Arithmetic Characteristic Classes of Automorphic Vector Bundles

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Abstract. We develop a theory of arithmetic characteristic classes of (fully decomposed) automorphic vector bundles equipped with an invariant hermitian metric. These characteristic classes have values in an arithmetic Chow ring constructed by means of differential forms with certain log-log type singularities. We first study the cohomological properties of log-log differential forms, prove a Poincaré lemma for them and construct the corresponding arithmetic Chow groups. Then, we introduce the notion of log-singular hermitian vector bundles, which is a variant of the good hermitian vector bundles introduced by Mumford, and we develop the theory of arithmetic characteristic classes. Finally we prove that the hermitian metrics of automorphic vector bundles considered by Mumford are not only good but also log-singular. The theory presented here provides the theoretical background which is required in the formulation of the conjectures of Maillot-Roessler in the semi-abelian case and which is needed to extend Kudla's program about arithmetic intersections on Shimura varieties to the non-compact case.

Contents

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1 Introduction

The main goal. The main purpose of this article is to extend the arithmetic intersection theory and the theory of arithmetic characteristic classes à la Gillet, Soulé to the category of (fully decomposed) automorphic vector bundles equipped with the natural equivariant hermitian metric on Shimura varieties of non-compact type. In order to achieve our main goal, an extension of the formalism by Gillet, Soulé taking into account vector bundles equipped with hermitian metrics allowing a certain type of singularities has to be provided. The main prerequisite for the present work is the article [10], where the foundations of cohomological arithmetic Chow groups are given. Before continuing to explain our main results and the outline of the paper below, let us fix some basic notations for the sequel.

Let B denote a bounded, hermitian, symmetric domain. By definition, $B = G/K$, where G is a semi-simple adjoint group and K a maximal compact subgroup of G with non-discrete center. Let Γ be a neat arithmetic subgroup of G; it acts properly discontinuously and fixed-point free on B. The quotient space $X = \Gamma\backslash B$ has the structure of a smooth, quasi-projective, complex variety. The complexification $G_{\mathbb{C}}$ of G yields the compact dual $\check{B}$ of B given by

$\check{B} = G_{\mathbb{C}}/P_+ \cdot K_{\mathbb{C}}$, where $P_+ \cdot K_{\mathbb{C}}$ is a suitable parabolic subgroup of G equipped with the Cartan decomposition of $\mathrm{Lie}(G)$ and P_+ is the unipotent radical of this parabolic subgroup. Every $G_{\mathbb{C}}$-equivariant holomorphic vector bundle $\check{E}$ on $\check{B}$ defines a holomorphic vector bundle E on X; E is called an *automorphic vector bundle*. An automorphic vector bundle E is called *fully decomposed*, if $E = E_\sigma$ is associated to a representation $\sigma : P_+ \cdot K_{\mathbb{C}} \longrightarrow \mathrm{GL}_n(\mathbb{C})$, which is trivial on P_+. Since K is compact, every fully decomposed automorphic vector bundle E admits a G-equivariant hermitian metric h.
Let us recall the following basic example. Let $\pi : \mathcal{B}_g^{(N)} \longrightarrow \mathcal{A}_g^{(N)}$ denote the universal abelian variety over the moduli space of principally polarized abelian varieties of dimension g with a level-N structure ($N \geq 3$); let $e : \mathcal{A}_g^{(N)} \longrightarrow \mathcal{B}_g^{(N)}$ be the zero section, and $\Omega = \Omega^1_{\mathcal{B}_g^{(N)}/\mathcal{A}_g^{(N)}}$ the relative cotangent bundle. The Hodge bundle $e^*\Omega$ is an automorphic vector bundle on $\mathcal{A}_g^{(N)}$, which is equipped with a natural hermitian metric h. Another example of an automorphic vector bundle on $\mathcal{A}_g^{(N)}$ is the determinant line bundle $\omega = \det(e^*\Omega)$; the corresponding hermitian automorphic line bundle $(\det(e^*\Omega), \det(h))$ is denoted by $\overline{\omega}$.

Background results. Let (E, h) be an automorphic hermitian vector bundle on $X = \Gamma \backslash B$, and $\overline{X}$ a smooth toroidal compactification of X. In [34], D. Mumford has shown that the automorphic vector bundle E admits a canonical extension E_1 to $\overline{X}$ characterized by a suitable extension of the hermitian metric h to E_1. However, the extension of h to E_1 is no longer a smooth hermitian metric, but inherits singularities of a certain type. On the other hand, it is remarkable that this extended hermitian metric behaves in many aspects like a smooth hermitian metric. In this respect, we will now discuss various definitions which were made in the past in order to extract basic properties for these extended hermitian metrics.
In [34], D. Mumford introduced the concept of good forms and good hermitian metrics. The good forms are differential forms, which are smooth on the complement of a normal crossing divisor and have certain singularities along this normal crossing divisor; the singularities are modeled by the singularities of the Poincaré metric. The good forms have the property of being locally integrable with zero residue. Therefore, they define currents, and the map from the complex of good forms to the complex of currents is a morphism of complexes. The good hermitian metrics are again smooth hermitian metrics on the complement of a normal crossing divisor and have logarithmic singularities along the divisor in question. Moreover, the entries of the associated connection matrix are good forms. The Chern forms for good hermitian vector bundles, i.e., of vector bundles equipped with good hermitian metrics, are good forms, and the associated currents represent the Chern classes in cohomology. Thus, in this sense, the good hermitian metrics behave like smooth hermitian metrics. In the same paper, D. Mumford proves that automorphic hermitian vector bundles are good hermitian vector bundles.
In [14], G. Faltings introduced the concept of a hermitian metric on line bundles

with logarithmic singularities along a closed subvariety. He showed that the heights associated to line bundles equipped with singular hermitian metrics of this type have the same finiteness properties as the heights associated to line bundles equipped with smooth hermitian metrics. The Hodge bundle $\overline{\omega}$ on $\mathcal{A}_g^{(N)}$ equipped with the Petersson metric provides a prominent example of such a hermitian line bundle; it plays a crucial role in Faltings's proof of the Mordell conjecture. Recall that the height of an abelian variety A with respect to $\overline{\omega}$ is referred to as the Faltings height of A. It is a remarkable fact that, if A has complex multiplication of abelian type, its Faltings height is essentially given by a special value of the logarithmic derivative of a Dirichlet L-series. It is conjectured by P. Colmez that in the general case the Faltings height is essentially given by a special value of the logarithmic derivative of an Artin L-series.

In [30], the third author introduced the concept of logarithmically singular hermitian line bundles on arithmetic surfaces. He provided an extension of arithmetic intersection theory (on arithmetic surfaces) adapted to such logarithmically singular hermitian line bundles. The prototype of such a line bundle is the automorphic hermitian line bundle $\overline{\omega}$ on the modular curve $\mathcal{A}_1^{(N)}$. J.-B. Bost and, independently, U. Kühn calculated its arithmetic self-intersection number $\overline{\omega}^2$ to

$$\overline{\omega}^2 = d_N \cdot \zeta_{\mathbb{Q}}(-1)\left(\frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + \frac{1}{2}\right);$$

here $\zeta_{\mathbb{Q}}(s)$ denotes the Riemann zeta function and d_N equals the degree of the classifying morphism of $\mathcal{A}_1^{(N)}$ to the coarse moduli space $\mathcal{A}_1^{(1)}$.

In [10], an abstract formalism was developed, which allows to associate to an arithmetic variety $\mathcal{X}$ arithmetic Chow groups $\widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{C})$ with respect to a cohomological complex $\mathcal{C}$ of a certain type. This formalism is an abstract version of the arithmetic Chow groups introduced in [8]. In [10], the arithmetic Chow ring $\widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}$ was introduced, where the cohomological complex $\mathcal{D}_{\mathrm{pre}}$ in question is built from pre-log and pre-log-log differential forms. This ring allows us to define arithmetic self-intersection numbers of automorphic hermitian line bundles on arithmetic varieties associated to $X = \Gamma\backslash B$. It is expected that these arithmetic self-intersection numbers play an important role for possible extensions of the Gross-Zagier theorem to higher dimensions (cf. conjectures of S. Kudla).

In [6], J. Bruinier, J. Burgos, and U. Kühn use the theory developed in [10] to obtain an arithmetic generalization of the Hirzebruch-Zagier theorem on the generating series for cycles on Hilbert modular varieties. Recalling that Hilbert modular varieties parameterize abelian surfaces with multiplication by the ring of integers $\mathcal{O}_K$ of a real quadratic field K, a major result in [6] is the following formula for the arithmetic self-intersection number of the automorphic hermitian line bundle $\overline{\omega}$ on the moduli space of abelian surfaces with multiplication by $\mathcal{O}_K$ with a fixed level-N structure

$$\overline{\omega}^3 = -d_N \cdot \zeta_K(-1)\left(\frac{\zeta_K'(-1)}{\zeta_K(-1)} + \frac{\zeta_{\mathbb{Q}}'(-1)}{\zeta_{\mathbb{Q}}(-1)} + \frac{3}{2} + \frac{1}{2}\log(D_K)\right);$$

here D_K is the discriminant of $\mathcal{O}_K$, $\zeta_K(s)$ is the Dedekind zeta function of K, and, as above, d_N is the degree of the classifying morphism obtained by forgetting the level-N structure.

As another application of the formalism developed in [10], we derived a height pairing with respect to singular hermitian line bundles for cycles in any codimension. Recently, G. Freixas in [15] has proved finiteness results for our height pairing, thus generalizing both Faltings's results mentioned above and the finiteness results of J.-B. Bost, H. Gillet and C. Soulé in [4] in the smooth case.

The main achievement of the present paper is to give constructions of arithmetic intersection theories, which are suited to deal with all of the above vector bundles equipped with hermitian metrics having singularities of a certain type such as the automorphic hermitian vector bundles on Shimura varieties of non-compact type.

For a perspective view of applications of the theory developed here, we refer to the conjectures of V. Maillot and D. Roessler [31], K. Köhler [26], and the program due to S. Kudla [28], [29], [27].

Arithmetic characteristic classes. We recall from [36] that the arithmetic K-group $\widehat{\mathrm{K}}_0(\mathcal{X})$ of an arithmetic variety $\mathcal{X}$ à la Gillet, Soulé is defined as the free group of pairs $(\overline{E}, \eta)$ of a hermitian vector bundle $\overline{E}$ and a smooth differential form η modulo the relation

$$(\overline{S}, \eta') + (\overline{Q}, \eta'') = (\overline{E}, \eta' + \eta'' + \widetilde{\mathrm{ch}}(\overline{\mathcal{E}})),$$

for every short exact sequence of vector bundles (equipped with arbitrary smooth hermitian metrics)

$$\overline{\mathcal{E}}:\, 0 \longrightarrow \overline{S} \longrightarrow \overline{E} \longrightarrow \overline{Q} \longrightarrow 0,$$

and for any smooth differential forms η', η''; here $\widetilde{\mathrm{ch}}(\overline{\mathcal{E}})$ denotes the (secondary) Bott-Chern form of $\overline{\mathcal{E}}$.

In [36], H. Gillet and C. Soulé attached to the elements of $\widehat{\mathrm{K}}_0(\mathcal{X})$, represented by hermitian vector bundles $\overline{E} = (E, h)$, arithmetic characteristic classes $\widehat{\phi}(\overline{E})$, which lie in the "classical" arithmetic Chow ring $\widehat{\mathrm{CH}}^*(\mathcal{X})_{\mathbb{Q}}$. A particular example of such an arithmetic characteristic class is the arithmetic Chern character $\widehat{\mathrm{ch}}(\overline{E})$, whose definition also involves the Bott-Chern form $\widetilde{\mathrm{ch}}(\overline{\mathcal{E}})$.

In order to be able to carry over the concept of arithmetic characteristic classes to the category of vector bundles E over an arithmetic variety $\mathcal{X}$ equipped with a hermitian metric h having singularities of the type considered in this paper, we proceed as follows: Letting h_0 denote an arbitrary smooth hermitian metric on E, we have the obvious short exact sequence of vector bundles

$$\overline{\mathcal{E}}:\ 0 \longrightarrow 0 \longrightarrow (E,h) \longrightarrow (E,h_0) \longrightarrow 0,$$

to which is attached the Bott-Chern form $\widetilde{\phi}(\overline{\mathcal{E}})$ being no longer smooth, but having certain singularities. Formally, we then set

$$\widehat{\phi}(E,h) := \widehat{\phi}(E,h_0) + \mathrm{a}\left(\widetilde{\phi}(\overline{\mathcal{E}})\right),$$

where a is the morphism mapping differential forms into arithmetic Chow groups. In order to give meaning to this definition, we need to know the singularities of $\widetilde{\phi}(\overline{\mathcal{E}})$; moreover, we have to show the independence of the (arbitrarily chosen) smooth hermitian metric h_0.

Once we can control the singularities of $\widetilde{\phi}(\overline{\mathcal{E}})$, the abstract formalism developed in [10] reduces our task to find a cohomological complex $\mathcal{C}$, which contains the elements $\widetilde{\phi}(\overline{\mathcal{E}})$, and has all the properties we desire for a reasonable arithmetic intersection theory. Once the complex $\mathcal{C}$ is constructed, we obtain an arithmetic K-theory with properties depending on the complex $\mathcal{C}$, of course.

The most naive way to construct an arithmetic intersection theory for automorphic hermitian vector bundles would be to only work with good forms and good metrics. This procedure is doomed to failure for the following two reasons: First, the complex of good forms is not a Dolbeault complex. However, this first problem can be easily solved by imposing that it is also closed under the differential operators ∂, $\bar{\partial}$, and $\partial\bar{\partial}$. The second problem is that the complex of good forms is not big enough to contain the singular Bott-Chern forms which occur. For example, if $\mathcal{L}$ is a line bundle, h_0 a smooth hermitian metric and h a singular metric, which is good along a divisor D (locally, in some open coordinate neighborhood, given by the equation $z=0$), the Bott-Chern form (associated to the first Chern class) $\widetilde{\mathrm{c}}_1(\mathcal{L};h,h_0)$ encoding the change of metrics grows like $\log\log(1/|z|)$, whereas the good functions are bounded.

The solution of these problems led us to consider the $\mathcal{D}_{\log}$-complexes $\mathcal{D}_{\mathrm{pre}}$ made by pre-log and pre-log-log forms and its subcomplex $\mathcal{D}_{l,ll}$ consisting of log and log-log forms. We emphasize that neither the complex of good forms nor the complex of pre-log-log forms are contained in each other. We also note that if one is interested in arithmetic intersection numbers, the results obtained by both theories agree.

Discussion of results. The $\mathcal{D}_{\log}$-complex $\mathcal{D}_{\mathrm{pre}}$ made out of pre-log and pre-log-log forms could be seen as the complex that satisfies the minimal requirements needed to allow log-log singularities along a fixed divisor as well as to have an arithmetic intersection theory with arithmetic intersection numbers in the proper case (see [10]). As we will show in theorem 4.55, the Bott-Chern forms associated to the change of metrics between a smooth hermitian metric and a good metric belong to the complex of pre-log-log forms. Therefore, we can define arithmetic characteristic classes of good hermitian vector bundles in the arithmetic Chow groups with pre-log-log forms. If our arithmetic variety is proper, we can use this theory to calculate arithmetic Chern numbers of automorphic hermitian vector bundles of arbitrary rank. However, the main disadvantage of $\mathcal{D}_{\text{pre}}$ is that we do not know the size of the associated cohomology groups.

The $\mathcal{D}_{\log}$-complex $\mathcal{D}_{l,ll}$ made out of log and log-log forms is a subcomplex of $\mathcal{D}_{\text{pre}}$. The main difference is that *all* the derivatives of the component functions of the log and log-log forms have to be bounded, which allows us to use an inductive argument to prove a Poincaré lemma, which implies that the associated Deligne complex computes the usual Deligne-Beilinson cohomology (see theorem 2.42). For this reason we have better understanding of the arithmetic Chow groups with log-log forms (see theorem 3.17).

Since a good form is in general not a log-log form, it is not true that the Chern forms for a good hermitian vector bundle are log-log forms. Hence, we introduce the notion of log-singular hermitian metrics, which have, roughly speaking, the same relation to log-log forms as the good hermitian metrics to good forms. We then show that the Bott-Chern forms associated to the change of metrics between smooth hermitian metrics and log-singular hermitian metrics are log-log forms. As a consequence, we can define the Bott-Chern forms for short exact sequences of vector bundles equipped with log-singular hermitian metrics. These Bott-Chern forms have an axiomatic characterization similar to the Bott-Chern forms for short exact sequences of vector bundles equipped with smooth hermitian metrics. The Bott-Chern forms are the main ingredients in order to extend the theory of arithmetic characteristic classes to log-singular hermitian vector bundles.

The price we have to pay in order to use log-log forms is that it is more difficult to prove that a particular form is log-log: we have to bound all derivatives. Note however that most pre-log-log forms which appear are also log-log forms (see for instance section 6). On the other hand, we point out that the theory of log-singular hermitian vector bundles is not optimal for several other reasons. The most important one is that it is not closed under taking sub-objects, quotients and extensions. For example, let

$$0 \longrightarrow (E', h') \longrightarrow (E, h) \longrightarrow (E'', h'') \longrightarrow 0$$

be a short exact sequence of hermitian vector bundles such that the metrics h' and h'' are induced by h. Then, the assumption that h is a log-singular hermitian metric does not imply that the hermitian metrics h' and h'' are log-singular, and vice versa. In particular, automorphic hermitian vector bundles that are not fully decomposed can always be written as successive extensions of fully decomposed automorphic hermitian vector bundles, whose metrics are in general not log-singular. A related question is that the hermitian metric of a unipotent variation of polarized Hodge structures induced by the polarization is in general not log-singular. These considerations suggest that one should further enlarge the notion of log-singular hermitian metrics.

Since the hermitian vector bundles defined on a quasi-projective variety may have arbitrary singularities at infinity, we also consider differential forms with arbitrary singularities along a normal crossing divisor. Using these kinds of differential forms we are able to recover the arithmetic Chow groups à la Gillet, Soulé for quasi-projective varieties.

Finally, another technical difference between this paper and [10] is the fact that in the previous paper the complex $\mathcal{D}_{\log}(X,p)$ is defined by applying the Deligne complex construction to the Zariski sheaf $E_{\log}$, which, in turn, is defined as the Zariski sheaf associated to the pre-sheaf $E^{\circ}_{\log}$. In theorem 3.6, we prove that the pre-sheaf $E^{\circ}_{\log}$ is already a sheaf, which makes it superfluous to take the associated sheaf. Moreover, the proof is purely geometric and can be applied to other similar complexes like $\mathcal{D}_{\mathrm{pre}}$ or $\mathcal{D}_{l,ll}$.

Outline of paper. The set-up of the paper is as follows. In section 2, we introduce several complexes of singular differential forms and discuss their relationship. Of particular importance are the complexes of log and log-log forms for which we prove a Poincaré lemma allowing us to characterize their cohomology by means of their Hodge filtration. In section 3, we introduce and study arithmetic Chow groups with differential forms which are log-log along a fixed normal crossing divisor D. We also consider differential forms having arbitrary singularities at infinity; in particular, we prove that for D being the empty set, the arithmetic Chow groups defined by Gillet, Soulé are recovered. In section 5, we discuss several classes of singular hermitian metrics; we prove that the Bott-Chern forms associated to the change of metrics between a smooth hermitian metric and a log-singular hermitian metric are log-log forms. We also show that the Bott-Chern forms associated to the change of metrics between a smooth hermitian metric and a good hermitian metric are pre-log-log. This allows us to define arithmetic characteristic classes of log-singular hermitian vector bundles. Finally, in section 6, after having given a brief recollection of the basics of Shimura varieties, we prove that the fully decomposed automorphic vector bundles equipped with an equivariant hermitian metric are log-singular hermitian vector bundles. In this respect many examples are provided to which the theory developed in this paper can be applied.

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2 Log and log-log differential forms

In this section, we will introduce several complexes of differential forms with singularities along a normal crossing divisor D, and we will discuss their basic properties.

The first one $\mathscr{E}_X^*\langle D\rangle$ is a complex with logarithmic growth conditions in the spirit of [22]. Unlike in [22], where the authors consider only differential forms of type $(0,q)$, we consider here the whole Dolbeault complex and we show that it is an acyclic resolution of the complex of holomorphic forms with logarithmic poles along the normal crossing divisor D, i.e., this complex computes the cohomology of the complement of D. Another difference with [22] is that, in order to be able to prove the Poincaré lemma for such forms, we need to impose growth conditions to all derivatives of the functions. Note that a similar condition has been already considered in [24].

The second complex $\mathscr{E}_X^*\langle\langle D\rangle\rangle$ contains differential forms with singularities of log-log type along a normal crossing divisor D, and is related with the complex of good forms in the sense of [34]. As the complex of good forms, it contains the Chern forms for fully decomposed automorphic hermitian vector bundles and is functorial with respect to certain inverse images. Moreover all the differential forms belonging to this complex are locally integrable with zero residue. The new property of this complex is that it satisfies a Poincaré lemma that implies that this complex is quasi-isomorphic to the complex of smooth differential forms, i.e., this complex computes the cohomology of the whole variety. The main interest of this complex, as we shall see in subsequent sections, is that it contains also the Bott-Chern forms associated to fully decomposed automorphic vector bundles. Note that neither the complex of good forms in the sense of [34] nor the complex of log-log forms are contained in each other.

The third complex $\mathscr{E}_X^*\langle D_1\langle D_2\rangle\rangle$ that we will introduce is a mixture of the previous complexes. It is formed by differential forms which are log along a normal crossing divisor D_1 and log-log along another normal crossing divisor D_2. This complex computes the cohomology of the complement of D_1.

By technical reasons we will introduce several other complexes.

2.1 Log forms

General notations. Let X be a complex manifold of dimension d. We will denote by $\mathscr{E}_X^*$ the sheaf of complex smooth differential forms over X.

Let D be a normal crossing divisor on X. Let V be an open coordinate subset of X with coordinates $z_1,\dots,z_d$; we put $r_i=|z_i|$. We will say that V is adapted to D, if the divisor $D\cap V$ is given by the equation $z_1\cdots z_k=0$, and the coordinate neighborhood V is small enough; more precisely, we will assume that all the coordinates satisfy $r_i\leq 1/e^e$, which implies that $\log(1/r_i)>e$ and $\log(\log(1/r_i))>1$.

We will denote by $\Delta_r\subseteq\mathbb{C}$ the open disk of radius r centered at 0, by $\overline{\Delta}_r$ the

closed disk, and we will write $\Delta_r^* = \Delta_r \setminus \{0\}$ and $\overline{\Delta}_r^* = \overline{\Delta}_r \setminus \{0\}$.
If f and g are two functions with non-negative real values, we write $f \prec g$, if there exists a real constant $C > 0$ such that $f(x) \leq C \cdot g(x)$ for all x in the domain of definition under consideration.

multi-indices. We collect here all the conventions we will use about multi-indices.

Notation 2.1. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d$, we write

$$|\alpha| = \sum_{i=1}^{d} \alpha_i, \qquad z^\alpha = \prod_{i=1}^{d} z_i^{\alpha_i}, \qquad \bar{z}^\alpha = \prod_{i=1}^{d} \bar{z}_i^{\alpha_i},$$
$$r^\alpha = \prod_{i=1}^{d} r_i^{\alpha_i}, \qquad (\log(1/r))^\alpha = \prod_{i=1}^{d} (\log(1/r_i))^{\alpha_i},$$
$$\frac{\partial^{|\alpha|}}{\partial z^\alpha} f = \frac{\partial^{|\alpha|}}{\prod_{i=1}^{d} \partial z_i^{\alpha_i}} f, \qquad \frac{\partial^{|\alpha|}}{\partial \bar{z}^\alpha} f = \frac{\partial^{|\alpha|}}{\prod_{i=1}^{d} \partial \bar{z}_i^{\alpha_i}} f.$$

If α and β are multi-indices, we write $\beta \geq \alpha$, if, for all $i = 1, \dots, d$, $\beta_i \geq \alpha_i$. We denote by $\alpha + \beta$ the multi-index with components $\alpha_i + \beta_i$. If $1 \leq i \leq d$, we will denote by γ^i the multi-index with all the entries zero except the i-th entry that takes the value 1. More generally, if I is a subset of $\{1, \dots, d\}$, we will denote by γ^I the multi-index

$$\gamma_i^I = \begin{cases} 1, \&i \in I, \\ 0, \&i \notin I. \end{cases}$$

We will denote by $\underline{n}$ the constant multi-index

$$\underline{n}_i = n.$$

In particular, $\underline{0}$ is the multi-index $\underline{0} = (0, \dots, 0)$.
If α is a multi-index and $k \geq 1$ is an integer, we will denote by $\alpha^{\leq k}$ the multi-index

$$\alpha_i^{\leq k} = \begin{cases} \alpha_i, & i \leq k, \\ 0, \&i > k. \end{cases}$$

For a multi-index α, the order function associated to α,

$$\Phi_\alpha : \{1, \dots, |\alpha|\} \longrightarrow \{1, \dots, d\}$$

is given by

$$\Phi_\alpha(i) = k, \text{ if } \sum_{j=1}^{k-1} \alpha_j < i \leq \sum_{j=1}^{k} \alpha_j.$$

Log forms. We introduce now a complex of differential forms with logarithmic growth along a normal crossing divisor. This complex can be used to compute the cohomology of a non-compact algebraic complex manifold with its usual Hodge filtration. It contains the C^{∞} logarithmic Dolbeault complex defined in [7], but it is much bigger and, in particular, it contains also the log-log differential forms defined later. In contrast to the pre-log forms introduced in [10], in the definition given here we impose growth conditions to the differential forms and to all their derivatives.

The problem of the weight filtration of the complex of log forms will not be treated here.

Let X be a complex manifold of dimension d, D a normal crossing divisor, $U = X \setminus D$, and $\iota : U \longrightarrow X$ the inclusion.

Definition 2.2. Let V be a coordinate neighborhood adapted to D. For every integer $K \geq 0$, we say that a smooth complex function f on $V \setminus D$ has *logarithmic growth along D of order K*, if there exists an integer N_K such that, for every pair of multi-indices $\alpha, \beta \in \mathbb{Z}^d_{\geq 0}$ with $|\alpha + \beta| \leq K$, it holds the inequality

$$\left|\frac{\partial^{|\alpha|}}{\partial z^{\alpha}}\frac{\partial^{|\beta|}}{\partial \bar{z}^{\beta}} f(z_1, \dots, z_d)\right| \prec \frac{\left|\prod_{i=1}^{k} \log(1/r_i)\right|^{N_K}}{\left|z^{\alpha^{\leq k}} \bar{z}^{\beta^{\leq k}}\right|}. \tag{2.3}$$

We say that f has *logarithmic growth along D of infinite order*, if it has logarithmic growth along D of order K for all $K \geq 0$. The *sheaf of differential forms on X with logarithmic growth of infinite order along D*, denoted by $\mathscr{E}^*_X\langle D\rangle$, is the subalgebra of $\iota_*\mathscr{E}^*_U$ generated, in each coordinate neighborhood adapted to D, by the functions with logarithmic growth of infinite order along D and the differentials

$$\begin{aligned} &\frac{\mathrm{d}\, z_i}{z_i}, \frac{\mathrm{d}\, \bar{z}_i}{\bar{z}_i}, && \text{for } i = 1, \dots, k, \\ &\mathrm{d}\, z_i, \mathrm{d}\, \bar{z}_i, && \text{for } i = k+1, \dots, d. \end{aligned} \tag{2.4}$$

As a shorthand, a differential form with logarithmic growth of infinite order along D is called log along D or, if D is understood, a log form.

The Dolbeault algebra of log forms. The sheaf $\mathscr{E}^*_X\langle D\rangle$ inherits from $\iota_*\mathscr{E}^*_U$ a real structure and a bigrading. Moreover, it is clear that, if ω is a log form, then $\partial\omega$ and $\bar{\partial}\omega$ are also log forms. Therefore, $\mathscr{E}^*_X\langle D\rangle$ is a sheaf of Dolbeault algebras. We will use all the notations of [10], §5, concerning Dolbeault algebras. For the convenience of the reader we will recall these notations in section 3.1. In particular, from the structure of Dolbeault algebra, there is a well defined Hodge filtration denoted by F.

Pre-log forms. Recall that, in [10], section 7.2, there is introduced the sheaf of pre-log forms denoted $\mathscr{E}^*_X\langle D\rangle_{\mathrm{pre}}$. It is clear that there is an inclusion of sheaves

$$\mathscr{E}^*_X\langle D\rangle \subseteq \mathscr{E}^*_X\langle D\rangle_{\mathrm{pre}}.$$

The cohomology of the complex of log forms. Let $\Omega^*_X(\log D)$ be the sheaf of holomorphic forms with logarithmic poles along D (see [12]). Then, the more general theorem 2.42 implies

Theorem 2.5. *The inclusion*

$$\Omega^*_X(\log D) \longrightarrow \mathscr{E}^*_X\langle D\rangle$$

is a filtered quasi-isomorphism with respect to the Hodge filtration.

In other words, this complex is a resolution of the sheaf of holomorphic forms with logarithmic poles along D, $\Omega^*_X(\log D)$. Thus, if X is a compact Kähler manifold, the complex of global sections $\Gamma(X, \mathscr{E}^*_X\langle D\rangle)$ computes the cohomology of the open complex manifold $U = X \setminus D$ with its Hodge filtration.
Note that corollary 2.5 implies that there is an isomorphism in the derived category $R\iota_*\underline{\mathbb{C}}_U \longrightarrow \mathscr{E}^*_X\langle D\rangle$. This isomorphism is compatible with the real structures. Hence, the complex $\mathscr{E}^*_X\langle D\rangle$ also provides the real structure of the cohomology of U.

Inverse images. The complex of log forms is functorial with respect to inverse images. More precisely, we have the following result.

Proposition 2.6. *Let $f : X \longrightarrow Y$ be a morphism of complex manifolds of dimension d and d'. Let D_X, D_Y be normal crossing divisors on X, Y, respectively, satisfying $f^{-1}(D_Y) \subseteq D_X$. If η is a section of $\mathscr{E}^*_Y\langle D_Y\rangle$, then $f^*\eta$ is a section of $\mathscr{E}^*_X\langle D_X\rangle$.*

Proof. Let p be a point of X. Let V and W be open coordinate neighborhoods of p and $f(p)$, respectively, adapted to D_X and D_Y, and such that $f(V) \subseteq W$. Let k and k' be the number of components of $V \cap D_X$ and $W \cap D_Y$, respectively. Then, the condition $f^{-1}(D_Y) \subseteq D_X$ implies that f can be written as

$$f(x_1, \dots, x_d) = (z_1, \dots, z_{d'}) \tag{2.7}$$

with

$$z_i = \begin{cases} x_1^{a_{i,1}} \cdots x_k^{a_{i,k}} u_i, & \text{if } i \leq k', \\ w_i, & \text{if } i > k', \end{cases}$$

where $u_1, \dots, u_{k'}$ are holomorphic functions that do not vanish in V, the $a_{i,j}$ are non negative integers and $w_{k'+1}, \dots, w_{d'}$ are holomorphic functions. Shrinking V, if necessary, we may assume that the functions u_j are holomorphic and do not vanish in a neighborhood of the adherence of V.
For $1 \leq i \leq k'$, we have

$$f^*\left(\frac{\mathrm{d}\, z_i}{z_i}\right) = \sum_{j=1}^{k} a_{i,j} \frac{\mathrm{d}\, x_j}{x_j} + \frac{\mathrm{d}\, u_i}{u_i}.$$

Since the function $1/u_i$ is holomorphic in a neighborhood of the adherence of V, the function $1/u_i$ and all its derivatives are bounded. If follows that $f^*(\mathrm{d}\, z_i/z_i)$

is a log form (along D_X). The same argument shows that $f^*(\mathrm{d}\,\bar{z}_i/\bar{z}_i)$ is a log form.

If a function g on W satisfies

$$|g(z_1,\dots,z_{d'})| \prec \left|\prod_{i=1}^{k'} \log(1/|z_i|)\right|^N,$$

then f^*g satisfies

$$\begin{aligned}|f^*g(x_1,\dots,x_d)| &\prec \left|\prod_{i=1}^{k'}\left(\sum_{j=1}^{k} a_{i,j}\log(1/|x_j|)+\log(1/|u_i|)\right)\right|^N\\ &\prec \left|\prod_{j=1}^{k}\log(1/|x_j|)\right|^{Nk'}.\end{aligned}$$

Therefore, f^*g has logarithmic growth. It remains to bound the derivatives of f^*g. To ease notation, we will bound only the derivatives with respect to the holomorphic coordinates, the general case being analogous.

For any multi-index $\alpha\in\mathbb{Z}^d_{\geq 0}$, the function $\partial^{|\alpha|}/\partial x^{\alpha}(f^*g)$ is a linear combination of the functions

$$\left\{\frac{\partial^{|\beta|}}{\partial z^{\beta}}g\prod_{i=1}^{|\beta|}\frac{\partial^{|\alpha^i|}}{\partial x^{\alpha^i}}z_{\Phi_\beta(i)}\right\}_{\beta,\{\alpha_i\}}, \tag{2.8}$$

where β runs over all multi-indices $\beta\in\mathbb{Z}^{d'}_{\geq 0}$ such that $|\beta|\leq|\alpha|$, and $\{\alpha_i\}$ runs over all families of multi-indices $\alpha^i\in\mathbb{Z}^d_{\geq 0}$ such that

$$\sum_{i=1}^{|\beta|}\alpha^i=\alpha.$$

The function Φ_α is the order function introduced in 2.1.

Then, since g is a log function,

$$\begin{aligned}\left|\frac{\partial^{|\beta|}}{\partial z^{\beta}}g\prod_{i=1}^{|\beta|}\frac{\partial^{|\alpha^i|}}{\partial x^{\alpha^i}}z_{\Phi_\beta(i)}\right| &\prec \frac{\left|\prod_{j=1}^{k'}\log(1/|z_j|)\right|^{N_{|\beta|}}}{|z^{\beta^{\leq k'}}|}\left|\prod_{i=1}^{|\beta|}\frac{\partial^{|\alpha^i|}}{\partial x^{\alpha^i}}z_{\Phi_\beta(i)}\right|\\ &\prec \left|\prod_{j=1}^{k}\log(1/|x_j|)\right|^{N_{|\beta|}k'}\prod_{i=1}^{|\beta^{\leq k'}|}\left|\frac{1}{z_{\Phi_\beta(i)}}\frac{\partial^{|\alpha^i|}}{\partial x^{\alpha^i}}z_{\Phi_\beta(i)}\right|.\end{aligned}$$

But, by the assumption on the map f, it is easy to see that, for $1\leq j\leq k'$, we have

$$\left|\frac{1}{z_j}\frac{\partial^{|\alpha^i|}}{\partial x^{\alpha^i}}z_j\right|\prec\frac{1}{|x^{(\alpha^i)^{\leq k}}|}, \tag{2.9}$$

which implies the proposition. ☐

Polynomial growth in the local universal cover. We can characterize log forms as differential forms that have polynomial growth in a local universal cover. Let $M > 1$ be a real number and let $U_M \subseteq \mathbb{C}$ be the subset given by

$$U_M = \{x \in \mathbb{C} \mid \operatorname{Im} x > M\}.$$

Let K be an open subset of $\mathbb{C}^{d-k}$. We consider the space $(U_M)^k \times K$ with coordinates $(x_1, \dots, x_d)$.

Definition 2.10. A function f on $(U_M)^k \times K$ is said to have *imaginary polynomial growth*, if there is a sequence of integers $\{N_n\}_{n \geq 0}$ such that for every pair of multi-indices $\alpha, \beta \in \mathbb{Z}^d_{\geq 0}$, the inequality

$$\left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial^{|\beta|}}{\partial \bar{x}^\beta} f(x_1, \dots, x_d) \right| \prec \left| \prod_{i=1}^k \operatorname{Im} x_i \right|^{N_{|\alpha|+|\beta|}} \tag{2.11}$$

holds. The space of differential forms on $(U_M)^k \times K$ with *imaginary polynomial growth* is generated by the functions with imaginary polynomial growth and the differentials

$$\mathrm{d}\, x_i,\ \mathrm{d}\, \bar{x}_i, \qquad \text{for } i = 1, \dots, d.$$

Let X, D, U, and ι be as in definition 2.2.

Definition 2.12. Let W be an open subset of X and $\omega \in \Gamma(W, \iota_*(\mathscr{E}^*_U))$ be a differential form. For every point $p \in W$, there is an open coordinate neighborhood $V \subseteq W$, which is adapted to D and such that the coordinates of V induce an identification $V \cap U = (\Delta^*_r)^k \times K$. We choose $M > \log(1/r)$ and denote by $\pi : (U_M)^k \times K \longrightarrow V$ the covering map given by

$$\pi(x_1, \dots, x_d) = (e^{2\pi i x_1}, \dots, e^{2\pi i x_k}, x_{k+1}, \dots, x_d).$$

We say that ω has *polynomial growth in the local universal cover*, if for every V and π as above, $\pi^*\omega$ has imaginary polynomial growth.

It is easy to see that the differential forms with polynomial growth in the local universal cover form a sheaf of Dolbeault algebras.

Theorem 2.13. *A differential form has polynomial growth in the local universal cover, if and only if, it is a log form.*

Proof. We start with the case of a function. So let f be a function with polynomial growth in the local universal cover and let V be a coordinate neighborhood as in definition 2.12. Let $g = \pi^* f$. By definition, g satisfies

$$g(\dots, x_i + 1, \dots) = g(\dots, x_i, \dots), \qquad \text{for } 1 \leq i \leq k. \tag{2.14}$$

We write formally

$$f(z_1,\dots,z_d)=g(x_1(z_1),\dots,x_d(z_d))$$

with

$$x_i(z_i)=\begin{cases}\frac{1}{2\pi i}\log z_i, &\text{for } i\le k,\\ z_i, &\text{for } i>k.\end{cases}$$

Note that this makes sense because of the periodicity properties (2.14). Then, we have

$$\frac{\partial^{|\alpha|}}{\partial z^{\alpha}}\frac{\partial^{|\beta|}}{\partial \bar{z}^{\beta}}f(z_1,\dots,z_d)=\sum_{\substack{\alpha'\le\alpha\\ \beta'\le\beta}}C_{\alpha,\beta}^{\alpha',\beta'}\frac{\partial^{|\alpha'|}}{\partial x^{\alpha'}}\frac{\partial^{|\beta'|}}{\partial \bar{x}^{\beta'}}g(x_1,\dots,x_d)\cdot$$
$$\cdot\frac{\partial^{|\alpha-\alpha'|}}{\partial z^{\alpha-\alpha'}}\left(\frac{\partial x}{\partial z}\right)^{\alpha'}\frac{\partial^{|\beta-\beta'|}}{\partial \bar{z}^{\beta-\beta'}}\left(\frac{\partial \bar{x}}{\partial \bar{z}}\right)^{\beta'}, \quad (2.15)$$

for certain constants $C_{\alpha,\beta}^{\alpha',\beta'}$. But the estimates

$$\left|\frac{\partial^{|\alpha'|}}{\partial x^{\alpha'}}\frac{\partial^{|\beta'|}}{\partial \bar{x}^{\beta'}}g(x_1,\dots,x_d)\right|\prec\left|\prod_{i=1}^{k}|x_i|\right|^{N_{\alpha',\beta'}}\prec\left|\prod_{i=1}^{k}\log(1/|z_i|)\right|^{N_{\alpha',\beta'}}$$

and

$$\frac{\partial^{|\alpha-\alpha'|}}{\partial z^{\alpha-\alpha'}}\left(\frac{\partial x}{\partial z}\right)^{\alpha'}\frac{\partial^{|\beta-\beta'|}}{\partial \bar{z}^{\beta-\beta'}}\left(\frac{\partial \bar{x}}{\partial \bar{z}}\right)^{\beta'}\prec\frac{1}{\left|z^{\alpha^{\le k}}\bar{z}^{\beta^{\le k}}\right|} \quad (2.16)$$

imply the bounds of f and its derivatives. The converse is proven in the same way.

To prove the theorem for differential forms, observe that, for $1\le i\le k$,

$$\pi^*\left(\frac{\mathrm{d}\,z_i}{z_i}\right)=2\pi i\,\mathrm{d}\,x_i.$$

□

2.2 Log-log forms

log-log growth forms. Let X, D, U, and ι be as in definition 2.2.

Definition 2.17. Let V be a coordinate neighborhood adapted to D. For every integer $K\ge 0$, we say that a smooth complex function f on $V\setminus D$ has *log-log growth along D of order K*, if there exists an integer N_K such that, for every pair of multi-indices $\alpha,\beta\in\mathbb{Z}^d_{\ge 0}$ with $|\alpha+\beta|\le K$, it holds the inequality

$$\left|\frac{\partial^{|\alpha|}}{\partial z^{\alpha}}\frac{\partial^{|\beta|}}{\partial \bar{z}^{\beta}}f(z_1,\dots,z_d)\right|\prec\frac{\left|\prod_{i=1}^{k}\log(\log(1/r_i))\right|^{N_K}}{\left|z^{\alpha^{\le k}}\bar{z}^{\beta^{\le k}}\right|}. \quad (2.18)$$

We say that f has *log-log growth along D of infinite order*, if it has log-log growth along D of order K for all $K \geq 0$. The *sheaf of differential forms on X with log-log growth along D of infinite order* is the subalgebra of $\iota_*\mathscr{E}^*_U$ generated, in each coordinate neighborhood V adapted to D, by the functions with log-log growth along D and the differentials

$$\begin{aligned} &\frac{\mathrm{d}\, z_i}{z_i \log(1/r_i)}, \frac{\mathrm{d}\, \bar{z}_i}{\bar{z}_i \log(1/r_i)}, && \text{for } i = 1, \dots, k, \\ &\mathrm{d}\, z_i,\ \mathrm{d}\, \bar{z}_i, && \text{for } i = k+1, \dots, d. \end{aligned}$$

A differential form with log-log growth along D of infinite order will be called a *log-log growth form*. The sheaf of differential forms on X with log-log growth along D of infinite order will be denoted by $\mathscr{E}^*_X\langle\langle D\rangle\rangle_{\mathrm{gth}}$.

The following characterization of differential forms with log-log growth of infinite order is left to the reader.

Lemma 2.19. *Let V be an open coordinate subset adapted to D and let I, J be two subsets of $\{1, \dots, d\}$. Then, the form $f\, \mathrm{d}\, z_I \wedge \mathrm{d}\, \bar{z}_J$ is a log-log growth form of infinite order, if and only if, for every pair of multi-indices $\alpha, \beta \in \mathbb{Z}^d_{\geq 0}$, there is an integer $N_{\alpha,\beta} \geq 0$ such that*

$$\left|\frac{\partial^{|\alpha|}}{\partial z^\alpha}\frac{\partial^{|\beta|}}{\partial \bar{z}^\beta} f(z_1, \dots, z_d)\right| \prec \frac{\left|\prod_{i=1}^k \log(\log(1/r_i))\right|^{N_{\alpha,\beta}}}{r^{(\gamma^I+\gamma^J+\alpha+\beta)^{\leq k}}(\log(1/r))^{(\gamma^I+\gamma^J)^{\leq k}}}. \tag{2.20}$$

□

Definition 2.21. A function that satisfies the bound (2.20) for any pair of multi-indices α, β with $\alpha + \beta \leq K$ will be called a *(I, J)-log-log growth function of order K*. If it satisfies the bound (2.20) for any pair multi-indices α, β, it will be called a *(I, J)-log-log growth function of infinite order.*

Log-log forms. Unlike the case of log growth forms, the fact that ω is a log-log growth form does not imply that its differential $\partial\omega$ is a log-log growth form.

Definition 2.22. We say that a smooth complex differential form ω is *log-log along D*, if the differential forms ω, $\partial\omega$, $\bar{\partial}\omega$, and $\partial\bar{\partial}\omega$ have log-log growth along D of infinite order. The sheaf of differential forms log-log along D will be denoted by $\mathscr{E}^*_X\langle\langle D\rangle\rangle$. As a shorthand, if D is clear from the context, a differential form which is log-log along D, will be called a log-log form.

From the definition, it is clear that the sheaf of log-log forms is contained in the sheaf of log forms.
Let V be a coordinate subset adapted to D. For $i = 1, \dots, k$, the function $\log(\log(1/r_i))$ is a log-log function and the differential forms

$$\frac{\mathrm{d}\, z_i}{z_i \log(1/r_i)}, \frac{\mathrm{d}\, \bar{z}_i}{\bar{z}_i \log(1/r_i)}, \qquad \text{for } i = 1, \dots, k,$$

are log-log forms.

The Dolbeault algebra of log-log forms. As in the case of log forms, the sheaf $\mathscr{E}^*_X\langle\langle D\rangle\rangle$ inherits a real structure and a bigrading. Moreover, we have forced the existence of operators ∂ and $\bar{\partial}$. Therefore, $\mathscr{E}^*_X\langle\langle D\rangle\rangle$ is a sheaf of Dolbeault algebras (see section 3.1). In particular, there is a well defined Hodge filtration, denoted by F.

Pre-log-log forms Recall that, in [10], section 7.1, there is introduced the sheaf of pre-log-log forms, denoted by $\mathscr{E}^*_X\langle\langle D\rangle\rangle_{\mathrm{pre}}$. It is clear that there is an inclusion of sheaves

$$\mathscr{E}^*_X\langle\langle D\rangle\rangle \subseteq \mathscr{E}^*_X\langle\langle D\rangle\rangle_{\mathrm{pre}}.$$

The cohomology of the complex of log-log differential forms. Let Ω^*_X be the sheaf of holomorphic forms. Then, theorem 2.42, which will be proved later, implies

Theorem 2.23. *The inclusion*

$$\Omega^*_X \longrightarrow \mathscr{E}^*_X\langle\langle D\rangle\rangle$$

is a filtered quasi-isomorphism with respect to the Hodge filtration.

In other words, this complex is a resolution of Ω^*_X, the sheaf of holomorphic differential forms on X. Therefore, if X is a compact Kähler manifold, the complex of global sections $\Gamma(X, \mathscr{E}^*_X\langle\langle D\rangle\rangle)$ computes the cohomology of X with its Hodge filtration. As in the case of log forms it also provides the usual real structure of the cohomology of X. One may say that the singularities of the log-log complex are so mild that they do not change the cohomology.

Inverse images. As in the case of pre-log-log forms, the sheaf of log-log forms is functorial with respect to inverse images. More precisely, we have the following result.

Proposition 2.24. *Let $f : X \longrightarrow Y$ be a morphism of complex manifolds of dimension d and d'. Let D_X, D_Y be normal crossing divisors on X, Y, respectively, satisfying $f^{-1}(D_Y) \subseteq D_X$. If η is a section of $\mathscr{E}^*_Y\langle\langle D_Y\rangle\rangle$, then $f^*\eta$ is a section of $\mathscr{E}^*_X\langle\langle D_X\rangle\rangle$.*

Proof. Since the differential operators ∂ and $\bar{\partial}$ are compatible with inverse images, we have to show that the pre-image of a form with log-log growth of infinite order has log-log growth of infinite order. We may assume that, locally, f can be written as in equation (2.7). If a function g satisfies

$$|g(z_1,\dots,z_{d'})| \prec \left|\prod_{i=1}^{k'} \log(\log(1/|z_i|))\right|^N,$$

we then estimate

$$\begin{aligned}
|(f^*g)(x_1,\dots,x_d)| &\prec \left|\prod_{i=1}^{k'} f^*\log(\log(1/|z_i|))\right|^N \\
&\prec \left|\prod_{i=1}^{k'}\sum_{j=1}^{k}\log(\log(1/|x_j|))\right|^N \\
&\prec \left|\sum_{j=1}^{k}\log(\log(1/|x_j|))\right|^{Nk'}.
\end{aligned}$$

Therefore, f^*g has log-log growth.

Next we have to bound the derivatives of f^*g. As in the proof of proposition 2.6, we will bound only the derivatives with respect to the holomorphic coordinates. Again, we observe that, for any multi-index $\alpha\in\mathbb{Z}^d_{\geq 0}$, the function $\partial^{|\alpha|}/\partial x^\alpha(f^*g)$ is a linear combination of the functions (2.8). But, using that g is a log-log growth function, we can further estimate

$$\begin{aligned}
\left|\frac{\partial^{|\beta|}}{\partial z^\beta}g\prod_{i=1}^{|\beta|}\frac{\partial^{|\alpha^i|}}{\partial x^{\alpha^i}}z_{\Phi_\beta(i)}\right| &\prec \frac{\left|\prod_{j=1}^{k'}\log(\log(1/|z_j|))\right|^{N_{|\beta|}}}{|z^{\beta^{\leq k'}}|}\left|\prod_{i=1}^{|\beta|}\frac{\partial^{|\alpha^i|}}{\partial x^{\alpha^i}}z_{\Phi_\beta(i)}\right| \\
&\prec \left|\prod_{j=1}^{k}\log(\log(1/|x_j|))\right|^{N_{|\beta|}k'}\prod_{i=1}^{|\beta^{\leq k'}|}\left|\frac{1}{z_{\Phi_\beta(i)}}\frac{\partial^{|\alpha^i|}}{\partial x^{\alpha^i}}z_{\Phi_\beta(i)}\right|.
\end{aligned}$$

By (2.9), this yields

$$\left|\frac{\partial^{|\beta|}}{\partial z^\beta}g\prod_{i=1}^{|\beta|}\frac{\partial^{|\alpha^i|}}{\partial x^{\alpha^i}}z_{\Phi_\beta(i)}\right| \prec \left|\prod_{j=1}^{k}\log(\log(1/|x_j|))\right|^{N_{|\beta|}k'}\prod_{i=1}^{|\beta^{\leq k'}|}\frac{1}{|x^{\alpha^{\leq k}}|}.$$

Thus, f^*g has log-log growth of infinite order.

Finally, we are led to study the inverse image of the differential forms

$$\frac{\mathrm{d}\,z_i}{z_i\log(1/|z_i|)},\ \frac{\mathrm{d}\,\bar z_i}{\bar z_i\log(1/|z_i|)},\ \text{for } i=1,\dots,k'.$$

We have

$$f^*\left(\frac{\mathrm{d}\,z_i}{z_i\log(1/z_i\bar z_i)}\right)=\frac{1}{\log(1/z_i\bar z_i)}\left(\sum_{i=j}^{k}a_{i,j}\frac{\mathrm{d}\,x_j}{x_j}+\frac{\mathrm{d}\,u_i}{u_i}\right).$$

Since we have assumed that u_i is a non-vanishing holomorphic function in a neighborhood of the adherence of V (see the proof of proposition 2.6), the function $1/u_i$ and all its derivatives are bounded. Therefore, it only remains to show that the functions

$$f^*\left(\frac{1}{\log(1/|z_i|)}\right) \text{ and } \log(1/|x_j|)f^*\left(\frac{1}{\log(1/|z_i|)}\right), \text{ for } a_{i,j} \neq 0, \tag{2.25}$$

have log-log growth of infinite order, which is left to the reader. □

INTEGRABILITY. Since the sheaf of log-log forms is contained in the sheaf of pre-log-log forms, then [10], proposition 7.6, implies

PROPOSITION 2.26. *(i) Any log-log form is locally integrable.*

(ii) If η is a log-log form, and $[\eta]_X$ is the associated current, then

$$[\mathrm{d}\,\eta]_X = \mathrm{d}[\eta]_X.$$

The same holds true for the differential operators ∂, $\bar{\partial}$, and $\partial\bar{\partial}$.

□

LOGARITHMIC GROWTH IN THE LOCAL UNIVERSAL COVER. We will define a new class of singular forms closely related to the log-log forms. The discussion will be parallel to the one at the end of the previous section.
Let U_M, K, and $(x_1, \dots, x_d)$ be as in definition 2.10.

DEFINITION 2.27. A function f on $(U_M)^k \times K$ is said to have *imaginary logarithmic growth*, if there is a sequence of integers $\{N_n\}_{n\geq 0}$ such that for every pair of multi-indices $\alpha, \beta \in \mathbb{Z}^d_{\geq 0}$, the inequality

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial \bar{x}^\beta} f(x_1, \dots, x_d)\right| \prec \frac{\left|\prod_{i=1}^k \log(\operatorname{Im} x_i)\right|^{N_{|\alpha|+|\beta|}}}{|x^{\alpha^{\leq k}}\bar{x}^{\beta^{\leq k}}|} \tag{2.28}$$

holds. The space of differential forms with *imaginary logarithmic growth* is generated by the functions with imaginary logarithmic growth and the differentials

$$\begin{aligned} &\frac{\mathrm{d}\,x_i}{\operatorname{Im} x_i}, \frac{\mathrm{d}\,\bar{x}_i}{\operatorname{Im} x_i}, &&\text{for } i = 1, \dots, k,\\ &\mathrm{d}\,x_i,\ \mathrm{d}\,\bar{x}_i, &&\text{for } i = k+1, \dots, d. \end{aligned}$$

Let X, D, U, and ι be as in definition 2.2.

DEFINITION 2.29. Let W be an open subset of X and let ω be a differential form in $\Gamma(W, \iota_*(\mathscr{E}_U)^*)$. For every point $p \in W$, there is an open coordinate neighborhood $V \subseteq W$, which is adapted to D and such that the coordinates of

V induce an identification $V \cap U = (\Delta_r^*)^k \times K$. We choose $M > \log(1/r)$ and denote by $\pi : (U_M)^k \times K \longrightarrow V$ the covering map given by

$$\pi(x_1, \dots, x_d) = (e^{2\pi i x_1}, \dots, e^{2\pi i x_k}, x_{k+1}, \dots, x_d).$$

We say that ω has *logarithmic growth in the local universal cover*, if, for every V and π as above, $\pi^*\omega$ has imaginary logarithmic growth.

It is easy to see that the differential forms with logarithmic growth in the local universal cover form a sheaf of Dolbeault algebras.

Theorem 2.30. *The sheaf of differential forms with logarithmic growth in the local universal cover is contained in the sheaf of log-log forms.*

Proof. Since the forms with logarithmic growth in the local universal cover form a Dolbeault algebra, it is enough to check that a differential form with logarithmic growth in the local universal cover has log-log growth of infinite order. We start with the case of a function. So let f and g be as in the proof of theorem 2.13. To bound the derivatives of f we use equation (2.15). But in this case

$$\begin{aligned}\left|\frac{\partial^{|\alpha'|}}{\partial x^{\alpha'}}\frac{\partial^{|\beta'|}}{\partial \bar{x}^{\beta'}} g(x_1, \dots, x_d)\right| &\prec \frac{\left|\prod_{i=1}^k \log(|x_i|)\right|^{N_{\alpha',\beta'}}}{|x^{\alpha'^{\leq k}}\bar{x}^{\beta'^{\leq k}}|} \\ &\prec \frac{\left|\prod_{i=1}^k \log(\log(1/|z_i|))\right|^{N_{\alpha',\beta'}}}{|\log(1/|z|)^{\alpha'^{\leq k}+\beta'^{\leq k}}|}.\end{aligned} \tag{2.31}$$

Note that now the different terms of equation (2.15) have slightly different bounds. If we combine the worst bounds of (2.31) with (2.16), we obtain

$$\left|\frac{\partial^{|\alpha|}}{\partial z^{\alpha}}\frac{\partial^{|\beta|}}{\partial \bar{z}^{\beta}} f(z_1, \dots, z_d)\right| \prec \frac{\left|\prod_{i=1}^k \log(\log(1/r_i))\right|^{N_K}}{|z^{\alpha^{\leq k}}\bar{z}^{\beta^{\leq k}}| \prod_{i=1}^k |\log(1/r_1)|^{\min(\alpha_i,1)+\min(\beta_i,1)}}, \tag{2.32}$$

which implies the bounds of f and its derivatives.
To prove the statement for differential forms, we observe that for $1 \leq i \leq k$,

$$\pi^* \frac{\mathrm{d}\, z_i}{z_i \log(1/|z_i|)} = \frac{i \,\mathrm{d}\, x_i}{\operatorname{Im} x_i}.$$

□

Remark 2.33. The differential forms that interest us are the forms with logarithmic growth in the local universal cover. We have introduced the log-log forms because it is easier to work with bounds of the function and its derivatives in usual coordinates than with the condition of logarithmic growth in the local universal cover. This is particularly true in the proof of the Poincaré lemma. Note however that theorem 2.30 provides us only with an inclusion of sheaves and does not give us a characterization of differential forms with logarithmic growth in the local universal cover. This can be seen by the fact that the bounds (2.32) are sharper than the bounds of definition 2.21. We have chosen the bounds of definition 2.21 because the sharper bounds (2.32) are not functorial. Moreover, they do not characterize forms with logarithmic growth in the local universal cover. In fact, it does not exist a characterization of forms with logarithmic growth in the local universal cover in terms of bounds of the function and its derivatives in usual coordinates.

2.3 Log and log-log mixed forms

For the general situation which we are interested in, we need a combination of the concepts of log and log-log forms.

Mixed growth forms. Let X, D, U, and ι be as in the previous section. Let D_1 and D_2 be normal crossing divisors, which may have common components, and such that $D = D_1 \cup D_2$. We denote by D_2' the union of the components of D_2 that are not contained in D_1. We say that an open coordinate subset V, with coordinates $z_1, \dots, z_d$, is adapted to D_1 and D_2, if $D_1 \cap V$ has equation $z_1 \cdots z_k = 0$ and $D_2' \cap V$ has equation $z_{k+1} \cdots z_l = 0$; we put $r_i := |z_i| < 1/e^e$ for $i = 1, \dots, d$.

Definition 2.34. Let V be a coordinate neighborhood adapted to D_1 and D_2. For every integer $K \geq 0$, we say that a smooth complex function f on $V \setminus D$ has *log growth along D_1 and log-log growth along D_2 of order K*, if there exists an integer $N_K \geq 0$ such that, for every pair of multi-indices $\alpha, \beta \in \mathbb{Z}^d_{\geq 0}$, with $|\alpha + \beta| \leq K$, it holds the inequality

$$\left| \frac{\partial^{|\alpha|}}{\partial z^\alpha} \frac{\partial^{|\beta|}}{\partial \bar{z}^\beta} f(z_1, \dots, z_d) \right| \prec \frac{\left| \prod_{i=1}^k \log(1/r_i) \prod_{j=k+1}^l \log(\log(1/r_j)) \right|^{N_K}}{|z^{\alpha^{\leq l}} \bar{z}^{\beta^{\leq l}}|}. \quad (2.35)$$

We say that f has *log growth along D_1 and log-log growth along D_2 of infinite order*, if it has log growth along D_1 and log-log growth along D_2 of order K for all $K \geq 0$. The *sheaf of differential forms on X with log growth along D_1 and log-log growth along D_2 of infinite order* is the subalgebra of $\iota_* \mathscr{E}^*_U$ generated, in each coordinate neighborhood V adapted to D_1 and D_2, by the functions with log growth along D_1 and log-log growth along D_2, and the differentials

$$\begin{aligned}
&\frac{\mathrm{d}\, z_i}{z_i}, \frac{\mathrm{d}\, \bar{z}_i}{\bar{z}_i}, && \text{for } i = 1, \dots, k, \\
&\frac{\mathrm{d}\, z_i}{z_i \log(1/r_i)}, \frac{\mathrm{d}\, \bar{z}_i}{\bar{z}_i \log(1/r_i)}, && \text{for } i = k+1, \dots, l, \\
&\mathrm{d}\, z_i, \mathrm{d}\, \bar{z}_i, && \text{for } i = l+1, \dots, d.
\end{aligned}$$

When the normal crossing divisors D_1 and D_2 are clear from the context, a differential form with log growth along D_1 and log-log growth along D_2 of infinite order will be called a *mixed growth form.* The sheaf of differential forms on X with log growth along D_1 and log-log growth along D_2 of infinite order will be denoted $\mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}$.

It is clear that

$$\mathscr{E}^*_X\langle D_1\rangle \wedge \mathscr{E}^*_X\langle\langle D_2\rangle\rangle_{\mathrm{gth}} \subseteq \mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}. \tag{2.36}$$

Observe moreover that, by definition,

$$\mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}} = \mathscr{E}^*_X\langle D_1\langle D_2'\rangle\rangle_{\mathrm{gth}}.$$

We leave it to the reader to state the analogue of lemma 2.19.

Partial differentials. Let V be an open coordinate system adapted to D_1 and D_2. In this coordinate system we may decompose the operators ∂ and $\bar\partial$ as

$$\partial = \sum_j \partial_j \quad \text{and} \quad \bar\partial = \sum_j \bar\partial_j, \tag{2.37}$$

where ∂_j and $\bar\partial_j$ contain only the derivatives with respect to z_j.
The following lemma follows directly from the definition.

Lemma 2.38. *Let E_j denote the divisor given by $z_j = 0$. If $\omega \in \mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(V)$, then*

$$\partial_j\omega \in \begin{cases} \mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(V), & \text{if } j \le k \text{ or } j > l, \\ \mathscr{E}^*_X\langle D_1 \cup E_j\langle D_2\rangle\rangle_{\mathrm{gth}}(V), & \text{if } k < j \le l, \end{cases}$$

and the same is true for the operator $\bar\partial_j$.

□

Mixed Forms.

Definition 2.39. We say that a section ω of $\iota_*\mathscr{E}^*_U$ is *log along D_1 and log-log along D_2*, if the differential forms ω, $\partial\omega$, $\bar\partial\omega$, and $\partial\bar\partial\omega$ are sections of $\mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}$. The sheaf of differential forms log along D_1 and log-log along D_2 will be denoted by $\mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle$. As a shorthand, a differential form which is log along D_1 and log-log along D_2, will be called a mixed form.

As the complexes we have defined in the previous sections, the complex $\mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle$ is a sheaf of Dolbeault algebras.

Inverse images. We can generalize propositions 2.6 and 2.24, with the same technique, to the case of mixed forms.

Proposition 2.40. *Let $f : X \longrightarrow Y$ be a morphism of complex manifolds. Let D_1, D_2 and E_1, E_2 be normal crossing divisors on X and Y respectively, such that $D_1 \cup D_2$ and $E_1 \cup E_2$ are also normal crossing divisors. Furthermore, assume that $f^{-1}(E_1) \subseteq D_1$ and $f^{-1}(E_2) \subseteq D_1 \cup D_2$. If η is a section of $\mathscr{E}^*_Y\langle E_1\langle E_2\rangle\rangle$, then $f^*\eta$ is a section of $\mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle$.* □

Integrability. Let X be a complex manifold and D a normal crossing divisor. Let y be a p-codimensional cycle of X and let $Y = \operatorname{supp} y$. Let $\pi : \widetilde{X} \longrightarrow X$ be an embedded resolution of singularities of Y, with normal crossing divisors $D_Y = \pi^{-1}(Y)$ and $\widetilde{D} = \pi^{-1}(D)$ and such that $D_Y \cup \widetilde{D}$ is also a normal crossing divisor.

Lemma 2.41. *Assume that $g \in \Gamma(\widetilde{X}, \mathscr{E}^n_{\widetilde{X}}\langle D_Y\langle \widetilde{D}\rangle\rangle)$. Then, the following statements hold:*

(i) If $n < 2p$, then g is locally integrable on the whole of X. We denote by $[g]_X$ the current associated to g.

(ii) If $n < 2p-1$, then $\mathrm{d}[g]_X = [\mathrm{d}\, g]_X$.

Proof. This is a particular case of [10], lemma 7.13. □

The cohomology of the complex of mixed forms. We are now in position to state the main result of this section.

Theorem 2.42. *The inclusion*

$$\Omega^*_X(\log D_1) \longrightarrow \mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle$$

is a filtered quasi-isomorphism with respect to the Hodge filtration.

Proof. To prove the theorem we will use the classical argument for proving the Poincaré lemma in several variables. We will state here the general argument and we will delay the specific analytic lemmas that we need until the next section.

The theorem is equivalent to the exactness of the sequence of sheaves

$$0 \longrightarrow \Omega^p_X(\log D_1) \xrightarrow{\ i\ } \mathscr{E}^{p,0}_X\langle D_1\langle D_2\rangle\rangle \xrightarrow{\ \bar\partial\ } \mathscr{E}^{p,1}_X\langle D_1\langle D_2\rangle\rangle \xrightarrow{\ \bar\partial\ } \dots$$

The exactness in the first step is clear because a holomorphic form on $X \setminus (D_1 \cup D_2)$ that satisfies the growth conditions imposed in the definitions can only have logarithmic poles along D_1.

For the exactness in the other steps we choose a point $x \in X$. Let V be a coordinate neighborhood of x adapted to D_1 and D_2, and such that x has coordinates $(0, \dots, 0)$.

Let $0 < \epsilon \ll 1$, we denote by $\Delta^d_{x,\epsilon}$ the poly-cylinder

$$\Delta^d_{x,\epsilon} = \{(z_1, \dots, z_d) \in V \mid r_i < \epsilon,\ i = 1, \dots, d\}.$$

In the next section we will prove that, for $j = 1, \dots, d$ and $0 < \epsilon' < \epsilon \ll 1$, there exist operators

$$\begin{aligned} K^{\epsilon',\epsilon}_j &: \mathscr{E}^{p,q}_X\langle D_1\langle D_2\rangle\rangle(\Delta^d_{x,\epsilon}) \longrightarrow \mathscr{E}^{p,q-1}_X\langle D_1\langle D_2\rangle\rangle(\Delta^d_{x,\epsilon'}),\\ P^{\epsilon',\epsilon}_j &: \mathscr{E}^{p,q}_X\langle D_1\langle D_2\rangle\rangle(\Delta^d_{x,\epsilon}) \longrightarrow \mathscr{E}^{p,q}_X\langle D_1\langle D_2\rangle\rangle(\Delta^d_{x,\epsilon'}),\end{aligned}$$

that satisfy the following conditions

(A) If the form ω does not contain any term with $\mathrm{d}\,\bar{z}_i$ for $i > j$, then $K_j^{\epsilon',\epsilon}\omega$ and $P_j^{\epsilon',\epsilon}\omega$ do not contain any term with $\mathrm{d}\,\bar{z}_i$ for $i \geq j$.

(B) $\bar{\partial}K_j^{\epsilon',\epsilon} + K_j^{\epsilon',\epsilon}\bar{\partial} + P_j^{\epsilon',\epsilon} = \mathrm{id}$.

Let $q > 0$ and let $\omega \in \mathscr{E}_X^{p,q}\langle D_1\langle D_2\rangle\rangle_x$ be a germ of a closed form. Assume that ω is defined in a poly-cylinder $\Delta_{x,\epsilon}^d$. By abuse of notation we will not distinguish between sections and germs. Therefore, ω will denote also a closed differential form over $\Delta_{x,\epsilon}^d$ that represents this germ. Moreover, as the open set of definition of each section will be clear from the context, we will not make it explicit. We choose real numbers $0 < \epsilon_1 < \ldots < \epsilon_d < \epsilon$. Then, by property (B), we have

$$\omega = \bar{\partial}K_d^{\epsilon,\epsilon_d}(\omega) + P_d^{\epsilon,\epsilon_d}(\omega).$$

We write $\omega_1 = P_d^{\epsilon,\epsilon_d}(\omega)$. Then, ω_1 does not contain $\mathrm{d}\,\bar{z}_d$ and $\omega - \omega_1$ is a boundary. We define inductively $\omega_{j+1} = P_{d-j}^{\epsilon_{d-j+1},\epsilon_{d-j}}(\omega_j)$. Then, for all j, $\omega - \omega_j$ is a boundary and ω_j does not contain $\mathrm{d}\,\bar{z}_i$ for $i > d - j$. Therefore, $\omega_{d-q+1} = 0$ and ω is a boundary. $\square$

2.4 Analytic lemmas

In this section we will prove the analytic lemmas needed to prove theorem 2.42 and we will define the operators K and P that appear in the proof of this theorem.

Primitive functions with growth conditions. Let f be a smooth function on Δ_ϵ^*, which is integrable on any compact subset of Δ_ϵ. Then, for $\epsilon' < \epsilon$ and $z \in \Delta_{\epsilon'}^*$, we write

$$I_{\epsilon'}(f)(z) = \frac{1}{2\pi\sqrt{-1}}\int_{\Delta_{\epsilon'}} f(w)\frac{\mathrm{d}\,w \wedge \mathrm{d}\,\bar{w}}{w - z}.$$

We denote $r = |z|$.

Lemma 2.43. *(i) If f is a smooth function on Δ_ϵ^* such that*

$$|f(z)| \prec \frac{|\log(\log(1/r))|^N}{(r\log(1/r))^2},$$

then f is integrable in each compact subset of Δ_ϵ and

$$\frac{\partial}{\partial\bar{z}}I_{\epsilon'}(f)(z) = f(z).$$

(ii) If f is a smooth function on Δ_ϵ^* such that

$$|f(z)| \prec \frac{|\log(\log(1/r))|^N}{r\log(1/r)} \text{ and } \left|\frac{\partial}{\partial\bar{z}}f(z)\right| \prec \frac{|\log(\log(1/r))|^N}{(r\log(1/r))^2},$$

then

$$2\pi\sqrt{-1}f(z)=\int_{\partial\Delta_{\epsilon'}}f(w)\frac{\mathrm{d}\,w}{w-z}+\int_{\overline{\Delta}_{\epsilon'}}\frac{\partial}{\partial\bar w}f(w)\frac{\mathrm{d}\,w\wedge\mathrm{d}\,\bar w}{w-z}.$$

(iii) *If f is a smooth function on Δ^*_ϵ such that*

$$|f(z)|\prec\frac{|\log(\log(1/r))|^N}{r\log(1/r)}\quad\textit{and}\quad\left|\frac{\partial}{\partial z}f(z)\right|\prec\frac{|\log(\log(1/r))|^N}{(r\log(1/r))^2},$$

then

$$\frac{\partial}{\partial z}\int_{\overline{\Delta}_{\epsilon'}}f(w)\frac{\mathrm{d}\,w\wedge\mathrm{d}\,\bar w}{w-z}=-\int_{\partial\Delta_{\epsilon'}}f(w)\frac{\mathrm{d}\,\bar w}{w-z}+\int_{\overline{\Delta}_{\epsilon'}}\frac{\partial}{\partial w}f(w)\frac{\mathrm{d}\,w\wedge\mathrm{d}\,\bar w}{w-z}.$$

Proof. We start by proving the integrability of f. Viewed as a function of ϵ, we estimate

$$\begin{aligned}\left|\int_{\Delta_\epsilon}\frac{(\log(\log(1/r)))^N}{r^2(\log(1/r))^2}\,\mathrm{d}\,z\wedge\mathrm{d}\,\bar z\right|&\prec\left|\int_0^\epsilon\frac{(\log(\log(1/r)))^N}{r^2(\log(1/r))^2}r\,\mathrm{d}\,r\right|\\&\prec\left|\int_0^\epsilon\frac{1}{r(\log(1/r))^{3/2}}\,\mathrm{d}\,r\right|\\&\prec\frac{1}{(\log(1/\epsilon))^{1/2}},\end{aligned}$$

which proves the integrability. Then, the claimed formulas are proven as in [20], pp. 24–26. The only new point one has to care about is that the singularities at $z=0$ do not contribute to Stokes theorem. □

Lemma 2.44. *Let $0<\epsilon\ll 1$ be a real number and let f be a smooth function on Δ^*_ϵ. Let $\epsilon'<\epsilon$.*

(i) *If $\omega=f\,\mathrm{d}\,\bar z\in\mathscr{E}^{0,1}_{\Delta_\epsilon}\langle 0\rangle(\Delta_\epsilon)$, then the function f is integrable on any compact subset of Δ_ϵ. We write $g=I_{\epsilon'}(f)$. Then, $g\in\mathscr{E}^{0,0}_{\Delta_{\epsilon'}}\langle 0\rangle(\Delta_{\epsilon'})$ and*

$$\bar\partial g=\omega. \tag{2.45}$$

(ii) *If, moreover, $\omega\in\mathscr{E}^{0,1}_{\Delta_\epsilon}\langle\langle 0\rangle\rangle_{\mathrm{gth}}(\Delta_\epsilon)$, then $g\in\mathscr{E}^{0,0}_{\Delta_{\epsilon'}}\langle\langle 0\rangle\rangle_{\mathrm{gth}}(\Delta_{\epsilon'})$.*

(iii) *If $\omega=f\,\mathrm{d}\,\bar z\wedge\mathrm{d}\,z\in\mathscr{E}^{1,1}_{\Delta_\epsilon}\langle\langle 0\rangle\rangle_{\mathrm{gth}}(\Delta_\epsilon)$, then the function f is integrable on any compact subset of Δ_ϵ. If we write $g=I_{\epsilon'}(f)$ as before, then $g\,\mathrm{d}\,z\in\mathscr{E}^{1,0}_{\Delta_{\epsilon'}}\langle\langle 0\rangle\rangle_{\mathrm{gth}}(\Delta_{\epsilon'})$ and*

$$\bar\partial(g\wedge\mathrm{d}\,z)=\omega. \tag{2.46}$$

Proof. The integrability in the three cases and equations (2.45) and (2.46) are in lemma 2.43. Therefore, it remains only to prove the necessary bounds.
Proof of (i). The condition on ω is equivalent to the inequalities

$$\left|\frac{\partial^{\alpha+\beta}}{\partial z^{\alpha}\partial \bar{z}^{\beta}} f(z)\right| \prec \frac{|\log(1/r)|^{N_{\alpha+\beta}}}{r^{\alpha+\beta+1}} \tag{2.47}$$

for a certain family of integers $\{N_n\}_{n\in\mathbb{Z}_{\geq 0}}$. We may assume that these integers satisfy for $a \leq b$ the inequality $N_a \leq N_b$. We can apply [22], lemma 1, to conclude that g is smooth on $\Delta^*_{\epsilon'}$ and that

$$|g(z)| \prec |\log(1/r)|^{N'_0}$$

for some integer N'_0.
Thus, to prove statement (i), it remains to bound the derivatives of g. Equation (2.45) implies the bound for the derivatives

$$\frac{\partial^{\alpha+\beta}}{\partial z^{\alpha}\partial \bar{z}^{\beta}} g,$$

when $\beta \geq 1$. Therefore, we may assume $\beta = 0$ and $\alpha \geq 1$.
Let $\rho : \mathbb{C} \longrightarrow [0,1]$ be a smooth function such that

$$\rho|_{B(0,1)} = 1, \qquad \rho|_{\mathbb{C}\setminus B(0,2)} = 0,$$

where $B(p,\delta)$ is the open ball of center p and radius δ. Fix $z_0 \in \Delta^*_{\epsilon'}$. Since we want to bound the derivatives of $g(z)$ as z goes to zero, we may assume $z_0 \in \Delta^*_{\epsilon'/2}$. Write $r_0 = |z_0|$, and put

$$\rho_{z_0}(z) = \rho\left(3\frac{z-z_0}{r_0}\right).$$

Then, we have

$$\rho|_{B(z_0,r_0/3)} = 1, \qquad \rho|_{\mathbb{C}\setminus B(z_0,2r_0/3)} = 0.$$

Moreover, we have

$$\frac{\partial^{\alpha}}{\partial z^{\alpha}}\rho_{z_0}(z) \leq \frac{C_{\alpha}}{r_0^{\alpha}} \tag{2.48}$$

for some constants C_{α}.
By the choice of z_0, we have that $\operatorname{supp}(\rho_{z_0}) \subseteq \Delta^*_{\epsilon'}$. We write $f_1 = \rho_{z_0} f$ and $f_2 = (1-\rho_{z_0})f$. Then, for $z \in B(z_0, r_0/3)$, we introduce the auxiliary functions

$$g_1(z) = \frac{1}{2\pi\sqrt{-1}}\int_{\Delta_\epsilon} f_1(w)\frac{\mathrm{d}\,w\wedge \mathrm{d}\,\bar{w}}{w-z}, \quad g_2(z) = \frac{1}{2\pi\sqrt{-1}}\int_{\Delta_\epsilon} f_2(w)\frac{\mathrm{d}\,w\wedge \mathrm{d}\,\bar{w}}{w-z}.$$

These functions satisfy

$$g = g_1 + g_2.$$

Therefore, we can bound the derivatives of g_1 and g_2 separately. We first bound the derivatives of g_1.

$$\begin{aligned}\frac{\partial^\alpha}{\partial z^\alpha}\int_{\Delta_\epsilon} f_1(w)\frac{\mathrm{d}\,w\wedge\mathrm{d}\,\bar w}{w-z} &= \frac{\partial^\alpha}{\partial z^\alpha}\int_{\mathbb{C}} f_1(w)\frac{\mathrm{d}\,w\wedge\mathrm{d}\,\bar w}{w-z}\\ &= \frac{\partial^\alpha}{\partial z^\alpha}\int_{\mathbb{C}} f_1(u+z)\frac{\mathrm{d}\,u\wedge\mathrm{d}\,\bar u}{u}\\ &= \int_{\mathbb{C}}\frac{\partial^\alpha}{\partial z^\alpha} f_1(u+z)\frac{\mathrm{d}\,u\wedge\mathrm{d}\,\bar u}{u}\\ &= \int_{\mathbb{C}}\frac{\partial^\alpha}{\partial w^\alpha} f_1(w)\frac{\mathrm{d}\,w\wedge\mathrm{d}\,\bar w}{w-z}\\ &= \int_{B(z_0,2r_0/3)}\frac{\partial^\alpha}{\partial w^\alpha} f_1(w)\frac{\mathrm{d}\,w\wedge\mathrm{d}\,\bar w}{w-z}.\end{aligned}$$

Hence, using the bounds for the derivatives of f and equation (2.48), we find the inequality

$$\begin{aligned}\left|\frac{\partial^\alpha}{\partial z^\alpha}g_1(z_0)\right| &\prec \frac{|\log(1/r_0)|^{N_\alpha}}{r_0^{\alpha+1}}\left|\int_{B(z_0,2r_0/3)}\frac{\mathrm{d}\,w\wedge\mathrm{d}\,\bar w}{w-z_0}\right|\\ &\prec \frac{|\log(1/r_0)|^{N_\alpha}}{r_0^{\alpha}}.\end{aligned}$$

Now we bound the derivatives of g_2. Since for $z\in B(z_0,r_0/3)$, the function $f_2(w)$ is identically zero in a neighborhood of the point $w=z$, we have

$$\frac{\partial^\alpha}{\partial z^\alpha}\int_{\Delta_\epsilon} f_2(w)\frac{\mathrm{d}\,w\wedge\mathrm{d}\,\bar w}{w-z} = \int_{\Delta_\epsilon} f_2(w)\alpha!\frac{\mathrm{d}\,w\wedge\mathrm{d}\,\bar w}{(w-z)^{\alpha+1}}.$$

Let $A=B(0,r_0/2)$. Then, for $w\in A$, we have $|w-z_0|\geq r_0/2$. Thus, we obtain

$$\begin{aligned}\left|\int_A f_2(w)\frac{\mathrm{d}\,w\wedge\mathrm{d}\,\bar w}{(w-z_0)^{\alpha+1}}\right| &\prec \frac{1}{r_0^{\alpha+1}}\int_0^{r_0/2}\frac{|\log(1/\rho)|^{N_0}}{\rho}\rho\,\mathrm{d}\,\rho\\ &\prec \frac{1}{r_0^{\alpha}}|\log(1/r_0)|^{N_0}.\end{aligned}$$

Here we use that

$$\int(\log x)^N\,\mathrm{d}\,x = x\sum_{i=0}^{N}(-1)^i\frac{N!}{(N-i)!}(\log x)^{N-i}.$$

We write $B=\Delta_{\epsilon'}\setminus(A\cup B(z_0,r_0/3))$. In this region $|w-z_0|\geq|w/4|$. Therefore, we get

$$\left|\int_B f_2(w)\frac{\mathrm{d}\,w\wedge\mathrm{d}\,\bar w}{(w-z_0)^{\alpha+1}}\right| \prec \int_{r_0/2}^{\epsilon'}\frac{|\log(1/\rho)|^{N_0}}{\rho}\frac{\rho\,\mathrm{d}\,\rho}{\rho^{\alpha+1}} \prec \frac{1}{r_0^{\alpha}}|\log(1/r_0)|^{N_0+1}.$$

Here we use that

$$\int (\log x)^n \frac{1}{x^m}\,\mathrm{d}\,x = \begin{cases} \frac{1}{n+1}(\log x)^{n+1} & \text{if } m=1, \\ \frac{1}{x^{m-1}} P_{n,m}(\log x) & \text{if } m>1, \end{cases}$$

where $P_{n,m}$ is a polynomial of degree n. Summing up, we obtain

$$\left|\frac{\partial^\alpha}{\partial z^\alpha} g(z_0)\right| \prec \frac{|\log(1/r_0)|^{N_\alpha+1}}{r_0^\alpha}$$

Observe that, for $\alpha = 0$, this is the proof of [22], lemma 1.
Proof of (ii). In this case, by lemma 2.19, the condition on ω is equivalent to the inequalities

$$\left|\frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta} f(z)\right| \prec \frac{|\log(\log(1/r))|^{N_{\alpha+\beta}}}{r^{\alpha+\beta+1}\log(1/r)} \tag{2.49}$$

for a certain increasing family of integers $\{N_n\}_{n\in\mathbb{Z}_{\geq 0}}$. Again, by lemma 2.19, to prove statement (ii), we have to show

$$\left|\frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta} g(z)\right| \prec \frac{|\log(\log(1/r))|^{N'_{\alpha+\beta}}}{r^{\alpha+\beta}} \tag{2.50}$$

for a certain family of integers $\{N'_n\}_{n\in\mathbb{Z}_{\geq 0}}$.
By (2.45), the functions

$$\frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta} g,$$

when $\beta \geq 1$, satisfy the required bounds. Thus, it remains to bound $\partial^\alpha/\partial z^\alpha g$ for $\alpha \geq 0$. As in the proof of statement (i), we fix z_0 and write $g = g_1 + g_2$. For g_1, we work as before and get for $\alpha \geq 0$

$$\left|\frac{\partial^\alpha}{\partial z^\alpha} g_1(z_0)\right| \prec \frac{|\log(\log(1/r_0))|^{N_\alpha}}{r_0^\alpha \log(1/r_0)}.$$

To bound g_2, we integrate over the regions A and B as before. We first bound the integral over the region $A = B(0, r_0/2)$.

$$\left|\int_A f_2(w) \frac{\mathrm{d}\,w \wedge \mathrm{d}\,\bar{w}}{(w-z_0)^{\alpha+1}}\right| \prec \frac{1}{r_0^{\alpha+1}} \int_0^{r_0/2} \frac{|\log(\log(1/\rho))|^{N_0}}{\log(1/\rho)}\,\mathrm{d}\,\rho.$$

Since, for $\rho < 1/e^{e^{N_0}}$, the function

$$\frac{(\log(\log(1/\rho)))^{N_0}}{\log(1/\rho)}$$

is an increasing function, we have

$$\begin{aligned} \frac{1}{r_0^{\alpha+1}} \int_0^{r_0/2} \frac{|\log(\log(1/\rho))|^{N_0}}{\log(1/\rho)}\,\mathrm{d}\,\rho &\prec \frac{|\log(\log(1/r_0))|^{N_0}}{r_0^{\alpha+1}\log(1/r_0)} \int_0^{r_0/2} \mathrm{d}\,\rho \\ &\prec \frac{|\log(\log(1/r_0))|^{N_0}}{r_0^{\alpha}\log(1/r_0)} \end{aligned}$$

in the domain $0 < r_0 \leq 2/e^{e^{N_0}}$.
If f and g are two continuous functions with g strictly positive, defined on a compact set, then $f \prec g$. Therefore, the above inequality extends to the domain $0 \leq r_0 \leq \epsilon'/2$.
We now bound the integral over the region $B = \Delta_{\epsilon'} \setminus (A \cup B(z_0, r_0/3))$. By the bound of the function f, we have

$$\left| \int_B f_2(w) \frac{\mathrm{d}\, w \wedge \mathrm{d}\, \bar{w}}{(w - z_0)^{\alpha+1}} \right| \prec \int_{r_0/2}^{\epsilon'} \frac{|\log(\log(1/\rho))|^{N_0}}{\rho^{\alpha+1} \log(1/\rho)} \,\mathrm{d}\, \rho.$$

Thus, in the case $\alpha = 0$, we have

$$\int_{r_0/2}^{\epsilon'} \frac{|\log(\log(1/\rho))|^{N_0}}{\rho \log(1/\rho)} \,\mathrm{d}\, \rho \prec |\log(\log(1/r_0))|^{N_0+1}.$$

In the case $\alpha > 0$, since, for $\rho < 1/e^e$, the function

$$\frac{(\log(\log(1/\rho)))^{N_0}}{\rho^{1/2} \log(1/\rho)}$$

is a decreasing function, we have

$$\begin{aligned} \int_{r_0/2}^{\epsilon'} \frac{|\log(\log(1/\rho))|^{N_0}}{\rho^{\alpha+1} \log(1/\rho)} \,\mathrm{d}\, \rho &\prec \frac{|\log(\log(1/r_0))|^{N_0}}{r_0^{1/2} \log(1/r_0)} \int_{r_0/2}^{\epsilon'} \frac{1}{\rho^{\alpha+1/2}} \,\mathrm{d}\, \rho \\ &\prec \frac{|\log(\log(1/r_0))|^{N_0}}{r_0^{\alpha} \log(1/r_0)}. \end{aligned}$$

Summing up, we obtain

$$\left| \frac{\partial^\alpha}{\partial z^\alpha} g(z_0) \right| \prec \frac{|\log(\log(1/r_0))|^{N_\alpha+1}}{r_0^\alpha}.$$

This finishes the proof of the second statement.
Proof of (iii). In this case, again by lemma 2.19, the condition on ω is equivalent to the conditions

$$\left| \frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta} f(z) \right| \prec \frac{|\log(\log(1/r))|^{N_{\alpha+\beta}}}{r^{\alpha+\beta+2} (\log(1/r))^2}$$

for a certain increasing family of integers $\{N_n\}_{n \in \mathbb{Z}_{\geq 0}}$, and the inequalities we have to prove are

$$\left| \frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta} g(z) \right| \prec \frac{|\log(\log(1/r))|^{N'_{\alpha+\beta}}}{r^{\alpha+\beta+1} \log(1/r)}$$

for a certain family of integers $\{N'_n\}_{n \in \mathbb{Z}_{\geq 0}}$.

First we note that, by equation (2.46), for $\beta \geq 1$, the functions

$$\frac{\partial^{\alpha+\beta}}{\partial z^{\alpha}\partial \bar{z}^{\beta}} g$$

satisfy the required bounds. Thus, it remains to bound the functions $\partial^{\alpha} g/\partial z^{\alpha}$ for $\alpha \geq 0$. The proof is similar as before. We decompose again $g = g_1 + g_2$. In this case

$$\left|\frac{\partial^{\alpha}}{\partial z^{\alpha}} g_1(z_0)\right| \prec \frac{|\log(\log(1/r_0))|^{N_\alpha}}{\log(1/r_0)^2 r_0^{\alpha+1}}.$$

Whereas the integral of g_2 over A is bounded as

$$\begin{aligned}\left|\int_A f_2(w)\frac{\mathrm{d}\, w\wedge \mathrm{d}\,\bar{w}}{(w-z_0)^{\alpha+1}}\right| &\prec \frac{1}{r_0^{\alpha+1}}\int_0^{r_0/2}\frac{|\log(\log(1/\rho))|^{N_0}}{\rho\log(1/\rho)^2}\,\mathrm{d}\,\rho\\ &\prec \frac{|\log(\log(1/r_0))|^{N_0}}{r_0^{\alpha+1}\log(1/r_0)},\end{aligned}$$

and the integral of g_2 over B is bounded as

$$\begin{aligned}\left|\int_B f_2(w)\frac{\mathrm{d}\, w\wedge \mathrm{d}\,\bar{w}}{(w-z_0)^{\alpha+1}}\right| &\prec \int_{r_0/2}^{\epsilon'}\frac{|\log(\log(1/\rho))|^{N_0}}{\rho^{\alpha+2}\log(1/\rho)^2}\,\mathrm{d}\,\rho\\ &\prec \frac{|\log(\log(1/r_0))|^{N_0}}{r_0^{\alpha+1}\log(1/r_0)^2}.\end{aligned}$$

Summing up, we obtain for $\alpha \geq 0$

$$\left|\frac{\partial^{\alpha}}{\partial z^{\alpha}} g(z_0)\right| \prec \frac{|\log(\log(1/r_0))|^{N_\alpha}}{r_0^{\alpha+1}\log(1/r_0)}.$$

This finishes the proof of the lemma. □

Remark 2.51. Observe that, in general, a section of $\mathscr{E}^{1,1}_{\Delta_\epsilon}\langle 0\rangle(\Delta_\epsilon)$ is not locally integrable (see remark 2.55). Therefore, the analogue of lemma 2.44 (iii) is not true for log forms.

The operators K and P. Let X, U, D, ι, D_1, and D_2 be as in definition 2.39.

Notation 2.52. Let $x \in X$. Let V be an open coordinate neighborhood of x with coordinates $z_1, \dots, z_d$, adapted to D_1 and D_2, such that x has coordinates $(0, \dots, 0)$. Thus, D_1 has equation $z_1 \cdots z_k = 0$ and D_2' has equation $z_{k+1} \cdots z_l = 0$. Once this coordinate neighborhood is chosen, we put

$$\begin{aligned}\zeta_i &= \frac{\mathrm{d}\, z_i}{z_i}, &&\text{if } 1 \leq i \leq k,\\ \zeta_i &= \mathrm{d}\, z_i, &&\text{if } i > k.\end{aligned}$$

For any subset $I \subseteq \{1, \dots, d\}$, we denote

$$\zeta_I = \bigwedge_{i \in I} \zeta_i, \quad \mathrm{d}\,\bar{z}_I = \bigwedge_{i \in I} \mathrm{d}\,\bar{z}_i.$$

Given any differential form ω, let

$$\omega = \sum_{I,J} f_{I,J} \zeta_I \wedge \mathrm{d}\,\bar{z}_J$$

be the decomposition of ω in monomials. Then, we write

$$\omega_{I,J} = f_{I,J} \zeta_I \wedge \mathrm{d}\,\bar{z}_J.$$

For any subset $I \subseteq \{1, \dots, d\}$ and $i \in I$, we will write

$$\sigma(I,i) = \sharp\{j \in I \mid j < i\} \qquad \text{and} \qquad I_i = I \setminus \{i\}.$$

Definition 2.53. Let $0 < \epsilon' < \epsilon \ll 1$. Let $\Delta^d_{x,\epsilon}$ be the poly-cylinder centered at x of radius ϵ. Let $\omega \in \mathscr{E}^{p,q}_X \langle D_1 \langle D_2 \rangle \rangle_{\mathrm{gth}}(\Delta^d_{x,\epsilon})$, and let

$$\omega = \sum_{I,J} f_{I,J} \zeta_I \wedge \mathrm{d}\,\bar{z}_J \tag{2.54}$$

be the decomposition of ω into monomials. We define

$$\begin{aligned} K^{\epsilon',\epsilon}_j(\omega) = \sum_I & (-1)^{\# I} \zeta_I \wedge \\ & \sum_{J \mid j \in J} \frac{(-1)^{\sigma(J,j)}}{2\pi\sqrt{-1}} \int_{\Delta_{\epsilon'}} f_{I,J}(\dots, z_{j-1}, w, z_{j+1}, \dots) \frac{\mathrm{d}\,w \wedge \mathrm{d}\,\bar{w}}{w - z_j}\, \mathrm{d}\,\bar{z}_{J_j}, \end{aligned}$$

$$\begin{aligned} P^{\epsilon',\epsilon}_j(\omega) = \sum_I & \zeta_I \wedge \\ & \sum_{J \mid j \notin J} \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta_{\epsilon'}} f_{I,J}(\dots, z_{j-1}, w, z_{j+1}, \dots) \frac{\mathrm{d}\,w}{w - z_j}\, \mathrm{d}\,\bar{z}_J. \end{aligned}$$

To ease notation, if ϵ and ϵ' are clear from the context, we will drop them and write K_j, resp. P_j instead of $K^{\epsilon',\epsilon}_j$, resp. $P^{\epsilon',\epsilon}_j$.

Remark 2.55. The reason why we use the differentials ζ_I instead of $\mathrm{d}\,z_I$ in the definition of K and P, is that, in general, a log form is not locally integrable. For instance, if $d = k = 1$ and $\omega = f\ \mathrm{d}\,z \wedge \mathrm{d}\,\bar{z}$ is a section of $\mathscr{E}^{1,1}_{\Delta_\epsilon}\langle 0 \rangle(\Delta_\epsilon)$, then f satisfies

$$|f(z)| \prec \frac{|\log(1/r)|^N}{r^2},$$

and the integral

$$\int_{\overline{\Delta}_{\epsilon'}} \frac{|\log(1/|w|)|^N}{|w|^2} \frac{\mathrm{d}\, w \wedge \mathrm{d}\, \bar{w}}{w - z}$$

does not converge. But, by the definition we have adopted, $K^{\epsilon',\epsilon}(\omega) = g \,\mathrm{d}\, z$, where

$$g(z) = \frac{1}{z} I_{\epsilon'}(z \cdot f) = \frac{1}{2\pi\sqrt{-1}} \frac{1}{z} \int_{\overline{\Delta}_{\epsilon'}} w f(w) \frac{\mathrm{d}\, w \wedge \mathrm{d}\, \bar{w}}{w - z}.$$

This integral is absolutely convergent and

$$\frac{\partial}{\partial \bar{z}} g(z) = \frac{1}{z} \frac{\partial}{\partial \bar{z}} I_{\epsilon'}(z \cdot f)(z) = \frac{z f(z)}{z} = f(z).$$

This trick will force us to be careful when studying the compatibility of K with the operator ∂ because, for a log form ω, the definitions of $K(\omega)$ and of $K(\partial\omega)$ use different kernels in the integral operators.

Theorem 2.56. *Let $\omega \in \mathscr{E}_X^{p,q}\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta_{x,\epsilon})$. Then, we have*

$$K_j^{\epsilon',\epsilon}(\omega) \in \mathscr{E}_X^{p,q-1}\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta_{x,\epsilon'}), \text{ and}$$
$$P_j^{\epsilon',\epsilon}(\omega) \in \mathscr{E}_X^{p,q}\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta_{x,\epsilon'}).$$

These operators satisfy

(i) If the form ω does not contain any term with $\mathrm{d}\, \bar{z}_i$ for $i > j$, then $K_j\omega$ and $P_j\omega$ do not contain any term with $\mathrm{d}\, \bar{z}_i$ for $i \geq j$.

(ii) If $\omega \in \mathscr{E}_X^{p,q}\langle D_1\langle D_2\rangle\rangle(\Delta_{x,\epsilon})$, then

$$K_j^{\epsilon',\epsilon}(\omega) \in \mathscr{E}_X^{p,q-1}\langle D_1\langle D_2\rangle\rangle(\Delta_{x,\epsilon'}), \text{ and}$$
$$P_j^{\epsilon',\epsilon}(\omega) \in \mathscr{E}_X^{p,q}\langle D_1\langle D_2\rangle\rangle(\Delta_{x,\epsilon'}).$$

(iii) In this case, $\bar{\partial} K_j + K_j \bar{\partial} + P_j = \mathrm{id}$.

Proof. By lemma 2.44 and the theorem of taking derivatives under the integral sign, we have that $K_j(\omega) \in \mathscr{E}_X^{p,q-1}\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta_{x,\epsilon'})$, and it is clear that $P_j(\omega) \in \mathscr{E}_X^{p,q}\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta_{x,\epsilon'})$. Then, property (i) follows from the definition and it is easy to see that, if $\bar{\partial}\omega$, $\partial\omega$, and $\partial\bar{\partial}\omega$ belong to $\mathscr{E}_X^{*}\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta_{x,\epsilon})$, the same is true for $\bar{\partial} P_j(\omega)$, $\partial P_j(\omega)$, and $\partial\bar{\partial} P_j(\omega)$.

In the sequel of the proof, we will denote by E_m the divisor given by $z_m = 0$. Assume now that $\bar{\partial}\omega \in \mathscr{E}_X^{*}\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta_{x,\epsilon})$. We will prove property (iii). We write

$$\omega = \sum_{I,J} f_{I,J}\zeta_I \wedge \mathrm{d}\,\bar{z}_J,$$

$$\omega_1 = \sum_{I,j\in J} f_{I,J}\zeta_I \wedge \mathrm{d}\,\bar{z}_J,$$

$$\omega_2 = \sum_{I,j\notin J} f_{I,J}\zeta_I \wedge \mathrm{d}\,\bar{z}_J.$$

Recall that we have introduced the operator $\bar{\partial}_j$ in equation (2.37). We write $\bar{\partial}_{\neq j} = \bar{\partial} - \bar{\partial}_j$, and we decompose

$$\bar{\partial}K_j(\omega) = \bar{\partial}K_j(\omega_1) = \bar{\partial}_{\neq j}K_j(\omega_1) + \bar{\partial}_j K_j(\omega_1),$$

and

$$K_j(\bar{\partial}\omega) = K_j(\bar{\partial}_{\neq j}\omega_1 + \bar{\partial}_j\omega_2)$$

The difficulty at this point is that, when $k < j \leq l$, the form ω is log-log along E_j but, according to lemma 2.38, $\bar{\partial}_j\omega$ only needs to be log along E_j, and the integral operator K_j for log-log forms may diverge when applied to log forms. The key point is to observe that the extra hypothesis about $\bar{\partial}\omega$ allows us to apply the operator K_j to the differential forms $\bar{\partial}_{\neq j}\omega_1$ and $\bar{\partial}_j\omega_2$ individually: Fix I and J with $j \in J$ and $m \neq j$. We consider first the problematic case $k < j \leq l$. By lemma 2.38, we have

$$\bar{\partial}_m\omega_{I,J_m} \in \begin{cases} \mathscr{E}^*_X\langle D_1 \cup E_m\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta^d_{x,\epsilon}), & \text{if } k < m \leq l, \\ \mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta^d_{x,\epsilon}), & \text{otherwise.} \end{cases}$$

Therefore, if we denote by D' the union of all the components of D different from E_j, then

$$(\bar{\partial}_{\neq j}\omega_1)_{I,J} \in \mathscr{E}^*_X\langle D'\langle E_j\rangle\rangle_{\mathrm{gth}}(\Delta^d_{x,\epsilon}).$$

Since, by hypothesis, $(\bar{\partial}\omega)_{I,J} \in \mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta_{x,\epsilon})$ and $(\bar{\partial}_j\omega_2)_{I,J} = (\bar{\partial}\omega - \bar{\partial}_{\neq j}\omega_1)_{I,J}$, then

$$(\bar{\partial}_j\omega_2)_{I,J} \in \mathscr{E}^*_X\langle D'\langle E_j\rangle\rangle_{\mathrm{gth}}(\Delta^d_{x,\epsilon}),$$

and we can apply the operator K_j for log-log forms to the differential forms $\bar{\partial}_{\neq j}\omega_1$ and $\bar{\partial}_j\omega_2$ individually. If $j \leq k$, then ω is log along E_j; the same is true for the differential forms $\bar{\partial}_{\neq j}\omega_1$ and $\bar{\partial}_j\omega_2$. But in this case the operator K_j is the operator for log forms and can be applied to $\bar{\partial}_{\neq j}\omega_1$ and $\bar{\partial}_j\omega_2$ individually. The case $j > l$ is similar. Thus, we can write

$$K_j(\bar{\partial}_{\neq j}\omega_1 + \bar{\partial}_j\omega_2) = K_j(\bar{\partial}_{\neq j}\omega_1) + K_j(\bar{\partial}_j\omega_2).$$

But by the theorem of taking derivatives under the integral sign, we now obtain

$$\bar{\partial}_{\neq j} K_j(\omega_1) + K_j(\bar{\partial}_{\neq j}\omega_1) = 0.$$

By lemma 2.44, we have

$$\bar{\partial}_j K_j(\omega_1) = \omega_1,$$

and by the generalized Cauchy integral formula (lemma 2.43 (ii)), we note

$$K_j(\bar{\partial}_j \omega_2) = \omega_2 - P_j(\omega_2) = \omega_2 - P_j(\omega).$$

Summing up, we obtain

$$\bar{\partial} K_j(\omega) + K_j(\bar{\partial}\omega) = \omega - P_j(\omega). \tag{2.57}$$

By (2.57) and the fact that $K_j(\bar{\partial}\omega), P_j(\omega) \in \mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta_{x,\epsilon'})$, we obtain that

$$\bar{\partial} K_j(\omega) \in \mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta_{x,\epsilon'}).$$

Assume now that $\partial\omega \in \mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta_{x,\epsilon})$. We fix $I, J \subseteq \{1, \dots, d\}$, with $j \in J$. If $j \notin I$, then

$$(\partial K_j(\omega))_{I,J_j} = \sum_{m \neq j} \partial_m K_j(\omega_{I_m,J}) = K_j\left(\sum_{m \neq j} \partial_m \omega_{I_m,J}\right) = K_j((\partial\omega)_{I,J}).$$

Therefore, it belongs to $\mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta^d_{x,\epsilon})$. If $j \in I$, we write

$$(\partial K_j(\omega))_{I,J_j} = \sum_{m \neq j} \partial_m K_j(\omega_{I_m,J}) + \partial_j K_j(\omega_{I_j,I}). \tag{2.58}$$

The theorem of taking derivatives under the integral sign implies for $m \neq j$

$$\partial_m K_j(\omega_{I_m,J}) = -K_j(\partial_m \omega_{I_m,J}).$$

Note that the term on the right hand side is well defined by lemma 2.38. We first treat the case $j \leq k$. We have to be careful because the integral kernels appearing in the expressions $\partial_j K_j(\omega_{I_j,J})$ and $K_j(\partial_j \omega_{I_j,J})$ are different in each term.

Again by lemma 2.38,

$$\partial_j \omega_{I_j,J} \in \mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta^d_{x,\epsilon}.$$

Since, moreover, $\partial\omega \in \mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta^d_{x,\epsilon})$,

$$\sum_{m \neq j} \partial_m \omega_{I_m,J} = (\partial\omega)_{I,J} - \partial_j \omega_{I_j,J} \in \mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta^d_{x,\epsilon}).$$

Hence, by lemma 2.44

$$K_j\left(\sum_{m\neq j}\partial_m\omega_{I_m,J}\right)\in\mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta^d_{x,\epsilon}).$$

By the same lemma it follows that

$$\partial_j K_j(\omega_{I_j,I})\in\mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta^d_{x,\epsilon}).$$

Now we treat the case $j>k$. In this case the expressions $\partial_j K_j(\omega_{I_j,J})$ and $K_j(\partial_j\omega_{I_j,J})$ use the same integral kernel. By lemma 2.43 (iii), we have

$$\partial_j K_j(\omega_{I_j,J})=-K_j(\partial_j\omega_{I_j,J})+\frac{(-1)^{\#I+\sigma(J,j)+\sigma(I,j)}}{2\pi\sqrt{-1}}\int_{\gamma_{\epsilon'}}f_{I_j,J}\frac{\mathrm{d}\,\bar{w}}{w-z}\zeta_I\wedge\mathrm{d}\,\bar{z}_{J_j}.$$

Hence, we arrive at

$$(\partial K_j(\omega))_{I,J_j}=-(K_j(\partial\omega))_{I,J_j}+\frac{(-1)^{\#I+\sigma(J,j)+\sigma(I,j)}}{2\pi\sqrt{-1}}\int_{\gamma_{\epsilon'}}f_{I_j,J}\frac{\mathrm{d}\,\bar{w}}{w-z}\zeta_I\wedge\mathrm{d}\,\bar{z}_{J_j}.$$

Thus, it belongs to $\mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta^d_{x,\epsilon})$.
Finally, assume that $\partial\omega,\bar{\partial}\omega,\partial\bar{\partial}\omega\in\mathscr{E}^*_X\langle D_1\langle D_2\rangle\rangle_{\mathrm{gth}}(\Delta^d_{x,\epsilon})$. By equation (2.57), we have

$$\partial\bar{\partial}K_j(\omega)=-\partial K_j(\bar{\partial}\omega)+\partial\omega-\partial P_j(\omega);$$

therefore, the result follows from the previous cases. □

2.5 Good forms.

In this section we recall the definition of good forms in the sense of [34]. We introduce the complex of Poincaré singular forms that is contained in both, the complex of good forms and the complex of log-log forms.

Poincaré growth. Let X, D, U, and ι be as in definition 2.2.

Definition 2.59. Let V be a coordinate neighborhood adapted to D. We say that a smooth complex function f on $V\setminus D$ has *Poincaré growth (along D)*, if it is bounded. We say that it has *Poincaré growth (along D) of infinite order*, if for all multi-indices $\alpha,\beta\in\mathbb{Z}^d_{\geq 0}$

$$\left|\frac{\partial^{|\alpha|}}{\partial z^\alpha}\frac{\partial^{|\beta|}}{\partial\bar{z}^\beta}f(z_1,\dots,z_d)\right|\prec\frac{1}{|z^{\alpha^{\leq k}}\bar{z}^{\beta^{\leq k}}|}. \qquad (2.60)$$

The *sheaf of differential forms on X with Poincaré growth (resp. of infinite order)* is the subalgebra of $\iota_*\mathscr{E}^*_U$ generated, in each coordinate neighborhood V

adapted to D, by the functions with Poincaré growth (resp. of infinite order) and the differentials

$$\begin{aligned} &\frac{\mathrm{d}\, z_i}{z_i \log(1/r_i)}, \frac{\mathrm{d}\, \bar{z}_i}{\bar{z}_i \log(1/r_i)}, &&\text{for } i=1,\dots,k,\\ &\mathrm{d}\, z_i,\, \mathrm{d}\, \bar{z}_i, &&\text{for } i=k+1,\dots,d. \end{aligned}$$

Good forms. We recall that a smooth form ω on $X \setminus D$ is *good (along D)*, if ω and $\mathrm{d}\,\omega$ have Poincaré growth along D (see [34]). Observe that, since the operator d is not bi-homogeneous, the sheaf of good forms is not bigraded. Although good forms are very similar to pre-log-log forms, there is no inclusion between both sheaves. Nevertheless, we have the following easy

Lemma 2.61. *If ω is a good form of pure bidegree, then it is a pre-log-log form, if and only if, $\partial\bar{\partial}\omega$ has log-log growth of order 0.* □

Poincaré singular forms.

Definition 2.62. We will say that ω is *Poincaré singular (along D)*, if ω, $\partial\omega$, $\bar{\partial}\omega$, and $\partial\bar{\partial}\omega$ have Poincaré growth of infinite order.

Note that the sheaf of Poincaré singular forms is contained in both, the sheaf of good forms and the sheaf of log-log forms. Observe moreover that we cannot expect to have a Poincaré lemma for the complex of Poincaré singular forms, precisely due to the absence of the functions $\log(\log(1/r_i))$.

Functoriality. The complex of Poincaré singular forms share some of the properties of the complex of log-log forms. For instance, we have the following compatibility with respect to inverse images which is proven as in proposition 2.24.

Proposition 2.63. *Let $f : X \longrightarrow Y$ be a morphism of complex manifolds of dimension d and d'. Let D_X, D_Y be normal crossing divisors on X, Y, respectively, satisfying $f^{-1}(D_Y) \subseteq D_X$. If η is a Poincaré singular form on Y, then $f^*\eta$ is a Poincaré singular form on X.* □

3 Arithmetic Chow rings with log-log growth conditions

In this section we use the theory of abstract cohomological arithmetic Chow rings developed in [10] to obtain a theory of arithmetic Chow rings with log-log forms. Since we have computed the cohomology of the complex of log-log forms, we have a more precise knowledge of the size of these arithmetic Chow rings than of the arithmetic Chow rings with pre-log-log forms considered in [10].

3.1 Dolbeault algebras and Deligne algebras

In this section we recall the notion of Dolbeault algebra and the properties of the associated Deligne algebra.

Dolbeault algebras.

Definition 3.1. A *Dolbeault algebra* $A = (A^*_{\mathbb{R}}, \mathrm{d}_A, \wedge)$ is a real differential graded commutative algebra which is bounded from below and equipped with a bigrading on $A_{\mathbb{C}} := A_{\mathbb{R}} \otimes \mathbb{C}$,

$$A^n_{\mathbb{C}} = \bigoplus_{p+q=n} A^{p,q},$$

satisfying the following properties:

(i) The differential d_A can be decomposed as the sum of operators $\mathrm{d}_A = \partial + \bar{\partial}$ of type $(1,0)$, resp. $(0,1)$.

(ii) It satisfies the symmetry property $\overline{A^{p,q}} = A^{q,p}$, where $\overline{}$ denotes complex conjugation.

(iii) The product induced on $A_{\mathbb{C}}$ is compatible with the bigrading:

$$A^{p,q} \wedge A^{p',q'} \subseteq A^{p+p',q+q'}.$$

By abuse of notation, we will also denote by A^* the complex differential graded commutative algebra $A^*_{\mathbb{C}}$.

Notation 3.2. Given a Dolbeault algebra A we will use the following notations. The Hodge filtration F of A^* is the decreasing filtration given by

$$F^p A^n_{\mathbb{C}} = \bigoplus_{p' \geq p} A^{p',n-p'}.$$

The filtration $\overline{F}$ is the complex conjugate of F, i.e.,

$$\overline{F}^p A^n = \overline{F^p A^n}.$$

For an element $x \in A$, we write $x^{i,j}$ for its component in $A^{i,j}$. For $k, k' \geq 0$, we define an operator $F^{k,k'} : A \longrightarrow A$ by the rule

$$F^{k,k'}(x) := \sum_{l \geq k, l' \geq k'} x^{l,l'}.$$

We note that the operator $F^{k,k'}$ is the projection of A^* onto the subspace $F^k A^* \cap \overline{F}^{k'} A^*$. We will write $F^k = F^{k,-\infty}$.
We denote by $A^n_{\mathbb{R}}(p)$ the subgroup $(2\pi i)^p \cdot A^n_{\mathbb{R}} \subseteq A^n$, and we define the operator

$$\pi_p : A \longrightarrow A_{\mathbb{R}}(p)$$

by setting $\pi_p(x) := \frac{1}{2}(x + (-1)^p \bar{x})$.

The Deligne complex.

Definition 3.3. Let A be a Dolbeault algebra. Then, the *Deligne complex* $(\mathcal{D}^*(A,*), \mathrm{d}_{\mathcal{D}})$ *associated to* A is the graded complex given by

$$\mathcal{D}^n(A,p) = \begin{cases} A_{\mathbb{R}}^{n-1}(p-1) \cap F^{n-p,n-p}A^{n-1}, & \text{if} \quad n \leq 2p-1, \\ A_{\mathbb{R}}^{n}(p) \cap F^{p,p}A^{n}, & \text{if} \quad n \geq 2p, \end{cases}$$

with differential given by $(x \in \mathcal{D}^n(A,p))$

$$\mathrm{d}_{\mathcal{D}}\, x = \begin{cases} -F^{n-p+1,n-p+1}\, \mathrm{d}_A\, x, & \text{if} \quad n < 2p-1, \\ -2\partial\bar{\partial}x, & \text{if} \quad n = 2p-1, \\ \mathrm{d}_A\, x, & \text{if} \quad n \geq 2p. \end{cases}$$

The Deligne algebra.

Definition 3.4. Let A be a Dolbeault algebra. The *Deligne algebra associated to* A is the Deligne complex $\mathcal{D}^*(A,*)$ together with the graded commutative product $\bullet : \mathcal{D}^n(A,p) \times \mathcal{D}^m(A,q) \longrightarrow \mathcal{D}^{n+m}(A,p+q)$, given by

$$x \bullet y =$$
$$\begin{cases} (-1)^n r_p(x) \wedge y + x \wedge r_q(y), & \text{if } n < 2p,\ m < 2q, \\ F^{l-r,l-r}(x \wedge y), & \text{if } n < 2p,\ m \geq 2q,\ l < 2r, \\ F^{r,r}(r_p(x) \wedge y) + 2\pi_r(\partial(x \wedge y)^{r-1,l-r}), & \text{if } n < 2p,\ m \geq 2q,\ l \geq 2r, \\ x \wedge y, & \text{if } n \geq 2p,\ m \geq 2q, \end{cases}$$

where we have written $l = n+m$, $r = p+q$, and $r_p(x) = 2\pi_p(F^p\, \mathrm{d}_A\, x)$.

Specific degrees. In the sequel we will be interested in some specific degrees, where we can give simpler formulas. Namely, we consider

$$\begin{aligned} \mathcal{D}^{2p}(A,p) &= A_{\mathbb{R}}^{2p}(p) \cap A^{p,p}, \\ \mathcal{D}^{2p-1}(A,p) &= A_{\mathbb{R}}^{2p-2}(p-1) \cap A^{p-1,p-1}, \\ \mathcal{D}^{2p-2}(A,p) &= A_{\mathbb{R}}^{2p-3}(p-1) \cap (A^{p-2,p-1} \oplus A^{p-1,p-2}). \end{aligned}$$

The corresponding differentials are given by

$$\begin{aligned} \mathrm{d}_{\mathcal{D}}\, x &= \mathrm{d}_A\, x, && \text{if } x \in \mathcal{D}^{2p}(A,p), \\ \mathrm{d}_{\mathcal{D}}\, x &= -2\partial\bar{\partial}x, && \text{if } x \in \mathcal{D}^{2p-1}(A,p), \\ \mathrm{d}_{\mathcal{D}}(x,y) &= -\partial x - \bar{\partial}y, && \text{if } (x,y) \in \mathcal{D}^{2p-2}(A,p). \end{aligned}$$

Moreover, the product is given as follows: for $x \in \mathcal{D}^{2p}(A,p)$, $y \in \mathcal{D}^{2q}(A,q)$ or $y \in \mathcal{D}^{2q-1}(A,q)$, we have

$$x \bullet y = x \wedge y,$$

and for $x \in \mathcal{D}^{2p-1}(A,p)$, $y \in \mathcal{D}^{2q-1}(A,q)$, we have

$$x \bullet y = -\partial x \wedge y + \bar{\partial} x \wedge y + x \wedge \partial y - x \wedge \bar{\partial} y.$$

Deligne complexes and Deligne-Beilinson cohomology. The main interest in Deligne complexes is expressed by the following theorem which is proven in [8] in a particular case, although the proof is valid in general.

Theorem 3.5. *Let X be a complex algebraic manifold, $\overline{X}$ a smooth compactification of X with $D = \overline{X} \setminus X$ a normal crossing divisor, and denote by $j : X \longrightarrow \overline{X}$ the natural inclusion. Let $\mathscr{A}^*$ be a sheaf of Dolbeault algebras over $\overline{X}^{\mathrm{an}}$ such that, for every n, p the sheaves $\mathscr{A}^*$ and $F^p\mathscr{A}^*$ are acyclic, $\mathscr{A}^*_{\mathbb{R}}$ is a multiplicative resolution of $Rj_*\mathbb{R}$ and $(\mathscr{A}^*, F)$ is a multiplicative filtered resolution of $(\Omega^*_{\overline{X}}(\log D), F)$. Putting $A^* = \Gamma(\overline{X}, \mathscr{A}^*)$, we have a natural isomorphism of graded algebras*

$$H^*_{\mathcal{D}}(X, \mathbb{R}(p)) \cong H^*(\mathcal{D}(A,p)).$$

□

Notation. In the sequel we will use the following notation. The sheaves of differential forms will be denoted by the italic letter $\mathscr{E}$, and the corresponding spaces of global sections will be denoted by the same letter in roman typography E. For instance, we have

$$E^n_X\langle D_1\langle D_2\rangle\rangle = \Gamma(X, \mathscr{E}^n_X\langle D_1\langle D_2\rangle\rangle).$$

Logarithmic singularities at infinity Let X be a quasi-projective complex manifold. Let $E_{\log}(X)$ be the Dolbeault algebra of differential forms with logarithmic singularities at infinity (see [10], §5). Recall that in [10], $E_{\log}$ is defined as the Zariski sheaf associated to the pre-sheaf $E_{\log}{}^\circ$, which associates to any quasi-projective complex manifold X

$$E^*_{\log}(X)^\circ = \varinjlim E^*_{\overline{X}_\alpha}(\log D_\alpha),$$

where the limit is taken over all possible compactifications $\overline{X}_\alpha$ of X with $D_\alpha = \overline{X}_\alpha \setminus X$ a normal crossing divisor. Nevertheless, the step of taking the associated Zariski sheaf is not necessary by the following result. See [10], definition 3.1, for the definition of a totally acyclic sheaf.

Theorem 3.6. *For every pair of integers p, q, the pre-sheaf $E^{p,q\circ}_{\log}$ is a totally acyclic sheaf.*

Proof. Let U and V be two open subsets of X. We have to prove the exactness of the sequence

$$0 \longrightarrow E^{p,q}_{\log}(U \cup V)^\circ \xrightarrow{\phi} E^{p,q}_{\log}(U)^\circ \oplus E^{p,q}_{\log}(V)^\circ \xrightarrow{\psi} E^{p,q}_{\log}(U \cap V)^\circ \longrightarrow 0.$$

The injectivity of ϕ and the fact that $\psi \circ \phi = 0$ are obvious.
Put $Y = X \setminus U$ and $Z = X \setminus V$. Let $\pi_{Y\cap Z} : \widetilde{X}_{Y\cap Z} \longrightarrow X$ be an embedded resolution of singularities of $Y \cap Z$ such that the strict transform of Y, denoted by $\widehat{Y}$, and the strict transform of Z, denoted by $\widehat{Z}$, do not meet. Let $\{\sigma_{Y,Z}, \sigma_{Z,Y}\}$ be a partition of unity subordinate to the open cover $\{\widetilde{X} \setminus \widehat{Z}, \widetilde{X} \setminus \widehat{Y}\}$. If $\omega \in E^{p,q}_{\log}(U \cap V)^{\circ}$, then $\sigma_{Y,Z}\omega \in E^{p,q}_{\log}(U)^{\circ}$ and $\sigma_{Z,Y}\omega \in E^{p,q}_{\log}(V)^{\circ}$. Therefore, we get

$$\omega = \psi(-\sigma_{Y,Z}\omega, \sigma_{Z,Y}\omega),$$

which proves the surjectivity of ψ.
Let now $(\omega, \eta) \in E^{p,q}_{\log}(U)^{\circ} \oplus E^{p,q}_{\log}(V)^{\circ}$ be such that $\psi(\omega, \eta) = 0$. Then, ω and η agree on $U \cap V$. Therefore, they define a smooth form on $U \cup V$; by abuse of notation, we denote it by ω. The subtle point here is to know that, after some blow-ups with centers contained in Y, ω will have logarithmic singularities along the exceptional divisor, and the same is true after some blow-ups with centers contained in Z. We have to prove that ω has logarithmic singularities after blowing-up only centers contained in $Y \cap Z$.
To this end we need the following easy lemma, which follows from Hironaka's resolution of singularities.

Lemma 3.7. *Let X be a regular variety over a field of characteristic zero and let C_1 and C_2 be two closed subsets. Let $\pi : \widetilde{X} \longrightarrow X$ be a proper birational morphism, which is an isomorphism in the complement of $C_1 \cup C_2$. Then, there is a factorization*

$$\begin{tikzcd}
& \widetilde{X}_2 \arrow[dl, "\pi_2"'] \arrow[dr] & \\
\widetilde{X}_1 \arrow[dr, "\pi_1"'] & & \widetilde{X} \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

where $\widetilde{X}_1$ and $\widetilde{X}_2$ are regular varieties, π_1 and π_2 are proper birational morphisms, π_1 is an isomorphism over the complement of C_1 and π_2 is an isomorphism over the complement of the strict transform of C_2 in $\widetilde{X}_1$. Moreover, it is possible to choose the factorization in such a way that $\pi_1^{-1}(C_1)$ and $(\pi_2^{-1} \circ \pi_1^{-1})(C_1 \cup C_2)$ are normal crossing divisors. □

Let $\pi_{Y\cap Z} : \widetilde{X}_{Y\cap Z} \longrightarrow X$ be as before, and denote by $D_{Y\cap Z}$ the exceptional divisor. Since $\omega \in E^{p,q}_{\log}(U)^{\circ}$, there exists an embedded resolution of singularities $\widetilde{X}_Y$ of Y with exceptional divisor D_Y, which we can assume to factor through a proper birational morphism $\widetilde{X}_Y \longrightarrow \widetilde{X}_{Y\cap Z}$, and $\omega \in E^{p,q}_{\widetilde{X}_Y}(\log D_Y)$. We apply the previous lemma to the morphism $\widetilde{X}_Y \longrightarrow \widetilde{X}_{Y\cap Z}$ and the closed subsets

$D_{Y\cap Z}$ and $\widehat{Y}$. In this way we obtain a diagram

$$\begin{tikzcd} & \widetilde{X}'_Y \arrow[dl] \arrow[dr] & \\ \widetilde{X}'_{Y\cap Z} \arrow[dr, "\pi'_{Y\cap Z}"'] & & \widetilde{X}_Y \arrow[dl] \\ & \widetilde{X}_{Y\cap Z} & \end{tikzcd}$$

In $\widetilde{X}'_{Y\cap Z}$ we denote by $\widehat{Y}'$ and by $\widehat{Z}'$ the strict transforms of Y and Z, respectively, and by $D'_{Y\cap Z}$ the exceptional divisor. Now, since $\omega \in E^{p,q}_{\log}(V)^{\circ}$, we can repeat the process. There exists an embedded resolution of singularities $\widetilde{X}'_Z$ of Z in X with exceptional divisor D'_Z that factors through a proper birational morphism $\widetilde{X}'_Z \longrightarrow \widetilde{X}'_{Y\cap Z}$. Then, $\omega \in E^{p,q}_{\widetilde{X}'_Z}(\log D'_Z)$. We apply the previous lemma to this last morphism and the closed subsets $D'_{Y\cap Z}$ and $\widehat{Z}'$ to obtain the diagram

$$\begin{tikzcd} & \widetilde{X}''_Z \arrow[dl] \arrow[dr] & \\ \widetilde{X}''_{Y\cap Z} \arrow[dr] & & \widetilde{X}'_Z \arrow[dl] \\ & \widetilde{X}'_{Y\cap Z} & \end{tikzcd}$$

In $\widetilde{X}''_{Y\cap Z}$ we denote by $\widehat{Y}''$ and $\widehat{Z}''$ the strict transforms of Y and Z, respectively, and by $D''_{Y\cap Z}$ the exceptional divisor. To conclude the proof of the theorem, it is enough to show that

$$\omega \in E^{p,q}_{\widetilde{X}''_{Y\cap Z}}(\log D''_{Y\cap Z}).$$

This condition can be checked locally.
If $x \notin D''_{Y\cap Z}$, by hypothesis, ω_x is the germ of a smooth form.
Assume now that $x \in D''_{Y\cap Z} \setminus \widehat{Z}''$. We write D'_Z and D''_Z for the preimages of Z in $\widetilde{X}'_Z$ and $\widetilde{X}''_Z$, respectively. By construction, both are normal crossing divisors. By hypothesis, $\omega \in E^{p,q}_{\widetilde{X}'}(\log D'_Z)$. By the functoriality of logarithmic singularities, $\omega \in E^{p,q}_{\widetilde{X}''}(\log D''_Z)$. Let W be a neighborhood of x, whose intersection with $\widehat{Z}''$ is empty. Therefore, it is isomorphic to an open subset of $\widetilde{X}''_Z$, hence

$$\omega|_W \in \Gamma(W, \mathscr{E}^{p,q}_{\widetilde{X}''_{Y\cap Z}}(\log D''_{Y\cap Z})) = \Gamma(W, \mathscr{E}^{p,q}_{\widetilde{X}''_Z}(\log D''_Z)).$$

Finally, if $x \in D''_{Y\cap Z} \cap \widehat{Z}''$, we use a similar argument. $\square$

Remark 3.8. The argument of the previous theorem applies also to the complex $E_{\mathrm{pre}}(\underline{X})$ of [10], definition 7.16. Therefore, it that case, the morphism between the pre-sheaf and the associated sheaf is an isomorphism. Observe moreover that the same argument will apply to all the Zariski sheaves that we will introduce in this paper.

The Deligne complex with logarithmic singularities. We will denote

$$\mathcal{D}^*_{\log}(X,p) = \mathcal{D}^*(E_{\log}(X),p).$$

Then, theorem 3.5 implies that

$$H^*_{\mathcal{D}}(X,\mathbb{R}(p)) \cong H^*(\mathcal{D}_{\log}(X,p)).$$

3.2 The $\mathcal{D}_{\log}$-complex of log-log forms

$\mathcal{D}_{\log}$-complexes. Recall that, to define the arithmetic Chow groups of an arithmetic variety X as in [10], we need first an auxiliary complex of graded abelian sheaves on the Zariski site of smooth real schemes that satisfies Gillet axioms. As in [10], we will use the complex of sheaves $\mathcal{D}_{\log}$. This sheaf is given, for any smooth real scheme $U_{\mathbb{R}}$, by

$$\mathcal{D}_{\log}(U_{\mathbb{R}},p) = \mathcal{D}_{\log}(U_{\mathbb{C}},p)^{\sigma},$$

where σ is the involution that acts as complex conjugation on the space and on the coefficients (see [10], §5.3).
Then, we need to choose a $\mathcal{D}_{\log}$-complex over $X_{\mathbb{R}}$. That is, a complex $\mathcal{C}^*_{X_{\mathbb{R}}}(*)$ of graded abelian sheaves on the Zariski topology of $X_{\mathbb{R}}$ together with a morphism

$$\mathcal{D}_{\log,X_{\mathbb{R}}} \longrightarrow \mathcal{C}_{X_{\mathbb{R}}}$$

such that all the sheaves $\mathcal{C}^n_{X_{\mathbb{R}}}(p)$ are totally acyclic (see [10], definitions 3.1 and 3.4). The $\mathcal{D}_{\log}$-complex $\mathcal{C}$ plays the role of the fiber over the archimedean places of the arithmetic ring A. The aim of this section is to construct a $\mathcal{D}_{\log}$-complex by mixing log and log-log forms.

Varieties with a fixed normal crossing divisor. We will follow the notations of [10], §7.4, that we recall shortly. Let X be a complex algebraic manifold of dimension d, and D a normal crossing divisor. We will denote by $\underline{X}$ the pair (X,D). If $W \subseteq X$ is an open subset, we will write $\underline{W} = (W, D\cap W)$. In the sequel we will consider all operations adapted to the pair $\underline{X}$. For instance, if $Y \subsetneq X$ is a closed algebraic subset and $W = X \setminus Y$, then an embedded resolution of singularities of Y in $\underline{X}$ is a proper modification $\pi : \widetilde{X} \longrightarrow X$ such that $\pi\big|_{\pi^{-1}(W)} : \pi^{-1}(W) \longrightarrow W$ is an isomorphism, and

$$\pi^{-1}(Y),\ \pi^{-1}(D),\ \pi^{-1}(Y\cup D)$$

are normal crossing divisors on $\widetilde{X}$. Using Hironaka's theorem on the resolution of singularities [25], one can see that such an embedded resolution of singularities exists.
Analogously, a normal crossing compactification of $\underline{X}$ will be a smooth compactification $\overline{X}$ such that the adherence $\overline{D}$ of D, the subsets $B_{\overline{X}} = \overline{X} \setminus X$ and $B_{\overline{X}} \cup \overline{D}$ are normal crossing divisors.

Logarithmic growth along infinity. Given a diagram of normal crossing compactifications of $\underline{X}$

$$\begin{tikzcd} \overline{X}' \arrow[r, "\varphi"] & \overline{X} \\ & X \arrow[ul] \arrow[u] \end{tikzcd}$$

with divisors $B_{\overline{X}'}$ and $B_{\overline{X}}$ at infinity, respectively, proposition 2.40 gives rise to an induced morphism

$$\varphi^* : \mathscr{E}^*_{\overline{X}}\langle B_{\overline{X}}\langle \overline{D}\rangle\rangle \longrightarrow \mathscr{E}^*_{\overline{X}'}\langle B_{\overline{X}'}\langle \overline{D}'\rangle\rangle.$$

In order to have a complex that is independent of the choice of a particular compactification we take the limit over all possible compactifications.

Definition 3.9. Let $\underline{X} = (X, D)$ be as above. Then, we define the *complex* $E^*_{\mathrm{l,ll}}(\underline{X})$ *of differential forms on* X *log along infinity and log-log along* D as

$$E^*_{\mathrm{l,ll}}(\underline{X}) = \varinjlim \Gamma(\overline{X}, \mathscr{E}^*_{\overline{X}}\langle B_{\overline{X}}\langle \overline{D}\rangle\rangle),$$

where the limit is taken over all normal crossing compactifications $\overline{X}$ of $\underline{X}$.

A $\mathcal{D}_{\log}$-complex. Let X be a smooth real variety and D a normal crossing divisor defined over $\mathbb{R}$; as before, we write $\underline{X} = (X, D)$. For any $U \subseteq X$, the complex $E^*_{\mathrm{l,ll}}(\underline{U}_{\mathbb{C}})$ is a Dolbeault algebra with respect to the wedge product.

Definition 3.10. For any Zariski open subset $U \subseteq X$, we put

$$\mathcal{D}^*_{\mathrm{l,ll},\underline{X}}(U, p) = (\mathcal{D}^*_{\mathrm{l,ll},\underline{X}}(U, p), \mathrm{d}_{\mathcal{D}}) = (\mathcal{D}^*(E_{\mathrm{l,ll}}(\underline{U}_{\mathbb{C}}), p)^{\sigma}, \mathrm{d}_{\mathcal{D}}),$$

where the operator $\mathcal{D}$ is as in definition 3.4 and σ is the involution that acts as complex conjugation in the space and in the coefficients (see [10], 5.55). When the pair $\underline{X}$ is understood, we write $\mathcal{D}^*_{\mathrm{l,ll}}$ instead of $\mathcal{D}^*_{\mathrm{l,ll},\underline{X}}$. The complex $\mathcal{D}^*_{\mathrm{l,ll}}$ will be called *the* $\mathcal{D}_{\log}$*-complex of log-log forms* or just *the complex of log-log forms.*

Then, the analogue of [10], theorem 7.18, holds.

Theorem 3.11. *The complex* $\mathcal{D}_{\mathrm{l,ll},\underline{X}}$ *is a* $\mathcal{D}_{\log}$*-complex on* X*. Moreover, it is a pseudo-associative and commutative* $\mathcal{D}_{\log}$*-algebra.* □

The cohomology of the complex $\mathcal{D}_{\mathrm{l,ll},\underline{X}}$. The main advantage of the complex $\mathcal{D}_{\mathrm{l,ll},\underline{X}}$ over the complex $\mathcal{D}_{\mathrm{pre},\underline{X}}$ of [10] is the following result that is a consequence of theorem 2.42 and theorem 3.5 (see [10], theorem 5.19, and [8]).

Theorem 3.12. *The inclusion $\mathcal{D}_{\log,X} \longrightarrow \mathcal{D}_{\mathrm{l,ll},\underline{X}}$ is a quasi-isomorphism. Therefore, the hypercohomology over X of the complex of sheaves $\mathcal{D}_{\mathrm{l,ll},\underline{X}}$, as well as the cohomology of its complex of global sections, is naturally isomorphic to the Deligne-Beilinson cohomology of X.* □

3.3 Properties of Green objects with values in $\mathcal{D}_{\mathrm{l,ll}}$.

We start by noting that theorem 3.11 together with [10], section 3, provides us with a theory of Green objects with values in $\mathcal{D}_{\mathrm{l,ll},\underline{X}}$.

Mixed forms representing the class of a cycle. Since we know the cohomology of the complex of mixed forms, we obtain the analogue of proposition 5.48 in [10], which is more precise than the analogue of proposition 7.20 in [10]. In particular, we have

Proposition 3.13. *Let X be a smooth real variety and D a normal crossing divisor. Put $\underline{X}=(X,D)$. Let y be a p-codimensional cycle on X with support Y. Then, we have that the class of the cycle (ω, g) in $H^{2p}_{\mathcal{D}_{\mathrm{l,ll}},Y}(X,p)$ is equal to the class of y, if and only if*

$$-2\partial\bar{\partial}[g]_X = [\omega] - \delta_y. \tag{3.14}$$

Proof. The proof is completely analogous to the proof of [10], 5.48, using theorem 3.12 and lemma 2.41. □

Inverse images.

Proposition 3.15. *Let $f : X \longrightarrow Y$ be a morphism of smooth real varieties, let D_X, D_Y be normal crossing divisors on X, Y respectively, satisfying $f^{-1}(D_Y) \subseteq D_X$. Put $\underline{X}=(X,D_X)$ and $\underline{Y}=(Y,D_Y)$. Then, there exists a contravariant f-morphism*

$$f^{\#} : \mathcal{D}_{\mathrm{l,ll},\underline{Y}} \longrightarrow f_*\mathcal{D}_{\mathrm{l,ll},\underline{X}}.$$

Proof. By proposition 2.40, the pull-back of differential forms induces a morphism of the corresponding Dolbeault algebras of mixed forms. This morphism is compatible with the involution σ. Thus, this morphism gives rise to an induced morphism between the corresponding Deligne algebras. □

Push-forward. We will only state the most basic property concerning direct images, which is necessary to define arithmetic degrees. Note however that we expect that the complex of log-log forms will be useful in the study of non smooth, proper, surjective morphisms. By proposition 2.26, we have

PROPOSITION 3.16. *Let $\underline{X} = (X, D)$ be a proper, smooth real variety with fixed normal crossing divisor D. Let $f : X \longrightarrow \operatorname{Spec}(\mathbb{R})$ denote the structural morphism. Then, there exists a covariant f-morphism*

$$f_{\#} : f_* \mathcal{D}_{\mathrm{l,ll},\underline{X}} \longrightarrow \mathcal{D}_{\log,\operatorname{Spec}(\mathbb{R})}.$$

In particular, if X has dimension d, we obtain a well defined morphism

$$f_{\#} : \widehat{H}^{2d+2}_{\mathcal{D}_{\mathrm{l,ll}},\mathcal{Z}^{d+1}}(X, d) \longrightarrow \widehat{H}^{2}_{\mathcal{D}_{\log},\mathcal{Z}^{1}}(\operatorname{Spec}(\mathbb{R}), 1) = \mathbb{R}.$$

Note that, by dimension reasons, we have $\mathcal{Z}^{d+1} = \emptyset$, and

$$\widehat{H}^{2d+2}_{\mathcal{D}_{\mathrm{l,ll}},\mathcal{Z}^{d+1}}(X, d) = H^{2d+1}(\mathcal{D}_{\mathrm{l,ll}}(X, d+1)) = H^{2d+1}_{\mathcal{D}}(X, \mathbb{R}(d+1).$$

Thus, every element of $\widehat{H}^{2d+2}_{\mathcal{D}_{\mathrm{l,ll}},\mathcal{Z}^{d+1}}(X, d)$ is represented by a pair $\mathfrak{g} = (0, \widetilde{g})$. The morphism $f_{\#}$ mentioned above, is then given by

$$\mathfrak{g} = (0, \widetilde{g}) \longmapsto \left(0, \frac{1}{(2\pi i)^d} \int_X g\right).$$

3.4 ARITHMETIC CHOW RINGS WITH LOG-LOG FORMS

ARITHMETIC CHOW GROUPS. We are now in position to apply the machinery of [10]. Let (A, Σ, F_∞) be an arithmetic ring and let X be a regular arithmetic variety over A. Let D be a fixed normal crossing divisor of X_Σ stable under F_∞. As in the previous section, we will denote by $\underline{X}$ the pair $(X_{\mathbb{R}}, D)$. The natural inclusion $\mathcal{D}_{\log} \longrightarrow \mathcal{D}_{\mathrm{l,ll}}$ induces a $\mathcal{D}_{\log}$-complex structure in $\mathcal{D}_{\mathrm{l,ll}}$. Then, $(X, \mathcal{D}_{\mathrm{l,ll}})$ is a $\mathcal{D}_{\log}$-arithmetic variety. Therefore, applying the theory of [10], section 4, we define the arithmetic Chow groups $\widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\mathrm{l,ll}})$. These groups will be called log-log arithmetic Chow groups.

EXACT SEQUENCES. We start the study of these arithmetic Chow groups by writing the exact sequences of [10], theorem 4.13. Observe that, since we have better control on the cohomology of $\mathcal{D}_{\mathrm{l,ll}}$, we obtain better results than in [10], §7.

THEOREM 3.17. *The following sequences are exact:*

$$\mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} \widetilde{\mathcal{D}}^{2p-1}_{\mathrm{l,ll}}(X, p) \xrightarrow{\mathrm{a}} \widehat{\mathrm{CH}}^p(X, \mathcal{D}_{\mathrm{l,ll}}) \xrightarrow{\zeta} \mathrm{CH}^p(X) \longrightarrow 0,$$

$$\begin{aligned} \mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) \xrightarrow{\mathrm{a}} \widehat{\mathrm{CH}}^p(X, \mathcal{D}_{\mathrm{l,ll}}) &\xrightarrow{(\zeta,-\omega)} \\ \mathrm{CH}^p(X) \oplus \mathrm{Z}\mathcal{D}^{2p}_{\mathrm{l,ll}}(X, p) &\xrightarrow{\mathrm{cl}+h} H^{2p}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) \longrightarrow 0, \end{aligned}$$

$$\mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) \xrightarrow{\mathrm{a}} \widehat{\mathrm{CH}}^p(X, \mathcal{D}_{\mathrm{l,ll}})_0 \xrightarrow{\zeta} \mathrm{CH}^p(X)_0 \longrightarrow 0.$$

□

Multiplicative properties. Since $\mathcal{D}_{\mathrm{l,ll}}$ is a pseudo-associative and commutative $\mathcal{D}_{\log}$-algebra, we have

Theorem 3.18. *The abelian group*

$$\widehat{\mathrm{CH}}^{*}(X,\mathcal{D}_{\mathrm{l,ll}})_{\mathbb{Q}}=\bigoplus_{p\geq 0}\widehat{\mathrm{CH}}^{p}(X,\mathcal{D}_{\mathrm{l,ll}})\otimes\mathbb{Q}$$

is an associative and commutative $\mathbb{Q}$*-algebra with a unit.* □

Inverse images. By proposition 2.40, there are some cases, where we can define the inverse image for the log-log arithmetic Chow groups.

Theorem 3.19. *Let* $f:X\longrightarrow Y$ *be a morphism of arithmetic varieties over* A*. Let* E *be a normal crossing divisor on* $Y_{\mathbb{R}}$ *and* D *a normal crossing divisor on* $X_{\mathbb{R}}$ *such that* $f^{-1}(E)\subseteq D$*. Write* $\underline{X}=(X_{\mathbb{R}},D)$ *and* $\underline{Y}=(Y_{\mathbb{R}},E)$*. Then, there is defined an inverse image morphism*

$$f^{*}:\widehat{\mathrm{CH}}^{*}(Y,\mathcal{D}_{\mathrm{l,ll}})\longrightarrow\widehat{\mathrm{CH}}^{*}(X,\mathcal{D}_{\mathrm{l,ll}}).$$

Moreover, it is a morphism of rings after tensoring with $\mathbb{Q}$*.* □

Push-forward. We will state only the consequence of the integrability of log-log forms.

Theorem 3.20. *If* X *is projective over* A*, then there is a direct image morphism of groups*

$$f_{*}:\widehat{\mathrm{CH}}^{d+1}(X,\mathcal{D}_{\mathrm{l,ll}})\longrightarrow\widehat{\mathrm{CH}}^{1}(\operatorname{Spec}A),$$

where d *is the relative dimension of* X*.* □

Relationship with other arithmetic Chow groups. Since we know the cohomology of the complex $\mathcal{D}_{\mathrm{l,ll}}$, we can make a comparison statement more precise than in [10], theorem 6.23.

Theorem 3.21. *The structural morphism*

$$\mathcal{D}_{\log,X}\longrightarrow\mathcal{D}_{\mathrm{l,ll},\underline{X}}$$

induces a morphism

$$\widehat{\mathrm{CH}}^{*}(X,\mathcal{D}_{\log})\longrightarrow\widehat{\mathrm{CH}}^{*}(X,\mathcal{D}_{\mathrm{l,ll}})$$

that is compatible with inverse images, intersection products and arithmetic degrees. If X *is projective, the isomorphism between* $\widehat{\mathrm{CH}}^{*}(X,\mathcal{D}_{\log})$ *and the* *arithmetic Chow groups defined by Gillet and Soulé (denoted by* $\widehat{\mathrm{CH}}^*(X)$*) induce morphisms*

$$\widehat{\mathrm{CH}}^*(X) \longrightarrow \widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\mathrm{l,ll}}) \tag{3.22}$$

also compatible with inverse images, intersection products and arithmetic degrees. Moreover, if D *is empty and* X *is projective, then the above morphisms are isomorphisms.* □

3.5 The $\mathcal{D}_{\log}$-complex of log-log forms with arbitrary singularities at infinity

The arithmetic Chow groups defined by Gillet and Soulé for quasi-projective varieties use differential forms with arbitrary singularities in the boundary. Therefore, in order to be able to recover the arithmetic Chow groups of Gillet and Soulé, we have to introduce another variant of arithmetic Chow groups, where we allow the differential forms to have arbitrary singularities in certain directions.

Mixing log, log-log and arbitrary singularities. Let X be a complex algebraic manifold and D a fixed normal crossing divisor of X. We write $\underline{X} = (X, D)$.

Definition 3.23. For every Zariski open subset U of X, we write

$$E^*_{\mathrm{l,ll,a},\underline{X}}(U) = \varinjlim_{\overline{U}} \Gamma(\overline{U}, \mathscr{E}^*_{\overline{U}}\langle B_{\overline{U}}\langle \overline{D}\rangle\rangle),$$

where the limit is taken over all diagrams

$$\begin{tikzcd} U \arrow[r, "\bar{\iota}"] \arrow[dr, "\iota"'] & \overline{U} \arrow[d, "\beta"] \\ & X \end{tikzcd}$$

such that $\bar{\iota}$ is an open immersion, β is a proper morphism and $B_{\overline{U}} = \overline{U} \setminus U$, $\overline{D} = \beta^{-1}(D)$, $B_{\overline{U}} \cup \overline{D}$ are normal crossing divisors.

Definition 3.24. Let X be a complex algebraic manifold and D a fixed normal crossing divisor of X. We write $\underline{X} = (X, D)$ as before. For any Zariski open subset $U \subseteq X$, we put

$$\mathcal{D}^*_{\mathrm{l,ll,a},\underline{X}}(U,p) = (\mathcal{D}^*_{\mathrm{l,ll,a},\underline{X}}(U,p), \mathrm{d}_{\mathcal{D}}) = (\mathcal{D}^*(E_{\mathrm{l,ll,a},\underline{X}}(U_{\mathbb{C}}), p), \mathrm{d}_{\mathcal{D}}).$$

If X is a smooth algebraic variety over $\mathbb{R}$, and D, U are defined over $\mathbb{R}$, we put

$$\mathcal{D}^*_{\mathrm{l,ll,a},\underline{X}}(U,p) = (\mathcal{D}^*_{\mathrm{l,ll,a},\underline{X}}(U,p), \mathrm{d}_{\mathcal{D}}) = (\mathcal{D}^*(E_{\mathrm{l,ll,a},\underline{X}}(U_{\mathbb{C}}), p)^{\sigma}, \mathrm{d}_{\mathcal{D}}),$$

where σ is as in section 3.2.

Note that, when X is quasi-projective, the varieties $\overline{U}$ of definition 3.23 are not compactifications of U, but only *partial compactifications*. Therefore, the sections of $\mathcal{D}^*_{\mathrm{l,ll,a},\underline{X}}(U,p)$ have three different kinds of singularities. We can see this more concretely as follows. Let Y be a closed subset of X with $U = X\setminus Y$, and let $\overline{X}$ be a smooth compactification of X with $Z = \overline{X}\setminus X$. Let η be a section of $\mathcal{D}^*_{\mathrm{l,ll,a},\underline{X}}(U,p)$. If we consider η as a singular form on $\overline{X}$, then η is log along Y (in the sense that it is log along a certain resolution of singularities of Y), log-log along D and has arbitrary singularities along Z. Therefore, in general, we have

$$\mathcal{D}^*_{\mathrm{l,ll,a},\underline{X}}(U,p) \neq \mathcal{D}^*_{\mathrm{l,ll,a},\underline{U}}(U,p).$$

Nevertheless, when $\underline{X}$ is clear from the context, we will drop it from the notation.

Remark 3.25. If X is projective, the complexes of sheaves $\mathcal{D}^*_{\mathrm{l,ll,a},\underline{X}}$ and $\mathcal{D}^*_{\mathrm{l,ll},\underline{X}}$ agree. In contrast, they do not agree, when X is quasi-projective. Note, moreover, that, when X is quasi-projective, the complex $\mathcal{D}^*_{\mathrm{l,ll,a},\underline{X}}$ does not compute the Deligne-Beilinson cohomology of X, but a mixture between Deligne-Beilinson cohomology and analytic Deligne cohomology. Nevertheless, as we will see, the local nature of the purity property of Deligne-Beilinson cohomology implies also a purity property for these complexes.

Logarithmic singularities and Blow-ups. Let X be a complex manifold, $D \subseteq X$ a normal crossing divisor, and $Y \subseteq X$ an e-codimensional smooth subvariety such that the pair (D,Y) has normal crossings. Let $\pi : \widetilde{X} \longrightarrow X$ be the blow-up of X along Y. Write $\widetilde{D} = \pi^{-1}(D)$ and $\widetilde{Y} = \pi^{-1}(Y)$. Let $i : Y \longrightarrow X$ and $j : \widetilde{Y} \longrightarrow \widetilde{X}$ denote the inclusions, and let $g : \widetilde{Y} \longrightarrow Y$ denote the induced morphism. Observe that g is a projective bundle.

Proposition 3.26. *Let $p \geq 0$ be an integer. Then, we have:*

(i) If $Y \subseteq D$, then the morphism $\Omega^p_X(\log D) \longrightarrow R\pi_*\Omega^p_{\widetilde{X}}(\log \widetilde{D})$ is a quasi-isomorphism, i.e.,

$$\begin{aligned} \pi_*\Omega^p_{\widetilde{X}}(\log \widetilde{D}) &\cong \Omega^p_X(\log D), \quad \textit{and} \\ R^q\pi_*\Omega^p_{\widetilde{X}}(\log \widetilde{D}) &= 0, \qquad\qquad \textit{for } q > 0. \end{aligned}$$

(ii) If $Y \not\subseteq D$ and $e > 1$, then

$$\begin{aligned} \pi_*\Omega^p_{\widetilde{X}}(\log \widetilde{D}\cup\widetilde{Y}) &\cong \Omega^p_X(\log D), \\ R^q\pi_*\Omega^p_{\widetilde{X}}(\log \widetilde{D}\cup\widetilde{Y}) &= 0, \qquad \textit{for } q \neq 0, e-1, \textit{ and} \\ R^{e-1}\pi_*\Omega^p_{\widetilde{X}}(\log \widetilde{D}\cup\widetilde{Y}) &\cong i_*(R^{e-1}g_*\Omega^{p-1}_{\widetilde{Y}}(\log \widetilde{D}\cap\widetilde{Y})) \\ &\cong i_*(\Omega^{p-e}_Y(\log D\cap Y)\otimes R^{e-1}g_*\Omega^{e-1}_{\widetilde{Y}/Y}). \end{aligned}$$

(iii) If $Y \not\subseteq D$ and $e = 1$, then $\pi = \mathrm{id}$, and there is a short exact sequence

$$0 \longrightarrow \Omega^p_X \longrightarrow \Omega^p_X(\log Y) \longrightarrow i_*\Omega^{p-1}_Y \longrightarrow 0.$$

Proof. The third statement is standard; the first statement is [19], Proposition 4.4 (ii).
Using [19], Proposition 4.4 (i), the fact that i_* is an exact functor and that g is a projective bundle, we obtain

$$\begin{aligned}
\pi_*\Omega^p_{\widetilde{X}}(\log \widetilde{D}) &\cong \Omega^p_X(\log D),\\
\pi_* j_*\Omega^{p-1}_{\widetilde{Y}}(\log \widetilde{D}\cap \widetilde{Y}) &\cong i_* g_*\Omega^{p-1}_{\widetilde{Y}}(\log \widetilde{D}\cap \widetilde{Y})\\
&\cong i_*\Omega^{p-1}_Y(\log D\cap Y),\\
R^q\pi_*\Omega^p_{\widetilde{X}}(\log \widetilde{D}) &\cong R^q(\pi\circ j)_*\Omega^p_{\widetilde{Y}}(\log \widetilde{D}\cap \widetilde{Y})\\
&\cong i_* R^q g_*\Omega^p_{\widetilde{Y}}(\log \widetilde{D}\cap \widetilde{Y})\\
&\cong \begin{cases} i_*(\Omega^{p-q}_Y(\log D\cap Y)\otimes R^q g_*\Omega^q_{\widetilde{Y}/Y}), & \text{if } 1\le q<e,\\ 0, & \text{if } g\ge e.\end{cases}
\end{aligned}$$

Let $\mathscr{O}(1)$ be the ideal sheaf of $\widetilde{Y}$ in $\widetilde{X}$. We consider the exact sequence

$$0 \longrightarrow \Omega^p_{\widetilde{X}}(\log \widetilde{D}) \longrightarrow \Omega^p_{\widetilde{X}}(\log \widetilde{D}\cup \widetilde{Y}) \xrightarrow{\mathrm{Res}} j_*\Omega^{p-1}_{\widetilde{Y}}(\log \widetilde{D}\cap \widetilde{Y}) \longrightarrow 0$$

and the corresponding long exact sequence obtained by applying the functor $R\pi_*$. The connecting morphism of this long exact sequence

$$\begin{aligned}
R^{q-1}\pi_* j_*\Omega^{p-1}_{\widetilde{Y}}(\log \widetilde{D}\cap \widetilde{Y}) &\cong i_*(\Omega^{p-q}_Y(\log D\cap Y)\otimes R^{q-1}g_*\Omega^{q-1}_{\widetilde{Y}/Y})\\
&\longrightarrow R^q\pi_*\Omega^p_{\widetilde{X}}(\log \widetilde{D}) \cong i_*(\Omega^{p-q}_Y(\log D\cap Y)\otimes R^q g_*\Omega^q_{\widetilde{Y}/Y})
\end{aligned}$$

can be identified with the product by $\mathrm{c}_1(\mathscr{O}_{\widetilde{X}}(1))$, which is an isomorphism for $0<q\le e-1$. The result now follows from this exact sequence. $\square$

This proposition has the following consequence.

COROLLARY 3.27. *Let X be a complex algebraic manifold and Y a complex subvariety of codimension e. Let $\widetilde{X} \longrightarrow X$ be an embedded resolution of singularities of Y obtained as in [25]. Then, we have*

$$R^q\pi_*\Omega^p_{\widetilde{X}}(\log D) \cong \begin{cases} \Omega^p_X, & \textit{if } q=0,\\ 0, & \textit{if } p<e \textit{ or } 0<q<e-1.\end{cases}$$

Proof. According to [25], $\widetilde{X}$ is obtained by a series of elementary steps

$$\widetilde{X} = \widetilde{X}_N \longrightarrow \widetilde{X}_{N-1} \longrightarrow \dots \longrightarrow \widetilde{X}_0 = X,$$

where $\widetilde{X}_k$ is the blow-up of $\widetilde{X}_{k-1}$ along a smooth subvariety W_{k-1}, contained in the strict transform of Y, therefore of codimension greater or equal than e. Moreover, if D_k is the union of exceptional divisors up to the step k, then the pair (D_k, W_k) has normal crossings. The result follows by applying proposition 3.26 to each blow-up. □

The following theorem implies in particular the weak purity condition for the complex $\mathcal{D}_{\mathrm{l,ll,a},\underline{X}}$.

Theorem 3.28. *Let $\underline{X} = (X, D)$ be as above. Let $Y \subseteq X$ be a Zariski closed subset of codimension greater or equal than p. Let c be the number of connected components of Y of codimension p. Then, the natural morphisms*

$$H^n_{\mathcal{D}_{\mathrm{l,ll,a}},Y}(X,p) \longrightarrow H^n_Y(X,\mathbb{R}(p))$$

are isomorphisms for all integers n. Therefore, we have

$$\begin{aligned} &H^n_{\mathcal{D}_{\mathrm{l,ll,a}},Y}(X,p) = 0, \qquad \text{for } n < 2p\,, \\ &H^{2p}_{\mathcal{D}_{\mathrm{l,ll,a}},Y}(X,p) \cong \mathbb{R}(p)^c. \end{aligned}$$

Proof. We fix a diagram

$$\begin{tikzcd} U \arrow[r, "\bar{\iota}"] \arrow[dr, "\iota"'] & \overline{U} \arrow[d, "\beta"] \\ & X \end{tikzcd}$$

such that $\bar{\iota}$ is an open immersion, β is a proper morphism, and $B = \overline{U} \setminus U$, $\overline{D} = \beta^{-1}(D)$, $B \cup \overline{D}$ are normal crossing divisors. Hence, $\overline{U}$ is an embedded resolution of singularities of Y. We assume moreover that $\overline{U}$ is obtained from X as $\widetilde{X}$ is obtained from X in corollary 3.27.
By theorem 2.42, the complexes $\mathcal{D}^*_{\mathrm{l,ll,a},\underline{X}}(X,p)$ and $\mathcal{D}^*_{\mathrm{l,ll,a},\underline{X}}(U,p)$ are quasi-isomorphic to the complexes $\mathcal{D}^*(E^*_X, p)$ and $\mathcal{D}^*(E^*_{\overline{U}}\langle B\rangle, p)$, respectively.
By the definition of the Deligne complex and theorem 2.6.2 in [8], there are quasi-isomorphisms

$$\begin{aligned} \mathcal{D}^*(E^*_X, p) &\longrightarrow s\left(E^*_{X,\mathbb{R}}(p) \to E^*_X \,/F^p E^*_X\,\right), \\ \mathcal{D}^*(E^*_{\overline{U}}\langle B\rangle, p) &\longrightarrow s\left(E^*_{\overline{U}}\langle B\rangle_{\mathbb{R}}(p) \to E^*_{\overline{U}}\langle B\rangle \,/F^p E^*_{\overline{U}}\langle B\rangle\,\right). \end{aligned}$$

By corollary 3.27 and theorem 2.5, the natural morphism

$$E^*_X \,/F^p E^*_X \longrightarrow E^*_{\overline{U}}\langle B\rangle \,/F^p E^*_{\overline{U}}\langle B\rangle$$

is a quasi-isomorphism. Hence, the morphism

$$s\left(\mathcal{D}^*(E^*_X, p) \to \mathcal{D}^*(E^*_{\overline{U}}\langle B\rangle, p)\right) \longrightarrow s\left(E^*_{X,\mathbb{R}}(p) \to E^*_{\overline{U}}\langle B\rangle_{\mathbb{R}}(p)\right)$$

is a quasi-isomorphism. Since the left hand complex computes $H^n_{\mathcal{D}_{\mathrm{l,ll,a}},Y}(X,p)$ and the right hand complex computes $H^n_Y(X,\mathbb{R}(p))$, we obtain the first statement of the theorem. The second statement follows form the purity of singular cohomology. □

Summing up the properties of the complex $\mathcal{D}_{\mathrm{l,ll,a},\underline{X}}$, we obtain

Theorem 3.29. *The complex $\mathcal{D}_{\mathrm{l,ll,a},\underline{X}}$ is a $\mathcal{D}_{\log}$-complex on X. Moreover, it is a pseudo-associative and commutative $\mathcal{D}_{\log}$-algebra and satisfies the weak purity condition (see [10], definition 3.1).* □

3.6 Arithmetic Chow rings with arbitrary singularities at infinity

Let A, X, D, and $\underline{X}$ be as at the beginning of section 3.4. Applying [10], section 4, we define the arithmetic Chow groups $\widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\mathrm{l,ll,a}})$. Then, theorems 3.18, 3.19, and 3.21 are also true for these groups. For theorem 3.20 to be true, we need X to be projective, but in this case there is no difference between $\widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\mathrm{l,ll,a}})$ and $\widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\mathrm{l,ll}})$.
Since $\mathcal{D}_{\mathrm{l,ll,a},\underline{X}}$ satisfies the weak purity property, the analogue of theorem 3.17 reads as follows.

Theorem 3.30. *The following sequence is exact:*

$$\mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} \widetilde{\mathcal{D}}^{2p-1}_{\mathrm{l,ll,a}}(X,p) \xrightarrow{\mathrm{a}} \widehat{\mathrm{CH}}^p(X, \mathcal{D}_{\mathrm{l,ll,a}}) \xrightarrow{\zeta} \mathrm{CH}^p(X) \longrightarrow 0.$$

□

Another consequence of theorem 3.28 is the analogue of proposition 3.13, which is proved in the same way.

Proposition 3.31. *Let X be a smooth real variety and D a normal crossing divisor. Put $\underline{X} = (X, D)$. Let y be a p-codimensional cycle on X with support Y. Then, the class of the cycle (ω, g) in $H^{2p}_{\mathcal{D}_{\mathrm{l,ll,a}},Y}(X,p)$ is equal to the class of y, if and only if*

$$-2\partial\bar{\partial}[g]_X = [\omega] - \delta_y. \tag{3.32}$$

□

From this proposition and theorem 3.30, we obtain the analogue of theorem 6.23 in [10]:

Theorem 3.33. *Let $\widehat{\mathrm{CH}}^p(X)$ be the arithmetic Chow groups defined by Gillet and Soulé. If $D = \emptyset$, the assignment*

$$[y, (\omega_y, \widetilde{g}_y)] \mapsto [y, 2(2\pi i)^{d-p+1}[g_y]_X]$$

induces a well defined isomorphism

$$\Psi : \widehat{\mathrm{CH}}^p(X, \mathcal{D}_{\mathrm{l,ll,a}}) \longrightarrow \widehat{\mathrm{CH}}^p(X),$$

which is compatible with products and pull-backs.

Remark 3.34. Note that, if $f : X \longrightarrow Y$ is a proper morphism between arithmetic varieties over A and such that $f_{\mathbb{R}} : X_{\mathbb{R}} \longrightarrow Y_{\mathbb{R}}$ is smooth, then there is a covariant f-pseudo morphism (see [10], definition 3.71) that induces a push-forward morphism

$$f_* : \widehat{\mathrm{CH}}^p(X, \mathcal{D}_{\mathrm{l,ll,a}}) \longrightarrow \widehat{\mathrm{CH}}^p(Y, \mathcal{D}_{\mathrm{l,ll,a}}).$$

This push-forward is compatible with the push-forward defined by Gillet and Soulé.

Remark 3.35. We can define $\mathcal{D}_{\mathrm{l,ll,a,pre},\underline{X}}$ in the same way as $\mathcal{D}_{\mathrm{l,ll,a},\underline{X}}$ by replacing pre-log and pre-log-log forms for log and log-log forms. We then obtain a theory of arithmetic Chow groups $\widehat{\mathrm{CH}}^p(X, \mathcal{D}_{\mathrm{l,ll,a,pre}})$ with analogous properties. Note however that since we have not established the weak purity property of pre-log forms, we do not have the analogue of theorem 3.33.

4 Bott-Chern forms for log-singular hermitian vector bundles

The arithmetic intersection theory of Gillet and Soulé is complemented by an arithmetic K-theory and a theory of characteristic classes. A main ingredient of the theory of arithmetic characteristic classes are the Chern forms and Bott-Chern forms for hermitian vector bundles. In this section, after defining the class of singular metrics considered in this paper, we will generalize the theory of Chern forms and Bott-Chern forms to include this class of singular metrics.

4.1 Chern forms for hermitian metrics

Here we recall the Chern-Weil theory of characteristic classes for hermitian vector bundles. By a hermitian metric we will always mean a smooth hermitian metric.

Chern forms. Let $B \subseteq \mathbb{R}$ be a subring, let $\phi \in B[[T_1, \dots, T_n]]$ be any symmetric power series in n variables, and let $M_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices. For every $k \geq 0$, let $\phi^{(k)}$ be the homogeneous component of ϕ of degree k. We will denote also by $\phi^{(k)} : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$ the unique polynomial map which is invariant under conjugation by $\mathrm{GL}_n(\mathbb{C})$ and whose value in the diagonal matrix $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$, $\lambda_i \in \mathbb{C}$, is $\phi^{(k)}(\lambda_1, \dots, \lambda_n)$. More generally, if A is any B-algebra, $\phi^{(k)}$ defines a map $\phi^{(k)} : M_n(A) \longrightarrow A$, and if $I \subseteq A$ is a nilpotent subalgebra, we can define $\phi = \sum_k \phi^{(k)} : M_n(I) \longrightarrow A$.
Let $\overline{E} = (E, h)$ be a hermitian vector bundle of rank n on a complex manifold X. Let $\xi = \{\xi_1, \dots, \xi_n\}$ be a frame for E in an open subset $V \subseteq X$. We denote by $h(\xi) = (h_{ij}(\xi))$ the matrix of h in the frame ξ. Let $K(\xi)$ be the curvature matrix $K(\xi) = \bar{\partial}(\partial h(\xi) \cdot h(\xi)^{-1})$. The Chern form associated to ϕ and $\overline{E}$ is the form

$$\phi(\overline{E}) = \phi(-K(\xi)) \in E^*_V.$$

Basic properties. The following properties of the Chern forms are well known.

Theorem 4.1. *(i) By the invariance of the $\phi^{(k)}$, the Chern form $\phi(\overline{E})$ is independent of the choice of the frame ξ. Therefore, it globalizes to a differential form $\phi(\overline{E}) \in E_X^*$.*

(ii) The Chern forms are closed.

(iii) The component $\phi^{(k)}$ belongs to $\mathcal{D}^{2k}(E_X, k) = E_X^{k,k} \cap E_{X,\mathbb{R}}^{2k}(k)$.

(iv) If $X_{\mathbb{R}} = (X, F_\infty)$ is a real manifold, the vector bundle E is defined over $\mathbb{R}$, and the hermitian metric h is invariant under F_∞, then $\phi^{(k)}(E,h) \in \mathcal{D}^{2k}(E_X, k)^\sigma$, where σ is as in definition 3.10.

□

Chern classes. Since the Chern forms are closed, they represent cohomology classes $\phi(E) = [\phi(E,h)] \in \bigoplus_k H^{2k}(\mathcal{D}(E_X,k))$. If X is projective, then $\bigoplus_k H^{2k}(\mathcal{D}(E_X,k)) = \bigoplus_k H^{2k}_{\mathcal{D}}(X, \mathbb{R}(k))$, hence we obtain classes in Deligne-Beilinson cohomology

$$\phi(E) \in \bigoplus_k H^{2k}_{\mathcal{D}}(X, \mathbb{R}(k)).$$

Note that, to simplify notations, the function ϕ will have different meanings according to its arguments. For instance, $\phi(E,h) = \phi(\overline{E})$ will mean the Chern form that depends on the bundle and the metric, whereas $\phi(E)$ will mean the Chern class that depends only on the bundle.

When X is quasi-projective, by means of smooth at infinity hermitian metrics, the Chern-Weil theory also allows us to construct Chern classes in Deligne-Beilinson cohomology.

Let E be an algebraic vector bundle on the quasi-projective complex manifold X. By [11], proposition 2.2, there exists a compactification $\widetilde{X}$ of X and a vector bundle $\widetilde{E}$ on $\widetilde{X}$ such that $\widetilde{E}|_X = E$. Let $\widetilde{h}$ be a smooth hermitian metric on $\widetilde{E}$ and let h be the induced metric on E. The hermitian metric h is said to be smooth at infinity.

With these notation, we write

$$\phi(E,h) = \phi(\widetilde{E}, \widetilde{h})|_X.$$

By [11], the class represented by $\phi(E,h)$ does not depend on the choice of $\widetilde{X}$, $\widetilde{E}$, nor $\widetilde{h}$.

Recall that there are Chern classes defined in the Chow ring $\phi(E)_{\mathrm{CH}} \in \mathrm{CH}^*(X)$; they are compatible with the Chern classes in cohomology. More precisely, we have

Proposition 4.2. *The composition*

$$\mathrm{CH}^k(X) \xrightarrow{\mathrm{cl}} H^{2k}_{\mathcal{D}}(X,\mathbb{R}(k)) \longrightarrow H^{2k}(\mathcal{D}(E_X,k))$$

sends $\phi^{(k)}(E)_{\mathrm{CH}}$ *to* $\phi^{(k)}(E)$.

Proof. If X is projective, then $H^{2k}_{\mathcal{D}}(X,\mathbb{R}(k)) = H^{k,k}(X,\mathbb{C}) \cap H^{2k}(X,\mathbb{R}(k))$. Therefore, the result follows from the compatibility of Chern classes on the Chow ring and on ordinary cohomology (see, e.g., [16], §19). If X is quasi-projective, the result follows from the projective case by functoriality. □

4.2 Bott-Chern forms for hermitian metrics

Here we recall the theory of Bott-Chern forms. For more details we refer to [36], [11], [9].

Bott-Chern forms. Let

$$\overline{\mathcal{E}} : 0 \longrightarrow (E',h') \longrightarrow (E,h) \longrightarrow (E'',h'') \longrightarrow 0$$

be a short exact sequence of hermitian vector bundles; by this we mean a short exact sequence of vector bundles, where each vector bundle is equipped with an arbitrarily chosen hermitian metric. Let ϕ be as in 4.1 and assume E has rank n.
The Chern classes behave additively with respect to exact sequences, i.e.,

$$\phi(E) = \phi(E' \oplus E'').$$

In general, this is not true for the Chern forms. This lack of additivity on the level of Chern forms is measured by the Bott-Chern forms.
The fundamental result of the theory of Bott-Chern forms is the following theorem (see [5], [2], [17]).

Theorem 4.3. *There is a unique way to attach to every sequence* $\overline{\mathcal{E}}$ *as above, a form* $\widetilde{\phi}(\overline{\mathcal{E}})$ *in*

$$\bigoplus_k \widetilde{\mathcal{D}}^{2k-1}(E_X,k) = \bigoplus_k \mathcal{D}^{2k-1}(E_X,k)/\operatorname{Im}(\mathrm{d}_{\mathcal{D}})$$

satisfying the following properties

(i) $\mathrm{d}_{\mathcal{D}}\,\widetilde{\phi}(\overline{\mathcal{E}}) = \phi(E' \oplus E'', h' \oplus h'') - \phi(E,h)$.

(ii) $f^*\widetilde{\phi}(\overline{\mathcal{E}}) = \widetilde{\phi}(f^*\overline{\mathcal{E}})$, *for every holomorphic map* $f : X \longrightarrow Y$.

(iii) If $(E,h) = (E',h') \overset{\perp}{\oplus} (E'',h'')$, *then* $\widetilde{\phi}(\overline{\mathcal{E}}) = 0$.

There are different methods to construct Bott-Chern forms. We will introduce a variant of the method used in [17] and that is the dual of the construction used in [11].

The first transgression bundle. Let $\mathcal{O}(1)$ be the dual of the tautological line bundle of $\mathbb{P}^1$ with the standard metric. If $(x:y)$ are projective coordinates of $\mathbb{P}^1_{\mathbb{C}}$, then x and y are generating global sections of $\mathcal{O}(1)$ with norms

$$\|x\|^2 = \frac{x\bar{x}}{x\bar{x}+y\bar{y}} \quad \text{and} \quad \|y\|^2 = \frac{y\bar{y}}{x\bar{x}+y\bar{y}}.$$

Let

$$\overline{\mathcal{E}} : 0 \longrightarrow (E', h') \longrightarrow (E, h) \longrightarrow (E'', h'') \longrightarrow 0$$

be a short exact sequence of hermitian vector bundles such that h' is induced by h.
Let p_1, p_2, be the first and the second projection of $X \times \mathbb{P}^1_{\mathbb{C}}$, respectively. We write $E(n) = p_1^* E \otimes p_2^* \mathcal{O}(n)$. On this vector bundle we consider the metric induced by h and the standard metric of $\mathcal{O}(n)$, and we denote by $\overline{E}(n)$ this hermitian vector bundle. Analogously, we write $E''(n) = p_1^* E'' \otimes p_2^* \mathcal{O}(n)$ and denote by $\overline{E}''(n)$ the corresponding hermitian vector bundle.

Definition 4.4. The *first transgression bundle* $\operatorname{tr}_1(\overline{\mathcal{E}})$ is the kernel of the morphism

$$\begin{array}{ccc} \overline{E}(1) \oplus \overline{E}''(1) & \longrightarrow & E''(2) \\ (s,t) & \longmapsto & s \otimes x - t \otimes y \end{array}$$

with the induced metric.

Note that the definition of $\operatorname{tr}_1(\overline{\mathcal{E}})$ includes the metric; therefore, the expression $\phi(\operatorname{tr}_1(\overline{\mathcal{E}}))$ means the Chern form of the hermitian vector bundle $\operatorname{tr}_1(\overline{\mathcal{E}})$ and not its Chern class.
The key property of the first transgression bundle is the following. We denote by i_0 and i_∞ the morphisms $X \longrightarrow X \times \mathbb{P}^1$ given by

$$\begin{aligned} i_0(p) &= (p, (0:1)), \\ i_\infty(p) &= (p, (1:0)). \end{aligned}$$

Then, $i_0^*(\operatorname{tr}_1(\overline{\mathcal{E}}))$ is isometric to (E,h) and $i_\infty^*(\operatorname{tr}_1(\overline{\mathcal{E}}))$ is isometric to $(E', h') \overset{\perp}{\oplus} (E'', h'')$.

The construction of Bott-Chern forms. Let $t = x/y$ be the absolute coordinate of $\mathbb{P}^1$. Let us consider the current $W_1 = \left[-\frac{1}{2}\log(t\bar{t})\right]$ on $\mathbb{P}^1$ given by

$$W_1(\eta) = \left[-\frac{1}{2}\log(t\bar{t})\right](\eta) = -\frac{1}{2\pi i}\int_{\mathbb{P}^1} \frac{\eta}{2}\log(t\bar{t}).$$

By the Poincaré Lelong equation

$$-2\partial\bar{\partial}\left[-\frac{1}{2}\log(t\bar{t})\right] = \delta_{(1:0)} - \delta_{(0:1)}. \tag{4.5}$$

Definition 4.6. Let X be a complex manifold, $\overline{\mathcal{E}}$ an exact sequence of hermitian vector bundles

$$\overline{\mathcal{E}} : 0 \longrightarrow (E', h') \longrightarrow (E, h) \longrightarrow (E'', h'') \longrightarrow 0,$$

such that the metric h' is induced by the metric h. The *Bott-Chern form* associated to the exact sequence $\overline{\mathcal{E}}$ is the differential form over X given by

$$\phi(\overline{\mathcal{E}}) = W_1(\phi(\operatorname{tr}_1(\overline{\mathcal{E}}))) = -\frac{1}{2\pi i}\int_{\mathbb{P}^1} \phi(\operatorname{tr}_1(\overline{\mathcal{E}}))\frac{1}{2}\log(t\bar{t}).$$

Note that we use also the letter ϕ to denote the Bott-Chern form associated to a power series ϕ because the meaning of $\phi(\overline{\mathcal{E}})$ is determined again by the argument $\overline{\mathcal{E}}$, which, in this case, is an exact sequence of hermitian vector bundles.

Definition 4.7. If $\overline{\mathcal{E}}$ is an exact sequence as above, but such that h' is not the metric induced by h, then we consider the exact sequences

$$\lambda^1\overline{\mathcal{E}} : 0 \longrightarrow (E', \widetilde{h'}) \longrightarrow (E, h) \longrightarrow (E'', h'') \longrightarrow 0,$$

where $\widetilde{h'}$ is the hermitian metric induced by h, and

$$\lambda^2\overline{\mathcal{E}} : 0 \longrightarrow 0 \longrightarrow (E' \oplus E'', \widetilde{h'} \oplus h'') \longrightarrow (E' \oplus E'', h' \oplus h'') \longrightarrow 0.$$

The *Bott-Chern form* associated to the exact sequence $\overline{\mathcal{E}}$ is

$$\phi(\overline{\mathcal{E}}) = \phi(\lambda^1\overline{\mathcal{E}}) + \phi(\lambda^2\overline{\mathcal{E}}).$$

Proposition 4.8. *If $\overline{\mathcal{E}}$ is an exact sequence as above with h' induced by h, then the Bott-Chern forms obtained from definition 4.6 and definition 4.7 agree.*

Proof. In this case we have $\lambda^1\overline{\mathcal{E}} = \overline{\mathcal{E}}$. Thus, we have to show that $\phi(\lambda^2\overline{\mathcal{E}}) = 0$. But $\operatorname{tr}_1(\lambda^2(\overline{\mathcal{E}}))$ is the bundle $p_1^*(E' \oplus E'')$ with the hermitian metric $h' \oplus h''$, which does not depend on the coordinate of $\mathbb{P}^1$. Therefore, we have

$$\phi(\lambda^2\overline{\mathcal{E}}) = -\frac{1}{2\pi i}\int_{\mathbb{P}^1} \phi(E' \oplus E'', h' \oplus h'')\frac{1}{2}\log(t\bar{t}) = 0.$$

□

It is easy to see that the forms $\phi(\overline{\mathcal{E}})$ belong to $\bigoplus_k \mathcal{D}^{2k-1}(E_X, k)$. We will denote by $\widetilde{\phi}(\overline{\mathcal{E}})$ the class of $\phi(\overline{\mathcal{E}})$ in the group

$$\bigoplus_k \widetilde{\mathcal{D}}^{2k-1}(E_X, k) = \bigoplus_k \mathcal{D}^{2k-1}(E_X, k)/\operatorname{Im}(\mathrm{d}_{\mathcal{D}}).$$

PROPOSITION 4.9. *The classes $\widetilde{\phi}(\overline{\mathcal{E}})$ satisfy the properties of theorem 4.3.*

Proof. The first property follows from the Poincaré lemma (see, e.g., [36]). The second property is clear, because all the ingredients of the construction are functorial. We prove the third property. If $\overline{\mathcal{E}}$ is a split exact sequence with $(E,h) = (E',h') \overset{\perp}{\oplus} (E'',h'')$ and the obvious morphisms, then

$$\operatorname{tr}_1(\overline{\mathcal{E}}) = \overline{E}'(1) \oplus \overline{E}''(0)$$

with the induced metrics. Let ω be the first Chern form of the line bundle $\overline{\mathcal{O}}_{\mathbb{P}^1}(1)$. Then, we find

$$\phi(\overline{E}'(1) \oplus \overline{E}''(0)) = p_1^*(a) + p_1^*(b) \wedge p_2^*(\omega),$$

where a and b are suitable forms on X. Now we get

$$\begin{aligned}\phi(\overline{\mathcal{E}}) &= -\frac{1}{2\pi i}\int_{\mathbb{P}^1} (p_1^*(a) + p_1^*(b) \wedge p_2^*(\omega))\frac{1}{2}\log(t\bar{t}) \\ &= -\frac{1}{2\pi i}\, a \wedge \int_{\mathbb{P}^1} \frac{1}{2}\log(t\bar{t}) - \frac{1}{2\pi i}\, b \wedge \int_{\mathbb{P}^1} \frac{\omega}{2}\log(t\bar{t}) = 0.\end{aligned}$$

□

CHANGE OF METRICS. Of particular importance is the Bott-Chern form associated to a change of hermitian metrics. Let E be a holomorphic vector bundle of rank n with two hermitian metrics h and h'. We denote by $\operatorname{tr}_1(E,h,h')$ the first transgression bundle associated to the short exact sequence

$$0 \longrightarrow 0 \longrightarrow (E,h) \longrightarrow (E,h') \longrightarrow 0.$$

Explicitly, $\operatorname{tr}_1(E,h,h')$ is isomorphic to p_1^*E with the embedding

$$\begin{array}{ccc} p_1^*E & \longrightarrow & \overline{E}(1) \oplus \overline{E}'(1) \\ s & \longmapsto & (s \otimes y, s \otimes x); \end{array}$$

here $\overline{E} = (E,h)$ and $\overline{E}' = (E,h')$. Therefore, if ξ is a local frame for E on an open set U, it determines a local frame for $\operatorname{tr}(E,h,h')$, also denoted by ξ, on $U \times \mathbb{P}^1$. In this frame the metric is given by the matrix

$$\frac{y\bar{y}h(\xi) + x\bar{x}h'(\xi)}{x\bar{x} + y\bar{y}}. \tag{4.10}$$

DEFINITION 4.11. Let X be a complex manifold, E be a complex vector bundle of rank n, h, h' two hermitian metrics on E, and ϕ as in section 4.1. The *Bott-Chern form* associated to the change of metric (E,h,h') is the Bott-Chern form associated to the short exact sequence

$$0 \longrightarrow 0 \longrightarrow (E,h) \longrightarrow (E,h') \longrightarrow 0.$$

We will denote this form by $\phi(E,h,h')$ or, if E is understood, by $\phi(h,h')$. This form satisfies

$$\mathrm{d}_{\mathcal{D}}\, \phi(E,h,h') = -2\partial\bar{\partial}\phi(E,h,h') = \phi(E,h') - \phi(E,h). \tag{4.12}$$

4.3 Iterated Bott-Chern forms for hermitian metrics

The theory of Bott-Chern forms can be iterated defining higher Bott-Chern forms for exact k-cubes of hermitian vector bundles. This theory provides explicit representatives of characteristic classes for higher K-theory (see [11], [9]).

Exact squares. Let $\langle -1,0,1\rangle$ be the category associated to the ordered set $\{-1,0,1\}$.

Definition 4.13. A *square of vector bundles* over X is a functor from the category $\langle -1,0,1\rangle^2$ to the category of vector bundles over X. Given a square of vector bundles $\mathcal{F}$ and numbers $i \in \{1,2\}$, $j \in \{-1,0,1\}$, then the (i,j)*-face* of $\mathcal{F}$, denoted by $\partial_i^j \mathcal{F}$, is the sequence

$$\begin{aligned} \partial_1^j \mathcal{F} &: \mathcal{F}_{j,-1} \longrightarrow \mathcal{F}_{j,0} \longrightarrow \mathcal{F}_{j,1}, \\ \partial_2^j \mathcal{F} &: \mathcal{F}_{-1,j} \longrightarrow \mathcal{F}_{0,j} \longrightarrow \mathcal{F}_{1,j}. \end{aligned}$$

A square of vector bundles is called *exact*, if all the faces are short exact sequences. A *hermitian exact square* $\overline{\mathcal{F}}$ is an exact square $\mathcal{F}$ such that the vector bundles $\mathcal{F}_{i,j}$ are equipped with arbitrarily chosen hermitian metrics. If $\overline{\mathcal{F}}$ is a hermitian exact square, then the faces of $\overline{\mathcal{F}}$ are equipped with the induced hermitian metrics. The reader is referred to [11] for the definition of exact n-cubes.

Let ϕ be as before and let $\overline{\mathcal{F}}$ be a hermitian exact square of vector bundles over X such that $\mathcal{F}_{0,0}$ has rank n. Then, the form

$$\phi(\partial_1^{-1}\overline{\mathcal{F}} \oplus \partial_1^{1}\overline{\mathcal{F}}) - \phi(\partial_1^{0}\overline{\mathcal{F}}) - \phi(\partial_2^{-1}\overline{\mathcal{F}} \oplus \partial_2^{1}\overline{\mathcal{F}}) + \phi(\partial_2^{0}\overline{\mathcal{F}})$$

is closed in the complex $\bigoplus_p \mathcal{D}^*(E_X,p)$. The iterated Bott-Chern form is a differential form

$$\phi(\overline{\mathcal{F}}) \in \bigoplus_p \mathcal{D}^{2p-2}(E_X,p)$$

satisfying

$$\mathrm{d}_{\mathcal{D}}\, \phi(\overline{\mathcal{F}}) = \phi(\partial_1^{-1}\overline{\mathcal{F}} \oplus \partial_1^{1}\overline{\mathcal{F}}) - \phi(\partial_1^{0}\overline{\mathcal{F}}) - \phi(\partial_2^{-1}\overline{\mathcal{F}} \oplus \partial_2^{1}\overline{\mathcal{F}}) + \phi(\partial_2^{0}\overline{\mathcal{F}}).$$

The second transgression bundle.

Definition 4.14. Let $\overline{\mathcal{F}}$ be a hermitian exact square such that for $j=-1,0,1$, the hermitian metrics of the vector bundles $\mathcal{F}_{j,-1}$ and $\mathcal{F}_{-1,j}$ are induced by the metrics of $\mathcal{F}_{j,0}$ and $\mathcal{F}_{0,j}$, respectively. The *second transgression bundle* associated to $\overline{\mathcal{F}}$ is the hermitian vector bundle on $X \times \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\mathrm{tr}_2(\overline{\mathcal{F}}) = \mathrm{tr}_1\left(\mathrm{tr}_1(\partial_2^{-1}\overline{\mathcal{F}}) \longrightarrow \mathrm{tr}_1(\partial_2^{0}\overline{\mathcal{F}}) \longrightarrow \mathrm{tr}_1(\partial_2^{1}\overline{\mathcal{F}})\right).$$

The second transgression bundle satisfies

$$\begin{aligned}
\operatorname{tr}_2(\overline{\mathcal{F}})|_{X\times\mathbb{P}^1\times(0:1)} &= \operatorname{tr}_1(\partial_2^0\overline{\mathcal{F}}),\\
\operatorname{tr}_2(\overline{\mathcal{F}})|_{X\times\mathbb{P}^1\times(1:0)} &= \operatorname{tr}_1(\partial_2^{-1}\overline{\mathcal{F}}) \overset{\perp}{\oplus} \operatorname{tr}_1(\partial_2^1\overline{\mathcal{F}}),\\
\operatorname{tr}_2(\overline{\mathcal{F}})|_{X\times(0:1)\times\mathbb{P}^1} &= \operatorname{tr}_1(\partial_1^0\overline{\mathcal{F}}),\\
\operatorname{tr}_2(\overline{\mathcal{F}})|_{X\times(1:0)\times\mathbb{P}^1} &= \operatorname{tr}_1(\partial_1^{-1}\overline{\mathcal{F}}) \overset{\perp}{\oplus} \operatorname{tr}_1(\partial_1^1\overline{\mathcal{F}}).
\end{aligned} \tag{4.15}$$

The second Wang current. On $\mathbb{P}^1\times\mathbb{P}^1$ we put homogeneous coordinates $((x_1:y_1),(x_2:y_2))$; let $t_1=x_1/y_1$ and $t_2=x_2/y_2$.

Definition 4.16. The *second Wang current* is the current on $\mathbb{P}^1\times\mathbb{P}^1$ given by

$$W_2=\frac{1}{4}\left[\log(t_1\bar{t}_1)\left(\frac{\mathrm{d}\,t_2}{t_2}-\frac{\mathrm{d}\,\bar{t}_2}{\bar{t}_2}\right)-\log(t_2\bar{t}_2)\left(\frac{\mathrm{d}\,t_1}{t_1}-\frac{\mathrm{d}\,\bar{t}_1}{\bar{t}_1}\right)\right].$$

Observe that $W_2\in\mathcal{D}^2(D^*_{(\mathbb{P}^1)^2},2)$, where $D^*_{(\mathbb{P}^1)^2}$ is the Dolbeault complex of currents on $\mathbb{P}^1\times\mathbb{P}^1$. Moreover, we can write

$$W_2=\left[\left(-\frac{1}{2}\log(t_1\bar{t}_1)\right)\bullet\left(-\frac{1}{2}\log(t_2\bar{t}_2)\right)\right], \tag{4.17}$$

where $\bullet$ is the product in the Deligne complex (see definition 3.4).
For $p=(x_0:y_0)\in\mathbb{P}^1$, $i=1,2$, let $\iota_{i,p}:\mathbb{P}^1\longrightarrow\mathbb{P}^1\times\mathbb{P}^1$ be the inclusion given by

$$\begin{aligned}
\iota_{1,p}(x:y)&=(x_0:y_0)\times(x:y),\\
\iota_{2,p}(x:y)&=(x:y)\times(x_0:y_0).
\end{aligned}$$

Proposition 4.18. *We have the equality*

$$\mathrm{d}_{\mathcal{D}}\,W_2=(\iota_{1,(1:0)})_*W_1-(\iota_{1,(0:1)})_*W_1-(\iota_{2,(1:0)})_*W_1+(\iota_{2,(0:1)})_*W_1.$$

Proof. This proposition follows easily from a residue computation. Formally, we can interpret it as the Leibniz rule for the Deligne complex and equations (4.5), (4.17). □

The iterated Bott-Chern form.

Definition 4.19. Let $\overline{\mathcal{F}}$ be a hermitian exact square satisfying the condition of definition 4.14. The *iterated Bott-Chern form* associated to $\overline{\mathcal{F}}$ is the differential form given by

$$\begin{aligned}
\phi(\overline{\mathcal{F}})=W_2(\phi(\overline{\mathcal{F}}))=&\frac{1}{(4\pi i)^2}\int_{\mathbb{P}^1\times\mathbb{P}^1}\phi(\operatorname{tr}_2(\overline{\mathcal{F}}))\wedge\log(t_1\bar{t}_1)\left(\frac{\mathrm{d}\,t_2}{t_2}-\frac{\mathrm{d}\,\bar{t}_2}{\bar{t}_2}\right)-\\
&\frac{1}{(4\pi i)^2}\int_{\mathbb{P}^1\times\mathbb{P}^1}\phi(\operatorname{tr}_2(\overline{\mathcal{F}}))\wedge\log(t_2\bar{t}_2)\left(\frac{\mathrm{d}\,t_1}{t_1}-\frac{\mathrm{d}\,\bar{t}_1}{\bar{t}_1}\right).
\end{aligned}$$

When $\overline{\mathcal{F}}$ does not satisfy the condition of definition 4.14 we proceed as follows. Let $\lambda_i^k \overline{\mathcal{F}}$, $i = 1, 2$, $k = 1, 2$, be the hermitian exact square determined by

$$\partial_i^j (\lambda_i^k \overline{\mathcal{F}}) = \lambda^k (\partial_i^j \overline{\mathcal{F}}) \qquad (j = -1, 0, 1);$$

here $\lambda^k(\cdot)$ is as in definition 4.7.

Definition 4.20. Let $\overline{\mathcal{F}}$ be a hermitian exact square. Then, the *iterated Bott-Chern form* associated to $\overline{\mathcal{F}}$ is the differential form given by

$$\phi(\overline{\mathcal{F}}) = \phi(\lambda_1^1 \lambda_2^1 \overline{\mathcal{F}}) + \phi(\lambda_1^1 \lambda_2^2 \overline{\mathcal{F}}) + \phi(\lambda_1^2 \lambda_2^1 \overline{\mathcal{F}}) + \phi(\lambda_1^2 \lambda_2^2 \overline{\mathcal{F}}).$$

As in the case of exact sequences, if $\overline{\mathcal{F}}$ satisfies the condition of definition 4.14, then the iterated Bott-Chern forms obtained by means of definition 4.19 and definition 4.20 agree.

Theorem 4.21. *The second iterated Bott-Chern form satisfies*

$$\mathrm{d}_{\mathcal{D}}\, \phi(\overline{\mathcal{F}}) = \phi(\partial_1^{-1} \overline{\mathcal{F}} \oplus \partial_1^1 \overline{\mathcal{F}}) - \phi(\partial_1^0 \overline{\mathcal{F}}) - \phi(\partial_2^{-1} \overline{\mathcal{F}} \oplus \partial_2^1 \overline{\mathcal{F}}) + \phi(\partial_2^0 \overline{\mathcal{F}}).$$

Proof. This follows from (4.15) and proposition 4.18. □

The case of three different metrics. Let X be a complex manifold, E a holomorphic vector bundle on X and h, h', h'' smooth hermitian metrics on E. We will denote by $\overline{\mathcal{F}}(E, h, h', h'')$ the hermitian exact square

$$\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
0 & \longrightarrow & (E, h) & \longrightarrow & (E, h'') \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
0 & \longrightarrow & (E, h') & \longrightarrow & (E, h'')
\end{array}$$

where the faces ∂_1^j are the rows and the faces ∂_2^j are the columns. As a shorthand, we will denote the hermitian vector bundle $\mathrm{tr}_2(\overline{\mathcal{F}}(E, h, h', h''))$ by $\mathrm{tr}_2(E, h, h', h'')$, or simply by $\mathrm{tr}_2(h, h', h'')$, if E is understood.

Definition 4.22. The *iterated Bott-Chern form* associated to the metrics h, h', h'' is the differential form given by

$$\phi(E, h, h', h'') = \phi(\overline{\mathcal{F}}(E, h, h', h'')).$$

Proposition 4.23. *The iterated Bott-Chern form satisfies*

$$\mathrm{d}_{\mathcal{D}}\, \phi(E, h, h', h'') = \phi(E, h, h') + \phi(E, h', h'') + \phi(E, h'', h).$$

Proof. By theorem 4.21, we have

$$\mathrm{d}_{\mathcal{D}}\, \phi(E, h, h', h'') = \phi(E, h', h'') - \phi(E, h, h'') - \phi(E, h'', h'') + \phi(E, h, h').$$

A direct computation shows that $\phi(E, h'', h'') = 0$ and that $\phi(E, h, h'') = -\phi(E, h'', h)$, which implies the result. □

4.4 Chern forms for singular hermitian metrics

There are various successful concepts of singular metrics in Arithmetic and Diophantine Geometry, see [3], [14], [33], and [34]. For our purposes the most important are: Faltings's notion of a metric with logarithmic singularities along a divisors with normal crossings (see [14]) and Mumford's notion of a good metric (see [34]). Both concepts have in common nature that automorphic vector bundles (equipped with their natural metrics) have the required local behavior. And, in fact, the application to automorphic vector bundles was the driving motivation to establish these definitions. For our purposes we will need a more precise description of the kind of metrics that appear when studying automorphic vector bundles.

Faltings's logarithmic singular metric. Let X be a complex manifold and let D be a normal crossing divisor. Put $U = X \setminus D$, and let $j : U \longrightarrow X$ be the inclusion. Let L be a line bundle on X and L_0 the restriction to U. A smooth metric h on L_0 is said to have logarithmic singularities along D, if, for any coordinate open subset V adapted to D and every non vanishing local section s, there exists a number $N \in \mathbb{N}$ such that

$$\max\{h(s), h^{-1}(s)\} \prec \left| \min_{j=1,..,k} \{\log |r_j|\} \right|^N . \tag{4.24}$$

Observe that this definition does not give any information on the behavior of the Chern form associated to the metric.

Good metrics in the sense of Mumford. We recall the notion of a good metric in the sense of Mumford, see [34].

Definition 4.25. Let E be a rank n vector bundle on X and E_0 the restriction to U. A smooth metric h on E_0 is said to be *good* on X, if, for all $x \in D$, there exist a neighborhood V adapted D and a holomorphic frame $\xi = \{e_1, \dots, e_n\}$ such that, writing $h(\xi)_{ij} = h(e_i, e_j)$, we have:

(i) $|h(\xi)_{ij}|, \det(h)^{-1} \prec \left(\prod_{i=1}^k \log(r_i)\right)^N$ for some $N \in \mathbb{N}$.

(ii) The 1-forms $(\partial h(\xi) \cdot h(\xi)^{-1})_{ij}$ are good.

A vector bundle provided with a good hermitian metric will be called a *good hermitian vector bundle.*

Lemma 4.26. *If (E, h) is a good hermitian vector bundle, then the 1-forms $(\partial h(\xi) \cdot h(\xi)^{-1})_{ij}$ are pre-log-log forms.*

Proof. Since a differential form with Poincaré growth has log-log growth (see [10], §7.1), we have that $(\partial h(\xi) \cdot h(\xi)^{-1})_{ij}$ and $\mathrm{d}(\partial h(\xi) \cdot h(\xi)^{-1})_{ij}$ have log-log growth. Since the condition of having log-log growth is bihomogeneous and

$(\partial h(\xi)\cdot h(\xi)^{-1})_{ij}$ has pure bidegree $(1,0)$, we have that $\partial(\partial h(\xi)\cdot h(\xi)^{-1})_{ij}$ and $\bar{\partial}(\partial h(\xi)\cdot h(\xi)^{-1})_{ij}$ have log-log growth. Finally, since

$$\partial(\partial h(\xi)\cdot h(\xi)^{-1}) = \partial h(\xi)\cdot h(\xi)^{-1}\wedge \partial h(\xi)\cdot h(\xi)^{-1},$$

the form $\partial\bar{\partial}(\partial h(\xi)\cdot h(\xi)^{-1})_{i,j}$ also has log-log growth. □

A fundamental property of the concept of good metrics is the following result of Mumford, see [34].

Proposition 4.27. *Let X, D, and U be as before.*

(i) Let (E_0, h) be a vector bundle over U. Then, it has at most one extension to a vector bundle E to X such that h is good along D.

(ii) If (E, h) is a good hermitian vector bundle, then, for any power series ϕ, the Chern form $\phi(E, h)$ is good. Moreover, its associated current $\left[\phi^{(k)}(E,h)\right]_X$ represents the Chern class $\phi(E)$ of E. □

Good Metrics of infinite order. Note that with the concept of good metric we have control on the local behavior of the Chern forms and of the cohomology class represented by its associated currents. As we will see later, we can also control the local behavior of the Bott-Chern forms. In order to have control on the cohomology classes represented by the Chern forms we need a slightly stronger definition, that is the analogue of our definition 2.62 of Poincaré singular forms.

Definition 4.28. Let X, D, and U be as before. Let E be a rank n vector bundle on X and let E_0 be the restriction of E to U. A smooth metric on E_0 is said to be *good of infinite order (along D)*, if, for every $x\in D$, there exist a trivializing open coordinate neighborhood V adapted to D and a holomorphic frame $\xi=\{e_1,\dots,e_n\}$ such that, writing $h(\xi)_{ij}=h(e_i,e_j)$, we have:

(i) The functions $h(\xi)_{ij}$, $\det(h(\xi))^{-1}$ belong to $\Gamma(V, \mathscr{E}^0_X\langle D\rangle)$.

(ii) The 1-forms $(\partial h(\xi)\cdot h(\xi)^{-1})_{ij}$ are Poincaré singular.

A vector bundle equipped with a good hermitian metric of infinite order will be called a ∞-*good hermitian vector bundle.*

Log-singular hermitian metrics. Although the hermitian metrics we are interested in, the automorphic hermitian metrics, are ∞-good, we will consider a slightly bigger set of singular metrics, the log-singular metrics, for which we will be able to define arithmetic characteristic classes.

Definition 4.29. Let X, D, and U be as before. Let E be a rank n vector bundle on X and let E_0 be the restriction of E to U. A smooth metric on E_0 is said to be *log-singular (along D)*, if, for every $x\in D$, there exist a trivializing open coordinate neighborhood V adapted to D and a holomorphic frame $\xi=\{e_1,\dots,e_n\}$ such that, writing $h(\xi)_{ij}=h(e_i,e_j)$, we have

(i) The functions $h(\xi)_{ij}$, $\det(h(\xi))^{-1}$ belong to $\Gamma(V, \mathscr{E}^0_X\langle D\rangle)$.

(ii) The 1-forms $(\partial h(\xi) \cdot h(\xi)^{-1})_{ij}$ belong to $\Gamma(V, \mathscr{E}^{1,0}_X\langle\langle D\rangle\rangle)$.

A vector bundle equipped with a log-singular hermitian metric will be called a *log-singular hermitian vector bundle.*

Note that, if a smooth metric on E_0 is log-singular, then the conditions of definition 4.29 are satisfied in every holomorphic frame in every trivializing open coordinate neighborhood V adapted to D.

Remark 4.30. By the very definition of log-singular metrics, the Chern forms $\phi(E,h)$ belong to the group $\oplus_k \mathcal{D}^{2k}(E_X\langle\langle D\rangle\rangle, k)$, if (E,h) is a log-singular hermitian vector bundle. Moreover, as we will see in proposition 4.61, the form $\phi(E,h)$ represents the Chern class $\phi(E)$ in $H^*_{\mathcal{D}}(X, \mathbb{R}(*))$.

Basic properties of log-singular hermitian metrics. The following properties are easily verified.

Proposition 4.31. *Let X, D, and U be as before. Let E and F be vector bundles on X, and let E_0 and F_0 be their restrictions to U. Let h_E and h_F be smooth hermitian metrics on E_0 and F_0. Write $\overline{E} = (E, h_E)$ and $\overline{F} = (F, h_F)$.*

(i) The hermitian vector bundle $\overline{E} \overset{\perp}{\oplus} \overline{F}$ is log-singular along D, if and only if, $\overline{E}$ and $\overline{F}$ are log-singular along D.

(ii) If $\overline{E}$ and $\overline{F}$ are log-singular along D, then the tensor product $\overline{E} \otimes \overline{F}$, the exterior and symmetric powers $\Lambda^n \overline{E}$, $S^n \overline{E}$, the dual bundle $\overline{E}^\vee$, and the bundle of homomorphisms $\mathrm{Hom}(\overline{E}, \overline{F})$, with their induced metrics, are log-singular along D.

□

Remark 4.32. Note however that the condition of being log-singular is not stable under taking general quotients and subbundles. That is, if (E,h) is a hermitian vector bundle, log-singular along a normal crossing divisor D, and E' is a subbundle or a quotient bundle, then the induced metric on E' need not be log-singular along D. For instance, let $X = \mathbb{A}^2$ with coordinates (t,z). Let $E = \mathcal{O}_X \oplus \mathcal{O}_X$ be the trivial rank two vector bundle with hermitian metric given, in the frame $\{e_1, e_2\}$, by the matrix

$$H = \begin{pmatrix} (\log(1/|z|))^{-1} & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.33}$$

This hermitian metric is log-singular along the divisor $D = \{z = 0\}$. But the subbundle generated by the section $e_1 + te_2$ with the induced metric does not satisfy the second condition of definition 4.29. Namely, let $h(t,z) = \|e_1 + te_2\|^2$. Then, we find

$$h(t,z) = t\bar{t} + (\log(1/|z|))^{-1},$$
$$\partial h/h = \frac{\bar{t}\,\mathrm{d}\,t}{t\bar{t} + (\log(1/|z|))^{-1}} + \frac{\mathrm{d}\,z}{z(\log(1/|z|))^2(t\bar{t} + (\log(1/|z|))^{-1})}.$$

But the function $\bar{t}/(t\bar{t} + (\log(1/|z|))^{-1})$ is not log-log along D, as can be seen by considering the set of points

$$t = \sqrt{(\log(1/|z|))^{-1}}.$$

In this concrete case, the induced metric is not far from being log-singular: If $\widetilde{X}$ is the blow-up of X along the point $(0,0)$ and $\widetilde{D}$ is the pre-image of D, then the metric h is log-singular along $\widetilde{D}$. See also proposition 4.59 for a related example.

Remark 4.34. The condition of being log-singular is also not stable under extensions. That is, let

$$0 \longrightarrow (E', h') \longrightarrow (E, h) \longrightarrow (E'', h'') \longrightarrow 0$$

be a short exact sequence with h' and h'' the hermitian metrics induced by h. If h' and h'' are log-singular, then h need not be log-singular.

Functoriality of log-singular metrics. The following result is a direct consequence of the definition and the functoriality of log forms and log-log forms.

Proposition 4.35. *Let X, X' be complex manifolds and let D, D' be normal crossing divisors of X, X', respectively. If $f : X' \longrightarrow X$ is a holomorphic map such that $f^{-1}(D) \subseteq D'$ and (E,h) is a log-singular hermitian vector bundle on X, then (f^*E, f^*h) is a log-singular hermitian vector bundle on X'.* $\square$

4.5 Bott-Chern forms for singular hermitian metrics

Bott-Chern forms for log-singular hermitian metrics. In order to define characteristic classes of log-singular hermitian metrics with values in the log-log arithmetic Chow groups, we have to show that the Bott-Chern forms associated to a change of metric between a smooth metric and a log-singular metric is a log-log form. By the proof of the next theorem, it is clear that, even if we restrict ourselves to ∞-good hermitian metrics, the Bott-Chern forms are not necessarily Poincaré singular, but log-log. Therefore, the log-log forms are an essential ingredient of the theory and not only a technical addition to have the Poincaré lemma.

Theorem 4.36. *Let X be a complex manifold and let D be a normal crossing divisor. Put $U = X \setminus D$. Let E be a vector bundle on X.*

(i) If h is a smooth hermitian metric on E and h' is a smooth hermitian metric on $E|_U$, which is log-singular along D, then the Bott-Chern form $\phi(E, h, h')$ belongs to the group $\bigoplus_k \mathcal{D}^{2k-1}(E_X\langle\langle < D\rangle\rangle, k)$.

(ii) If h and h' are smooth hermitian metrics on E and h'' is a smooth hermitian metric on $E|_U$, which is log-singular along D, the iterated Bott-Chern form $\phi(E, h, h', h'')$ belongs to the group $\bigoplus_k \mathcal{D}^{2k-2}(E_X\langle\langle D\rangle\rangle, k)$.

Proof. Let V be a trivializing coordinate subset adapted to D with coordinates $(z_1, \dots, z_d)$. Thus, D has equation $z_1 \cdots z_k = 0$; we put $r_i = |z_i|$. We may also assume that V is contained in a compact subset of X. Let $\xi = \{e_i\}$ be a local holomorphic frame for E. Let g be the hermitian metric of $\operatorname{tr}_1(E, h, h')$. Since the vector bundle $\operatorname{tr}_1(E, h, h')$ is isomorphic to $p_1^* E$, the holomorphic frame ξ induces a holomorphic frame (also denoted by ξ) of $\operatorname{tr}_1(E, h, h')$.
For the rest of the proof the frame ξ will be fixed. Therefore, we drop it from the notation and we write

$$H = h(\xi), \qquad H_{ij} = h(\xi)_{ij} = h(e_i, e_j).$$

We use the same notation for the metrics h' and g.
Let $(x : y)$ be homogeneous coordinates of $\mathbb{P}^1$. Write $t = x/y$. We decompose $\mathbb{P}^1$ into two closed sets

$$\mathbb{P}^1_+ = \{(x : y) \in \mathbb{P}^1 \mid |x| \geq |y|\} \text{ and } \mathbb{P}^1_- = \{(x : y) \in \mathbb{P}^1 \mid |x| \leq |y|\}.$$

Then, we write

$$\phi(h, h') = \phi_+(h, h') + \phi_-(h, h'),$$

with

$$\phi_\pm(h, h') = \frac{-1}{4\pi i} \int_{\mathbb{P}^1_\pm} \phi(\operatorname{tr}_1(h, h')) \log(t\bar{t}). \tag{4.37}$$

We first show that the form $\phi_-(h, h')$ is log-log along D. One technical difficulty that we have to solve at this point is that the differential form $\phi(\operatorname{tr}_1(h, h'))$ is, in general, not a log-log form along $D \times \mathbb{P}^1$, because the vector bundle $\operatorname{tr}_1(h, h')$ need not be log-singular along $D \times \mathbb{P}^1$. This is the reason why we have to introduce a new class of singular functions.

Definition 4.38. For any pair of subsets $I, J \subseteq \{1, \dots, d\}$ and integers $n, K \geq 0$, we say that a smooth complex function f on $(V \setminus D) \times \mathbb{P}^1_-$ has *singularities of type* (n, α, β) *of order* K if there is an integer $N \geq 0$ such that, for any pair of multi-indices $\alpha, \beta \in \mathbb{Z}^d_{\geq 0}$ and integers $a, b \geq 0$ with $|\alpha + \beta| + a + b \leq K$, it holds the estimate

$$\left|\frac{\partial^{|\alpha|}}{\partial z^{\alpha}}\frac{\partial^{|\beta|}}{\partial z^{\beta}}\frac{\partial^{a}}{\partial t}\frac{\partial^{b}}{\partial \bar{t}}f(z_1,\dots,z_d,t)\right| \prec \left(\frac{1}{|t|+(\prod_{i=1}^{k}\log(1/r_i))^{-N}}\right)^{n+a+b} \cdot \\ \cdot \frac{\left|\prod_{i=1}^{k}\log(\log(1/r_i))\right|^{N}}{r^{(\gamma^I+\gamma^J+\alpha+\beta)^{\leq k}}(\log(1/r))^{(\gamma^I+\gamma^J)^{\leq k}}}.$$

We say that f has *singularities of type* (n,α,β) *of infinite order*, if it has singularities of type (n,α,β) of order K for all $K\geq 0$.

The singularities of the differential form $\phi(\operatorname{tr}_1(h,h'))$ are controlled by the following result.

Lemma 4.39. *Let*

$$\phi(\operatorname{tr}_1(h,h')) = \sum_{\substack{0\leq a,b\leq 1\\ I,J}} f_{I,J,a,b}\,\mathrm{d}\,z^I\wedge \mathrm{d}\,\bar{z}^J\wedge \mathrm{d}\,t^a\wedge \mathrm{d}\,\bar{t}^b$$

be the decomposition of $\phi(\operatorname{tr}_1(h,h'))$ *into monomials over* $V\times\mathbb{P}^1_-$. *Then, the function* $f_{I,J,a,b}$ *has singularities of type* $(a+b,I,J)$ *of infinite order.*

Proof. On $V\times\mathbb{P}^1_-$, the matrix of g in the holomorphic frame ξ is

$$G=\frac{1}{1+t\bar{t}}(H+t\bar{t}H').$$

We write $G_1=H+t\bar{t}H'$. The differential form $\phi(\operatorname{tr}_1(h,h'))$ is a polynomial in the entries of the matrix $\bar{\partial}(\partial GG^{-1})$. Since

$$\partial GG^{-1}=\frac{-\bar{t}\,\mathrm{d}\,t}{1+t\bar{t}}\operatorname{id}+\partial G_1G_1^{-1},$$

and the first summand of the right term is smooth, we are led to study the singularities of the matrices $\partial G_1G_1^{-1}$ and $\bar{\partial}(\partial G_1G_1^{-1})$. This will be done in the subsequent lemmas.
We write $G_2=(H'^{-1}+t\bar{t}H^{-1})$. The following lemma is easy.

Lemma 4.40. *The matrices* H, H', G_1 *and* G_2 *satisfy the rules*

(1) $HG_1^{-1}=G_2^{-1}H'^{-1}$, (2) $G_1^{-1}H=H'^{-1}G_2^{-1}$,
(3) $H'G_1^{-1}=G_2^{-1}H^{-1}$, (4) $G_1^{-1}H'=H^{-1}G_2^{-1}$.

□

In order to bound the entries of $\partial G_1G_1^{-1}$ and the other matrices, we need the following estimates.

Lemma 4.41. *(i) The entries of the matrix G_1^{-1} are bounded. In particular, they have singularities of type $(0,\emptyset,\emptyset)$ of order 0.*

(ii) The entries of the matrix G_2^{-1} have singularities of type $(2,\emptyset,\emptyset)$ of order 0. Therefore, the entries of the matrices $t\,G_2^{-1}$ and $\bar t\,G_2^{-1}$ have singularities of type $(1,\emptyset,\emptyset)$ of order 0 and the entries of the matrix $t\bar t\,G_2^{-1}$ are bounded.

Proof. Let $H=T^+T$ be the Cholesky decomposition of H, where $(\bullet)^+$ denotes conjugate-transpose. Since H is smooth, the same is true for T. We can write

$$G_1^{-1}=T^{-1}(\mathrm{id}+t\bar t(T^{-1})^+H'T^{-1})^{-1}(T^{-1})^+.$$

But for any symmetric definite positive matrix A, the entries of $(\mathrm{id}+A)^{-1}$ have absolute value less that one. Therefore, the entries of the matrix G_1^{-1} are bounded. This proves the first statement.
To prove the second statement, we write

$$G_2^{-1}=T^+(TH'^{-1}T^++t\bar t\ \mathrm{id})^{-1}T.$$

By the first condition of a log-singular metric, we can decompose

$$TH'^{-1}T^+=U^+DU$$

with U unitary and D diagonal with all the diagonal elements bounded from above by $(\prod_{i=1}^k\log(1/r_i))^N$ and bounded from below by $(\prod_{i=1}^k\log(1/r_i))^{-N}$ for some integer N. Then, we find

$$G_2^{-1}=(UT)^+(D+t\bar t\ \mathrm{id})^{-1}(UT).$$

Now the lemma follows from the fact that the norm of any entry of a unitary matrix is less or equal than one. □

The remainder of the proof of lemma 4.39 is based on lemma 4.41.

Lemma 4.42. *Let $\sum\psi_{I,J,a,b}\,\mathrm{d}\,z^I\wedge\mathrm{d}\,\bar z^J\wedge\mathrm{d}\,t^a\wedge\mathrm{d}\,\bar t^b$ be the decomposition into monomials of an entry of any of the matrices $\partial G_1G_1^{-1}$, $\bar\partial(\partial G_1G_1^{-1})$, $\partial(\partial G_1G_1^{-1})$, and $\partial\bar\partial(\partial G_1G_1^{-1})$. Then, $\psi_{I,J,a,b}$ has singularities of type $(a+b,I,J)$ of order 0.*

Proof. We start with the entries of $\partial G_1G_1^{-1}$. Using lemma 4.40, we have

$$\begin{aligned}\partial G_1G_1^{-1}&=\partial HG_1^{-1}+\bar t\,\mathrm{d}\,tH'G_1^{-1}+t\bar t\partial H'G_1^{-1}\\&=\partial HG_1^{-1}+(\bar t\,\mathrm{d}\,t+t\bar t\partial H'H'^{-1})G_2^{-1}H^{-1}.\end{aligned}\tag{4.43}$$

Therefore, the bound of the entries of $\partial G_1G_1^{-1}$ follows from lemma 4.41 and the fact that h' is log-singular.
The bound of the entries of $\partial(\partial G_1G_1^{-1})$ follows from the previous case and the formula

$$\partial(\partial G_1G_1^{-1})=\partial G_1G_1^{-1}\wedge\partial G_1G_1^{-1}.\tag{4.44}$$

Before bounding $\bar\partial(\partial G_1 G_1^{-1})$, we compute

$$\bar\partial G_1^{-1} = -G_1^{-1}\bar\partial G_1 G_1^{-1} = -(\partial G_1 G_1^{-1})^+ G_1^{-1}$$

and

$$\begin{aligned}\bar\partial G_2^{-1} &= -G_2^{-1}\bar\partial G_2 G_2^{-1}\\ &= -G_2^{-1}(\bar\partial H'^{-1} + t\,\mathrm{d}\,\bar t H^{-1} + t\bar t\bar\partial H^{-1})G_2^{-1}\\ &= G_2^{-1}H'^{-1}\bar\partial H' H'^{-1}G_2^{-1} - G_2^{-1}(t\,\mathrm{d}\,\bar t H^{-1} + t\bar t\bar\partial H^{-1})G_2^{-1}\\ &= G_2^{-1}(\partial H' H'^{-1})^+ G_1^{-1}H - G_2^{-1}(t\,\mathrm{d}\,\bar t H^{-1} + t\bar t\bar\partial H^{-1})G_2^{-1}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\bar\partial(\partial G_1 G_1^{-1}) =& \bar\partial\partial H G_1^{-1} + \partial H \wedge (\partial G_1 G_1^{-1})^+ G_1^{-1}\\ &+ \bar\partial(\bar t\,\mathrm{d}\,t + t\bar t\partial H' H'^{-1})G_2^{-1}H^{-1}\\ &- (\bar t\,\mathrm{d}\,t + t\bar t\partial H' H'^{-1})G_2^{-1} \wedge (\partial H' H'^{-1})^+ G_1^{-1}\\ &+ (\bar t\,\mathrm{d}\,t + t\bar t\partial H' H'^{-1})G_2^{-1} \wedge (t\,\mathrm{d}\,\bar t H^{-1} + t\bar t\bar\partial H^{-1})G_2^{-1}H^{-1}\\ &- (\bar t\,\mathrm{d}\,t + t\bar t\partial H' H'^{-1})G_2^{-1} \wedge \bar\partial H^{-1}.\end{aligned} \tag{4.45}$$

Thus, the bound for the entries of $\bar\partial(\partial G_1 G_1^{-1})$ follows again by lemma 4.41 and the assumptions on H and H'.
Finally, the case of $\partial\bar\partial(\partial G_1 G_1^{-1})$ follows from the formula

$$\partial\bar\partial(\partial G_1 G_1^{-1}) = -\bar\partial(\partial G_1 G_1) \wedge \partial G_1 G_1^{-1} + \partial G_1 G_1^{-1} \wedge \bar\partial(\partial G_1 G_1). \tag{4.46}$$

□

As a direct consequence of the previous lemma, we obtain that the functions $f_{I,J,a,b}$ of lemma 4.39 have singularities of type $(a+b, I, J)$ of order 0. But we have to show that they have singularities of type $(a+b, I, J)$ of infinite order. Thus, we have to bound all of their derivatives. As before, it is enough to bound the derivatives of the components of the entries of the matrix $\bar\partial(\partial G_1 G_1^{-1})$. By the formulas (4.43) and (4.45), it is enough to bound the derivatives of the entries of the matrices G_1^{-1} and G_2^{-1}. The idea to accomplish this task is to use induction, because the derivatives of these matrices can be written in terms of the same matrices and the derivatives of H and H', which we can control. The inductive step is provided by the next lemmas.

Lemma 4.47. *If the entries of the matrices G_1^{-1} and G_2^{-1} have singularities of type $(0,\emptyset,\emptyset)$ and $(2,\emptyset,\emptyset)$, respectively, of order K, then, for every $i = 1,\dots,d$, the entries of $\frac{\partial}{\partial z_i}G_1 G_1^{-1}$ have singularities of type $(0,\{i\},\emptyset)$ of order K and the entries of $\frac{\partial}{\partial t}G_1 G_1^{-1}$ have singularities of type $(1,\emptyset,\emptyset)$ of order K.*

Proof. The result is a consequence of the formulas

$$\frac{\partial}{\partial z_i} G_1 G_1^{-1} = \frac{\partial}{\partial z_i} H G_1^{-1} + t\bar{t}\left(\frac{\partial}{\partial z_i} H' H'^{-1}\right) G_2^{-1} H^{-1}, \tag{4.48}$$

$$\frac{\partial}{\partial t} G_1 G_1^{-1} = \bar{t}\,\mathrm{d}\,t G_2^{-1} H^{-1}, \tag{4.49}$$

which follow from equation (4.43). □

Lemma 4.50. *If the entries of the matrix G_1^{-1} have singularities of type $(0, \emptyset, \emptyset)$ of order K for all $i = 1, \dots, d$, the entries of the matrix $\frac{\partial}{\partial z_i} G_1 G_1^{-1}$ have singularities of type $(0, \{i\}, \emptyset)$ of order K, and the entries of the matrix $\frac{\partial}{\partial t} G_1 G_1^{-1}$ have singularities of type $(1, \emptyset, \emptyset)$ of order K, then the entries of the matrix G_1^{-1} have singularities of type $(0, \emptyset, \emptyset)$ of order $K + 1$.*

Proof. The result follows from formulas

$$\frac{\partial}{\partial z_i} G_1^{-1} = -G_1^{-1}\left(\frac{\partial}{\partial z_i} G_1 G_1^{-1}\right), \qquad \frac{\partial}{\partial \bar{z}_i} G_1^{-1} = -\left(\frac{\partial}{\partial z_i} G_1 G_1^{-1}\right)^{+} G_1^{-1},$$
$$\frac{\partial}{\partial t} G_1^{-1} = -G_1^{-1}\left(\frac{\partial}{\partial t} G_1 G_1^{-1}\right), \qquad \frac{\partial}{\partial \bar{t}} G_1^{-1} = -\left(\frac{\partial}{\partial t} G_1 G_1^{-1}\right)^{+} G_1^{-1}.$$

□

Lemma 4.51. *If the entries of the matrices G_1^{-1} and G_2^{-1} have singularities of type $(0, \emptyset, \emptyset)$ and $(2, \emptyset, \emptyset)$, respectively, of order K, then the entries of the matrix G_2^{-1} have singularities of type $(2, \emptyset, \emptyset)$ of order $K + 1$.*

Proof. This result is consequence of the equations

$$\begin{aligned}
\frac{\partial}{\partial z_i} G_2^{-1} &= -t\bar{t} G_2^{-1} \frac{\partial}{\partial z_i} H^{-1} G_2^{-1} + H G_1^{-1}\left(\frac{\partial}{\partial z_i} H' H'^{-1}\right) G_2^{-1},\\
\frac{\partial}{\partial \bar{z}_i} G_2^{-1} &= -t\bar{t} G_2^{-1} \frac{\partial}{\partial \bar{z}_i} H^{-1} G_2^{-1} + G_2^{-1}\left(\frac{\partial}{\partial z_i} H' H'^{-1}\right)^{+} G_1^{-1} H\\
\frac{\partial}{\partial t} G_2^{-1} &= -G_2^{-1}(\bar{t}\,\mathrm{d}\,t H^{-1}) G_2^{-1},\\
\frac{\partial}{\partial \bar{t}} G_2^{-1} &= -G_2^{-1}(t\,\mathrm{d}\,\bar{t} H^{-1}) G_2^{-1}.
\end{aligned}$$

□

Summing up lemmas 4.41, 4.42, 4.47, 4.50, 4.51 and equations (4.43), (4.44), (4.45), (4.46), we obtain

Lemma 4.52. *Let $\sum \psi_{I,J,a,b}\,\mathrm{d}\,z^I \wedge \mathrm{d}\,\bar{z}^J \wedge \mathrm{d}\,t^a \wedge \mathrm{d}\,\bar{t}^b$ be the decomposition into monomials of an entry of any of the matrices $\partial G_1 G_1^{-1}$, $\bar{\partial}(\partial G_1 G_1^{-1})$, $\partial(\partial G_1 G_1^{-1})$, and $\partial\bar{\partial}(\partial G_1 G_1^{-1})$. Then, $\psi_{I,J,a,b}$ has singularities of type $(a + b, I, J)$ of infinite order.* □

End of proof of lemma 4.39. This finishes the proof of lemma 4.39. □

Once we have bounded the components of $\phi(\mathrm{tr}_1(h,h'))$ over $V\times\mathbb{P}^1_-$, in order to bound the components of $\phi_-(h,h')$, we have to estimate the integral (4.37).

Lemma 4.53. *Let $0\leq a\leq 1$ be a real number. Then, we have*

$$\int_0^1 \frac{\log(1/r)}{r+a}\,\mathrm{d}\,r \leq 1+\log(1/a)+\frac{1}{2}\log^2(1/a),$$
$$\int_0^1 \frac{r\log(1/r)}{(r+a)^2}\,\mathrm{d}\,r \leq 1+\log(1/a)+\frac{1}{2}\log^2(1/a).$$

Proof. We have the following estimates

$$\begin{aligned}\int_0^1 \frac{r\log(1/r)}{(r+a)^2}\,\mathrm{d}\,r &\leq \int_0^1 \frac{\log(1/r)}{r+a}\,\mathrm{d}\,r\\ &= \int_0^a \frac{\log(1/r)}{r+a}\,\mathrm{d}\,r+\int_a^1 \frac{\log(1/r)}{r+a}\,\mathrm{d}\,r\\ &\leq \int_0^a \frac{\log(1/r)}{a}\,\mathrm{d}\,r+\int_a^1 \frac{\log(1/r)}{r}\,\mathrm{d}\,r\\ &= \left.\frac{r\log(1/r)+r}{a}\right|_0^a-\left.\frac{1}{2}\log^2(1/r)\right|_a^1\\ &= \log(1/a)+1+\frac{1}{2}\log^2(a).\end{aligned}$$

□

We are now in position to bound the components of $\phi_-(h,h')$. Let

$$\phi_-(h,h')=\sum_{I,J} g_{I,J}\,\mathrm{d}\,z^I\wedge\mathrm{d}\,\bar{z}^J$$

be the decomposition of $\phi_-(h,h')$ into monomials. Then, using lemma 4.39 and lemma 4.53, we have

$$\begin{aligned}|g_{I,J}| &= \left|\frac{1}{4\pi i}\int_{\mathbb{P}^1_-} f_{I,J,1,1}\log(t\bar{t})\,\mathrm{d}\,t\wedge\mathrm{d}\,\bar{t}\right|\\ &\prec \frac{\left|\prod_{i=1}^k \log(\log(1/r_i))\right|^N}{r^{(\gamma^I+\gamma^J)^{\leq k}}(\log(1/r))^{(\gamma^I+\gamma^J)^{\leq k}}}\cdot\\ &\qquad\cdot\int_{\mathbb{P}^1_-}\left(\frac{1}{|t|+(\prod_{i=1}^k\log(1/r_i))^{-N}}\right)^2\log(t\bar{t})\,\mathrm{d}\,t\wedge\mathrm{d}\,\bar{t}\\ &\prec \frac{\left|\prod_{i=1}^k \log(\log(1/r_i))\right|^{N'}}{r^{(\gamma^I+\gamma^J)^{\leq k}}(\log(1/r))^{(\gamma^I+\gamma^J)^{\leq k}}}.\end{aligned}$$

The derivatives of $g_{I,J}$ are bounded in the same way using the theorem of taking derivatives under the integral sign. The components of $\partial\phi_-(h,h')$ and $\bar\partial\phi_-(h,h')$ and their derivatives are bounded in a similar way using that

$$\partial\phi_-(h,h') = \frac{-1}{4\pi i}\int_{\mathbb{P}^1_-} \phi(\operatorname{tr}_1(h,h')) \wedge \frac{\mathrm{d}\,t}{t},$$

and

$$\bar\partial\phi_-(h,h') = \frac{-1}{4\pi i}\int_{\mathbb{P}^1_-} \phi(\operatorname{tr}_1(h,h'))\frac{\mathrm{d}\,\bar t}{\bar t}.$$

To bound the components of $\phi_+(h,h')$, $\partial\phi_+(h,h')$ and $\bar\partial\phi_+(h,h')$ and their derivatives, we will use the same technique. Let $s=1/t$ be a local coordinate in $\mathbb{P}^1_+$. In these coordinates, we have

$$G = \frac{1}{1+s\bar s}(H' + s\bar s H).$$

We write

$$G_3 = (H'+s\bar s H), \qquad G_4 = (H^{-1} + s\bar s H'^{-1}).$$

In this case, using the adequate variant of definition 4.38, the analogue of lemma 4.41 is

Lemma 4.54. *(i) The entries of the matrix G_3^{-1} have singularities of type $(2,\emptyset,\emptyset)$ of order* 0. *Therefore, the entries of the matrices $s\,G_3^{-1}$ and $\bar s\,G_3^{-1}$ have singularities of type $(1,\emptyset,\emptyset)$ of order* 0 *and the entries of the matrix $t\bar s\,G_3^{-1}$ are bounded.*

(ii) The entries of the matrix G_4^{-1} are bounded. In particular, they have singularities of type $(0,\emptyset,\emptyset)$ of order 0.

□

Note that the bounds for G_3^{-1} and G_4^{-1} are not the same as the bounds for G_1^{-1} and G_2^{-1}, but they are switched. To bound the entries of $\partial G_3 G_3^{-1}$, we use

$$\partial G_3 G_3^{-1} = \partial H' H'^{-1} G_4^{-1} H^{-1} + \bar s \,\mathrm{d}\, s H G_3^{-1} + s\bar s \partial H G_3^{-1}.$$

We leave the remaining details to the reader.
Finally, to bound $\partial\bar\partial\phi(h,h')$, we use equation (4.12). This completes the proof of the first statement.
We now prove the second statement. By definition, we have

$$\operatorname{tr}_2(h,h',h'') = \operatorname{tr}_1(0 \longrightarrow \operatorname{tr}_1(h,h') \longrightarrow \operatorname{tr}_1(h'',h'')).$$

But $\operatorname{tr}_1(h,h')$ is a smooth hermitian vector bundle on $X\times\mathbb{P}^1$ and $\operatorname{tr}_1(h'',h'')$ is isometric to $p_1^*(E,h'')$ and, in consequence, log-singular along $D\times\mathbb{P}^1$. Therefore, we can apply lemma 4.39 to $\phi(\operatorname{tr}_2(h,h',h''))$; by lemma 4.53, the form $\phi(h,h',h'')$ has log-log growth of infinite order. To conclude that it is a log-log form we still have to control $\partial\phi(h,h',h'')$, $\bar{\partial}\phi(h,h',h'')$, and $\partial\bar{\partial}\phi(h,h',h'')$. A residue computation shows

$$\begin{aligned}\partial\phi(h,h',h'') =& \frac{1}{2}(\phi(h,h')+\phi(h',h'')+\phi(h'',h))\\ &+\frac{2}{(4\pi i)^2}\int_{\mathbb{P}^1\times\mathbb{P}^1}\phi(\mathrm{tr}_2(h,h',h''))\wedge\frac{\mathrm{d}\,t_1}{t_1}\wedge\frac{\mathrm{d}\,t_2}{t_2}\\ &-\frac{1}{(4\pi i)^2}\int_{\mathbb{P}^1\times\mathbb{P}^1}\phi(\mathrm{tr}_2(h,h',h''))\wedge\left(\frac{\mathrm{d}\,\bar{t}_1}{\bar{t}_1}\wedge\frac{\mathrm{d}\,t_2}{t_2}+\frac{\mathrm{d}\,t_1}{t_1}\wedge\frac{\mathrm{d}\,\bar{t}_2}{\bar{t}_2}\right),\end{aligned}$$

and

$$\begin{aligned}\bar{\partial}\phi(h,h',h'') =& \frac{1}{2}(\phi(h,h')+\phi(h',h'')+\phi(h'',h))\\ &-\frac{2}{(4\pi i)^2}\int_{\mathbb{P}^1\times\mathbb{P}^1}\phi(\mathrm{tr}_2(h,h',h''))\wedge\frac{\mathrm{d}\,\bar{t}_1}{\bar{t}_1}\wedge\frac{\mathrm{d}\,\bar{t}_2}{\bar{t}_2}\\ &+\frac{1}{(4\pi i)^2}\int_{\mathbb{P}^1\times\mathbb{P}^1}\phi(\mathrm{tr}_2(h,h',h''))\wedge\left(\frac{\mathrm{d}\,\bar{t}_1}{\bar{t}_1}\wedge\frac{\mathrm{d}\,t_2}{t_2}+\frac{\mathrm{d}\,t_1}{t_1}\wedge\frac{\mathrm{d}\,\bar{t}_2}{\bar{t}_2}\right).\end{aligned}$$

Hence, again by lemma 4.53, the forms $\partial\phi(h,h',h'')$ and $\bar{\partial}\phi(h,h',h'')$ have log-log growth of infinite order. Finally, since

$$\partial\bar{\partial}\phi(h,h',h'') = (\partial-\bar{\partial})(\phi(h,h')+\phi(h',h'')+\phi(h'',h))$$

by the first statement, the form $\partial\bar{\partial}\phi(h,h',h'')$ also has log-log growth of infinite order; therefore, $\phi(h,h',h'')$ is a log-log form.

End of proof of theorem 4.36. This finishes the proof of theorem 4.36. □

Bott-Chern forms for good hermitian metrics. All the theory we have developed so far is also valid for good hermitian vector bundles with the obvious changes. For instance, if the hermitian vector bundle is good instead of log-singular, we obtain that the Bott-Chern forms are pre-log-log instead of log-log.

Theorem 4.55. *Let X be a complex manifold and let D be a normal crossing divisor. Put $U = X\setminus D$. Let E be a vector bundle on X. If h and h' are smooth hermitian metrics on E and h'' is a smooth hermitian metric on $E|_U$, which is good along D, then the Bott-Chern form $\phi(E,h,h'')$ and the iterated Bott-Chern form $\phi(E,h,h',h'')$ are pre-log-log forms.*

Proof. Observe that lemma 4.40, lemma 4.41, and lemma 4.42 are true in the case of good hermitian metrics by lemma 4.26, and these results are enough to prove that $\phi(E,h,h')$ and $\phi(E,h,h',h'')$ are pre-log-log forms by the same arguments as before. □

The singularities of the first transgression bundle. With the notation of theorem 4.36, observe that the hermitian vector bundle $\mathrm{tr}_1(E, h, h')$ need not be log-singular along the divisor $D \times \mathbb{P}^1$ (see remark 4.32). Nevertheless, as we will see in the following results, it is close to be log-singular. For instance, it is log-singular along $D \times \mathbb{P}^1 \cup X \times \{(0:1), (1:0)\}$, or it can be made a log-singular hermitian vector bundle after some blow-ups. This second statement will be useful in the axiomatic characterization of Bott-Chern classes.

Lemma 4.56. *Let a, b be real numbers with $a > 0$ and $b > e^{1/e}$. Then, we have*

$$\frac{\log(a)}{\log(b)} < 1 + \frac{a}{b}.$$

Proof. If $b \geq a$, then the statement is obvious. If $a > b$, we write $a = cb$ with $c > 1$. Then, the inequality of the lemma is equivalent to

$$\frac{\log(c)}{c} < \log(b).$$

But the function $\log(c)/c$ is a bounded function that has a maximum at $c = e$ with value $1/e$. Therefore, the result is a consequence of the condition on b. □

Corollary 4.57. *With the notation of theorem 4.36, the first transgression hermitian vector bundle $\mathrm{tr}_1(E, h, h')$ is log-singular along the divisor*

$$D \times \mathbb{P}^1 \cup X \times \{(0:1), (1:0)\}.$$

Proof. The first condition of definition 4.29 is easy to prove. We will prove the second condition. Lemma 4.56 implies that, for $a, b \gg 0$, the inequality

$$\frac{1}{1/a + 1/b} < \frac{\log(b)}{(1/a)\log(a)} \tag{4.58}$$

holds. Applying this equation to $a = 1/|t|$ and $b = (\prod_{i=1}^{k} \log(1/r_i))^N$, we obtain

$$\frac{1}{|t| + (\prod_{i=1}^{k} \log(1/r_i))^{-N}} < \frac{\log((\prod_{i=1}^{k} \log(1/r_i))^N)}{|t|\log(1/|t|)} \prec \frac{\sum_{i=1}^{k} \log(\log(1/r_i))}{|t|\log(1/|t|)}.$$

Therefore, lemma 4.52 implies that on $V \times \mathbb{P}^1_-$ the entries of $\partial G G^{-1}$ are log-log along $D \times \mathbb{P}^1 \cup X \times \{(0:1), (1:0)\}$. The proof for the bound on $V \times \mathbb{P}^1_+$ is analogous. □

Proposition 4.59. *With the same hypothesis of theorem 4.36, let $D = D_1 \cup \ldots \cup D_n$ be the decomposition of D in smooth irreducible components. Let $\widetilde{Z}$ be the variety obtained from $X \times \mathbb{P}^1$ by blowing-up $D_1 \times (1:0)$ and then, successively, the strict transforms of $D_2 \times (1:0), \ldots, D_n \times (1:0), D_1 \times (0:1), \ldots, D_n \times (0:1)$. Let $\pi : \widetilde{Z} \longrightarrow X \times \mathbb{P}^1$ be the morphism induced by the blow-ups and let $C \subseteq \widetilde{Z}$ be the pre-image by π of $D \times \mathbb{P}^1$. Then, we have* *(i)* *C is a normal crossing divisor.*

(ii) *The closed immersions $i_0, i_\infty : X \longrightarrow X \times \mathbb{P}^1$, given by*

$$i_0(p) = (p, (0:1)), \qquad i_\infty(p) = (p, (1:0)),$$

can be lifted to closed immersions $j_0, j_\infty : X \longrightarrow \widetilde{Z}$.

(iii) *The hermitian vector bundle $\pi^* \operatorname{tr}_1(E, h, h')$ is log-singular along the divisor C.*

Proof. The first statement is obvious and the second is a direct consequence of the universal property of the blow-up and the fact that the intersection of the center of every blow-up with the transform of $X \times (1:0)$ or $X \times (0:1)$ is either empty or a divisor.

To prove the third statement, we will use the same notations as in the proof of theorem 4.36. Let U be the subset of $V \times \mathbb{P}^1_-$, where $|t| < 1/e^e$. For simplicity, we assume that the components of D meeting V are $D_1, \dots, D_k$ and that the component D_i has equation $z_i = 0$. Then, U, with coordinates $(z_1, \dots, z_d, t)$, is a coordinate neighborhood adapted to $D \times \mathbb{P}^1$. The open subset $\pi^{-1}(U)$ can be covered by $k+1$ coordinate neighborhoods, denoted by $\widetilde{U}_1, \dots, \widetilde{U}_{k+1}$. The coordinates of these subsets, the expression of π and the equation of C in these coordinates are given in the following table:

Subset	Coordinates	π	
U_1	$(u, x_1, \dots, x_n)$	$t = u$ $z_1 = ux_1$ $z_i = x_i \ (i \neq 1)$	$ux_1 \cdots x_k = 0$
U_j $(1 < j < k+1)$	$(u, x_1, \dots, x_n)$	$t = ux_1 \cdots x_{j-1}$ $z_j = ux_j$ $z_i = x_i \ (i \neq j)$	$ux_1 \cdots x_k = 0$
U_{k+1}	$(u, x_1, \dots, x_n)$	$t = ux_1 \cdots x_k$ $z_i = x_i \ (i = 1, \dots, d)$	$x_1 \cdots x_k = 0$

Since, for $j = 1, \dots, k$, we have

$$\pi^{-1}(D \times \mathbb{P}^1 \cup X \times \{(0:1), (1:0)\}) \cap U_j = C \cap U_j,$$

we know by corollary 4.57 and the functoriality of log-singular metrics that the hermitian vector bundle $\pi^* \operatorname{tr}_1(E, h, h')|_{U_j}$ is log-singular. Hence, we only have to prove that $\pi^* \operatorname{tr}_1(E, h, h')|_{U_{k+1}}$ is log-singular. The first condition of definition 4.29 follows easily from the definition of the metric g. To prove the second condition of definition 4.29, we can proceed in two ways. The first method is to derive this result directly from lemma 4.52 applying the chain rule. But, since we have to bound all derivatives, this is a notational nightmare. The second method is to bound the derivatives inductively mimicking the proof of lemma 4.53. To this end, instead of lemma 4.41, we use the following substitute.

Lemma 4.60. *(i) The entries of the matrix $\pi^* G_1^{-1}|_{U_{k+1}}$ are bounded in every compact subset of U_{k+1}. In particular, they are $(\emptyset,\emptyset)$-log-log growth functions of order* 0 *(see definition 2.21).*

(ii) If ψ is an entry of the matrix G_2^{-1}, then we have

$$|(\pi^*\psi|_{U_{k+1}})(x_1,\dots,x_d,u)| \prec \left|\prod_{i=1}^{k}\log(1/|x_i|)\right|^N$$

for some integer N. Therefore, $\pi^*(t\psi)$ and $\pi^*(\bar{t}\psi)$ are bounded in any compact subset of U_{k+1} and, for $i=1,\dots,k$, the function

$$\prod_{j\neq i}|x_j|\pi^*\psi$$

is a $(\{i\},\emptyset)$-log-log growth function of order 0.

□

We leave it to the reader to make explicit the analogues of lemmas 4.47, 4.50, and 4.51 in this case.
The proof that it is also log-singular in the pre-image of an open subset of $\mathbb{P}^1_+$ is analogous. □

Chern forms for log-singular hermitian bundles.

Proposition 4.61. *Let X be a complex projective manifold, D a normal crossing divisor of X, (E,h) a hermitian vector bundle log-singular along D. Let ϕ be any symmetric power series. Then, the Chern form $\phi(E,h)$ represents the Chern class $\phi(E)$ in $H^*_{\mathcal{D}}(X,\mathbb{R}(*))$.*

Proof. By theorem 2.42 and theorem 3.5, the inclusion

$$\mathcal{D}^*(E_X,*) \longrightarrow \mathcal{D}^*(E_X\langle\langle D\rangle\rangle,*)$$

is a quasi-isomorphism. Moreover, if h' is a smooth hermitian metric on E, then, in the complex $\mathcal{D}^*(E_X\langle\langle D\rangle\rangle,*)$, we have

$$\phi(E,h)-\phi(E,h') = \mathrm{d}_{\mathcal{D}}\,\phi(E,h',h).$$

Therefore, both forms represent the same class. □

Bott-Chern classes.

Definition 4.62. Let X be a complex manifold and D a normal crossing divisor. Let

$$\overline{\mathcal{E}}: 0\longrightarrow (E',h')\longrightarrow (E,h)\longrightarrow (E'',h'')\longrightarrow 0$$

be an exact sequence of hermitian vector bundles log-singular along D. Let h'_s, h_s, and h''_s be smooth hermitian metrics on E', E, and E'', respectively. We denote by $\overline{\mathcal{E}}_s$ the corresponding exact sequence of smooth vector bundles. Let ϕ be a symmetric power series. Then, the *Bott-Chern class* associated to $\overline{\mathcal{E}}$ is the class represented by

$$\phi(\overline{\mathcal{E}}_s) + \phi(E' \oplus E'', h'_s \oplus h''_s, h' \oplus h') - \phi(E, h_s, h)$$

in the group

$$\bigoplus_k \widetilde{\mathcal{D}}^{2k-1}(E_X\langle\langle D\rangle\rangle, k) = \bigoplus_k \mathcal{D}^{2k-1}(E_X\langle\langle D\rangle\rangle, k) \,/\mathrm{d}_{\mathcal{D}}\, \mathcal{D}^{2k-2}(E_X\langle\langle D\rangle\rangle, k)\ .$$

This class is denoted by $\widetilde{\phi}(\overline{\mathcal{E}})$.

Proposition 4.63. *The Bott-Chern classes are well defined.*

Proof. The fact that the Bott-Chern forms belong to the group

$$\bigoplus_k \mathcal{D}^{2k-1}(E_X\langle\langle D\rangle\rangle)$$

is proven in theorem 4.36.
Let h'_{sa}, h_{sa} and h''_{sa} be another choice of smooth hermitian metrics and let $\overline{\mathcal{E}}_{sa}$ be the corresponding exact sequence. We denote by $\overline{\mathcal{C}}$ the exact square of smooth hermitian vector bundles

$$0 \longrightarrow 0 \longrightarrow \overline{\mathcal{E}}_{sa} \longrightarrow \overline{\mathcal{E}}_s \longrightarrow 0.$$

Then, we have

$$\begin{aligned}
\phi(\overline{\mathcal{E}}_s) + \phi(E' \oplus E'', h'_s \oplus h''_s, h' \oplus h') - \phi(E, h_s, h)&\\
- \phi(\overline{\mathcal{E}}_{sa}) - \phi(E' \oplus E'', h'_{sa} \oplus h''_{sa}, h' \oplus h') + \phi(E, h_{sa}, h) &=\\
\mathrm{d}_{\mathcal{D}}\, \phi(\overline{\mathcal{C}}) - \mathrm{d}_{\mathcal{D}}\, \phi(E' \oplus E'', h'_s \oplus h''_s, h'_{sa} \oplus h''_{sa}, h' \oplus h'') + \mathrm{d}_{\mathcal{D}}\, \phi(E, h_s, h_{sa}, h).&
\end{aligned}$$

Therefore, the Bott-Chern classes do not depend on the choice of the smooth metrics. □

Axiomatic characterization of Bott-Chern classes.

Theorem 4.64. *The Bott-Chern classes satisfy the following properties. If X is a complex manifold, D is a normal crossing divisor, and*

$$\overline{\mathcal{E}} : 0 \longrightarrow (E', h') \longrightarrow (E, h) \longrightarrow (E'', h'') \longrightarrow 0$$

is a short exact sequence of hermitian vector bundles, log-singular along D, then we have *(i)* $\mathrm{d}_{\mathcal{D}}\,\widetilde{\phi}(\overline{\mathcal{E}}) = \phi(E' \oplus E'', h' \oplus h'') - \phi(E, h)$.

(ii) *If* $(E, h) = (E' \oplus E'', h' \oplus h'')$, *then* $\widetilde{\phi}(\overline{\mathcal{E}}) = 0$.

(iii) *If* X' *is another complex manifold,* D' *is a normal crossing divisor in* X', *and* $f : X' \longrightarrow X$ *is a holomorphic map such that* $f^{-1}(D) \subseteq D'$, *then* $\widetilde{\phi}(f^{*}\overline{\mathcal{E}}) = f^{*}\widetilde{\phi}(\overline{\mathcal{E}})$.

(iv) *If* $\overline{\mathcal{F}}$ *is a hermitian exact square of vector bundles on* X, *log-singular along* D, *then*

$$\widetilde{\phi}(\partial_1^{-1}\overline{\mathcal{F}} \oplus \partial_1^{1}\overline{\mathcal{F}}) - \widetilde{\phi}(\partial_1^{0}\overline{\mathcal{F}}) - \widetilde{\phi}(\partial_2^{-1}\overline{\mathcal{F}} \oplus \partial_2^{1}\overline{\mathcal{F}}) + \widetilde{\phi}(\partial_2^{0}\overline{\mathcal{F}}) = 0.$$

Moreover, these properties determine the Bott-Chern classes.

Proof. First we prove the unicity. By [17], 1.3.2 (see also [36], IV.3.1) properties (1) to (3) characterize the Bott-Chern classes in the case $D = \emptyset$. By functoriality, the Bott-Chern classes are determined for short exact sequences, when the three metrics are smooth. Let E be a vector bundle, h a smooth hermitian metric on E and h' a hermitian metric log-singular along D. The vector bundle $\widetilde{E} = \mathrm{tr}_1(E, h, h')$ over $X \times \mathbb{P}^1$ is isomorphic (as a vector bundle) to $p_1^{*}E$. Let h_1 be the hermitian metric on $\widetilde{E}$ induced by h and this isomorphism. Then, h_1 is a smooth hermitian metric. Let h_2 be the metric of definition 4.4. Let $\pi : Z \longrightarrow X \times \mathbb{P}^1$ and C be as in proposition 4.59. By this proposition $\pi^{*}(\widetilde{E}, h_2)$ is log-singular along C. Therefore, we can assume the existence of the Bott-Chern class $\widetilde{\phi}(\widetilde{E}, h_1, h_2)$. Write $\pi' = p_1 \circ \pi$. We consider the integral

$$I = -\frac{1}{2\pi i}\int_{\pi'} -2\partial\bar{\partial}\widetilde{\phi}(\widetilde{E}, h_1, h_2)\pi^{*}(\frac{1}{2}\log(t\bar{t})).$$

By property (1), we have

$$\begin{aligned} I &= -\frac{1}{2\pi i}\int_{\pi'} \phi(\widetilde{E}, h_2)\pi^{*}(\frac{1}{2}\log(t\bar{t})) + \frac{1}{2\pi i}\int_{\pi'} \phi(\widetilde{E}, h_1)\pi^{*}(\frac{1}{2}\log(t\bar{t})) \\ &= -\frac{1}{2\pi i}\int_{\mathbb{P}^1} \phi(\mathrm{tr}_1(E, h, h'))\frac{1}{2}\log(t\bar{t}), \end{aligned}$$

because the second integral vanishes. But using Stokes theorem and properties (2) and (3) as in [17], 1.3.2, or [36], IV.3.1, we get

$$\begin{aligned} I &\sim j_{\infty}^{*}\widetilde{\phi}(\widetilde{E}, h_1, h_2) - j_0^{*}\widetilde{\phi}(\widetilde{E}, h_1, h_2) \\ &= \widetilde{\phi}(E, h, h') - \widetilde{\phi}(E, h, h) \\ &= \widetilde{\phi}(E, h, h'), \end{aligned}$$

where the symbol $\sim$ means equality up to the image of $\mathrm{d}_{\mathcal{D}}$. Therefore, the class $\widetilde{\phi}(E, h, h')$ is also determined by properties (1) to (3). Finally, for an arbitrary exact sequence $\overline{\mathcal{E}}$ of hermitian vector bundles log-singular along D, property (4) implies that $\widetilde{\phi}(\overline{\mathcal{E}})$ is given by definition 4.62.

Next we prove the existence. By proposition 4.63 it only remains to show that the Bott-Chern classes defined by 4.62 satisfy properties (1) to (4). Property (1) is known for smooth metrics. If E is a vector bundle, h is a smooth hermitian metric and h' a hermitian metric log-singular along D, then, since two differential forms that agree in an open dense subset are equal, by the smooth case

$$\mathrm{d}_{\mathcal{D}}\,\widetilde{\phi}(E,h,h') = \phi(E,h') - \phi(E,h).$$

The general case follows from these two cases. Property (2) follows directly from the case of smooth metrics and definition 4.62. Property (3) is obvious from the functoriality of the definition. To prove property (4), we consider $\overline{\mathcal{F}}'$, an exact square with the same vector bundles as $\overline{\mathcal{F}}$, but with smooth metrics. Then, if we use the definition of Bott-Chern classes in the expression

$$\widetilde{\phi}(\partial_1^{-1}\overline{\mathcal{F}} \oplus \partial_1^1\overline{\mathcal{F}}) - \widetilde{\phi}(\partial_1^0\overline{\mathcal{F}}) - \widetilde{\phi}(\partial_2^{-1}\overline{\mathcal{F}} \oplus \partial_2^1\overline{\mathcal{F}}) + \widetilde{\phi}(\partial_2^0\overline{\mathcal{F}}) = 0,$$

all the change of metric terms appear twice with opposite sign. Therefore, this property follows from the smooth case. $\square$

Real varieties. The following result follows easily.

Proposition 4.65. *Let $X_{\mathbb{R}} = (X, F_\infty)$ be a real variety, D a normal crossing divisor on $X_{\mathbb{R}}$, E a complex vector bundle defined over $\mathbb{R}$, h, h' (resp. h") smooth hermitian metrics (resp. log-singular hermitian metric) on E invariant under complex conjugation. Then, the forms $\phi(E,h'')$, $\widetilde{\phi}_1(E,h,h'')$ and $\widetilde{\phi}_2(E,h,h',h'')$ belong to the group*

$$\bigoplus_k \mathcal{D}^{2k-1}(E_{X_{\mathbb{R}}}\langle\langle D\rangle\rangle, k) = \bigoplus_k \mathcal{D}^{2k-1}(E_{X_{\mathbb{R}}}\langle\langle D\rangle\rangle, k)^{\sigma},$$

where σ is the involution that acts as complex conjugation on the space and on the coefficients. $\square$

5 Arithmetic K-theory of log-singular hermitian vector bundles

The arithmetic intersection theory of Gillet and Soulé is complemented by an arithmetic K-theory and a theory of characteristic classes. In this section we will generalize both theories to cover the kind of singular hermitian metrics that appear naturally when considering (fully decomposed) automorphic vector bundles. If E be a vector bundle over a quasi-projective complex manifold X, then a hermitian metric h on E may have arbitrary singularities near the boundary of X. Therefore, the associated Chern forms will also have arbitrary singularities "at infinity". Thus, in order to define arithmetic characteristic classes for this kind of hermitian vector bundles, we are led to use the complex $\mathcal{D}_{\mathrm{l,ll,a}}$.

5.1 Arithmetic Chern classes of log-singular herm. vector bundles

Arithmetic Chow groups with coefficients. Let A be an arithmetic ring. Let $\widehat{X} = (X, \mathcal{C})$ be a $\mathcal{D}_{\log}$-arithmetic variety over A. Let B be a subring of $\mathbb{R}$. We will define the arithmetic Chow groups of $\widehat{X}$ with coefficients in B using the same method as in [3]. We follow the notations of [10], §4.2.
For an integer $p \geq 0$, let $\mathrm{Z}^p(X)_B = \mathrm{Z}^p(X) \otimes B$ be the group of algebraic cycles of X with coefficients in B. Then, the *group of p-codimensional arithmetic cycles of $\widehat{X} = (X, \mathcal{C})$ with coefficients in B* is given by

$$\widehat{\mathrm{Z}}^p_B(X, \mathcal{C}) = \left\{ (y, \mathfrak{g}_y) \in \mathrm{Z}^p_B(X) \oplus \widehat{H}^{2p}_{\mathcal{C}, \mathcal{Z}^p}(X, p) \,\middle|\, \mathrm{cl}(y) = \mathrm{cl}(\mathfrak{g}_y) \right\}.$$

Let $\widehat{\mathrm{Rat}}^p_B(X, \mathcal{C})$ be the B-submodule of $\widehat{\mathrm{Z}}^p_B(X, \mathcal{C})$ generated by $\widehat{\mathrm{Rat}}^p(X, \mathcal{C})$. We define *the p-th arithmetic Chow group of $\widehat{X} = (X, \mathcal{C})$ with coefficients in B* by

$$\widehat{\mathrm{CH}}^p_B(X, \mathcal{C}) = \widehat{\mathrm{Z}}^p_B(X, \mathcal{C}) \big/ \widehat{\mathrm{Rat}}^p_B(X, \mathcal{C}).$$

There is a canonical morphism

$$\widehat{\mathrm{CH}}^p_B(X, \mathcal{C}) \longrightarrow \widehat{\mathrm{CH}}^p(X, \mathcal{C}) \otimes B.$$

For instance, if $B = \mathbb{Q}$, this morphism an isomorphism, but in general, if $B = \mathbb{R}$, it is not an isomorphism.

The main theorem. Let X be a regular scheme, flat and quasi-projective over $\mathrm{Spec}(A)$. Let D be a normal crossing divisor on $X_{\mathbb{R}}$. Write $\underline{X} = (X, D)$. Then, $(X, \mathcal{D}_{\mathrm{l,ll,a},\underline{X}})$ is a quasi-projective $\mathcal{D}_{\log}$-arithmetic variety over A. A log-singular hermitian vector bundle over X is a vector bundle E over X together with a metric on E_∞, which is smooth over $X_\infty \setminus D_\infty$, log-singular along D_∞, and invariant under complex conjugation.

Theorem 5.1. *Let $\phi \in B[[T_1, \dots, T_n]]$ be a symmetric power series with coefficients in a subring B of $\mathbb{R}$. Then, there is a unique way to attach to every log-singular hermitian vector bundle $\overline{E} = (E, h)$ of rank n over $\underline{X} = (X, D)$ a characteristic class*

$$\widehat{\phi}(\overline{E}) \in \widehat{\mathrm{CH}}^*_B(X, \mathcal{D}_{\mathrm{l,ll,a}})$$

having the following properties:

(i) Functoriality. *When $f : Y \longrightarrow X$ is a morphism of regular schemes, flat and quasi-projective over A, D' a normal crossing divisor on $Y_{\mathbb{R}}$ with $f^{-1}(D) \subseteq D'$, and $\overline{E}$ a log-singular hermitian vector bundle on X, then*

$$f^*(\widehat{\phi}(\overline{E})) = \widehat{\phi}(f^*\overline{E}).$$

(ii) Normalization. *When $\overline{E}=\overline{L}_1\oplus\dots\oplus\overline{L}_n$ is an orthogonal direct sum of hermitian line bundles, then*

$$\widehat{\phi}(\overline{E})=\phi(\widehat{\mathrm{c}}_1(\overline{L}_1),\dots,\widehat{\mathrm{c}}_1(\overline{L}_n)).$$

(iii) Twist by a line bundle. *Let*

$$\phi(T_1+T,\dots,T_n+T)=\sum_{i\geq 0}\phi_i(T_1,\dots,T_n)T^i.$$

Let $\overline{L}$ be a log-singular hermitian line bundle, then

$$\widehat{\phi}(\overline{E}\otimes\overline{L})=\sum_i\widehat{\phi}_i(\overline{E})\,\widehat{\mathrm{c}}_1(\overline{L}).$$

(iv) Compatibility with characteristic forms.

$$\omega(\widehat{\phi}(\overline{E}))=\phi(E,h).$$

(v) Compatibility with the change of metrics. *If h' is another log-singular hermitian metric, then*

$$\widehat{\phi}(E,h)=\widehat{\phi}(E,h')+\mathrm{a}(\widetilde{\phi}_1(E,h',h)).$$

(vi) Compatibility with the definition of Gillet and Soulé. *If D is empty, let ψ be the isomorphism $\widehat{\mathrm{CH}}^*(X,\mathcal{D}_{\mathrm{l,ll,a}})\longrightarrow\widehat{\mathrm{CH}}^*(X)$ of theorem 3.33 and let $\widehat{\phi}_{\mathrm{GS}}(\overline{E})\in\widehat{\mathrm{CH}}^*(X)$ be the characteristic class defined in [17]. Then*

$$\psi(\widehat{\phi}(\overline{E}))=\widehat{\phi}_{\mathrm{GS}}(\overline{E}).$$

Proof. If D is empty, we define $\widehat{\phi}(\overline{E})=\psi^{-1}(\widehat{\phi}_{\mathrm{GS}}(\overline{E}))$. If D is not empty, but h is smooth on the whole of $X_{\mathbb{R}}$, then we define $\widehat{\phi}(\overline{E})$ by functoriality, using the tautological morphism $(X,D)\longrightarrow(X,\emptyset)$.
If D is not empty and $\overline{E}=(E,h)$ is a log-singular hermitian vector bundle, we choose any smooth metric h', invariant under F_∞. Then, we define

$$\widehat{\phi}(\overline{E})=\widehat{\phi}(E,h')+\mathrm{a}(\widetilde{\phi}(E,h',h)).$$

This definition is independent of the choice of the metric h', because, if h'' is another smooth F_∞-invariant metric, then

$$\begin{aligned}\widehat{\phi}(E,h')+{}&\mathrm{a}(\widetilde{\phi}(E,h',h))-\widehat{\phi}(E,h'')-\mathrm{a}(\widetilde{\phi}(E,h'',h))\\&=\mathrm{a}(\widetilde{\phi}(E,h'',h'))+\mathrm{a}(\widetilde{\phi}(E,h',h))+\mathrm{a}(\widetilde{\phi}(E,h,h''))\\&=\mathrm{a}(\mathrm{d}_{\mathcal{D}}(E,h'',h',h))\\&=0.\end{aligned}$$

All the properties stated in the theorem can be checked as in [17]. □

Remark 5.2. If X is projective, the groups $\widehat{\mathrm{CH}}^{*}(X,\mathcal{D}_{\mathrm{l,ll}})$ and $\widehat{\mathrm{CH}}^{*}(X,\mathcal{D}_{\mathrm{l,ll,a}})$ agree. Therefore, the arithmetic characteristic classes also belong to the former group. When X is quasi-projective, in order to define characteristic classes in the group $\widehat{\mathrm{CH}}^{*}(X,\mathcal{D}_{\mathrm{l,ll}})$, we have to impose conditions on the behavior of the hermitian metrics at infinity. For instance, one may consider smooth at infinity hermitian metrics (see [11]).

Remark 5.3. If we replace good hermitian vector bundle by log hermitian vector bundle and pre-log-log forms by log-log forms (implicit in the definition of $\mathcal{D}_{\mathrm{l,ll,a}}$) in theorem 5.1, the result remains true.

5.2 Arithmetic K-theory of log-singular hermitian vector bundles

Log-singular arithmetic K-theory. We want to generalize the definition of arithmetic K-theory given by Gillet and Soulé in [17] to cover log-singular hermitian metrics.
We write

$$\widetilde{\mathcal{D}}_{\mathrm{l,ll,a}}(X)=\bigoplus_{p}\widetilde{\mathcal{D}}^{2p-1}_{\mathrm{l,ll,a}}(X,p),$$

$$\mathrm{Z}\mathcal{D}_{\mathrm{l,ll,a}}(X)=\bigoplus_{p}\mathrm{Z}\mathcal{D}^{2p}_{\mathrm{l,ll,a}}(X,p).$$

Let ch be the power series associated with the Chern character. In particular, it induces Bott-Chern forms $\widetilde{\mathrm{ch}}$ and arithmetic characteristic classes $\widehat{\mathrm{ch}}$.

Definition 5.4. Let $\underline{X}$ be as in theorem 5.1. Then, the group $\widehat{\mathrm{K}}_0(X,\mathcal{D}_{\mathrm{l,ll,a}})$ is the group generated by pairs $(\overline{E},\eta)$, where $\overline{E}$ is a log-singular hermitian metric on X and $\eta\in\widetilde{\mathcal{D}}_{\mathrm{l,ll,a}}(X)$ satisfying the relations

$$(\overline{S},\eta')+(\overline{Q},\eta'')=(\overline{E},\eta'+\eta''+\widetilde{\mathrm{ch}}(\overline{\mathcal{E}}))$$

for every $\eta',\eta''\in\widetilde{\mathcal{D}}_{\mathrm{l,ll,a}}(X)$ and every short exact sequence of log-singular hermitian vector bundles

$$\overline{\mathcal{E}}:0\longrightarrow\overline{S}\longrightarrow\overline{E}\longrightarrow\overline{Q}\longrightarrow 0.$$

If D is empty, then this definition agrees with the definition of Gillet and Soulé in [17].

Basic properties. The following theorem summarizes the basic properties of the arithmetic K-theory groups. They are a consequence of the corresponding results of [17] together with theorem 5.1.

Theorem 5.5. *Let $\underline{X}=(X,D)$ be an arithmetic variety over A with a fixed normal crossing divisor. Then, we have*

(i) *There are natural maps*

$$\begin{aligned}
\mathrm{a} : \widetilde{\mathcal{D}}_{\mathrm{l,ll,a}}(X) &\longrightarrow \widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}}),\\
\mathrm{ch} : \widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}}) &\longrightarrow \mathrm{Z}\mathcal{D}_{\mathrm{l,ll,a}}(X),\\
\mathrm{v} : \widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}}) &\longrightarrow \mathrm{K}_0(X),\\
\widehat{\mathrm{ch}} : \widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}}) &\longrightarrow \bigoplus \widehat{\mathrm{CH}}^p_{\mathbb{Q}}(X, \mathcal{D}_{\mathrm{l,ll,a}}),
\end{aligned}$$

given by

$$\begin{aligned}
\mathrm{a}(\eta) &= (0, \eta),\\
\mathrm{ch}([\overline{E}, \eta]) &= \mathrm{ch}(\overline{E}) + \mathrm{d}_{\mathcal{D}}\, \eta,\\
\mathrm{v}([\overline{E}, \eta]) &= [E],\\
\widehat{\mathrm{ch}}([\overline{E}, \eta]) &= \widehat{\mathrm{ch}}(\overline{E}) + \mathrm{a}(\eta).
\end{aligned}$$

(ii) *The product*

$$(\overline{E}, \eta) \otimes (\overline{E}', \eta') = \Big(\overline{E} \otimes \overline{E}', (\mathrm{ch}(\overline{E}) + \mathrm{d}_{\mathcal{D}}\, \eta) \bullet \eta' + \eta \bullet \mathrm{ch}(\overline{E}')\Big)$$

induces a commutative and associative ring structure on $\widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}})$. *The maps* v, ch, *and* $\widehat{\mathrm{ch}}$ *are compatible with this ring structure.*

(iii) *If* $\underline{Y} = (Y, D')$ *is another arithmetic variety over* A *with a fixed normal crossing divisor and* $f : X \longrightarrow Y$ *is a morphism such that* $f^{-1}(D') \subseteq D$, *then there is a pull-back morphism*

$$f^* : \widehat{\mathrm{K}}_0(Y, \mathcal{D}_{\mathrm{l,ll,a}}) \longrightarrow \widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}}),$$

compatible with the maps a, ch, v *and* $\widehat{\mathrm{ch}}$.

(iv) *There are exact sequences*

$$\mathrm{K}_1(X) \xrightarrow{\rho} \widetilde{\mathcal{D}}_{\mathrm{l,ll,a}}(X) \xrightarrow{\mathrm{a}} \widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}}) \xrightarrow{\mathrm{v}} \mathrm{K}_0(X) \to 0, \tag{5.6}$$

and

$$\begin{aligned}
\mathrm{K}_1(X) \xrightarrow{\rho} \bigoplus_p H^{2p-1}_{\mathcal{D}_{\mathrm{l,ll,a}}}(X, p) \xrightarrow{\mathrm{a}} \widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}}) \xrightarrow{\mathrm{v}+\mathrm{ch}}&\\
\mathrm{K}_0(X) \oplus \mathrm{Z}\mathcal{D}_{\mathrm{l,ll,a}}(X) \to \bigoplus_p H^{2p}_{\mathcal{D}_{\mathrm{l,ll,a}}}(X, p) \to 0.&
\end{aligned} \tag{5.7}$$

In these exact sequences the map ρ *is the composition*

$$\mathrm{K}_1(X) \to \bigoplus_p H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) \to \bigoplus_p H^{2p-1}_{\mathcal{D}_{\mathrm{l,ll,a}}}(X, p) \subseteq \widetilde{\mathcal{D}}_{\mathrm{l,ll,a}}(X),$$

where the first map is Beilinson's regulator.

(v) *The Chern character*

$$\widehat{\mathrm{ch}} : \widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}}) \otimes \mathbb{Q} \longrightarrow \bigoplus \widehat{\mathrm{CH}}^p_{\mathbb{Q}}(X, \mathcal{D}_{\mathrm{l,ll,a}})$$

is a ring isomorphism. □

5.3 Variant for non regular arithmetic varieties

Since there is no general theorem of resolution of singularities, it is useful to extend the theory of arithmetic Chow groups to the case of non regular arithmetic varieties.

Arithmetic Chow groups for non regular arithmetic varieties. Let (A, Σ, F_∞) be an arithmetic ring with fraction field F. We will assume that A is equidimensional and Jacobson. In contrast to the rest of the paper, in this section, an arithmetic variety over A will be a scheme X that is quasi-projective and flat over $\mathrm{Spec}(A)$, and such that the generic fiber X_F is smooth, but that not need be regular. Since X_F is smooth, the analytic variety X_Σ is a disjoint union of connected components X_i that are equidimensional of dimension d_i. For every cohomological complex of sheaves $\mathcal{F}^*(*)$ on X_Σ we write

$$\mathcal{F}_n(p)(U) = \bigoplus_i \mathcal{F}^{2d_i - n}(d_i - p)(U \cap X_i).$$

Then, the definition of Green objects and of arithmetic Chow groups of [10] can easily be adapted to the grading by dimension.
In this way we can define, for X regular, homological Chow groups with respect to any $\mathcal{D}_{\log}$-complex $\mathcal{C}$. These homological Chow groups will be denoted by $\widehat{\mathrm{CH}}_*(X, \mathcal{C})$. In particular, we are interested in the groups $\widehat{\mathrm{CH}}_*(X, \mathcal{D}_{\mathrm{l,ll,a}})$. But now we can proceed as in [18] and we can extend the definition to the case of non regular arithmetic varieties.

Basic properties of homological Chow groups. Following [18], we can extend some of the properties of the arithmetic Chow groups to the non regular case. The proof of the next results are as in [18], 2.2.7, 2.3.1, and 2.4.2 for the algebraic cycles, but using the techniques of [10] for the Green objects.

Theorem 5.8. *Let $f : X \longrightarrow Y$ be a morphism of irreducible arithmetic varieties over A which is flat or l.c.i. Let D_Y be a normal crossing divisor on $Y_{\mathbb{R}}$ and D_X a normal crossing divisor on $X_{\mathbb{R}}$ such that $f^{-1}(D_Y) \subseteq D_X$. Write $\underline{X} = (X_{\mathbb{R}}, D_X)$ and $\underline{Y} = (Y_{\mathbb{R}}, D_Y)$. Then, there is defined an inverse image morphism*

$$f^* : \widehat{\mathrm{CH}}_p(Y, \mathcal{D}_{\mathrm{l,ll,a}}) \longrightarrow \widehat{\mathrm{CH}}_{p+d}(X, \mathcal{D}_{\mathrm{l,ll,a}}),$$

where d is the relative dimension. □

Theorem 5.9. *Let $f : X \longrightarrow Y$ be a map of arithmetic varieties with Y regular. Let D_Y be a normal crossing divisor on $Y_{\mathbb{R}}$ and D_X a normal crossing divisor* *on $X_{\mathbb{R}}$ such that $f^{-1}(D_Y) \subseteq D_X$. Write $\underline{X} = (X_{\mathbb{R}}, D_X)$ and $\underline{Y} = (Y_{\mathbb{R}}, D_Y)$. Then, there is a cap product*

$$\widehat{\mathrm{CH}}^p(Y, \mathcal{D}_{\mathrm{l,ll,a}}) \otimes \widehat{\mathrm{CH}}_q(X, \mathcal{D}_{\mathrm{l,ll,a}}) \longrightarrow \widehat{\mathrm{CH}}_{q-p}(X, \mathcal{D}_{\mathrm{l,ll,a}});$$

for $x \in \widehat{\mathrm{CH}}^p(Y, \mathcal{D}_{\mathrm{l,ll,a}})$ and $y \in \widehat{\mathrm{CH}}_q(X, \mathcal{D}_{\mathrm{l,ll,a}})$ we denote it by $x._f y$. This cap product turns $\widehat{\mathrm{CH}}_*(X, \mathcal{D}_{\mathrm{l,ll,a}})$ into a graded $\widehat{\mathrm{CH}}^*(Y, \mathcal{D}_{\mathrm{l,ll,a}})$-module. Moreover, it is compatible with inverse images (when defined). □

Arithmetic K-theory. The definition of arithmetic K-theory carries over to the case of non regular arithmetic varieties without modification (see [18], 2.4.2). Thus, we obtain a contravariant functor $(X, D) \longmapsto \widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}})$ from arithmetic varieties with a fixed normal crossing divisor to rings.

Theorem 5.10. *Let X be an arithmetic variety. Let D_X a normal crossing divisor on $X_{\mathbb{R}}$. Write $\underline{X} = (X_{\mathbb{R}}, D_X)$. Then, there is a biadditive pairing*

$$\widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}}) \otimes \widehat{\mathrm{CH}}_*(X, \mathcal{D}_{\mathrm{l,ll,a}}) \longrightarrow \widehat{\mathrm{CH}}_*(X, \mathcal{D}_{\mathrm{l,ll,a}})_{\mathbb{Q}},$$

which we write as $\alpha \otimes x \mapsto \widehat{\mathrm{ch}}(\alpha) \cap x$, with the following properties

(i) Let $f : X \longrightarrow Y$ be a morphism of arithmetic varieties, with Y regular. Let D_Y be a normal crossing divisor on $Y_{\mathbb{R}}$ such that $f^{-1}(D_Y) \subseteq D_X$. Write $\underline{Y} = (Y_{\mathbb{R}}, D_Y)$. If $\alpha \in \widehat{\mathrm{K}}_0(Y, \mathcal{D}_{\mathrm{l,ll,a}})$ and $x \in \widehat{\mathrm{CH}}_*(X, \mathcal{D}_{\mathrm{l,ll,a}})$, then

$$\widehat{\mathrm{ch}}(f^*\alpha) \cap x = \widehat{\mathrm{ch}}(\alpha)._f x.$$

(ii) If $(0, \eta) \in \widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}})$ and $x \in \widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\mathrm{l,ll,a}})$, then

$$\widehat{\mathrm{ch}}((0, \eta)) \cap x = \mathrm{a}(\eta\omega(x)).$$

(iii) If $\alpha \in \widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}})$ and $x \in \widehat{\mathrm{CH}}_*(X, \mathcal{D}_{\mathrm{l,ll,a}})$, then

$$\omega(\widehat{\mathrm{ch}}(\alpha) \cap x) = \mathrm{ch}(\alpha) \wedge \omega(x).$$

(iv) The pairing makes $\widehat{\mathrm{CH}}_*(X, \mathcal{D}_{\mathrm{l,ll,a}})_{\mathbb{Q}}$ into a $\widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}})$-module, i.e., for all $\alpha, \beta \in \widehat{\mathrm{K}}_0(X)$, and $x \in \widehat{\mathrm{CH}}^*(X)$, we have

$$\widehat{\mathrm{ch}}(\alpha) \cap (\widehat{\mathrm{ch}}(\beta) \cap x) = \widehat{\mathrm{ch}}(\alpha\beta) \cap x.$$

(v) If $f : X \longrightarrow Y$ is a flat or l.c.i. morphism of arithmetic varieties, let $\alpha \in \widehat{\mathrm{K}}_0(X, \mathcal{D}_{\mathrm{l,ll,a}})$ and $x \in \widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\mathrm{l,ll,a}})$. Then

$$\widehat{\mathrm{ch}}(f^*\alpha) \cap f^*x = f^*(\widehat{\mathrm{ch}}(\alpha) \cap x).$$

Proof. Follow [18], 2.4.2, but using theorem 4.64 to prove the independence of the choices. □

5.4 Some remarks on the properties of $\widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\mathrm{l,ll,a}})$

In [31], V. Maillot and D. Roessler have announced a preliminary version of the theory developed in this paper. The final theory has some minor differences that do not affect the heart of [31]. The aim of this section is to compare both theories.

We fix an arithmetic ring (A, Σ, F_∞), and we consider pairs $\underline{X} = (X, D)$, where X is an arithmetic variety over A and D is a normal crossing divisor of X_Σ, invariant under F_∞.

A log-singular hermitian vector bundle $\overline{E}$ is a pair (E, h), where E is a vector bundle over X and h is a hermitian metric on E_Σ, invariant under F_∞ and log-singular along D. Observe that the notion of log-singular hermitian metric is not the same as the notion of good hermitian metric. This is not important by two reasons. First, as we will see in the next section, the main examples of good hermitian vector bundles, the fully decomposed automorphic vector bundles, are good and log-singular. Second, if one insists in using good hermitian vector bundles, one can replace pre-log and pre-log-log forms by log and log-log forms to obtain an analogous theory. This alternative theory has worse cohomological properties (we have not proven the Poincaré lemma for pre-log and pre-log-log forms), but the arithmetic intersection numbers computed by both theories agree.

To each pair $\underline{X} = (X, D)$, we have assigned an $\mathbb{N}$-graded abelian group $\widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\mathrm{l,ll,a}})$ that satisfies, among others, the following properties:

(i) The group $\widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\mathrm{l,ll,a}})$ is equipped with an associative, commutative and unitary ring structure, compatible with the grading.

(ii) If X is proper over $\operatorname{Spec} A$, there is a direct image group homomorphism $f_* : \widehat{\mathrm{CH}}^{d+1}(X, \mathcal{D}_{\mathrm{l,ll,a}}) \longrightarrow \widehat{\mathrm{CH}}^1(\operatorname{Spec} A)$, where d is the relative dimension.

(iii) For every integer $r \geq 0$ and every log-singular hermitian vector bundle there is defined the arithmetic r-th Chern class $\widehat{\mathrm{c}}_r(\overline{E}) \in \widehat{\mathrm{CH}}^r(X, \mathcal{D}_{\mathrm{l,ll,a}})$.

(iv) Let $g : X \longrightarrow Y$ be a morphism of arithmetic varieties over A, and let D and E be normal crossing divisors on $X_\mathbb{R}$ and $Y_\mathbb{R}$, respectively, such that $g^{-1}(E) \subseteq D$. Write $\underline{X} = (X_\mathbb{R}, D)$ and $\underline{Y} = (Y_\mathbb{R}, E)$. Then, there is defined an inverse image morphism

$$g^* : \widehat{\mathrm{CH}}^*(Y, \mathcal{D}_{\mathrm{l,ll,a}}) \longrightarrow \widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\mathrm{l,ll,a}}).$$

Moreover, it is a morphism of rings after tensoring with $\mathbb{Q}$.

(v) For every $r \geq 0$, it holds the equality $g^*(\widehat{\mathrm{c}}_r(\overline{E})) = \widehat{\mathrm{c}}_r(g^*(\overline{E}))$.

(vi) There is a forgetful morphism $\zeta : \widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\mathrm{l,ll,a}}) \longrightarrow \mathrm{CH}^*(X)$, compatible with inverse images and Chern classes.

(vii) There is a complex of groups

$$H_{\mathcal{D}}^{2p-1}(X_{\mathbb{R}},\mathbb{R}(p)) \xrightarrow{\mathrm{a}} \widehat{\mathrm{CH}}^{p}(X,\mathcal{D}_{\mathrm{l,ll,a}}) \xrightarrow{(\zeta,\omega)} \mathrm{CH}^{p}(X) \oplus \mathrm{Z}\mathcal{D}_{\mathrm{l,ll,a}}^{2p}(X,p)$$

that is an exact sequence when X_{Σ} is projective. Observe that the group $\mathrm{Z}\mathcal{D}_{\mathrm{l,ll,a}}^{2p}(X,p)$ does not agree with the group denoted by $\mathrm{Z}^{p,p}(X(\mathbb{C}),D)$ in [31], §1 (7). The former is made of forms that are log-log along D and the latter by forms that are good along D. Again, this is not important by two reasons. First, the image by ω of the arithmetic Chern classes of fully decomposed automorphic vector bundles lies in the intersection of the good and log-log forms. Second, the complex of log-log forms shares all the important properties of the complex of good forms (see proposition 2.26).

(viii) The morphism (ζ,ω) is a ring homomorphism; the image of a is a square zero ideal. Moreover, it holds the equality

$$\mathrm{a}(x)\cdot y = \mathrm{a}(x\cdot \mathrm{cl}(\zeta(y))),$$

where $x \in H_{\mathcal{D}}^{2p-1}(X_{\mathbb{R}},\mathbb{R}(p))$, $y \in \widehat{\mathrm{CH}}^{p}(X,\mathcal{D}_{\mathrm{l,ll,a}})$, cl is the class map, the product on the left hand side is the product in the arithmetic Chow groups and the product on the right hand side is the product in Deligne-Beilinson cohomology.

(ix) When D is empty, there is a canonical isomorphism $\widehat{\mathrm{CH}}^{*}(X,\mathcal{D}_{\mathrm{l,ll,a}}) \longrightarrow \widehat{\mathrm{CH}}^{*}(X)$, compatible with the previously discussed structures. Note that we have dropped the projectivity assumption in [31], §1 (9). Observe, moreover, that, if we use the alternative theory with pre-log-log forms, then this property is not established.

6 Automorphic vector bundles

6.1 Automorphic bundles and log-singular hermitian metrics

Fully decomposed automorphic vector bundles. Let B be a bounded, hermitian, symmetric domain. Then, by definition $B = G/K$, where G is a semi-simple adjoint group and K is a maximal compact subgroup. Inside the complexification $G_{\mathbb{C}}$ of G, there is a suitable parabolic subgroup of the form $P_{+}\cdot K_{\mathbb{C}}$, with P_{+} its unipotent radical and such that $K = G \cap P_{+}\cdot K_{\mathbb{C}}$ and $G\cdot(P_{+}\cdot K_{\mathbb{C}})$ are open in $G_{\mathbb{C}}$. This induces an open G-equivariant immersion

$$\begin{array}{ccc} B & \overset{\iota}{\hookrightarrow} & \check{B} \\ \| & & \| \\ G/K & \hookrightarrow & G_{\mathbb{C}}/P_{+}\cdot K_{\mathbb{C}}. \end{array}$$

Here, $\check{B} = G_{\mathbb{C}}/P_+ \cdot K_{\mathbb{C}}$ is a projective rational variety, and the immersion ι is compatible with the complex structure of B.
Let $\sigma : K \longrightarrow \mathrm{GL}(n, \mathbb{C})$ be a representation of K. Then, σ defines a G-equivariant vector bundle E_0 on B. We complexify σ and extend it trivially to $P_+ \cdot K_{\mathbb{C}}$ by letting it kill P_+. Then, σ defines a holomorphic $G_{\mathbb{C}}$-equivariant vector bundle $\check{E}_0$ on $\check{B}$ with $E_0 = \iota^*(\check{E}_0)$. This induces a holomorphic structure on E_0. Observe that different extensions of σ to $P_+ \cdot K_{\mathbb{C}}$ will define different holomorphic structures on E_0.
Let Γ be a neat arithmetic subgroup of G acting on B. Then, $X = \Gamma \backslash B$ is a smooth quasi-projective complex variety, and E_0 defines a holomorphic vector bundle E on X. Following [24], the vector bundles obtained in this way (with σ extended trivially) will be called *fully decomposed automorphic vector bundles*. Since we will not treat more general automorphic vector bundles in this paper, we will just call them automorphic vector bundles.
Let h_0 be a G-equivariant hermitian metric on E_0. Such metrics exist by the compactness of K. Then, h_0 determines a hermitian metric h on E.

Definition 6.1. A hermitian vector bundle (E, h) as above will be called an *automorphic hermitian vector bundle.*

Let $\overline{X}$ be a smooth toroidal compactification of X with $D = \overline{X} \setminus X$ a normal crossing divisor. We recall the following result of Mumford (see [34], theorem 3.1).

Theorem 6.2. *The automorphic vector bundle E admits a unique extension to a vector bundle E_1 over $\overline{X}$ such that h is a singular hermitian metric which is good along D.* □

By abuse of notation, the extension (E_1, h) will also be called an automorphic hermitian vector bundle.
Our task now is to improve slightly Mumford's theorem.

Theorem 6.3. *The automorphic hermitian vector bundle (E_1, h) is a ∞-good hermitian vector bundle; therefore, it is log-singular along D.*

Proof. The proof of this result will take the rest of this section. The technique of proof used follows closely the proof of theorem 3.1 in [34]. Instead of repeating the whole proof of Mumford, we will only point out the results needed to bound all the derivatives of the functions involved.

Cones and Jordan algebras. Let V be a real vector space and let $C \subseteq V$ be a homogeneous self-adjoint cone. We refer to [1] for the theory of homogeneous self-adjoint cones and their relationship with Jordan algebras. We will recall here only some basic facts.
Let $G \subseteq \mathrm{GL}(V)$ be the group of linear maps that preserve C. Since C is homogeneous, G acts transitively on C. We will denote by $\mathfrak{g}$ the Lie algebra of

G. For any point $x \in C$, let $K_x = \mathrm{Stab}(x)$. It is a maximal compact subgroup of G. Let $\mathfrak{k}_x$ be the Lie algebra of K_x and let

$$\mathfrak{g} = \mathfrak{k}_x \oplus \mathfrak{p}_x$$

be the associated Cartan decomposition. Let σ_x be the Cartan involution. Let us choose a point $e \in C$. Let $\langle\, ,\, \rangle = \langle\, ,\, \rangle_e$ be a positive definite scalar product such that $\sigma_e(g) = {}^t g^{-1}$ for all $g \in G$. Then, C is self-adjoint with respect to this inner product. For any point $x \in C$, let us choose $g \in G$ such that $x = ge$. We will identify V with $T_{C,x}$. For $t_1, t_2 \in V$, we will write

$$\langle t_1, t_2 \rangle_x = \langle g^{-1} t_1, g^{-1} t_2 \rangle_e.$$

The right hand side is independent of g because $\langle\, ,\, \rangle_e$ is K_e-invariant. These products define a G-invariant Riemannian metric on C, which is denoted by ds^2_C.

The elements of $\mathfrak{g}$ act on V by endomorphisms. This action can be seen as the differential of the G action at $e \in V$, or given by the inclusion $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. For any $x \in C$ there are isomorphisms

$$\mathfrak{p}_x \xrightarrow{\cong} \mathfrak{p}_x . x = V \quad \text{and} \quad P_x = \exp(\mathfrak{p}_x) \xrightarrow{\cong} P_x . x = C.$$

The elements of $\mathfrak{p}_x$ act on V by self-adjoint endomorphisms with respect to $\langle\, ,\, \rangle_x$.

Every $\mathfrak{p}_x$ has a structure of Jordan algebra defined by

$$(\pi . \pi') . x = \pi . (\pi' . x).$$

The isomorphism $\mathfrak{p}_x \longrightarrow V$ defines a Jordan algebra structure on V, which we denote by $t_1 \underset{x}{.} t_2$. Observe that x is the unit element for this Jordan algebra structure.

We summarize the compatibility relations between the objects defined so far and the action of the group. Let $x = g.e$:

$$\begin{aligned} K_x &= \mathrm{Ad}(g) K_e = g K_e g^{-1}, \\ \mathfrak{k}_x &= \mathrm{ad}(g) \mathfrak{k}_e = g \mathfrak{k}_e g^{-1}, \\ \mathfrak{p}_x &= \mathrm{ad}(g) \mathfrak{p}_e = g \mathfrak{p}_e g^{-1}. \end{aligned}$$

There is a commutative diagram

$$\begin{tikzcd} \mathfrak{p}_e \arrow[r, "\mathrm{ad}(g)"] \arrow[d, ".e"'] & \mathfrak{p}_x \arrow[d, ".x"] \\ V \arrow[r, "g."] & V \end{tikzcd}$$

The horizontal arrows in the above diagram are morphisms of Jordan algebras. In particular

$$g.(t_1 \underset{e}{.} t_2) = g t_1 \underset{x}{.} g t_2.$$

When a unit element e is chosen, we will write $t_1.t_2$ and $\langle\, ,\, \rangle$ instead of $t_1 \underset{e}{.} t_2$ and $\langle\, ,\, \rangle_e$.

Derivatives with respect to the base point. We now study the derivatives of the scalar product and the Jordan algebra product when we move the base point.

Lemma 6.4. *Let $t_1, t_2, t_3 \in V$. Then, we have*

(i) $D_{t_1}(\langle t_2, x^{-1}\rangle_e) = -\langle t_2, t_1\rangle_x$.

(ii) $D_{t_3}\langle t_1, t_2\rangle_x = -(\langle t_3 \underset{x}{.} t_1, t_2\rangle_x + \langle t_1, t_3 \underset{x}{.} t_2\rangle_x) = -2\langle t_1, t_3 \underset{x}{.} t_2\rangle_x$.

(iii) $D_{t_3}(t_1 \underset{x}{.} t_2) = -((t_3 \underset{x}{.} t_1) \underset{x}{.} t_2 + t_1 \underset{x}{.} (t_3 \underset{x}{.} t_2))$.

Proof. The proof of 1 is in [34], p. 244. To prove 2, write $t_3 = M.x$ with $M \in \mathfrak{p}_x$. Then, $\alpha(\delta) = \exp(\delta M).x$ is a curve with $\alpha(0) = x$ and $\alpha'(0) = t_3$. Therefore, we find

$$\begin{aligned} D_{t_3}\langle t_1, t_2\rangle_x &= \frac{\mathrm{d}}{\mathrm{d}\,\delta}\langle t_1, t_2\rangle_{\exp(\delta M).x}|_{\delta=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\,\delta}\langle \exp(\delta M)^{-1}.t_1, \exp(\delta M)^{-1}.t_2\rangle_x|_{\delta=0} \\ &= -(\langle M.t_1, t_2\rangle_x + \langle t_1, M.t_2\rangle_x) \\ &= -(\langle t_3.t_1, t_2\rangle_x + \langle t_1, t_3.t_2\rangle_x). \end{aligned}$$

The second equality of 2 follows from the fact that M acts by an endomorphism which is self-adjoint with respect to $\langle\, ,\, \rangle_x$.
The proof of 3 is completely analogous. □

We will denote by $\|\;\|_x$ the norm associated to the inner product $\langle\, ,\, \rangle_x$.

Lemma 6.5. *There is a constant $K > 0$ such that, for all $x \in C$ and $t_1, t_2 \in V$,*

$$\|t_1 \underset{x}{.} t_2\|_x \leq K\|t_1\|_x\|t_2\|_x.$$

Proof. On $\mathfrak{p}_e$ we may define the norm

$$\|M\|'_e = \sup_{t\in V}\frac{\|M.t\|_e}{\|t\|_e}.$$

Via the isomorphism $\mathfrak{p}_e \longrightarrow V$ it induces a norm on V given by

$$\|t_1\|'_e = \sup_{t\in V}\frac{\|t_1 \underset{e}{.} t\|_e}{\|t\|_e}.$$

Since any two norms in a finite dimensional vector space are equivalent, there is a constant $K > 0$ such that

$$\|t\|'_e \leq K\|t\|_e$$

for all t. Therefore, we get

$$\|t_1 \underset{e}{.} t_2\|_e \leq \|t_1\|'_e \|t_2\|_e \leq K\|t_1\|_e\|t_2\|_e.$$

But for any $x = ge$, we have

$$\|t_1 \underset{x}{.} t_2\|_x = \|g^{-1}t_1 \underset{e}{.} g^{-1}t_2\|_e \leq K\|g^{-1}t_1\|_e\|g^{-1}t_2\|_e = K\|t_1\|_x\|t_2\|_x.$$

□

Maximal $\mathbb{R}$-split torus. We fix a unit element $e \in C$. This fixes also the Jordan algebra structure of V, and we write $K = K_e$ and $\mathfrak{p} = \mathfrak{p}_e$. Let $A \subseteq \exp(\mathfrak{p})$ be a maximal $\mathbb{R}$-*split* torus with $A = \exp(\mathfrak{a})$. Then, $\exp(\mathfrak{p}) = K.A.K^{-1}$ and $C = K.A.e$. A useful result, which is proven in [1], II, §3, is the following

Proposition 6.6. *There exist a maximal set of mutually orthogonal idempotents $\epsilon_1, \dots, \epsilon_r$ of V with $e = \epsilon_1 + \ldots + \epsilon_r$ such that*

$$\mathfrak{a}.e = \sum_{i=1}^{r} \mathbb{R}\epsilon_i \text{ and } A.e = \sum_{i=1}^{r} \mathbb{R}^+\epsilon_i.$$

Moreover, $C \cap \mathfrak{a}.e = A.e$. □

On A, we can introduce the coordinates given by

$$A \cong A.e = \sum_{i=1}^{r} \mathbb{R}^+\epsilon_i \cong (\mathbb{R}^+)^r.$$

As an application of the previous result we prove a bound for the norm of x^{-1}.

Lemma 6.7. *Let $\sigma \in C$. There exists a constant K such that $\|x^{-1}\| \leq K$ for all $x \in \sigma + C$.*

Proof. Since $\bigcup_{\lambda>0}(\lambda e + C) = C$, we may assume that $\sigma = \lambda e$ for some $\lambda > 0$. Since K is compact and

$$\lambda e + C = K(\lambda e + A.e),$$

it is enough to bound x^{-1} for $x \in \lambda e + A.e$. If $x \in \lambda e + A.e$, then we can write, using the above coordinates of A, $x = a.e$ with $a = (a_1, \dots, a_r)$ and all $a_i \geq \lambda$. Then, $x^{-1} = a^{-1}.e$. Since on a finite dimensional vector space any two norms are equivalent, we obtain

$$\|x^{-1}\|^2 \leq K_1(a_1^{-2} + \ldots + a_k^{-2}) \leq K_2/\lambda^2.$$

□

EQUIVARIANT SYMMETRIC REPRESENTATIONS. Let C_n be the cone of positive definite $n \times n$ hermitian matrices. An equivariant symmetric representation of dimension n is a pair (ρ, H), where $\rho : G \longrightarrow \mathrm{GL}(n, \mathbb{C})$ is a representation and $H : C \longrightarrow C_n$ is a map such that

(i) (Equivariance) $H(gx) = \rho(g)H(x)^t\overline{\rho(g)}$ for all $x \in C$, $g \in G$.

(ii) (Symmetry) $\rho(g^*) = H(e).{}^t\overline{\rho(g)}^{-1}.H(e)^{-1}$ for all $g \in G$.

We will consider an equivariant symmetric representation (ρ, H_t) with H_t depending differentiably on a parameter $t \in T$ with T compact as in [34], pp. 245, 246.

BOUNDS OF H AND $\det H^{-1}$. The first step is to bound the entries of H_t and $\det H(t)^{-1}$. This is done in [34], proposition 2.3.

PROPOSITION 6.8. *For all $\sigma \in C$, there is a constant $K > 0$ and an integer N such that*

$$\|H_t(x)\|,\ |\det H_t(x)|^{-1} \leq K\langle x, x\rangle^N \text{ for all } x \in \sigma + C.$$

□

The following results of Mumford (see [34], propositions 2.4 and 2.5) are the starting point to bound the entries of $D_v H_t.H_t^{-1}$; they will also be used to bound the derivatives of H_t.

PROPOSITION 6.9. *Let $\xi \in V$. For all $1 \leq \alpha, \beta \leq n$, let $(D_\xi H_t.H_t^{-1})_{\alpha,\beta}$ be the (α, β)-th entry of this matrix. There is a linear map*

$$C_{\alpha\beta,t} : V \longrightarrow V$$

depending differentiably on t such that

$$(D_\xi H_t.H^{-1})_{\alpha\beta}(x) = \langle C_{\alpha\beta,t}(\xi), x^{-1}\rangle.$$

Moreover, $C_{\alpha\beta,t}$ has the property

$$\left.\begin{matrix} \xi, \eta \in \overline{C} \\ \langle \xi, \eta\rangle = 0 \end{matrix}\right\} \Rightarrow \langle C_{\alpha\beta,t}(\xi), \eta\rangle = 0.$$

□

PROPOSITION 6.10. *For all vector fields δ on T, $\delta H_t.H_t^{-1}(x)$ is independent of x.* □

PROPOSITION 6.11. *Let $\sigma \in C$, let P be a differential operator on T and let $\xi_1, \dots, \xi_d \in V$. Then, there is a constant $K > 0$ and an integer N such that*

$$\|D_{\xi_1} \dots D_{\xi_d} P H_t(x)\|,\ |D_{\xi_1} \dots D_{\xi_d} P \det H_t(x)| \leq K\langle x, x\rangle^N \quad (x \in \sigma + C).$$

Proof. In view of proposition 6.10 and since T is compact, it is enough to consider the case $P = \mathrm{id}$. Now, by proposition 6.9 and the fact that

$$D_{\xi_i} x^{-1} = -x^{-1}.(x^{-1}.\xi_i),$$

we can prove by induction that

$$D_{\xi_1} \dots D_{\xi_d} H_t(x) = M(C(\xi_1, \dots, \xi_d, x)).H_t(x),$$

where $M : V \longrightarrow M_n(\mathbb{C})$ is linear and $C : V \longrightarrow V$ is linear on $\xi_1, \dots, \xi_d$ and polynomial in x^{-1}.
Then, the proposition follows from proposition 6.8 and lemma 6.7. □

Bounds of $\delta H.H^{-1}$. Let $e = \epsilon_1 + \ldots + \epsilon_r$ be a maximal set of orthogonal idempotents, and let A be the corresponding $\mathbb{R}$-split maximal torus. Let C_i be the boundary component containing $\epsilon_{i+1} + \ldots + \epsilon_r$ (see [1], II, §3). Let $\widetilde{C} = C \cup C_1 \cup \ldots \cup C_r \cup 0$ and let P be the parabolic subgroup stabilizing the flag $\{C_i\}$.
In order to be able to use proposition 6.9 to bound $D_v H_t.H_t^{-1}$ and its derivatives, we will need the following result (see [34], proposition 2.6).

Proposition 6.12. *Let $\xi_1, \xi_2 \in \widetilde{C}$, and let $\xi_1' \in V$ satisfy*

$$\left.\begin{array}{r} \eta \in \overline{C} \\ \langle \xi_1, \eta \rangle = 0 \end{array}\right\} \Rightarrow \langle \xi_1', \eta \rangle = 0.$$

Then, for every compact subset $\omega \subseteq P$, there is a constant $K > 0$ such that

(i) $|\langle \xi_1', x^{-1} \rangle| \leq K \|\xi_1\|_x$ *for all $x \in \omega.A.e$.*

(ii) $|\langle \xi_1', \xi_2 \rangle| \leq K \|\xi_1\|_x \|\xi_2\|_x$ *for all $x \in \omega.A.e$.*

Now we can bound the derivatives of $D_v H_t.H_t^{-1}$ in terms of the Riemannian metric ds^2_C. Let $N = \dim V$ and let $\xi_1, \dots, \xi_N \in \widetilde{C}$ span V.

Proposition 6.13. *Let δ be a vector field in T, let P be a differential operator, which is a product of vector fields in T, let $(j_i)_{i=1}^n$ be a finite sequence of elements of $\{1, \dots, N\}$, and let ω be a compact subset of P. Then, there is a constant $K > 0$ such that*

$$\|D_{\xi_{j_1}} \dots D_{\xi_{j_n}} P(D_\delta H_t.H_t^{-1})\| \leq K \|\xi_{j_1}\|_x \dots \|\xi_{j_n}\|_x,$$
$$\|D_{\xi_{j_1}} \dots D_{\xi_{j_{n-1}}} P(D_{\xi_{j_n}} H_t.H_t^{-1})\| \leq K \|\xi_{j_1}\|_x \dots \|\xi_{j_n}\|_x$$

for all $x \in \omega.A.e$.

Proof. Since T is compact and in view of proposition 6.10, it is enough to prove the second inequality for $P = \mathrm{id}$. In this case, the lemma follows from propositions 6.12 and 6.9, and lemmas 6.4 and 6.5. □

Let $\sigma \subseteq C$ be the simplicial cone

$$\sigma = \sum_{i=1}^{N} \mathbb{R}^{+}\xi_i.$$

Let $\{l_i\}$ be the dual basis of $\{\xi_i\}$.

Proposition 6.14. *Let δ be a vector field in T, let P be a differential operator, which is a product of vector fields in T, let $(i_j)_{j=1}^{n}$ be a finite sequence of elements of $\{1,\dots,N\}$, and let $a \in \overline{C}$. Then, there is a constant $K > 0$ such that*

$$\left|\prod_{j=2}^{r} D_{\xi_{i_j}} P(D_{\xi_{i_1}} H_t . H_t^{-1}(x))_{\alpha,\beta}\right| \leq \frac{K}{\prod_{j=1}^{r} l_{i_j}(x) - l_{i_j}(a)},$$

$$\left|\prod_{j=1}^{r} D_{\xi_{i_j}} P(\delta H_t . H_t^{-1}(x))_{\alpha,\beta}\right| \leq K$$

for all integers $1 \leq \alpha, \beta \leq n$ and $x \in \mathrm{Int}(\sigma + a)$.

Proof. The proof is as in [34], proposition 2.7, but using proposition 6.13 to estimate the higher derivatives. □

End of the proof. Now the proof of theorem 6.3 goes exactly as the proof of [34], theorem 3.1, but using propositions 6.11 and 6.14 to bound the higher derivatives. □

Remark 6.15. Observe that we really have proven that, if $\{e_1,\dots,e_r\}$ is a holomorphic frame of E_1 and $H = (h_{e_i,e_j})$ is the matrix of h in this frame, then the entries of H and $\det H^{-1}$ are of polynomial growth in the local universal cover (which, by theorem 2.13, is equivalent of being log forms) and that the entries of $\partial H \cdot H^{-1}$ are of logarithmic growth in the local universal cover (which, by theorem 2.30, is stronger than being log-log forms).

6.2 Shimura varieties and automorphic vector bundles

A wealth of examples where the theory developed in this paper can be applied is provided by non-compact Shimura varieties. In fact, the concrete examples developed so far are modular curves (see [30]) and Hilbert modular surfaces (see [6]), which are examples of Shimura varieties of non-compact type.
For an algebraic group G, $G(\mathbb{R})^{+}$ is the identity component of the topological group $G(\mathbb{R})$ and $G(\mathbb{R})_{+}$ is the inverse image of $G^{\mathrm{ad}}(\mathbb{R})^{+}$ in $G(\mathbb{R})$; also $G(\mathbb{Q})^{+} = G(\mathbb{Q}) \cap G(\mathbb{R}))^{+}$ and $G(\mathbb{Q})_{+} = G(\mathbb{Q}) \cap G(\mathbb{R}))^{+}$.

Definition of Shimura varieties. Let $\underline{S}$ be the real algebraic torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. Following Deligne [13] (see also [32]) one considers the data:

(1) G a connected reductive group defined over $\mathbb{Q}$,

(2) X a $G(\mathbb{R})$-conjugacy class of morphisms $h_x : \underline{S} \longrightarrow G_{\mathbb{R}}$ of real algebraic groups $(x \in X)$,

satisfying the properties:

(a) The Hodge structure on $\operatorname{Lie} G_{\mathbb{R}}$ defined by $\operatorname{Ad} \circ h_x$ is of type

$$\{(-1,1),(0,0),(1,-1)\}.$$

(b) The involution $\operatorname{int} h_x(i)$ induces a Cartan involution on the adjoint group $G^{\mathrm{ad}}(\mathbb{R})$.

(c) Let $w : \mathbb{G}_{m,\mathbb{R}} \longrightarrow \underline{S}$ be the canonical conorm map. The weight map $h_x \circ w$ (whose image is central by (a)) is defined over $\mathbb{Q}$.

(d) Let Z'_G be the maximal $\mathbb{Q}$-split torus of Z_G, the center of G. Then, $Z_G(\mathbb{R})/Z'_G(\mathbb{R})$ is compact.

Under the above assumptions X is a product of hermitian symmetric domains corresponding to the simple non-compact factors of $G^{\mathrm{ad}}(\mathbb{R})$. Denote by $\mathbb{A}^f$ the finite adèles of $\mathbb{Q}$ and let $K \subseteq G(\mathbb{A}^f)$ be a neat (see, e.g., [35] for the definition of neat) open compact subgroup. With these data the Shimura variety $M_K(\mathbb{C})$ is defined by

$$M_K(\mathbb{C}) = M_K(G,X)(\mathbb{C}) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}^f)/K.$$

Connected components of Shimura varieties. Let X^+ be a connected component of X, and for each $x \in X^+$, let h'_x be the composite of h_x with $G_{\mathbb{R}} \longrightarrow G^{\mathrm{ad}}_{\mathbb{R}}$. Then, $x \longmapsto h'_x$, identifies X^+ with a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class of morphisms $\underline{S} \longrightarrow G^{\mathrm{ad}}_{\mathbb{R}}$ that satisfy the axioms of a connected Shimura variety. In particular, X^+ is a bounded symmetric domain and X is a finite disjoint union of bounded symmetric domains (indexed by $G(\mathbb{R})/G(\mathbb{R})_+$).
Let $\mathcal{C}$ be a set of representatives of the finite set $G(\mathbb{R})_+\backslash G(\mathbb{A}^f)/K$ and, for each $g \in \mathcal{C}$, let Γ_g be the image in $G^{\mathrm{ad}}(\mathbb{R})^+$ of the subgroup $\Gamma'_g = gKg^{-1} \cap G(\mathbb{Q})_+$ of $G(\mathbb{Q})_+$. Then, Γ_g is a torsion free arithmetic subgroup of $G^{\mathrm{ad}}(\mathbb{R})^+$ and $M_K(\mathbb{C})$ is a finite disjoint union

$$M_K(\mathbb{C}) = \coprod_{g \in \mathcal{C}} \Gamma_g \backslash X^+.$$

The connected component $\Gamma_g\backslash X^+$ will be denoted by M_{Γ_g}.

Algebraic models of Shimura varieties. Every Shimura variety is a quasi-projective variety. It has a "minimal" compactification, the Baily-Borel compactification, which is highly singular. The theory of toroidal compactifications provides us with various other compactifications; among them we can choose non-singular ones whose boundaries are normal crossing divisors. Moreover, it has a model over a number field E, called the reflex field, and the toroidal compactifications are also defined over E (see [35]). This model can be extended to a proper regular model defined over $\mathcal{O}_E[N^{-1}]$, where $\mathcal{O}_E$ is the ring of integers of E and N is a suitable natural number.

Automorphic vector bundles. Let K_x be the subgroup of $G(\mathbb{R})$ stabilizing a point $x \in X$ and let P_x be the parabolic subgroup of $G(\mathbb{C})$ arising from the Cartan decomposition of $\mathrm{Lie}(G)$ associated to K_x. Let $\lambda : K_x \longrightarrow \mathrm{GL}_n$ be a finite dimensional representation of K_x. It can be extended trivially to a representation of P_x and defines a $G(\mathbb{C})$-equivariant vector bundle $\check{V}$ on the compact dual $\check{M}(\mathbb{C}) = G(\mathbb{C})/P_x$. Let $\beta : X \longrightarrow \check{M}(\mathbb{C})$ be the Borel embedding, then $V = \beta^*(\check{V})$ is a $G(\mathbb{R})$-equivariant vector bundle on X. For any neat open compact subgroup $K \subseteq G(\mathbb{A}^f)$ it defines a vector bundle

$$V_K = G(\mathbb{Q})\backslash V \times G(\mathbb{A}^f)/K$$

on the Shimura variety M_K. This vector bundle is algebraic and it is defined over the reflex field E. Following [23], the vector bundles obtained in this way, will be called *fully decomposed automorphic vector bundles.*
The restriction to any component M_{Γ_g} will be denoted by V_{Γ_g}. It is a fully decomposed automorphic vector bundle in the sense of the previous section.

Canonical extensions. Let $M_{K,\Sigma}$ be a smooth toroidal compactification of M_K and let V_K be an automorphic vector bundle on M_K. Then, there exists a canonical extension of V_K to a vector bundle $V_{K,\Sigma}$ over $M_{K,\Sigma}$ (see [34], [32], [21]). This canonical extension can be characterized in terms of an invariant hermitian metric on V.

Let M_K be a Shimura variety defined over the reflex field E. Let $M_{K,\Sigma}$ be a smooth toroidal compactification of M_K defined over E such that $D_E = M_{K,\Sigma} \setminus M_K$ is a normal crossing divisor. Let V_K be an automorphic vector bundle defined over E with canonical extension $V_{K,\Sigma}$. Let h be a $G^{\mathrm{der}}(\mathbb{R})$-invariant hermitian metric on V; it induces a hermitian metric on V_K, also denoted by h. We denote again by h the singular hermitian metric induced on $V_{K,\Sigma}$. Let $\mathcal{M}_{K,\Sigma}$ be a regular model of $M_{K,\Sigma}$ over $\mathcal{O}_E[N^{-1}]$. Assume that $V_{K,\Sigma}$ can be extended to a vector bundle $\mathcal{V}_{K,\Sigma}$ over $\mathcal{M}_{K,\Sigma}$. Then, theorem 6.3 implies

Theorem 6.16. *The pair $(\mathcal{V}_{K,\Sigma}, h)$ is a log-singular hermitian vector bundle on $\mathcal{M}_{K,\Sigma}$.* □

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