alpha, \beta)| \gg_n \frac{1}{\max\limits_{i+j \le n} |a_{i,j}|^{K_1}},$$

whenever $p(\alpha, \beta) \neq 0$. This holds for almost all real pairs (α, β), see section 2.2.

Theorem 1.1. *Let $\Lambda = \langle 1, \alpha + i\beta \rangle$ where (α, β) is algebraically independent and strongly Diophantine pair of real numbers. Assume that $\rho = \rho(T) \to 0$, but for every $\delta > 0$, $\rho \gg T^{-\delta}$. Then for every interval $\mathcal{A}$,*

$$\lim_{T \to \infty} \frac{1}{T} meas \left\{ t \in [T, 2T] : \frac{S_\Lambda(t, \rho)}{\sigma} \in \mathcal{A} \right\} = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathcal{A}} e^{-\frac{x^2}{2}} dx, \tag{1}$$

where the variance is given by

$$\sigma^2 := \frac{4\pi}{\beta} \cdot \rho. \tag{2}$$

Remark: Note that the variance σ^2 is α-independent, since the determinant $\det(\Lambda) = \beta$.
One of the features of a rectangular lattice is that it is quite easy to show that the number of so-called close pairs of lattice points or pairs of points lying within a narrow annulus is bounded by essentially its average (see lemma 5.2 of [W]). This particular feature of the rectangular lattices was exploited while reducing the computation of the moments to the ones of a smooth counting function (we call it "unsmoothing"). In order to prove an analogous bound for a general lattice, we extend a result from Eskin, Margulis and Mozes [EMM] for our needs to obtain proposition 3.1. We believe that this proposition is of independent interest.

2 The distribution of $\tilde{S}_{\Lambda,\,M,\,L}$

We apply the same smoothing as in [HR] and [W]: let χ be the indicator function of the unit disc and ψ a nonnegative, smooth, even function on the real line, of total mass unity, whose Fourier transform, $\hat{\psi}$ is smooth and has compact support. Introduce a rotationally symmetric function Ψ on $\mathbb{R}^2$ by setting $\hat{\Psi}(\vec{y}) = \hat{\psi}(|\vec{y}|)$, where $|\cdot|$ denotes the standard Euclidian norm. For $\epsilon > 0$, set

$$\Psi_\epsilon(\vec{x}) = \frac{1}{\epsilon^2}\Psi\left(\frac{\vec{x}}{\epsilon}\right).$$

Define a *smooth* counting function

$$\tilde{N}_{\Lambda,M}(t) = \sum_{\vec{n}\in\Lambda} \chi_\epsilon\left(\frac{\vec{n}}{t}\right), \tag{3}$$

with $\epsilon = \epsilon(M)$ and $\chi_\epsilon = \chi * \Psi_\epsilon$, the convolution of χ with Ψ_ϵ. In what will follow,

$$\epsilon = \frac{1}{t\sqrt{M}}, \tag{4}$$

where $M = M(T)$ is the smoothing parameter, which tends to infinity with t. In this section, we are interested in the distribution of the smooth version of $S_\Lambda(t,\,\rho)$, denoted $\tilde{S}_{\Lambda,\,M,\,L}(t)$, where $L := \frac{1}{\rho}$, defined by

$$\tilde{S}_{\Lambda,\,M,\,L}(t) = \frac{\tilde{N}_{\Lambda,\,M}(t+\frac{1}{L}) - \tilde{N}_{\Lambda,\,M}(t) - \frac{\pi}{d}(\frac{2t}{L}+\frac{1}{L^2})}{\sqrt{t}}, \tag{5}$$

We assume that for every $\delta > 0$, $L = L(T) = O(T^\delta)$, which corresponds to the assumption of theorem 1.1 regarding $\rho := \frac{1}{L}$.
Rather than drawing t at random from $[T,\,2T]$ with a uniform distribution, we prefer to work with smooth densities: introduce $\omega \geq 0$, a smooth function of total mass unity, such that both ω and $\hat{\omega}$ are rapidly decaying, namely

$$|\omega(t)| \ll \frac{1}{(1+|t|)^A},\quad |\hat{\omega}(t)| \ll \frac{1}{(1+|t|)^A},$$

for every $A > 0$. Define the averaging operator

$$\langle f\rangle_T = \frac{1}{T}\int\limits_{-\infty}^{\infty} f(t)\omega(\frac{t}{T})dt,$$

and let $\mathbb{P}_{\omega,\,T}$ be the associated probability measure:

$$\mathbb{P}_{\omega,\,T}(f\in\mathcal{A}) = \frac{1}{T}\int\limits_{-\infty}^{\infty} 1_{\mathcal{A}}(f(t))\omega(\frac{t}{T})dt.$$

Remark: In what follows, we will suppress the explicit dependency on T, whenever convenient.

Theorem 2.1. *Suppose that $M(T)$ and $L(T)$ are increasing to infinity with T, such that $M = O(T^{\delta})$ for all $\delta > 0$, and $L/\sqrt{M} \to 0$. Then if (α, β) is an* algebraically independent strongly Diophantine *pair, we have for $\Lambda = \langle 1, \alpha + i\beta \rangle$,*

$$\lim_{T\to\infty} \mathbb{P}_{\omega, T}\left\{ \frac{\tilde{S}_{\Lambda, M, L}}{\sigma} \in \mathcal{A} \right\} = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathcal{A}} e^{-\frac{x^2}{2}} dx,$$

for any interval $\mathcal{A}$, where

$$\sigma^2 := \frac{4\pi}{\beta L}. \tag{6}$$

Definition: A tuple of real numbers $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ is called *Diophantine*, if there exists a number $K > 0$, such that for every integer tuple $\{a_i\}_{i=0}^{n}$,

$$\left| a_0 + \sum_{i=1}^{n} a_i \alpha_i \right| \gg \frac{1}{q^K}, \tag{7}$$

with $q = \max\limits_{0 \le i \le n} |a_i|$, whenever the LHS of the inequality doesn't vanish. Khintchine proved that *almost all* tuples in $\mathbb{R}^n$ are Diophantine (see, e.g. [S], pages 60-63).

Denote the dual lattice

$$\Lambda^* = \langle 1, \gamma + i\delta \rangle$$

with $\gamma = -\frac{\alpha}{\beta}$ and $\delta = \frac{1}{\beta}$. In the rest of the current section, we assume, that, unless specified otherwise, the set of the squared lengths of vectors in Λ^* satisfy the Diophantine property. That means, that $(\alpha^2, \alpha\beta, \beta^2)$ is a Diophantine triple of real numbers. We may assume $(\alpha^2, \alpha\beta, \beta^2)$ being Diophantine, since theorem 1.1 (and theorem 2.1) assume (α, β) is *strongly Diophantine*, which is, obviously, a stronger assumption.

We use the following approximation to $\tilde{N}_{\Lambda,M}(t)$ (see e.g [W], lemma 4.1), which holds unconditionally on any Diophantine assumption:

Lemma 2.2. *As $t \to \infty$,*

$$\tilde{N}_{\Lambda,M}(t) = \frac{\pi t^2}{\beta} - \frac{\sqrt{t}}{\beta\pi} \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{\cos\left(2\pi t|\vec{k}| + \frac{\pi}{4}\right)}{|\vec{k}|^{\frac{3}{2}}} \cdot \hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right) + O\left(\frac{1}{\sqrt{t}}\right), \tag{8}$$

where, again, Λ^* is the dual lattice.

By the definition of $\tilde{S}_{\Lambda, M, L}$ in (5) and appropriately manipulating the sum in (8) we obtain the following

Corollary 2.3.

$$\tilde{S}_{\Lambda,\,M,\,L}(t)=\frac{2}{\beta\pi}\sum_{\vec{k}\in\Lambda^*\setminus\{0\}}\frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^{\frac{3}{2}}}\sin\left(2\pi\big(t+\frac{1}{2L}\big)|\vec{k}|+\frac{\pi}{4}\right)\hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right)+O\left(\frac{1}{\sqrt{t}}\right). \tag{9}$$

One should note that $\hat{\psi}$ being compactly supported means that the sum essentially truncates at $|\vec{k}|\approx\sqrt{M}$.
Unlike the standard lattice, clearly there are no nontrivial multiplicities in Λ, that is

Lemma 2.4. *Let $\vec{a_j}=m_j+n_j(\alpha+i\beta)\in\Lambda$, $j=1,\,2$, with an irrational α such that $\beta\notin\mathbb{Q}(\alpha)$. Then if $|\vec{a_1}|=|\vec{a_2}|$, either $n_1=n_2$ and $m_1=m_2$ or $n_1=-n_2$ and $n_2=-m_2$.*

Proof of theorem 2.1. We will show that the moments of $\tilde{S}_{\Lambda,\,M,\,L}$ corresponding to the smooth probability space converge to the moments of the normal distribution with zero mean and variance which is given by theorem 2.1. This allows us to deduce that the distribution of $\tilde{S}_{\Lambda,\,M,\,L}$ converges to the normal distribution as $T\to\infty$, precisely in the sense of theorem 2.1.
First, we show that the mean is $O(\frac{1}{\sqrt{T}})$. Since ω is real,

$$\left|\left\langle\sin\left(2\pi\big(t+\frac{1}{2L}\big)|\vec{k}|+\frac{\pi}{4}\right)\right\rangle\right|=\left|\Im m\left\{\hat{\omega}\big(-T|\vec{k}|\big)e^{i\pi(\frac{|\vec{k}|}{L}+\frac{1}{4}}\right\}\right|\ll\frac{1}{T^A|\vec{k}|^A}$$

for any $A>0$, where we have used the rapid decay of $\hat{\omega}$. Thus

$$\left|\left\langle\tilde{S}_{\Lambda,\,M,\,L}\right\rangle\right|\ll\sum_{\vec{k}\in\Lambda^*\setminus\{0\}}\frac{1}{T^A|\vec{k}|^{A+3/2}}+O\left(\frac{1}{\sqrt{T}}\right)\ll O\left(\frac{1}{\sqrt{T}}\right),$$

due to the convergence of $\sum\limits_{\vec{k}\in\Lambda^*\setminus\{0\}}\frac{1}{|\vec{k}|^{A+3/2}}$, for $A>\frac{1}{2}$
Now define

$$\mathcal{M}_{\Lambda,\,m}:=\left\langle\left(\frac{2}{\beta\pi}\sum_{\vec{k}\in\Lambda^*\setminus\{0\}}\frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^{\frac{3}{2}}}\sin\left(2\pi\big(t+\frac{1}{2L}\big)|\vec{k}|+\frac{\pi}{4}\right)\hat{\psi}\big(\frac{|\vec{k}|}{\sqrt{M}}\big)\right)^m\right\rangle \tag{10}$$

Then from (9), the binomial formula and the Cauchy-Schwartz inequality,

$$\left\langle\big(\tilde{S}_{\Lambda,\,M,\,L}\big)^m\right\rangle=\mathcal{M}_{\Lambda,\,m}+O\left(\sum_{j=1}^{m}\binom{m}{j}\frac{\sqrt{\mathcal{M}_{2m-2j}}}{T^{j/2}}\right)$$

Proposition 2.5 together with proposition 2.8 allow us to deduce the result of theorem 2.1 for an algebraically independent strongly Diophantine $(\xi,\,\eta) := (-\frac{\alpha}{\beta},\,\frac{1}{\beta})$. Clearly, $(\alpha,\,\beta)$ being algebraically independent and strongly Diophantine is sufficient. □

2.1 The variance

The computation of the variance is done in two steps. First, we reduce the main contribution to the *diagonal* terms, using the assumption on the pair $(\alpha,\,\beta)$ (i.e. $(\alpha^2,\,\alpha\beta,\,\beta^2)$ is *Diophantine*). Then we compute the contribution of the *diagonal* terms. Both these steps are very close to the corresponding ones in [W].
Suppose that the triple $(\alpha^2,\,\alpha\beta,\,\beta^2)$ satisfies (7).

Proposition 2.5. *If $M = O\big(T^{1/(K+1/2+\delta)}\big)$ for fixed $\delta > 0$, then the variance of $\tilde{S}_{\Lambda,\,M,\,L}$ is asymptotic to*

$$\sigma^2 := \frac{4}{\beta^2\pi^2}\sum_{\vec{k}\in\Lambda^*\setminus\{0\}}\frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3}\hat{\psi}^2\left(\frac{|\vec{k}|}{\sqrt{M}}\right)$$

If $L\to\infty$, but $L/\sqrt{M}\to 0$, then

$$\sigma^2 \sim \frac{4\pi}{\beta L} \tag{11}$$

Remark: In the formulation of proposition 2.5, K is implicitly given by (7).

Proof. Expanding out (10), we have

$$\begin{aligned}\mathcal{M}_{\Lambda,\,2} = \frac{4}{\beta^2\pi^2}\sum_{\vec{k},\vec{l}\in\Lambda^*\setminus\{0\}}&\frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)\sin\left(\frac{\pi|\vec{l}|}{L}\right)\hat{\psi}(\frac{|\vec{k}|}{\sqrt{M}})\hat{\psi}(\frac{|\vec{l}|}{\sqrt{M}})}{|\vec{k}|^{\frac{3}{2}}|\vec{l}|^{\frac{3}{2}}}\\ &\times\left\langle\sin\left(2\pi\left(t+\frac{1}{2L}\right)|\vec{k}|+\frac{\pi}{4}\right)\sin\left(2\pi\left(t+\frac{1}{2L}\right)|\vec{l}|+\frac{\pi}{4}\right)\right\rangle\end{aligned} \tag{12}$$

It is easy to check that the average of the second line of the previous equation is:

$$\begin{aligned}\frac{1}{4}\Big[&\hat{\omega}\big(T(|\vec{k}|-|\vec{l}|)\big)e^{i\pi(1/L)(|\vec{l}|-|\vec{k}|)}+\\ &\hat{\omega}\big(T(|\vec{l}|-|\vec{k}|)\big)e^{i\pi(1/L)(|\vec{k}|-|\vec{l}|)}+\\ &\hat{\omega}\big(T(|\vec{k}|+|\vec{l}|)\big)e^{-i\pi(1/2+(1/L)(|\vec{k}|+|\vec{l}|))}-\\ &\hat{\omega}\big(-T(|\vec{k}|+|\vec{l}|)\big)e^{i\pi(1/2+(1/L)(|\vec{k}|+|\vec{l}|))}\Big]\end{aligned} \tag{13}$$

Recall that the support condition on $\hat{\psi}$ means that $\vec{k}$ and $\vec{l}$ are both constrained to be of length $O(\sqrt{M})$. Thus the off-diagonal contribution (that is for $|\vec{k}| \neq |\vec{l}|$) of the first two lines of (13) is

$$\ll \sum_{\substack{\vec{k},\vec{l}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|,\,|\vec{k}'|\leq\sqrt{M}}} \frac{M^{A(K+1/2)}}{T^A} \ll \frac{M^{A(K+1/2)+2}}{T^A} \ll T^{-B},$$

for every $B>0$, since $(\alpha,\,\alpha\beta,\,\beta^2)$ is Diophantine.
Obviously, the contribution to (12) of the two last lines of (13) is negligible both in the diagonal and off-diagonal cases, justifying the diagonal approximation of (12) in the first statement of the proposition. To compute the asymptotics, we write we take a large parameter $Y = Y(T) > 0$ (to be chosen later), and write:

$$\sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3}\hat{\psi}^2\left(\frac{|\vec{k}|}{\sqrt{M}}\right) = \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|^2\leq Y}} + \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|^2> Y}} := I_1 + I_2,$$

Now for $Y = o(M)$, $\hat{\psi}^2\big(\frac{|\vec{k}|}{\sqrt{M}}\big) \sim 1$ within the constraints of I_1, and so

$$I_1 \sim \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|^2\leq Y}} \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3}.$$

Recall that $\Lambda^* = \langle 1,\, \gamma + i\delta\rangle$. The sum in

$$\sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|^2\leq Y}} \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3} = \frac{1}{L}\sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|^2\leq Y}} \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{\big(\frac{|\vec{k}|}{L}\big)^3}\frac{1}{L^2}.$$

is a 2-dimensional Riemann sum of the integral

$$\iint\limits_{1/L^2 \ll (x+y\gamma)^2+(\delta y)^2 \leq Y/L^2} \frac{\sin^2\left(\pi\sqrt{(x+y\gamma)^2+(\delta y)^2}\right)}{|(x+y\gamma)^2+(\delta y)^2|^{3/2}}dxdy$$

$$\sim \frac{2\pi}{\delta}\int\limits_{\frac{1}{L}}^{\frac{\sqrt{Y}}{L}} \frac{\sin^2(\pi r)}{r^2}dr \to \beta\pi^3,$$

provided that $Y/L^2 \to \infty$, since $\int\limits_0^\infty \frac{\sin^2(\pi r)}{r^2} dr = \frac{\pi^2}{2}$. We changed the coordinates appropriately. And so,

$$I_1 \sim \frac{\beta\pi^3}{L}$$

Next we will bound I_2. Since $\hat{\psi} \ll 1$, we may use the same change of variables to obtain:

$$\begin{aligned} I_2 &\ll \frac{1}{L} \iint\limits_{(x+y\gamma)^2+(\delta y)^2 \geq Y/L^2} \frac{\sin^2\left(\pi\sqrt{(x+y\gamma)^2+(\delta y)^2}\right)}{|(x+y\gamma)^2+(\delta y)^2|^{3/2}} dxdy \\ &\ll \frac{1}{L} \int\limits_{\sqrt{Y}/L}^{\infty} \frac{dr}{r^2} = o\left(\frac{1}{L}\right). \end{aligned}$$

This concludes the proposition, provided we have managed to choose Y with $L^2 = o(Y)$ and $Y = o(M)$. Such a choice is possible by the assumption of the proposition regarding L.

□

2.2 The higher moments

In order to compute the higher moments we will prove that the main contribution comes from the so-called *diagonal* terms (to be explained later). Our bound for the contribution of the *off-diagonal* terms holds for a *strongly Diophantine* pair of real numbers, which is defined below. In order to show that the strongly Diophantine pairs are "generic", we use theorem 2.6 below, which is a consequence of the work of Kleinbock and Margulis [KM]. The contribution of the diagonal terms is computed exactly in the same manner it was done in [W], and so we will omit it here.

Definition: We call the pair (ξ, η) *strongly Diophantine*, if for all natural n there exists a number $K_1 = K_1(\xi, \eta, n) \in \mathbb{N}$ such that for every integral polynomial of 2 variables $p(x, y) = \sum\limits_{i+j\leq n} a_{i,j} x^i y^j$ of degree $\leq n$, we have

$$\left|p(\xi, \eta)\right| \gg h^{-K_1}, \tag{14}$$

where $h = \max\limits_{i+j\leq n} |a_{i,j}|$ is the height of p. The constant involved in the " $\gg$ " notation may depend only on ξ, η, n and K_1.

Theorem 2.6. *Let an integer n be given. Then almost all pairs of real numbers $(\xi, \eta) \in \mathbb{R}^2$ satisfy the following property: there exists a number $K_1 = K_1(n) \in \mathbb{N}$ such that for every integer polynomial of 2 variables $p(x, y) = \sum\limits_{i+j\leq n} a_{i,j} x^i y^j$ of degree $\leq n$, (14) is satisfied.*

Theorem 2.6 states that almost all real pairs of numbers are strongly Diophantine.

Remark: Theorem A in [KM] is much more general than the result we are using. As a matter of fact, we have the inequality

$$\left|b_0 + b_1 f_1(x) + \ldots + b_n f_n(x)\right| \gg_\epsilon \frac{1}{h^{n+\epsilon}}$$

with $b_i \in \mathbb{Z}$ and

$$h := \max_{0\le i\le n} |b_i|.$$

The inequality above holds for every $\epsilon > 0$ for a wide class of functions $f_i : U \to \mathbb{R}$, for almost all $x \in U$, where $U \subset \mathbb{R}^m$ is an open subset. Here we use this inequality for the monomials.

Remark: Simon Kristensen [KR] has recently shown, that the set of all pairs $(\xi, \eta) \in \mathbb{R}^2$ which fail to be strongly Diophantine has Hausdorff dimension 1. Obviously, if (ξ, η) is strongly Diophantine, then any n-tuple of real numbers, which consists of a set of monomials in ξ and η, is Diophantine. Moreover, (ξ, η) is strongly Diophantine iff $(-\frac{\xi}{\eta}, \frac{1}{\eta})$ is such.
We have the following analogue of lemma 4.7 in [W], which will eventually allow us to exploit the strong Diophantine assumption of (α, β).

Lemma 2.7. *If (ξ, η) is strongly Diophantine, then it satisfies the following property: for any fixed natural m, there exists $K \in \mathbb{N}$, such that if*

$$z_j = a_j^2 + b_j^2\xi^2 + 2a_j b_j \xi + b_j^2\eta^2 \ll M,$$

and $\epsilon_j = \pm 1$ for $j = 1, \ldots, m$, with integral a_j, b_j and if $\sum\limits_{j=1}^{m} \epsilon_j \sqrt{z_j} \ne 0$, then

$$\Big|\sum_{j=1}^{m} \epsilon_j \sqrt{z_j}\Big| \gg M^{-K}, \tag{15}$$

where the constant involved in the "$\gg$" notation depends only on η and m.

The proof is essentially the same as the one of lemma 4.7 from [W], considering the product Q of numbers of the form $\sum\limits_{j=1}^{m} \delta_j \sqrt{z_j}$ over all possible signs δ_j. Here we use the Diophantine condition of the real tuple (ξ, η) rather than of a single real number.

Proposition 2.8. *Let $m \in \mathbb{N}$ be given. Suppose that $\Lambda = \langle 1, \alpha + i\beta\rangle$, such that the pair $(\xi, \eta) := (-\frac{\alpha}{\beta}, \frac{1}{\beta})$ is algebraically independent strongly Diophantine, which satisfy the property of lemma 2.7 for the given m, with $K = K_m$. Then if $M = O\big(T^{\frac{1-\delta}{K_m}}\big)$ for some $\delta > 0$, and if $L \to \infty$ such that $L/\sqrt{M} \to 0$, the following holds:*

$$\frac{\mathcal{M}_{\Lambda, m}}{\sigma^m} = \begin{cases} \frac{m!}{2^{m/2}\left(\frac{m}{2}\right)!} + O\big(\frac{\log L}{L}\big), & m \text{ is even} \\ O\big(\frac{\log L}{L}\big), & m \text{ is odd} \end{cases}$$

Proof. Expanding out (10), we have

$$\mathcal{M}_{\Lambda,\, m} = \frac{2^m}{\beta^m \pi^m} \sum_{\vec{k_1},\ldots,\,\vec{k_m} \in \Lambda^* \setminus \{0\}} \prod_{j=1}^{m} \frac{\sin\left(\frac{\pi|\vec{k_j}|}{L}\right)\hat{\psi}\left(\frac{|\vec{k_j}|}{\sqrt{M}}\right)}{|\vec{k_j}|^{\frac{3}{2}}} \times \left\langle \prod_{j=1}^{m} \sin\left(2\pi\left(t+\frac{1}{2L}\right)|\vec{k_1}| + \frac{\pi}{4}\right)\right\rangle \tag{16}$$

Now,

$$\left\langle \prod_{j=1}^{m} \sin\left(2\pi\left(t+\frac{1}{2L}\right)|\vec{k_1}| + \frac{\pi}{4}\right)\right\rangle = \sum_{\epsilon_j=\pm 1} \frac{\prod\limits_{j=1}^{m}\epsilon_j}{2^m i^m}\hat{\omega}\left(-T\sum_{j=1}^{m}\epsilon_j|\vec{k_j}|\right)e^{\pi i \sum\limits_{j=1}^{m}\epsilon_j\left((1/L)|\vec{k_j}|+1/4\right)}$$

We call a term of the summation in (16) with $\sum\limits_{j=1}^{m}\epsilon_j|\vec{k_j}| = 0$ *diagonal*, and *off-diagonal* otherwise. Due to lemma 2.7, the contribution of the *off-diagonal* terms is:

$$\ll \sum_{\substack{\vec{k_1},\ldots,\,\vec{k_m}\in\Lambda^*\setminus\{0\}\\ |\vec{k_1}|,\,\ldots,\,|\vec{k_m}|\le\sqrt{M}}} \left(\frac{T}{M^{K_m}}\right)^{-A} \ll M^m T^{-A\delta},$$

for every $A > 0$, by the rapid decay of $\hat{\omega}$ and our assumption regarding M. Since m is constant, this allows us to reduce the sum to the *diagonal terms*. In order to be able to sum over all the diagonal terms we need the following analogue of a well-known theorem due to Besicovitch [BS] about incommensurability of square roots of integers.

Proposition 2.9. *Suppose that ξ and η are algebraically independent, and*

$$z_j = a_j^2 + 2a_jb_j\xi + b_j^2(\xi^2+\eta), \tag{17}$$

such that $(a_j,\, b_j) \in \mathbb{Z}_+^2$ are all different primitive vectors, for $1 \le j \le m$. Then $\{\sqrt{z_j}\}_{j=1}^m$ are linearly independent over $\mathbb{Q}$.

The last proposition is an immediate consequence of a theorem proved in the appendix of [BL2].

DEFINITION: We say that a term corresponding to $\{\vec{k_1},\ldots,\vec{k_m}\} \in \Big(\Lambda^* \setminus \{0\}\Big)^m$ and $\{\epsilon_j\} \in \{\pm 1\}^m$ is a *principal diagonal* term if there is a partition $\{1,\ldots,m\} = \bigsqcup_{i=1}^{l} S_i$, such that for each $1 \le i \le l$ there exists a primitive $\vec{n_i} \in \Lambda^* \setminus \{0\}$, with non-negative coordinates, that satisfies the following property: for every $j \in S_i$, there exist $f_j \in \mathbb{Z}$ with $|\vec{k_j}| = f_j |\vec{n_i}|$. Moreover, for each $1 \le i \le l$, $\sum_{j\in S_i} \epsilon_j f_j = 0$.

Obviously, the principal diagonal is contained within the diagonal. However, the meaning of proposition 2.9 is, that in our situation, the converse also is true:

COROLLARY 2.10. *Every* $\underline{\textit{diagonal}}$ *term is a* $\underline{\textit{principle diagonal}}$ *term whenether* ξ *and* η *are algebraically independent.*

Computing the contribution of the principal diagonal terms is done literally the same way it was done in [W], and we sketch it here. As in [W], one can show that the contribution of a particular partition $\{1,\ldots,m\} = \bigsqcup_{i=1}^{l} S_i$ is negligible, unless $m = 2l$ is even and $\#S_i = 2$ for all $1 \le i \le l$.

In the latter case, the contribution is asymptotic to 1. Therefore, the m-th moment is asymptotic to 0, if m is *odd*, and to the number of partitions $\{1,\ldots,m\} = \bigsqcup_{i=1}^{l} S_i$ with $\#S_i = 2$ for all i, $m = 2l$. This number equals to $\frac{m!}{2^{m/2}\left(\frac{m}{2}\right)!}$, which is also the m-th moment of the standard Gaussian distribution.

□

3 BOUNDING THE NUMBER OF CLOSE PAIRS OF LATTICE POINTS

Roughly speaking, we say that a pair of lattice points, n and n' is *close*, if $\big||n| - |n'|\big|$ is *small*. We would like to show that this phenomenon is *rare*. This is closely related to the Oppenheim conjecture, as $|n|^2 - |n'|^2$ is a quadratic form on the coefficients of n and n'.

In order to establish a quantative result, we use a technique developed in a paper by Eskin, Margulis and Mozes [EMM]. Note that the proof is unconditional on any Diophantine assumptions.

3.1 STATEMENT OF THE RESULTS

The ultimate goal of this section is to establish the following

PROPOSITION 3.1. *Let* Λ *be a lattice and denote*

$$A(R,\delta) := \{(\vec{k},\vec{l}) \in \Lambda \times \Lambda :\ R \le |\vec{k}|^2 \le 2R,\ |\vec{k}|^2 \le |\vec{l}|^2 \le |\vec{k}|^2 + \delta\}. \tag{18}$$

Then if $\delta > 1$, such that $\delta = o(R)$, we have

$$\#A(R, \delta) \ll R\delta \cdot \log R$$

In order to prove this result, we note that evaluating the size of $A(R, \delta)$ is equivalent to counting integer points $\vec{v} \in \mathbb{R}^4$ with $T \leq \|\vec{v}\| \leq 2T$ such that

$$0 \leq Q_1(v) \leq \delta,$$

where Q_1 is a quadratic form of signature (2, 2), given explicitly by

$$Q_1(\vec{v}) = (v_1 + v_2\alpha)^2 + (v_2\beta)^2 - (v_3 + v_4\alpha)^2 - (v_4\beta)^2. \tag{19}$$

For a fixed $\delta > 0$ and a large R, this situation was considered extensively by Eskin, Margulis and Mozes [EMM]. The authors give an asymptotical upper bound in this situation. We will examine how the constants involved in their bound depend on δ, and find out that there is a linear dependency, which is what we essentially need. The author wishes to thank Alex Eskin for his assistance with this matter.

Remarks: 1. In a more recent paper, Eskin Margulis and Mozes [EMM1] prove that for "generic" lattice Λ, there is a constant $c > 0$, such that for any *fixed* $\delta > 0$, as $R \to \infty$, $\#A(R, \delta)$ is asymptotic to $c\delta R$.
2. For our purposes we need a weaker result:

$$\#A(R, \delta) \ll_\epsilon R\delta \cdot R^\epsilon,$$

for every $\epsilon > 0$. If Λ is a rectangular lattice (i.e. $\alpha = 0$), then this result follows from properties of the divisor function (see e.g. [BL], lemma 3.2).
Theorem 2.3 in [EMM] considers a more general setting than proposition 3.1. We state here theorem 2.3 from [EMM] (see theorem 3.2). It follows from theorem 3.3 from [EMM], which will be stated as well (see theorem 3.3). Then we give an outline of the proof of theorem 2.3 of [EMM], and inspect the dependency on δ of the constants involved.

3.2 Theorems 2.3 and 3.3 from [EMM]

Let Δ be a lattice in $\mathbb{R}^n$. We say that a subspace $L \subset \mathbb{R}^n$ is Δ-rational, if $L \cap \Delta$ is a lattice in L. We need the following definitions:

Definitions:

$$\alpha_i(\Delta) := \sup \left\{ \frac{1}{d_\Delta(L)} \,\middle|\, L \text{ is a } \Delta - \text{rational subspace of dimension } i \right\},$$

where

$$d_\Delta(L) := vol(L/(L \cap \Delta)).$$

Also

$$\alpha(\Delta) := \max_{0\le i\le n} \alpha_i(\Delta).$$

Since the space of unimodular lattices is canonically isomorphic to $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$, the notation $\alpha(g)$ makes sense for $g \in G := SL(n, \mathbb{R})$. For a bounded function $f : \mathbb{R}^n \to \mathbb{R}$, with $|f| \le M$, which vanishes outside a ball $B(0, R)$, define $\tilde{f} : SL(n, \mathbb{R}) \to \mathbb{R}$ by the following formula:

$$\tilde{f}(g) := \sum_{v\in\mathbb{Z}^n} f(gv).$$

Lemma 3.1 in [S2] implies that

$$\tilde{f}(g) < c\alpha(g), \tag{20}$$

where $c = c(f)$ is an explicit constant constant

$$c(f) = c_0 M \max(1, R^n),$$

for some constant $c_0 = c_0(n)$, independent on f. In section 3.4 we prove a stronger result, assuming some additional information about the support of f. Let Q_0 be a quadratic form defined by

$$Q_0(\vec{v}) = 2v_1v_n + \sum_{i=2}^{p} v_i^2 - \sum_{i=p+1}^{n-1} v_i^2.$$

Since

$$v_1v_n = \frac{(v_1+v_n)^2 - (v_1-v_n)^2}{2},$$

Q_0 is of signature p, q. Obviously, $G := SL(n,\mathbb{R})$ acts on the space of quadratic forms of signature (p, q), and discriminant ± 1, $\mathcal{O} = \mathcal{O}(p, q)$ by:

$$Q^g(v) := Q(gv).$$

Moreover, by the well known classification of quadratic forms, $\mathcal{O}$ is the orbit of Q_0 under this action.

In our case the signature is $(p, q) = (2, 2)$ and $n = 4$. We fix an element $h_1 \in G$ with $Q^{h_1} = Q_1$, where Q_1 is given by (19). There exists a constant $\tau > 0$, such that for every $v \in \mathbb{R}^4$,

$$\tau^{-1}\|v\| \le \|h_1 v\| \le \tau\|v\|. \tag{21}$$

We may assume, with no loss of generality that $\tau \ge 1$.

Let $H := Stab_{Q_0}(G)$. Then the natural mophism $H\backslash G \to \mathcal{O}(p,q)$ is a homeomorphism. Define a 1-parameter family $a_t \in G$ by:

$$a_t e_i = \begin{cases} e^{-t}e_1, & i = 1 \\ e_i, & i = 2,\dots, n-1 \\ e^t e_n, & i = n \end{cases}.$$

Clearly, $a_t \in H$. Furthermore, let $\hat{K}$ be the subgroup of G consisting of orthogonal matrices, and denote $K := H \cap \hat{K}$.
Let $(a, b) \in \mathbb{R}^2$ be given and let $Q : \mathbb{R}^n \to \mathbb{R}$ be any quadratic form. The object of our interest is:

$$V_{(a,\,b)}(\mathbb{Z}) = V^{Q}_{(a,\,b)}(\mathbb{Z}) = \{x \in \mathbb{Z}^n :\ a < Q(x) < b\}.$$

Theorem 2.3 states, in our case:

Theorem 3.2 (Theorem 2.3 from [EMM]). *Let $\Omega = \{v \in \mathbb{R}^4 |\ \|v\| < \nu(v/\|v\|)\}$, where ν is a nonnegative continuous function on S^3. Then we have:*

$$\# V^{Q_1}_{(a,\,b)}(\mathbb{Z}) \cap T\Omega < cT^2 \log T,$$

where the constant c depends only on (a, b).

The proof of theorem 3.2 relies on theorem 3.3 from [EMM], and we give here a particular case of this theorem

Theorem 3.3 (Theorem 3.3 from [EMM]). *For any (fixed) lattice Δ in $\mathbb{R}^4$,*

$$\sup_{t>1} \frac{1}{t} \int_K \alpha(a_t k\Delta) dm(k) < \infty,$$

where the upper bound is universal.

3.3 Outline of the proof of theorem 3.2:

Step 1: Define

$$J_f(r, \zeta) = \frac{1}{r^2} \int_{\mathbb{R}^2} f(r, x_2,\, x_3,\, x_4) dx_2 dx_3, \tag{22}$$

where

$$x_4 = \frac{\zeta - x_2^2 + x_3^2}{2r}$$

Lemma 3.6 in [EMM] states that J_f is approximable by means of an integral over the compact subgroup K. More precisely, there is some constant $C > 0$, such that for every $\epsilon > 0$,

$$\left| C \cdot e^{2t} \int_K f(a_t k v) \nu(k^{-1} e_1) dm(k) - J_f\big(\|v\| e^{-t},\, Q_0(v)\big) \nu(\frac{v}{\|v\|}) \right| < \epsilon \tag{23}$$

with e^t, $\|v\| > T_0$ for some $T_0 > 0$.

Step 2: Choose a continuous nonnegative function f on $\mathbb{R}^4_+ = \{x_1 > 0\}$ which vanishes outside a compact set so that

$$J_f(r, \zeta) \geq 1 + \epsilon$$

on $[\tau^{-1},\, 2\tau] \times [a,\, b]$. We will show later, how one can choose f.

STEP 3: Denote $T = e^t$, and suppose that $T \leq \|v\| \leq 2T$ and $a \leq Q_0(h_1 v) \leq b$. Then by (21), $J_f\big(\|h_1 v\| T^{-1}, Q_0(h_1 v)\big) \geq 1+\epsilon$, and by (23), for a sufficiently large t,

$$C \cdot T^2 \int\limits_K f(a_t k h_1 v) dm(k) \geq 1, \tag{24}$$

for $T \leq \|v\| \leq 2T$ and

$$a \leq Q_0^x(v) \leq b. \tag{25}$$

STEP 4: Summing (24) over all $v \in \mathbb{Z}^4$ with (25) and $T \leq \|v\| \leq 2T$, we obtain:

$$\begin{aligned} \#V_{(a,\,b)}(\mathbb{Z}) \cap [T,\, 2T]S^3 &\leq \sum_{v \in \mathbb{Z}^n} C \cdot T^2 \int\limits_K f(a_t k h_1 v) dm(k) \\ &= C \cdot T^2 \int\limits_K \tilde{f}(a_t k h_1) dm(k) \end{aligned} \tag{26}$$

using the nonnegativity of f.

STEP 5: By (20), (26) is

$$\leq C \cdot c(f) \cdot T^2 \int\limits_K \alpha(a_t k h_1) dm(k).$$

STEP 6: The result of theorem 2.3 is obtained by using theorem 3.3 on the last expression.

3.4 δ-DEPENDENCY:

In this section we assume that $(a,\, b) = (0,\, \delta)$, which suits the definition of the set $A(R,\, \delta)$, (18). One should notice that there only 3 δ-dependent steps:

- Choosing f in step 2, such that $J_f \geq 1+\epsilon$ on $[\tau^{-1},\, 2\tau] \times [0,\, \delta]$. We will construct a family of functions f_δ with an universal bound $|f_\delta| \leq M$, such that f_δ vanishes outside of a compact set which is only slightly larger than

$$V(\delta) = [\tau^{-1},\, 2\tau] \times [-1,\, -1]^2 \times [0,\, \frac{\delta\tau}{2}]. \tag{27}$$

This is done in section 3.4.1.
- The dependency of T_0 of step 3, so that the usage of lemma 3.6 in [EMM] is legitimate. For this purpose we will have to examine the proof of this lemma. This is done in section 3.4.2.
- The constant c in (20). We would like to establish a *linear* dependency on δ. This is straightforward, once we are able to control the number of integral points in a domain defined by (27). This is done in section 3.4.3.

3.4.1 Choosing f_δ:

Notation: For a set $U \subset \mathbb{R}^n$, and $\epsilon > 0$, denote

$$U_\epsilon := \{x \in \mathbb{R}^n : \max_{1 \le i \le n} |x_i - y_i| \le \epsilon, \text{ for some } y \in U\}.$$

Choose a nonnegative continuous function f_0, on $\mathbb{R}^4_+$, which vanishes outside a compact set, such that its support, E_{f_0}, slightly exceeds the set $V(1)$. More precisely, $V(1) \subset E_{f_0} \subset V(1)_{\delta_0}$ for some $\delta_0 > 0$. By the uniform continuity of f, there are ϵ_0, $\delta_0 > 0$, such that if $\max_{1 \le i \le 4} |x_i - x_i^0| \le \delta_0$, then $f(x) > \epsilon_0$, for every $x^0 = (x_1^0, 0, 0, x_4^0) \in V(1)$.
Thus for $(r, \zeta) \in [\tau^{-1}, 2\tau] \times [0, \delta]$, the contribution of $[-\delta_0, \delta_0]^2$ to J_{f_0} is $\ge \epsilon_0 \cdot (2\delta_0)^2$. Multiplying f_0 by a suitable factor, and by the linearity of J_{f_0}, we may assume that this contribution is at least $1 + \epsilon$.
Now define $f_\delta(x_1, \ldots, x_4) := f_0(x_1, x_2, x_3, \frac{x_4}{\delta})$. We have for $\delta \ge 1$

$$\frac{\zeta - x_2^2 + x_3^2}{2r\delta} = \frac{\zeta/2r}{\delta} - \frac{(x_2/\sqrt{\delta})^2}{2r} + \frac{(x_3/\sqrt{\delta})^2}{2r}.$$

Thus for $\delta \ge 1$, if $(r, \zeta) \in [\tau^{-1}, 2\tau] \times [0, \delta]$ and for $i = 2, 3$, $|x_i| < \delta_0$, f_δ satisfies:

$$f_\delta(r, x_2, x_3, x_4) > \epsilon_0,$$

and therefore the contribution of this domain to J_{f_δ} is

$$\ge \epsilon_0 (2\delta)^2 \ge 1 + \epsilon$$

by our assumption.
By the construction, the family $\{f_\delta\}$ has a universal upper bound M which is the one of f_0.

3.4.2 How large is T_0

The proof of lemma 3.6 from [EMM] works well along the same lines, as long as

$$f(a_t x) \neq 0 \tag{28}$$

implies that for $t \to \infty$, $x/\|x\|$ converges to $e_1 = (1, 0, 0, 0)$. Now, since a_t preserves $x_1 x_4$, (28) implies for the particular choice of $f = f_\delta$ in section 3.4.1:

$$|x_1 x_4| = O(\delta); \quad x_1 \gg T.$$

Thus

$$\|x\| = x_1 + O\left(\frac{\delta}{T}\right) + O(1),$$

and so, as long as $\delta = O(T)$, $x/\|x\|$ indeed converges to e_1.

3.4.3 BOUNDING INTEGRAL POINTS IN V_δ:

LEMMA 3.4. *Let $V(\delta)$ defined by*

$$V(\delta) = [\tau^{-1},\, 2\tau] \times [-1,\, -1]^{n-2} \times [0,\, \frac{\delta\beta}{2}]. \tag{29}$$

for some constant τ and $n \geq 3$. Let $g \in SL(n,\, \mathbb{R})$ and denote

$$N(g,\, \delta) := \# V(\delta) \cap g\mathbb{Z}^n.$$

Then for $\delta \geq 1$,

$$\left| N(g,\, \delta) - \frac{2^{n-2}(2\tau - \tau^{-1})\delta}{\det g} \right| \leq c_5 \delta \sum_{i=1}^{n-1} \frac{1}{vol(L_i/(g\mathbb{Z}^n \cap L_i)}$$

for some g-rational subspaces L_i of $\mathbb{R}^4$ of dimension i, where $c_5 = c_5(n)$ depends only on n.

A direct consequence of lemma 3.4 is the following

COROLLARY 3.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative function which vanishes outside a compact set E. Suppose that $E \subset V_\epsilon(\delta)$ for some $\epsilon > 0$. Then for $\delta \geq 1$, (20) is satisfied with*

$$c(f) = c_3 \cdot M\delta,$$

where the constant c_3 depends on n only.

In order to prove lemma 3.4, we shall need the following:

LEMMA 3.6. *Let $\Lambda \subset \mathbb{R}^n$ be a m-dimensional lattice, and let*

$$A_t = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & t \end{pmatrix} \tag{30}$$

an n-dimensional linear transformation. Then for $t > 0$ we have

$$\det A_t\Lambda \leq t \det \Lambda. \tag{31}$$

Proof. We may assume that $m < n$, since if $m = n$, we obviously have an equality. Let $v_1,\, \ldots,\, v_m$ the basis of Λ and denote for every i, $u_i \in \mathbb{R}^{n-1}$ the vector, which consists of first $n-1$ coordinates of v_i. Also, let $x_i \in \mathbb{R}$ be the last coordinate of v_i. By switching vectors, if necessary, we may assume $x_1 \neq 0$. We consider the function

$$f(t) := (\det A_t\Lambda)^2,$$

as a function of $t \in \mathbb{R}$.
Obviously,

$$f(t) = \det\left(\langle u_i,\, u_j\rangle + x_i x_j t^2\right)_{1\le i,\, j\le m}.$$

Substracting $\frac{x_i}{x_1}$ times the first row from any other, we obtain:

$$f(t) = \begin{vmatrix} \langle u_1,\, u_j\rangle + x_1 x_j t^2 \\ \langle u_2,\, u_j\rangle - \frac{x_2}{x_1}\langle u_1,\, u_j\rangle \\ \vdots \\ \langle u_m,\, u_j\rangle - \frac{x_m}{x_1}\langle u_1,\, u_j\rangle \end{vmatrix},$$

and by the multilinearity property of the determinant, f is a linear function of t^2. Write

$$f(t) = a(t^2 - 1) + bt^2.$$

Thus

$$b = f(1); \quad a = -f(0),$$

and so $b = \det \Lambda$, and $a = -\det\left(\langle u_i,\, u_j\rangle\right) \le 0$, being minus the determinant of a Gram matrix. Therefore,

$$(\det A_t\Lambda)^2 - t^2 \det\Lambda = a(t^2-1) \le 0$$

for $t \ge 1$, implying (30). □

Proof of lemma 3.4. We will prove the lemma, assuming $\beta = 2$. However, it implies the result of the lemma for any β, affecting only c_5. Let $\delta > 0$. Trivially,

$$N(g,\, \delta) = N(g_0,\, 1),$$

where $g_0 = A_\delta^{-1} g$ with A_δ given by (30). Let $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ be the successive minima of g_0, and pick linearly independent lattice points $v_1, \ldots, v_n$ with $\|v_i\| = \lambda_i$. Denote M_i the linear space spanned by $v_1, \ldots, v_i$ and the lattice $\Lambda_i = g_0\mathbb{Z}^n \cap M_i$.
First, assume that $\lambda_n \le \sqrt{\tau^2 + (n-1)} =: r$. Now, by Gauss' argument,

$$\left| N(g_0,\, 1) - \frac{2^{n-1}(2\tau - \tau^{-1})\delta}{\det g} \right| \le \frac{1}{\det g_0} vol(\Sigma),$$

where

$$\Sigma := \{x :\ dist(x,\, \partial V(1)) \le n\lambda_n\}.$$

Now, for $\lambda_n \le r$,

$$vol(\Sigma) \ll \lambda_n,$$

where the constant implied in the " $\ll$ "-notation depends on n only (this is obvious for $\lambda_n \le \frac{1}{2n}$, and trivial otherwise, since for $\lambda_n \le r$, $vol(\Sigma) = O(1)$).

Thus,

$$\left|N(g_0,\,1)-\frac{2^{n-1}(2\tau-\tau^{-1})\delta}{\det g}\right|\ll\frac{\lambda_n}{\det g_0}\ll\frac{1}{\det\Lambda_{n-1}}$$
$$=\frac{1}{vol(M_{n-1}/M_{n-1}\cap g_0\mathbb{Z}^n)}\le\frac{\delta}{vol(A_\delta M_{n-1}/A_\delta M_{n-1}\cap g\mathbb{Z}^n)}$$

Next, suppose that $\lambda_n>r$. Then,

$$V(\delta)\cap g_0\mathbb{Z}^n\subset V(\delta)\cap\Lambda_{n-1}.$$

Thus, by the induction hypothesis, the number of such points is:

$$\le c_4\sum_{i=0}^{k-1}\frac{1}{\det(\Lambda_i)}=\sum_{i=0}^{k-1}\frac{1}{vol(M_i/M_i\cap g_0\mathbb{Z}^n)}$$
$$\le\delta\sum_{i=0}^{k-1}\frac{1}{vol(A_\delta M_i/A_\delta\mathcal{M}_i\cap g\mathbb{Z}^n)}.$$

Since $\lambda_n>r$, we have

$$\frac{1}{\det g}=\frac{1}{\lambda_n}\frac{1}{\det g/\lambda_n}\ll\frac{1}{\det g/\lambda_n}\ll\frac{1}{\lambda_1\cdot\ldots\cdot\lambda_{n-1}},$$

and we're done by defining $L_i:=A_\delta M_i$. □

4 Unsmoothing

4.1 An asymptotic formula for N_Λ

We need an asymptotic formula for the *sharp* counting function N_Λ. Unlike the case of the standard lattice, $\mathbb{Z}^2$, in order to have a good control over the error terms we should use some Diophantine properties of the lattice we are working with. We adapt the following notations:
Let $\Lambda=\langle 1,\,\alpha+i\beta\rangle$, be a lattice, $d:=\det\Lambda=\beta$ its determinant, and $t>0$ a real variable. Denote the set of squared norms of Λ by

$$SN_\Lambda=\{|\vec{n}|^2:\,n\in\Lambda\}.$$

Suppose we have a function $\delta_\Lambda:SN_\Lambda\to\mathbb{R}$, such that given $\vec{k}\in\Lambda$, there are no vectors $\vec{n}\in\Lambda$ with $0<||\vec{n}|^2-|\vec{k}|^2|<\delta_\Lambda(|\vec{k}|^2)$. That is,

$$\Lambda\cap\{\vec{n}\in\Lambda:\,|\vec{k}|^2-\delta_\Lambda(|\vec{k}|^2)<|\vec{n}|^2<|\vec{k}|^2+\delta_\Lambda(|\vec{k}|^2)\}=A_{|\vec{k}|},$$

where

$$A_y:=\{\vec{n}\in\Lambda:\,|\vec{n}|=y\}.$$

Extend δ_Λ to $\mathbb{R}$ by defining $\delta_\Lambda(x):=\delta_\Lambda(|\vec{k}|^2)$, where $\vec{k}\in\Lambda$ minimizes $|x-|\vec{k}|^2|$ (in the case there is any ambiguity, that is if $x=\frac{|\vec{n_1}|^2+|\vec{n_2}|^2}{2}$ for vectors $\vec{n_1},\,\vec{n_2}\in\Lambda$ with consecutive increasing norms, choose $\vec{k}:=\vec{n_1}$). We have the following lemma:

Lemma 4.1. *For every $a > 0$, $c > 1$,*

$$N_\Lambda(t) = \frac{\pi}{\beta}t^2 - \frac{\sqrt{t}}{\beta\pi} \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|\le\sqrt{N}}} \frac{\cos\left(2\pi t|\vec{k}| + \frac{\pi}{4}\right)}{|\vec{k}|^{\frac{3}{2}}} + O(N^a)$$
$$+ O\bigg(\frac{t^{2c-1}}{\sqrt{N}}\bigg) + O\bigg(\frac{t}{\sqrt{N}} \cdot \big(\log t + \log(\delta_\Lambda(t^2))\big)\bigg)$$
$$+ O\bigg(\log N + \log(\delta_{\Lambda^*}(t^2))\bigg)$$

As a typical example of such a function, δ_Λ, for $\Lambda = \langle 1, \alpha + i\beta\rangle$, with a Diophantine $(\alpha, \alpha^2, \beta^2)$, we may choose $\delta_\Lambda(y) = \frac{c}{y^K}$, where c is a constant. In this example, if $\Lambda \ni \vec{k} = (a, b)$, then by lemma 2.4, $A_{|\vec{k}|} = \pm(a, b)$, provided that $\beta \notin \mathbb{Q}(\alpha)$.

Sketch of proof. The proof of this lemma is essentially the same as the one of lemma 5.1 in [W]. We start from

$$\mathcal{Z}_\Lambda(s) := \frac{1}{2}\sum_{\vec{k}\in\Lambda\setminus 0} \frac{1}{|\vec{k}|^{2s}} = \sum_{(m,\,n)\in\mathbb{Z}^2_+\setminus 0} \frac{1}{\big((m+n\alpha)^2 + (\beta n)^2\big)^s},$$

where the series is convergent for $\Re s > 1$.
The function $\mathcal{Z}_\Lambda$ has an analytic continuation to the whole complex plane, except for a single pole at $s = 1$, defined by the formula

$$\Gamma(s)\pi^{-s}\mathcal{Z}_\Lambda(s) = \int_1^\infty x^{s-1}\psi_\Lambda(x)dx + \frac{1}{d}\int_1^\infty x^{-s}\psi_{\Lambda^*}(x)dx - \frac{s - d(s-1)}{2ds(1-s)},$$

where

$$\psi_\Lambda(x) := \frac{1}{2}\sum_{\vec{k}\in\Lambda\setminus 0} e^{-\pi|\vec{k}|^2 x}.$$

Moreover, $\mathcal{Z}_\Lambda$ satisfies the following functional equation:

$$\mathcal{Z}_\Lambda(s) = \frac{1}{d}\chi(s)\mathcal{Z}_{\Lambda^*}(1-s), \tag{32}$$

with

$$\chi(s) = \pi^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}. \tag{33}$$

The connection between N_Λ and $\mathcal{Z}_\Lambda$ is given in the following formula, which is satisfied for every $c > 1$:

$$\frac{1}{2}N_\Lambda(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{Z}_\Lambda(s)\frac{x^s}{s}ds.$$

The result of the current lemma follows from moving the contour of the integration to the left, collecting the residue at $s = 1$ (see [W] for details). □

PROPOSITION 4.2. *Let a lattice $\Lambda = \langle 1,\, \alpha + i\beta\rangle$ with a Diophantine triple of numbers $(\alpha^2, \alpha\beta,\, \beta^2)$ be given. Suppose that $L \to \infty$ as $T \to \infty$ and choose M, such that $L/\sqrt{M} \to 0$, but $M = O\big(T^\delta\big)$ for every $\delta > 0$ as $T \to \infty$. Suppose furthermore, that $M = O(L^{s_0})$ for some (fixed) $s_0 > 0$. Then*

$$\left\langle \left| S_\Lambda(t,\, \rho) - \tilde{S}_{\Lambda,\, M,\, L}(t) \right|^2 \right\rangle \ll \frac{1}{\sqrt{M}}$$

The proof of proposition 4.2 proceeds along the same lines as the one of proposition 5.1 in [W], using again an asymptotic formula for the sharp counting function, given by lemma 4.1. The only difference is that here we use proposition 3.1 rather than lemma 5.2 from [W].
Once we have proposition 4.2 in our hands, the proof of our main result, namely, theorem 1.1 proceeds along the same lines as the one of theorem 1.1 in [W].

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